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„Asymptotic problems in
non-spatially symmetric spacetimes“

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ABSTRACT

Many crucial problems in general relativity concern the asymptotic behavior of solutions to the Einstein equations, both in time and space. These are commonly studied under the assumption of symmetry conditions on the spatial slices of the spacetime, which greatly simplifies the analysis. In this thesis we investigate several such problems without relying on spatial symmetry.

The introduction of a conformal boundary at infinity is a useful tool to study the asymptotic behavior of spacetimes at large distances. This naturally leads to the question of what the possible configurations of this boundary are. In the first part of this thesis we start with asymptotic data, given on the conformal boundary, and construct stationary spacetimes solving the Einstein equations with a negative cosmological constant coupled to a wide variety of matter fields. Here the boundary is timelike and corresponds to the limit of large radii. As there are only mild restrictions on the asymptotic data, our solutions possess no spatial Killing vector fields in general. Spacetimes of this type are of interest both within general relativity, as they show strikingly different behavior than ones with positive or zero cosmological constant, and within the context of the AdS/CFT conjecture.

Another class of asymptotic problems of interest, at the "opposite end" of spacetimes, concerns their behavior near singularities. We know from the singularity theorems of Penrose and Hawking that singularities are a generic feature of general relativity, but their detailed structure is still unclear. Here we construct nakedly singular vacuum spacetimes without symmetries from asymptotic data at the singularity. Within the framework of the BKL conjecture, which aims to give a detailed description of generic singularities, their behavior is of the non-chaotic type, which provides the building blocks for the more complicated chaotic behavior expected in the general case.

A related issue of recent interest is the possible existence of lower regularity extensions through the singular boundary, which is connected to the cosmic censorship conjecture. For a class of "expanding singularities" (which includes spacetimes without any symmetries) we analyze the properties of C^0-extensions across the singularity by extending methods used in the Schwarzschild case.

ZUSAMMENFASSUNG

Viele wichtige Probleme in der Allgemeinen Relativitätstheorie betreffen das asymptotische Verhalten, sowohl in der Zeit als auch im Raum, von Lösungen der Einstein'schen Feldgleichungen. Solche Probleme werden meistens unter der Annahme von Symmetrien der raumartigen Schnitte studiert, was die Analyse stark vereinfacht. In dieser Dissertation betrachten wir mehrere solche Probleme ohne räumliche Symmetrien.

Die Definition einer konformen Grenzfläche im Unendlichen ist ein nützliches Werkzeug um das asymptotische Verhalten von Raumzeiten zu studieren. Es stellt sich die Frage was die möglichen Konfigurationen dieser Grenzfläche sind. Im ersten Teil dieser Dissertation konstruieren wir stationäre Lösungen der Einsteingleichungen mit negativer kosmologischer Konstante, gekoppelt an verschiedene Materiefelder, die ein vorgeschriebenes Verhalten im Unendlichen zeigen. Die konforme Grenzfläche ist in diesem Fall zeitartig und entspricht dem Limes hoher Entfernung vom Zentrum. Da die asymptotischen Daten weitgehend frei sind haben die konstruierten Lösungen im allgemeinen keine räumlichen Killing Vektorfelder. Raumzeiten dieser Art sind von Interesse, sowohl vom Standpunkt der allgemeinen Relativitätstheorie, da sie stark unterschiedliches Verhalten als im Fall von positiver oder verschwindender Kosmologischer Konstante zeigen, als auch im Rahmen der AdS/CFT Vermutung.

Eine andere Klasse von asymptotischen Problemen, sozusagen am "gegenüberliegenden Ende" von Raumzeiten, ist ihr Verhalten in der Nähe von Singularitäten. Die Singularitätentheoreme von Penrose und Hawking zeigen dass Singularitäten ein generisches Phänomen der allgemeinen Relativitätstheorie sind, aber ihre detaillierte Struktur ist noch immer unklar. Hier konstruieren wir vakuum Raumzeiten ohne Symmetrien basierend auf asymptotischen Daten an einer nackten Singularität. Im Rahmen der BKL-Vermutung, die eine detaillierte Beschreibung generischer Singularitäten anstrebt, zeigen unsere Lösungen sogenanntes nicht-chaotisches Verhalten. Dieses bildet den Baustein für das kompliziertere chaotische Verhalten das im allgemeinen Fall erwartet wird.

Ein verwandtes Problem ist die mögliche Existenz von Erweiterungen niedrigerer Regularität durch die singuläre Grenzfläche, im Zusammenhang mit der Cosmic Censorship Vermutung. Wir untersuchen die Eigenschaften von C^0-Erweiterungen durch die Singularität für eine Klasse von "expandierenden Singularitäten" (diese enthält Raumzeiten ohne Symmetrien) indem wir Methoden, die für den Schwarzschild-Fall entwickelt wurden, erweitern.

ACKNOWLEDGMENTS

I am deeply grateful to my supervisor, Piotr Chruściel, for his support during my master and PhD studies. His dedication and knowledge are an inspiration and he always found time for me.

I would like to thank all my friends and colleagues at the gravitational physics group for making my stay there a pleasant one.

This thesis, and my studies in general, would not have been possible without the unwavering support of my parents. I will always be grateful to them.

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CONTENTS

1 INTRODUCTION

1.1 GENERAL RELATIVITY

Albert Einstein's theory of *general relativity* [45] provides the most accurate mathematical model of gravity known to date. The fundamental interaction of gravity dominates the universe on large scales, governing the movement of planets, stars, and galaxies. Our best cosmological models, which describe the evolution of the universe as a whole, are formulated within the framework of general relativity. On smaller scales, such as those relevant to our day-to-day life, the other fundamental interactions—electromagnetism, the weak-, and the strong nuclear force—are much stronger than gravity. As the nuclear forces are short ranged and large bodies tend to collect positive and negative electric charges in equal measure, only gravity remains important at larger scales.

There is, as of yet, no theory which combines these quantum forces with the classical description of gravity given by general relativity. In their respective domains, however, both agree with experimental evidence to incredible precision. General relativity improves upon the Newtonian theory of gravity both in a quantitative sense, by e.g. allowing more precise calculation of orbital motion, and qualitatively, by encompassing entirely new phenomena such as black holes and gravitational waves.

More fundamentally, general relativity is a *geometric theory*. In contrast to Newtonian mechanics, gravity is not described as a force acting on bodies which move on a fixed background, like actors on a stage. Instead the stage becomes one of the actors as the fixed notions of space and time combine to form a dynamical *spacetime*. Test bodies moving under the influence of gravity simply follow straight lines, albeit a notion of "straightness" defined by the curvature of the underlying spacetime.

Using the mathematical tools of differential geometry, the theory of general relativity can be formulated in a remarkably concise and elegant way. Spacetime is described as a *Lorentzian manifold*, consisting of a set of points which represent events occurring at a specific time and place, and additional structure which specifies the relation of these events to each other. The crucial element is the *metric tensor* $g_{\mu\nu}$ which determines the causal relationships between events and makes it possible to calculate the (proper) time experienced by any observer. The trajectories of test bodies in general relativity are given by *geodesics*, which are extremal curves of the proper time defined from $g_{\mu\nu}$.

General relativity subsumes the earlier theory of *special relativity*, which postulates a fixed metric tensor $g_{\mu\nu} = \eta_{\mu\nu}$, the *Minkowski metric*, and, in fact, reduces to it in the limit of small curvature. For any spacetime it is possible to choose coordinates such that $g_{\mu\nu} = \eta_{\mu\nu}$ and $\partial_\alpha g_{\mu\nu} = 0$ at one point. The existence of these *local inertial coordinates* represents the *equivalence principle*, which states that local experiments are unaffected by the position or velocity of the laboratory.

The source of the curvature of spacetime is the *energy momentum tensor* $T_{\mu\nu}$, which describes the distribution of energy and momentum in the universe. The relation between the metric tensor $g_{\mu\nu}$, and $T_{\mu\nu}$ is given by *Einstein's field equations*

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \,. \tag{1}$$

Here the *Ricci tensor* $R_{\mu\nu}$ and the *Ricci scalar* R describe the curvature of spacetime, and are defined in terms of the metric $g_{\mu\nu}$. The constant Λ is the *cosmological constant* which corresponds to an energy density of empty spacetime. The constants c and G are the familiar speed of light and (Newtonian) gravitational constant, respectively.

Taking into account the symmetries of the various tensors, in the standard $1 + 3$ dimensions, (1) consists of 6 partial differential equations for 10 unknowns. The remaining 4 degrees of freedom correspond to the gauge invariance of the theory under diffeomorphism, i.e. the freedom of choosing coordinates to label the points of spacetime.

In contrast to many other equations of physical interest the Einstein equations are *nonlinear*. This means, for instance, that it is not possible to simply superimpose the gravitational fields of two bodies to obtain their combined effect. A fascinating consequence of this complicated structure is that even the vacuum Einstein equations, obtained by setting $T_{\mu\nu} \equiv 0$ in (1), admit a wide variety of non-trivial solutions. Our results in Chapter 4 (for specific choices of the free data) and Chapter 6 show existence of large classes of such vacuum solutions with prescribed asymptotic behavior.

1.2 SYMMETRIES

The use of symmetries to simplify complicated problems is a common approach in physics. As in many other areas, symmetries play an important role in general relativity: Exact solutions, such as the Schwarzschild and Kerr spacetimes, were found by searching for metrics consistent with a symmetry assumption. In cosmology, the assumption of a spatially homogeneous and isotropic universe is the basis for the highly successful ΛCDM models. In general, the end state of dynamical evolution is expected to be a stationary, i.e. time-symmetric, configuration.

Symmetries in general relativity are described by *isometries*, that is, for a spacetime (M, g), by diffeomorphisms $\phi : M \to M$ satisfying $\phi^* g = g$. Such symmetries may be discrete or continuous, i.e. part of a family parametrized by some continuous variables. The continuous symmetries of a spacetime form a Lie group, whose connected component of the identity is generated by *Killing vector fields* (KVFs). A KVF ξ^α satisfies

$$\mathcal{L}_\xi g_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0\,, \tag{2}$$

and the set of all KVFs form a Lie algebra using the vector field commutator. Any Killing vector field locally generates a continuous symmetry, and, if the vector field is complete, this extends to a full one parameter family of isometries.

1.3 asymptotic problems

The complicated nonlinear structure of the Einstein equations makes it impossible to give general solutions which encompass all possible situations. Many exact solutions are known, but these generally describe very specialized situations and cannot be expected to shed light on generic behavior. To apply the theory beyond these solutions one needs to either use numerical methods, i.e. to approximate (1) by a discrete system of equations that can be solved by a computer, or to simplify the problem. Common forms of simplification include restricting to symmetric configurations or only considering specific features of solutions instead of their complete form.

For many questions of physical interest it is sufficient to consider only the asymptotic behavior of solutions. Here "asymptotic behavior" can mean either the long-term evolution towards the past or future or the behavior at large distances from a more complicated central region. For example, one might be interested whether some matter configuration collapses to form a black hole or approaches some other stationary state. In many cases it is not necessary to treat the potentially complicated behavior at intermediate times in order to draw conclusions about the final state.

Stability results, such as the celebrated proof of the stability of Minkowski space, are one example of such an asymptotic problem. Starting from a known solution, consider arbitrary small perturbations of the associated initial data on a spacelike slice. If the perturbations can be shown to decay in time, assuming they are initially small enough, this characterizes the asymptotic behavior of an open set of initial data.

A different approach is to directly construct (i.e., to prove existence of) new solutions which show a desired type of asymptotic behavior. This is often easier than starting from initial data, but makes it harder to answer questions of genericity (c.f. Section 3.3.2 below). We use such an approach in Chapters 4 and 6 below, although the asymptotic boundary in these cases lies in a spacelike instead of timelike direction.

1.4 OUTLINE

This thesis consists of two parts:

- The first part contains the papers [31] (in collaboration with Piotr Chruściel and Erwann Delay) and [70], attached as Chapters 4 and 5. These include the crucial results from a sequence of papers [24, 31, 25, 70] by the same authors and [26] (in collaboration with Chruściel, Delay, Kriegl, Michor, and Rainer) which together construct a class of stationary spacetimes with negative cosmological constant containing a wide variety of matter fields. The solutions are parametrized by data given at conformal infinity and include both black holes and geodesically complete spacetimes. Except for the timelike Killing vector field of stationarity they posses no symmetries.

- The second part concerns the nature of singularities. In Chapter 6 (published as [71]) we construct a class of spacetimes containing timelike singularities which show so-called non-chaotic or AVTD (Asymptotically Velocity Term Dominated) behavior. The metric behaves asymptotically like a spatially homogeneous Kasner metric at each point, but with coefficients that vary in space (see Section 3.3.2). This extends the results of the author's master thesis (published as [69]) to the timelike case. As in [69] the construction does not use any symmetry assumption. In analogy to the first part the solutions are parametrized by free functions which define the asymptotics of the metric at the singularity.

  In Section 3.3.3 we provide a new analysis of the only known inhomogeneous solutions showing the more complicated chaotic behavior.

  In Chapter 7 (in collaboration with Piotr Chruściel, published as [29]) we investigate C^0 extensions across a class of singularities we call "expanding singularities", i.e. where space is "stretched" as the singularity is approached, similar to what happens in the Schwarzschild or Kasner spacetimes. We use methods introduced by Sbierski in his proof of the C^0-inextendibility of Schwarzschild [96], but do not restrict to the case of spherical symmetry. Our results apply, e.g., to the spacetimes constructed in [69], which have no symmetries.

In Chapters 2 and 3 we give relevant background for each part and reproduce the main theorems of the later chapters.

1.5 OVERVIEW OF THE ATTACHED PAPERS

Here we list the papers attached as Chapters 4 to 7 and give their publication status and abstracts. See page 7 for a statement on the co-authored papers.

1. "Non-singular spacetimes with a negative cosmological constant: IV. Stationary black hole solutions with matter fields",

   P. T. Chruściel, E. Delay, and P. Klinger,

   published in Classical and Quantum Gravity 35.3 (2018), p. 035007,

   DOI: 10.1088/1361-6382/aa9e0c, arXiv: 1708.04947.

   Abstract. *We use an elliptic system of equations with complex coefficients for a set of complex-valued tensor fields as a tool to construct infinite-dimensional families of non-singular stationary black holes, real-valued Lorentzian solutions of the Einstein–Maxwell–dilaton–scalar fields–Yang-Mills–Higgs–Chern-Simons–$f(R)$ equations with a negative cosmological constant. The families include an infinite-dimensional family of solutions with the usual AdS conformal structure at conformal infinity.*

2. "Non-degeneracy of Riemannian Schwarzschild-anti de Sitter metrics: Birkhoff-type results in linearized gravity",

   P. Klinger,

   submitted to Journal of mathematical Physics (2018-06-14),

   arXiv: 1806.05023.

