

## ALGEBRAIC SPECTRAL GAPS

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**Résumé.**

### INTRODUCTION

A spectral gap (or eigengap) of a self-adjoint operator is a closed real interval to which no eigenvalues belongs. In [2] we have presented a systematic method for finding gaps in the discrete part of the spectrum of a one-dimensional non-magnetic Schrödinger equation with a potential  $V(x)$ . When  $V$  is a polynomial half-bounded from below, the boundaries of the gaps are given by the real zeroes of a family of polynomials whose degree  $D$  may be arbitrary large. The construction of these polynomials is provided by an explicit and straightforward algorithm. For still not understood reasons, it happens that in every case we have considered, our method works surprisingly well when compared to numerical computations: when increasing  $D$  the more and more numerous intervals we compute resolve the spectrum from below (i.e. the lowest eigenvalues are separated by at least one gap) and the infima of each interval seem to converge quickly to the eigenvalues. For the moment, we have no clue to understand these two phenomena and our method comes without any estimation of the distance between the gaps and the spectrum.

Being as local as possible (no computation of integrals is required), our method differs strongly in spirit from other spectral approximations like the Rayleigh-Ritz variational methods or the Rayleigh-Schrödinger perturbation methods. Since generically the spectrum cannot be determined exactly, finding such gaps may offer a valuable piece of spectral information, complementary to the information obtained by other methods.

One natural issue is to try to extend our method to multi-dimensional systems. Then, the nature of the spectrum depends on the integrability properties of the system (among the vast literature on this subject see for instance [3]). Qualitatively the statistical distribution of the eigenvalues exhibits different correlations according to the nature of its symmetries. For instance, for non-integrable systems, the two point correlation function of the discrete spectrum exhibits a so-called “level repulsion” because, unlike what occurs in integrable cases, the probability of finding two successive eigenvalues whose distance is  $s$  vanishes when  $s$  tends to zero. Therefore we expect that this dichotomy between integrable and non-integrable cases should somehow appear in any general method for finding spectral gaps. However, most unfortunately, we have not been able to generalise our strategy in higher dimensions. Although there was *a priori* no obstacle in sight to such an attempt, it happened that the origin of the obstruction came from very subtle arguments that are deeply hidden. One aim of this note is to explain (in section 2) this negative result with the hope that it may help to find out some way to bypass the pitfalls or, at least, to help avoiding the same tracks.

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Our second aim is more optimistic but still rely on speculative grounds. After recalling the main ingredients of our method in section 1.1, I will introduce a systematic algebraic approach which at first sight seems to rephrase in a more elaborate way what we have done in [2]. However by associating with the Schrödinger equation the closed algebra of differential operators that will be introduced in §§ 1.2, 1.3, we can easily guess a fruitful strategy to deal with spectral problems associated with more general (1d) linear equations — for instance of order larger than two — or even with non-linear equation like the Gross-Pitaevskii equation ( $g$  is a real coupling constant)

$$\varphi'' = (V - E)\varphi + g\varphi^3. \quad (1)$$

## 1. THE ONE DIMENSIONAL CASE: AN ALGEBRAIC APPROACH

### 1.1. The principle of the method

The stationary one-dimensional non-magnetic Schrödinger equation can be written as follows

$$\frac{d^2\varphi}{dx^2} = 2(V(x) - E)\varphi. \quad (2)$$

where we will take the real potential  $V$  to be smoothly defined on  $\mathbf{R}$ . The real  $E$  will be an eigenvalue whenever the real function  $\varphi$  is square integrable on  $\mathbf{R}$ . The key idea of our method is to construct, for a given integer  $N \geq 1$ , a real function  $J_N(\varphi', \varphi, x, E)$  (the prime stands for the derivative  $d/dx$ ) such that

$$\begin{aligned} \text{(i)} \quad & \frac{d}{dx} \left( J_N(\varphi'(x), \varphi(x), x, E) \right) = (\varphi(x))^N F_N(x, E), \\ \text{(ii)} \quad & \lim_{|x| \rightarrow +\infty} J(\varphi'(x), \varphi(x), x, E) = 0. \end{aligned}$$

The real function  $F_N$  of both the spatial coordinate  $x$  and the energy  $E$  is  $\varphi$ -independent and is obtained from the potential  $V$  and its derivatives. For instance we will find the following expressions:

$$F_1 = -a_0'' + 2(V - E)a_0, \quad (3a)$$

$$F_2 = \frac{1}{2}a_0''' - 4(V - E)a_0' - 2V'a_0, \quad (3b)$$

$$F_3 = -\frac{1}{6}a_0^{(iv)} + \frac{10}{3}(V - E)a_0'' + \frac{10}{3}V'a_0' + (V'' - 6(V - E)^2)a_0, \quad (3c)$$

$$\begin{aligned} F_4 = & \frac{1}{24}a_0^{(v)} - \frac{5}{3}(V - E)a_0''' - \frac{5}{2}V'a_0'' + \left(-\frac{3}{2}V'' + \frac{32}{3}(V - E)^2\right)a_0' \\ & + \left(-\frac{1}{3}V''' + \frac{32}{3}V'(V - E)\right)a_0. \end{aligned} \quad (3d)$$

where  $a_0$  is any smooth function such that

$$\lim_{|x| \rightarrow \infty} a_0(x)|\varphi|^N = 0 \quad (4)$$

which is not very restrictive since we know using semiclassical arguments that  $\varphi$  itself is exponentially decreasing at infinity [1, chap.10]:

$$\varphi(x) \underset{|x| \rightarrow \infty}{\sim} e^{-\int^{|x|} \sqrt{2(V(x)-E)} dx} \quad (5)$$

Condition (i) is the cornerstone of our method and, before justifying how it can be obtained (we will see that condition (ii) is not so restrictive), let us first explain how gaps in the spectrum may be obtained.

When the conditions (i) and (ii) are simultaneously fulfilled, an immediate consequence is that the integral  $\int_{-\infty}^{+\infty} (\varphi(x))^N F_N(x, E) dx$  vanishes. This implies that, if  $E$  is truly an eigenenergy, the function  $x \mapsto (\varphi(x))^N F_N(x, E)$  should change its sign. If  $N$  is even, we obtain a  $\varphi$ -independent condition: *for any fixed energy  $x \mapsto F_N(x, E)$  must change its sign on the real axis.* For such a one-dimensional problem, and for a given  $N$ , we still can choose  $F_N$  in a wide continuous set of smooth functions on the real axis because we have a lot of freedom in choosing  $a_0$ . A forbidden value of  $E$  (*i.e.*  $E$  cannot be an eigenenergy) is obtained if we are able to chose  $a_0$  such that  $F_N$  remains positive on the whole  $x$ -axis. Once this property is achieved, it remains stable under small perturbations within the set of  $a_0$ 's, for instance by varying the control parameters  $\lambda$  on which  $a_0$  may depend, and we obtain a whole interval where no eigenenergy can exist. More precisely, if we introduce explicitly the  $\lambda$ -dependence in  $F_N$ , the boundaries of the gaps will necessary be given by some solutions of the system of equations

