

# Interaction with a field: a simple integrable model with backreaction

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## Abstract

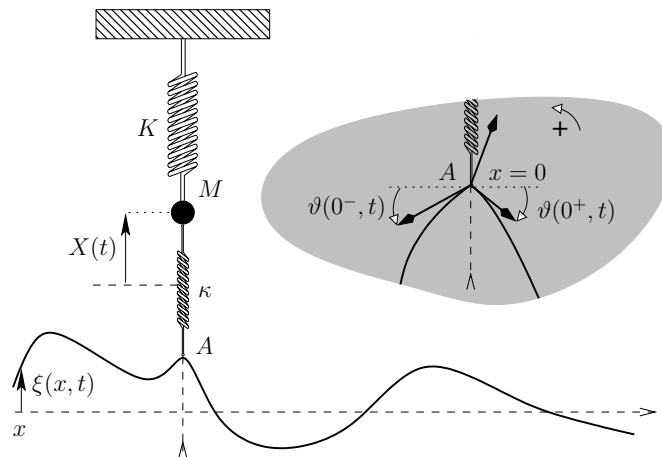
The classical model of an oscillator linearly coupled to a string captures, for a low price in technique, many general features of more realistic models for describing a particle interacting with a field or an atom in an electromagnetic cavity. The scattering matrix and the asymptotic in and out-waves on the string can be computed exactly and the phenomenon of resonant scattering can be introduced in the simplest way. The dissipation induced by the coupling of the oscillator to the string can be studied completely. In the case of a d'Alembert string, the backreaction leads to an Abraham–Lorentz–Dirac-like equation. In the case of a Klein–Gordon string, one can see explicitly how radiation governs the (meta)stability of the (quasi)bounded mode.

## 1. Introduction

Getting back to the spirit of the 19th century—when purely mechanical models were ubiquitous even for understanding systems involving electromagnetic fields—this paper discusses a simple model of an oscillator coupled to a string that presents a host of interesting features in a very accessible way. It will be presented in detail in section 2 (see figure 1). Even though the required technical background is on an upper undergraduate/early graduate level, if one remains within a classical (i.e. non-quantum) context<sup>1</sup>, this model encapsulates many relevant features of physical interest which can be found in more realistic models, for instance when considering the interaction of an atom (the oscillator) with light (the string).

First (section 3), when the string is infinite, it allows one to gain insight into (or to discover for the first time in academic studies) the scattering of waves by a dynamical system. It provides the opportunity to introduce some of the ingredients of scattering theory: when dealing with a one-dimensional mechanical system, the notion of asymptotic states (the so-called ‘in’ and ‘out’ states) is made as simple as possible and the  $S$  matrix takes a particular

<sup>1</sup> In the following, the starred sections indicate a more advanced level and may be omitted in a first reading.

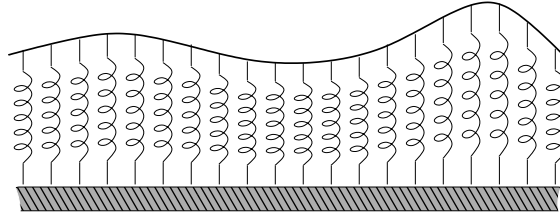


**Figure 1.** Model of a harmonic oscillator coupled to a string via a massless spring. The long-dashed lines refer to the equilibrium positions of the string and the oscillator. The inset shows the three forces that cancel at the massless attachment point  $A$  ( $x = 0$ ): the left and right tensions of the string and the elastic force of the coupling spring.

simple form. Unlike the transmission of light in a Fabry–Perot interferometer, the resonant scattering appears here explicitly in connection with the familiar resonance phenomenon of a forced oscillator and we can also understand how the motion of the oscillator affects in return the external excitation.

Second (section 4), from the point of view of the oscillator, this model constitutes the most elementary example of radiation. It shows concretely how an interaction not only induces a shift in the natural frequency of the oscillator (this can already be seen when only two degrees of freedom are coupled) but also that the coupling to a large number of degrees of freedom (the string being seen as a large collection of oscillators) induces a friction term for the oscillator, although no dissipation exists in the system as a whole (for an electric analogous phenomenon see section 22.6 of Feynman *et al* (1970), Krivine and Lesne (2003)). Indeed, as far as I know, there are very few places where this model is discussed and always in the specialized literature with a d’Alembert string (Sollfrey and Goertzel 1951, Dekker 1985). Some of its variants (Stevens 1961, Rubin 1963, Yurke 1984, 1986, Dekker 1984) are introduced precisely for studying dissipation at a quantum level. As we will see, a Klein–Gordon string allows one to keep one discrete mode (a bounded state) without dissolving it in the continuous spectrum of the string and therefore allows one to mimic the interaction of a field with a stable particle, not just a metastable one. This model is particularly relevant to see how backreaction works: for instance, for a d’Alembert string, the oscillator is governed by an Abraham–Lorentz–Dirac-like equation. Besides, we will be able to illustrate precisely the deep connection between the resonance and the poles of the  $S$  matrix (a major feature in high-energy particle physics and in condensed matter physics).

Before we give some guidelines for further developments in the conclusion (section 6), we will complete our classical study in section 5 by the detailed diagonalization of the Hamiltonian and the discussion of the completeness of the basis of modes that are used to describe the dynamics. This will be the occasion to sketch the finite size effects if one wants to use this model for describing an atom placed in a cavity. This study also prepares the ground for the quantization which will be proposed in a future paper.



**Figure 2.** A mechanical model that leads to a Klein–Gordon equation: the string is attached to an elastic support that can be seen as a tight collection of massless springs uniformly distributed per unit length along the string. For another mechanical example, see Crawford (1968), section 3.5.

## 2. The spring-string model

### 2.1. Description of the model

A very thin homogeneous string of linear mass  $\mu_0$  is considered to have only transverse displacements in one direction. At equilibrium it forms a straight line along the  $x$ -axis with the uniform tension  $T_0$  (we will never take into account the effects of gravity). The string is coupled to a system of mass  $M$  with one degree of freedom connected to a fixed support with a massless spring of stiffness  $K$  (see figure 1). The coupling is modeled by a second massless spring of stiffness  $\kappa$  attached to the string at the massless point  $A$  at  $x = 0$ . All the vibrations will be considered within the harmonic approximation of small amplitudes. We will denote by  $\xi(x, t)$  the transverse displacement at time  $t$  of the string element located at  $x$  at equilibrium. The displacement of  $M$  with respect to its equilibrium position will be denoted by  $X(t)$ .

### 2.2. Equations of motion

The equation of motion of the oscillator is

$$M \frac{d^2 X}{dt^2} + M \Omega_0^2 X = -\kappa (X - \xi_0), \quad (1a)$$

where  $\xi_0(t) \stackrel{\text{def}}{=} \xi(0, t)$  is the displacement of  $A$  and  $\Omega_0 \stackrel{\text{def}}{=} \sqrt{K/M}$  is the harmonic frequency of the free oscillator. For  $x \neq 0$ ,  $\xi$  fulfils the one-dimensional d'Alembert equation

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \xi}{\partial t^2} = 0, \quad (1b)$$

where the wave velocity on the string is given by  $c \stackrel{\text{def}}{=} \sqrt{T_0/\mu_0}$ . The derivation of (1b) is a major step in every introductory course on waves (Crawford 1968, section 2.2, for instance). Less common is perhaps the refinement of sticking the string to a ‘mattress’ (see figure 2) made of  $n$  massless springs per unit length along the  $x$ -axis, each of them having a stiffness  $\kappa$ . The restoring force per unit length due to the mattress,  $-n\kappa\xi$ , turns the d'Alembert equation into a Klein–Gordon equation

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \xi}{\partial t^2} - \frac{\omega_0^2}{c^2} \xi = 0, \quad (x \neq 0), \quad (1b')$$

where  $\omega_0 \stackrel{\text{def}}{=} c\sqrt{n\kappa/T_0}$ . This equation governs also the propagation of the electromagnetic field in a rectangular waveguide (Feynman *et al* 1970, chapter 24) and is also the relativistic equation of a quantum particle of mass  $\hbar\omega_0/c^2$ ,  $c$  being the velocity of light in the vacuum.

