Applications of Noether conservation theorem to Hamiltonian systems

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Abstract The Noether theorem connecting symmetries and conservation laws can be applied directly in a Hamiltonian framework without using any intermediate Lagrangian formulation. This requires a careful discussion about the invariance of the boundary conditions under a canonical transformation and this paper proposes to address this issue. Then, the unified treatment of Hamiltonian systems offered by Noether's approach is illustrated on several examples, including classical field theory and quantum dynamics.

1 Introduction

After its original publication in German in 1918, and even though it was first motivated by theoretical physics issues in General Relativity, it took a surprisingly long time for the physicists of the twentieth century to become aware of the profoundness of Noether's seminal article (see [22] for an English translation and a historical analysis of its impact, see also [20, § 7] and [7]). Since then, about the 1950's say, as far as theoretical physics is concerned, Noether's work spread widely from research articles in more general textbooks and, nowadays, it even reaches some online pages like Wikipedia's [8] intended to a (relatively) large audience including undergraduate students (see also [27] and [21, § 5.2]). However, the vast majority of these later presentations, unfortunately following the steps of [19] (see [22, § 4.7]), reduces drastically the

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scope of Noether's article[1]; (i) because they commonly refer to the first main theorem ("*The* Noether theorem") without even mentioning that Noether's 1918 paper contains more physically relevant material[2] and also (ii) because the connection between the existence of a conservation law and some invariance under a continuous group of transformation in a variational problem is predominantly illustrated in a Lagrangian framework, for instance [34, §7.3], (not to speak that the order of the derivatives involved in the Lagrangian do not generally exceed one, albeit Noether explicitly works with integrands of arbitrary orders). As a consequence, an enormous literature flourished that claimed to generalise Noether's results whereas it only generalised the secondary poor man's versions of it without acknowledging that these so-called generalisations were already present in Noether's original work [22, § 5.5] or in Bessel-Hagen's paper [4] — directly owed to a "an oral communication from Emmy Noether" (see also [28, § 4, footnote 20]) — where invariance of the integrand defining the functional is considered "up to a divergence".

Nevertheless, fortunately, the success of gauge theories in quantum field theory motivated several works where Noether's contribution was employed in (almost) all its powerful generality (for articles not concerned by (i) see for instance [2, 26] and the more epistemological approach proposed in [5]). To counterbalance (ii), the present paper is an attempt to provide a unified treatment of Noether's conservation laws in the Hamiltonian framework, i.e. where the canonical formalism is used. In this context, the advantages of the latter have already been emphasized by a certain number of works among which we can cite [18, 25, 12] where the main focus was naturally put on the Noether's second theorem (see footnote 2) but not necessarily, since classical mechanics was also considered — [32], regrettably suffering of flaw (i) — even with pedagogical purposes [23], [11, § 7.11]. The main advantage of the Hamiltonian approach over the standard Lagrangian one is that it incorporates more naturally a larger class of transformations, namely the canonical transformations (in phase-space), than the point transformations (in configuration space). To recover the constants of motion associated with the canonical transformations that cannot be reduced to some point transformations, one has to consider some symmetry transformations of the Lagrangian action that depend on the time derivative of the degrees of freedom. Anyway, these so called "dynamical", "accidental" or "hidden" symmetries (the best known example being the Laplace-Runge-Lenz vector for the two-body Coulombian model [24, § 5A]) are completely covered by Noether's original treatment, even if we stick to a Lagrangian framework.

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[1] Obviously, the common fact that research articles are more quoted than read is all the more manifest for rich fundamental papers.

[2] There is a second main theorem establishing a one-to-one correspondence between Gauge invariance and some identities between the Euler-Lagrange equations and their derivatives (see § 4.3 below). These Noether identities render that a gauge-invariant model is necessarily a constrained Hamiltonian/Lagrangian system in Dirac's sense [13]. Furthermore, a by-product result also proven by Noether [28, § 5] is that the constants of motion associated, through the first theorem, with an invariance under a Lie group are themselves invariant under the transformations representing this group.

As a starting point I will explain in § 2, how the price to pay when working within the Hamiltonian framework is that special care is required concerning the boundary conditions imposed when formulating the variational principle: unlike what occurs in the configuration space, in phase-space not all the initial and final dynamical variables can be fixed arbitrarily but rather half of them; the choice of which ones should be fixed is an essential part of the model and therefore should be included in any discussion about its invariance under a group of transformations. As far as I know, in the literature where Noether's work is considered, including [28] itself or even when a Hamiltonian perspective is privileged, the invariance of the boundary conditions is not genuinely considered and only the invariance of the functional upon which the variational principle relies is examined. This may be understood because as far as we keep in mind a Lagrangian formulation, the boundary conditions are not generically constrained; on the other hand, in a Hamiltonian formulation, there are some constraints that fix half of the canonical variables and the invariance of the action under a canonical transformation does not guarantee that the constraints are themselves invariant under this transformation. Since the present paper intends to show how Noether's conservation laws can be directly applied in a Hamiltonian context, I will have to clarify this issue and for this purpose I propose to introduce (§ 2.3) a boundary function defined on phase space whose role is to encapsulate the boundary conditions. In § 3, for a classical Hamiltonian system we derive the conservation laws from the invariance under the most general canonical transformations. Then, before I show in § 5.1 that the same results can be obtained with Noether's approach, I will paraphrase Noether's original paper in § 4 for the sake of self-containedness and for defining the notations. Before I briefly conclude, I will show explicitly how Noether's method can be applied for models involving classical fields (§ 5.2) and in quantum theory (§ 6). For completeness the connection with the Lagrangian framework will be presented in § 5.3.

2 Hamiltonian variational principle and the boundary conditions

2.1 Formulation of the variational principle in a Hamiltonian context

We shall work with a Hamiltonian system described by the independent canonical variables (p, q) referring to a point in phase space. Whenever required, we will explicitly label the degrees of freedom by α that may be a set of discrete indices, a subset of continuous numbers or a mixture of both. For instance, for L degrees of freedom, we have $(p, q) = (p_\alpha, q_\alpha)_{\alpha \in \{1,...,L\}}$ whereas for a scalar field in a D-dimensional space we will take $\alpha = \mathbf{x} = (x^1, ..., x^D) = (x^i)_{i \in \{1,...,D\}}$ and then (p, q) will stand for the fields $\{\pi(\mathbf{x}), \varphi(\mathbf{x})\}_{\mathbf{x} \in \mathbb{R}^D}$. The dynamics of the system is based on a variational principle i.e. it corresponds to an evolution where the dynamical variables are functions of time[3] that extremalise some

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[3] We will never bother about the regularity of all the functions we will meet, assuming they are smooth enough for their derivative to be defined when necessary.

functional S called the action. In the standard presentation of the Hamilton principle in phase space, see [30] and its references, the action is defined as the functional $\int_{t_i}^{t_f} \big(p\mathrm{d}q/\mathrm{d}t - H(p,q,t)\big)\mathrm{d}t$ of the smooth functions of time $t \mapsto \big(p(t), q(t)\big)$ (the summation/integral on the degrees of freedom labeled by α is left implicit). When the Hamiltonian $H(p,q,t)$ depends explicitly on time t, it is often convenient to work in an extended phase space where $(-H, t)$ can be seen as an additional pair of canonical dynamical variables; we shall not use this possibility but still, we shall keep some trace of the similarity between q and t on one hand and between p and $-H$ on the other hand by considering the action

$$S_0[p(\cdot), q(\cdot), t(\cdot)] \stackrel{\text{def}}{=} \int_{s_i}^{s_f} \left(p(s)\frac{\mathrm{d}q}{\mathrm{d}s}(s) - H\big(p(s), q(s), t(s)\big)\frac{\mathrm{d}t}{\mathrm{d}s}(s) \right) \mathrm{d}s \quad (1)$$

as a functional of $s \mapsto p(s)$, $s \mapsto q(s)$ and $s \mapsto t(s)$ where s is a one-dimensional real parametrisation. An infinitesimal variation $p(s)+\delta p(s)$, $q(s)+\delta q(s)$, $t(s)+\delta t(s)$ induces the variation $S + \delta S$ of the value of the action where, to first order in $(\delta p, \delta q, \delta t)$, we have, with the customary use of integration by parts,

$$\begin{aligned}
\delta S_0 =\ & p(s_f)\delta q(s_f) - p(s_i)\delta q(s_i) \\
& - H\big(p(s_f), q(s_f), t(s_f)\big)\delta t(s_f) + H\big(p(s_i), q(s_i), t(s_i)\big)\delta t(s_i) \\
& + \int_{s_i}^{s_f} \left\{ \left[\frac{\mathrm{d}q}{\mathrm{d}s}(s) - \partial_p H\big(p(s), q(s), t(s)\big)\frac{\mathrm{d}t}{\mathrm{d}s}(s) \right] \delta p(s) \right. \\
& + \left[-\frac{\mathrm{d}p}{\mathrm{d}s}(s) - \partial_q H\big(p(s), q(s), t(s)\big)\frac{\mathrm{d}t}{\mathrm{d}s}(s) \right] \delta q(s) \\
& \left. + \left[\frac{\mathrm{d}}{\mathrm{d}s}H\big(p(s), q(s), t(s)\big) - \partial_t H\big(p(s), q(s), t(s)\big)\frac{\mathrm{d}t}{\mathrm{d}s}(s) \right] \delta t(s) \right\} \mathrm{d}s
\end{aligned} \quad (2)$$

and, then, the Hamilton variational principle can be formulated as follows: in the set of all phase-space paths connecting the initial position $q(s_i) = q_i$ at $t(s_i) = t_i$ to the final position $q(s_f) = q_f$ at $t(s_f) = t_f$ the dynamics of the system follows one for which S_0 is stationary[4]; in other words, the variation δS vanishes in first order provided we restrict the variations to those such that

$$\delta q(s_f) = \delta q(s_i) = 0 \ ; \quad (3\mathrm{a})$$

$$\delta t(s_f) = \delta t(s_i) = 0 \quad (3\mathrm{b})$$

whereas the other variation $\delta t(s)$, $\delta p(s)$ and $\delta q(s)$ remain arbitrary (but small), hence independent one from the other. Hamilton's equations

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\partial_q H(p,q,t) \ ; \quad (4\mathrm{a})$$

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \ \ \partial_p H(p,q,t) \ ; \quad (4\mathrm{b})$$

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[4] This classical path is not necessarily unique and may be even a degenerate critical path for S_0, see however the next footnote.

come from the cancellation of the two first brackets in the integrand of (2), then the cancellation of the third one follows. The restrictions (3) on the otherwise arbitrary variations $\delta p(s)$, $\delta q(s)$, $\delta t(s)$ provides sufficient conditions to cancel the boundary terms given by the two first lines of the right-hand side of (2) but they are not necessary, one could impose δq to be transversal to p both at t_i and t_f, or impose some periodic conditions (see footnote 10).

2.2 The differences concerning the boundary conditions between Lagrangian and Hamiltonian models

In the usual Lagrangian approach the q's constitute all the dynamical variables and a generic choice of (q_i, q_f, t_i, t_f) leads to a well-defined variational problem having one isolated solution[5]: no constraint on (q_i, q_f, t_i, t_f) is required and it is commonly assumed that the variations of all the dynamical variables vanish at the boundary; any point transformation $q \rightarrow q^{\mathrm{T}}(q)$ preserves this condition since then $\delta q^{\mathrm{T}} = (\partial_q q^{\mathrm{T}}) \delta q$ and we have $\delta q = 0 \Leftrightarrow \delta q^{\mathrm{T}} = 0$.