$$F_N(x, E, \lambda) = 0 ; \quad (6a)$$

$$\partial_x F_N(x, E, \lambda) = 0 ; \quad (6b)$$

$$\partial_\lambda F_N(x, E, \lambda) = 0 . \quad (6c)$$

Using the implicit function theorem where

$$\begin{vmatrix} \partial_E F_N & \partial_{xE}^2 F_N \\ \partial_x F_N & \partial_{xx}^2 F_N \end{vmatrix} = |\partial_E F_N| |\partial_{xx}^2 F_N| \neq 0 , \quad (6d)$$

the first two equations define implicitly  $x(\lambda)$  and  $E(\lambda)$ ; these are the conditions for a bifurcation in the zeroes of  $x \mapsto F_N(x, E, \lambda)$  to occur. On one side of the bifurcation  $x \mapsto F_N(x, E, \lambda)$  has locally a constant sign (and therefore the corresponding value  $E(\lambda)$  is forbidden) whereas on the other side  $x \mapsto F_N(x, E, \lambda)$  locally changes its sign and  $E(\lambda)$  cannot be ruled out from the spectrum. To put it differently, in the  $(x, E, \lambda)$ , the set of zeroes of  $F_N$  becomes tangent to the  $x$ -space. Then for each value of  $\lambda$ ,  $E(\lambda)$  is a candidate for being the boundary of a spectral gap. If this is the case, we can reach an extremal value provided  $0 = \partial E / \partial \lambda = -\partial_\lambda F_N / \partial_E F_N$  that leads to the equation (6c).

See [2] for an effective implementation of this method and for applications. In the following, we will remain at a more formal level and let us start by defining some notations.

## 1.2. Algebraic construction of condition (i) and classification of the possible $F_N$ 's

Denote by  $\hat{\mathcal{P}}_N$ , the vector space of smooth applications from  $\mathbf{R}$  to  $\mathbf{R}^{N+1}$ . Any element  $a$  of  $\hat{\mathcal{P}}_N$  may be represented by a vector field  $x \mapsto (a_n(x))_{n \in \{0, \dots, N\}} = (a_0(x), \dots, a_N(x))$  and can be associated in a one-to-one correspondence with the homogenous polynomial of degree  $N$  in the two variables  $\Phi$  and  $\Psi$ :

$$P_a(\Psi, \Phi, x) \stackrel{\text{def}}{=} \sum_{n=0}^N a_n(x) \Psi^{N-n} \Phi^n ; \quad (7)$$

then, it may be used to construct the real function on  $\mathbf{R}$  defined by

$$P_a(\varphi'(x), \varphi(x), x) \stackrel{\text{def}}{=} \sum_{n=0}^N a_n(x) (\varphi'(x))^{N-n} (\varphi(x))^n . \quad (8)$$

For computations we will distinguish the “total” derivative  $D$  of a function  $P(\varphi'(x), \varphi(x), x)$  from its partial derivatives  $\partial_{\varphi'}$ ,  $\partial_\varphi$  and  $\partial_x$ :

$$DP(\varphi'(x), \varphi(x), x) \stackrel{\text{def}}{=} v(x) \varphi(x) \partial_{\varphi'} P(\varphi'(x), \varphi(x), x) + \varphi'(x) \partial_\varphi P(\varphi'(x), \varphi(x), x) + \partial_x P(\varphi'(x), \varphi(x), x) \quad (9)$$

where the substitution  $\varphi'' = v\varphi$  has been made since we suppose that  $\varphi$  fulfills (2). For simplicity we have left implicit the  $E$ -dependence in

$$v(x) \stackrel{\text{def}}{=} 2(V(x) - E). \quad (10)$$

From its very definition, it is obvious that the set  $\mathcal{P}_N$  of homogenous polynomials of degree  $N$  in  $\varphi'(x)$  and  $\varphi(x)$  is stable under  $D$  and, moreover,  $D$  is represented by a linear operator  $\hat{D}$  in  $\hat{\mathcal{P}}_N$ :

$$\hat{D} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \\ \vdots \\ a_{N-2} \\ a_{N-1} \\ a_N \end{pmatrix} = \begin{pmatrix} \partial_x & 1 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ Nv & \partial_x & 2 & 0 & 0 & \ddots & & & \vdots \\ 0 & (N-1)v & \partial_x & 3 & 0 & \ddots & & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & (N-n+1)v & \partial_x & n+1 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 0 & 3v & \partial_x & N-1 & 0 \\ \vdots & \vdots & & \ddots & 0 & 0 & 2v & \partial_x & N \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & v & \partial_x \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \\ \vdots \\ a_{N-2} \\ a_{N-1} \\ a_N \end{pmatrix}. \quad (11)$$

For each  $n \in \{0, \dots, N\}$ , denote by  $\hat{\mathcal{Q}}_n$  the subspace of  $\hat{\mathcal{P}}_N$  defined by  $a_n = 0$  and  $\hat{\hat{\mathcal{Q}}}_n$  its complementary defined by the direct sum decomposition  $\hat{\mathcal{P}}_N = \hat{\mathcal{Q}}_n \oplus \hat{\hat{\mathcal{Q}}}_n$ . Looking for all the  $J_N$  that fulfill condition (i) can therefore be interpreted as the determination in  $\hat{\mathcal{P}}_N$  of the preimage  $\hat{D}^{-1}\hat{\hat{\mathcal{Q}}}_N$ . In [2] we have shown how to straightforwardly compute  $J_N$  and obtain  $F_N$  but let us propose a strategy based on a more algebraic formalism that may be useful as a warming up for higher dimensions.

From any  $a$ , we can systematically reduce the degree in  $\varphi'$  of  $P_a$  if we work up to a total derivative. Indeed, for any monomial characterised by  $n \in \{0, \dots, N-1\}$ , we use the identity

$$a_n \varphi'^{N-n} \varphi^n = a_n \varphi'^{N-n-1} D \left( \frac{1}{n+1} \varphi^{n+1} \right); \quad (12)$$

$$= -\frac{1}{n+1} a'_n \varphi'^{N-n-1} \varphi^{n+1} - \frac{N-n-1}{n+1} v a_n \varphi'^{N-n-2} \varphi^{n+2} + D \left( \frac{1}{n+1} a_n \varphi'^{N-n-1} \varphi^{n+1} \right), \quad (13)$$

where, again, we have substituted  $\varphi''$  by  $v\varphi$ . The operation that transforms  $a_n\varphi'^{N-n}\varphi^n$  to the two first terms in (13) may be linearly represented in  $\hat{\mathcal{P}}_N$  by the reduction operator defined by

$$\hat{\mathbf{R}} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \\ \vdots \\ a_{N-1} \\ a_N \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ -\partial_x & 0 & 0 & \ddots & & & & \vdots \\ (1-N)v & -\frac{1}{2}\partial_x & \ddots & \ddots & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \frac{n-1-N}{n-1}v & -\frac{1}{n}\partial_x & 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 & \frac{-2}{N-2}v & -\frac{1}{N-1}\partial_x & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & \frac{-1}{N-1}v & -\frac{1}{N}\partial_x & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \\ \vdots \\ a_{N-1} \\ a_N \end{pmatrix}. \quad (14)$$

