

Exam – February 2010 – 3h

- *Written-by-hands documents are allowed.*
- *Printed documents, computers, cellular phones are forbidden.*
- *The text is composed of 4 pages.*
- *Do not worry about the length of the text. It is not necessary to answer all questions to have the maximum mark.*
- *Answer seriously, rigorously and clearly the questions you choose to work.*
- *You may use without proof the results which were proven in the lecture.*
- *Exercises in Part I are independent. Part II, III and IV can be treated independently in any order.*
- *For the correction see my webpage: <http://www.lmpt.univ-tours.fr/~ley> (teaching)*

Notations: In \mathbb{R}^N , we consider the classical Euclidean inner product

$$\langle x, y \rangle = \sum_{i=1}^N x_i y_i \quad \text{for all } x = (x_1, x_2, \dots, x_N), y = (y_1, y_2, \dots, y_N) \in \mathbb{R}^N.$$

The Euclidean norm is written $|\cdot|$ or $\|\cdot\|$:

$$|x| = \|x\| = \langle x, x \rangle^{1/2} = \sqrt{\sum_{i=1}^N x_i^2}.$$

I. Preliminaries exercises

I.1. Let $f : \mathbb{R}^N \rightarrow \mathbb{R}$ be a *bounded* continuous function. For any $\alpha > 0$, we define

$$M_\alpha = \sup_{x \in \mathbb{R}^N} \{f(x) - \alpha \|x\|\} \quad \text{and} \quad M = \sup_{x \in \mathbb{R}^N} \{f(x)\}.$$

Prove that $\lim_{\alpha \rightarrow 0} M_\alpha = M$.

I.2. Let $f : \mathbb{R}^N \rightarrow \mathbb{R}$ be a continuous function. Suppose that f is *not* differentiable at $x_0 \in \mathbb{R}^N$. Prove that either $D^+ f(x_0) = \emptyset$ or $D^- f(x_0) = \emptyset$.

I.3. Let $f : \mathbb{R}^N \rightarrow \mathbb{R}$ be a K -Lipschitz continuous function ($K > 0$), that is,

$$|f(x) - f(y)| \leq K \|x - y\| \quad \text{for all } x, y \in \mathbb{R}^N.$$

Let $\bar{x} \in \mathbb{R}^N$. Prove that: if $\bar{p} \in D^+ f(\bar{x})$, then $\|\bar{p}\| \leq K$.

I.4. Let $\psi : \mathbb{R}^N \times \mathbb{R}^P \rightarrow \mathbb{R}$ be a C^1 function. Let C be a compact subset of \mathbb{R}^P . We define

$$\varphi(x) = \inf_{y \in C} \psi(x, y).$$

Let $x_0 \in \mathbb{R}^N$ $y_0 \in C$ be such that $\varphi(x_0) = \psi(x_0, y_0)$. Prove that

$$D_x \psi(x_0, y_0) \in D^+ \varphi(x_0)$$

($D_x \psi$ denotes the gradient of ψ with respect to the x variable).

II. The distance function is a viscosity solution to the Eikonal equation

Let $\Omega \subset \mathbb{R}^N$ be an *open bounded* subset. We consider the Eikonal equation

$$\begin{cases} \|Du\| = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1)$$

The subset $\partial\Omega = \overline{\Omega} \setminus \text{int}(\Omega)$ is the boundary of Ω , $\overline{\Omega}$ is its closure and $\text{int}(\Omega)$ is its interior. Notice that $\partial\Omega$ and $\overline{\Omega}$ are compact subset of \mathbb{R}^N .

We introduce the distance function to $\partial\Omega$:

$$d_{\partial\Omega}(x) = \inf_{y \in \partial\Omega} \|y - x\|.$$

II.1. Show that $d_{\partial\Omega}$ is 1-Lipschitz continuous in \mathbb{R}^N and that, $d_{\partial\Omega}(x) = 0$ if and only if $x \in \partial\Omega$.

II.2. Let $x \in \Omega$ and $p \in D^+ d_{\partial\Omega}(x)$. Prove that $\|p\| \leq 1$ and conclude that $d_{\partial\Omega}$ is a viscosity subsolution of (1) in Ω .

II.3. Let $x_0 \in \Omega$. We distinguish two cases.