Since the attachment point  $A$  is massless, the sum of the three forces applied to it vanishes as depicted in the inset of figure 1. To first order in the slope  $\vartheta(x, t) = \partial\xi/\partial x(x, t)$ , the transverse component of the right tension applied to  $A$  is given by  $T_0\vartheta(0^+, t)$  the limit when  $x \rightarrow 0$  keeping  $x > 0$ . On the other side, the transverse component of the left tension is  $-T_0\vartheta(0^-, t)$ . The restoring force of the coupling string corresponds to the opposite of the left-hand side of (1a), namely  $\kappa(X - \xi_0)$ . Therefore the coupling introduces a discontinuity of the slope of the string at  $x = 0$ :

$$T_0 \frac{\partial\xi}{\partial x}(0^+, t) - T_0 \frac{\partial\xi}{\partial x}(0^-, t) = -\kappa(X(t) - \xi_0(t)). \quad (1c)$$

### 2.3. Dimensionless quantities

The fundamental units will be chosen to be  $T \stackrel{\text{def}}{=} M/(\mu_0 c)$  for the times,  $L \stackrel{\text{def}}{=} M/\mu_0$  for the lengths and  $M$  for the masses. The model is therefore uniquely determined in terms of dimensionless quantities defined by the appropriate rescaling:  $x_{\text{eff}} \stackrel{\text{def}}{=} x/L$ ,  $t_{\text{eff}} \stackrel{\text{def}}{=} t/T$ ,  $\xi_{\text{eff}} \stackrel{\text{def}}{=} \xi/L$ ,  $X_{\text{eff}} \stackrel{\text{def}}{=} X/L$ ,  $\Omega_{0,\text{eff}} \stackrel{\text{def}}{=} \Omega_0/(T^{-1})$ ,  $\kappa_{\text{eff}} \stackrel{\text{def}}{=} \kappa/(MT^{-2})$ , etc. These conventions correspond to  $c_{\text{eff}} \stackrel{\text{def}}{=} c/(LT^{-1}) = 1$ . There is no need for any quantization as long as the effective Planck constant  $\hbar_{\text{eff}} \stackrel{\text{def}}{=} \hbar/(ML^2T^{-1}) = \hbar\mu_0/(M^2c) \ll 1$ . For simplifying the notations, in the following we will drop the ‘effective’ subscript and work directly with  $M = 1$ ,  $c = 1$ ,  $\mu_0 = 1$  and  $T_0 = 1$ . Introducing the shifted<sup>2</sup> frequency

$$\Omega_\kappa \stackrel{\text{def}}{=} \sqrt{\Omega_0^2 + \kappa}, \quad (2)$$

the equations governing the dynamics of the system become

$$\frac{d^2 X}{dt^2} + \Omega_\kappa^2 X = \kappa \xi_0, \quad (3a)$$

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{\partial^2 \xi}{\partial t^2} - \omega_0^2 \xi = 0, \quad (x \neq 0) \quad (3b)$$

and

$$\frac{\partial \xi}{\partial x}(0^+, t) - \frac{\partial \xi}{\partial x}(0^-, t) = \kappa(\xi_0(t) - X(t)) \quad (3c)$$

with the help of the Dirac distribution  $\delta$ , equations (3b) and (3c) can be combined in one equation, valid for all  $x$ ,

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{\partial^2 \xi}{\partial t^2} - \omega_0^2 \xi = \kappa \delta(x)(\xi - X). \quad (3bc)$$

Indeed, (3c) is recovered after integrating (3bc) between  $x = -\epsilon$  and  $x = +\epsilon$  when  $\epsilon \rightarrow 0^+$ , since  $\xi$  and its time derivatives are continuous (and finite) everywhere.

### 2.4. The Hamiltonian\*

If one wants to prepare the ground for some perturbative treatment of some nonlinear corrections, if one wants to quantize the model and/or to couple it to a thermal bath, one possible starting point is the Hamiltonian of the system expressed in terms of some canonical variables. The continuous part of the system (the string) corresponds to a Hamiltonian density (Goldstein 1980, section 12.4) involving a pair of canonically conjugate fields  $(\pi(x, t), \xi(x, t))$

<sup>2</sup> With more pedantry, one could speak of the renormalized frequency. Here the shift can be understood by the effective restoring force  $-(K + \kappa)X$  that  $M$  actually feels.

whereas the oscillator is described in terms of  $(P_X, X)$ . The Poisson bracket has also a mixed structure of continuous and discrete variables: for any two  $O_1, O_2$  that are functions of  $(P_X, X)$  and functionals of  $(\pi, \xi)$ ,

$$\{O_1, O_2\} \stackrel{\text{def}}{=} \frac{\partial O_1}{\partial P_X} \frac{\partial O_2}{\partial X} - \frac{\partial O_2}{\partial P_X} \frac{\partial O_1}{\partial X} + \int \left( \frac{\delta O_1}{\delta \pi} \frac{\delta O_2}{\delta \xi} - \frac{\delta O_2}{\delta \pi} \frac{\delta O_1}{\delta \xi} \right) dx. \quad (4)$$

The Hamiltonian of the whole system generates the evolution of any function(al)  $O$  via  $dO/dt = \partial_t O + \{H, O\}$ . For our system we have

$$H \stackrel{\text{def}}{=} \frac{1}{2} P_X^2 + \frac{1}{2} \Omega_0^2 X^2 + \frac{1}{2} \kappa (X - \xi_0)^2 + \frac{1}{2} \int \left[ \pi^2 + \left( \frac{\partial \xi}{\partial x} \right)^2 + \omega_0^2 \xi^2 \right] dx, \quad (5)$$

the corresponding Hamilton equations yield directly to (3). The interaction term is completely different from the one used in a recent one-dimensional pedagogical model (Boozer 2007). The latter emphasizes the recoil effect of the field on the mass  $M$  (unlike in the present work, the external (translational) degrees of freedom for  $M$  are considered) whereas we are more interested in the resonance effects.

### 3. Scattering

#### 3.1. Free modes

As for any vibrating system, a normal mode is defined to be a particular collective motion where all the degrees of freedom oscillate with the same frequency. In the absence of coupling ( $\kappa = 0$ ), for a given frequency  $\omega \geq 0$ , one can choose two independent normal modes (the free modes) on the string given by

$$\xi_{\pm k}^{\text{fr}}(x, t) = \frac{1}{\sqrt{2\pi}} e^{i(\pm kx - \omega t)}. \quad (6)$$

The wave number is obtained from the dispersion relation of the Klein–Gordon equation:

$$k(\omega) \stackrel{\text{def}}{=} \sqrt{\omega^2 - \omega_0^2} \iff \omega(k) = \sqrt{\omega_0^2 + k^2}. \quad (7)$$

When  $\omega < \omega_0$ ,  $k$  is purely imaginary with a positive imaginary part and does not correspond to any travelling wave. When  $\omega > \omega_0$ ,  $k$  is real positive and the two modes are two monochromatic counter-propagating waves.

A real field  $\xi$  obeying the Klein–Gordon equation (3b) carries a linear energy density  $\rho_e = \frac{1}{2} [(\partial \xi / \partial t)^2 + (\partial \xi / \partial x)^2 + \omega_0^2 \xi^2]$  and a linear density of energy current  $j_e = -\partial \xi / \partial t \partial \xi / \partial x$ . The conservation of energy takes the form  $\partial \rho_e / \partial t + \partial j_e / \partial x = 0$  for  $x \neq 0$ . For a monochromatic travelling wave of complex amplitude  $a$ ,  $\xi(x, t) = a e^{i(\pm kx - \omega t)}$ , an elementary calculation shows that the average current over one period is given by

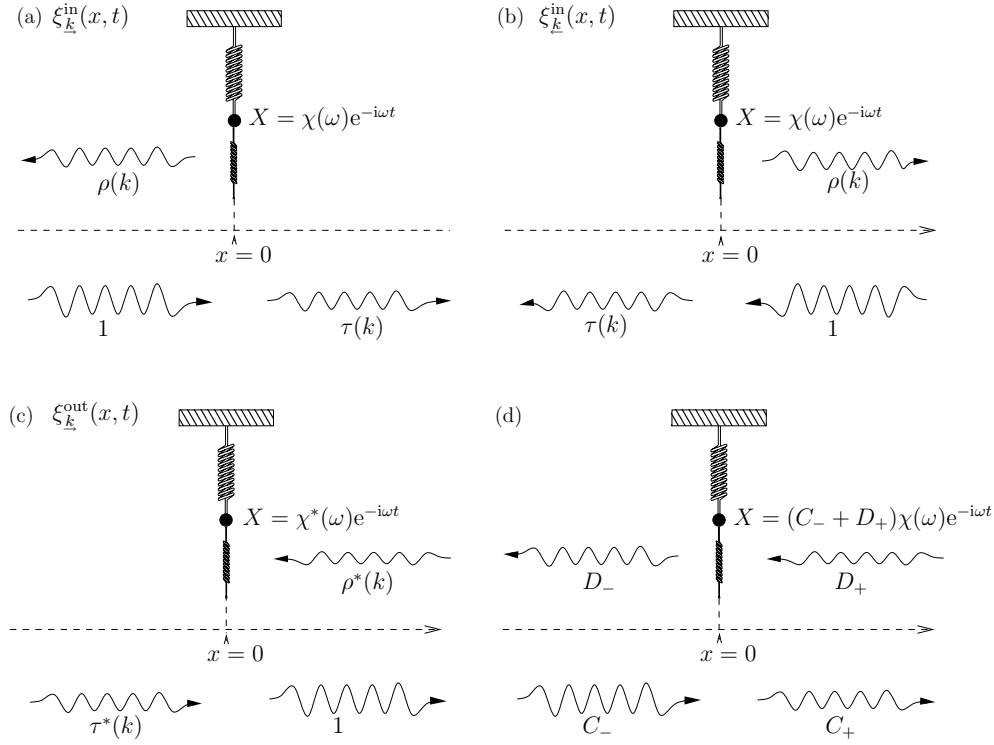
$$\langle j_e \rangle = \pm k \omega |a|^2 / 2 \quad (k \text{ real}), \quad (8)$$