For  $N = 1$ , we have  $\hat{\mathbf{R}} = \begin{pmatrix} 0 & 0 \\ -\partial_x & 1 \end{pmatrix}$ . By construction we have  $a - \hat{\mathbf{R}}a = 0$  for any vector  $a = (0, \dots, 0, a_N) \in \hat{\mathcal{Q}}_N$ ; by (13), we have  $a - \hat{\mathbf{R}}a \in \text{Im}\hat{\mathbf{D}}$  for any vector  $a \in \hat{\mathcal{Q}}_n$  with  $n \in \{0, \dots, N-1\}$ ; therefore, by linearity, we have  $a - \hat{\mathbf{R}}a \in \text{Im}\hat{\mathbf{D}}$  for any  $a \in \hat{\mathcal{P}}_N$ . Moreover, since  $\hat{\mathbf{R}}$  is a lower triangular matrix with all diagonal terms but one being zero, all the components of  $\hat{\mathbf{R}}^N a$  vanish but the last one  $A_N[a] \stackrel{\text{def}}{=} (\hat{\mathbf{R}}^N a)_N$  which is a differential operator on  $a$  of order  $N$ . For instance we have

$$A_1[a] = -a'_0 + a_1; \quad (15a)$$

$$A_2[a] = \frac{1}{2}a''_0 - va_0 - \frac{1}{2}a'_1 + a_2; \quad (15b)$$

$$A_3[a] = -\frac{1}{6}a'''_0 + \frac{7}{6}va'_0 + \frac{2}{3}v'a_0 + \frac{1}{6}a''_1 - \frac{1}{2}va_1 - \frac{1}{3}a'_2 + a_3; \quad (15c)$$

$$A_4[a] = \frac{1}{24}a^{(iv)}_0 - \frac{2}{3}va''_0 - \frac{3}{4}v'a'_0 - \left(\frac{1}{4}v'' - v^2\right)a_0 - \frac{1}{24}a'''_1 + \frac{5}{12}va'_1 + \frac{1}{4}v'a_1 + \frac{1}{12}a''_2 - \frac{1}{3}va_2 - \frac{1}{4}a'_3 + a_4. \quad (15d)$$

Now if we use the decomposition

$$1 = \hat{\mathbf{R}}^N + (1 - \hat{\mathbf{R}}) \sum_{n=0}^{N-1} \hat{\mathbf{R}}^n, \quad (16)$$

it can be seen immediately that any  $a$  can be uniquely decomposed in  $\hat{\mathbf{R}}^N a \in \hat{\mathcal{Q}}_N$  plus a vector in  $\text{Im}\hat{\mathbf{D}}$ . Translating this decomposition into the language of functions and taking for  $a$  the vector associated to  $\mathbf{D}K_N(\varphi'(x), \varphi(x), x)$  where  $K_N$  is any homogenous polynomial of degree  $N$  in  $(\varphi'(x), \varphi(x))$ , we have shown that there always exists a homogenous polynomial  $\tilde{K}_N(\varphi'(x), \varphi(x), x)$  of degree  $N$  in  $(\varphi'(x), \varphi(x))$ , such that

$$\mathbf{D}K_N(\varphi'(x), \varphi(x)) = F_N(x)(\varphi(x))^N - \mathbf{D}\tilde{K}_N(\varphi'(x), \varphi(x)) \quad (17)$$

and therefore, in order to recover (i), it is sufficient to choose  $J_N = K_N + \tilde{K}_N$ . The function  $F_N(x)$  is independent on  $\varphi$  and  $\varphi'$  and is just given by the action of the linear operator  $A_N$  on the coefficients of  $\mathbf{D}K_N$ .

A priori, we can start with any set of trial functions  $(a_n)_{n \in \{1, \dots, N\}}$  to build up our  $K_N$ , then compute  $F_N$  by computing the  $N^{\text{th}}$  power of  $\hat{\mathbf{R}}$ . Before we try to control the sign of  $F_N$  for even  $N$ , the only restriction

so far on the  $a$ 's is to preserve (ii):  $a_n$  should not increase faster than  $\varphi'^{N-n}\varphi^n$  at  $|x| \rightarrow \infty$ . However we will now show that our freedom is in fact restricted to the choice of one test function only. In other words, many different choices of  $(a_n)_n$  will lead to the same  $F_N$  and therefore will not help to gain any piece of information (in particular those leading to an identically vanishing  $F_N$ ). To put it very qualitatively,  $\mathcal{Q}_N$  is a very thin subspace in  $\mathcal{P}_N$  (of co-dimension  $N$  if seen as a vector space on smooth real functions) and the kernel of  $D$  is too small (given by the solutions of a linear ordinary differential equation of order  $N$ ) for  $D^{-1}\mathcal{Q}_N$  to decrease its codimension. To understand that, let us introduce the projector  $\hat{\Pi}_n$  on the  $n^{\text{th}}$  component of  $a$ . Our previous construction of condition (i) can therefore be re-written

$$\hat{D}a = \hat{R}^N \hat{D}a . \quad (18)$$

where  $a$  is the element of  $\hat{\mathcal{P}}_N$  associated with the function  $J_N$ . In section 1.3 we will show that

$$\hat{R}^N \hat{D}(1 - \hat{\Pi}_0) = 0 \quad (19)$$

which has the following consequence: adding to  $a$  any vector  $b = (b_n)_n$  whose  $b_0 = 0$  will not affect the left hand side of (18) from which  $F_N$  is computed. Therefore  $F_N$  depends only on one function, namely  $a_0$ . All the others can be canceled without loss of generality. Actually, if we start with  $a = (a_0, 0, \dots, 0)$ , we have  $\tilde{a} \stackrel{\text{def}}{=} \hat{D}a = (a'_0, Nva_0, 0, \dots, 0) = (\tilde{a}_0, \tilde{a}_1, 0, \dots, 0)$  and substituting  $\tilde{a}$  with  $a$  in (15) leads straightforwardly to (3). If we start with  $a = (0, a_1, 0, \dots, 0)$ , we have  $\tilde{a} \stackrel{\text{def}}{=} \hat{D}a = (a_1, a'_1, (N-1)va_1, 0, \dots, 0) = (\tilde{a}_0, \tilde{a}_1, 0, \dots, 0)$  and (19) can be (tediously) checked in the special cases  $N = 2, 3$  and  $4$ : by substituting  $\tilde{a}$  with  $a$  in (15),  $\Lambda_N$  identically vanishes. The same remains true for  $a = (0, 0, a_2, 0, \dots, 0)$ ,  $a = (0, 0, 0, a_3, 0, \dots, 0)$ , etc.