   Abstract. *We prove Birkhoff-type results showing that L^2 solutions of the linearized Einstein equations around Riemannian Kottler ("Schwarzschild-anti de Sitter") metrics in arbitrary dimension and horizon topology, which are not controlled by "master functions" are pure gauge. Together with earlier results this implies that the TT-gauge-fixed linearized Einstein operator for these metrics is non-degenerate for open ranges of the mass parameter.*

3. "Timelike singularities and Hamiltonian cosmological billiards",

   P. Klinger,

   published in Classical and Quantum Gravity 33.11 (2016), p. 117002,

   DOI: 10.1088/0264-9381/33/11/117002, arXiv: 1512.03302.

   Abstract. *We construct a large class of vacuum solutions of the Einstein equations without any symmetries and with controlled asymptotics near a timelike singularity. The solutions are obtained by a Fuchs analysis of the equations which evolve the metric in a spacelike direction. We further observe that the change of sign of some of the* *terms (walls) in the associated Hamiltonian invalidate the "cosmological billards" heuristic arguments for the existence of singularities of the mixmaster type in the current context.*

4. "The annoying null boundaries",

   P. T. Chruściel, and P. Klinger,

   published in Journal of Physics Conference Series 968.1 (2018), p. 012003,

   DOI: 10.1088/1742-6596/968/1/012003, arXiv: 1801.06037.

   Abstract. *We consider a class of globally hyperbolic space-times with "expanding singularities". Under suitable assumptions we show that no C^0-extensions across a compact boundary exist, while the boundary must be null wherever differentiable (which is almost everywhere) in the non-compact case.*

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PhD Committee
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Dear Colleagues,

Paul Klinger's thesis consists of an introduction to the topic, written by him, and several research papers, some of which were written together with co-authors. One of these papers is the product of a collaboration with Prof. Erwann Delay of the Université d'Avignon and me. Paul has contributed materially to all our joint papers (most of which are not included in this thesis), both to the ideas and the calculations required. It is impossible to determine precisely who is responsible for various parts of the papers, as they developed over many months in a collaborative process, with substantial and often key input from Paul. The paper "The annoying null boundaries", written in collaboration between Paul and me, is based mostly on his initiative and ideas. The papers solely authored by Paul are entirely his own work.

Piotr T. Chruściel
Professor of Gravitational Physics

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STATIONARY SPACETIMES WITH NEGATIVE COSMOLOGICAL CONSTANT

In Chapters 4 to 5 we construct stationary solutions to the Einstein equations with a negative cosmological constant and various types of matter fields. This chapter motivates these works and provides background information. In Section 2.1 we show how our results fit into the related literature. Section 2.2 defines stationary and static spacetimes and gives some of their properties. Section 2.3 discusses the case of negative cosmological constant and some of the new phenomena occurring there. In Section 2.4 we introduce the concept of the conformal boundary. This is important as our new solutions are parametrized by data given on this boundary. Finally, Section 2.5 describes the general approach used in our construction and gives references for the analytical details.

2.1 PREVIOUS WORK AND NEW RESULTS

In the case of zero cosmological constant there are strong rigidity theorems restricting the possible stationary and static configurations (see Section 2.3.1 below). It was believed that this would hold independently of the value of the cosmological constant [6]. Boucher, Gibbons, and Horowitz [16] gave arguments suggesting uniqueness of anti-de Sitter under strong conditions on the asymptotic behavior.

The question of the correct notion of asymptotic convergence is non-trivial (c.f. [36]), however, in contrast to the $\Lambda = 0$ case, such an additional assumption is essential: In [6, 5] Anderson, Chruściel, and Delay constructed static solutions to the vacuum Einstein equations with negative cosmological constant which had no Killing vectors except for the one given by the staticity condition.

The result was generalized to stationary (vacuum) spacetimes, and to static spacetimes with Maxwell fields by Chruściel and Delay in [30, 33] using somewhat different methods. In contrast to [6, 5] these new results are valid only for asymptotic data that is close to that of a seed metric, e.g. the Anti-de Sitter solution, which satisfies a non-degeneracy condition.

In [24], together with Chruściel and Delay, we extend these results to stationary spacetimes with a wider variety of matter fields. These include Maxwell with a Chern Simons term, a Dilaton coupled to the Maxwell field and additional minimally coupled scalar fields. Instead of the $U(1)$ Maxwell field we can also construct solutions with more general Yang-Mills and Higgs fields. As the matter fields are determined by the asymptotic data we can set any combination of them to zero by giving trivial data and choosing vanishing coupling constants. The inclusion of matter fields means that the results of [16] do not apply, i.e. it is possible to construct geodesically complete solutions with the same asymptotic behavior as AdS but which contain only the Killing vector of stationarity. Our results include a class of "Boson stars", solutions of the Einstein-complex scalar field equations where the metric is time-independent but the scalar field is multiplied with a time dependent phase factor $\exp(i\omega t)$.

In [24] we had to assume a negative mass for the scalar field. In [26], together with Chruściel, Delay, Kriegl, Michor, and Rainer, we use similar techniques in the vicinity of an eigenmode of the scalar field equations on the fixed background seed solution. This leads to Boson star solutions for positive mass scalar fields.

In [31] (Chapter 4) with Chruściel and Delay we construct stationary black hole solutions with the same matter fields as in [24]. As before, the construction relies on a non-degeneracy property of the seed metrics. We show this property for generalized Kottler (Schwarzschild-de Sitter) metrics for a subset of horizon geometries and mass parameters in [25] with Chruściel and Delay. The results of [70] (Chapter 5) extend this range of parameters.

Our final result, for Anti-de Sitter or generalized Kottler seed metrics, states (c.f. Chapter 4, Theorem 4.8 and Chapter 5, Theorem A)

THEOREM 2.1.1. *Consider a $n+1 \geq 4$ dimensional generalized Kottler metric $\mathring{\mathbf{g}}$ of the form*

$$\mathring{\mathbf{g}} = -\left(r^2 + K - \frac{2\mu}{r^{n-1}}\right)dt^2 + \frac{dr^2}{r^2 + K - \frac{2\mu}{r^{n-1}}} + r^2\gamma_K\,, \tag{3}$$

where γ_K is a metric of constant sectional curvature $K \in \{-1, 0, 1\}$ and the mass parameter μ satisfies

- $\mu \neq \mu_c := \frac{12}{25}\sqrt{\frac{3}{5}}$ *and* $\mu > \mu_{min}(n,K)$ *for* $n = 3$,
- $\mu \in (\mu_{min}(n,K), \mu(n,K))$ *for* $n > 3$, *where* $\mu(n,K) > \mu_{min}(n,K)$ *solves a polynomial equation,*

with

$$\mu_{min}(n,K) := \begin{cases} 0 & K \in \{0,1\}\,, \\ -\frac{1}{n}\left(\frac{n}{(n-2)}\right)^{-n/2} & K = -1\,. \end{cases} \tag{4}$$

We further assume that the associated Riemannian metric $\mathring{\mathfrak{g}}$ has no harmonic one forms in L^2, that $\mathscr{V}''(0)$ is not an L^2-eigenvalue of the operator $\Delta_{\mathring{\mathfrak{g}}}$ [1] *and that*

$$W(0) = 1\,, \quad \mathscr{V}(0) = 0 = \mathscr{V}'(0)\,, \quad \mathscr{V}''(0) > -n^2/4\,, \tag{5}$$

1 See Chapter 4, Remark 4.10 for comments on the kernel conditions.

where $\mathscr{V}$ and W are the potential of the scalar field and the dilaton coupling, respectively [2].

For all $k \in \mathbb{N} \setminus \{0\}$, $\alpha \in (0,1)$, $a \in \mathbb{R}$ with $|a|$ small enough, every smooth real-valued $\hat{\theta} \in C^{k+2,\alpha}(\partial M, \mathcal{T}_1)$ and $\hat{U} \in C^{k+2,\alpha}(\partial M)$ and

1. *$\mathscr{V}''(0) < 0$ with $\hat{A} \in C^{k+2,\alpha}(\partial M, \mathcal{T}_1)$, and $\hat{\Phi} \in \rho^{\sigma_-} C^{k+2,\alpha}(\partial M)$ (where $\sigma_- = n/2 - \sqrt{n^2/4 + \mathscr{V}''(0)}$) which are sufficiently small smooth fields on ∂M, or*

2. *$\hat{\Phi} \equiv 0$, and $\hat{A} \in C^{k+2,\alpha}(\partial M, \mathcal{T}_1)$ which is a sufficiently small smooth field on ∂M,*

there exists a unique, modulo diffeomorphisms which are the identity at the boundary, nearby stationary Lorentzian solution of the Einstein-Maxwell-dilaton-scalar fields-Chern-Simons equations, or of the Yang-Mills-Higgs-Chern-Simons-dilaton equations with a trivial principal bundle, so that, in local coordinates near ∂M, we have

$$\begin{gathered} g \to_{\rho\to 0} \mathring{g}\,,\ V \to_{\rho\to 0} \mathring{V}\,,\ \theta \to_{\rho\to 0} a\hat{\theta}\,, \\ U \to_{\rho\to 0} a\hat{U}\,,\ A \to_{\rho\to 0} \hat{A}_a dx^a\,,\ \Phi \to_{\rho\to 0} \hat{\Phi} \end{gathered} \tag{6}$$

with all convergences in $\mathring{g}$-norm. The hypothesis of non-existence of harmonic L^2-one-forms is not needed if $\hat{A} \equiv 0 \equiv \hat{U}$, in which case the Maxwell field or the Yang-Mills field are identically zero.

Parallel to these analytic constructions, similar solutions were investigated using numerical methods. Static Einstein-Yang-Mills solutions, including both geodesically complete and black hole ones, are constructed numerically in [105] for spacetime dimension 5, in [14] for dimension 4, and in [90] for all dimensions. Similar solutions for the Einstein-Maxwell-Chern-Simons equation were investigated in [15]. Stationary Boson stars were recently constructed numerically in [18]. Some of these numerical solutions are included in the classes we obtain in [31] (Chapter 4) and [26]. However, due to our restriction to asymptotic data which is close to that of vacuum anti-de Sitter space, we do not recover all of them.

Rotating Boson star solutions with a black hole were constructed numerically in [42], for a massless scalar field. Our use of a periodic time function in [31] means that we cannot construct black hole solutions with a time-dependent scalar field.

2.2 STATIONARY AND STATIC SPACETIMES

The concepts of stationarity and staticity make rigorous the notion of a spacetime that does not change in time. This is not an a priori trivial concept as it depends on a notion of time, which is not uniquely

2 See the action in equation 4.13 of Chapter 4 for the definition of W and $\mathscr{V}$.

defined in general relativity. Spacetimes of this type are of interest as they are natural candidates for endpoints of dynamical evolution.

A spacetime is called *strictly stationary* if it contains a globally timelike Killing vector field (KVF). The full Schwarzschild spacetime is not strictly stationary as the KVF ∂_t (in standard coordinates) is spacelike in the interior of the black hole. It is, however, *stationary*, meaning that it contains a Killing vector that is timelike for large enough r. This asymptotic notion of stationarity is only defined for spacetimes with an asymptotic end, e.g. asymptotically flat or asymptotically (anti-) de Sitter ones.

Let us assume we are in a region of spacetime where there exists a timelike Killing vector $\xi = \xi^\mu \partial_\mu$. Then we can always find (local) coordinates such that the metric g takes the form

$$g = -V(x)^2(\mathrm{d}t + \theta_i(x)\mathrm{d}x^i)^2 + g_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j\,, \tag{7}$$

where $\nabla t = \xi$, i.e. t is defined by integrating along the integral curves of ξ, and the indices i, j take values in $\{1, \dots, n\}$ with $n+1$ the spacetime dimension.

A stronger condition than stationarity is staticity: If there exists a spacelike hypersurface Σ such that the KVF ξ of a (strictly) stationary spacetime is everywhere normal to Σ the spacetime is called *(strictly) static*. This corresponds to $\theta_i \equiv 0$ in the coordinates (7).

The Kerr solution, which describes a rotating black hole, is an example of a stationary spacetime which is not static. The mixed $\mathrm{d}x^i\mathrm{d}t$ terms in the metric cause effects such as Lense-Thirring frame dragging which do not occur in the static case.

The solutions we construct in Chapter 4 include both strictly stationary geodesically complete ones, and stationary black hole solutions for which the Killing vector field becomes null on the horizon. For appropriate choices of the asymptotic data we obtain static solutions instead.

2.3 NEGATIVE COSMOLOGICAL CONSTANT

There is strong observational evidence that the cosmological constant Λ in our universe is positive. Assuming a standard ΛCDM cosmological model, this follows both from direct measurements of the accelerating expansion of the universe using supernova observations and from analysis of inhomogeneities in the cosmic microwave background (CMB). The most recent Planck collaboration data, obtained from CMB observations using the Planck space telescope, shows $\Omega_\Lambda = 0.6889(56)$ [89], corresponding to a cosmological constant $\Lambda = 1.1056 \times 10^{-52}\,\mathrm{m}^{-2}$.

The case of negative Λ is nevertheless interesting, both from a physical and mathematical point of view. On the physical side there has been a lot of interest in spacetimes with negative cosmological constant in recent years due to the so-called AdS/CFT correspondence [78, 106]. This is a conjectured relationship between theories of quantum gravity on asymptotically AdS spacetimes and conformal field theories defined on the conformal boundary of these spaces (see Section 2.4 for a definition of the boundary). Classical solutions of general relativity which are asymptotically AdS are useful in this context, as they should correspond to a low energy limit of the quantum theory. The AdS/CFT correspondence is a major field of research in theoretical high energy physics, but rigorous results are still scarce. We will not go into more details here, as this thesis focuses on classical general relativity.

On the mathematical side the $\Lambda < 0$ case is of interest as it shows strikingly different phenomena from the $\Lambda = 0$ or $\Lambda > 0$ cases. For instance, there are time periodic solutions close to anti-de Sitter space (such solutions were found numerically in [79] and similar ones constructed rigorously in [27]), while asymptotically flat ones do not exist [2]. The $\Lambda < 0$ case also allows for a much wider variety of possible stationary solutions, including both black holes and geodesically complete ones.

2.3.1 *Uniqueness theorems for $\Lambda = 0$ and differences in the $\Lambda < 0$ case*

In the case of zero cosmological constant it was shown by Lichnerowicz that geodesically complete asymptotically flat stationary vacuum solutions must be Minkowski [75]. Anderson [4] removed the condition of asymptotic flatness, showing

THEOREM 2.3.1. *[4, Theorem 0.1] Let (M, g) be a geodesically complete, chronological, stationary vacuum space-time. Then (M, g) is the Minkowski spacetime or a quotient of the Minkowski spacetime by a discrete group of isometries of $\mathbb{R}^3$, commuting with the symmetry group of stationarity.*

Similar results for black holes go under the name *black hole uniqueness* or *no-hair conjecture*, i.e. the problem of classifying all asymptotically stationary, asymptotically flat (or hyperbolic) black hole spacetimes. This is of interest as the end-state of black holes is expected to be in this class, including astrophysical black holes formed by gravitational collapse.

