Master de Mathématiques–Ho Chi Minh Ville–2011/12
"Optimal control and Hamilton-Jacobi Equations"–O.Ley (IRMAR, INSA de Rennes)

Exam – November 2011 – 4h

– *Written-by-hands documents are allowed.*
– *Printed documents, computers, cellular phones are forbidden.*
– *The text is composed of 4 pages.*
– *The 4 exercises are independent and can be treated in any order. Even in an exercise, most of the questions are independent.*
– *Do not worry about the length of the text. It is not necessary to answer all questions to have the maximum mark.*
– *Answer seriously, rigorously and clearly the questions you choose to work.*
– *You may use* without proof *the results which were proven in the lecture.*
– *For the correction see my webpage:* `http://ley.perso.math.cnrs.fr/` *(teaching)*

Notations: In $\mathbb{R}^N$, we consider the classical Euclidean inner product

$$\langle x, y\rangle = \sum_{i=1}^{N} x_i y_i \quad \text{for all } x = (x_1, x_2, \ldots, x_N), y = (y_1, y_2, \ldots, y_N) \in \mathbb{R}^N.$$

The Euclidean norm is written $|\cdot|$ (or $||\cdot||$): $|x| = ||x|| = \langle x, x\rangle^{1/2} = (\sum_{i=1}^{N} x_i^2)^{1/2}$.

Exercise I.

You are given two big sheets of metal, one with thickness e and the other with thickness $2e$ (e is fixed). You have to realize a can (a cylinder, see Figure 1) with maximal volume using the sheet with thickness e for the lateral part and the sheet with double tickness $2e$ for the top and the bottom of the can. Moreover the total volume of the metal is prescribed to be α.

I.1. Explain why the problem consists in solving

$$\sup x^2 y \quad \text{under the constraint } 2x^2 + xy = \text{constant}.$$

I.2. Prove that such a can exists.

I.3. Find the radius x and the height y of an optimal can. Is such a can unique?

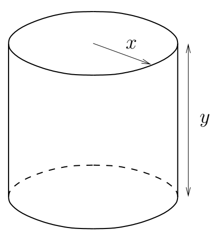

Figure 1: *The can*

Exercise II.
For $\varepsilon, C > 0$ and every $(x,t) \in \mathbb{R}^N \times (0,+\infty)$, we define

$$L^\varepsilon(x,t) = (|x|^2 + \varepsilon^2)^{1/2} - \varepsilon + Ct.$$

II.1. Prove that L^ε is a C^1 function on $\mathbb{R}^N \times (0,+\infty)$ for $\varepsilon > 0$. Is it still true for $\varepsilon = 0$?
II.2. Prove that L^ε, for $\varepsilon > 0$, is a viscosity supersolution of the equation

$$\frac{\partial u}{\partial t} - C|Du| = 0 \quad \text{in } \mathbb{R}^N \times (0,+\infty). \tag{1}$$

II.3. Prove by *two different methods* that L^0 is a viscosity supersolution of (1).
II.4. Prove that L^0 is a viscosity subsolution of (1).
II.5. Let $\psi : \mathbb{R}^N \to \mathbb{R}$ be a C^1 increasing function. Prove that $\psi(L^\varepsilon)$ is still a viscosity supersolution of (1).
II.6. If now $\psi : \mathbb{R}^N \to \mathbb{R}$ is a continuous nondecreasing function, is the result of II.5. still true?
[I do not ask for a complete proof: if you think that the result is not true anymore, explain briefly why. If you think it is still true, explain how you would prove it.]

Exercise III.
We consider the stationary Hamilton-Jacobi equation

$$H(x, u(x), Du(x)) = 0 \quad \text{in } \mathbb{R}^N. \tag{2}$$

We assume that H is coercive with respect to the gradient variable, that is:

$$H(x,r,p) \to +\infty \quad \text{when } |p| \to +\infty, \tag{3}$$

uniformly with respect to $x \in \mathbb{R}^N$ and $r \in [-R,R]$ for any $R > 0$.
We want to prove that every *bounded* continuous viscosity subsolution u of (2) is Lipschitz continuous in $\mathbb{R}^N$.
III.1. Give an example of H which satisfies (3).
III.2. Show that (3) implies that, for every $R > 0$, there exists a constant $C = C(H,R) > 0$ such that

$$\forall x \in \mathbb{R}^N,\ \forall r \in [-R,R],\ \forall p \in \mathbb{R}^N, \quad H(x,r,p) \le 0 \quad \Rightarrow \quad |p| \le C.$$

In order to prove the result, for a fixed $x \in \mathbb{R}^N$, we consider

$$\sup_{y \in \mathbb{R}^N} \{u(y) - K|y-x|\},$$

and we denote $\varphi_{x,K}(y) = K|y-x|$.
III.3. Explain why this supremum is well-defined and is achieved at some $\bar{y}$.