In a Hamiltonian framework, obviously, because the dynamical variables q and p are not treated on the same footing in the definition (1) of S_0, there is an imbalance in the boundary conditions and in their variations between δq and δp. More physically, this comes from the fact that the classical orbits, defined to be the solutions of (4), are generically determined by *half* of the set (p_i, q_i, p_f, q_f); in general, there will be no classical solution for a given a priori set (p_i, q_i, p_f, q_f) and a well-defined variational principle — that is, neither overdetermined nor underdetermined — requires some constraints that make half of these dynamical variables to be functions of half the independent other ones. Any canonical transformation, which usually shuffles the (p, q)'s, will not only affect the functional S_0 but also the boundary conditions required by the statement of variational principle. For a canonical transformation the transformed dynamical variables q^{T} and p^{T} are expected to be functions of both q and p and, then, as noted in [29], the conditions (3a) alone do not imply that $\delta q_i^{\mathrm{T}} = \delta q_f^{\mathrm{T}} = 0$ since neither δp_i^{T} nor δp_f^{T} vanish in general.

In any case, the behaviour of the initial conditions under a transformation should be included when studying the invariance of a variational model but this issue is made more imperative in a Hamiltonian than in a Lagrangian viewpoint.

2.3 The boundary function

To restore some sort of equal treatment between the q's and the p's in the Hamiltonian framework, one can tentatively add to S_0 a function A of the dy-

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[5] In the space of initial conditions, the singularities corresponding to bifurcation points, caustics, etc. are submanifolds of strictly lower dimension (higher co-dimension) and therefore outside the scope, by definition, of what is meant by "generic". In other words we consider as generic any property that is structurally stable, that is, unchanged under a small enough arbitrary transformation.

namical variables at the end points $(q_f, p_f, t_f; q_i, p_i, t_i)$ whose variations δA depend a priori on the variations of all the dynamical variables at the boundaries. Nevertheless we will restrict the choice of $A(q_f, p_f, t_f; q_i, p_i, t_i)$ to functions of the form $B(q_f, p_f, t_f) - B(q_i, p_i, t_i)$ in order to preserve the *concatenation property* according to which the value of the action of two concatenated paths is the sum of the actions of each of the two paths. This strategy is equivalent to add to the integrand of S_0 the total derivative of the boundary function B (see [9, § IV.5.1, footnote 1 p. 211]):

$$S_B[p(\cdot), q(\cdot), t(\cdot)] \overset{\text{def}}{=} \int_{s_i}^{s_f} \left(p \frac{\mathrm{d}q}{\mathrm{d}s} + \left(-H(p,q,t) + \frac{\mathrm{d}}{\mathrm{d}t} B(p,q,t) \right) \frac{\mathrm{d}t}{\mathrm{d}s} \right) \mathrm{d}s \ . \tag{5}$$

This modification does not alter Hamilton's equations $(4)^6$ but allows to reformulate the variational problem within the set of phase-space paths defined by the boundary conditions such that

$$\left[p\delta q - H\delta t + \delta B \right]_{s_i}^{s_f} = 0 \ . \tag{6}$$

For instance by choosing $B(p,q,t) = -pq$, the roles of the p's and the q's are exchanged and (3a) is replaced by $\delta p(s_f) = \delta p(s_i) = 0$ whereas if we take $B(p,q,t) = -pq/2$ the symmetry between p and q is (almost) obtained.

We see that the boundary function is defined up to a function of time only since the substitution

$$B'(p,q,t) \overset{\text{def}}{=} B(p,q,t) + b(t) \ ; \qquad H'(p,q,t) \overset{\text{def}}{=} H(p,q,t) + \frac{\mathrm{d}b}{\mathrm{d}t}(t) \tag{7}$$

leaves unchanged both the action (5) and the boundary conditions (6). A dependence of b on the other dynamical variables is unacceptable since it would introduce time derivatives of p and q in the Hamiltonian.

3 Transformation, invariance and conservation laws

3.1 Canonical transformation of the action, the Hamiltonian and the boundary function

In the present paper we refrain to use the whole concepts and formalism of symplectic geometry that has been developed for dynamical systems and prefer to keep a "physicist touch" without referring to fiber bundles, jets, etc. even though the latter allow to work with a completely coordinate-free formulation. With this line of thought, we follow a path closer to Noether's original formulation. However, keeping a geometrical interpretation in mind, if we consider the action (5) as a scalar functional of a geometrical path in phase space, any

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[6] The fact that a total derivative can be added to a Lagrangian without changing the evolution equations is well-known for a long-time. As already noticed above it is mentioned by Noether [28, § 4, footnote 20] and this flexibility has been used for many purposes ; in particular in Bessel-Hagen's paper [4, § 1], see also the discussion in [5, § 3].

canonical transformation $(q, p, t) \mapsto (q^{\mathrm{T}}, p^{\mathrm{T}}, t^{\mathrm{T}})$ can be seen as a change of coordinate patch (the so-called passive transformation on which the geometrical concept of manifold relies) that does not affect the *value* of the action for the considered path, so we should have

$$S_{B^{\mathrm{T}}}^{\mathrm{T}}[p^{\mathrm{T}}(\cdot), q^{\mathrm{T}}(\cdot), t^{\mathrm{T}}(\cdot)] \stackrel{\mathrm{def}}{=} S_B[p(\cdot), q(\cdot), t(\cdot)] ; \tag{8}$$

in this point of view, the latter relation is a *definition* of the transformed functional, not an expression of the invariance of the model. The canonical character of the transformation guarantees that $S_{B^{\mathrm{T}}}^{\mathrm{T}}$ takes the same form as (5), namely

$$S_{B^{\mathrm{T}}}^{\mathrm{T}}[p^{\mathrm{T}}(\cdot), q^{\mathrm{T}}(\cdot), t^{\mathrm{T}}(\cdot)] = \int_{s_i}^{s_f} \left(p^{\mathrm{T}}(s) \frac{\mathrm{d}q^{\mathrm{T}}}{\mathrm{d}s}(s) - H^{\mathrm{T}}\big(p^{\mathrm{T}}(s), q^{\mathrm{T}}(s), t^{\mathrm{T}}(s)\big) \frac{\mathrm{d}t^{\mathrm{T}}}{\mathrm{d}s}(s) \right.$$
$$\left. + \frac{\mathrm{d}}{\mathrm{d}s} \Big[B^{\mathrm{T}}\big(p^{\mathrm{T}}(s), q^{\mathrm{T}}(s), t^{\mathrm{T}}(s)\big) \Big] \right) \mathrm{d}s , \quad (9)$$

which leads to a *definition* of H^{T} and B^{T} up to a function of time only (see (7)). Since the equality (8) holds for any phase-space path (whether classical or not), a necessary (and sufficient) condition is that

$$p^{\mathrm{T}} \, \mathrm{d}q^{\mathrm{T}} - H^{\mathrm{T}}(p^{\mathrm{T}}, q^{\mathrm{T}}, t^{\mathrm{T}}) \, \mathrm{d}t^{\mathrm{T}} + \mathrm{d}\big(B^{\mathrm{T}}(p^{\mathrm{T}}, q^{\mathrm{T}}, t^{\mathrm{T}})\big)$$
$$= p \, \mathrm{d}q - H(p, q, t) \, \mathrm{d}t + \mathrm{d}\big(B(p, q, t)\big) , \quad (10)$$

which provides an explicit expression for $H^{\mathrm{T}}(p^{\mathrm{T}}, q^{\mathrm{T}}, t^{\mathrm{T}})$ and $B^{\mathrm{T}}(p^{\mathrm{T}}, q^{\mathrm{T}}, t^{\mathrm{T}})$ according to the choice of the independent coordinates in phase-space. For instance, if we pick up p^{T}, q and t and assume that the transformation of time is given by a general function $t^{\mathrm{T}}(p^{\mathrm{T}}, q, t)$[7], the expression (10) in terms of the corresponding differential forms is

$$q^{\mathrm{T}} \, \mathrm{d}p^{\mathrm{T}} + p \, \mathrm{d}q + H^{\mathrm{T}}(p^{\mathrm{T}}, q^{\mathrm{T}}, t^{\mathrm{T}}) \, \mathrm{d}t^{\mathrm{T}} - H(p, q, t) \, \mathrm{d}t$$
$$= \mathrm{d}\big(p^{\mathrm{T}}q^{\mathrm{T}} + B^{\mathrm{T}}(p^{\mathrm{T}}, q^{\mathrm{T}}, t^{\mathrm{T}}) - B(p, q, t)\big) , \quad (11)$$

which is the differential of a generating function $F(p^{\mathrm{T}}, q, t)$ of the canonical transformation implicitly defined (up to a function of time only) by

$$p = \frac{\partial F}{\partial q} - H^{\mathrm{T}}(p^{\mathrm{T}}, q^{\mathrm{T}}, t^{\mathrm{T}}) \frac{\partial t^{\mathrm{T}}}{\partial q} ; \tag{12a}$$

$$q^{\mathrm{T}} = \frac{\partial F}{\partial p^{\mathrm{T}}} - H^{\mathrm{T}}(p^{\mathrm{T}}, q^{\mathrm{T}}, t^{\mathrm{T}}) \frac{\partial t^{\mathrm{T}}}{\partial p^{\mathrm{T}}} . \tag{12b}$$

Then, we get

$$H^{\mathrm{T}}(p^{\mathrm{T}}, q^{\mathrm{T}}, t^{\mathrm{T}}) \frac{\partial t^{\mathrm{T}}}{\partial t}(p^{\mathrm{T}}, q, t) = H(p, q, t) + \frac{\partial F}{\partial t}(p^{\mathrm{T}}, q, t) \tag{13}$$

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[7] A notable case where t^{T} depends on q is provided by the Lorentz transformations.

and
$$B^{\mathrm{T}}(p^{\mathrm{T}}, q^{\mathrm{T}}, t^{\mathrm{T}}) = B(p, q, t) - p^{\mathrm{T}} q^{\mathrm{T}} + F(p^{\mathrm{T}}, q, t) \ . \tag{14}$$

The substitution (7) corresponds to the alternative choice $F' \stackrel{\text{def}}{=} F - b$. From the latter relation, we understand why a boundary function B has to be introduced in the definition of the action when discussing the effects of a general canonical transformation. Even if we start with a B that vanishes identically, a canonical transformation turns $B \equiv 0$ into $-\breve{F}(q^{\mathrm{T}}, q)$ where $\breve{F}$ is the generating function given by the following Legendre transform of F

$$\breve{F}(q^{\mathrm{T}}, q) \stackrel{\text{def}}{=} p^{\mathrm{T}} q^{\mathrm{T}} - F(p^{\mathrm{T}}, q) \ , \tag{15}$$

and therefore $B^{\mathrm{T}} \not\equiv 0$ in general (this special case is the point raised in [29]). In the particular case of point transformations $q^{\mathrm{T}} = f(q, t)$, the boundary function can remain unchanged since we can always choose $F(p^{\mathrm{T}}, q, t) = p^{\mathrm{T}} f(q, t)$ for which $\breve{F} \equiv 0$.