II.3.1. Case 1: there exists a *unique* $y_0 \in \partial\Omega$ such that $d_{\partial\Omega}(x_0) = \|x_0 - y_0\|$. Prove that $d_{\partial\Omega}$ is then differentiable at x_0 and that $\|Dd_{\partial\Omega}(x_0)\| = 1$.

[You may use, without proof, some results of an exercise solved during the lecture.]

II.3.2. Case 2: assume that y_0 in Case 1 is not unique. Prove that $(x_0 - y_0) / \|x_0 - y_0\| \in D^+ d_{\partial\Omega}(x_0)$. Conclude that, either $d_{\partial\Omega}(x_0)$ is differentiable at x_0 with $\|Dd_{\partial\Omega}(x_0)\| = 1$, or $d_{\partial\Omega}$ is not differentiable at x_0 and $D^- d_{\partial\Omega}(x_0) = \emptyset$. Conclude that $d_{\partial\Omega}$ is a supersolution of (1).

[You may use I.4 and I.2.]

II.3.3. Draw a quick picture which illustrates Case 1 and Case 2 in II.3.1 and II.3.2.

II.4. Conclude that $d_{\partial\Omega}$ is a viscosity solution of (1).

III. The distance function is the unique viscosity solution of (1).

We use the notations and the definitions of Section II.

III.1. Write the Hamiltonian $H(x, r, p)$ associated with (1). Does this Hamiltonian satisfy (H1)? (H2)? ((H1) and (H2) are the assumptions introduced in the Section “Uniqueness” in the lecture).

It follows that we cannot apply directly the uniqueness theorem (stationary case) proven in the lecture.

III.2. We make the change of function $v = \Phi(u) = -\exp(-u)$ in (1). Prove that v satisfies a new Hamilton-Jacobi equation

$$\begin{cases} F(v, Dv) = 0 & \text{in } \Omega, \\ v = c & \text{on } \partial\Omega. \end{cases} \quad (2)$$

Give $F(r, p)$ and the constant c .

[You can use the result of the lecture about the change of functions in Hamilton-Jacobi equations without proof. But you have to justify that Φ is an admissible change of function.]

III.3. Prove that the new Hamiltonian F satisfies (H1) and (H2) and conclude that (2) (and therefore (1)) has a unique viscosity solution.

IV. Optimal control

You can apply, without proof, all the results about optimal control which were proven in the lecture.

We consider the controlled ordinary differential equation

$$\begin{cases} \dot{X}_x(s) = \alpha(s) & \text{for } 0 < s \leq T, \\ X_x(0) = x \in \mathbb{R}, \end{cases} \quad (3)$$

where the control $\alpha \in L^\infty([0, T])$ is such that

$$\alpha(s) \in [0, 1] \quad \text{for almost all } s \in [0, T].$$

We consider the control problem of minimizing, among all admissible control, the cost

$$J(x, t, \alpha(\cdot)) = \int_0^t ds + u_0(X_x(t)),$$

where $x \in \mathbb{R}$, $t \in [0, T]$ and u_0 is a given real-valued bounded Lipschitz continuous function. We introduce as usual the value function

$$V(x, t) = \inf_{\alpha \in L^\infty([0, T]), |\alpha| \leq 1 \text{ a.e.}} \{J(x, t, \alpha(\cdot))\}$$

IV.1. Write the general form of the solution of (3). What is the trajectory $X_x(t)$ when $T = 1$, $\alpha(s) = 1$ for $0 \leq s \leq 1/2$ and $\alpha(s) = -s$ for $1/2 < s \leq 1$?

IV.2. Find a Hamilton-Jacobi-Bellman equation such that the value function V is the unique viscosity solution.

IV.3. Check (formally) that $W(x, t) = V(x, t) - t$ is the unique viscosity solution of

$$\begin{cases} \frac{\partial u}{\partial t} + \left| \frac{\partial u}{\partial x} \right| = 0 & \text{in } \mathbb{R} \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}. \end{cases}$$

Deduce a formula for $V(x, t)$.

IV.4. Let $u_0(x) = 1$. What is $V(x, t)$ in this case?

IV.5. Let $u_0(x) = \frac{1}{1+x^2}$. What is $V(x, t)$ in this case?

IV.6. Let $u_0(x) = \sin(x)$. Prove that $V(x, t) = -1 + t$ if $t \geq \pi$.