In the $\Lambda = 0$ case it turns out that, up to some technical assumptions, all four-dimensional, asymptotically stationary, asymptotically flat vacuum black hole spacetimes lie in the Kerr (or Kerr-Newman for Einstein-Maxwell) family (see [32] for a review). This implies that they can be described using only three parameters: mass, angular momentum, and charge. [3]

3 In fact this property is of great importance for the detection of gravitational waves: Detectors such as LIGO use waveform templates—analytical and numerical predictions of the expected waveforms from sources such as the collision of two black

For the case of Einstein-Maxwell the relevant theorem was proven by Costa, Chruściel, and Nguyen [77, 35]:

THEOREM 2.3.2. *Let (M, g) be an asymptotically stationary, asymptotically-flat (and therefore $\Lambda = 0$), $\mathcal{I}^+$-regular, electrovacuum, four-dimensional analytic spacetime. If the event horizon is connected and either mean non-degenerate or rotating, then the exterior of the event horizon is isometric to the domain of outer communications of a Kerr–Newman spacetime.*

It is conjectured that the assumptions of analyticity and the conditions on the horizon can be dropped. A similar theorem of Alexakis, Ionescu, and Klainerman does not require analyticity but only holds for spacetimes close to Kerr [1].

The situation is completely different if the cosmological constant is negative: Anderson, Chruściel, and Delay [6] showed that there are infinite dimensional families of static black hole solutions without any additional Killing vectors. In Chapter 4 we construct a much larger family of asymptotically stationary (instead of static) black hole solutions, including both vacuum solutions and ones with various matter fields. Our results could therefore be interpreted as a black hole “non-uniqueness” theorem.

Even in the $\Lambda = 0$ case black hole uniqueness does not hold in higher dimensions, or for all types of matter fields: The $4+1$ dimensional “black ring” spacetimes of Emparan and Reall [46], with horizon topology $S^1 \times S^2$ cannot be distinguished from the standard Myers-Perry solutions (with S^3 horizon) by their conserved charges. In $3+1$ dimensions large families of black holes with non-abelian Yang-Mills fields [101] and complex scalar fields [62] have been found.

The Birkhoff theorem

The *Birkhoff theorem* is a result in the opposite direction of those mentioned above. Instead of assuming stationarity or staticity and obtaining some symmetry of the spatial slices, it shows that spherical symmetry implies staticity. The classical form of the Birkhoff theorem (first proven by Jebsen [66]) states that a spherically symmetric solution of the vacuum Einstein equations is static, and, in fact, isometric to the Schwarzschild solution. Equivalent results apply for nonzero cosmological constant, leading to the Schwarzschild-(anti-)de Sitter or Nariai solutions [44].

In Chapter 5 we show analogous results to the Birkhoff theorem in the context of linearized gravity. These are needed to prove the non-degeneracy property of (generalized) Kottler metrics, which are used as seed metrics in Chapter 4.

holes—to extract the signal from noise which is orders of magnitude greater. As the parameter space of possible configurations of black holes is of low dimension it can be covered using a reasonable number of such templates.

2.4 CONFORMAL COMPACTIFICATION

The spacetimes we construct in Chapter 4 are parametrized by their asymptotic behavior at large radii. A convenient way to represent this behavior is the *conformal method* of Penrose [87, 86]. We start with an $n+1$ dimensional spacetime $(\mathcal{M}, g)$, called the *physical spacetime*. The idea is to construct an *unphysical spacetime* $(\tilde{\mathcal{M}}, \tilde{g})$ which includes the boundary at infinity as a regular part of the manifold. The metric $\tilde{g}$ is obtained by conformally rescaling g, in order to "pull in" the infinity and to attach a boundary there.

More precisely, we consider an embedding $\phi : \mathcal{M} \to \tilde{\mathcal{M}}$, where $\tilde{\mathcal{M}}$ is an $n+1$ dimensional manifold with boundary. From now on we identify subsets $A \subseteq \mathcal{M}$ with their images $\phi(A) \subset \tilde{\mathcal{M}}$.

We call a function $\rho : \tilde{\mathcal{M}} \to \mathbb{R}$ a *defining function* for the boundary $\partial\mathcal{M}$ of $\phi(\mathcal{M})$ in $\tilde{\mathcal{M}}$ if ρ vanishes on $\partial\mathcal{M}$, $\rho > 0$ on $\tilde{\mathcal{M}} \setminus \partial\mathcal{M}$, and $d\rho$ is nonzero on $\partial\mathcal{M}$. If there exists a smooth defining function ρ of the boundary $\partial\mathcal{M}$ such that $\tilde{g} := \rho^2 g$ is a C^k metric on $\tilde{\mathcal{M}}$ then we call $(\mathcal{M}, g)$ *conformally compact of class* C^k. This is a well defined concept, as it is independent of the choice of defining function ρ. The set $\{\rho = 0\} = \partial\mathcal{M} \subset \tilde{\mathcal{M}}$ is called the *conformal boundary*. The induced metric on the conformal boundary depends on the defining function used, but its conformal class $[\rho^2 g|_{\partial\mathcal{M}}]$ does not. It is called the *conformal infinity*.

Let us consider a simple example: The anti-de Sitter metric in standard coordinates is given by

$$g = -(1+r^2)\mathrm{d}t^2 + \frac{1}{1+r^2}\mathrm{d}r^2 + r^2\gamma$$

where γ is the round metric on S^2. A conformal transformation with $\rho = 1/r$ gives

$$\begin{aligned}\tilde{g} &= -\frac{1+r^2}{r^2}\mathrm{d}t^2 + \frac{1}{r^2(1+r^2)}\mathrm{d}r^2 + \gamma \\ &= -(1+\rho^2)\mathrm{d}t^2 + \frac{1}{1+\rho^2}d\rho^2 + \gamma\,,\end{aligned}$$

i.e. the compactified metric is smooth up to the boundary and is given by $\tilde{g}|_{\partial\mathcal{M}} = -dt^2 + \gamma$ there.

Note that, unless the coordinate t is periodic, $\tilde{\mathcal{M}}$ is not actually compact. In our applications we will only apply the conformal compactification to stationary spacetimes, where the metric does not depend on t in any case. We will switch between periodic t coordinate and $t \in \mathbb{R}$ where convenient.

2.4.1 *Asymptotically hyperbolic manifolds and the Fefferman-Graham expansion*

In Chapters 4 to 5 we will consider conformal compactifications of *Riemannian* manifolds. Mazzeo showed that Riemannian metrics which are conformally compact of class at least C^2 are *asymptotically hyperbolic*, i.e. their sectional curvature approaches -1 (if ρ is normalized to $|d\rho|^2_{\tilde{g}}|_{\partial\mathcal{M}} = 1$) as $\rho \to 0$ [80]. This is the reason why the methods we use to construct solutions only apply to the case of negative cosmological constant.

If we assume that a conformally compact metric is, in addition, Einstein, i.e. $R_{ij} \propto g_{ij}$, then asymptotic hyperbolicity fixes the constant of proportionality to

$$R_{ij} = n g_{ij}\,,$$

where $n+1$ is the dimension of the manifold.

Fefferman and Graham introduced a method of expanding conformally compact metrics in a formal power series in ρ, possibly containing logarithmic terms [47]. The precise dependence of the regularity of asymptotically hyperbolic Einstein metrics on the dimension was shown in [28]:

Let us assume that we have a d-dimensional, C^2 conformally compact Riemannian Einstein manifold $(\mathcal{M}, g)$ with smooth conformal infinity. Riemannian metrics satisfying $R_{ij} \propto g_{ij}$ are always real analytic [41], so the compactified metric is smooth in the interior. Thus, its overall regularity is determined by the behavior at the boundary. We consider a neighborhood $O_Y = [0,1) \times Y \subset \tilde{\mathcal{M}}$ of a compact boundary component Y. By the results of [28], if $\rho^2 g|_{\partial\mathcal{M}}$ is smooth, then there exists a diffeomorphism which leaves the boundary invariant such that, in O_Y, $\tilde{g}$ takes the form

$$\tilde{g} = \rho^2 g = \begin{cases} d\rho^2 + \varphi(\rho)\,, & \text{for } d = 3 \text{ and } d \text{ even}\,, \\ d\rho^2 + \varphi(\rho, \rho^n \log \rho)\,, & \text{for } d \geq 5 \text{ odd}\,, \end{cases} \tag{8}$$

where φ is a family of metrics on Y that is smooth in all its arguments. Therefore, for $d = 3$ and for even d, $(\mathcal{M}, g)$ is conformally compact of class C^∞ while for odd $d \geq 5$ it is only of class C^{d-1} in general.

2.5 METHOD

As Chapter 4 is one in a series of papers [30, 33, 24] not all steps of the construction are discussed in detail. Here we give an overview of the method and refer to previous works where appropriate.

The main innovation in Chapter 4 is the method of transforming the problem of constructing real Lorentzian solutions of a PDE system to constructing complex Riemannian ones. This is described in detail there.

Let us assume that we are already in the Riemannian setting. We start with a static vacuum solution and use an implicit function theorem argument to construct stationary non-vacuum solutions close to it. We recall

Theorem 2.5.1 (Implicit function theorem for Banach spaces [72]). *Consider a C^1 map $f : U \times V \to C$, where $U \subset A$, $V \subset B$ and C are Banach spaces. If the partial derivative of f with respect to the second factor in $U \times V$ is invertible at a point $p = (p_u, p_v) \in U \times V$ such that $f(p) = 0$ then there exists an open neighborhood $O \subset U$ of p_u and a function $g : O \to C$ such that $f(x, g(x)) = 0$ for all $x \in O$.*

We will work in weighted Hölder spaces $C^{k,\alpha}_s(M; E)$, where E is some tensor bundle, following the definitions of [73, Section 3]. The weight s gives the decay behavior of the tensors by

$$C^{k,\alpha}_s(M; E) := \rho^s C^{k,\alpha}_0(M; E) = \{\rho^s u \mid u \in C^{k,\alpha}_0(M; E)\}\,, \tag{9}$$

where $C^{k,\alpha}_0(M; E)$ are the standard (unweighted) Hölder spaces of k-times differentiable tensors such that the highest derivative is Hölder continuous of degree α. Note that there are multiple inequivalent definitions of the Hölder norm on tensor fields, depending on how tensor values are transported to different points. The definition in [73] relies on constructing *Möbius charts* which map from (M, g) to subsets of hyperbolic space.

The behavior of the components of a tensor field in these spaces depends on the tensor rank, due to the metric terms appearing in the norm: For example a 2-covariant tensor field $u \in C^{k,\alpha}_s$ has components $u_{\alpha\beta} = O(\rho^{s-2})$ in coordinates near the boundary.

We use the implicit function theorem in these spaces to construct solutions of the Einstein equations with specified asymptotics. This works as follows: If the equations are of the form $E(g) = 0$ we consider solutions $g = \mathring{g} + \hat{g} + \bar{g}$ where $\mathring{g}$ is a given seed metric (e.g. the anti-de Sitter or Schwarzschild-anti de Sitter metric), $\hat{g}$ are the given asymptotics that parametrize the desired solution and $\bar{g}$ is the remaining part that is solved for. We define $f(\mathring{g} + \hat{g}, \bar{g}) := E(g)$. As the seed metric satisfies the equations we have $f(\mathring{g}, 0) = 0$. If we can show that $\partial_{\bar{g}} f(\mathring{g} + \hat{g}, 0)|_{\hat{g}=0}$ is an isomorphism then the implicit function theorem guarantees that for all $\hat{g}$ that are small enough there exists a $\bar{g}$ such that $f(\mathring{g} + \hat{g}, \bar{g}) = 0$.

The main difficulty in the construction is proving the isomorphism property. This requires adding gauge terms to both the Einstein and matter equations to obtain an elliptic system. The gauge terms themselves satisfy an elliptic system as a consequence of the Bianchi equations and the continuity equation for the energy momentum tensor. These equations ensure that the gauge terms vanish if they decay fast enough as $\rho \to 0$. As they are defined as differences between the Christoffel symbols of the new and seed solutions, this is equivalent to certain asymptotic conditions on the behavior of the new metric.

The gauge-fixed version of the linearized Einstein operator turns out to be the *Lichnerowicz Laplacian* (Equation (3.5) in Chapter 4). We can thus apply results of Lee [73] in the setting of asymptotically hyperbolic Riemannian manifolds. This is described in detail in [30] for the Einstein equations and similarly in [33, 24] for the matter fields. The results of Lee require that the seed manifold satisfy certain additional conditions, e.g. for the Einstein equations the L^2 kernel of the Lichnerowicz Laplacian needs to vanish. We call manifolds which satisfy this condition "non-degenerate". The class of non-degenerate seed solutions includes Anti-de Sitter space, which allows us to construct new solutions with asymptotics close to those of AdS. In [25] and Chapter 5 we show that Schwarzschild anti-de Sitter spacetimes, and their counterparts with other horizon topologies, also satisfy this property, at least for some values of the mass parameter.

We thus obtain, for suitable seed metrics and asymptotic data, solutions of the gauge-fixed Einstein-matter equations. Plugging the resulting behavior of the metric- and matter functions into the definition of the gauge fixing terms shows that they have the required decay, i.e. they must vanish. Therefore the constructed solutions satisfy the (Riemannian) Einstein(-matter) equations, and, finally, their Lorentzian counterparts satisfy the original Lorentzian equations.

3 THE BEHAVIOR OF SINGULARITIES

3.1 HISTORY & MOTIVATION

Singularities are one of the most perplexing features of general relativity. Their investigation has accompanied the development of the theory since its conception: Even the first known non-trivial solution, the Schwarzschild spacetime, which was discovered less than a month after Einstein presented general relativity for the first time, is singular.

In standard Schwarzschild coordinates the solution contains two apparently singular sets: The horizon at $r = 2m$ and the central singularity at $r = 0$. It took a long time until the nature of these infinities was understood [1]: At first the coordinate singularity at the horizon was seen as a physical singularity. Painleve [83] and Gullstrand [58] independently discovered solutions which extended smoothly through the horizon, but it was not realized at the time that these were just extensions of the Schwarzschild solution in different coordinates. Georges Lemaître was the first to find a coordinate transformation that removed the coordinate singularity at $r = 2m$, although its interpretation was still unclear at that time [74]. Synge [100] found a different set of such coordinates, which actually cover the full maximal analytic extension of the Schwarzschild spacetime, though in a less elegant way than the now standard coordinates of Kruskal and Szekeres.

At that point only a few exact solutions of the Einstein equations were known, and it was unclear if singularities would occur outside of these highly non-generic symmetric examples. The singularity theorems of Penrose and Hawking showed that, indeed, singularities are an essential feature of the theory [88, 59]. There are various versions of these theorems, but they share a common structure (see the review [97]). Under the assumption of

1. a causality condition,
2. an energy (or, equivalently, curvature) condition,
3. and an initial or boundary condition,

they show that the spacetime is geodesically incomplete, i.e. that there exist causal geodesics which stop at some finite value of the affine parameter and cannot be extended further. The classical result is the Penrose singularity theorem, which states

THEOREM 3.1.1 (Penrose singularity theorem [88], in the formulation of [97]). *If a spacetime contains a non-compact Cauchy surface Σ and a*

1 For a more detailed account of the historical interpretation of singularities see [43].

closed future trapped surface, and satisfies the null energy condition (i.e. $R_{\mu\nu}u^\mu u^\nu \geq 0$ for all null vectors u^μ) then it is future null geodesically incomplete.