3.2 What is meant by invariance

When talking about the invariance of a Hamiltonian model under a transformation, one may imply (at least) three non-equivalent conditions: the invariance of the form of the action (5), the invariance of the form of Hamilton's equations (4) or the invariance of the form of Newton equations derived from the latter. As far as only classical dynamics is concerned, the invariance of the action appears to be a too strong condition: if only the critical points of a function(nal) are relevant, there is no need to impose the invariance of the function(nal) itself outside some neighbourhood of its critical points and, provided no bifurcation occurs, one may substantially transform the function(nal) without impacting the location and the properties of its critical points. For instance the transformation $S \mapsto S^{\mathrm{T}} = S + \epsilon \sinh S$, with ϵ being a the real parameter, would actually lead to the same critical points[8]. However, by considering that quantum theory is a more fundamental theory than the classical one, from its formulation in terms of path integrals due to Feynman[9] we learn that the value of the action is relevant beyond its stationary points all the more than we leave the (semi-)classical domain and reach a regime where the typical value of the action of the system is of order $\hbar$. Therefore we will retain the invariance of the form of the action as a fundamental expression of the invariance of a model:

$$S_{B^{\mathrm{T}}}^{\mathrm{T}}[p^{\mathrm{T}}(\cdot), q^{\mathrm{T}}(\cdot), t^{\mathrm{T}}(\cdot)] = S_B[p^{\mathrm{T}}(\cdot), q^{\mathrm{T}}(\cdot), t^{\mathrm{T}}(\cdot)] \ . \tag{16}$$

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[8] It is also easy to construct an example for which not only the critical points are preserved but also their stability as well as the higher orders of the functional derivatives of S evaluated on the classical solutions.

[9] The original Feynman's formulation has a Lagrangian flavour and introduces integrals over paths in the configuration space [16]. An extension to integrals over phase-space paths has been done in [15, Appendix B] (see also [33, 10, 17]).

This means the invariance of the boundary function up to a function of time only

$$B^{\mathrm{T}}(p^{\mathrm{T}}, q^{\mathrm{T}}, t^{\mathrm{T}}) = B(p^{\mathrm{T}}, q^{\mathrm{T}}, t^{\mathrm{T}}) + b(t^{\mathrm{T}}) \tag{17}$$

and the invariance of the Hamiltonian function up to $\dot{b}$

$$H^{\mathrm{T}}(p^{\mathrm{T}}, q^{\mathrm{T}}, t^{\mathrm{T}}) = H(p^{\mathrm{T}}, q^{\mathrm{T}}, t^{\mathrm{T}}) + \frac{\mathrm{d}b}{\mathrm{d}t^{\mathrm{T}}}(t^{\mathrm{T}}) \tag{18}$$

that both assure the invariance of the boundary conditions (6). When, on the one hand, we put (18) into (13) and, on the other hand, when we put (17) into (14), the invariance of the model under the canonical transformation T is equivalent to

$$H(p^{\mathrm{T}}, q^{\mathrm{T}}, t^{\mathrm{T}}) \frac{\partial t^{\mathrm{T}}}{\partial t}(p^{\mathrm{T}}, q, t) = H(p, q, t) + \frac{\partial F}{\partial t}(p^{\mathrm{T}}, q, t) \tag{19}$$

for the Hamiltonian and

$$B(p^{\mathrm{T}}, q^{\mathrm{T}}, t^{\mathrm{T}}) = B(p, q, t) - p^{\mathrm{T}} q^{\mathrm{T}} + F(p^{\mathrm{T}}, q, t) \tag{20}$$

for the boundary function, once we have absorbed the irrelevant term b in an alternative definition of F.

3.3 Conservation of the generators

From the Hamilton's equations, the classical evolution of any function $O(p, q, t)$ is given by

$$\frac{\mathrm{d}O}{\mathrm{d}t} = \{H, O\} + \frac{\partial O}{\partial t}\ . \tag{21}$$

where the Poisson brackets between two phase-space functions are defined by

$$\{O_1, O_2\} \overset{\text{def}}{=} \partial_p O_1 \partial_q O_2 - \partial_p O_2 \partial_q O_1 \tag{22}$$

(recall that the summation/integral on the degrees of freedom is left implicit).

Consider a continuous set of canonical transformations parametrised by a set of essential real parameters $\epsilon = (\epsilon^a)_a$ where $\epsilon = 0$ corresponds to the identity. The generators $G = (G_a)_a$ of this transformation are, by definition, given by the terms of first order in ϵ in the Taylor expansion of the generating function $F(p^{\mathrm{T}}, q, t; \epsilon)$

$$F(p^{\mathrm{T}}, q, t; \epsilon) = p^{\mathrm{T}} q + \epsilon\, G(p^{\mathrm{T}}, q, t) + \mathrm{O}(\epsilon^2) \tag{23}$$

(in addition to the implicit summation/integral on the degrees of freedom α, there is also an implicit sum on the labels a of the essential parameters of the Lie group, those being continuous for a gauge symmetry). We shall consider the general canonical transformations where t^{T} is a function of (p^{T}, q, t) whose infinitesimal form is

$$t^{\mathrm{T}}(p^{\mathrm{T}}, q, t) = t + \epsilon \tau(p^{\mathrm{T}}, q, t) + \mathrm{O}(\epsilon^2). \tag{24}$$

Now with $H^{\mathrm{T}}(p,q,t) = H(p,q,t)$, using the form (23) in equations (12) one obtains the canonical transformation explicitly to first order

$$p^{\mathrm{T}} = p - \epsilon \partial_q G(p,q,t) + \epsilon H(p,q,t) \frac{\partial \tau}{\partial q}\Big|_{p,q,t} + \mathrm{O}(\epsilon^2) \; ; \qquad (25\mathrm{a})$$

$$q^{\mathrm{T}} = q + \epsilon \partial_p G(p,q,t) - \epsilon H(p,q,t) \frac{\partial \tau}{\partial p}\Big|_{p,q,t} + \mathrm{O}(\epsilon^2) \; . \qquad (25\mathrm{b})$$

Reporting (23) and (25) in (19), the identification of the first order terms in ϵ leads, with help of (21), to

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(G(p,q,t) - \tau(p,q,t)H(p,q,t)\Big) = 0. \qquad (26)$$

Similarly, from (20), we get

$$\tau \frac{\mathrm{d}B}{\mathrm{d}t} + \{G - \tau H, B\} + p(\partial_p G - H \partial_p \tau) - G = 0 \qquad (27)$$

where the arguments of all the functions that appear are (p,q,t).

As a special case, first consider the invariance with respect to time translations $p^{\mathrm{T}} = p$, $q^{\mathrm{T}} = q$, $t^{\mathrm{T}} = t + \epsilon$ for any real ϵ, then with $F(p^{\mathrm{T}}, q, t) = p^{\mathrm{T}}q$ corresponding to the identity, the relations (19) and (20) read respectively $H(p,q,t+\epsilon) = H(p,q,t)$ and $B(p,q,t+\epsilon) = B(p,q,t)$ that is $\partial_t H = 0$ and $\partial_t B = 0$. The identity (21) considered for $O = H$ and $O = B$ leads respectively to

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0 \qquad (28)$$

and

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \{H, B\} \qquad (29)$$

which of course are also obtained from (26) and (27) with $G \equiv 0$ and $\tau \equiv 1$. Now consider a continuous set of canonical transformations such that $t^{\mathrm{T}} = t$, then from (26) with $\tau \equiv 0$ we get

$$\frac{\mathrm{d}G}{\mathrm{d}t} = 0. \qquad (30)$$

Not only the conservation law follows straightforwardly from (19) but the constant of motion are precisely the generators of the continuous canonical transformations [1]. Similarly, from (27) with $\tau \equiv 0$ we get a relation

$$\{G, B\} = G - p\,\partial_p G \qquad (31)$$

that must be fulfilled by B to have the invariance of the boundary conditions.

4 Noether's original formulation

4.1 General variational principle

The above result is actually completely embedded in Noether's original formulation except the discussion on the boundary conditions. Indeed, being more Lagrangian in flavour, [28] works systematically with a variational principle where the variations of all the dynamical variables u vanish (as well as the derivatives of δu if necessary, see below). To illustrate this let us first follow Noether's steps and paraphrase her analysis. The variational principle applies to any functional whose general form is

$$S[u(\cdot)] \overset{\text{def}}{=} \int_{\mathcal{D}} f\big(x, u(x), \partial_x u_{|x}, \partial^2_{xx} u_{|x}, \partial^3_{xxx} u_{|x}, \dots \big)\, \mathrm{d}^d x \qquad (32)$$

where the functions $u(x) = \big(u_1(x), \dots, u_N(x)\big) = \big(u_n(x)\big)_n$ (the dependent variables in Noether's terminology) are defined on a d-dimensional domain $\mathcal{D}$ in $\mathbb{R}^d$ where some coordinates (the independent variables) $x = (x^0, \dots, x^{d-1}) = (x^\mu)_\mu$ are used. Physically, one may think the u's to be various fields defined on some domain $\mathcal{D}$ of space-time and x to be a particular choice of space-time coordinates. The function f depends on x, on $u(x)$ and on their higher derivatives in x (the dots in its argument refer to derivatives of u of order four or more).

An infinitesimal variation $u(x) + \delta u(x)$ implies the first-order variation

$$\delta S \overset{\text{def}}{=} S[u(\cdot) + \delta u(\cdot)] - S[u(\cdot)] = \int_{\mathcal{D}} \delta f\, \mathrm{d}^d x \qquad (33)$$

where δf, with the help of integration by parts, takes the form

$$\delta f = \sum_{n=1}^{N} E^n \delta u_n + \sum_{\mu=0}^{d-1} \mathrm{d}_\mu \delta X^\mu = E \cdot \delta u + \mathrm{d}_x \cdot \delta X \qquad (34)$$

where E stands for the N-dimensional vector whose components are

$$E^n = \frac{\partial f}{\partial u_n} - \mathrm{d}_\mu \left(\frac{\partial f}{\partial(\partial_\mu u_n)} \right) + \mathrm{d}^2_{\mu\nu} \left(\frac{\partial f}{\partial(\partial^2_{\mu\nu} u_n)} \right) - \mathrm{d}^3_{\mu\nu\rho} \left(\frac{\partial f}{\partial(\partial^3_{\mu\nu\rho} u_n)} \right) + \cdots \tag{35}$$

(from now on we will work with an implicit summation over the repeated space-time indices or field indices and the same notation "$\cdot$" will be indifferently used for a — possibly Minkowskian — scalar product between d-dimensional space-time vectors or between N-dimensional fields) and δX a d-dimensional infinitesimal vector in first order in δu and its derivatives which

appears through a divergence:

$$
\begin{aligned}
\delta X^\mu = {} & \left[\frac{\partial f}{\partial(\partial_\mu u_n)} - \mathrm{d}_\nu \left(\frac{\partial f}{\partial(\partial^2_{\mu\nu} u_n)} \right) + \mathrm{d}^2_{\nu\rho} \left(\frac{\partial f}{\partial(\partial^3_{\mu\nu\rho} u_n)} \right) - \cdots \right] \delta u_n \\
& + \left[\frac{\partial f}{\partial(\partial^2_{\mu\nu} u_n)} \quad - \quad \mathrm{d}_\rho \left(\frac{\partial f}{\partial(\partial^3_{\mu\nu\rho} u_n)} \right) + \cdots \right] \partial_\nu(\delta u_n) \\
& \qquad\qquad + \left[\frac{\partial f}{\partial(\partial^3_{\mu\nu\rho} u_n)} \quad - \cdots \right] \partial^2_{\nu\rho}(\delta u_n) \\
& \qquad\qquad\qquad\qquad + \cdots .
\end{aligned}
\tag{36}
$$

The notation d_μ distinguishes the total derivative from the partial derivative ∂_μ:

$$
\mathrm{d}_\mu = \partial_\mu + \partial_\mu u_n \frac{\partial}{\partial u_n} + \partial^2_{\mu\nu} u_n \frac{\partial}{\partial(\partial_\nu u_n)} + \cdots .
\tag{37}
$$

The stationarity conditions of S when computed for the functions u_{cl} imply the Euler-Lagrange equations

$$
E_{|u_{\mathrm{cl}}} = 0 .
\tag{38}
$$

Then, remains

$$
\delta S[u_{\mathrm{cl}}(\cdot)] = \int_{\mathcal{D}} \mathrm{d}_x \cdot \delta X_{|u_{\mathrm{cl}}} \, \mathrm{d}^d x = \int_{\partial\mathcal{D}} \delta X_{|u_{\mathrm{cl}}} \cdot \mathrm{d}^{d-1}\sigma
\tag{39}
$$

(Stokes' theorem leads to the second integral which represents the outgoing flux of the vector δX through the boundary $\partial\mathcal{D}$ whose surface element is denoted by $\mathrm{d}\sigma$) and S will be indeed stationary if we restrict the variations δu on the boundaries such that the last integral vanishes[10] (and Noether assumes that all the variations δu_n, $\partial_\nu(\delta u_n)$, $\partial^2_{\nu,\rho}(\delta u_n)$ ... appearing in the right-hand side of (36) vanish on $\partial\mathcal{D}$).