III.4. Prove that the supremum cannot be achieved at some $\bar{y} \neq x$ if K is chosen larger than some $\bar{K}$. Explain that $\bar{K}$ depend on C (see III.2) and $||u||_\infty = \sup_{\mathbb{R}^N} |u|$ but not on x.

[Indication: show that, if $\bar{y} \neq x$, then $\varphi_{x,K}$ is C^1 in a neighborhood of $\bar{y}$ and use $\varphi_{x,K}$ as a test-function for the subsolution u.]

III.5. Write that the supremum is achieved at $\bar{y} = x$ and conclude.

Exercise IV.

We consider the controlled ordinary differential equation

$$\begin{cases} \dot{X}_x(s) = b(X_x(s), \alpha(s)), & s > 0, \\ X_x(0) = x & x \in \mathbb{R}^N, \end{cases} \tag{4}$$

where the control $\alpha(\cdot) \in L^\infty([0, +\infty); \overline{B}(0,1))$ (the set of controls is the closed ball $\overline{B}(0,1)$) and $b \in C(\mathbb{R}^N \times \overline{B}(0,1), \mathbb{R}^N)$ is Lispchitz continuous and bounded with respect to x, that is, there exists $C_b > 0$ such that

$$|b(x, \alpha)| \leq C_b \quad \text{and} \quad |b(x, \alpha) - b(y, \alpha)| \leq C_b |x - y| \quad \text{for all } x, y \in \mathbb{R}^N, \alpha \in \overline{B}(0,1). \tag{5}$$

We recall that, for every $\alpha(\cdot) \in L^\infty([0, +\infty); \overline{B}(0,1))$ and $x \in \mathbb{R}^N$, (4) has a unique solution $X_x \in AC([0, +\infty))$.

We introduce the cost

$$J(x, \alpha(\cdot)) = \int_0^{+\infty} e^{-s} f(X_x(s), \alpha(s)) ds$$

where $f \in C(\mathbb{R}^N \times \overline{B}(0,1), \mathbb{R})$ is Lispchitz continuous with respect to x, that is, there exists $C_f > 0$ such that

$$|f(x, \alpha) - f(y, \alpha)| \leq C_f |x - y| \quad \text{for all } x, y \in \mathbb{R}^N, \alpha \in \overline{B}(0,1). \tag{6}$$

We define the value function of the related infinite horizon problem by

$$V(x) = \inf_{\alpha(\cdot) \in L^\infty([0,+\infty); \overline{B}(0,1))} J(x, \alpha(\cdot)).$$

We admit that Theorems 4 and 6 of the lecture are true (even if the cost f is not bounded with respect to x) that is, V is a viscosity solution of the stationary Hamilton-Jacobi equation

$$H(x, u(x), Du(x)) = 0 \quad \text{in } \mathbb{R}^N, \tag{7}$$

where

$$H(x, r, p) = \sup_{\alpha \in \overline{B}(0,1)} \{-\langle b(x, \alpha), p \rangle + r - f(x, \alpha)\} \quad \text{for all } x \in \mathbb{R}^N, r \in \mathbb{R}, p \in \mathbb{R}^N. \tag{8}$$

We say that a function $u : \mathbb{R}^N \to \mathbb{R}$ has linear growth if u satisfies:

$$\exists C_1, C_2 > 0 \text{ such that } |u(x)| \leq C_1 + C_2|x|. \tag{9}$$

IV.1. Prove that (6) implies that f has linear growth uniformly with respect to α, that is, there exists $C_1, C_2 > 0$ such that $|f(x,\alpha)| \leq C_1 + C_2|x|$ for all $x \in \mathbb{R}^N$, $\alpha \in \overline{B}(0,1)$.

IV.2. Prove that the value function has linear growth (see (9)).

[You can prove that $|X_x(t)| \leq |x| + Ct$ for some $C > 0$ and then use IV.1 to obtain an estimate of V.]