Here the causality condition is given by the existence of a Cauchy surface (which implies the spacetime is *globally hyperbolic* and therefore topologically $I \times \Sigma$ for some interval $I \subset \mathbb{R}$) and the null energy condition provides the curvature condition, which ensures that geodesics are focused. The initial condition is provided by the existence of a "closed future trapped surface", which captures the intuitive notion that a singularity is formed when the gravitational attraction becomes strong enough. Such a surface is defined as a compact spacelike 2-surface without boundary such that the future-directed null geodesics which start orthogonally to the surface pointing inward and outward converge. The importance of the singularity theorems comes from the fact that trapped surfaces are preserved under small perturbations, and therefore singularities occur *generically*, not just in highly symmetric situations.

The conclusion of these theorems, the existence of incomplete geodesics, is a very different concept of a singularity than that occurring in the Schwarzschild solution, where we can directly see that curvature invariants are unbounded. Unfortunately the singularity theorems do not give any more information about the behavior of the metric near the singularity, e.g. they don't say anything about the behavior of curvature invariants there. It is therefore of interest to further investigate the behavior of singularities in the general case.

3.2 CAUSAL CHARACTER OF SINGULARITIES

One might start by trying to classify singularities according to their causal character. As singularities are not themselves part of the spacetime manifold the standard approach to assigning a causal character to subsets does not work. Instead we call a singularity timelike if it lies both in the future and in the past of points in the spacetime. More precisely a singularity is timelike if the future of a past incomplete geodesic is contained in the future of some point in the manifold. This definition excludes the big bang singularity, as there is no point "before" it.

Such singularities are sometimes called "naked" (e.g. in [84]), but this is not equivalent to the standard definition. A *naked singularity* is usually defined to be a singularity that is visible from infinity, i.e. one that can be reached by past directed causal curves starting at future null infinity [104]. On the other hand, timelike singularities can be hidden behind event horizons such that they are not visible at infinity. (An artificial example of such a singularity can be constructed by simply removing some points inside the event horizon from the Schwarzschild spacetime.) In [85] Penrose calls them "locally naked" to make the distinction clear. Timelike singularities are thus compatible with the weak cosmic censorship conjecture (which states that there are no naked singularities) but violate the strong cosmic censorship conjecture, which is the topic of Section 3.4. In Chapter 6, we construct timelike AVTD (see Section 3.3.2 below) singularities without symmetries. This provides the first examples of spacetimes without any symmetries in this class.

Singularities which are not timelike should be either spacelike or null. To distinguish between these cases one needs to somehow extend the manifold to also include the endpoints of previously incomplete curves, and to define the causal character of this boundary. The simplest method to achieve this, which works at least for highly symmetric exact solutions, is the conformal method of Penrose (see Section 2.4). Various extensions and generalizations of this method have been proposed (see [51] for a review) but there is no strategy that works in general.

It might seem plausible that the existence of a sequence of spacelike hypersurfaces $\{r = \text{const.}\}$ for some time function r, approaching a singularity "at" $\{r = 0\}$ (as, e.g., in the Schwarzschild case), means that the singularity is spacelike, but this does not suffice: If we assume we have some extension of the manifold which is isometric in the interior, the boundary could well be null. The hypersurfaces could simply tilt as they approach the singularity and become null in the limit. In fact this is the main difficulty in Sbierski's proof of the C^0-inextendibility of the Schwarzschild spacetime across the $r = 0$ singularity in [96], and the inspiration for the title of Chapter 7. In the Schwarzschild case spherical symmetry is crucial to conclude the argument.

The singularities in two of the most important exact solutions, the Schwarzschild and Kerr spacetimes, show very different behavior: At the Schwarzschild singularity curvature invariants blow up and, by [96], the metric cannot be continued even in the continuous class. In contrast, the physically relevant boundary of the Kerr spacetime is a (null) Cauchy horizon, which is not a curvature singularity. As the spacetime can be extended through the Cauchy horizon the null character of this boundary is well defined.

It is expected that a perturbation of Kerr would turn the Cauchy horizon into a singular boundary, but the type of that boundary was unclear until recently. In [38] Dafermos and Luk show that, starting with characteristic initial data which is close to that on the Kerr event horizon, the causal structure of interior Kerr, including the Cauchy horizon, is preserved. They conjecture that the boundary in fact becomes a *weak null singularity*, characterized by the existence of an extension with continuous metric up to the boundary but singular in the sense that the Christoffel symbols are no longer locally square integrable there. As there is a metric defined on the boundary, it is clear what it means for it to be null. The results of [38], together with the conjectured but widely believed stability of the exterior Kerr solution, imply that there is an open set of intial data (given on spacelike Cauchy surfaces) such that the maximal globally hyperbolic development is bounded by a null surface.

3.3 MIXMASTER AND BKL

A completely different theory of generic singularities is provided by the Belinski-Khalatnikov-Lifschitz (BKL) framework. This encompasses a vast spectrum of heuristic and numerical results, but rather fewer rigorous ones, mostly confined to symmetric or at least non-generic situations.

The basic idea is given by the slogan that *generic singularities are spacelike, local, and oscillatory*. The BKL approach grew out of the work of the three authors in the 60's and 70's [76, 9, 8]. These papers argue heuristically that close to a (spacelike) singularity the Einstein equations simplify enormously, as spatial derivative terms become irrelevant compared to time derivatives. Dropping the spatial derivatives turns the Einstein equations into a system of ODEs at each point, similar to the equations for (anisotropic) spatially homogeneous spacetimes. This picture is supported by various heuristic arguments (e.g. the cosmological billiards approach [39]) and by numerical simulations [11, 53, 52]. In fact, these simulations have shown that the situation is somewhat more complicated than envisioned by BKL, as spatial derivatives can become significant again for short periods of time, during so-called "spikes".

3.3.1 *Homogeneous Bianchi cosmologies*

Assuming the BKL framework applies, the analysis of spatially homogeneous spacetimes gives information about the behavior in the inhomogeneous case. Such spacetimes are characterized by the existence of a group of isometries acting simply transitively (i.e. such that for every two points there exists a unique group element mapping one to the other) on each spatial slice. This means that there exists a Lie-algebra of Killing vector fields ξ_i, with commutators

$$[\xi_i, \xi_j] = \gamma_{ij}^k \xi_k \,, \tag{10}$$

where the structure constants γ_{ij}^k can be decomposed as

$$\gamma_{ij}^k = \hat{n}^{k\ell}\epsilon_{ij\ell} + a_i\delta_j^k - a_j\delta_i^k \,, \tag{11}$$

with ϵ the antisymmetric Levi-Civita symbol and $\hat{n}^{k\ell}$, a_i constants. The Jacobi identity for the commutator implies

$$\hat{n}^{k\ell}a_\ell = 0 \,. \tag{12}$$

Bianchi classified the three-dimensional Lie-algebras into 11 (or 9, or 10 depending on how they are counted) classes, according to the signs of the eigenvalues of $\hat{n}^{k\ell}$ (denoted $\hat{n}_i \in \{-1, 0, 1\}$ here) and whether $a_i \neq 0$ [12, 13].

The simplest anistropic solutions are the Kasner spacetimes in Bianchi class I ($a_i = 0$, $n_i = 0$). They are given explicitly by the metric

$$-dt^2 + \sum_i t^{2p_i} (\mathrm{d}x^i)^2 , \tag{13}$$

where the p_i are called *Kasner exponents* and satisfy $\sum_i p_i = \sum_i p_i^2 = 1$. In spacetime dimension $3+1$ these conditions imply that, unless one of the p_i is equal to 1 and the others vanish, there is exactly one negative Kasner exponent. This corresponds to one direction of space that expands as the singularity is approached, while the orthogonal directions contract. The Kasner spacetimes contain a curvature singularity at $\{t = 0\}$ where the Kretschmann scalar $K := R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 16t^{-4}(p_3^2 - p_3^3)$ diverges. While the Bianchi classification only applies to spacetime dimension $3+1$, the Kasner solution exists for all dimensions.

The Kasner solution is important, as it provides a building block for the more complicated behavior in the higher Bianchi classes, and, conjecturally, for that of inhomogeneous singular spacetimes. For the homogeneous case this is made clear using the dynamical systems approach of Wainwright and Ellis [102]. Their analysis uses variables which are normalized with respect to the overall expansion of the spacetime. This was first done by Wainwright and Hsu in [103], where they introduced the variables Σ_i, N_i, and θ which satisfy a system of first order evolution equations following from the Einstein equations.

The variable θ represents the overall expansion, and its evolution equation decouples from the remaining ones. It can be separately integrated after the others are determined. The constraint equations fix one of the Σ_i, leaving two which can be expressed as $\Sigma_+ := -\Sigma_1/2$, $\Sigma_- := (\Sigma_3 - \Sigma_2)/(2\sqrt{3})$. Thus there are five variables of interest: Σ_+, Σ_-, and the three N_i.

In these variables the Kasner solutions correspond to the circle $\Sigma_+^2 + \Sigma_-^2 = 1$ of fixed points in the $\{N_i = 0\}$ plane. Bianchi II solutions ($a_i = 0$ and one n_i equal to 1 with the others vanishing) lie in the hemispheres $\Sigma_+^2 + \Sigma_-^2 + N_k^2 = 1$ for $k = 1, 2, 3$. The trajectories of these solutions are straight lines in the Σ_+, Σ_- variables which start and end on the Kasner circle. In the language of dynamical systems theory Bianchi II solutions follow *heteroclinic orbits*, i.e. their future and past asymptotics are given by two different fixed points – in this case by two Kasner solutions.

The most general solutions in the Bianchi class, and therefore the ones that are expected to model generic singularities in the BKL picture, are those of Bianchi classes VIII and IX. An overview of what is known about these classes is provided by [60] (most of the conclusions about Bianchi IX mentioned there also apply to Bianchi VIII by later results of Brehm [17]). They show more complicated behavior consisting of a sequence of near-Kasner states and Bianchi II transitions. It is widely believed (but unproven) that this behavior is *chaotic*, and, more specifically, that generic Bianchi VIII and IX solutions asymptotically switch between (almost) Kasner states according to a discrete map, the so-called "Mixmaster map", which maps the Kasner circle to itself according to the Bianchi II trajectories.

The behavior of the Mixmaster map itself is well understood: For a generic starting point on the Kasner circle it leads to a non-periodic sequence of Kasner states which are dense on the Kasner circle. What is less well understood is how this is related to the behavior of the full dynamical system for the Bianchi VIII and IX solutions. The expectation is that generic solutions should asymptotically "shadow" a sequence of Bianchi II transitions which connect Kasner states related by the Mixmaster map. This would imply that the α-limit set [2] of such solutions includes the whole Kasner circle. The best results in this direction are the attractor theorems of Ringström for Bianchi IX [92] and their generalization to Bianchi VIII by Brehm [17]. They show that generic solutions in these classes have at least three α-limit points on the Kasner circle, but they don't rule out the possibility that the solution converges to a periodic orbit.

3.3.2 *Results in the inhomogeneous case*

There are far fewer rigorous results in the inhomogeneous case. The only known class of inhomogeneous solutions which show Bianchi IX type behavior is that of Berger and Moncrief, which we will analyze in Section 3.3.3. This is only a finite dimensional space of solutions and therefore doesn't give much information about the generic case.

There are several heuristic approaches which aim to formalize the arguments of BKL. Chitre [19] and Misner [81] introduced the so-called cosmological billiards approach, which was later extended by Damour, Henneaux, and Nicolai [39] (this last reference provides a good overview). They start with a Hamiltonian formulation of the Einstein equations. The Hamiltonian can, for each spatial point, be brought to the form of that of a particle in an auxiliary space (whose position represents some functions appearing in the metric) moving in a complicated potential. Assuming that the limiting behavior of the metric is as conjectured by BKL, this potential simplifies enormously in the limit. In fact it approaches the sum of a number of sharp potential walls, which constrain the region where the particle can move. In the interior region the particle travels along straight lines, corresponding

2 A point x lies in the α-limit set of a solution $u(t)$ if there exists a sequence $t_i \to 0$ such that $u(t_i) \to x$.

to Kasner behavior of the metric. On contact with the walls it is reflected elastically, which corresponds to the transition to a new Kasner phase. This type of "billiards" behavior is well understood, e.g. it is known that a finite volume of the allowed region leads to chaotic trajectories. The heuristic part of the argument is contained in the step from the full Hamiltonian to the simplified asymptotic one.

A different approach is that of Heinzle, Uggla, and Röhr [61]. They extend the dynamical systems approach for homogeneous models, described in the previous section, to the inhomogeneous case. Various subsets of the full phase space can be identified, which correspond, e.g., to the assumption of locality by BKL.

More rigorous results are available in cases which show a simpler type of behavior, called *asymptotically velocity term dominated* (AVTD). Solutions of this type asymptotically show Kasner-like behavior at each spatial point, with the Kasner exponents depending on the spatial location. As they are mostly constructed from asymptotic data (using Fuchsian methods) their behavior before this phase is not clear. It seems likely that there are AVTD solutions which undergo a number of Bianchi II type transitions at each point before settling down to a final Kasner state.

In the cosmological billiards picture AVTD behavior occurs if some of the potential walls are suppressed, e.g. by imposing symmetries or asymptotic conditions. This allows the existence of trajectories which continue indefinitely without hitting a wall, i.e. ones which stay in one Kasner state (at each spatial point) until the singularity. In this setting the billiards equations can be used to obtain rigorous results [69].

AVTD-type behavior is not expected to be generic, but occurs in several different cases:

- When there is some symmetry present:
  - The Gowdy spacetimes, which are characterized by a 2-dimensional symmetry group with spacelike orbits that are orthogonal to the vector field ∂_t. The coordinate t is the area time function, given at each point by the area of the group orbit passing through that point. AVTD behavior was shown for the subclass of polarized Gowdy spacetimes in [64, 34] and for general ones in [67, 93].
  - More general classes of T^2-symmetric spacetimes which do not fulfill the orthogonality condition [63, 37, 3]. Note that it is expected that generic T^2 solutions show chaotic BKL behavior; these papers analyze different non-generic subclasses.
  - Some subclasses of $U(1)$-symmetric spacetimes [65, 23, 22].
- When asymptotic conditions are imposed, e.g. the spacetimes we constructed in [69] which are determined by asymptotic

conditions (see also their counterparts with timelike singularities in Chapter 6).

- In high spacetime dimension ($d \geq 12$) [40].
- If specific types of matter are present, e.g. a stiff fluid or scalar field [7] or p-form fields for some values of the coupling constant [40].