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CORRECTION of the exam of February 2010

I.1. $\alpha \in [0, +\infty) \mapsto M_\alpha \in \mathbb{R}$ is a nonincreasing function bounded from above by $M = M_0 < +\infty$ since f is bounded. It follows that $\lim_{\alpha \rightarrow 0} M_\alpha$ exists and is less than M . For any $\epsilon > 0$, there exists $x_\epsilon \in \mathbb{R}^N$ such that $M \leq f(x_\epsilon) + \epsilon$. Therefore

$$M \leq f(x_\epsilon) - \alpha \|x_\epsilon\| + \alpha \|x_\epsilon\| + \epsilon \leq M_\alpha + \alpha \|x_\epsilon\| + \epsilon.$$

Sending $\alpha \rightarrow 0$, we get $M \leq \lim_{\alpha \rightarrow 0} M_\alpha + \epsilon$. Sending $\epsilon \rightarrow 0$, we conclude.

I.2. It suffices to prove that, if $D^+f(x_0)$ and $D^-f(x_0)$ are both nonempty, then f is differentiable at x_0 . Suppose that $p \in D^+f(x_0)$ and $q \in D^-f(x_0)$. By definition, for all $h \in \mathbb{R}^N$, we have

$$f(x_0 + h) - f(x_0) - \langle p, h \rangle \leq o(\|h\|) \quad \text{and} \quad f(x_0 + h) - f(x_0) - \langle q, h \rangle \geq o(\|h\|). \quad (4)$$

It follows $\langle q - p, h \rangle \leq o(\|h\|)$. For $h = \lambda|q - p|$ (for small $\lambda > 0$), we get $\|q - p\|^2 \leq o(\lambda)$ which implies, by sending $\lambda \rightarrow 0$, $p = q$. Coming back to (4), we finally obtain $f(x_0 + h) - f(x_0) - \langle p, h \rangle = o(\|h\|)$ which proves that f is differentiable at x_0 with $Df(x_0) = p$.

I.3. Using that $\bar{p} \in D^+f(\bar{x})$ and that f is K -Lipschitz, we have, for $h \in \mathbb{R}^N$,

$$-K\|h\| - \langle \bar{p}, h \rangle \leq f(\bar{x} + h) - f(\bar{x}) - \langle \bar{p}, h \rangle \leq o(\|h\|).$$

If $\bar{p} = 0$, the inequality is obvious. Otherwise, we can choose $h = -\lambda p / \|p\|$ for small $\lambda > 0$. It follows $\lambda(-K + \|p\|) \leq o(\lambda)$. Dividing by λ and sending $\lambda \rightarrow 0$, we obtain the desired inequality.

I.4. Writing that ψ is differentiable with respect to x at x_0 , that $\varphi(x_0) = \psi(x_0, y_0)$ and that $\varphi(x_0 + h) \leq \psi(x_0 + h, y_0)$, we have

$$\varphi(x_0 + h) - \varphi(x_0) - \langle D_x \psi(x_0, y_0), h \rangle \leq \psi(x_0 + h, y_0) - \psi(x_0, y_0) - \langle D_x \psi(x_0, y_0), h \rangle = o(\|h\|).$$

The inequality $\varphi(x_0 + h) - \varphi(x_0) - \langle D_x \psi(x_0, y_0), h \rangle \leq o(\|h\|)$ proves that $D_x \psi(x_0, y_0) \in D^+ \varphi(x_0)$.

II.1. Using that “ $\inf(a) - \inf(b) \leq \sup(a - b)$ ”, we have, for all $x, x' \in \mathbb{R}^N$,

$$d_{\partial\Omega}(x) - d_{\partial\Omega}(x') \leq \inf_{y \in \partial\Omega} \|y - x - (y - x')\| = \|x - x'\|.$$

Let $\bar{x} \in \mathbb{R}^N$. Since $\partial\Omega$ is a compact subset and $y \mapsto \|y - \bar{x}\|$ is continuous in \mathbb{R}^N , there exists $\bar{y} \in \partial\Omega$ such that $d_{\partial\Omega}(\bar{x}) = \|\bar{y} - \bar{x}\|$. It follows that $d_{\partial\Omega}(\bar{x}) = 0$ if and only if $\bar{y} = \bar{x}$, i.e., $\bar{x} \in \partial\Omega$.

II.2. It is an immediate consequence of II.1, I.3 and the equivalent definition of viscosity solution using the sub- and superdifferentials.