whereas for an evanescent wave

$$\langle j_e \rangle = 0 \quad (k \text{ imaginary}). \quad (9)$$

#### 3.2. Definitions of the in and out asymptotic modes

When a coupling is present ( $\kappa > 0$ ), the two free waves (6) are no longer solutions of (3). A monochromatic travelling wave will be partially reflected (respectively, transmitted) by the oscillator with a reflection (respectively, transmission) coefficient  $\rho$  (respectively,  $\tau$ ) that is a



**Figure 3.** Several choices for the normal modes with frequency  $\omega$ . The wavy arrows symbolize monochromatic travelling waves whose complex amplitude is specified below; (a) the in-state given by (10a); (b) the in-state given by (10b); (c) the out-state given by (15a) and (d) shows the general coefficients that will be linked by the  $S$  or  $T$  matrix defined, respectively, by (21) and (23).

complex function of  $k$  or  $\omega$ . The linearity of equations (3) guarantees that the frequency will be unchanged by scattering. Indeed, to describe such a scattering process, a relevant choice for the two modes of frequency  $\omega$  is to look for in-states, defined (for real positive  $k$ ) to be of the form (see figure 3(a))

$$\xi_{\vec{k}}^{\text{in}}(x, t) = \frac{1}{\sqrt{2\pi}} \begin{cases} e^{i(kx - \omega t)} + \rho e^{i(-kx - \omega t)} & \text{for } x \leq 0, \\ \tau e^{i(kx - \omega t)} & \text{for } x \geq 0, \end{cases} \quad (10a)$$

and, since the scattering is symmetric with respect to  $x \mapsto -x$ ,

$$\xi_{\vec{k}}^{\text{in}}(x, t) \stackrel{\text{def}}{=} \xi_{\vec{k}}^{\text{in}}(-x, t) = \frac{1}{\sqrt{2\pi}} \begin{cases} \tau e^{i(-kx - \omega t)} & \text{for } x \leq 0, \\ e^{i(-kx - \omega t)} + \rho e^{i(kx - \omega t)} & \text{for } x \geq 0, \end{cases} \quad (10b)$$

with, for both modes, the same amplitude  $\chi(\omega)$  for the oscillator that can be interpreted, using the language of linear response theory, as a susceptibility:

$$X^{\text{in}}(t) = \chi(\omega) e^{-i\omega t}. \quad (11)$$

These modes form a set of waves that is suited for constructing localized wave-packets that look like free wave-packets when  $t \rightarrow -\infty$  (i.e. far away from the oscillator). For instance, when considering a localized wave-packet travelling to the right, we shall have<sup>3</sup>

$$\int \tilde{\varphi}(k) \xi_{\rightarrow}^{\text{in}}(x, t) dk \xrightarrow{t \rightarrow -\infty} \int \tilde{\varphi}(k) \xi_k^{\text{fr}}(x, t) dk = \frac{1}{\sqrt{2\pi}} \int \tilde{\varphi}(k) e^{i(kx - \omega t)} dk. \quad (12)$$

The continuity of  $\xi$  at  $x = 0$  implies that

$$1 + \rho = \tau. \quad (13)$$

Conservation of energy implies that the average energetic current is conserved in the stationary regime. From (8), we must have

$$1 = |\rho|^2 + |\tau|^2. \quad (14)$$

Moreover, the equations are real and invariant under the time reversal; therefore if  $\xi(x, t)$  is a solution, so is its complex conjugate  $(\xi(x, t))^*$  and  $\xi(x, -t)$ . If we consider the solution  $(\xi_{\leftarrow}^{\text{in}}(x, -t))^*$ , then we obtain the mode depicted in figure 3(c) obtained from figure 3(b) by reversing the orientation of the arrows and by conjugating the amplitudes. This procedure defines the out-modes that behave like free modes for remote future times when packed in localized superpositions. We will have (see figure 3(c))

$$\xi_{\rightarrow}^{\text{out}}(x, t) \stackrel{\text{def}}{=} (\xi_{\leftarrow}^{\text{in}}(x, -t))^* = \frac{1}{\sqrt{2\pi}} \begin{cases} \tau^* e^{i(kx - \omega t)} & \text{for } x \leq 0, \\ e^{i(kx - \omega t)} + \rho^* e^{i(-kx - \omega t)} & \text{for } x \geq 0, \end{cases} \quad (15a)$$

and

$$\xi_{\leftarrow}^{\text{out}}(x, t) \stackrel{\text{def}}{=} (\xi_{\rightarrow}^{\text{in}}(x, -t))^* = \frac{1}{\sqrt{2\pi}} \begin{cases} e^{i(-kx - \omega t)} + \rho^* e^{i(kx - \omega t)} & \text{for } x \leq 0, \\ \tau^* e^{i(-kx - \omega t)} & \text{for } x \geq 0, \end{cases} \quad (15b)$$

with

$$X^{\text{out}}(t) = \chi^*(\omega) e^{-i\omega t}. \quad (16)$$

If we interpret  $\xi_{\rightarrow}^{\text{out}}$  as a superposition of in-modes coming from both sides that conspire to product no wave travelling to the right for  $x < 0$ , we get

$$\tau^* = \frac{\tau}{\rho + \tau}, \quad \rho^* = -\frac{\rho}{\rho + \tau}. \quad (17)$$

More mathematically,  $\xi_{\rightarrow}^{\text{out}}(x, t)$  can be seen as the continuation of  $\xi_{\leftarrow}^{\text{in}}(x, t)$  to the domain of negative  $k$ 's. Indeed, we have  $\xi_{\rightarrow}^{\text{out}}(x, t) = \xi_{\leftarrow}^{\text{in}}(x, t)$  provided that we define

$$\tau(-k) \stackrel{\text{def}}{=} (\tau(k))^*, \quad \rho(-k) \stackrel{\text{def}}{=} (\rho(k))^*. \quad (18)$$

### 3.3. Definitions of the scattering and transfer matrices

The in-modes and the out-modes are two possible bases for describing a scattered wave-packet. These bases can be obtained one from each other by linear transformations; the linearity of the equations of our model implies that they connect waves with the same frequency only, which is a major simplification. A typical scattering experiment consists in preparing one wave-packet travelling towards the scatterer (the oscillator). Long before the diffusion, this ingoing

<sup>3</sup> This can be understood with the stationary phase approximation. If  $\tilde{\varphi}$  is concentrated around  $k_0 > 0$ , the wave-packets travel with the group velocity  $\pm d\omega/dk(k_0) \gtrless 0$ . When  $t \rightarrow -\infty$ , the dominant contributions to the integrals in (12) form one wave-packet located in  $x < 0$  and travelling to the right.

wave-packet is a simple superposition of in-modes. Long after the diffusion, we get two outgoing wave-packets that are naturally described in terms of out-modes. The passage from the in-basis to the out-basis is described in terms of the scattering matrix  $S$  that encapsulates all the information about the possible scattering processes<sup>4</sup>. It is made of  $2 \times 2$  blocks  $S(\omega)$  defined by

$$\begin{pmatrix} \xi_k^{\text{out}} \\ \xi_k^{\text{out}} \end{pmatrix} = S(\omega) \begin{pmatrix} \xi_k^{\text{in}} \\ \xi_k^{\text{in}} \end{pmatrix}. \quad (19)$$

The decomposition of each out-mode in terms of the two in-modes for, say,  $x \leq 0$ , leads to

$$S(\omega) = \begin{pmatrix} \frac{\tau}{\rho + \tau} & -\frac{\rho}{\rho + \tau} \\ -\frac{\rho}{\rho + \tau} & \frac{\tau}{\rho + \tau} \end{pmatrix} = \begin{pmatrix} \tau^* & \rho^* \\ \rho^* & \tau^* \end{pmatrix}. \quad (20)$$

In other words, for the general monochromatic wave

$$\xi(x, t) = \begin{cases} C_- e^{i(kx - \omega t)} + D_- e^{i(-kx - \omega t)} & \text{for } x \leq 0, \\ C_+ e^{i(kx - \omega t)} + D_+ e^{i(-kx - \omega t)} & \text{for } x \geq 0, \end{cases} \quad (21)$$

the  $S$  matrix connects linearly the coefficients

$$\begin{pmatrix} C_- \\ D_+ \end{pmatrix} = S(\omega) \begin{pmatrix} C_+ \\ D_- \end{pmatrix}. \quad (22)$$

In the absence of scattering ( $\tau = 1, \rho = 0$ ),  $S$  simply reduces to the identity. The unitarity of  $S$ , which can be checked on (20), can be seen as a direct consequence of the conservation of energy since, from (8), the norm of the two vectors involved in (22) is preserved:  $|C_+|^2 + |D_-|^2 = |C_-|^2 + |D_+|^2$ .