In Chapter 6 we construct AVTD solutions with timelike singularities. Our result states (c.f. Chapter 6, Theorem 1 for more detailed asymptotics)

THEOREM 3.3.1. *For any choice of $J \in \{1,2,3\}$ and analytic functions $\beta_\circ^2$, $\beta_\circ^3$ and ${P_\circ}^2{}_1$ depending on coordinates x^i, $i \in \{1,2,3\}$, and for any two analytic functions, $p_\circ^2$ and $p_\circ^3$ depending on x^i, which satisfy the inequalities*

$$0 < p_\circ^2 < (\sqrt{2}-1)p_\circ^3 \,. \tag{14}$$

we obtain a solution of the vacuum Einstein equations with arbitrary cosmological constant given by the metric

$$g = e^{-2\sum_{a=1}^{3}\beta^a} \mathrm{d}\tau^2 + \sum_{a=1}^{3} m_a^J e^{-2\beta^a} \mathcal{N}^a{}_i \mathcal{N}^a{}_j \, \mathrm{d}x^i \mathrm{d}x^j \,. \tag{15}$$

Here $m_a^J = 1 - 2\delta_{Ja}$, β^a and $\mathcal{N}^a{}_i$, $i, a \in \{1,2,3\}$ depend on all coordinates τ, x^i and behave asymptotically as

$$\beta^a = \beta_\circ^a + \tau p_\circ^a + O(e^{-\tau\nu}) \quad \text{and} \quad \mathcal{N}^a{}_i = \delta_i^a + O(e^{-\tau\nu}) \,, \tag{16}$$

where ν is a positive constant, the $\beta_\circ^a$'s and $p_\circ^a$'s depend only upon x^i and ${\mathcal{N}_s}^a{}_i = 0$ for $a \geq i$. The functions $p_\circ^1$ and $\beta_\circ^1$, which are not part of the initial data, are determined from the asymptotic constraint equations.

Finally the Kretschmann scalar behaves as

$$\begin{aligned} R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \Bigg(& \frac{16 e^{4(\beta_\circ^1+\beta_\circ^2+\beta_\circ^3)} \left(p_\circ^2 p_\circ^3\right)^2}{(p_\circ^2+p_\circ^3)^2} \left((p_\circ^2)^2 + p_\circ^2 p_\circ^3 + (p_\circ^3)^2\right) \\ & + O(e^{-\nu\tau}) \Bigg) e^{\tau 4(p_\circ^1+p_\circ^2+p_\circ^3)} \end{aligned} ,$$

and therefore, since $p_\circ^2 p_\circ^3 > 0$, the curvature diverges as $\tau \to \infty$.

Most of the results mentioned above, including Theorem 3.3.1, are obtained using Fuchsian methods. The construction starts with asymptotic data, given as some number of free functions which specify the asymptotic behavior of the metric. Using Fuchsian results (see e.g. [20, Appendix V.3] for the analytic case and [3] for lower regularity) it can be shown that solutions with the specified asymptotics exist. It is tempting to think that, if the number of free functions is high enough, the solutions represent the general behavior in these classes. To actually prove that it is generic in the standard sense would, however, require mapping asymptotic data to Cauchy data given on a spacelike Cauchy surface, which is extremely difficult. In fact the only case where genericity has been established is the class of Gowdy spacetimes, where this is shown in the proof of strong cosmic censorship by Ringström [93].

There is a recent result by Rodnianski and Speck proving the existence of an open (but not full measure) set of initial data leading to AVTD behavior in high spacetime dimension ($d \geq 38$) [94]. They show the nonlinear stability of Kasner solutions which are close to locally rotationally symmetric (i.e. where the difference between the Kasner exponents is small enough). The Fuchsian results mentioned above suggest that this might hold also for large data and down to $d \geq 12$.

3.3.3 *The Berger-Moncrief construction*

The only known examples of inhomogenous solutions to the vacuum Einstein equations which show the chaotic behavior expected in the general case are those constructed by Berger and Moncrief in [10]. This is a three-parameter family of solutions with only one Killing vector which is obtained by applying a solution-generating technique of Geroch [54] to a spatially homogeneous Bianchi IX metric. It is, of course, highly non-generic and in fact behaves somewhat differently than the expectation for a generic BKL-type solution: As all the time dependence comes from that of the original Bianchi IX metric, the bounces between different Kasner epochs happen simultaneously at all spatial points (in a suitable slicing). The heuristic arguments of BKL (and numerical simulations) predict that in the generic case the timing of bounces would be independent at each spatial point.

The Berger-Moncrief construction starts with a Bianchi IX metric in the form

$$g = -A(t)B(t)C(t)\mathrm{d}t^2 + A^2(t)(\hat{\sigma}^1)^2 + B^2(t)(\hat{\sigma}^2)^2 + C^2(t)(\hat{\sigma}^3)^2 \quad (17)$$

on $\mathbb{R} \times S^3$, where the (time-independent) one forms $\hat{\sigma}^i$ are adapted to the symmetry of the spacetime and the singularity is approached as $t \to 0$. In [10] the associated inhomogeneous solutions are investigated using numerical methods. The authors numerically solve the Bianchi IX equations, obtaining functions A, B, C, and apply the solution generating transformation. To analyze the behavior of the inhomogeneous metric they compare the norm γ_g of the single preserved Killing vector of the new solution with the norm γ of the corresponding Killing vector in the original homogeneous solution. The time evolution of γ and γ_g is qualitatively similar, showing the typical oscillatory behavior. However, γ_g depends on the spatial coordinates in a non-trivial way.

Using the later results of Ringström [92] and Brehm [17], it is now possible to do a similar analysis based on analytical knowledge of the behavior of Bianchi IX solutions. Our aim here is to translate those results, formulated in terms of the Wainwright-Hsu variables, to the quantities A, B, and C defined above. The following argument is based on [91].

We will use the time coordinate τ, defined in Section 21 of [92], instead of the t appearing in (17), but this does not affect the conclusions. In these coordinates the singularity is located at $\tau = -\infty$.

From the proof of Lemma 21.2 in [92] we see that

$$A \propto (n_2 n_3)^{-1/2}, \quad B \propto (n_1 n_3)^{-1/2}, \quad C \propto (n_1 n_2)^{-1/2}, \tag{18}$$

where $n_i := \theta N_i$ are the quantities defined there. Let us consider the expression $\sqrt{N_1 n_2 n_3}$. Using the Wainwright-Hsu equations [92, (9)] we find that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sqrt{N_1 n_2 n_3} = \sqrt{N_1 n_2 n_3}\,(\Sigma_+^2 + \Sigma_-^2 - 1)\,. \tag{19}$$

Lemma 3.6 of [17] states that the product $N_1 N_2 N_3$ converges to 0 exponentially as $\tau \to -\infty$. Therefore, by [92, Lemma 7.3], $\Sigma_+^2 + \Sigma_-^2 - 1$ is bounded above by an exponentially decaying function, so the positive part of $\Sigma_+^2 + \Sigma_-^2 - 1$ is integrable up to the singularity. Corollary 18.1 of [92] shows that generic, i.e. non-NUT, Bianchi IX solutions have at least 3 α-limit points on the Kasner circle. In fact, by Proposition 6.1 of [92], the Bianchi II transitions leading to these limit points are themselves part of the α-limit set. During such transitions $\Sigma_+^2 + \Sigma_-^2 - 1 < 0$ and, since there are infinitely many of them and they take a finite amount of τ time,

$$\int_{-\infty}^{\tau_0} (\Sigma_+^2 + \Sigma_-^2 - 1)\mathrm{d}\tau = -\infty\,. \tag{20}$$

Together with (19) this shows that $\sqrt{N_1 n_2 n_3} \to \infty$ as $\tau \to -\infty$ ($t \to 0$). As the N_i are bounded by [92, Theorem 19.2], we find $\sqrt{n_2 n_3} \to \infty$, i.e. $A \to 0$. The same argument applies for B and C.

On the other hand, for each ratio A/B, B/A, C/B,... there exists a sequence of times such that it diverges. This follows from the fact that the α-limit set includes Bianchi II transitions (for which two of the N_i are zero).

We now come back to the class of non-homogeneous solutions constructed in [10]. There for each choice of $h := (a, b, c) \in \mathbb{R}^3$ a new metric g_h is obtained which inherits one of the Killing vectors of g. The metric g_h is *not* spatially homogeneous and lives either on $\mathbb{R} \times S^3$ as the original one, or on another S^1 bundle over $\mathbb{R} \times S^2$.

The components of the new metric are not purely algebraic expressions of A, B, and C as they contain a quantity $\beta_{(h)a}$, defined in [10], which is determined by solving a Poisson equation on S^2. However, the component $g_h(\partial_\psi, \partial_\psi) =: \gamma_g$, i.e. the norm of the inherited Killing vector ∂_ψ, does not depend on $\beta_{(h)a}$.

For the original spatially homogeneous Bianchi IX metric g we have

$$\gamma := g(\partial_\psi, \partial_\psi) = C^2 \cos^2\theta + (B^2 \cos^2\varphi + A^2 \sin^2\varphi)\sin^2\theta\,, \qquad (21)$$

after replacing the one forms $\hat{\sigma}^i$ with their representations in Euler angle coordinates.

The expression for $g_h(\partial_\psi, \partial_\psi)$ is somewhat complicated, but using $A, B, C \to 0$ we find

$$g_h(\partial_\psi, \partial_\psi) \to \frac{A^2B^2(C^2\cos^2\theta + (B^2\cos^2\varphi + A^2\sin^2\varphi)\sin^2\theta)}{c^2(BA' + AB')^2}\,. \qquad (22)$$

Bianchi IX solutions spend most of their time near the Kasner circle, where A'/A etc. is approximately constant, and there the previous expression simplifies to

$$g_h(\partial_\psi, \partial_\psi) \precsim C^2 \cos^2\theta + (B^2\cos^2\varphi + A^2\sin^2\varphi)\sin^2\theta\,, \qquad (23)$$

i.e. exactly the same as in the homogeneous case. This is consistent with the numerical results in Figure 1 of [10].

It is clear that the behavior of (at least this component of) the metric is similar to that of the original mixmaster solution. As it is still open if these are chaotic in a stronger sense than provided by the attractor theorem of Ringström [92] (i.e. having at least three α-limit points on the Kasner circle) not much more can be said here.

There are similar solution generating procedures to those used in [10] which transform solutions of the vacuum Einstein equations to solutions of the Einstein-Maxwell system with nonzero electromagnetic field (see [99, Part IV, Chapter 34]). Applying these to a Bianchi IX seed should give solution of a similar type to the Berger-Moncrief ones in this setting. As the equations are more complicated, their behavior would probably have to be investigated using numerical methods.

3.4 STRONG COSMIC CENSORSHIP AND C^0-INEXTENDIBILITY

The fundamental theorems of Choquet-Bruhat and Geroch [48, 21] establish existence of a unique, maximal, globally hyperbolic development (MGHD) of initial data for the vacuum Einstein equations, given on a spacelike hypersurface. However, just because this development is maximal in the class of globally hyperbolic spacetimes doesn't mean that it can't be extended outside this class. This leads to the question of what happens at the boundary, if any, of the maximal globally hyperbolic development. The *strong cosmic censorship conjecture*, introduced by Penrose [85], proposes an answer to this question.

It states, roughly speaking,

CONJECTURE 3.4.1 (cosmic censorship). The maximal globally hyperbolic development of generic asymptotically flat or compact initial data is inextendible in some suitable sense.

This is connected to the notion of predictability: The MGHD is exactly the maximal set for which the Einstein equations uniquely determine the evolution of the metric. If strong cosmic censorship holds this would mean that we cannot leave this region of predictability, at least not without crossing a singularity of some type.

The above formulation of the conjecture leaves open what type of inextendibility we should expect. In general, a spacetime (M, g) is extendible if there exists a spacetime $(\tilde{M}, \tilde{g})$ and an isometric embedding $\phi : M \to \tilde{M}$ such that $\phi(M)$ is a proper subset of $\tilde{M}$. An extension is C^k if $\tilde{g}$ is a C^k metric.

One option for Conjecture 3.4.1 might be C^2-inextendibility, which is implied by the blowup of curvature invariants. However this formulation would exclude situations where the Einstein equations still guarantee the existence of solutions: The bounded L^2 curvature conjecture, proven by Klainerman, Rodnianski, and Szeftel [68], shows that a solution of the Einstein equations can be continued as long as the curvature and first derivatives of the second fundamental form stay locally square integrable. Pointwise blowup of the second derivatives of the metric would violate the C^2 definition of strong cosmic censorship while still satisfying this condition. On the physical side, a natural condition for a singular boundary of a spacetime is that observers trying to cross it would be destroyed, e.g. by infinite tidal forces. This is also not guaranteed by C^2-inextendibility [82].

The strongest reasonable sense of inextendibility would be C^0-inextendibility, i.e. extensions where the metric is only assumed to be continuous. The study of such extensions was initiated by Sbierski in [96], where he proved the C^0-inextendibility of the Minkowski and Schwarzschild spacetimes (a simplified version of the proof is given in [95]). The idea of the proof is based on the following observation [96, Lemma 2.17]: If an extension exists then it contains a timelike curve crossing from the original manifold into the new region.

This naturally splits the proof of the inextendibility of Schwarzschild into two parts: One part considers curves leaving through null or timelike infinity and the other those leaving across the $r = 0$ singularity. For the first case the essential ingredient is the fact that the timelike diameter (the supremum over the distance beween points) of sets of the form $I^-(p) \cap I^+(q)$, where p is a point on the boundary and q lies in the interior, is infinite. For curves leaving through $r = 0$ Sbierski introduces the notion of a spacelike diameter, defined as the supremum over Cauchy surfaces of the (Riemannian) diameter in each surface. As the tt component of the Schwarzschild metric blows up on approach to the singularity the distance between points separated in the t direction diverges.

In both cases a contradiction is obtained by considering a small neighborhood of the point where the timelike curve crosses the boundary. It is always possible to choose coordinates there such that the metric is close to Minkowski. This can be used to show that timelike or spacelike diameters are bounded, contradicting the results obtained in the interior region.

The main difficulty in the proof, which was glossed over in the above, is to identify the sets whose diameter is investigated in the interior and boundary neighborhood. This involves showing that, in the $r = 0$ case, the future of a point in the interior which is sufficiently close to the boundary lies completely within the small neighborhood mentioned above. Sbierski's argument to this effect relies on the high degree of symmetry present in the interior Schwarzschild spacetime. It seems that this is an essential condition which cannot be removed without the development of new methods.

In Chapter 7 we investigate C^0-extensions of spacetimes without assuming symmetry conditions. We define the concept of a "(globally) expanding singularity" which is characterized by the type of behavior seen in the Schwarzschild case, i.e. an unbounded stretching of space in at least one direction.

More precisely, a globally expanding singularity is defined by

DEFINITION 3.4.2 (Chapter 7, Definitions 1.1). We shall say that a globally hyperbolic space-time $(\mathscr{M} = (0, \infty) \times \mathscr{S}, g)$, with a Cauchy time function t such that $\{t = \text{const.}\} = \mathscr{S}$, contains a *globally expanding singularity towards the past* if for every open set $A \subset \mathscr{S}$ there exists a sequence t_i decreasing to zero such that the (Riemannian) diameter of $\{t_i\} \times A$ within $\{t_i\} \times \mathscr{S}$ tends to infinity as $t_i \to 0$.

(For the slightly more involved notion of an expanding singularity see Chapter 7, Definitions 2.4.)