II.3.1. We use the exercise we did during the lecture about the properties of the function φ defined in I.4. Since $x_0 \in \Omega$, the function $y \mapsto \|y - x_0\|$ is C^1 in a neighborhood of $\partial\Omega$ (in fact, in any open subset containing $\partial\Omega$ which does not contain x_0). The uniqueness of y_0 for which the minimum is achieved then implies that $d_{\partial\Omega}$ is differentiable at x_0 and

$$Dd_{\partial\Omega}(x_0) = D(\|y_0 - \cdot\|)(x_0) = \frac{x_0 - y_0}{\|x_0 - y_0\|} \quad \text{thus} \quad \|Dd_{\partial\Omega}(x_0)\| = \left\| \frac{x_0 - y_0}{\|x_0 - y_0\|} \right\| = 1.$$

II.3.2. If the minimum is not achieved at some unique y_0 , we saw in the exercise that $d_{\partial\Omega}$ may be not differentiable at x_0 and we cannot proceed as in II.3.1. In this case, $(x, y) \mapsto \|y - x\|$ is C^1 in the open set $B(x_0, \epsilon) \times \mathcal{O}$ (\mathcal{O} is a neighborhood of $\partial\Omega$ and ϵ is small enough in order that $\mathcal{O} \cap B(x_0, \epsilon) = \emptyset$). From I.4., we obtain that $D(\|y_0 - \cdot\|)(x_0) = (x_0 - y_0) / \|x_0 - y_0\|$ belongs to $D^+d_{\partial\Omega}(x_0)$. In particular $D^+d_{\partial\Omega}(x_0)$ is not empty. Now, either $D^-d_{\partial\Omega}(x_0)$ is empty, or $D^-d_{\partial\Omega}(x_0)$ is also not empty. In the first case, the condition “ $d_{\partial\Omega}$ is a supersolution at x_0 ” is automatically fulfilled. In the second case, from I.2, we have that $d_{\partial\Omega}(x_0)$ is differentiable at x_0 with $Dd_{\partial\Omega}(x_0) = (x_0 - y_0) / \|x_0 - y_0\|$. It follows that $\|Dd_{\partial\Omega}(x_0)\| = 1$ which proves, as in II.3.1, that $d_{\partial\Omega}$ is a supersolution at x_0 .

II.3.3. On the picture 1, the open set Ω can be divided in $\Omega = D \cup (\Omega \setminus D)$. For all x_0 in the open set $\Omega \setminus D$, there is a unique $y_0 \in \partial\Omega$ such that $d_{\partial\Omega}(x_0) = \|x_0 - y_0\|$ and $d_{\partial\Omega}$ is differentiable. On the contrary, if x_0 lies on the line D , there are two different $y_0, y'_0 \in \partial\Omega$ such that $d_{\partial\Omega}(x_0) = \|x_0 - y_0\| = \|x_0 - y'_0\|$. At such points x_0 , we can prove that $d_{\partial\Omega}$ is not differentiable.

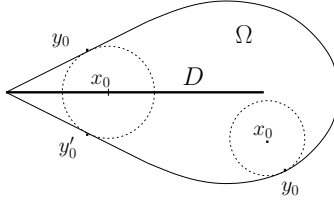


Figure 1: Illustration of the cases in II.3.1 and II.3.2.

II.3.4. From II.2 and II.3, we know that $d_{\partial\Omega}$ is a viscosity solution in Ω . From II.1, we obtain that $d_{\partial\Omega} = 0$ on $\partial\Omega$ and therefore the boundary conditions are satisfied. Finally $d_{\partial\Omega}$ is a viscosity solution of (1).

III.1. The Hamiltonian H associated with (1) is $H(x, r, p) = \|p\| - 1$ for all $x, p \in \mathbb{R}^N$ and $r \in \mathbb{R}$. This Hamiltonian does not depend on x and r . It satisfies obviously (H1) since $H(x, r, p) - H(y, r, p) = 0$ but it does not satisfy (H2) since, if $r \geq s$, then $H(x, r, p) - H(y, s, p) = 0$ (in other word (H2) holds with $\gamma = 0$ whereas one requires $\gamma > 0$ in the uniqueness theorem of the lecture). It follows that we cannot apply directly the uniqueness result of the lecture. We have to extend the uniqueness result to this “degenerate” case. There are several ways to proceed. One, which is left as an exercise, consists in repeating the uniqueness proof of the lecture by considering

$$M_{\mu, \varepsilon} = \sup_{x, y \in \Omega} \left\{ \mu u(x) - v(y) - \frac{\|x - y\|^2}{\varepsilon^2} \right\}$$

for $\varepsilon > 0$ and some fixed $0 < \mu < 1$. The idea is to do an analogous proof with the fixed parameter μ which will help, and to send $\mu \rightarrow 1$ at the end of the proof to conclude. Here we choose an alternative way to overcome the difficulty: we do a change of function.