If one wants to calculate the diffusion by several scatterers, it is more convenient to introduce the transfer matrix  $T$  whose  $2 \times 2$  blocks are defined to connect the left coefficients to the right coefficients,

$$\begin{pmatrix} C_+ \\ D_+ \end{pmatrix} = T(\omega) \begin{pmatrix} C_- \\ D_- \end{pmatrix}. \quad (23)$$

Then we have

$$T(\omega) = \begin{pmatrix} 1 + \frac{\rho}{\tau} & \frac{\rho}{\tau} \\ -\frac{\rho}{\tau} & 1 \end{pmatrix} \quad (24)$$

whose determinant is one. The addition of one scatterer on the string corresponds to a multiplication by a  $T$  matrix.

### 3.4. Physical interpretation of the solutions—resonant scattering

The definitions and the general properties presented in sections 3.2 and 3.3 are valid for any non-dissipative punctual scatterer. As far as our model is concerned, inserting the expression

<sup>4</sup> In the literature, specially within the context of scattering of quantum waves (Taylor 1972, section 2c, for instance), the matrices that connect the free waves to the in-waves on the one hand and the free waves to the out-waves on the other hand are often introduced under the name of Møller operators with the caveat that unlike the free states, the set of scattering states may be incomplete, that is insufficient to construct *all* the states. As we will see in section 4.2, to get a complete basis one may add to the in-states (10) the bounded modes when existing.

(10a) with the oscillation (11) in equations (3) yields to a linear system that can be solved straightforwardly:

$$\tau(\omega) = \frac{1}{2} + \frac{1}{2} e^{-2i\eta(\omega)} = \frac{1}{1 + i \frac{\kappa}{2\sqrt{\omega^2 - \omega_0^2}} \frac{\omega^2 - \Omega_0^2}{\omega^2 - \Omega_\kappa^2}}, \quad (25)$$

$$\rho(\omega) = -\frac{1}{2} + \frac{1}{2} e^{-2i\eta(\omega)} = \frac{-1}{1 - i \frac{\kappa}{2\sqrt{\omega^2 - \omega_0^2}} \frac{\omega^2 - \Omega_\kappa^2}{\omega^2 - \Omega_0^2}}, \quad (26)$$

with

$$\eta(\omega) \stackrel{\text{def}}{=} \arctan \left( \frac{\kappa}{2\sqrt{\omega^2 - \omega_0^2}} \frac{\omega^2 - \Omega_0^2}{\omega^2 - \Omega_\kappa^2} \right) \quad (27)$$

and

$$\chi(\omega) = \frac{\kappa/\sqrt{2\pi}}{\Omega_\kappa^2 - \omega^2 - i \frac{\kappa}{2\sqrt{\omega^2 - \omega_0^2}} (\omega^2 - \Omega_0^2)}. \quad (28)$$

Even though the coefficients  $\rho$  and  $\tau$  were first defined for travelling waves, i.e. for  $\omega > \omega_0$ , the above expressions can be continued for  $\omega < \omega_0$ . We will understand the physical interpretation of this procedure when we study radiation in section 4. As long as  $\omega > \omega_0$ , (14), (17) and (18) hold. We can check that the ultra-violet limit  $\omega \rightarrow \infty$  is equivalent to the limit of weak coupling where the oscillator becomes transparent:  $\tau \rightarrow 1$  and  $\rho \rightarrow 0$ . Another case where the coupling is inefficient is when  $A$  and  $X$  both oscillate in phase with  $\omega = \Omega_0$  since the coupling spring remains unstretched. More interesting is the resonant scattering that occurs, provided that  $\omega_0 < \Omega_\kappa$ , when the ingoing wave that forces the oscillator has precisely the same frequency as the shifted frequency of the latter, that is  $\omega = \Omega_\kappa$ . Then, the scattering is the most efficient since no transmission occurs ( $\rho = -1$ ,  $\tau = 0$ ). The resonance spike can be seen in figure 4 and its quality factor can be evaluated from its width  $\Delta\omega$  when  $|\rho(\Omega_\kappa \pm \Delta\omega/2)| = 1/\sqrt{2}$ : for a small coupling,

$$Q \stackrel{\text{def}}{=} \frac{\Omega_\kappa}{\Delta\omega} = \frac{2\Omega_0^2}{\kappa^2} \sqrt{\Omega_0^2 - \omega_0^2} (1 + O(\kappa)), \quad (29)$$

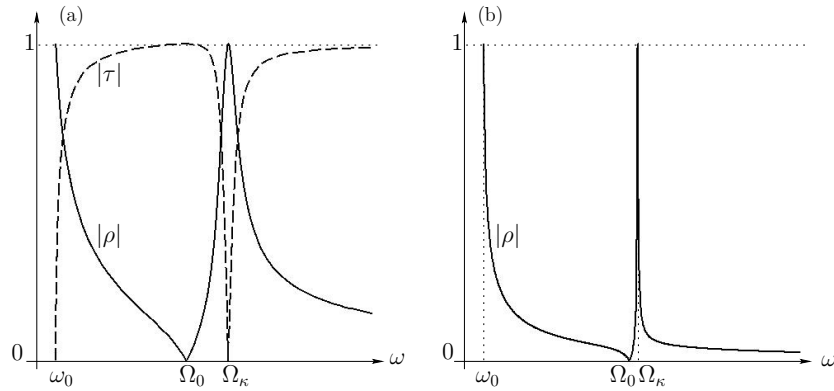
and therefore, the smaller  $\kappa$ , the better the quality of the resonance.

#### 4. Radiation, damping and bounded mode

The general idea that damping and therefore irreversibility emerge because of the interaction with a large number of degrees of freedom can be illustrated explicitly on our model. If we choose initial conditions such that the spring is at rest at  $t = 0$ , the entire energy being contained in the oscillator, for instance

$$\xi(x, 0) = 0, \quad \frac{\partial \xi}{\partial t}(x, 0) = 0, \quad X(0) = X_0, \quad \dot{X}(0) = 0, \quad (30)$$

the energy transfer to the string will damp the oscillations of  $M$  and the latter may completely lose its energy far before the energy can get back from the string if its boundary is far away from the oscillator (for a string of length  $\ell$ , Poincaré recurrence time is of order  $\ell/c$  if there



**Figure 4.** Graphs of  $|\tau(\omega)|$  (dashed line) and  $|\rho(\omega)|$  (solid line) given by (25) and (26) when  $\omega_0 < \Omega_0$ . One can observe a resonance scattering spike for  $\omega = \Omega_k$  and an antiresonance for  $\omega = \Omega_0$  where no scattering occurs. The quality factor is (a)  $Q \simeq 15$  and (b)  $Q \simeq 200$ . Let us mention that when  $\Omega_0 < \omega_0 < \Omega_k$ , the antiresonance has vanished and the local maximum  $|\rho(\Omega_k)| = 1$  is too soft to be called a resonance. When  $\Omega_k < \omega_0$ , no scattering resonance occurs and  $|\rho|$  decreases monotonically from 1 at  $\omega = \omega_0$  to 0 when  $\omega \rightarrow +\infty$ .

were no dispersions). The equation of motion of the oscillator is particularly simple when the string is non-dispersive ( $\omega_0 = 0$ ) and therefore we will start by studying this case. However, we will also consider the case of the Klein–Gordon string because when  $\omega_0 > \Omega_0$ , we will see that there exists a stable mode of the oscillator at a frequency  $\omega_b > 0$  whose dissipation is blocked because  $\omega_b < \omega_0$ . Its vibration does not decay because at this frequency, only evanescent waves can exist on the string, which do not carry away energy current on average (see (9)).