The simplest example of a (globally) expanding singularity is the Kasner spacetime (13). As it is spatially homogeneous the methods of Sbierski apply directly and show its C^0-inextendibility. However, as described in Section 3.3.2 above, there is a large class of AVTD-type spacetimes which show Kasner-like behavior but are not spatially homogeneous. In particular these include the spacetimes we construct in [69], which have no symmetries at all.

We find that C^0-extensions of such spacetimes are only possible across non-compact (almost-everywhere-)null boundaries of a particular structure:

THEOREM 3.4.3 (Chapter 7, Theorem 1.2, Proposition 1.3 & 1.4). *Suppose that $(\mathscr{M}, g)$ contains a globally expanding singularity towards the past. Then in every continuous past extension of $\mathscr{M}$ the boundary $\partial\iota(\mathscr{M})$ of the image of $\mathscr{M}$ in the extension is non-compact, achronal, and null at all its differentiability points.*

Further, every spacelike hypersurface in the extension which intersects the boundary $\partial\iota(\mathscr{M})$ also intersects $\iota(\mathscr{M})$ itself.

Here a *past extension* is one that can only be reached by past-directed timelike curves starting in the original manifold (see Chapter 7, Definition 2.3). As achronality implies Lipschitz continuity the boundary is differentiable almost everywhere by Rademacher's theorem. For further results on more general extensions see Section 3 of Chapter 7.

It would be of interest to consider also extensions across timelike boundaries, such as the timelike singularities we construct in Chapter 6 or the ring singularity past the Cauchy horizon of the Kerr spacetime. A difficulty in this case is that the futures and pasts of points cannot be used to identify subsets between the original and extended manifold.

There are a number of recent results on C^0-(in)extendibility, inspired by [96]: In [49] Galloway and Ling introduce a class of C^0-extendible spacetimes (called "Milne-like") and prove inextendibility of some FLRW spacetimes in the spherically symmetric class. In [50] Galloway Ling and Sbierski show that global hyperbolicity together with timelike geodesic completeness implies C^0-inextendibility. The proof uses similar methods as that of the inextendibility of Minkowski space in [96] but requires much less restrictive conditions. Graf and Ling show in [55] that timelike geodesic completeness alone implies $C^{0,1}$-inextendibility (i.e. no extensions with Lipschitz continuous metric). In [56] Grant, Kunzinger, and Sämann investigate C^0-inextendibility from the viewpoint of synthetic Lorentzian geometry.

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Non-singular spacetimes with a negative cosmological constant: IV. Stationary black hole solutions with matter fields

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Abstract

We use an elliptic system of equations with complex coefficients for a set of complex-valued tensor fields as a tool to construct infinite-dimensional families of non-singular stationary black holes, real-valued Lorentzian solutions of the Einstein–Maxwell-dilaton-scalar fields-Yang–Mills–Higgs–Chern–Simons-$f(R)$ equations with a negative cosmological constant. The families include an infinite-dimensional family of solutions with the usual AdS conformal structure at conformal infinity.

Keywords: black holes, stationary spacetime, negative cosmological constant

1. Introduction

There is currently considerable interest in the literature in space-times with a negative cosmological constant. This is fueled on one hand by studies of the AdS-CFT conjecture and of the implications thereof: Indeed, this problem is of immediate physical interest in the context of the weakly coupled supergravity limit of the AdS/CFT correspondence. On the other hand, these solutions are interesting because of a rich dynamical morphology: existence of periodic or quasi-periodic solutions, and of instabilities. All this leads naturally to the question

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of existence of stationary solutions of the Einstein equations with $\Lambda < 0$, with or without sources, and of properties thereof.

This manuscript is the fourth in a series of papers, starting with [13], which are devoted to proving existence of a large class of solutions to the Einstein equations with negative cosmological constant by perturbation of known ones. All of these papers are further related by the fact that the field equations can be transposed to an elliptic system on a conformally compact *Riemannian* manifold. The system is solved by an implicit function theorem argument under a non degeneracy hypothesis. This can be traced back to earlier work of Graham and Lee [20] on constructing Einstein metrics on $(n+1)$-dimensional balls with S^n boundary, as generalized to more general infinities by Lee [25] (compare [7] for more general symmetric spaces). Using such methods, in [14, 15] we have constructed infinite dimensional families of non-singular *strictly stationary* space times, solutions of the Einstein equations with a negative cosmological constant and with various matter sources. These families include an infinite-dimensional family of solutions with the usual AdS conformal structure at conformal infinity. The construction there did not provide any black hole solutions, as strict stationarity is incompatible with existence of horizons. However, black hole solutions are of special interest. In fact, various such solutions have already been constructed numerically: For example, static Einstein–Yang–Mills black holes have been constructed in space-time dimension five in [29], with four-dimensional solutions constructed in [9], and higher dimensional ones in [28]. In [27] an explicit five-dimensional such solution has been given. Rotating Einstein–Maxwell-Chern–Simons solutions have been presented in [10]. In [19] a family of five-dimensional black holes was constructed satisfying the Einstein-complex scalar field equations, with a stationary geometry and time-periodic scalar field; compare [5].

The object of this work is to provide a rigorous existence proof for large families of such solutions. The idea is to use a 'Wick rotation' to construct suitable solutions of a system of *elliptic equations with complex coefficients for a complex valued 'Riemannian metric'*. In a nutshell, we show that Lee's theorem on existence of perturbed Poincaré–Einstein Riemannian metrics [25, theorem A] can be extended to complex valued 'metrics', and to more general equations, and that this can be used to construct stationary Lorentzian black hole solutions with large classes of matter sources. This proceeds as follows:

We wish to construct a Lorentzian metric $\mathbf{g}$ in any space-dimension $n \geqslant 3$, with Killing vector $X = \partial/\partial t$, satisfying the Einstein–Maxwell-Chern–Simons-Yang–Mills-dilaton-scalar fields equations, with a stationary geometry but possibly time-periodic complex fields, or the $f(R)$ equations. In adapted coordinates the metric can be written as

$$\mathbf{g} = -V^2(\mathrm{d}t + \underbrace{\theta_i \mathrm{d}x^i}_{=:\theta})^2 + \underbrace{g_{ij}\mathrm{d}x^i \mathrm{d}x^j}_{=:g}, \tag{1.1}$$

$$\partial_t V = \partial_t \theta = \partial_t g = 0\,. \tag{1.2}$$

Let us introduce a *complex parameter* $a \in \mathbb{C}$ and consider the complex-valued tensor field

$$\mathbf{g} = -V^2(\mathrm{d}t + a\theta)^2 + g\,, \tag{1.3}$$

satisfying (1.2). We will say that a complex valued symmetric tensor field g is a *complex metric* if g is symmetric and invertible. Replacing $\mathrm{d}t$ by $-\mathrm{i}\,\mathrm{d}t$ in (1.3), where $\mathrm{i} = \sqrt{-1}$, we obtain a complex metric with Riemannian real part:

$$\mathfrak{g} := V^2(\mathrm{d}t + a\mathrm{i}\theta)^2 + g\,. \tag{1.4}$$

Under such a substitution the field equations transform in a controlled way, for example t-independent vacuum metrics lead to t-independent, possibly complex valued, tensor fields satisfying the vacuum equations, etc.

Working near a (real-valued) static Einstein metric $\mathring{\mathfrak{g}} = \mathring{V}^2 \mathrm{d}t^2 + \mathring{g}$ satisfying a non-degeneracy condition (as defined in the paragraph after equation (3.5) below) we will

(1) construct complex metrics reminiscent of (1.4) which solve the vacuum Einstein equations for small $|a|$, and
(2) show that V, g and θ are real-valued if $a \in \mathbb{R}$.

(Incidentally, we will also show that $\mathfrak{g}$, and hence $\mathbf{g}$, is analytic in a, an interesting property of the stationary metrics at hand which has does not seem to have been noticed so far.) After 'undoing the Wick rotation' leading from (1.3) to (1.4), we will show that the resulting Lorentzian space-time has a smooth event horizon at suitable zeros of V.

The construction guarantees that V has zeros when $\mathring{V}$ did, and leads indeed to the desired Lorentzian black-hole solution of the Einstein, or Einstein-matter equations.

Our non-degeneracy condition is satisfied by large classes of metrics, including all four-dimensional Kottler metrics except the spherical ones with a single critical value of the mass parameter [16] (see also [4, proposition D.2]). It is clear that the method opens further possibilities, which remain to be explored. For example, the technique is used in [17] to construct boson star solutions.

2. Elliptic equations with complex principal symbol

Consider an n-dimensional (real) manifold M. Complex-valued tensor fields over M are defined as sections of the usual (real) tensor bundles over M tensored with $\mathbb{C}$. In other words, all coordinate transformations are real but we allow tensors to have complex components. We emphasise that the 'Wick rotation' above is *not* considered to be a coordinate transformation, but a useful device mapping one set of equations and fields to another, more convenient, one.

As already mentioned, we will say that a two-covariant complex valued tensor g is a *complex metric* if g is symmetric and invertible.

Let $\Phi = (\Phi^A)$, $A = 1, \ldots N$, be a collection of complex valued fields, forming a section of a complex bundle over M. Let g be a complex metric and consider a collection of N equations of the form

$$g^{ij}\partial_i\partial_j\Phi^A = F^A(g, \partial g, \Phi, \partial\Phi), \tag{2.1}$$

with some functions F^A which will be assumed to depend smoothly upon their arguments. This can be rewritten as the following collection of $2N$ equations for $2N$ real fields $(\Re\Phi, \Im\Phi)$:

$$\Re g^{ij}\partial_i\partial_j\Re\Phi^A = \Re\big(F^A(g, \partial g, \Phi, \partial\Phi)\big) + \Im g^{ij}\partial_i\partial_j\Im\Phi^A, \tag{2.2}$$

$$\Re g^{ij}\partial_i\partial_j\Im\Phi^A = \Im\big(F^A(g, \partial g, \Phi, \partial\Phi)\big) - \Im g^{ij}\partial_i\partial_j\Re\Phi^A. \tag{2.3}$$

We will say that (2.1) is elliptic if the system (2.2) and (2.3) is elliptic in the usual sense for PDEs involving real-valued functions. The principal symbol of (2.2) and (2.3) is bloc-diagonal, built out of blocs of the form

$$\begin{pmatrix} \Re g^{ij}k_ik_j & -\Im g^{ij}k_ik_j \\ \Im g^{ij}k_ik_j & \Re g^{ij}k_ik_j \end{pmatrix}. \tag{2.4}$$

This is an isomorphism for $k \neq 0$ if and only if

$$(\Re g^{ij}k_ik_j)^2 + (\Im g^{ij}k_ik_j)^2 > 0. \tag{2.5}$$

Hence, if $\Re g^{ij}$ is positive-definite then (2.1) will be elliptic regardless of $\Im g^{ij}$. More importantly for us, when $\Im g^{ij}$ is small enough all the usual elliptic estimates, as needed for our analysis below, apply to (2.2) and (2.3), and hence to (2.1). Likewise, isomorphism properties for a real-valued g carry over to nearby complex-valued g's. As we will be using an implicit function theorem around real valued Riemannian metrics, our perturbation of $\Re g_{ij}$, as well as $\Im g^{ij}$ will always be sufficiently small for the estimates and the isomorphism properties to remain valid.

3. The setup

We work in space-time dimension $d := n + 1$ and we normalise the cosmological constant to

$$\Lambda = -\frac{n(n-1)}{2}; \tag{3.1}$$

this can always be achieved by a constant rescaling of the metric.

Let $\mathring{\nabla}$ denote the covariant derivative associated with the metric $\mathring{\mathfrak{g}}$, set

$$\lambda^\mu := \frac{1}{\sqrt{\det \mathfrak{g}}}\mathring{\nabla}_\alpha(\sqrt{\det \mathfrak{g}}\mathfrak{g}^{\alpha\mu}). \tag{3.2}$$

(In (3.2) the derivative $\mathring{\nabla}$ is of course understood as a covariant derivative operator acting on tensor densities.) Denoting by $\mathfrak{R}^\mu{}_{\alpha\beta\gamma}$ the Riemann tensor of $\mathfrak{g}$, similarly for the Ricci tensor, we set

$$\mathfrak{R}^H{}_{\alpha\beta} := \mathfrak{R}_{\alpha\beta} + \frac{1}{2}\big(\mathfrak{g}_{\alpha\mu}\mathring{\nabla}_\beta\lambda^\mu + \mathfrak{g}_{\beta\mu}\mathring{\nabla}_\alpha\lambda^\mu\big). \tag{3.3}$$

Then the linearisation with respect to the metric, at $\mathfrak{g} = \mathring{\mathfrak{g}}$, in dimension $d = n + 1$, of the map

$$\mathring{\mathfrak{g}} \mapsto \mathfrak{R}^H_{\alpha\beta} + (d-1)\mathfrak{g}_{\alpha\beta}$$

is the operator

$$\mathring{P} := \frac{1}{2}(\mathring{\Delta}_L + 2n), \tag{3.4}$$

where the *Lichnerowicz Laplacian* $\mathring{\Delta}_L$ acts on symmetric two-tensor fields h as

$$\mathring{\Delta}_L h_{\alpha\beta} := -\mathring{\nabla}^\gamma\mathring{\nabla}_\gamma h_{\alpha\beta} + \mathring{\mathfrak{R}}_{\alpha\gamma}h^\gamma{}_\beta + \mathring{\mathfrak{R}}_{\beta\gamma}h^\gamma{}_\alpha - 2\mathring{\mathfrak{R}}_{\alpha\gamma\beta\delta}h^{\gamma\delta}. \tag{3.5}$$

We will say that a metric $\mathring{\mathfrak{g}}$ is *non-degenerate* if $\mathring{\Delta}_L + 2n$ has no L^2-kernel. This should not be confused with the notion of *non-degenerate black holes*, also called *extreme black holes*, which is the requirement of non-zero surface gravity.

Large classes of non-degenerate Einstein metrics are described in [1, 2, 4, 25], see also remark 4.10 below.

It follows immediately from the openness of the set of invertible operators that if $\mathring{\mathfrak{g}}$ is a real-valued non-degenerate Riemannian metric, then all nearby (in a suitable topology, as determined by the problem at hand) complex valued metrics will also be non-degenerate.

The following is well-known (see, e.g. the proof of theorem A at the end of [25], compare [20] for the Poincaré ball):

Proposition 3.1. *Suppose that $\mathring{\mathfrak{g}}$ is non-degenerate and that $\mathring{P}h = 0$, for a tensor field h satisfying*

$$|h|_{\mathring{\mathfrak{g}}} = o(1) \tag{3.6}$$

as the conformal boundary is approached. Then $h \equiv 0$.

Our solutions will be perturbations of a space-time $(\mathscr{M}, \mathring{\mathbf{g}})$ with a static metric $\mathring{\mathbf{g}}$ solving the vacuum Einstein equations with a negative cosmological constant. By definition of staticity, near every point in $(\mathscr{M}, \mathring{\mathbf{g}})$ at which the Killing vector is timelike there exist coordinates in which the metric takes the form (1.1) and (1.2) with $\theta \equiv 0$,

$$\mathring{\mathbf{g}} = -\mathring{V}^2 \mathrm{d}t^2 + \underbrace{\mathring{g}_{ij}\mathrm{d}x^i \mathrm{d}x^j}_{=:\mathring{g}}, \quad \partial_t \mathring{V} = \partial_t \mathring{g} = 0\,. \tag{3.7}$$

The solutions we are about to construct will be defined in the domain of outer communications, where the representation (3.7) is in fact global.