### 1.3. The closed algebra

Let us prove (19) by introducing an algebra of operators that may be represented by their action on  $\hat{\mathcal{P}}_N$  or on the functions in  $\mathcal{P}_N$ . The operator  $S^\downarrow \stackrel{\text{def}}{=} \varphi \partial_{\varphi'}$  when multiplied by  $v$  allow to formalise the substitution of  $\varphi''$  by  $v\varphi$ . We also define  $S^\uparrow \stackrel{\text{def}}{=} \varphi' \partial_\varphi$ . The arrows recall that these operators raise and lower the component of  $a$ :

$$\hat{S}^\downarrow \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{N-2} \\ a_{N-1} \\ a_N \end{pmatrix} = \begin{pmatrix} 0 \\ Na_0 \\ (N-1)a_1 \\ \vdots \\ 3a_{N-3} \\ 2a_{N-2} \\ a_{N-1} \end{pmatrix} ; \quad \hat{S}^\uparrow \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{N-2} \\ a_{N-1} \\ a_N \end{pmatrix} = \begin{pmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ \vdots \\ (N-1)a_{N-1} \\ Na_N \\ 0 \end{pmatrix} . \quad (20)$$

Both operators are nilpotent ( $\hat{S}^\downarrow)^{N+1} = 0$ ,  $(\hat{S}^\uparrow)^{N+1} = 0$ . Some products with the projectors on the first and last component vanish:

$$\hat{S}^\downarrow \hat{\Pi}_N = 0, \quad \hat{\Pi}_0 \hat{S}^\downarrow = 0, \quad \hat{S}^\uparrow \hat{\Pi}_0 = 0, \quad \hat{\Pi}_N \hat{S}^\uparrow = 0 . \quad (21)$$

Within this formalism, the operator (11) is

$$\hat{D} = 1\partial_x + v\hat{S}^\downarrow + \hat{S}^\uparrow . \quad (22)$$

We can also express the reduction operator  $\hat{R}$  given by (14) in terms of the shift operators  $\hat{S}$ . The simplest way is to work within  $\mathcal{P}_N$  and express the first two terms of (13) in terms of the basic monomial  $a_n \varphi'^{N-n} \varphi^n$ .

For  $n \in \{0, \dots, N-1\}$ , we have (note that  $S^\downarrow$  and  $1\partial_x$  commute).

$$R(a_n \varphi'^{N-n} \varphi^n) = -\frac{1}{n+1} a'_n \varphi'^{N-n-1} \varphi^{n+1} - \frac{N-n-1}{n+1} v a_n \varphi'^{N-n-2} \varphi^{n+2}; \quad (23)$$

$$= \frac{1}{(n+1)(n-N)} (\partial_x \varphi \partial_{\varphi'} + v(\varphi \partial_{\varphi'})^2) a_n \varphi'^{N-n} \varphi^n; \quad (24)$$

$$= \frac{1}{(n+1)(n-N)} (S^\downarrow \partial_x + v(S^\downarrow)^2) a_n \varphi'^{N-n} \varphi^n. \quad (25)$$

For when  $n = N$ , we have

$$R(a_N \varphi^N) = a_N \varphi^N. \quad (26)$$

If we introduce the diagonal matrix

$$\hat{\Lambda} \stackrel{\text{def}}{=} \begin{pmatrix} -\frac{1}{N} & 0 & 0 & & & \\ 0 & \frac{1}{2(1-N)} & 0 & & & \\ & & \ddots & & & \\ & & & 0 & \frac{1}{(n+1)(n-N)} & 0 \\ & & & & \ddots & \\ & & & & 0 & -\frac{1}{N} & 0 \\ & & & & 0 & 0 & 1 \end{pmatrix}, \quad (27)$$

we can sum up the action of  $R$  by

$$\hat{R} = (1\partial_x + v\hat{S}^\downarrow) \hat{S}^\downarrow \hat{\Lambda} (1 - \hat{\Pi}_N) + \hat{\Pi}_N. \quad (28)$$

The commutation of  $\hat{\Lambda}$  and  $\hat{\Pi}_N$  and the first identity of (21) show that

$$\hat{S}^\downarrow \hat{\Lambda} \hat{\Pi}_N = 0 \quad (29)$$

and allow to get a simplified expression

$$\hat{R} = (1\partial_x + v\hat{S}^\downarrow) \hat{S}^\downarrow \hat{\Lambda} + \hat{\Pi}_N. \quad (30)$$

Now, for the same reason, in the expansion of the product

$$\hat{R}^N = [(1\partial_x + v\hat{S}^\downarrow) \hat{S}^\downarrow \hat{\Lambda} + \hat{\Pi}_N] [(1\partial_x + v\hat{S}^\downarrow) \hat{S}^\downarrow \hat{\Lambda} + \hat{\Pi}_N] \cdots [(1\partial_x + v\hat{S}^\downarrow) \hat{S}^\downarrow \hat{\Lambda} + \hat{\Pi}_N] \quad (31)$$

many terms vanish. Moreover with  $(\hat{\Pi}_N)^n = \hat{\Pi}_N$  for  $n \geq 1$ ,

$$\hat{R}^N = [(1\partial_x + v\hat{S}^\downarrow) \hat{S}^\downarrow \hat{\Lambda}]^N + \hat{\Pi}_N \sum_{n=0}^{N-1} [(1\partial_x + v\hat{S}^\downarrow) \hat{S}^\downarrow \hat{\Lambda}]^n. \quad (32)$$

In the expansion of  $[(1\partial_x + v\hat{S}^\downarrow) \hat{S}^\downarrow \hat{\Lambda}]^N$  every term in  $v$  comes with a product of at least  $N+1$  operators that lower by one unity the components of each vector they are acting on. Therefore these contributions vanish and we are left with

$$\hat{R}^N = [\partial_x \hat{S}^\downarrow \hat{\Lambda}]^N + \hat{\Pi}_N \sum_{n=0}^{N-1} [(1\partial_x + v\hat{S}^\downarrow) \hat{S}^\downarrow \hat{\Lambda}]^n. \quad (33)$$

Faire agir ensuite chaque terme sur

$$\hat{D}(1 - \hat{\Pi}_0) = [1\partial_x + v\hat{S}^\downarrow + \hat{S}^\uparrow](1 - \hat{\Pi}_0) \quad (34)$$

en utilisant le genre d'argument précédent...

Peut-être faut-il finalement plutôt (ou en plus) introduire les opérateurs de création et d'annihilation

$$\hat{S}^+ \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{N-2} \\ a_{N-1} \\ a_N \end{pmatrix} = \begin{pmatrix} 0 \\ a_0 \\ a_1 \\ \vdots \\ a_{N-3} \\ a_{N-2} \\ a_{N-1} \end{pmatrix} ; \quad \hat{S}^- \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{N-2} \\ a_{N-1} \\ a_N \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{N-1} \\ a_N \\ 0 \end{pmatrix} ; \quad (35)$$

et de dérivation  $S_0 \stackrel{\text{def}}{=} \varphi \partial_\varphi$ ,  $S_N \stackrel{\text{def}}{=} \varphi' \partial_{\varphi'}$

$$\hat{S}_0 \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{N-2} \\ a_{N-1} \\ a_N \end{pmatrix} = \begin{pmatrix} 0 \\ a_1 \\ 2a_2 \\ \vdots \\ (N-2)a_{N-2} \\ (N-1)a_{N-1} \\ Na_N \end{pmatrix} ; \quad \hat{S}_N \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{N-2} \\ a_{N-1} \\ a_N \end{pmatrix} = \begin{pmatrix} Na_0 \\ (N-1)a_1 \\ (N-2)a_2 \\ \vdots \\ 2a_{N-2} \\ a_{N-1} \\ 0 \end{pmatrix} . \quad (36)$$

et exprimer tous les opérateurs précédents avec uniquement ceux-là et  $\partial_x$ ... Même les projecteurs peuvent s'exprimer à l'aide de séries infinies de ces opérateurs (ce qui montre qu'ils engendrent tout ce que l'on veut). Par exemple,  $\Pi_0$  n'est autre que l'évaluation en  $\varphi = 0$  qui s'écrit

$$\Pi_0 = \sum_{n=0}^{\infty} \frac{(-\varphi)^n}{n!} \partial_\varphi^n \quad (37)$$

pour n'importe quelle fonction analytique. En utilisant  $[\partial_\varphi, \varphi] = 1$ , on peut exprimer chaque terme de la somme en fonction de  $S_0$ . Ressemble à une exponentielle en ordre normal.