#### 4.1. The d'Alembert string

With the initial conditions (30), the general form of the radiated waves on the string will be  $\xi(x, t) = \xi_0(t - |x|)$ : each of the two wave-packets travels away from  $x = 0$  without distortion when  $\omega_0 = 0$ . Equation (3c) becomes

$$2\dot{\xi}_0 + \kappa \xi_0 = \kappa X \quad (31)$$

and the elimination of  $\xi_0$  from (31) and (3a) yields

$$\ddot{X} + \Omega_0^2 X = -\frac{2\Omega_0^2}{\kappa} \dot{X} - \frac{2}{\kappa} \ddot{X}. \quad (32)$$

The two terms in the right-hand side are dissipative forces. The first one has the familiar taste of the viscous resistive force whereas the second has the flavour of the Schott term  $2e^2 \ddot{x}/3$  (Rohrlich 2000, equation (2.7b)) in the Abraham–Lorentz–Dirac equation which governs the dynamics of an electric charge  $e$  that takes into account the electromagnetic self-force of the charge. The major difference is the sign of the coefficient in front of the third derivative. Unlike the Schott term, the negative sign in (32) prevents the spurious exponentially accelerating solutions. It can be clearly seen that the irreversibility due to dissipation comes straightforwardly from the choice of initial conditions (30) that break the time-reversal symmetry under which the original equations are invariant.

Many models of an oscillator coupled to one-dimensional waves are recovered in the limit of strong coupling  $\kappa \rightarrow +\infty$  (see the references given in the introduction, for instance when

the mass is directly attached to the string). In that case, only the viscous force remains in (32) and we immediately get the well-known damped oscillator  $\ddot{X} + 2\dot{X} + \Omega_0^2 X = 0$ .

Looking for exponential solutions  $X(t) = X(0)e^{zt}$  yields to the characteristic equation of (32):

$$z^3 + \frac{1}{2}\kappa z^2 + \Omega_\kappa^2 z + \frac{1}{2}\kappa\Omega_0^2 = 0. \quad (33)$$

There are three solutions, one real  $z_0$  and two complex  $z_+$ ,  $z_-$  all having a strictly negative real part. Perturbatively in  $\kappa$ , we have

$$z_0 = -\frac{\kappa}{2} + \frac{\kappa^2}{2\Omega_0^2} + \mathcal{O}(\kappa^3), \quad (34a)$$

$$z_+ = z_-^* = -\frac{\kappa^2}{4\Omega_0^2} + i\left(\Omega_0 + \frac{\kappa}{2\Omega_0} - \frac{\kappa^2}{8\Omega_0^3}\right) + \mathcal{O}(\kappa^3). \quad (34b)$$

For generic initial conditions, including (30), where the string is at rest the energy of the oscillator will exponentially decay like  $e^{-\Gamma t}$  at the rate

$$\Gamma = \frac{\Omega_0}{Q} = \frac{\kappa^2}{2\Omega_0^2} + \mathcal{O}(\kappa^3). \quad (35)$$

In the language of particle physics, the stable non-interacting particle (the mode of the free oscillator) has been destabilised into a metastable particle of lifetime  $\Gamma^{-1}$  because of its interactions.

#### 4.2. The Klein–Gordon string

When  $\omega_0 > 0$ , one cannot get a differential equation for  $X(t)$  but must keep working with its temporal Fourier transform

$$\tilde{X}(\omega) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int X(t) e^{i\omega t} dt, \quad (36)$$

together with a superposition of purely radiated waves of the form  $\xi(x, t) = (\sqrt{2\pi})^{-1} \int \tilde{\xi}(\omega) e^{i(k|x| - \omega t)} d\omega$ . Inserting them in (3),  $\tilde{X}$  must satisfy<sup>5</sup>

$$[\kappa(\omega^2 - \Omega_0^2) - 2i\sqrt{\omega^2 - \omega_0^2}(\omega^2 - \Omega_\kappa^2)]\tilde{X}(\omega) = 0. \quad (37)$$

Therefore  $\tilde{X}$  vanishes everywhere but at the frequencies that cancel the brackets. These are precisely the poles of  $\tau$  and therefore of  $\rho = \tau - 1$  given by (25) and (26). Indeed, for pure radiative modes, the ingoing waves vanish ( $C_- = D_+ = 0$ ) and therefore the matrix element  $T_{22}$  must go to infinity in order to keep  $D_-$  finite (see figure 3(d)) and equations (23) and (24)). Letting  $Z = -\omega^2$ , we look for the solutions of the cubic equation

$$(Z + \omega_0^2)(Z + \Omega_0^2 + \kappa)^2 - \frac{\kappa^2}{4}(Z + \Omega_0^2)^2 = 0. \quad (38)$$

Perturbatively in  $\kappa$ , those are

$$Z_0 = -\omega_0^2 + \frac{\kappa^2}{4} - \frac{\kappa^3}{2(\Omega_0^2 - \omega_0^2)} + \mathcal{O}(\kappa^4), \quad (39a)$$

<sup>5</sup> The presence of the square root in (37) is the reason that prevents us from obtaining a local differential operator for  $X(t)$ .

$$Z_+ = Z_-^* = -\Omega_0^2 - \kappa - i \frac{\kappa^2}{2\sqrt{\Omega_0^2 - \omega_0^2}} + O(\kappa^3). \quad (39b)$$

When  $\omega_0 \rightarrow 0$ , we recover  $Z_0 \rightarrow z_0^2$  and  $Z_{\pm} \rightarrow z_{\pm}^2$ . The physical frequencies will be the three square roots  $i\sqrt{Z_0}$ ,  $i\sqrt{Z_{\pm}}$  whose imaginary part is not positive: the typical decay rate of energy will be given by the nearest root  $\omega_{\min}$  to the real axis:  $\Gamma = -2\text{Im}(\omega_{\min})$ . As long as  $2\omega_0 < \kappa \ll \Omega_0^2$ , all the three frequencies have strictly negative real part. The decay rate is given by

$$\Gamma = \frac{\Omega_0}{Q} = \frac{\kappa^2}{2\Omega_0\sqrt{\Omega_0^2 - \omega_0^2}} + O(\kappa^3). \quad (40)$$

It is a very general feature that the poles of the  $S$  matrix are associated with resonances and, more precisely, that their imaginary parts provide the decay rates, which are proportional to the inverse of the quality factor of the resonances.

When  $\kappa < 2\omega_0$ ,  $Z_0$  is negative, one residual oscillation persists at frequency  $\omega_b \stackrel{\text{def}}{=} \sqrt{-Z_0}$ . For  $\omega_b < \omega_0$ , no transfer of energy is allowed; only evanescent waves are created and those do not carry any average energy current. Unlike the scattering states, this non-decaying mode is spatially localized. More generally, any bounded mode has a purely real frequency  $\omega_b$  that must be less than  $\omega_0$  since  $Z_b + \omega_0^2 = \omega_0^2 - \omega_b^2 > 0$  in order to fulfil (38). From (7),  $k(\omega_b)$  is therefore purely imaginary. Moreover, in order to cancel the bracket in (37),  $\omega_b^2$  must lie in between  $\Omega_0^2$  and  $\Omega_{\kappa}^2$ . For simplicity, let us introduce the auxiliary parameter  $\Upsilon \stackrel{\text{def}}{=} (\Omega_0^2 - \omega_0^2)/\kappa$  and the real positive variable  $u \stackrel{\text{def}}{=} |k|/\sqrt{\kappa}$ ; a stable mode will exist if the cubic equation

$$2u(u^2 + \Upsilon + 1) + \sqrt{\kappa}(u^2 + \Upsilon) = 0 \quad (41)$$

has a positive real solution. This can be achieved for  $\Upsilon < 0$  only i.e. in a regime where  $\Omega_0 < \omega_0$ . For  $\Upsilon < 0$ , the product of the roots of the left-hand side of (41),  $u_+ u_- u_b$  is  $-\sqrt{\kappa}\Upsilon > 0$ . If two roots are complex conjugated, the third one is necessarily positive. If the three roots are real, either only one is positive or all three of them are. The latter case must be ruled out since the sum  $u_+ + u_- + u_b = -\sqrt{\kappa}$  is strictly negative. We have therefore proved that a sufficient and necessary condition for a stable mode to exist is that  $\omega_0 > \Omega_0$ . Its frequency is given by

$$\omega_b = \sqrt{\omega_0^2 - \kappa u_b^2}, \quad (42)$$

where  $u_b$  is the unique positive real solution of (41). Perturbatively in  $\kappa$ , we have

$$\omega_b = \Omega_0 + \frac{\kappa}{2\Omega_0} - \frac{2\Omega_0^2 + \sqrt{\omega_0^2 - \Omega_0^2} \kappa^2}{\Omega_0^3 \sqrt{\omega_0^2 - \Omega_0^2}} \frac{\kappa^2}{8} + O(\kappa^3) \quad (43)$$

and the corresponding bounded mode is given by

$$\xi^b(x, t) = C_b e^{-\sqrt{\omega_0^2 - \omega_b^2}|x|} e^{-i\omega_b t}, \quad (44a)$$

$$X_b(t) = \frac{\kappa C_b}{\Omega_{\kappa}^2 - \omega_b^2} e^{-i\omega_b t}. \quad (44b)$$

The choice of the normalization,

$$C_b = \left( \frac{1}{\sqrt{\omega_0^2 - \omega_b^2}} + \frac{\kappa^2}{(\Omega_{\kappa}^2 - \omega_b^2)^2} \right)^{-1/2} \quad (45)$$

will be justified below (equation (54)).