In this work we will consider two cases:

H1. $\mathring{V}$ is strictly positive, $\mathscr{M}$ is diffeomorphic to $\mathbb{R} \times M$, where the coordinate t along the $\mathbb{R}$ factor labels the static slices of $\mathring{\mathbf{g}}$ in $\mathscr{M}$. We set

$$\mathfrak{M} = S^1 \times M,$$

thus time translations in $\mathscr{M}$ become rotations of the S^1-factor of $\mathfrak{M}$.

In this case our analysis below provides an alternative proof of the results in [13, 15].

H2. We allow $\mathring{\mathbf{g}}$ to describe a static vacuum black-hole metric with a Killing horizon with non-zero surface gravity and with global structure similar to that of the domain of outer communications in the Schwarzschild–anti de Sitter (S–AdS) black holes. More precisely, we assume that the Lorentzian manifold $\mathscr{M}$ takes the form

$$\mathscr{M} = \mathbb{R} \times [R_0, \infty) \times {}^{n-1}\mathcal{N},$$

for some $R_0 > 0$, where ${}^{n-1}\mathcal{N}$ is a compact $(n-1)$-dimensional boundaryless manifold. We require that $\mathbb{R} \times \{R_0\} \times {}^{n-1}\mathcal{N}$ coincides with the zero-level set of $\mathring{V}$ which, in a suitable extension of $\mathscr{M}$, becomes an event horizon with non-zero surface gravity. The coordinate t along the $\mathbb{R}$-factor labels the static slices of $\mathscr{M}$. We further assume that after a 'Wick rotation', where $\mathrm{d}t^2$ is replaced by $-\mathrm{d}t^2$, the resulting Riemannian metric

$$\mathring{\mathfrak{g}} := \mathring{V}^2 \mathrm{d}t^2 + \mathring{g}. \tag{3.8}$$

extends to a smooth metric on

$$\mathfrak{M} = \mathbb{R}^2 \times {}^{n-1}\mathcal{N},$$

with the action of the flow of the vector field

$$X := \partial_t \tag{3.9}$$

being rotations of the $\mathbb{R}^2$ factor.

In this case our analysis generalises the results in [13, 15] to black-hole solutions.

An example of H2 above is given by the $(n+1)$-dimensional Schwarzschild–anti de Sitter metrics with non-vanishing surface gravity, where ${}^{n-1}\mathcal{N}$ is the $(n-1)$-dimensional sphere S^{n-1} and $\mathfrak{M} = \mathbb{R}^2 \times S^{n-1}$. More generally, the $(n+1)$-dimensional Birmingham metrics [8], where ${}^{n-1}\mathcal{N}$ is an $(n-1)$-dimensional Einstein manifold, with non-extreme horizons are of this form.

4. The construction

To avoid a discussion of the technicalities associated with the matter fields, we will start by describing in some detail the construction of the vacuum solutions. Note, however, that the argument is essentially the same in both cases, once the isomorphisms needed to handle matter fields have been established. The key difference is in the boundary conditions: the vacuum stationary solutions are determined by their asymptotic data at the conformal boundary, and might have a non-standard conformal infinity when these data are *not* the usual AdS ones[6]. On the other hand, our non-vacuum solutions are determined by both the asymptotic data for matter fields and for the metric, which allows existence of nontrivial solutions with the manifestly standard AdS conformal structure at timelike infinity.

4.1. Vacuum solutions

We denote by ρ a coordinate near ∂M which vanishes at ∂M, and by $C^{\ell,\alpha}(\partial M, \mathcal{T}_1)$ the space of one-forms on ∂M of $C^{\ell,\alpha}$-differentiability class.

We have the following:

Theorem 4.1. *Let $n = \dim M \geqslant 3$, $k \in \mathbb{N} \setminus \{0\}$, $\alpha \in (0,1)$, and consider a static Lorentzian real-valued Einstein metric $\mathring{\mathbf{g}}$ of the form (1.1) and (1.2) as described in section 3, such that the associated Riemannian metric $\mathring{\mathfrak{g}}$ is C^2 compactifiable and non-degenerate, with smooth conformal infinity. We further assume that the hypotheses H1 or H2 of section 3 hold. For all $a \in \mathbb{R}$ with $|a|$ small enough and every smooth real-valued $\widehat{\theta} \in C^{k+2,\alpha}(\partial M, \mathcal{T}_1)$ there exists a unique, modulo diffeomorphisms which are the identity at the boundary, nearby stationary Lorentzian real-valued vacuum metric of the form (1.1) and (1.2) such that, in local coordinates near the conformal boundary ∂M,*

$$V - \mathring{V} = O(\rho), \quad \theta_i = a\widehat{\theta}_i + O(\rho), \quad g_{ij} - \mathring{g}_{ij} = O(1). \tag{4.1}$$

The Lorentzian solutions with $V > 0$ (case H1) are globally stationary, in the sense that they have a globally timelike Killing vector. We show in section 5 below that the Lorentzian solutions with V vanishing (case H2) describe smooth black holes.

Remark 4.2. Large families of static vacuum metrics $\mathbf{g}$ satisfying the conditions of the theorem have been constructed in [3, 4]. In particular if $\mathbf{g}$ is non-degenerate, then the nearby metrics as constructed in [3, 4] also are.

Remark 4.3. The decay rates in (4.1) have to be compared with the leading order behavior ρ^{-2} both for $\mathring{V}^2$ and $\mathring{g}_{ij}$ in local coordinates near the conformal boundary. A precise version of (4.1) in terms of weighted function spaces reads (our notation for function spaces follows [25])

$$(V - \mathring{V}) \in C_1^{k+2,\alpha}(\mathfrak{M}), \quad (g - \mathring{g}) \in C_2^{k+2,\alpha}(\mathfrak{M}, \mathcal{S}_2), \tag{4.2}$$

[6] Note that some non-trivial asymptotic data are compatible with the usual locally conformally flat structure of the conformal boundary. An example is provided by the Demiański–Carter 'Kerr–anti-de Sitter' solutions, see [22, appendix B].

$$\theta - a\widehat{\theta} \in C_2^{k+2,\alpha}(\mathfrak{M}, \mathcal{T}_1), \tag{4.3}$$

and the norms of the differences above are small in those spaces. If the boundary data are smooth, then the solution has a complete polyhomogeneous expansion at the conformal boundary.

Proof. We start by solving on $\mathfrak{M}$ the 'harmonically-reduced Riemannian Einstein equations',

$$\mathfrak{R}^H_{\alpha\beta} + n\mathfrak{g}_{\alpha\beta} = 0, \tag{4.4}$$

for a complex-valued tensor-field $\mathfrak{g}$, with the asymptotic conditions

$$V - \mathring{V} = O(\rho), \quad \theta_k = \mathrm{i}a\widehat{\theta}_k + O(\rho), \quad g_{k\ell} - \mathring{g}_{k\ell} = O(1). \tag{4.5}$$

Here we have extended $\widehat{\theta}$ from ∂M to $S^1 \times \partial M$ by imposing invariance under rotations of the S^1 factor.

The existence of a solution, for all $a \in \mathbb{C}$ with $|a|$ small enough, follows by rewriting the equations as in (2.2) and (2.3) (with $(\Phi^A) = (\mathfrak{g}_{\mu\nu})$), and applying the implicit function theorem. This can be done because of our hypothesis of non-degeneracy of $\mathring{\mathfrak{g}}$; see [4, 13] for the analytical details. In particular (4.2) and (4.3) hold.

The implicit function theorem guarantees that the solutions sufficiently close to $\mathring{\mathfrak{g}}$ with the asymptotics (4.5) are uniquely determined by $a\widehat{\theta}$. We denote by $\mathfrak{g}(a)$ this solution.

The usual argument, spelled-out in detail e.g. in [15, section 4], applies to show that $\lambda^\mu \equiv 0$, so that:

Lemma 4.4. *The complex metrics $\mathfrak{g}(a)$ solve the Riemannian Einstein equations.* □

We continue by showing that:

Lemma 4.5. *The complex metrics $\mathfrak{g}(a)$ are invariant under rotations of the S^1 factor of $\mathfrak{M}$ in the case H1, or of the $\mathbb{R}^2$ factor in the case H2.*

Proof. Let us denote by $P(a)$ the operator obtained by linearising (4.4) at $\mathfrak{g}(a)$; compare (3.4). The Lie derivative of (4.4) with respect to X gives

$$P(a)\mathscr{L}_X\mathfrak{g}(a) = 0, \tag{4.6}$$

where $\mathscr{L}_X$ is the Lie-derivative with respect to the vector field X generating rotations of the S^1 factor of $\mathfrak{M}$ in the case H1, or of the $\mathbb{R}^2$ factor in the case H2; we have also used the fact that $\mathscr{L}_X\mathring{\mathfrak{g}} = 0$. It follows from (4.2) and (4.3) and polyhomogeneity of the solutions that

$$\mathscr{L}_X V \in C_1^{k+1,\alpha}(\mathfrak{M}), \quad \mathscr{L}_X g \in C_2^{k+1,\alpha}(\mathfrak{M}, \mathcal{S}_2), \tag{4.7}$$

$$\mathscr{L}_X\theta \in C_2^{k+1,\alpha}(\mathfrak{M}, \mathcal{T}_1). \tag{4.8}$$

This, together with proposition 3.1, implies $\mathscr{L}_X\mathfrak{g} \equiv 0$, as desired. □

Denoting by t the usual angular coordinate on the $\mathbb{R}^2$ factor (H2 case), or the parameter along S^1 (H1 case), we can thus write the metrics $\mathfrak{g}(a)$ in coordinates adapted to the flow of X in the form

$$\mathfrak{g}(a) := V(a)^2(\mathrm{d}t + a\mathrm{i}\theta(a)_k\mathrm{d}x^k)^2 + g(a)_{ij}\mathrm{d}x^i\mathrm{d}x^j\,. \tag{4.9}$$

Lemma 4.6. *In coordinates as in (4.9), the functions $V(a)$, $g_{ij}(a)$ and $\theta(a)_i$ are even functions of a.*

Proof. Let $\psi : \mathfrak{M} \to \mathfrak{M}$ denote the map which, in the coordinates of (4.9) changes t to its negative, leaving the remaining coordinates unchanged. Then ψ is a smooth isometry of $(\mathfrak{M}, \mathring{\mathfrak{g}})$. The metric $\psi^*\mathfrak{g}(a)$ satisfies the same equation, with same asymptotic data, as $\mathfrak{g}(-a)$, and is close to $\mathring{\mathfrak{g}}$ for $|a|$ sufficiently small, so that uniqueness gives

$$\psi^*\mathfrak{g}(a) = \mathfrak{g}(-a), \tag{4.10}$$

which implies the claim. □

Lemma 4.7. *The metrics $\mathfrak{g}(a)$ are holomorphic functions of a.*

Proof. It is standard to show that the metrics $\mathfrak{g}(a)$ are continuously differentiable functions of a. Differentiating (4.4) with respect to $\overline{a}$ gives

$$P(a)\frac{\partial \mathfrak{g}(a)}{\partial \overline{a}} = 0,$$

where $\partial/\partial\overline{a}$ is the usual complex-derivative operator with respect to the complex conjugate $\overline{a}$ of a. The vanishing of the asymptotic data for $\frac{\partial \mathfrak{g}(a)}{\partial \overline{a}}$ gives $\frac{\partial \mathfrak{g}(a)}{\partial \overline{a}} \equiv 0$. □

Now, if $a \in \mathrm{i}\mathbb{R}$, we can repeat the above construction in the space of real-valued Riemannian metrics. Uniqueness implies then that the corresponding metrics $i\mathbb{R} \ni a \to \mathfrak{g}(a)$ are real-valued. Hence all the coefficients $\mathfrak{g}(x^\alpha)_{\mu\nu k}$ in the convergent Taylor expansions

$$\mathfrak{g}(a, x^\alpha)_{\mu\nu} = \sum_{k\in\mathbb{N}} \mathfrak{g}(x^\alpha)_{\mu\nu k}(\mathrm{i}a)^k \tag{4.11}$$

are real. Lemma 4.6 implies that $V(a)$, the $g_{ij}(a)$'s, and the $\theta(a)_i$'s are real for real a. It follows that for real a the real-valued Lorentzian metrics

$$\mathbf{g}(a) := -V(a)^2(\mathrm{d}t + a\theta(a)_k\mathrm{d}x^k)^2 + g(a)_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{4.12}$$

satisfy all our claims. □

4.2. Matter fields

We now seek solutions to the Einstein–Yang–Mills–Higgs–Maxwell-dilaton-scalar fields-Chern–Simons equations defined by the action

$$S = \int \mathrm{d}^{n+1}x\frac{\sqrt{-\det \mathbf{g}}}{16\pi G}\left[R(\mathbf{g}) - 2\Lambda - W(\Phi)|F|^2 - \frac{1}{2}(\nabla\Phi)^2 - \mathscr{V}(\Phi)\right] + S_{\mathrm{CS}}. \tag{4.13}$$

Here $R(\mathbf{g})$ is the Ricci scalar of the metric $\mathbf{g}$, W and $\mathscr{V}$ are smooth functions, $|F|$ is the gauge-invariant norm of a possibly non-Abelian Yang–Mills field, Φ is allowed to be a section of a bundle associated to the possibly non-Abelian gauge-group, with $\nabla\Phi$ depending if desired upon the Yang–Mills gauge potential. Finally, in even space dimension n, S_{CS} is the Chern–Simons action which, in the Abelian case, takes the form:

$$S_{\rm CS} = \begin{cases} 0, & n \text{ is odd;} \\ \frac{\lambda}{16\pi G}\int A\wedge \underbrace{F\wedge\cdots\wedge F}_{k \text{ times}}, & n=2k, \end{cases} \tag{4.14}$$

for a constant $\lambda\in\mathbb{R}$. In the general (non-Abelian) case $S_{\rm CS}$ is given by [11, equation (3.5)]

$$S_{\rm CS} = \frac{\lambda}{16\pi G}\int \operatorname{Tr}\left(\sum_{i=0}^{k} C_{k,i}A\wedge[A,A]^i\wedge F^{k-i}\right), \tag{4.15}$$

when $n=2k$ and

$$C_{k,i} = \frac{(-1)^i(k+1)!k!}{2^i(k+1+i)!(k-i)!}. \tag{4.16}$$

We obtain:

Theorem 4.8. *Let $n=\dim M\geqslant 3$, $k\in\mathbb{N}\setminus\{0\}$, $\alpha\in(0,1)$, and consider a static Lorentzian real-valued Einstein metric $\mathring{\mathbf{g}}$ of the form (1.1) and (1.2) as described in section 3, such that the associated Riemannian metric $\mathring{\mathfrak{g}}$ is C^2 compactifiable and non-degenerate, with smooth conformal infinity, has no harmonic one-forms which are in L^2 and $\mathscr{V}''(0)$ which is not an L^2-eigenvalue of the operator $\Delta_{\mathring{\mathfrak{g}}}$. We further assume that the hypotheses H1 or H2 of section 3 hold and that*

$$W(0)=1,\quad \mathscr{V}(0)=0=\mathscr{V}'(0),\quad \mathscr{V}''(0)>-n^2/4. \tag{4.17}$$