Adding the divergence of a d-vector $B\big(x, u(x), \partial_x u_{|x}, \partial^2_{xx} u_{|x}, \partial^3_{xxx} u_{|x}, \dots \big)$ to the integrand,

$$
f_B = f_0 + \mathrm{d}_\mu B^\mu
\tag{40}
$$

does not affect the expressions of the Euler-Lagrange vector E

$$
E_B = E_0
\tag{41}
$$

but adds to S a boundary term

$$
S_B[u(\cdot)] = S_0[u(\cdot)] + \int_{\partial\mathcal{D}} B \cdot \mathrm{d}^{d-1}\sigma
\tag{42}
$$

from which we have

$$
\delta X_B = \delta X_0 + \delta B
\tag{43}
$$

---

[10] Working with $\delta X_{|u_{\mathrm{cl}}}$ orthogonal to $\mathrm{d}^{d-1}\sigma$ is sufficient and generalises the transversality condition discussed in [9, §§ IV.5.2 and IV.12.9]. A radical way of getting rid of the discussion on boundary conditions is also to work with a model where $\mathcal{D}$ has no boundaries.

or, more explicitly,

$$\delta X_B^\mu = \delta X_0^\mu + \frac{\partial B^\mu}{\partial u_n}\delta u_n + \frac{\partial B^\mu}{\partial(\partial_\nu u_n)}\partial_\nu \delta u_n + \frac{\partial B^\mu}{\partial(\partial_{\nu\rho}^2 u_n)}\partial_{\nu\rho}^2 \delta u_n + \cdots \qquad (44)$$

where the "$\cdots$" stand for derivatives of B with respect to higher derivatives of u.

4.2 Invariance with respect to infinitesimal transformations and Noether currents

The most general transformation T comes with both a change of coordinates $x \mapsto x^\mathrm{T}$ and a change of functions $u \mapsto u^\mathrm{T}$. By definition the transformed action is given by

$$S^\mathrm{T}[u^\mathrm{T}(\cdot)] = \int_{\mathcal{D}^\mathrm{T}} f^\mathrm{T}\big(x^\mathrm{T}, u^\mathrm{T}(x^\mathrm{T}), \partial_{x^\mathrm{T}} u^\mathrm{T}_{|x^\mathrm{T}}, \partial_{x^\mathrm{T}x^\mathrm{T}}^2 u^\mathrm{T}_{|x^\mathrm{T}}, \dots\big)\, \mathrm{d}^d x^\mathrm{T} \qquad (45)$$

with $S^\mathrm{T}[u^\mathrm{T}(\cdot)] = S[u(\cdot)]$ for any u and for any domain $\mathcal{D}$. After the change of variables $x^\mathrm{T} \mapsto x$ that pulls back $\mathcal{D}^\mathrm{T}$ to $\mathcal{D}$, we get

$$f^\mathrm{T}\big(x^\mathrm{T}, u^\mathrm{T}(x^\mathrm{T}), \partial_{x^\mathrm{T}} u^\mathrm{T}_{|x^\mathrm{T}}, \partial_{x^\mathrm{T}x^\mathrm{T}}^2 u^\mathrm{T}_{|x^\mathrm{T}}, \dots\big)\left|\det\left(\frac{\partial x^\mathrm{T}}{\partial x}\right)\right|$$
$$= f\big(x, u(x), \partial_x u_{|x}, \partial_{xx}^2 u_{|x}, \dots\big) \qquad (46)$$

which provides a *definition* of f^T. We have an *invariance* when the same computation rules are used to evaluate S and S^T that is $f^\mathrm{T} = f$. Then we have

$$f\big(x^\mathrm{T}, u^\mathrm{T}(x^\mathrm{T}), \partial_{x^\mathrm{T}} u^\mathrm{T}_{|x^\mathrm{T}}, \partial_{x^\mathrm{T}x^\mathrm{T}}^2 u^\mathrm{T}_{|x^\mathrm{T}}, \dots\big)\left|\det\left(\frac{\partial x^\mathrm{T}}{\partial x}\right)\right|$$
$$- f\big(x, u(x), \partial_x u_{|x}, \partial_{xx}^2 u_{|x}, \dots\big) = 0\,. \qquad (47)$$

The Noether conservation theorem comes straightforwardly from the computation of the left-hand side of (47) when the transformation T is infinitesimal[11]:

$$x^\mathrm{T} = x + \delta x\,; \qquad (48\mathrm{a})$$
$$u^\mathrm{T}(x) = u(x) + \delta u(x)\,. \qquad (48\mathrm{b})$$

---

[11] In Noether's spirit the transformation of all the dependent and independent variables can be as general as possible and therefore she first considers the case where δx is a function of both x and u; her two theorems indeed apply in this very general situation. Physically this corresponds to a transformation where the variations of the space-time coordinates δx depend not only on x, as this is the case in General Relativity where all the diffeomorphisms of space-time are considered, but also on the fields u. I do not know any relevant model in physics where this possibility has been exploited. In the following we will restrict δx to depend on x only, this simplification is eventually done by Noether from § 5 in [28].

To first order in δx and δu, (47) reads

$$f\,\partial_x\!\cdot\!\delta x+(\partial_x f)\!\cdot\!\delta x+\frac{\partial f}{\partial u_n}Du_n+\frac{\partial f}{\partial(\partial_\mu u_n)}D(\partial_\mu u_n)+\frac{\partial f}{\partial(\partial^2_{\mu\nu}u_n)}D(\partial^2_{\mu\nu}u_n)+\cdots$$
$$\cdots+\mathrm{O}(\delta^2)=0\;,\quad(49)$$

where $\mathrm{O}(\delta^2)$ denotes terms of order at least equal to two. The first term of the left-hand side comes from the Jacobian

$$\left|\det\left(\frac{\partial x^{\mathrm{T}}}{\partial x}\right)\right|=1+\partial_x\cdot\delta x+\mathrm{O}(\delta^2)\;.\tag{50}$$

The infinitesimal quantity δu denotes the variation of the field u while staying at the same point x and Du stands for the infinitesimal variation "following the transformation"[12]

$$Du(x)\overset{\text{def}}{=}u^{\mathrm{T}}(x^{\mathrm{T}})-u(x)=\delta u(x^{\mathrm{T}})+u(x^{\mathrm{T}})-u(x)=\delta u(x)+(\partial_x u)\cdot\delta x+\mathrm{O}(\delta^2)\;.\tag{51}$$

The chain rule for a composite function reads

$$\partial_{x^{\mathrm{T}}}u^{\mathrm{T}}(x^{\mathrm{T}})=\partial_{x^{\mathrm{T}}}x\,\partial_x\big(u^{\mathrm{T}}(x^{\mathrm{T}})\big)=\partial_{x^{\mathrm{T}}}x\,\partial_x\big(u(x)+Du(x)\big)\tag{52}$$

where the $d\times d$ Jacobian matrix of the transformation is

$$\partial_x x^{\mathrm{T}}=(\partial_{x^{\mathrm{T}}}x)^{-1}=1+\partial_x\delta x+\mathrm{O}(\delta^2)\;.\tag{53}$$

By putting (51) and (53) in (52), we obtain[13]

$$D(\partial_x u)\overset{\text{def}}{=}\partial_{x^{\mathrm{T}}}u^{\mathrm{T}}_{|x^{\mathrm{T}}}-\partial_x u_{|x}=\partial_x(\delta u)+\partial_x(\partial_x u)\cdot\delta x+\mathrm{O}(\delta^2)\;.\tag{54}$$

In the same way,

$$D(\partial^2_{xx}u)\overset{\text{def}}{=}\partial^2_{x^{\mathrm{T}}x^{\mathrm{T}}}u^{\mathrm{T}}_{|x^{\mathrm{T}}}-\partial^2_{xx}u_{|x}=\partial^2_{xx}(\delta u)+\partial_x(\partial^2_{xx}u)\cdot\delta x+\mathrm{O}(\delta^2)\tag{55}$$

and so on for the derivatives of u of higher orders. By reporting $D(\cdots)$ in (49) we get

$$f\,\partial_x\cdot\delta x+(\partial_x f)\cdot\delta x+\frac{\partial f}{\partial u_n}\,(\partial_x u_n)\cdot\delta x$$
$$+\frac{\partial f}{\partial(\partial_\mu u_n)}\,\partial_x(\partial_\mu u_n)\cdot\delta x+\frac{\partial f}{\partial(\partial^2_{\mu\nu}u_n)}\,\partial_x(\partial^2_{\mu\nu}u_n)\cdot\delta x \tag{56}$$
$$+\frac{\partial f}{\partial u_n}\,\delta u_n+\frac{\partial f}{\partial(\partial_\mu u_n)}\,\partial_\mu\delta u_n+\frac{\partial f}{\partial(\partial^2_{\mu\nu}u_n)}\,\partial^2_{\mu\nu}\delta u_n+\cdots+\mathrm{O}(\delta^2)=0\;.$$

---

[12] Borrowing the usual notation of fluid dynamics, this variation corresponds to the derivative following the motion often known as the convective/particle/material/Lagrangian derivative.

[13] If one prefers a notation where the indices are made explicit, the equations (54) and (55) can be respectively re-written as $D(\partial_\nu u_n)=\partial_\nu(\delta u_n)+(\partial^2_{\nu\mu}u_n)\delta x^\mu+\mathrm{O}(\delta^2)$ and $D(\partial^2_{\mu\nu}u_n)=\partial^2_{\mu\nu}(\delta u_n)+(\partial^3_{\mu\nu\rho}u_n)\delta x^\rho+\mathrm{O}(\delta^2)$.