## 5. Some like it diagonal\*

### 5.1. Normal coordinates

What makes the model completely tractable is of course that it remains linear. However, the direct diagonalization of the quadratic Hamiltonian (5) remains particularly difficult. In that case, the trick is to solve the equations of motion to determine the normal modes first—this is precisely what we have done in the previous paragraphs—and then write the Hamiltonian in its diagonal form

$$H = \frac{1}{2} \sum_{\alpha} (p_{\alpha}^2 + \omega_{\alpha}^2 q_{\alpha}^2) = \sum_{\alpha} \omega_{\alpha} a_{\alpha}^* a_{\alpha} \quad (46)$$

in terms of some (real) canonical coordinate  $\{p_{\alpha}, q_{\alpha}\}_{\alpha}$  or (complex) normal coordinates  $\{a_{\alpha}\}_{\alpha}$  that are associated with modes labelled by the discrete and/or continuous index  $\alpha$ .<sup>6</sup> We have

$$a_{\alpha} = \sqrt{\frac{\omega_{\alpha}}{2}} q_{\alpha} + \frac{i}{\sqrt{2\omega_{\alpha}}} p_{\alpha}, \quad (47)$$

$$q_{\alpha} = \frac{1}{\sqrt{2\omega_{\alpha}}} (a_{\alpha}^* + a_{\alpha}), \quad p_{\alpha} = i\sqrt{\frac{\omega_{\alpha}}{2}} (a_{\alpha}^* - a_{\alpha}), \quad (48)$$

and, for each pair  $\{\alpha_1, \alpha_2\}$ ,

$$\{a_{\alpha_1}, a_{\alpha_2}\} = 0, \quad \{a_{\alpha_1}, a_{\alpha_2}^*\} = i\delta_{\alpha_1, \alpha_2}, \quad (49)$$

$$\{p_{\alpha_1}, p_{\alpha_2}\} = 0, \quad \{q_{\alpha_1}, q_{\alpha_2}\} = 0, \quad \{p_{\alpha_1}, q_{\alpha_2}\} = \delta_{\alpha_1, \alpha_2}, \quad (50)$$

where  $\delta$  stands for the Kronecker symbol or the Dirac distribution. The second step consists in determining the canonical transformation that expresses  $a_{\alpha}$  in terms of some *a priori* known normal coordinates, namely some free normal modes  $a_{\text{fr}, \alpha}$ . In our case this transformation is linear and will be transposed directly into the quantum theory by replacing the complex number  $a_{\alpha}$  (respectively,  $a_{\alpha}^*$ ) by the creation (respectively, annihilation) operator  $\hat{a}_{\alpha}$  (respectively, its Hermitian conjugate  $\hat{a}_{\alpha}^*$ ) of the  $\alpha$ th one-particle eigenstate whose energy is  $\hbar\omega_{\alpha}$ . As we have seen, all the scattering states are twice degenerate, in the sense that each normal frequency  $\omega$  is associated with two independent states labelled by  $k$  and  $-k$ . These modes both diagonalize the Hamiltonian (5). An infinite number of pairs of eigenvectors can be chosen to constitute a basis, among them, the in- and out-states, which are particularly relevant as soon as we get into a quantum field theory<sup>7</sup>. But in order to get (46) properly one must check that the set of modes is actually complete—i.e. that any kind of motion of our system can be described as a linear superposition of modes—and orthonormalized correctly in order to deal with canonical complex coordinates. Fourier analysis assures that the free states (6) constitute a complete set for describing the waves on the string. When interacting with the oscillator, if  $\omega_0 > \Omega_0$  one bounded state exists that must be added to the in-modes (or to the out-modes) to get a genuine basis. Then, including the normal coordinates  $A_b$  of the bounded mode if there is any, (46) reads

$$H = \omega_b A_b^* A_b + \int \omega(k) a_{\text{in}}^*(k) a_{\text{in}}(k) dk = \omega_b A_b^* A_b + \int \omega(k) a_{\text{out}}^*(k) a_{\text{out}}(k) dk. \quad (51)$$

<sup>6</sup> To avoid ambiguities we will often subscript the brace describing a set like  $\{\dots\}_{\alpha \in \mathcal{A}}$  to recall which indices are running and what is their range  $\mathcal{A}$  if the latter does matter.

<sup>7</sup> The normal coordinates  $a_{\text{in}}(k)$  and  $a_{\text{out}}(k)$  constructed from the scattering modes, once quantized into  $\hat{a}_{\text{in}}(k)$  and  $\hat{a}_{\text{out}}(k)$ , allow the interpretation of the quantum states in terms of asymptotic (quasi-)particles; more precisely the linear transformations from the free  $\hat{a}_{\text{fr}}(k)$  provide the explicit connection between the non-interacting states (the Fock space for bare particles including the free vacuum) and the interacting states (the Fock space for dressed particles including the interacting vacuum).

We chose the convention that, when  $k > 0$ ,  $a_{\text{in}}(k)$  (respectively,  $a_{\text{out}}(k)$ ) is constructed from  $\xi_k^{\text{in}} \rightarrow$  (respectively,  $\xi_k^{\text{out}} \rightarrow$ ) while  $a_{\text{in}}(-k)$  (respectively,  $a_{\text{out}}(-k)$ ) is constructed from  $\xi_k^{\text{in}} \leftarrow$  (respectively,  $\xi_k^{\text{out}} \leftarrow$ ).

### 5.2. Orthonormalization

Having a complete set of modes does not guarantee that they are orthogonal. Indeed, it may happen that two eigenvectors having a common eigenfrequency are not. For instance, one must check in one way or another that the modes (10) are orthogonal and properly normalized. If we denote by  $\Xi(t)$  a *classical* state represented by the displacement  $X(t)$  of the oscillator and the wave  $\xi(x, t)$  on the string, the scalar product between two states  $\Xi_1(t)$  and  $\Xi_2(t)$  is

$$\Xi_1(t) \cdot \Xi_2(t) \stackrel{\text{def}}{=} X_1^*(t)X_2(t) + \int \xi_1^*(x, t)\xi_2(x, t) dx. \quad (52)$$

It is shown in the appendix that if  $\Xi_k^{\text{in}}$  (respectively,  $\Xi_{-k}^{\text{in}}$ ) stands for the mode  $\xi_k^{\text{in}}(x, t)$  (respectively,  $\xi_{-k}^{\text{in}}(x, t)$ ) both with  $X^{\text{in}}(t) = \chi(\omega) e^{-i\omega t}$ , then we have, for any (positive and/or negative) real pair  $(k_1, k_2)$ ,

$$\Xi_{k_1}^{\text{in}}(t) \cdot \Xi_{k_2}^{\text{in}}(t) = \delta(k_1 - k_2). \quad (53)$$

It is easy to see that we chose the normalization (45) in order to get

$$\Xi^b(t) \cdot \Xi^b(t) = 1. \quad (54)$$

The free states  $\Xi_{\pm k}^{\text{fr}}$  represented by  $X=0$  and (6) are clearly orthonormalized,  $\Xi_{k_1}^{\text{fr}} \cdot \Xi_{k_2}^{\text{fr}} = \delta(k_1 - k_2)$ , and form a complete basis if we add the state that allows us to describe the motion of the oscillator, namely  $\Xi_{\text{osc}}^{\text{fr}}$  represented by  $X=1$  and  $\xi \equiv 0$ .