IV.3. Prove that H given by (8) satisfies

$$(H1) \qquad \exists \gamma > 0 \quad \text{such that} \quad H(x,r,p) - H(x,s,p) \geq \gamma(r-s) \quad \text{for all } r \geq s,\ x \in \mathbb{R}^N,\ p \in \mathbb{R}^N;$$

$$(H2) \qquad \exists C > 0 \quad \text{such that} \quad |H(x,r,p) - H(y,r,p)| \leq C(1+|p|)|x-y| \quad \text{for all } x, y \in \mathbb{R}^N,\ r \in \mathbb{R},\ p \in \mathbb{R}^N;$$

$$(H4') \qquad \exists C > 0 \quad \text{such that} \quad |H(x,r,p) - H(x,r,q)| \leq C|p-q| \quad \text{for all } x \in \mathbb{R}^N,\ r \in \mathbb{R},\ p, q \in \mathbb{R}^N.$$

Remark: under the assumptions (H1)-(H2)-(H4'), we can prove as in Theorem 1 of the lecture that (7) has a unique viscosity solution u with linear growth.

IV.4. Prove that V is Lipschitz continuous if $C_b < 1$.

Now we assume that:

$$b(x,\alpha) = B(x) + \alpha$$

with B bounded Lipschitz continuous (with Lispchitz constant $C_B < 1$) and $B(x) = -B(-x)$, $B(0) = 0$;

$$f(x,\alpha) = |x| + |\alpha|^2.$$

IV.5. Compute precisely H given by (8).

IV.6. Show that $V(0) = 0$ and prove by two different methods that $V(x) = V(-x)$ for all $x \in \mathbb{R}^N$.

[1st method: you can start to prove that $J(x,\alpha(\cdot)) = J(-x,-\alpha(\cdot))$; second method: what is the equation satisfied by $V(-x)$? and use uniqueness of the solution with linear growth to (7).]

END

CORRECTION of the exam of November 2011

Exercise I.
I.1. The volume of the can of Figure 1 is
$$V(x,y) = \pi x^2 y.$$

The volume of the metal used to build the can is

$$\underbrace{2}_{\text{top+bottom}} \times \pi x^2 \times \underbrace{2e}_{\text{double thickness}} + 2\pi xye = \alpha.$$

It follows that, to find optimal cans, we have to maximize $f(x,y) = x^2y$ (maximizing x^2y or $\pi x^2 y$ is the same) under the contraint $g(x,y) = 2x^2 + xy - C = 0$ where the constant $C = \alpha/(2\pi e)$. Note that both f and g are C^1 functions.
I.2. We want to prove that there exists a solution to the problem. Let

$$A = \{(x,y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0 \text{ et } 2x^2 + xy = C\}.$$

The set A is not a compact subset of $\mathbb{R}^2$ but setting $a = x$ et $b = xy$ and

$$\tilde{A} = \{(a,b) \in \mathbb{R}^2 : a \geq 0,\ b \geq 0 \text{ et } 2a^2 + b = C\},$$

we obtain a compact subset. Since $f(x,y) = ab$, the initial problem is equivalent to solve

$$\sup_{(a,b)\in\tilde{A}} ab.$$

By compactness and continuity, there exists at least one solution $(\bar{a},\bar{b}) \in \tilde{A}$ to the problem. Note $\bar{a},\bar{b} > 0$ (otherwise $V = 0$ which would be a contradiction) It follows that there exists a solution $(\bar{x},\bar{y})$ to the original problem with $\bar{x} = \bar{a}$ and $\bar{y} = \bar{b}/\bar{x}$.
I.3. We look for necessary conditions of optimality. Since f and g are C^1, if $(\bar{x},\bar{y})$ is a solution, then $g(\bar{x},\bar{y}) = 0$ and there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that $Df(\bar{x},\bar{y}) + \lambda Dg(\bar{x},\bar{y}) = 0$. It leads to the system

$$\begin{cases} 2\bar{x}^2 + \bar{x}\bar{y} & = & C, \\ 2\bar{x}\bar{y} + \lambda(4\bar{x} + \bar{y}) & = & 0, \\ \bar{x}^2 + \lambda\bar{x} & = & 0. \end{cases}$$

The first equation shows that $\bar{x} > 0$. It follows from the last equation that $\lambda = -\bar{x}$. Then we can solve the system finding a unique solution

$$\bar{x} = \frac{1}{2}\sqrt{\frac{\alpha}{3\pi e}} \quad \text{et} \quad \bar{y} = 4\bar{x} = 2\sqrt{\frac{\alpha}{3\pi e}} = \sqrt{\frac{\alpha}{\pi e}}\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right).$$

The necessary conditions give a unique candidate for our problem and, from I.2, we know that there exists a solution. We can conclude that $(\bar{x},\bar{y})$ is the unique solution to our problem, $V = \frac{1}{6\sqrt{3\pi}}\left(\frac{\alpha}{e}\right)^{3/2}$.