The first two lines provide the divergence $\mathrm{d}_x \cdot \left(f\left(x, u(x), \partial_x u_{|x}, \partial_{xx}^2 u_{|x}, \dots \right) \delta x \right)$ and at the last line we recognise the variation δf given by (34). Then

$$E \cdot \delta u + \mathrm{d}_x \cdot (\delta X + f \delta x) = 0 \ . \tag{57}$$

With the help of (38), we deduce Noether's conservation law for the infinitesimal current: *If the functional* (32) *is invariant under a continuous family of transformations having, in the neighbourhood of the identity the form* (48), *then for any solution* u_{cl} *such that S is stationary, the (infinitesimal) Noether current*

$$\delta J \ \overset{\mathrm{def}}{=} \ \delta X + f \delta x \tag{58}$$

with δX given by (36) *is conserved; that is*

$$\mathrm{d}_x \cdot \delta J_{|u_{\mathrm{cl}}} = \mathrm{d}_\mu \delta J^\mu_{|u_{\mathrm{cl}}} = 0 \ . \tag{59}$$

More explicitly we have

$$\delta J^\mu = f \delta x^\mu + \frac{\partial f}{\partial(\partial_\mu u_n)} \delta u_n - \mathrm{d}_\nu \left(\frac{\partial f}{\partial(\partial_{\mu\nu}^2 u_n)} \right) \delta u_n + \frac{\partial f}{\partial(\partial_{\mu\nu}^2 u_n)} \partial_\nu (\delta u_n) + \cdots$$

$$\tag{60a}$$

$$= \left[f \delta^\mu_\nu - \frac{\partial f}{\partial(\partial_\mu u_n)} \partial_\nu u_n + \mathrm{d}_\rho \left(\frac{\partial f}{\partial(\partial_{\mu\rho}^2 u_n)} \right) \partial_\nu u_n - \frac{\partial f}{\partial(\partial_{\mu\rho}^2 u_n)} \partial_{\nu\rho}^2 u_n + \cdots \right] \delta x^\nu$$

$$+ \frac{\partial f}{\partial(\partial_\mu u_n)} D u_n - \mathrm{d}_\nu \left(\frac{\partial f}{\partial(\partial_{\mu\nu}^2 u_n)} \right) D u_n + \frac{\partial f}{\partial(\partial_{\mu\nu}^2 u_n)} D(\partial_\nu u_n) + \cdots \tag{60b}$$

where the Kronecker symbol δ is used and "$\cdots$" stands for terms involving the derivatives of f with respect to third order or higher derivatives of u. Since the invariance of the variational problem depends on the choice of boundary function, so will the Noether current as we can see from (40) and (43):

$$\delta J_B = \delta J_0 + (\mathrm{d}_x \cdot B) \delta x + \delta B \ . \tag{61}$$

In fact, Noether currents δJ are defined up to a divergence-free current since adding such a term does not affect (59). For instance

$$\delta J'^\mu = \delta J^\mu + \mathrm{d}_\nu \left[\left(\frac{\partial B^\mu}{\partial(\partial_\nu u_n)} - \frac{\partial B^\nu}{\partial(\partial_\mu u_n)} \right) \delta u_n \right] \tag{62}$$

would be also an acceptable Noether current associated with the symmetry under the scope.

4.3 Aside remarks about the two Noether theorems

The result established in the previous section is neither the first Noether theorem nor the second one but encapsulates both of them; the conservation of the *infinitesimal* current δJ occurs for any global or local symmetry. Noether's first theorem follows from the computation of δX for a global symmetry i.e. when the number of the essential parameters $\epsilon = (\epsilon^a)_a$ of the Lie group of transformations is finite. In that case

$$\delta J = \mathscr{J}\epsilon + \mathrm{O}(\epsilon^2) \tag{63}$$

or in terms of coordinates

$$\delta J^\mu = \mathscr{J}_a^{\ \mu}\epsilon^a + \mathrm{O}(\epsilon^2) \tag{64}$$

and the first Noether theorem states the conservation of the *non infinitesimal* $\mathscr{J}$

$$\mathrm{d}_x \cdot \mathscr{J}_a = \partial_\mu \mathscr{J}_a^{\ \mu} = 0 \tag{65}$$

obtained immediately from the infinitesimal conservation law (59) since ϵ is arbitrary and x-independent.

Noether's second theorem (see footnote 2) follows from the computation of δX for a local symmetry i.e. when the essential parameters are functions $\epsilon(x)$ and, in that case, the proportionality relation (63) does not hold anymore ; the right-hand side now includes the derivatives of ϵ:

$$\delta J^\mu = \mathscr{J}^\mu \epsilon + \mathscr{F}^{\mu\nu}\partial_\nu\epsilon + \mathscr{K}^{\mu\nu\rho}\partial^2_{\nu\rho}\epsilon + \cdots + \mathrm{O}(\epsilon^2) \;. \tag{66}$$

By expanding the variation of the fields according to

$$\delta u = \frac{\partial u}{\partial \epsilon}\epsilon + \frac{\partial u}{\partial(\partial_\mu \epsilon)}\partial_\mu\epsilon + \frac{\partial^2 u}{\partial(\partial^2_{\mu\nu}\epsilon)}\partial^2_{\mu\nu}\epsilon + \cdots + \mathrm{O}(\epsilon^2) \;, \tag{67}$$

then (57) reads

$$\left[E \cdot \frac{\partial u}{\partial \epsilon} + \mathrm{d}_\mu \mathscr{J}^\mu \right]\epsilon + \left[E \cdot \frac{\partial u}{\partial(\partial_\mu \epsilon)} + \mathscr{J}^\mu + \mathrm{d}_\nu \mathscr{F}^{\nu\mu} \right]\partial_\mu\epsilon$$
$$+ \left[E \cdot \frac{\partial^2 u}{\partial(\partial^2_{\mu\nu}\epsilon)} + \frac{1}{2}(\mathscr{F}^{\nu\mu} + \mathscr{F}^{\mu\nu}) + \mathrm{d}_\rho\mathscr{K}^{\rho\nu\mu} \right]\partial^2_{\mu\nu}\epsilon + \cdots = 0 \;. \tag{68}$$

Since the functions ϵ are arbitrary, all the brackets vanish separately. When evaluated on the stationary solutions u_{cl}, we get

$$\mathrm{d}_\mu \mathscr{J}^\mu = 0 \;; \quad \mathrm{d}_\mu\mathscr{F}^{\mu\nu} = -\mathscr{J}^\nu \;, \quad \mathrm{d}_\rho\mathscr{K}^{\rho\nu\mu} = -\frac{1}{2}(\mathscr{F}^{\nu\mu} + \mathscr{F}^{\mu\nu}) \;, \quad \text{etc.} \tag{69}$$

For a constant ϵ we recover the first theorem from the first equality. The second theorem stipulates that to each a there is one identity connecting the E's:

$$E \cdot \frac{\partial u}{\partial \epsilon} - \mathrm{d}_\mu\left(E \cdot \frac{\partial u}{\partial(\partial_\mu \epsilon)} \right) + \mathrm{d}_\mu\mathrm{d}_\nu\left(E \cdot \frac{\partial^2 u}{\partial(\partial^2_{\mu\nu}\epsilon)} \right) + \cdots = 0 \;. \tag{70}$$

Those can be obtained from the vanishing brackets of (68) or directly from the following re-writing of (57):

$$
\left[E \cdot \frac{\partial u}{\partial \epsilon} \; -\mathrm{d}_\mu \left(E \cdot \frac{\partial u}{\partial(\partial_\mu \epsilon)} \right) + \mathrm{d}_\mu \mathrm{d}_\nu \left(E \cdot \frac{\partial^2 u}{\partial(\partial_{\mu\nu}^2 \epsilon)} \right) + \cdots \right] \epsilon
\tag{71}
$$

$$
+\mathrm{d}_\mu \left[\delta J^\mu + E \cdot \frac{\partial u}{\partial(\partial_\mu \epsilon)} \, \epsilon - \mathrm{d}_\nu \left(E \cdot \frac{\partial^2 u}{\partial(\partial_{\mu\nu}^2 \epsilon)} \right) \epsilon + E \cdot \frac{\partial^2 u}{\partial(\partial_{\mu\nu}^2 \epsilon)} \, \partial_\nu \epsilon + \cdots \right] = 0 \; .
$$

By an integration on any arbitrary volume and choosing ϵ and its derivatives vanishing on its boundary, on can get rid of the integral of the second term of the left-hand-side. Since ϵ can be chosen otherwise arbitrarily within this volume, the first bracket vanishes which is exactly the Noether identity (70)[14]. If one had to speak of just one theorem connecting symmetries and conservation laws, one could choose the cancellation of all the brackets of (68) from which Noether's theorems I and II are particular cases.

Eventually, let us mention that both Noether's theorems include also a reciprocal statement: the invariance in the neighbourhood of $\epsilon = 0$ implies an invariance for any finite ϵ and this comes from the properties of the underlying Lie structure of the transformation group and its internal composition law that allow to naturally map any neighbourhood of $\epsilon = 0$ to a neighbourhood of any other element of the group.

5 Applications

5.1 Finite number of degrees of freedom

From the general formalism in § 4 it is straightforward to show that the conservation law we obtained within the Hamiltonian framework in § 3.2 is encapsulated in Noether's original approach. For L degrees of freedom $q = (q_\alpha)_{\alpha \in \{1,\dots,\mathrm{L}\}}$ we have $u = (p, q, t)$ with $N = 2\mathrm{L} + 1$, S is of course S_B given by equation (5), $\mathcal{D}$ is $[s_i, s_f]$, x is identified with s $(d = 1)$ and only the first derivatives of q, t, and possibly p through $\mathrm{d}B/\mathrm{d}s$ are involved. We are considering transformations where s is unchanged: $\delta x = \delta s = 0$, and then, $D = \delta$. There-

---

[14] As a consequence, the cancellation of the first bracket in (68) allows to write the first term of (70) as a total derivative and this leads to the conservation of a current

$$
\mathrm{d}_\mu \left[\mathscr{J}^\mu + E \cdot \frac{\partial u}{\partial(\partial_\mu \epsilon)} - \mathrm{d}_\nu \left(E \cdot \frac{\partial^2 u}{\partial(\partial_{\mu\nu}^2 \epsilon)} \right) + \cdots \right] = 0
\tag{72}
$$

which is qualified as a "strong" [2, § 6 and its references] because this constraint holds even if the Euler-Lagrange equations are not satisfied (a primary constraint in Dirac's terminology [13]).

fore, with $f_B(p, q, t, \mathrm{d}p/\mathrm{d}s, \mathrm{d}q/\mathrm{d}s, \mathrm{d}t/\mathrm{d}s) = p\,\mathrm{d}q/\mathrm{d}s - H(p, q, t)\,\mathrm{d}t/\mathrm{d}s + \mathrm{d}B/\mathrm{d}s$,

$$\delta J_B = \delta X_B = \frac{\partial f_B}{\partial (\mathrm{d}p/\mathrm{d}s)}\delta p + \frac{\partial f_B}{\partial (\mathrm{d}q/\mathrm{d}s)}\delta q + \frac{\partial f_B}{\partial (\mathrm{d}t/\mathrm{d}s)}\delta t \; ; \qquad (73\mathrm{a})$$

$$= \partial_p B\,\delta p + (p + \partial_q B)\,\delta q - (H - \partial_t B)\,\delta t \; ; \qquad (73\mathrm{b})$$

$$= p\delta q - H\delta t + \delta B \; , \qquad (73\mathrm{c})$$

which is a particular case of (61). Precisely because of the invariance, the variations coming from the infinitesimal transformation under the scope naturally satisfy the boundary conditions (6) used to formulate the variational principle that now can be interpreted as the conservation of δJ_B between s_i and s_f. The invariance (20) of B reads $\delta B = -p^{\mathrm{T}}q^{\mathrm{T}} + F(p^{\mathrm{T}}, q, t) = -p^{\mathrm{T}}(q^{\mathrm{T}} - q) + \epsilon G(p^{\mathrm{T}}, q, t) + \mathrm{O}(\epsilon^2) = -p\delta q + \epsilon G(p, q, t) + \mathrm{O}(\epsilon^2)$ and

$$\delta J_B = \epsilon G(p, q, t) - H\delta t \; . \qquad (74)$$

For an arbitrary pure time translation δt is s-independent and $\epsilon = 0$, then (59), which reads $\mathrm{d}\delta J_B/\mathrm{d}s = 0$, just expresses the constancy of H. For a canonical transformation that does not affect the time, the latter equation shows that its generator G is an integral of motion. Thus, with a presentation much closer to Noether's original spirit we actually recover the results of section § 3.2. What is remarkable is that, in the latter case, the Noether constants are independent of H and B whereas, a priori, the general expression of the current (60a) depends on f (see also (61)): only the canonical structure, intimately bound to the structure of the action (1), leaves its imprint whereas the explicit forms of the Hamiltonian and the boundary function have no influence on the expression of the conserved currents (as soon as the invariance is maintained of course). In other words, it is worth noticed that the Noether currents keep the same expression for all the (infinite class of) actions that are invariant under the associated transformations.