Est-ce que ce genre d'algèbre d'opérateurs nilpotents (plus l'identité) est connue ?

#### 1.4. Generalization

Pour des équations linéaires d'ordre plus élevé  $\varphi^n = \dots$ , on peut introduire des opérateurs de substitution, etc. en restant dans  $\hat{\mathcal{P}}_N$ . Pour Gross-Pitaevskii, on sort de  $\hat{\mathcal{P}}_N$  mais on peut toujours essayer de travailler dans  $\bigoplus_N \hat{\mathcal{P}}_N$  en faisant la somme directe des opérateurs définis précédemment. On devra alors certainement relâcher la condition (i) en factorisant par un polynôme strictement positif en  $\varphi$  et non plus uniquement  $\varphi^N$ .

## 2. HIGHER DIMENSIONAL CASES

One natural generalization to dimension  $d$  is to consider the Schrödinger equation

$$\Delta_d \varphi = \sum_{\mu=1}^d \partial_\mu^2 \varphi = 2(V - E)\varphi = v\varphi \quad (38)$$



where  $x = (x^\mu)_\mu \in \mathbf{R}^d$  (we keep definition (10)). In the following, greek indices always label the dimension and we will follow the usual convention of letting implicit the sum from 1 to  $d$  over repeated greek indices unless the opposite is specified. The partial derivative with respect to the  $\mu^{\text{th}}$  coordinate is denoted by  $\partial_\mu \stackrel{\text{def}}{=} \partial/\partial x^\mu$ .

We will work within the space  $\hat{\mathcal{P}}_{N,d} \stackrel{\text{def}}{=} \bigotimes_{N=1}^d \hat{\mathcal{P}}_N$  of smooth real functions  $(a_n)_n$  where  $n = (n_1, \dots, n_d)$  is a multi-index with  $n_\mu \in \{0, \dots, N\}$  from which we can construct the set  $\mathcal{P}_{N,d}$  of functions built from homogenous polynomials, namely having the form

$$P_a(\partial\varphi(x), \varphi(x), x) \stackrel{\text{def}}{=} \sum_n a_n (\partial\varphi)^{N-n} \varphi^{N(1-d)+|n|} \Big|_x \quad (39)$$

where  $|n| = n_1 + \dots + n_d$ ,  $(\partial\varphi)^{N-n}$  stands for  $(\partial_1\varphi)^{N-n_1}(\partial_2\varphi)^{N-n_2} \dots (\partial_d\varphi)^{N-n_d}$ . To apply the same reasoning that led to gaps in the spectrum, condition (i) will be extended in  $d$  dimension by looking for a current  $J_N = (J_N^\mu)_\mu$  whose divergence can be factorised by a  $\varphi$ -independent function times a positive function. More precisely, each  $J_N^\mu$  is associated with an element of  $\hat{\mathcal{P}}_{N,d}$  and is constructed in order to fulfill

$$(i)_d \quad D_\mu \left( J_N^\mu(\partial\varphi(x), \varphi(x), x, E) \right) = \varphi^N F_N(x, E) . \quad (40)$$

Then by integrating it on the whole  $\mathbf{R}^d$ , provided that

$$(ii)_d \quad \int_{\mathcal{V}} D_\mu J_N^\mu d^d x = \int_{\partial\mathcal{V}} J_N^\mu d^{d-1} \sigma_\mu \rightarrow 0 \quad (41)$$

where  $\mathcal{V}$  is a closed radius whose typical length  $R$  tend to infinity ( $d^{d-1}\sigma_\mu$  is the measure on its boundary  $\partial\mathcal{V}$  whose surface grows algebraically with  $R$ , therefore any exponential decrease of  $J_N$  will guarantee  $(ii)_d$ ), the condition that  $F_N(x, E)$  should change its sign for even  $N$  will hopefully lead to some constraints on  $E$ . The total derivative is defined as the linear operator in  $\mathcal{P}_{N,d}$

$$D_\mu \left( P_a(\partial\varphi(x), \varphi(x), x) \right) \stackrel{\text{def}}{=} \frac{\partial P_a}{\partial(\partial_\nu\varphi)} \partial_{\nu,\mu}^2 \varphi + \frac{\partial P_a}{\partial\varphi} \partial_\mu \varphi + \partial_\mu P_a . \quad (42)$$

When  $d > 1$ , we cannot get rid of the second derivatives of  $\varphi$  as easily as for  $d = 1$  because (38) provides us with only one substitution rule<sup>1</sup>: it is only when grouped into a Laplacian, that the substitution  $\sum_{\mu=1}^d \partial_\mu^2 \varphi = v\varphi$  can be done. If we start looking for a  $J_N$  from a generic  $a$ , grouping the second derivative in  $D_\mu J_N^\mu$  into Laplacians will eventually impose some relations on the  $(a_n)_n$ . Following what we have explained in the previous section, we will however systematically work up to a total derivative. We can also extend the factorisation in  $(i)_d$  to other functions  $\varphi$  and its derivative whose sign is fixed. Rather than  $\varphi^N$ , we still can apply the argument if we manage to obtain

$$(i)'_d \quad D_\mu \left( J_N^\mu(\partial\varphi(x), \varphi(x), x, E) \right) = \left( B_{N/2}(\partial\varphi(x), \varphi(x), x, E) \right)^2 F_N(x, E) . \quad (43)$$

for even  $N$  where  $B_{N/2}$  is an homogenous polynomial in  $(\partial\varphi, \varphi)$  of degree  $N/2$ .