III.2. The function $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ is C^1 and increasing. It is therefore an admissible change of function (result in the lecture). An easy computation shows that, if $v = -\exp(-u)$, then $Du = -Dv/v$ and therefore, using that $v \leq 0$,

$$\|Du\| - 1 = 0 \Leftrightarrow \left\| \frac{Dv}{v} \right\| - 1 = 0 \Leftrightarrow \|Dv\| + v = 0.$$

Moreover, $u = 0 \Leftrightarrow v = -1$. It follows that v is solution to (2) with $F(x, r, p) = \|p\| + r$ and $c = -1$. The new Hamiltonian F satisfies now clearly (H1) and (H2) (with $\gamma = 1 > 0$). Therefore (2) has at most one solution (uniqueness) but, since, if u is solution to (1), then $\Phi(u)$ is solution to (2); we have also uniqueness for (1). We conclude that (1) has a unique viscosity solution which is $d_{\partial\Omega}$ by Part II.

IV.1. We have (see the lecture), $X_x(t) = x + \int_0^t \alpha(s) ds$. If $T = 1$, $\alpha(s) = 1$ for $0 \leq s \leq 1/2$ and $\alpha(s) = -s$ for $1/2 < s \leq 1$, we have

$$X_x(t) = x + t \quad \text{for } 0 \leq t \leq \frac{1}{2}, \quad X_x(t) = x + \frac{5}{8} - \frac{t^2}{2} \quad \text{for } \frac{1}{2} \leq t \leq 1.$$

The trajectory is absolutely continuous (and even C^1 except at $t = 1/2$).

IV.2. Using the notations of the lecture, we have that: the control set A is $[-1, 1]$ which is a compact subset of \mathbb{R} ; $b(x, \alpha) = \alpha$ is bounded by 1 and obviously Lipschitz continuous with respect to x ; $f(x, \alpha) = 1$ which is obviously bounded and Lipschitz continuous. Finally u_0 is bounded Lipschitz continuous by assumption. From the lecture, we then know that the value function V of the control problem is the *unique viscosity solution* of the Hamilton-Jacobi-Bellman equation

$$\begin{cases} \frac{\partial u}{\partial t} + \left| \frac{\partial u}{\partial x} \right| = 1 & \text{in } \mathbb{R} \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R} \end{cases} \quad (H(x, p) = \sup_{\alpha \in [-1, 1]} \{-b(x, \alpha)p - f(x, \alpha)\} = |p| - 1).$$

IV.3. Set $W(x, t) = V(x, t) - t$. Then

$$\frac{\partial V}{\partial t} = \frac{\partial W}{\partial t} + 1, \quad \frac{\partial V}{\partial x} = \frac{\partial W}{\partial x} \quad \text{and} \quad V(x, 0) = u_0(x) = W(x, 0).$$

It follows that W is the unique viscosity solution of the equation in IV.3. In the lecture, we gave a formula for the unique viscosity solution W of this equation: we have, for all $x \in \mathbb{R}$, $t \in [0, T]$,

$$V(x, t) = W(x, t) + t = \inf_{|y-x| \leq t} \{u_0(y)\} + t. \quad (5)$$

IV.4,5,6. These questions are easy computations from the formula (5). We find $V(x, t) = 1 + t$ when $u_0 = 1$. When $u_0(x) = 1/(1 + x^2)$ (which is increasing on \mathbb{R}_- and decreasing on \mathbb{R}_+), then $V(x, t) = 1/(1 + (x + t)^2) + t$ when $x \geq 0$ and $V(x, t) = 1/(1 + (x - t)^2) + t$ when $x \leq 0$. The last example looks more complicated but the minimum of sinus is -1 and, for any $x_0 \in \mathbb{R}$, there is y_0 such that $|x_0 - y_0| \leq \pi$ and $\sin(y_0) = -1$. It follows from (5) that, if $t \geq \pi$, then $\inf_{|y-x| \leq t} \{\sin(y)\} = -1$ and therefore $V(x, t) = -1 + t$.