5.2 Examples in field theory

The discussion of the previous paragraph still holds at the limit $\mathrm{L} \to \infty$ but it is worth to adapt it to the case of field models. A field involves an infinite number of degrees of freedom that we shall take continuous and preferably labeled by the D-dimensional space coordinates $\alpha = \mathbf{x}$ rather than the dual wave-vectors $\mathbf{k}$. The additional discrete "internal" quantum numbers like those that distinguish the spin components are left implicit. Now the Hamiltonian appears to be a functional of the dynamical variables, namely the fields $\{\pi, \varphi\}$ and their spatial derivatives—restricted to order one for the sake of simplicity whereas we have seen from the general approach that this assumption is not mandatory—of the form

$$H[\pi(t, \cdot), \varphi(t, \cdot), t] = \int_{\mathcal{V}} \mathscr{H}\big(\pi(t, \mathbf{x}), \varphi(t, \mathbf{x}), \partial_{\mathbf{x}}\pi(t, \mathbf{x}), \partial_{\mathbf{x}}\varphi(t, \mathbf{x}), t, \mathbf{x}\big)\,\mathrm{d}^{\mathrm{D}}\mathbf{x}$$

$$\qquad (75)$$

where $\mathcal{V}$ is a D-dimensional spatial domain and $\mathscr{H}$, the Hamiltonian density that may a priori depend explicitly on $x = (t, \mathbf{x})$. The action

$$S_B[\pi(\cdot), \varphi(\cdot)] = \int_{\mathcal{V} \times [t_i, t_f]} (\pi \partial_t \varphi - \mathscr{H} + \mathrm{d}_t \mathscr{B}) \, \mathrm{d}^D \mathbf{x} \, \mathrm{d}t \qquad (76)$$

involves a boundary density $\mathscr{B}\big(\pi(t, \mathbf{x}), \varphi(t, \mathbf{x}), \partial_{\mathbf{x}} \pi(t, \mathbf{x}), \partial_{\mathbf{x}} \varphi(t, \mathbf{x}), t, \mathbf{x}\big)$ from which the boundary function(nal) is given by $\int_{\mathcal{V}} \mathscr{B} \mathrm{d}^D \mathbf{x}$ keeping the same locality principle as we used for H (we assume that neither H nor B involve non-local terms like $\varphi(\mathbf{x}) V(\mathbf{x}' - \mathbf{x}) \varphi(\mathbf{x})$). Whenever working in a relativistic framework, $\mathscr{B}$ can be seen as the 0th-component of a $(D + 1)$-vector $B = (B^0, B^1, \ldots, B^D) = (\mathscr{B}, 0, \ldots, 0)$ such that $\mathrm{d}_t \mathscr{B} = \mathrm{d}_\mu B^\mu$ and the space-time integral defining S_B can be seen as an integral over the $d = (D+1)$-dimensional domain $\mathcal{D}$ between two appropriate Cauchy surfaces. The action (76) takes the general expression form (32) with now $N = 2$ fields $u = (u_1, u_2) = (\pi, \varphi)$ and f given by

$$f_B\big(\pi, \varphi, \partial_x \pi, \partial_x \varphi, x\big) = \pi \partial_0 \varphi - \mathscr{H}\big(\pi, \varphi, \partial_{\mathbf{x}} \pi, \partial_{\mathbf{x}} \varphi, x\big) + \mathrm{d}_\mu B^\mu . \qquad (77)$$

By canceling the components E^1 and E^2 computed from (35) we obtain the evolution equations of the classical fields

$$\partial_t \varphi = \frac{\partial \mathscr{H}}{\partial \pi} - \frac{\mathrm{d}}{\mathrm{d}x^i}\left(\frac{\partial \mathscr{H}}{\partial(\partial_i \pi)}\right) ; \qquad (78\text{a})$$

$$\partial_t \pi = -\frac{\partial \mathscr{H}}{\partial \varphi} + \frac{\mathrm{d}}{\mathrm{d}x^i}\left(\frac{\partial \mathscr{H}}{\partial(\partial_i \varphi)}\right) . \qquad (78\text{b})$$

The Noether infinitesimal current is given by (61) with

$$\delta J_B^\mu = (\pi \partial_t \varphi - \mathscr{H}) \delta x^\mu + \left(\pi \delta_0^\mu - \frac{\partial \mathscr{H}}{\partial(\partial_\mu \varphi)}\right) \delta \varphi - \delta \pi \frac{\partial \mathscr{H}}{\partial(\partial_\mu \pi)} \qquad (79)$$

$$+ (\mathrm{d}_\rho B^\rho) \delta x^\mu + \frac{\partial B^\mu}{\partial \varphi} \delta \varphi + \delta \pi \frac{\partial B^\mu}{\partial \pi} + \frac{\partial B^\mu}{\partial(\partial_\rho \varphi)} \partial_\rho \delta \varphi + \partial_\rho \delta \pi \frac{\partial B^\mu}{\partial(\partial_\rho \pi)} .$$

As an illustration, let us specify the latter general expression in the special case of the space-time translations. We have $\varphi^{\mathrm{T}}(x) = \varphi(x - \delta x)$ and $\pi^{\mathrm{T}}(x) = \pi(x - \delta x)$ so the infinitesimal variations of the fields are

$$\delta \pi = -\partial \pi \cdot \delta x ; \qquad \delta \varphi = -\partial \varphi \cdot \delta x \qquad (80)$$

and then, since we take δx to be independent of x, we get (*cf* equation (64) with a being now the space-time label and $\epsilon = \delta x$)

$$\delta J_B^\mu = \mathscr{T}_{B|\nu}^\mu \delta x^\nu \qquad (81)$$

with the energy-momentum tensor given up to a divergence-free current[15] by

$$\mathscr{T}^\mu_{B|\nu} = \mathscr{T}^\mu_{0|\nu} + (\mathrm{d}_\rho B^\rho)\delta^\mu_\nu - \frac{\partial B^\mu}{\partial \varphi}\partial_\nu\varphi - \partial_\nu\pi\frac{\partial B^\mu}{\partial \pi} - \frac{\partial B^\mu}{\partial(\partial_\rho\varphi)}\partial^2_{\rho\nu}\varphi - \partial^2_{\rho\nu}\pi\frac{\partial B^\mu}{\partial(\partial_\rho\pi)} \; ; \tag{82}$$

$$= \mathscr{T}^\mu_{0|\nu} + (\mathrm{d}_\rho B^\rho)\delta^\mu_\nu - \mathrm{d}_\nu B^\mu + \partial_\nu B^\mu \; , \tag{83}$$

and

$$\mathscr{T}^\mu_{0|\nu} = (\pi\partial_t\varphi - \mathscr{H})\,\delta^\mu_\nu + \partial_\nu\pi\frac{\partial \mathscr{H}}{\partial(\partial_\mu\pi)} + \left(\frac{\partial \mathscr{H}}{\partial(\partial_\mu\varphi)} - \pi\delta^\mu_0\right)\partial_\nu\varphi \; . \tag{84}$$

The invariance of the boundary function under translations requires $\partial_\nu B^\mu = 0$ and the corresponding $(\mathrm{D} + 1)$-momentum contained in the volume $\mathcal{V}$ is therefore given by

$$P_{B|\nu} = \int_\mathcal{V} \mathscr{T}^0_{B|\nu}\,\mathrm{d}^\mathrm{D}\mathbf{x} = P_\nu + \Delta P_\nu \tag{85}$$

where

$$P_\nu = \int_\mathcal{V} \left[(\pi\partial_t\varphi - \mathscr{H})\,\delta^0_\nu - \pi\partial_\nu\varphi\right]\mathrm{d}^\mathrm{D}\mathbf{x} \; . \tag{86}$$

On can check that $P^0 = -P_0$ is given by (75). The boundary function brings some surface corrections

$$\Delta P_\nu = \int_\mathcal{V} \left[(\mathrm{d}_\rho B^\rho)\delta^0_\nu - \mathrm{d}_\nu B^0\right]\mathrm{d}^\mathrm{D}\mathbf{x} \tag{87}$$

that is

$$\Delta P_0 = \int_\mathcal{V} \mathrm{d}_i B^i\,\mathrm{d}^\mathrm{D}\mathbf{x} = \int_{\partial\mathcal{V}} B^i\,\mathrm{d}^{\mathrm{D}-1}\sigma_i \tag{88}$$

and

$$\Delta P_i = \int_\mathcal{V} \mathrm{d}_i B^0\mathrm{d}^\mathrm{D}\mathbf{x} = \int_{\partial\mathcal{V}} B^0\,\mathrm{d}^{\mathrm{D}-1}\sigma_i \tag{89}$$

where $\mathrm{d}^{\mathrm{D}-1}\sigma_i$ are the D components of the surface element defined on $\partial\mathcal{V}$. In any reasonable model these corrections are expected to vanish when $\partial\mathcal{V}$ is extended to infinity.

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[15] Adding a divergence-free current may be exploited to work with a symmetric tensor known as the Belinfante-Rosenfeld tensor since this was first proposed by [3,31].