## 2.1. Attempt for $N = 2$

Let us tentatively start with

$$J_2^\mu = g_0^{\mu\nu\rho} \partial_\nu \varphi \partial_\rho \varphi + g_1^{\mu\nu} \partial_\nu \varphi + g_2^\mu \quad (44)$$

<sup>1</sup>Without further information on  $V$ . In the non-generic case of a separable potential, there are in fact  $d$  independent substitution rules. The substitution rules may also implement some symmetries if there are any, like in the case of integrable systems.

where the  $g$ 's are smooth functions of  $\varphi$ ,  $x$  and  $E$ . We will start with  $d$  functions of type  $g_2$ ,  $d^2$  functions of type  $g_1$  and  $d^2(d+1)/2$  functions of type  $g_0$  such that, without loss of generality,

$$g_0^{\mu\nu\rho} = g_0^{\mu\rho\nu} . \quad (45)$$

All the terms in  $D_\mu J_2^\mu$  involving a second derivative in  $\varphi$  can be collected in

$$(2g_0^{\mu\nu\rho} \partial_\rho \varphi + g_1^{\mu\nu}) \partial_{\mu\nu}^2 \varphi . \quad (46)$$

To construct a Laplacian, we must impose the parenthesis to be anti-symmetric when  $\mu \neq \nu$  :

$$g_0^{\mu\nu\rho} = -g_0^{\nu\mu\rho} \quad (\mu \neq \nu) ; \quad (47)$$

$$g_1^{\mu\nu} = -g_1^{\nu\mu} \quad (\mu \neq \nu) , \quad (48)$$

and independent of  $\mu$  when  $\nu = \mu$ , that is there are  $d$  functions  $h_0^\rho$  and one function  $h_1$  such that

$$g_0^{\mu\mu\rho} = h_0^\rho \quad (\text{no summation on } \mu) \quad (49)$$

and

$$g_1^{\mu\mu} = h_1 \quad (\text{no summation on } \mu) . \quad (50)$$

Combining (45) with (47) leads to  $g_0^{\mu\nu\rho} = g_0^{\mu\rho\nu} = -g_0^{\rho\mu\nu} = -g_0^{\rho\nu\mu} = g_0^{\nu\rho\mu} = g_0^{\nu\mu\rho} = -g_0^{\mu\nu\rho}$  and therefore  $g_0^{\mu\nu\rho} = 0$  when  $(\mu, \nu, \rho)$  are pairwise distinct. Collecting in  $D_\mu J_2^\mu$  the cubic terms in  $\partial\varphi$  leads to

$$(\partial_\varphi g_0^{\mu\nu\rho}) \partial_\mu \varphi \partial_\nu \varphi \partial_\rho \varphi = (\partial_\varphi h_0^\rho) \partial_\rho \varphi \sum_{\mu=1}^d (\partial_\mu \varphi)^2 . \quad (51)$$

Therefore we will take

$$\partial_\varphi h_0^\rho = 0 . \quad (52)$$

Quadratic terms in  $\partial\varphi$  appearing in  $D_\mu J_2^\mu$  are

$$(\partial_\rho g_0^{\rho\mu\nu} + \partial_\varphi g_1^{\mu\nu}) \partial_\mu \varphi \partial_\nu \varphi = \sum_{\mu=1}^d \sum_{\nu=1}^d \partial_\mu h_0^\nu \partial_\mu \varphi \partial_\nu \varphi + \sum_{\mu=1}^d \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^d \partial_\nu h_0^\mu \partial_\mu \varphi \partial_\nu \varphi + \sum_{\mu=1}^d \left[ (\partial_\mu \varphi)^2 (\partial_\varphi h_1 - \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^d \partial_\nu h_0^\nu) \right] . \quad (53)$$

Cancelling each independent term requires

$$\partial_\mu h_0^\nu + \partial_\nu h_0^\mu = 0 \quad (\mu \neq \nu) \quad (54)$$

and

$$\partial_\mu h_0^\mu - \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^d \partial_\nu h_0^\nu + \partial_\varphi h_1 = 0 \quad (\text{no summation on } \mu) . \quad (55)$$

The last equation appears as a linear system of  $d$  equations that can be rewritten with the help of the  $d \times d$  matrix  $2 - A_d$  where  $(A_d)_{\mu,\nu} = 1$ . Then,  $\det(2 - A_d) = (-2)^{d-1}(d-2)$  and therefore when  $d \neq 2$ , it can be inversed and leads to

$$\partial_\mu h_0^\mu = \frac{1}{d-2} \partial_\varphi h_1 \quad (\text{no summation on } \mu, d \neq 2) . \quad (56)$$

Now  $h_0^\mu$  does not depend on  $\varphi$ , by (52), and then  $\partial_\varphi h_1$  neither, hence there exist two  $\varphi$  independent functions  $\tilde{h}_1$  and  $\tilde{\tilde{h}}_1$  such that

$$h_1 = \tilde{h}_1 \varphi + \tilde{\tilde{h}}_1 . \quad (57)$$

The relation (56) implies (with now an implicit summation on  $\mu$ )

$$\tilde{h}_1 = \frac{d-2}{d} \partial_\mu h_0^\mu . \quad (58)$$

In the special case  $d = 2$ , (55) leads to

$$\partial_1 h_0^1 = \partial_2 h_0^2 \quad (59)$$

and

$$\partial_\varphi h_1 = 0 . \quad (60)$$

Then we can keep (57) together with (58) even for  $d = 2$ .

Collecting all the previous relations we get

$$D_\mu J_2^\mu = (2h_0^\mu \Delta_d \varphi + \partial_\nu g_1^{\nu\mu} + \partial_\varphi g_2^\mu) \partial_\mu \varphi + h_1 \Delta_d \varphi + \partial_\mu g_2^\mu \quad (61)$$

where of course, we can use the substitution (38). Without loss of generality we can take  $g_1^{\nu\mu} = 0$  for  $\mu \neq \nu$  by possibly redefining

$$g_2^\mu(\varphi, x, E) \mapsto g_2^\mu(\varphi, x, E) - \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^d \int_0^\varphi \partial_\nu g_1^{\nu\mu}(\varphi', x, E) d\varphi' \quad (62)$$

since  $\partial_\mu g_2^\mu \mapsto \partial_\mu g_2^\mu - \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^d \int_0^\varphi \partial_\mu \partial_\nu g_1^{\nu\mu}(\varphi', x, E) d\varphi' = \partial_\mu g_2^\mu$  because the integrand cancels by (48). To cancel the parenthesis in (61) we must take

$$\partial_\varphi g_2^\mu = -\partial_\mu h_1 - 2h_0^\mu v \varphi \quad (63)$$

The dependence in  $\varphi$  appears only through (57) and we can immediately integrate the last relation

$$g_2^\mu = - \left( \frac{1}{2} \partial_\mu \tilde{h}_1 + h_0^\mu v \right) \varphi^2 - (\partial_\mu \tilde{h}_1) \varphi + \tilde{g}_2^\mu \quad (64)$$

where  $\tilde{g}_2$  is  $\varphi$ -independent. With this expression, (61) becomes

$$D_\mu J_2^\mu = -\varphi^2 \left[ \frac{1}{2} \Delta_d \tilde{h}_1 - v \tilde{h}_1 + \partial_\mu (v h_0^\mu) \right] - \varphi [\Delta_d \tilde{h}_1 - v \tilde{h}_1] + \partial_\mu \tilde{g}_2^\mu . \quad (65)$$

The last term is irrelevant because it is a total divergence and can be reabsorbed in the definition of  $J_2^\mu$ . The second term does not contribute also since by integration by part it can be converted to  $\tilde{h}_1 [\Delta_d - v] \varphi$  which vanishes. with the use of (58), the first term can be further simplified in order to keep  $h_0^\mu$  only.