### 5.3. The real symmetric modes

The potential in (5) is a real definite positive symmetric quadratic form and therefore can be diagonalized in an orthogonal basis of *real* vectors. The natural choice of retaining the real or the imaginary part of  $\Xi_{\pm k}^{\text{in}}$  actually provides two real modes but that are not orthogonal. A way to assure that we deal with an orthogonal basis, is to pick up a symmetry, say the parity  $x \mapsto -x$ , of the Hamiltonian and classify the eigenmodes accordingly. The bounded state, if there is any, remains even. The in- and out-modes are not symmetric under space inversion but it is straightforward to obtain eigenmodes that are also eigenvectors of parity. For any complex factors  $c_{\pm}$ , the combinations  $c_{\pm}(\Xi_k^{\text{in}} \pm \Xi_{-k}^{\text{in}})$  are symmetric/antisymmetric eigenvectors at any time with the eigenvalues  $\pm 1$ . After some algebraic manipulations using the expressions (26) of  $\rho$  in terms of  $\eta$  given by (27), the symmetric and antisymmetric modes are represented, for  $k > 0$ , by

$$c_+(\xi_k^{\text{in}}(x, t) + \xi_{-k}^{\text{in}}(x, t)) = c_+ e^{-i\eta} \sqrt{\frac{2}{\pi}} \cos(k|x| - \eta) e^{-i\omega t}, \quad (55a)$$

$$c_-(\xi_k^{\text{in}}(x, t) - \xi_{-k}^{\text{in}}(x, t)) = ic_- \sqrt{\frac{2}{\pi}} \sin(kx) e^{-i\omega t}. \quad (55b)$$

The choice  $c_+ = e^{i\eta}/\sqrt{2}$  and  $c_- = -i/\sqrt{2}$  leads to the real normalized symmetric modes, defined for  $k > 0$  by

$$\begin{pmatrix} \Xi_k^+ \\ \Xi_k^- \end{pmatrix} = R(\omega) \begin{pmatrix} \Xi_k^{\text{in}} \\ \Xi_{-k}^{\text{in}} \end{pmatrix} \quad (56)$$

with the unitary matrix

$$R(\omega) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\eta(\omega)} & e^{i\eta(\omega)} \\ -i & i \end{pmatrix}. \quad (57)$$

Constructing the real (anti)symmetric states from the out-modes at any time leads to the same  $\Xi_k^\pm$  since the common eigenspace to  $H$  and to the parity is of dimension one. To sum up, for  $k > 0$ ,  $\Xi_k^\pm$  is represented by

$$\xi_k^+(x, t) = \frac{1}{\sqrt{\pi}} \cos[k|x| - \eta(\omega)] e^{-i\omega t}, \quad X^+(t) = \frac{\kappa}{\sqrt{\pi}} \frac{\cos[\eta(\omega)]}{\Omega_\kappa^2 - \omega^2} e^{-i\omega t}, \quad (58a)$$

$$\xi_k^-(x, t) = \frac{1}{\sqrt{\pi}} \sin(kx) e^{-i\omega t}, \quad X^-(t) = 0, \quad (58b)$$

and  $\{\Xi_b(t)\} \cup \{\Xi_k^\pm(t)\}_{k>0}$  is, at any time, an orthonormalized eigenbasis of symmetric or antisymmetric eigenvectors of the Hamiltonian with eigenvalues given, respectively, by (42) and (7).

#### 5.4. An atom in a closed cavity

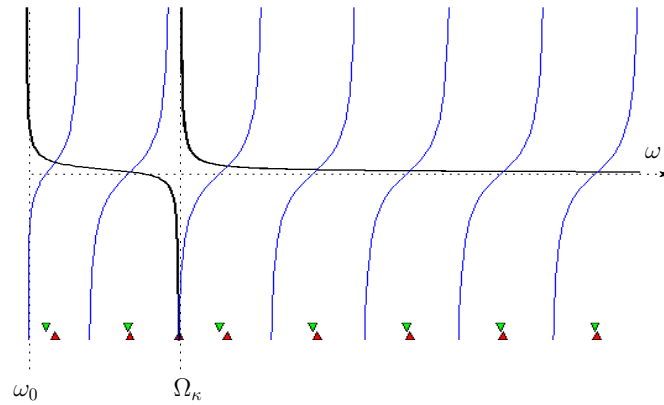
The explicit canonical linear transformation that connects the free canonical variables to the interacting ones (the in or out-modes via the real symmetric ones) is beyond the scope of this paper and will be given and extensively interpreted in a future paper where we will quantize our model. As explained above (see section 2.4 and also the note 7), this is really interesting and beyond a purely academic exercise only if one wants to switch to quantum theory and/or statistical physics. The quantum linear transformation between  $\hat{a}_{\text{in}}(k)$  and  $\hat{a}_{\text{fr}}(k)$  appears to be a generalized Bogoliubov transformation and our model provides an explicit construction of quasi-particles in terms of free particles.

However, the real symmetric modes that have been founded in the previous section remain interesting at the less advanced level of the present paper because they are the natural modes to work with when the finite size  $\ell$  of the string becomes relevant. Indeed, when  $\ell/c$  is not too large compared to the typical time  $\Gamma^{-1}$  characterizing the radiations of the oscillator, the discrete character of the spectrum of the non-interacting string can be ‘felt’ by the oscillator. When boundary conditions are imposed, say  $\xi(\ell/2, t) = \xi(-\ell/2, t) = 0$ , the discrete (even) spectrum of the whole system is modified by the presence of the oscillator and, from (58a) given by the  $\{k_n\}_{n \in \mathbb{Z}}$  that fulfil the equations

$$\frac{1}{2} k_n \ell - \eta(\omega(k_n)) = \frac{\pi}{2} + n\pi \iff \tan(\eta(k)) = \tan(k\ell/2 - \pi/2), \quad (59)$$

that can be solved graphically (figure 5). The frequency  $\Omega_0$  of the free oscillations of the mass comes into the spectrum of the string. The even spectrum will differ from the non-interacting case when  $e^{i\eta}$  is significantly different from one. For resonances with high quality, it will not affect the frequencies that are away from the resonant frequency.

What one gets here, for  $\omega_0 = 0$  is an elementary model of an atom in a (perfect) electrodynamics cavity of size  $\ell$  (some imperfections can be taken into account if we relax the Dirichlet boundary conditions and put partially reflectives ‘mirrors’ on the string). The field may or may not be quantized and, not to speak of lasers, we obtain a sort of primer for the widespread physics of quantum electrodynamics cavities that have been realised in laboratory to test successfully some fundamental concepts in quantum physics (Haroche and Raymond 2006). The purely mechanical model for infinite  $\kappa$  (the mass is directly attached on the string) has been carefully studied with experiments in Gómez *et al* (2007).



**Figure 5.** Graphical resolution of equations (59) that provide the even frequency spectrum when the oscillator is attached to the middle of string of finite size  $\ell$ . The frequencies, represented by small up triangles on a horizontal line at the bottom of the figure, are centred on the abscissae of the intersections of the graph  $\omega \mapsto \tan(\eta(\omega))$  (thick solid line) with the graphs  $\omega \mapsto \tan(k(\omega)\ell - \pi/2)$  (thin solid lines) for  $\omega > \omega_0$ . As in figure 4(a), the quality factor of the resonance is  $Q \simeq 15$ . The down triangles indicate the spectrum of vibration of the Klein–Gordon string alone of finite size  $\ell$  obtained for  $\tan(k(\omega)\ell/2 - \pi/2) = 0$ , that is for  $\omega_n = \sqrt{\omega_0^2 + \pi^2(2n+1)^2/\ell^2}$  with  $n$  being a positive integer.

(This figure is in colour only in the electronic version)

## 6. Conclusion

In addition to a more detailed study of the finite size effects, another natural development of the present work would be to deal with multiple scatterers. For instance, when there are two identical scatterers with  $\omega_0 > \Omega_0$ , we expect that the degeneracies of the two bounded modes are broken and that the splitting between the symmetric and the antisymmetric bounded modes decreases exponentially with the separation of the oscillators. Starting with initial conditions where only one oscillator has some energy, the beating between the two oscillators is an example of tunnelling due to the presence of evanescent waves connecting the two oscillators.

Even before we quantize the whole system, our model may be interesting to keep the field classical whereas only the oscillator is quantized. It would provide an illustration of say, the Fermi golden rule within the context of time-dependent perturbation theory (Cohen-Tannoudji *et al* 1980, chapter XIII). However, we have proven that this golden rule transpires in our classical model since the transition rate (35) to the continuum of the modes is proportionnal to the square of the coupling strength  $\kappa$  to first order in the perturbation.