5.3 Comparison with the Lagrangian approach

For the sake of completeness let us comment on the connection with the Lagrangian framework of a system with L degrees of freedom. Consider now (32) with f being $L(q, \dot{q}, t) + \mathrm{d}B/\mathrm{d}t$ where B is a function of q, $\dot{q}$ and t, the integration variable x is just the time t $(d = 1)$ and the number of dynamical variables $u = q$ is divided by two $(N = \mathrm{L})$ by comparison with the Hamiltonian framework. The derivative $\mathrm{d}B/\mathrm{d}t$ depends on $\ddot{q}$ and this must be taken into account when computing directly δX from (36)

$$\delta X = \frac{\partial f}{\partial \dot{q}} \delta q - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial f}{\partial \ddot{q}} \right) \delta q + \frac{\partial f}{\partial \ddot{q}} \frac{\mathrm{d}\delta q}{\mathrm{d}t} \; ; \tag{90a}$$

$$= \left[\frac{\partial L}{\partial \dot{q}} + \frac{\partial B}{\partial q} \underbrace{+ \frac{\partial^2 B}{\partial \dot{q} \partial \dot{q}} \ddot{q} + \frac{\partial^2 B}{\partial \dot{q} \partial q} \dot{q} + \frac{\partial^2 B}{\partial \dot{q} \partial t} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial B}{\partial \dot{q}} \right)}_{= 0} \right] \delta q + \frac{\partial B}{\partial \dot{q}} \frac{\mathrm{d}\delta q}{\mathrm{d}t}. \tag{90b}$$

Hence, since $\delta x = \delta t$, we have rederived a particular case of (61),

$$\delta J = \frac{\partial L}{\partial \dot{q}} \delta q + L \delta t + \frac{\partial B}{\partial q} \delta q + \frac{\partial B}{\partial \dot{q}} \frac{\mathrm{d}\delta q}{\mathrm{d}t} + \frac{\mathrm{d}B}{\mathrm{d}t} \delta t \; . \tag{91}$$

To reconcile (91) and (73c), one must be aware that δq has a different meaning in the two equations. Indeed, in the general expression (36) δu stands for a variation of u computed at the same x (see (48b)); within the Hamiltonian formalism, $\delta^{(\mathrm{ham})}q$ thus denotes a variation of q at the same parameter s whereas within the Lagrangian formalism, $\delta^{(\mathrm{lag})}q$ denotes a variation of q at the same time t. Precisely when the transformation modifies t, these two variations differs. To connect them one has to introduce the parametrisation s in the Lagrangian formalism

$$\delta^{(\mathrm{lag})}q\big(t(s)\big) = q^{\mathrm{T}}\big(t(s)\big) - q\big(t(s)\big) \tag{92}$$

and then

$$\delta^{(\mathrm{ham})}q\big(t(s)\big) = q^{\mathrm{T}}\big(t^{\mathrm{T}}(s)\big) - q\big(t(s)\big) = D^{(\mathrm{lag})}q \; , \tag{93}$$

with $t^{\mathrm{T}}(s) = t(s) + \delta t(s)$ [16]. Then,

$$\delta^{(\mathrm{lag})}q = \delta^{(\mathrm{ham})}q - \dot{q}\delta t \; . \tag{94}$$

Reporting this last expression in (91), we get

$$\delta J = \frac{\partial L}{\partial \dot{q}} \delta^{(\mathrm{ham})}q + \left(L - \frac{\partial L}{\partial \dot{q}} \dot{q} \right) \delta t + \frac{\partial B}{\partial t} \delta t + \frac{\partial B}{\partial q} \delta^{(\mathrm{ham})}q + \frac{\partial B}{\partial \dot{q}} \left(\frac{\mathrm{d}\delta^{(\mathrm{ham})}q}{\mathrm{d}t} - \dot{q} \frac{\mathrm{d}\delta t}{\mathrm{d}t} \right) \; . \tag{95}$$

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[16] Because $\delta^{(\mathrm{ham})}t = t^{\mathrm{T}}(s) - t(s) = t^{\mathrm{T}} - t = \delta^{(\mathrm{lag})}t$, we won't use two different notations for the variations of t.

Turning back to the parametrisation by s, the last parenthesis is

$$\delta^{\text{(ham)}}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right) = \delta^{\text{(ham)}}\left(\frac{1}{\mathrm{d}t/\mathrm{d}s}\frac{\mathrm{d}q}{\mathrm{d}s}\right) ; \tag{96}$$

$$= \underbrace{\frac{1}{\mathrm{d}t/\mathrm{d}s}\delta^{\text{(ham)}}\left(\frac{\mathrm{d}q}{\mathrm{d}s}\right)}_{= \frac{\mathrm{d}\delta^{\text{(ham)}}q}{\mathrm{d}s}} - \frac{1}{(\mathrm{d}t/\mathrm{d}s)^2}\frac{\mathrm{d}q}{\mathrm{d}s}\underbrace{\delta^{\text{(ham)}}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)}_{= \frac{\mathrm{d}\delta t}{\mathrm{d}s}} . \tag{97}$$

therefore one recovers

$$\delta J = \frac{\partial L}{\partial \dot{q}}\delta^{\text{(ham)}}q + \left(L - \frac{\partial L}{\partial \dot{q}}\dot{q}\right)\delta t + \delta^{\text{(ham)}}B \tag{98}$$

which coincides with (73c) using

$$L\left(q, \frac{\mathrm{d}q}{\mathrm{d}t}, t\right) \overset{\text{def}}{=} p\frac{\mathrm{d}q}{\mathrm{d}t} - H(p, q, t) . \tag{99}$$

We could also have obtained (73c) by working with the Lagrangian functional where all the functions are systematically computed with s

$$S_B = \int_{s_i}^{s_f}\left[L\left(q(s), \frac{1}{\mathrm{d}t/\mathrm{d}s}\frac{\mathrm{d}q}{\mathrm{d}s}, t(s)\right)\frac{\mathrm{d}t}{\mathrm{d}s} + \frac{\mathrm{d}B}{\mathrm{d}s}\right]\mathrm{d}s , \tag{100}$$

or, conversely, by eliminating all the references to s in the Hamiltonian functional

$$S_B = \int_{t_i}^{t_f}\left[p\frac{\mathrm{d}q}{\mathrm{d}t} - H + \frac{\mathrm{d}B}{\mathrm{d}t}\right]\mathrm{d}t . \tag{101}$$

For a Lagrangian field model we have $u = \varphi$ and (60a) reads

$$\delta J^\mu = \mathscr{L}\delta x^\mu + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi)}\delta\varphi . \tag{102}$$

Using the fact that $\mathscr{H}$ does not depend on $\partial_t\pi$ nor $\partial_t\varphi$ and with the help of

$$\mathscr{L} = \pi\partial_t\varphi - \mathscr{H} \tag{103}$$

and

$$\pi = \frac{\partial\mathscr{L}}{\partial(\partial_t\varphi)} , \tag{104}$$

the two first terms of the right-hand-side of (79) are identical to those appearing in (102):

$$\delta J_0^\mu = (\pi\partial_t\varphi - \mathscr{H})\delta x^\mu + \left(\pi\delta_0^\mu - \frac{\partial\mathscr{H}}{\partial(\partial_\mu\varphi)}\right)\delta\varphi - \delta\pi\frac{\partial\mathscr{H}}{\partial(\partial_\mu\pi)} . \tag{105}$$

The two currents coincide when $\mathscr{H}$ does not depend on $\partial_{\mathbf{x}}\pi$ which is a common case.

6 Quantum framework

6.1 Complex canonical formalism

In quantum theory, any state $|\psi\rangle$ can be represented by the list $z = (z_\alpha)_\alpha$ of its complex components $z_\alpha \overset{\text{def}}{=} \langle \phi_\alpha | \psi \rangle$ on a given orthonormal basis $\big(|\phi_\alpha\rangle \big)_\alpha$ labeled by the quantum numbers α. For simplicity we will work with discrete quantum numbers but this is not a decisive hypothesis here and what follows can be adapted to relativistic as well as non-relativistic quantum field theory. The quantum evolution is governed by a self-adjoint Hamiltonian[17] $\hat{H}(t)$ according to

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\psi\rangle = \hat{H}(t) |\psi\rangle \tag{106}$$

or equivalently

$$i\hbar \dot{z}_\alpha(t) = \sum_{\alpha'} H_{\alpha,\alpha'}(t) \, z_{\alpha'}(t) \tag{107}$$

with the matrix element

$$H_{\alpha,\alpha'}(t) \overset{\text{def}}{=} \langle \phi_\alpha | \, \hat{H}(t) \, | \phi_{\alpha'} \rangle \; . \tag{108}$$

Provided we accept to extend the classical Hamiltonian formalism to complex dynamical variables, one can see that the quantum dynamics described above can be derived from the "classical" quadratic Hamiltonian

$$\mathsf{H}(w, z, t) \overset{\text{def}}{=} \frac{1}{i\hbar} \sum_{\alpha,\alpha'} w_\alpha H_{\alpha,\alpha'}(t) z_{\alpha'} \tag{109}$$

where each couple (w_α, z_α) is now considered as a pair of complex canonical variables (p_α, q_α). The equation (107) corresponds to Hamilton's equations for q whereas Hamilton's equations for p are

$$i\hbar \dot{w}_\alpha(t) = -\sum_{\alpha'} w_{\alpha'}(t) H_{\alpha',\alpha}(t) \tag{110}$$

which can also be derived by complex conjugation of (107) since the hermiticity of $\hat{H}$ reads $H_{\alpha,\alpha'}^* = H_{\alpha',\alpha}$.

The quantum evolution between t_i and t_f can therefore be rephrased with a variational principle based on a functional having the classical form (5) with a boundary function $\mathsf{B}(w, z, t)$. Since in this context we will not consider transformations of time that depend on the dynamical variables, we can use t as the integration variable and work with

$$S_B[w(\cdot), z(\cdot)] \overset{\text{def}}{=} \int_{t_i}^{t_f} \left\{ \sum_\alpha w_\alpha(t) \, \dot{z}_\alpha(t) - \mathsf{H}(w, z, t) + \frac{\mathrm{d}\mathsf{B}}{\mathrm{d}t} \right\} \mathrm{d}t \tag{111}$$

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[17] For a non-isolated system, even in the Schrödinger picture, the Hamiltonian may depend on time.

where the complex functions $t \mapsto z_\alpha(t)$ and $t \mapsto w_\alpha(t)$ are considered to be independent one from the other. Together they constitute $u = (w, z)$ with $x = t$ ($d = 1$). Thus, all the classical analysis of § 2 and § 5.1 still holds. The variations of z and w are constrained by the boundary conditions

$$\left[\sum_\alpha w_\alpha \, \delta z_\alpha + \delta \mathsf{B} \right]_{t_i}^{t_f} = \left[\langle \chi | \left(\delta | \psi \rangle \right) + \delta \mathsf{B} \right]_{t_i}^{t_f} = 0 \tag{112}$$

where $\langle \chi |$ is such that $w_\alpha = \langle \chi | \phi_\alpha \rangle$. All the variations $\delta | \psi \rangle$ of the dynamical variables given by $| \psi \rangle$ cannot generically vanish at t_i and t_f since there is in general no solution of the Schrödinger equation (106) for an a priori given arbitrary choice of an initial *and* a final state. Due also to the linear dependence of the Hamiltonian H with respect to z and w, we cannot express $p = w$ as a function of $(q, \dot{q}) = (z, \dot{z})$ and therefore we cannot switch to a Lagrangian formulation unless we collect the variables w with the variables z into the same configuration space.