To sum up, condition (i)<sub>d</sub> can be obtained for  $N = 2$  with

$$F_2 = \frac{2-d}{2d} \Delta_d \partial_\mu h_0^\mu - \frac{2}{d} v \partial_\mu h_0^\mu - h_0^\mu \partial_\mu v \quad (66)$$

with  $h_0^\mu$  being any  $d$  smooth functions such that

$$\partial_1 h_0^1 = \partial_2 h_0^2 = \dots = \partial_d h_0^d ; \quad (67a)$$

$$\partial_\mu h_0^\nu = -\partial_\nu h_0^\mu \quad (\mu \neq \nu) . \quad (67b)$$

Our freedom of choosing  $J_2$  has therefore being reduced first because eliminating the second derivatives of  $\varphi$  through its Laplacian impose severe constraints and second because, as in the  $d = 1$  case, many different initial choices lead to the same  $F_2$ ; in others words the linear application from  $a$  to  $F$  has a non-zero kernel.

For  $d = 1$ , we have one test function  $h_0 = g_0 = a_0$  at our disposal but without any restriction on its derivative and we can easily checked that, for  $d = 1$ , (66) is (3b).

For  $d = 2$ , we have  $F_2 = -\partial_\mu(vh_0^\mu)$  and surprisingly (67) appears to be the Cauchy-Riemann conditions for  $h_0^1 + ih_0^2$  to be analytic. Anyway, if we compute

$$\int_S F_2 dx dy = \int_{\partial S} v \left( \frac{h_0^1}{h_0^2} \right) \cdot d\vec{\sigma} \quad (68)$$

on a surface  $S \in \mathbf{R}^2$  whose boundary  $\partial S$  coincide with the energy level  $V(x, y) = E$ , that is  $v = 0$ , the integrand of the right hand side vanishes ( $d\vec{\sigma}$  is an infinitesimal  $2d$ -vector normal to the curve  $\partial S$  pointing outwards say). Therefore whatever choice we make for  $h_0$ , for any energy  $E$  belonging to the image of  $V$  (where we know the spectrum lies),  $F_2$  changes its sign and no information can be obtained further.

For  $d \geq 3$ , the constraints (67) are so strong that they limit the choice of  $h_0$ 's to polynomials in  $x$  of degree at most 3. Indeed all the third derivatives of  $h_0$  must cancel (up to equation (72) included, any pair of distinct greek letters denote any pair of distinct values of indices and no summation over repeated indices is left implicit<sup>2</sup>). First,

$$\partial_{\mu\nu}^2 h_0^\sigma = -\partial_{\sigma\nu}^2 h_0^\mu = \partial_{\sigma\mu}^2 h_0^\nu = -\partial_{\nu\mu}^2 h_0^\sigma \quad (69)$$

and therefore  $\partial_{\mu\nu}^2 h_0^\sigma = 0$ ; then  $\partial_{\mu\nu\rho}^3 h_0^\sigma = 0$

$$\partial_{\mu\nu\rho}^3 h_0^\sigma = 0; \quad \partial_{\mu\nu\rho}^3 h_0^\mu = 0; \quad \partial_{\mu\mu\nu}^3 h_0^\rho = 0. \quad (70)$$

Furthermore,

$$\partial_{\mu\mu\mu}^3 h_0^\nu = -\partial_{\mu\mu\nu}^3 h_0^\mu = -\partial_{\mu\rho\nu}^3 h_0^\rho = 0 \quad (71)$$

since for  $d \geq 3$  we can always find an index  $\rho$  distinct from both  $\mu$  and  $\nu$ ; eventually we have

$$\partial_{\mu\mu\mu}^3 h_0^\mu = \partial_{\mu\mu\nu}^3 h_0^\nu = -\partial_{\mu\nu\nu}^3 h_0^\mu = -\partial_{\rho\nu\nu}^3 h_0^\rho = \partial_{\rho\rho\nu}^3 h_0^\nu = \partial_{\rho\rho\mu}^3 h_0^\mu = -\partial_{\rho\mu\mu}^3 h_0^\rho = -\partial_{\mu\mu\mu}^3 h_0^\mu \quad (72)$$

and all these third derivatives actually vanish as well. Now  $h_0(x)$  being a polynomial of degree at most two in  $x$ , the constraints (67) on its coefficients leads to the general form

$$h_0^\mu(x) = h_0^\mu(0) + kx^\mu + A^\mu{}_\nu x^\nu - \frac{1}{2} l^\mu x^2 + x^\mu l \cdot x \quad (73)$$

where  $A$  is a  $d \times d$  constant antisymmetric real matrix,  $k$  a real constant,  $l$  a real constant  $d$ -vector ; the cartesian product  $l \cdot x = l^\mu x^\mu$  is used. Then, from (66) we get:

$$F_2(x, E, \lambda) = 4(E - V(x))(k + l \cdot x) - (h_0^\mu(0) + kx^\mu + A^\mu{}_\nu x^\nu - \frac{1}{2} l^\mu x^2 + x^\mu l \cdot x) \partial_\mu V(x). \quad (74)$$

Unlike what occurs for  $d = 1$  where we are free to construct  $F_2$  from a whole set of test functions  $x \mapsto a_0(x)$ , for  $d \geq 3$  we are left with only  $(d^2 + 3d + 2)/2$  free  $x$ -independent parameters, namely  $\lambda = (h_0^\mu(0), k, A^\mu{}_\nu, l^\mu)$ .

Now the boundaries of the gaps must belong to the solutions of (6). The linearity of  $F_2$  in  $\lambda$  simplify considerably the computations. The conditions (6c) imply (6a) and are equivalent to

$$E = V(x); \quad \partial_x V = 0. \quad (75)$$

Condition (6b) is guaranteed if we choose for instance

$$h_0^\mu(0) = -kx_c^\mu + A^\mu{}_\nu x_c^\nu + \frac{1}{2} l^\mu x_c^2 + x_c^\mu l. \quad (76)$$

---

<sup>2</sup>I am grateful to Oleg Lisovyy [4] for providing the following arguments that concisely and rigourously proved my first guess of (73).

and condition (6d) is generically fulfilled.

Therefore, with our method, possible candidates for the gap boundaries are the critical points  $x_c$  of  $V$  which is not a surprise from a semiclassical point of view. With this method we cannot expect to find more interesting and more relevant piece of information. Actually, some inequalities concerning the global spectrum may be obtained if we are to maintain the sign of (74), specially once a specific  $V$  is given; but our ambition was, as we have shown in [2] for  $d = 1$ , to obtain some local information in the very core of the spectrum.