Exercise II.
II.1. The function L^ε is C^1 since $x \mapsto |x|^2 + \varepsilon^2$ is C^∞ and positive on $\mathbb{R}^N$ when $\varepsilon > 0$. For $\varepsilon = 0$, $L^0 = |x| + Ct$ which is nonsmooth at $(x,t) = (0,t)$.
II.2. Since L^ε is C^1, we just have to compute that L^ε is a classical supersolution: for every $x \in \mathbb{R}^N$, $t > 0$, we have

$$\frac{\partial u}{\partial t}(x,t) - C|Du(x,t)| = C - C\left|\frac{x}{(|x|^2 + \varepsilon^2)^{1/2}}\right| \geq 0.$$

II.3. *Method 1 (stability).* For every $(x,t) \in \mathbb{R}^N \times (0,+\infty)$,

$$|L^\varepsilon(x,t) - L^0(x,t)| \leq \frac{\varepsilon^2}{(|x|^2 + \varepsilon^2)^{1/2} + |x|} + \varepsilon \leq 2\varepsilon.$$

Therefore, L^ε converges uniformly to L^0 in $\mathbb{R}^N \times (0, +\infty)$. By stability, since L^ε is a supersolution, the limit L^0 is still a subsolution of (1).

Method 2 (direct computation). We check directly that L^0 is a supersolution by using the definition with subdifferentials at the points where L^0 is not differentiable. The function L^0 is C^1 on $(\mathbb{R}^N - \{0\}) \times (0, +\infty)$. On this set,

$$\frac{\partial u}{\partial t}(x,t) - C|Du(x,t)| = C - C|\frac{x}{|x|}| = C - C = 0,$$

hence L^0 is a (classical) solution (thus a viscosity supersolution). Let $(x,t) = (0,t)$. An easy computation shows that the subdifferential of L^0 at $(0,t)$ is $D^- L^0(0,t) = \bar{B}(0,1) \times \{C\}$. For every $p = (p_x, p_t) \in D^- L^0(0,t)$, we have

$$p_t - C|p_x| = C - C|p_x| \geq 0$$

since $|p_x| \leq 1$. Therefore the viscosity inequality for supersolution holds on $\{0\} \times (0, +\infty)$. We can conclude that L^0 is a supersolution everywhere.

II.4. On the set $(\mathbb{R}^N - \{0\}) \times (0, +\infty)$, we proved in II.3 that L^0 is a classical solution. At points $(x,t) = (0,t)$, the superdifferential $D^+ L^0(0,t)$ is empty and therefore the viscosity condition for subsolution is automatically fulfilled. We conclude that L^0 is a subsolution.

II.5. Suppose that, for $\varphi \in C^1(\mathbb{R}^N \times (0, +\infty))$, $\psi \circ L^\varepsilon - \varphi$ achieves a local minimum at some $(x,t) \in \mathbb{R}^N \times (0, +\infty)$ and that $\psi(L^\varepsilon(x,t)) = \varphi(x,t)$. It follows that for (y,s) close enough to (x,t), we have

$$\psi(L^\varepsilon(y,s)) \geq \varphi(y,s) \quad \Longrightarrow \quad L^\varepsilon(y,s) \geq \psi^{-1}(\varphi(y,s)),$$

where ψ^{-1} is the increasing inverse function of the C^1 increasing function ψ. Note that ψ^{-1} is still C^1 with $(\psi^{-1})'(r) = (\psi'(\psi^{-1}(r)))^{-1}$. Therefore $L^\varepsilon - \psi^{-1} \circ \varphi$ achieves a local minimum at (x,t). Writing that L^ε is a supersolution, we have, setting $r = \varphi(x,t)$,

$$(\psi^{-1})'(r)\frac{\partial \varphi}{\partial t}(x,t) - C|(\psi^{-1})'(r) D\varphi(x,t)| \geq 0.$$

Dividing the inequality by $(\psi^{-1})'(r) > 0$, we obtain the viscosity inequality proving that $\psi \circ L^\varepsilon$ is a supersolution at (x,t).