According to Wigner theorem, a (possibly time-dependent) continuous transformation is represented by a unitary operator $\hat{U}$ implemented as follows

$$^\mathrm{T}\langle \chi | \overset{\mathrm{def}}{=} \langle \chi | \hat{U}^* \; ; \qquad | \psi \rangle^\mathrm{T} \overset{\mathrm{def}}{=} \hat{U} | \psi \rangle \tag{113}$$

or with the canonical complex notation,

$$\sum_{\alpha'} w_{\alpha'}^\mathrm{T} \langle \phi_{\alpha'} | \hat{U} | \phi_\alpha \rangle = w_\alpha \; ; \qquad z_\alpha^\mathrm{T} = \sum_{\alpha'} \langle \phi_\alpha | \hat{U} | \phi_{\alpha'} \rangle z_{\alpha'} \; . \tag{114}$$

By straightforward identification with the complex version of (12) with vanishing derivatives of t^T, we have

$$w_\alpha = \frac{\partial \mathsf{F}}{\partial z_\alpha} \; ; \tag{115a}$$

$$z_\alpha^\mathrm{T} = \frac{\partial \mathsf{F}}{\partial w_\alpha^\mathrm{T}} \tag{115b}$$

with the generating function

$$\mathsf{F}(w^\mathrm{T}, z) = \sum_{\alpha, \alpha'} w_\alpha^\mathrm{T} \langle \phi_\alpha | \hat{U} | \phi_{\alpha'} \rangle z_{\alpha'} \tag{116}$$

or, equivalently,

$$\mathsf{F} = {}^\mathrm{T}\langle \chi | \hat{U} | \psi \rangle \; . \tag{117}$$

For a one-parameter transformation, its generator is a self-adjoint operator $\hat{G}$, possibly time-dependent, such that

$$\hat{U}(\epsilon) = 1 + \frac{\mathrm{i}\epsilon}{\hbar} \hat{G} + \mathrm{O}(\epsilon^2). \tag{118}$$

Then the generating function $\mathsf{F}(w, z, t; \epsilon)$ given by

$$\mathsf{F}(w^\mathrm{T}, z, t; \epsilon) = \sum_\alpha w_\alpha^\mathrm{T} z_\alpha + \frac{\mathrm{i}\epsilon}{\hbar} \sum_{\alpha, \alpha'} w_\alpha^\mathrm{T} \langle \phi_\alpha | \hat{G} | \phi_{\alpha'} \rangle z_{\alpha'} + \mathrm{O}(\epsilon^2) \tag{119}$$

from which, by identification with the complexification of (23), we read the "classical" generator

$$\mathsf{G} = \frac{\mathrm{i}}{\hbar} \sum_{\alpha,\alpha'} w_\alpha \langle \phi_\alpha | \hat{G} | \phi_{\alpha'} \rangle z_{\alpha'} = \frac{\mathrm{i}}{\hbar} \langle \chi | \hat{G} | \psi \rangle \tag{120}$$

of the transformation.

Now for an invariance we respect the time translations, (28) reads

$$0 = \frac{\mathrm{d}\mathsf{H}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \langle \chi | \hat{H} | \psi \rangle = \langle \chi | \frac{\mathrm{d}\hat{H}}{\mathrm{d}t} | \psi \rangle \tag{121}$$

for any $\langle \chi |$ and $| \psi \rangle$, that is we recover

$$\frac{\mathrm{d}\hat{H}}{\mathrm{d}t} = 0. \tag{122}$$

For an invariance with respect to a time-independent transformation, (30) reads

$$0 = \frac{\mathrm{d}\mathsf{G}}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar} \frac{\mathrm{d}}{\mathrm{d}t} \langle \chi | \hat{G} | \psi \rangle = \frac{\mathrm{i}}{\hbar} \langle \chi | \left(\frac{\mathrm{d}\hat{G}}{\mathrm{d}t} + \frac{\mathrm{i}}{\hbar}[\hat{H}, \hat{G}] \right) | \psi \rangle \tag{123}$$

where $[\ ,\ ]$ denotes the commutator between two operators. Then we get the identity

$$\frac{\mathrm{d}\hat{G}}{\mathrm{d}t} + \frac{\mathrm{i}}{\hbar}[\hat{H}, \hat{G}] = 0 \ . \tag{124}$$

In the Schrödinger picture the time-independence of the transformation is equivalent to $\mathrm{d}\hat{G}/\mathrm{d}t = 0$ and therefore the previous identity reduces to

$$[\hat{H}, \hat{G}] = 0 \tag{125}$$

which is of course the well-known consequence of the invariance of the quantum dynamics under the transformations generated by $\hat{G}$.

6.2 Following Noether's approach

It is instructive to check directly that the results of the previous section can be obtained with more Noether flavour by the method of § 5.1. In terms of bras and kets we rewrite (111) as

$$S_B[\chi, \psi] \overset{\mathrm{def}}{=} \int_{t_i}^{t_f} \left\{ \langle \chi | \frac{\mathrm{d}}{\mathrm{d}t} | \psi \rangle + \frac{\mathrm{i}}{\hbar} \langle \chi | \hat{H} | \psi \rangle + \frac{\mathrm{d}\mathsf{B}}{\mathrm{d}t} \right\} \mathrm{d}t \tag{126}$$

The general expression (60a) together with (61) provides

$$\delta J_B = \langle \chi | \left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{i}}{\hbar}\hat{H} \right) | \psi \rangle \, \delta t + \langle \chi | \left(\delta | \psi \rangle \right) + \frac{\mathrm{d}\mathsf{B}}{\mathrm{d}t} \delta t + \delta \mathsf{B} \ . \tag{127}$$

Moreover, in order to preserve the structure of S_B, we naturally choose the boundary function with the same structure as the Hamiltonian (109), that is

$$\mathsf{B} \overset{\text{def}}{=} \langle\chi|\,\hat{B}\,|\psi\rangle \tag{128}$$

for some operator $\hat{B}$. Then the infinitesimal current reads

$$\delta J_B = \delta J_0 + \delta t \frac{\mathrm{d}}{\mathrm{d}t}\Big(\langle\chi|\,\hat{B}\,|\psi\rangle \Big) + \langle\chi|\,\hat{B}\big(\delta\,|\psi\rangle\big) + \big(\delta\,\langle\chi|\big)\hat{B}\,|\psi\rangle\ . \tag{129}$$

with

$$\delta J_0 = \delta t \,\langle\chi|\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{i}}{\hbar}\hat{H}\right)|\psi\rangle + \langle\chi|\,\big(\delta\,|\psi\rangle\big)\ . \tag{130}$$

The action of (118) on $\langle\chi|$ and on $|\psi\rangle$ leads to

$$\delta\,\langle\chi| = -\frac{\mathrm{i}\epsilon}{\hbar}\,\langle\chi|\,\hat{G}\ ;\qquad \delta\,|\psi\rangle = \frac{\mathrm{i}\epsilon}{\hbar}\hat{G}\,|\psi\rangle\ . \tag{131}$$

Thus,

$$\delta J_B = \delta t\,\langle\chi|\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{i}}{\hbar}\hat{H}\right)|\psi\rangle + \frac{\mathrm{i}\epsilon}{\hbar}\,\langle\chi|\left(\hat{G} + [\hat{B},\hat{G}]\right)|\psi\rangle + \delta t\frac{\mathrm{d}}{\mathrm{d}t}\Big(\langle\chi|\,\hat{B}\,|\psi\rangle \Big) \tag{132}$$

The transformed boundary operator is defined to be such that

$$^{\mathrm{T}}\langle\chi|\,\hat{B}^{\mathrm{T}}(t^{\mathrm{T}})\,|\psi\rangle^{\mathrm{T}} = \langle\chi|\,\hat{B}(t)\,|\psi\rangle \tag{133}$$

for any $\langle\chi|$ and $|\psi\rangle$, that is, by using (113),

$$\hat{B}^{\mathrm{T}}(t^{\mathrm{T}}) = \hat{U}\hat{B}(t)\hat{U}^* \ . \tag{134}$$

This identity can also be recovered from (14) by using the complex canonical formalism of the previous section. The traduction of the invariance is simply $\hat{B}^{\mathrm{T}}(t^{\mathrm{T}}) = \hat{B}(t^{\mathrm{T}})$ and then, for an infinitesimal transformation characterized by $\delta t = t^{\mathrm{T}} - t$ and ϵ, we get

$$\delta t\frac{\mathrm{d}\hat{B}}{\mathrm{d}t} + \frac{\mathrm{i}\epsilon}{\hbar}[\hat{B},\hat{G}] = 0\ . \tag{135}$$

If we choose all the operators in the Heisenberg picture, this identity leads to

$$\delta t\frac{\mathrm{d}\hat{B}}{\mathrm{d}t} + \delta t\frac{\mathrm{i}}{\hbar}[\hat{H},\hat{B}] + \frac{\mathrm{i}\epsilon}{\hbar}[\hat{B},\hat{G}] = 0\ . \tag{136}$$

where all the operators are now considered in the Schrödinger picture[18]. When both $|\psi\rangle$ and $\langle\chi|$ satisfy the Schrödinger equation let us show how the infinitesimal current (132) simplifies. The first term in the right-hand side vanishes and the last term is given by

$$\delta t \frac{\mathrm{d}}{\mathrm{d}t}\Big(\langle\chi|\,\hat{B}\,|\psi\rangle\Big) = \delta t \,\langle\chi|\left(\frac{\mathrm{d}\hat{B}}{\mathrm{d}t} + \frac{\mathrm{i}}{\hbar}[\hat{H},\hat{B}]\right)|\psi\rangle = -\frac{\mathrm{i}\epsilon}{\hbar}\,\langle\chi|\,[\hat{B},\hat{G}]\,|\psi\rangle \quad (139)$$

where (136) has been used for the second equality. Eventually we obtain

$$\delta J_B = \frac{\mathrm{i}\epsilon}{\hbar}\,\langle\chi|\,\hat{G}\,|\psi\rangle \tag{140}$$

and the conservation law $\mathrm{d}\delta J_B/\mathrm{d}t = 0$ is exactly equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\langle\chi|\,\hat{G}\,|\psi\rangle\Big) = 0 \tag{141}$$

from which we already derived (122) for a model invariant under time-translations and (125) for a model invariant under a time-independent transformations.

In passing we note that the Noether constant associated with the invariance of S_B under a global change of phase $^{\mathrm{T}}\langle\chi| = \langle\chi|\,\mathrm{e}^{-\mathrm{i}\theta}$ together with $|\psi\rangle^{\mathrm{T}} = \mathrm{e}^{-\mathrm{i}\theta}\,|\psi\rangle$ for any constant θ corresponds to $\hat{G} = 1$ and therefore is given by the scalar product $\langle\chi|\psi\rangle$ which is indeed conserved by any unitary evolution.

7 Conclusion

Unlike what occurs generically in the Lagrangian context where one remains in the configuration space, the Hamiltonian variational principle cannot be formulated with keeping fixed all the dynamical variables at the boundaries in phase-space. Nevertheless, with the use of a boundary function that helps to manage the issues of boundary conditions, we have shown how Noether's seminal work [28] does cover the Hamiltonian variational principle and how the constant generators of the canonical—classical or quantum—transformations are indeed the corresponding Noether constants.

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[18] By using a label to distinguish the two pictures, for any operator $\hat{O}$ we have the connection

$$\hat{O}^{(\mathrm{H})}(t) = \hat{U}^{(\mathrm{S})}(t_0,t)\,\hat{O}^{(\mathrm{S})}(t)\hat{U}^{(\mathrm{S})}(t,t_0) \tag{137}$$

where t_0 denotes the time where the two pictures coincide and $\hat{U}^{(\mathrm{S})}(t,t_0)$ is the evolution operator between t_0 and t in the Schrödinger picture. Therefore we have

$$\frac{\mathrm{d}\hat{O}^{(\mathrm{H})}(t)}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}[\hat{H}^{(\mathrm{H})}(t),\hat{O}^{(\mathrm{H})}(t)] + \left(\frac{\mathrm{d}\hat{O}^{(\mathrm{S})}(t)}{\mathrm{d}t}\right)^{(\mathrm{H})}. \tag{138}$$

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