### 3. ENDING REMARKS

#### 3.1. Simplified starting point

For  $N = 2$  and any  $d$  we have shown directly that without loss of generality we could have started with no term in  $\varphi$  in  $J_2^\mu$ , that is with a current such that

$$\partial_\varphi J_2^\mu(\partial\varphi(x), \varphi(x), x, E) = 0. \quad (77)$$

Indeed, the  $h_0^\mu$  are  $\varphi$  independent, see (52), and we could have taken  $\tilde{h}_1 = 0$  with no consequence on the result (66). For  $d = 1$  and any  $N$ , we have proven this result through (19): we obtain all the possible  $F_N$ 's even if we restrict our self to  $a = (a_0, 0, \dots)$ . This result can be also obtained for any  $d$  and any  $N$  in another way. First remark that if we start with  $J_N^\mu(\partial\varphi(x), \varphi(x), x, E)$  the second derivatives in

$$D_\mu J_N^\mu = \partial_\mu J_N^\mu + \partial_\varphi J_N^\mu \partial_\mu \varphi + \partial_{\partial_\nu \varphi} J_N^\mu \partial_{\mu\nu} \varphi \quad (78)$$

can be eliminated with the help of (38) if and only if

$$\partial_{\partial_\nu \varphi} J_N^\mu = L \delta_\nu^\mu + W^\mu{}_\nu \quad (79)$$

where  $\delta$  is the Kronecker symbol,  $W(\partial\varphi(x), \varphi(x), x, E)$  an antisymmetric  $d \times d$  matrix and  $L(\partial\varphi(x), \varphi(x), x, E)$  a function. Then after the substitution of (38), an integration by part can be made and we have

$$D_\mu J_N^\mu = \partial_\mu J_N^\mu + L v \varphi - D_\mu(\partial_\varphi J_N^\mu) \varphi + D_\mu(\partial_\varphi J_N^\mu). \quad (80)$$

By computing  $D_\mu(\partial_\varphi J_N^\mu)$  in the same way, and iterating the process up to infinity, we find that  $D_\mu J_N^\mu$  can be written like

$$D_\mu J_N^\mu = v \varphi \sum_{n=0}^{\infty} \frac{(-\varphi)^n}{n!} \partial_\varphi^n L + \sum_{n=0}^{\infty} \frac{(-\varphi)^n}{n!} \partial_\varphi^n \partial_\mu J_N^\mu + D_\mu \left( J_N^\mu \varphi \sum_{n=0}^{\infty} \frac{(-\varphi)^n}{n!} \partial_\varphi^n J_N^\mu \right); \quad (81)$$

Up to a total derivative, starting from any  $J_N$ , we therefore are always led to

$$D_\mu(\Pi_0 J_N^\mu) = v \varphi \Pi_0 L + \Pi_0 \partial_\mu J_N^\mu \quad (82)$$

where  $\Pi_0$  in the evaluation at  $\varphi = 0$  that can be expressed as

$$\Pi_0 = \sum_{n=0}^{\infty} \frac{(-\varphi)^n}{n!} \partial_\varphi^n \quad (83)$$

for any analytic function of  $\varphi$ . Therefore even if we start with a  $J_N$  whose component takes the general form (39), working up to divergence terms, we will be led to the same identity as if we had started with all the  $a_n$  such that  $N(1-d) + |n| > 0$  being zero. In  $N = 2$ , all the  $g$ 's in (44) could have been taken independent of  $\varphi$  from the very beginning.

### 3.2. Remark about condition (i)'<sub>d</sub>

For  $d \neq 2$  and  $N = 2$ , condition (i)'<sub>d</sub> with

$$B_1 = b_1^\mu \partial_\mu \varphi + b_0 \varphi \quad (84)$$

where  $(b_1 = (b_1^\mu)_\mu, b_0)$  are  $d + 1$  smooth real functions of  $x$ , does not provide  $F_2$  explicitly but rather leads to a linear differential equation for it. For instance, when  $d = 1$  we get

$$\frac{1}{2} b_1^2 F_2'' + (2b_1 b_1' - b_1 b_0) F_2' + [(b_1 b_1')' - v b_1^2 - (b_1 b_0)' + b_0^2] F_2 = \frac{1}{2} a_0''' - 2v a_0' - v' a_0 \quad (85)$$

which reduces to (3b) when we make the simplest choice  $b_0 = 1$  and  $b_1 = 0$ . For  $d \geq 1$ , by reproducing the same line of reasoning as in § 2.1, we get a second order linear partial differential equation for  $F_2$

$$\begin{aligned} \frac{1}{2} (b_1)^2 \Delta_d F_2 + \left( \frac{2}{d} b_1 \cdot \partial_\mu b_1 - b_1^\mu b_0 \right) \partial_\mu F_2 + \left[ \frac{1}{2d} \Delta_d (b_1)^2 - \frac{1}{d} v (b_1)^2 - \partial_\mu (b_1^\mu b_0) + b_0^2 \right] F_2 \\ = \frac{2-d}{2d} \Delta_d \partial_\mu h_0^\mu - \frac{2}{d} v \partial_\mu h_0^\mu - h_0^\mu \partial_\mu v. \end{aligned} \quad (86)$$

instead of (66) obtained for  $b_0 = 1$  and  $b_1 = 0$ . The constraints on  $h_0^\mu$  now involve the  $b$ 's and are entangled with  $F_2$ :

$$\partial_1 h_0^1 - \frac{1}{2} (b_1^1)^2 F_2 = \partial_2 h_0^2 - \frac{1}{2} (b_1^2)^2 F_2 = \dots = \partial_d h_0^d - \frac{1}{2} (b_1^d)^2 F_2; \quad (87a)$$

$$\partial_\mu h_0^\mu + \partial_\nu h_0^\nu = b_1^\mu b_1^\nu F_2 \quad (\mu \neq \nu). \quad (87b)$$

For  $d \geq 1$ , the only way to get rid of  $F_2$  from (87a) is to take  $b_1^\mu$  independent of  $\mu$ , and eventually  $b_1 = 0$  if we want (87b) not to involve  $F_2$  either. Then, since  $F_2$  and  $b_0^2 F_2$  have the same sign, (86) take us back to (66) that is to case (i)<sub>d</sub> where  $b_0 = 1$ .

Even for  $d = 1$  and a specific  $v$ , I did not exploit further these possibilities, but the choice of  $b$ 's for which (85) can be solved explicitly is rather limited (not to speak of the control of the sign of its solutions). In any case of course, we ought to work with simpler differential equations than the Schrödinger equation itself !

### 3.3. Attempt for $d = 2, N = 4$

The second remark concerns an attempt to obtain (i)<sub>d</sub> for  $d = 2$  and  $N = 4$ . Using the same ideas as in the case  $N = 2$  and with

$$J_4^\mu = \sum_{n=0}^4 g_{4-m,m}^\mu (\partial_1 \varphi)^{4-n} (\partial_2 \varphi)^n \quad (88)$$

with, according to the last remark, taking  $g$  as  $\varphi$ -independent. The constraints (79) imposed by the elimination of the second derivatives of  $\varphi$  in  $D_\mu J_4^\mu$  lead to two independent functions instead of the ten  $g$ 's. But if we go further to eliminate the cubic terms in  $\partial \varphi$ , these functions must vanish identically and no non zero  $g$  can be found this way. Extending (i)<sub>d</sub> to (i)'<sub>d</sub> with

$$B_2 = b_{20} (\partial_1 \varphi)^2 + b_{11} \partial_1 \varphi \partial_2 \varphi + b_{02} (\partial_2 \varphi)^2 + b_{10} \varphi \partial_1 \varphi + b_{01} \varphi \partial_2 \varphi + b_{00} \varphi^2 \quad (89)$$

where the  $b$ 's are smooth function of  $x$ , leads to the same conclusion.

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