II.6. The result is still true and can be obtained by approximation. Given a nondecreasing function ψ, we find a sequence of C^1 increasing functions $(\psi_n)_n$ converging locally uniformly in $\mathbb{R}$ to ψ. For instance on may take $\psi_n(r) = \psi * \rho_n(r) + \frac{1}{n}\arctan(r)$ (the convolution with a standard C^∞ mollifier ρ_n gives a C^∞ function which is still nondecreasing. The term with arctan ensures that ψ_n is increasing). By II.5, $\psi_n(L^\varepsilon)$ is a supersolution and $\psi_n(L^\varepsilon)$ converges locally uniformly to $\psi(L^\varepsilon)$. We conclude by stability.

Exercise III.

III.1. A classical example of coercive Hamiltonian is $H(x,u,Du) = \lambda u + c(x)|Du|^m$ with $\lambda, m > 0$ and $c \in C(\mathbb{R}^N; \mathbb{R})$ such that $c(x) \geq c_0 > 0$ for all $x \in \mathbb{R}^N$.

III.2. Let $R > 0$. Since $H(x,r,p) \to +\infty$ when $|p| \to +\infty$ uniformly with respect to $x \in \mathbb{R}^N$, $r \in [-R, R]$, by the very definition, there exists $C > 0$ such that, for all $x \in \mathbb{R}^N$, $r \in [-R, R]$ and $p \in \mathbb{R}^N$ such that $|p| \geq C$, we have $H(x,r,p) > 0$. It is equivalent to the result.

III.3. For all $y \in \mathbb{R}^N$,

$$u(y) - K|y-x| \leq ||u||_\infty - K|y-x| \leq ||u||_\infty \tag{10}$$

Therefore the supremum is finite. The function $y \mapsto u(y) - K|y-x|$ is continuous on $\mathbb{R}^N$ and converges to $-\infty$ as $|y| \to +\infty$ by (10). This implies that the supremum is achieved at some $\bar{y} \in \mathbb{R}^N$.

III.4. Assume that the supremum is achieved at $\bar{y} \neq x$. It means that $u - \varphi_{x,K}$ has a local maximum at $\bar{y}$. Moreover, since $|\bar{y} - x| \neq 0$, $\varphi_{x,K}$ is C^1 in a neighborhood of $\bar{y}$. So we can use $\varphi_{x,K}$ as a test-function for the subsolution u at $\bar{y}$ and we obtain

$$H(\bar{y}, u(\bar{y}), D\varphi_{x,K}(\bar{y})) = H(\bar{y}, u(\bar{y}), K\frac{\bar{y}-x}{|\bar{y}-x|}) \leq 0.$$

From III.2, it follows that

$$|K\frac{\bar{y}-x}{|\bar{y}-x|}| = K \leq C(H, ||u||_\infty).$$

If we choose at the beginning $K \geq \bar{K} > C(H, ||u||_\infty)$, we obtain a contradiction. Therefore we cannot have $\bar{y} \neq x$ if K is large enough.

III.5. Choosing $K = \bar{K} > C(H, ||u||_\infty)$, necessarily $\bar{y} = x$ and $\sup_{y \in \mathbb{R}^N}\{u(y) - \bar{K}|y-x|\} = u(x)$. It follows that, for every $y \in \mathbb{R}^N$, $u(y) - u(x) \leq \bar{K}|y-x|$. Since this formula holds for any x (with the same $\bar{K}$ since it is independent of x), this proves that u is $\bar{K}$-Lipschitz continuous.

Exercise IV.

IV.1. From (6), we have $|f(x,\alpha)| \leq |f(0,\alpha)| + C_f|x|$. This implies that (9) holds true with $C_1 = \max_{\overline{B}(0,1)} |f(0,\alpha)|$ (recall that f is continuous) and $C_2 = C_f$.

IV.2. Since b is bounded by C_b, for every control $\alpha(\cdot)$, we have

$$|X_x(t)| - |x| \leq |X_x(t) - x| = |\int_0^t \dot{X}_x(s)ds| \leq \int_0^t |\dot{X}_x(s)|ds = \int_0^t |b(X_x(s), \alpha(s)|ds \leq C_b t.$$

Using IV.1 and the previous computation, it follows that

$$\begin{aligned} V(x) &= \inf_{\alpha(\cdot)} \int_0^{+\infty} e^{-s} f(X_x(s), \alpha(s))ds \leq \inf_{\alpha(\cdot)} \int_0^{+\infty} e^{-s}(C_1 + C_f|X_x(s)|)ds \\ &\leq \int_0^{+\infty} e^{-s}(C_1 + C_f|x| + C_f C_b s)ds = C_1 + C_f C_b + C_f|x|, \end{aligned}$$

which proves that V has linear growth.