As has been demonstrated, this model captures many fundamental phenomena that are important in many areas of physics and offers wide possibilities for pedagogical use. Above all, I hope it will help the reader to discover and/or to understand them more deeply.

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## Appendix. Normalization of the modes\*

The construction of the real symmetric modes presented in section 5.3 leads to an orthogonal basis  $\{\Xi_b(t)\} \cup \{\Xi_k^\pm(t)\}_{k>0}$ . When it exists ( $\omega_0 > \Omega_0$ ), the bounded state  $\Xi_b$  has a norm unity. That they are eigenvectors for different eigenvalues of  $H$  or of the parity guarantees that  $\Xi_{k_1}^\pm \cdot \Xi_{k_2}^\pm \propto \delta(k_1 - k_2)$  and this appendix proves that the proportionality factor is indeed unity.

Rewriting (10a) with the help of (13) and with the Heaviside step function  $\Theta$ ,

$$\xi_{\vec{k}}^{\text{in}}(x, t) = \frac{1}{\sqrt{2\pi}} [e^{ikx} + \rho(k) e^{-ikx} \Theta(-x) + \rho(k) e^{ikx} \Theta(x)] e^{-i\omega t}, \quad (\text{A.1})$$

we have, with  $\rho_1 \stackrel{\text{def}}{=} \rho(k_1)$  and  $\rho_2 \stackrel{\text{def}}{=} \rho(k_2)$ ,

$$\begin{aligned} \int \xi_{\vec{k}_1}^*(x, t) \xi_{\vec{k}_2}(x, t) dx &= \delta(k_1 - k_2) + \frac{i}{2\pi} \left[ \rho_2 \left( \frac{1}{k_2 + k_1 + i0^+} + \frac{1}{k_2 - k_1 + i0^+} \right) \right. \\ &\quad \left. + \rho_1^* \left( \frac{1}{k_2 - k_1 + i0^+} - \frac{1}{k_2 + k_1 - i0^+} \right) + \frac{2\rho_1^* \rho_2}{k_2 - k_1 + i0^+} \right]. \end{aligned} \quad (\text{A.2})$$

We have used the identity ( $0^+$  stands for the limit  $\epsilon \rightarrow 0$  keeping  $\epsilon > 0$ )

$$\int_{x_0}^{+\infty} e^{ikx} dx = \frac{i e^{ikx_0}}{k + i0^+} \quad (\text{A.3})$$

valid for any real  $k$ . The other identity

$$\frac{1}{k + i0^+} = \frac{\wp}{k} - i\pi \delta(k) \quad (\text{A.4})$$

allows one to convert (A.2) in terms of the  $\delta$  distribution and of the Cauchy principal value  $\wp$ :

$$\begin{aligned} \int \xi_{\vec{k}_1}^*(x, t) \xi_{\vec{k}_2}(x, t) dx &= \delta(k_1 - k_2) + \rho_1^* \delta(k_1 + k_2) \\ &\quad + \frac{i}{2\pi} \left[ (\rho_2 - \rho_1^*) \frac{\wp}{k_2 + k_1} + (\rho_1^* + \rho_2 + 2\rho_1^* \rho_2) \frac{\wp}{k_2 - k_1} \right]. \end{aligned} \quad (\text{A.5})$$

In fact, the coefficients of the principal values both vanish when the respective denominators cancel (we use (18), (13) and (14), then  $\rho + \rho^* + 2|\rho|^2 = 0$  follows) and we can drop the symbol  $\wp$ . The  $\delta(k_1 + k_2)$  can also be forgotten for, to constitute the basis, we retain only strictly positive values of  $k_1$  and  $k_2$ . A little bit of algebraic jugglery with (25) and (26) allows to check that

$$\frac{i}{2\pi} \left[ \frac{\rho_2 - \rho_1^*}{k_2 + k_1} + \frac{\rho_1^* + \rho_2 + 2\rho_1^* \rho_2}{k_2 - k_1} \right] = -\frac{\kappa^2}{2\pi} \frac{\tau_1^* \tau_2}{(\Omega_\kappa^2 - \omega_1^2)(\Omega_\kappa^2 - \omega_2^2)} = -\chi^*(\omega_1) \chi(\omega_2) \quad (\text{A.6})$$

with  $\tau_n \stackrel{\text{def}}{=} \tau(k_n)$ ,  $\omega_n \stackrel{\text{def}}{=} \omega(k_n)$  ( $n = 1, 2$ ). Then we have proved that, for  $k_1 > 0$  and  $k_2 > 0$ ,

$$\Xi_{k_1}^{\text{in}}(t) \cdot \Xi_{k_2}^{\text{in}}(t) = (X_{k_1}^{\text{in}}(t))^* X_{k_2}^{\text{in}}(t) + \int \xi_{\vec{k}_1}^*(x, t) \xi_{\vec{k}_2}(x, t) dx = \delta(k_1 - k_2). \quad (\text{A.7})$$

The space inversion of this identity leads immediately to  $\Xi_{-k_1}^{\text{in}}(t) \cdot \Xi_{-k_2}^{\text{in}}(t) = \delta(k_1 - k_2)$ . At last, with the same techniques we can obtain

$$\int \xi_{\vec{k}_1}^*(x, t) \xi_{\vec{k}_2}(x, t) dx = (1 + \rho_1^*) \delta(k_1 + k_2) + \frac{i}{2\pi} \left[ \frac{2\tau_2 \tau_1^* - \tau_2 - \tau_1^*}{k_2 - k_1} + \frac{\tau_2 - \tau_1^*}{k_2 + k_1} \right]. \quad (\text{A.8})$$

As above, for  $k_1$  and  $k_2$  both strictly positive,  $\delta(k_1 + k_2)$  vanishes whereas the second term on the right-hand side is precisely  $-(X_{k_1}^{\text{in}}(t))^* X_{-k_2}^{\text{in}}(t)$ . Therefore

$$\Xi_{k_1}^{\text{in}}(t) \cdot \Xi_{-k_2}^{\text{in}}(t) = (X_{k_1}^{\text{in}}(t))^* X_{-k_2}^{\text{in}}(t) + \int \xi_{\vec{k}_1}^*(x, t) \xi_{\vec{k}_2}(x, t) dx = 0. \quad (\text{A.9})$$

The complex conjugation and the time reversal  $t \mapsto -t$  of the above relations allow us to show that the out-modes are also orthonormalized. The orthonormalization of  $\{\Xi_k^\pm\}_{k>0}$  follows from (56) and (57). To sum up, we have obtained the following scalar products, for any  $k_1 > 0$  and  $k_2 > 0$ :

$$\Xi_{\pm k_1}^{\text{in}}(t) \cdot \Xi_{\pm k_2}^{\text{in}}(t) = \delta(k_1 - k_2), \quad \Xi_{\pm k_1}^{\text{in}}(t) \cdot \Xi_{\mp k_2}^{\text{in}}(t) = 0, \quad (\text{A.10})$$

$$\Xi_{\pm k_1}^{\text{out}}(t) \cdot \Xi_{\pm k_2}^{\text{out}}(t) = \delta(k_1 - k_2), \quad \Xi_{\pm k_1}^{\text{out}}(t) \cdot \Xi_{\mp k_2}^{\text{out}}(t) = 0, \quad (\text{A.11})$$

$$\Xi_{k_1}^\pm(t) \cdot \Xi_{k_2}^\pm(t) = \delta(k_2 - k_2), \quad \Xi_{k_1}^\pm(t) \cdot \Xi_{k_2}^\mp(t) = 0, \quad (\text{A.12})$$

and, when  $\omega_0 > \Omega_0$  for a unique bounded state to exist,

$$\Xi_b(t) \cdot \Xi_b(t) = 1, \quad (\text{A.13})$$

$$\Xi_{\pm k_1}^{\text{in}}(t) \cdot \Xi_b(t) = 0, \quad \Xi_{\pm k_1}^{\text{out}}(t) \cdot \Xi_b(t) = 0, \quad \Xi_{k_1}^\pm(t) \cdot \Xi_b(t) = 0. \quad (\text{A.14})$$

Three eigenbases for the Hamiltonian have been chosen:  $\{\Xi_b(t)\} \cup \{\Xi_k^{\text{in}}(t)\}_{k \in \mathbb{R}}, \{\Xi_b(t)\} \cup \{\Xi_k^{\text{out}}(t)\}_{k \in \mathbb{R}}$  and  $\{\Xi_b(t)\} \cup \{\Xi_k^\pm(t)\}_{k>0}$ . The passage from one to the other is done with unitary matrices made of independent  $2 \times 2$  unitary blocks of  $S(\omega)$  or  $R(\omega)$  given by (20) and (57), respectively.

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