IV.3. We recall that

$$H(x,r,p) = r + \sup_{\alpha}\{-\langle b(x,\alpha), p\rangle - f(x,\alpha)\}.$$

It follows that (H1) is obvious with $\gamma = 1 > 0$. Using "sup − sup ≤ sup", we have

$$\begin{aligned} H(x,r,p) - H(y,r,q) &\leq \sup_{\alpha}\{\langle b(y,\alpha), q\rangle - \langle b(x,\alpha), p\rangle + f(y,\alpha) - f(x,\alpha)\} \\ &\leq \sup_{\alpha}\{\langle b(y,\alpha), q - p\rangle + \langle b(y,\alpha) - b(x,\alpha), p\rangle + C_f|x-y|\} \\ &\leq \sup_{\alpha}\{C_b|q-p| + C_b|x-y||p| + C_f|x-y|\} \\ &\leq \max\{C_b, C_f\}(1+|p|)|x-y| + C_b|p-q|, \end{aligned}$$

which proves (H2) and (H3).

IV.4. From Gronwall Inequality, if X_x and X_y are two trajectories with same control $\alpha(\cdot)$ starting from x and y respectively, we have

$$|X_x(t) - X_y(t)| \leq e^{C_b t}|x-y|.$$

Using "inf − inf ≤ sup," we get

$$\begin{aligned} V(x) - V(y) &\leq \sup_{\alpha(\cdot)} \int_0^{+\infty} e^{-s}[f(X_x(s), \alpha(s)) - f(X_y(s), \alpha(s))]ds \\ &\leq \int_0^{+\infty} e^{-s} C_f|X_x(s) - X_y(s)|ds \\ &\leq C_f \int_0^{+\infty} e^{-(1-C_b)s}|x-y|ds = \frac{C_f}{1-C_b}|x-y|, \end{aligned}$$

which gives the conclusion.

IV.5. By Formula (8), solving an easy problem of optimization, we obtain

$$H(x,u,p) = \sup_{|\alpha|\leq 1}\{-\langle B(x) + \alpha, p\rangle + u - |x| - |\alpha|^2\} = \begin{cases} u + \frac{|p|^2}{4} - \langle B(x), p\rangle - |x| & \text{if } |p| \leq 2, \\ u + |p| - 1 - \langle B(x), p\rangle - |x| & \text{if } |p| \geq 2. \end{cases}$$

IV.6. Since the running cost $f \geq 0$, we have $V \geq 0$. But $V(0) \leq J(0,0) = 0$. It follows $V(0) = 0$.
Let $X_{x,\alpha}$ be the trajectory solution to $\dot{X}_{x,\alpha} = B(X_{x,\alpha}) + \alpha(t)$ starting from x and $X_{-x,-\alpha}$ be the trajectory solution to $\dot{X}_{-x,-\alpha} = B(X_{-x,-\alpha}) - \alpha(t)$ starting from $-x$. Since $B(-y) = -B(y)$, we have $-\dot{X}_{-x,-\alpha} = B(-X_{-x,-\alpha}) + \alpha(t)$. So $-X_{-x,-\alpha}$ satisfies the same equation as $X_{x,\alpha}$. By uniqueness $X_{x,\alpha} = -X_{-x,-\alpha}$. Using that $f(x,\alpha) = f(-x,-\alpha)$, it follows $J(x,\alpha) = J(-x,-\alpha)$. Moreover $\{\alpha(\cdot) : \alpha \in L^\infty([0,+\infty); \overline{B}(0,1))\} = \{-\alpha(\cdot) : \alpha \in L^\infty([0,+\infty); \overline{B}(0,1))\}$. We conclude

$$V(x) = \inf_{\alpha(\cdot)} J(x,\alpha) = \inf_{\alpha(\cdot)} J(-x,-\alpha) = \inf_{\alpha(\cdot)} J(-x,\alpha) = V(-x).$$

Another proof: we make the change of function $u(x) = V(-x)$, $Du(x) = -DV(-x)$, in the Hamilton-Jacobi (7). Formally (it is not difficult to write everything rigorously), we have:

$$0 = H(-x, V(-x), DV(-x)) = H(-x, u(x), -Du(x)) = H(x, u(x), Du(x))$$

since $H(x,u,p) = H(-x,u,-p)$ by IV.5. It follows that u is a viscosity solution of the same equation (7) as V and u, V have linear growth. By uniqueness, we obtain $u = V$ so $V(-x) = V(x)$.