For all $a\in\mathbb{R}$ with $|a|$ small enough, every smooth real-valued $\widehat{\theta}\in C^{k+2,\alpha}(\partial M,\mathcal{T}_1)$ and $\widehat{U}\in C^{k+2,\alpha}(\partial M)$ and

(1) $\mathscr{V}''(0)<0$ with $\widehat{A}\in C^{k+2,\alpha}(\partial M,\mathcal{T}_1)$, and $\widehat{\Phi}\in\rho^{\sigma_-}C^{k+2,\alpha}(\partial M)$ (where $\sigma_-=n/2-\sqrt{n^2/4+\mathscr{V}''(0)}$) which are sufficiently small smooth fields on ∂M, or
(2) $\widehat{\Phi}\equiv 0$, and $\widehat{A}\in C^{k+2,\alpha}(\partial M,\mathcal{T}_1)$ which is a sufficiently small smooth field on ∂M,

there exists a unique, modulo diffeomorphisms which are the identity at the boundary, nearby stationary Lorentzian solution of the Einstein–Maxwell-dilaton-scalar fields-Chern–Simons equations, or of the Yang–Mills–Higgs–Chern–Simons-dilaton equations with a trivial principal bundle, so that, in local coordinates near ∂M, we have

$$\begin{aligned} &g\to_{\rho\to 0}\mathring{g},\ V\to_{\rho\to 0}\mathring{V},\ \theta\to_{\rho\to 0}a\widehat{\theta},\\ &U\to_{\rho\to 0}a\widehat{U},\ A\to_{\rho\to 0}\widehat{A}_a\mathrm{d}x^a,\ \Phi\to_{\rho\to 0}\widehat{\Phi} \end{aligned} \tag{4.18}$$

with all convergences in $\mathring{g}$-norm. The hypothesis of non-existence of harmonic L^2-one-forms is not needed if $\widehat{A}\equiv 0\equiv\widehat{U}$, in which case the Maxwell field or the Yang–Mills field are identically zero.

Remark 4.9. The remarks in [15, section 7] concerning the energy and the asymptotics of the solutions remain valid word-for-word in the current setting.

Remark 4.10. For the convenience of the reader we repeat here the comments from [15] concerning the kernel conditions in the theorem.

First, it is shown in [15, appendix C] that the condition of non-existence of L^2-harmonic forms is satisfied near anti-de Sitter space-time in any case.

Next, it has been shown by Lee [24, theorem A] that there are no L^2-eigenvalues of $\Delta_{\mathring{\mathfrak{g}}}$ when the Yamabe invariant of the conformal infinity is positive, in particular near anti-de Sitter and Schwarzschild anti-de Sitter space-time. Furthermore, and quite generally, $\mathscr{V}''(0) = 0$ is never an eigenvalue by the maximum principle. Finally, again quite generally, the L^2 spectrum of $-\Delta_{\mathfrak{g}}$ for asymptotically hyperbolic manifolds is $[n^2/4, +\infty[$ together with possibly a finite set of eigenvalues, with finite multiplicity, between 0 and $n^2/4$ [21] (compare [26]), so our non-eigenvalue condition is true except for at most a finite number of values of $\mathscr{V}''(0) \in (-n^2/4, 0)$ for all asymptotic geometries. □

Proof. This follows directly from the arguments of [15]: The indicial exponents of the relevant equations remain unchanged, as terms containing θ, which are here multiplied by ia, are of lower order in ρ. Note that the solutions obtained below using the implicit function theorem might *a priori* depend upon the 'periodic time coordinate' t, but this is irrelevant for the calculation of the indicial exponents.

We start by sketching the argument in the case of a single real-valued scalar field Φ, which satisfies the equation

$$\nabla_\alpha \nabla^\alpha \Phi - \mathscr{V}'(\Phi) = 0.$$

Its indicial exponents are $\sigma_\pm = n/2 \pm \sqrt{n^2/4 + \mathscr{V}''(0)}$, unchanged from those in [15], as $\mathfrak{g}^{tt} = -V^{-2} = O(\rho^2)$ so that terms arising from t derivatives are of lower order. We assume $-n^2/4 < \mathscr{V}''(0) < 0$ so that the solutions show the desired asymptotics. By [15, theorem D.1], using the assumption that $\mathscr{V}''(0)$ is not an L^2 eigenvalue of $\Delta_{\mathring{\mathfrak{g}}}$, it follows that the linearisation $(\Delta_{\mathring{\mathfrak{g}}} - \mathscr{V}''(0))$ is an isomorphism from $C^{k+2,\alpha}_{\sigma_- + s}$ to $C^{k,\alpha}_{\sigma_- + s}$ for small $s > 0$.

Then $\mathscr{L}_X \Phi \equiv 0$ by the same argument as in the proof of lemma 4.5 above: Applying $\mathscr{L}_X$ to the equations and using $\mathscr{L}_X \mathring{\mathfrak{g}} = 0$ gives

$$\begin{cases} P(a)\mathscr{L}_X \mathfrak{g} = \mathscr{L}_X\big(q_1[V - \mathring{V}, g - \mathring{g}, \theta, \Phi]\big), \\ (\Delta_{\mathring{\mathfrak{g}}} - \mathscr{V}''(0))\mathscr{L}_X \Phi = q_2[V, g, \theta, \Phi]. \end{cases} \tag{4.19}$$

Here q_1 is a linear combination of the energy-momentum tensor of the scalar field and its trace times the metric, and is at least quadratic in its arguments and their derivatives, so that $\mathscr{L}_X q_1$ is a linear first-order differential operator in $(\mathscr{L}_X \mathfrak{g}, \mathscr{L}_X \Phi)$. Furthermore, each term in q_2 is linear in Φ or its derivatives and contains an $\mathscr{L}_X$ derivative of one of the arguments. Φ behaves asymptotically as in [15], i.e. $\Phi = \rho^{\sigma_-}\widehat{\Phi} + o(\rho^{\sigma_-})$ and, using $\mathscr{L}_X \widehat{\Phi} = 0$, we have $\mathscr{L}_X \Phi = o(\rho^{\sigma_-})$. As the coefficients of $\mathscr{L}_X \Phi$ and $\mathscr{L}_X \mathfrak{g}$, and of their first derivatives, on the right-hand side of (4.19) are small in the relevant spaces (e.g. the coefficients of the $\mathscr{L}_X \Phi$ terms on the right-hand side of the second equation are small in $C^{k,\alpha}_0$), we can use proposition 3.1 and the isomorphism properties of $(\Delta_{\mathring{\mathfrak{g}}} - \mathscr{V}''(0))$ described above to conclude that $\mathscr{L}_X \mathfrak{g} \equiv 0$ and $\mathscr{L}_X \Phi \equiv 0$.

Next, we show that V, g, θ, Φ are even functions of a: $\psi^*(\mathfrak{g}(a), \Phi(a))$ satisfy the same equations as $(\mathfrak{g}(-a), \Phi(-a))$, with identical asymptotic data (since $\widehat{\Phi}$ is independent of a, the only relevant terms are the asymptotic data for $\mathfrak{g}(a)^{0j} = -iag(a)^{jk}\theta(a)_k$ which are unchanged under $(t, a) \mapsto (-t, -a)$) and by uniqueness we have $\Phi(a) = \psi^*\Phi(a) = \Phi(-a)$, similarly for V, g, θ.

Therefore $\Phi(a)$, $V(a)$, $g_{ij}(a)$, $\theta_i(a)$ are real for real a by the same argument as for $V(a)$, $g_{ij}(a)$, $\theta_i(a)$ above.

Rather similar considerations apply for the whole system of Einstein–Maxwell-dilaton-Yang–Mills–Higgs–Chern–Simons-scalar fields equations: The matter equations arising from the action (4.13) are

$$\begin{cases} \frac{1}{V\sqrt{\det g}}\partial_\mu(V\sqrt{\det g}WF^{\mu\nu}) + B^\nu_{\rm CS} = 0, \\ \frac{1}{V\sqrt{\det g}}\partial_\mu(V\sqrt{\det g}\mathfrak{g}^{\mu\nu}\partial_\nu\Phi) - W'(\Phi)|F|^2 - \mathscr{V}'(\Phi) = 0, \end{cases} \tag{4.20}$$

where[7]

$$B^\nu_{\rm CS} = \begin{cases} 0, & n \text{ is odd;} \\ -\frac{\lambda}{2^{k+2}}\varepsilon^{\nu\alpha_1\beta_1\cdots\alpha_k\beta_k}F_{\alpha_1\beta_1}\cdots F_{\alpha_k\beta_k}, & n = 2k. \end{cases} \tag{4.21}$$

After replacing $\mathrm{d}t$ by $-\mathrm{i}\mathrm{d}t$, the asymptotic data for the Riemannian solution, say $\widehat{F}(a)$, take the form $\widehat{F}(a) = d(-\mathrm{i}a\widehat{U}\mathrm{d}t + \widehat{A}_i\mathrm{d}x^i)$. They are clearly invariant under

$$t \mapsto -t \text{ and } a \mapsto -a. \tag{4.22}$$

The only other asymptotic data that are possibly affected by (4.22) are those associated with the inverse metric components $\mathfrak{g}(a)^{0j}$. These change sign under each of $a \mapsto -a$ and $t \mapsto -t$. It follows

$$\psi^*(F(-a), \mathfrak{g}(-a), \Phi(-a)) = (F(a), \mathfrak{g}(a), \Phi(a)),$$

which again implies that $(U(a), A(a), \Phi(a), V(a), \theta(a), g(a))$ are even functions of a. As before, analyticity holds and we conclude that all these fields are real for real a.

Note that the implicit function theorem in the Riemannian regime produces essentially complex electric fields for real a and non-zero $\widehat{U}$'s, which will however be mapped to real ones when one returns to the Lorentzian setting. □

4.3. $\boldsymbol{f(R)}$ theories

Our method allows the construction of black-hole solutions to specific $f(R)$ theories: As described in e.g. [18, section 2.3] these can be reduced to the Einstein-scalar field equations with a specific potential $\mathscr{V}(\Phi)$ by a conformal transformation, if the function f fulfills certain conditions. These conditions are satisfied simultaneously with our assumptions on $\mathscr{V}$ (in theorem 4.8) if

$$f' > 0, \quad f'' \neq 0, \quad f'^{-1}(1) < 0, \quad f(f'^{-1}(1)) = f'^{-1}(1)/2. \tag{4.23}$$

(this is shown in detail in [15, section 5.5]). An example of a function f which fulfills these conditions is

[7] One can check by a direct time-and-space decomposition of the equations that the 'Wick rotation' $\mathrm{d}t \to -\mathrm{i}\mathrm{d}t$, $\partial_t \to \mathrm{i}\partial_t$, is consistent with the Chern–Simons terms in the equations by defining $\varepsilon^{\alpha_1\cdots\alpha_d}$ as $(-\det \mathbf{g})^{-1/2}\mathring{\varepsilon}^{\alpha_1\cdots\alpha_d}$, where $\mathring{\varepsilon}^{\alpha_1\cdots\alpha_k}$ is totally antisymmetric with values in $\{0, \pm 1\}$, with the cut in the definition of $\sqrt{z}$, $z \in \mathbb{C}$, lying e.g. on the positive imaginary axis, so that $\sqrt{z^2} = z$ both near $z = -i$ and near $z = 1$.

$$f(\tilde{R}) = \mathrm{d}\tilde{R} + c\tilde{R}^{\alpha+1} + e, \tag{4.24}$$

where $\tilde{R}$ is the Ricci scalar in the $f(\tilde{R})$ theory (i.e. before the conformal transformation) and $d < 1, c < 0, \alpha = 1, 3, 5, \ldots$, and

$$e = \frac{\alpha(1-2d)-1}{2(\alpha+1)} \sqrt[\alpha]{\frac{1-d}{c(\alpha+1)}},$$

are constants.

4.4. Time-periodic scalar fields

Similarly to [15, section 6.1] we can use the method there to construct solutions with a time-periodic complex scalar field $\Phi(t,x) = \mathrm{e}^{\mathrm{i}\omega t}\psi(x)$ where $\omega \in \mathbb{R}$ is a constant and $\psi(x)$ is allowed to be complex. We assume that

$$\mathscr{V}(\Phi) = G_{\mathscr{V}}(\Phi\bar{\Phi}) \text{ and } W(\Phi) = G_W(\Phi\bar{\Phi}) \tag{4.25}$$

for some differentiable functions $G_{\mathscr{V}}$ and G_W, and replace the term $(\nabla\Phi)^2$ in the action by $\nabla^\alpha\overline{\Phi}\nabla_\alpha\Phi$, where $\overline{\Phi}$ is the complex conjugate of Φ.

The Lorentzian Φ equation for a complex scalar field $\Phi = \mathrm{e}^{\mathrm{i}\omega t}\psi$ takes the form

$$\Delta_{\mathbf{g}}(\mathrm{e}^{\mathrm{i}\omega t}\psi) - G_W{}'(\psi\bar{\psi})\mathrm{e}^{\mathrm{i}\omega t}\psi|F|^2 - G_{\mathscr{V}}{}'(\psi\bar{\psi})\mathrm{e}^{\mathrm{i}\omega t}\psi = 0. \tag{4.26}$$

This leads to the following associated Riemannian equation

$$\begin{aligned} &\Delta_{\mathfrak{g}}(\mathrm{e}^{a\omega t}\psi) - G_W{}'(\psi\bar{\psi})\mathrm{e}^{a\omega t}\psi|F|^2 - G_{\mathscr{V}}{}'(\psi\bar{\psi})\mathrm{e}^{a\omega t}\psi = 0 \\ &\Longleftrightarrow \\ &0 = \Delta_{\mathfrak{g}}\psi + 2a\omega\mathfrak{g}(Dt, D\psi) + |\mathrm{d}t|^2_{\mathfrak{g}}a^2\omega^2\psi + a\omega\psi\Delta_{\mathfrak{g}}t \\ &\qquad -G_W{}'(\psi\bar{\psi})\psi|F|^2 - G_{\mathscr{V}}{}'(\psi\bar{\psi})\psi\psi, \end{aligned} \tag{4.27}$$

where the crucial difference to a naive replacement $t \mapsto -\mathrm{i}t$ (and therefore $\Phi \mapsto \mathrm{e}^{\omega t}\psi$) is that the argument of $G_W{}'$ and $G_{\mathscr{V}}{}'$ is $\psi\bar{\psi}$ instead of $\Phi\bar{\Phi} = \mathrm{e}^{2\omega t}\psi\bar{\psi}$. The equations (4.26) and (4.27), together with the respective Lorentzian and Riemannian equations for the other variables, are equivalent: The bijection

$$(V, \theta, g, U, A, \omega, \psi) \mapsto (iV, \mathrm{i}\theta, g, -\mathrm{i}U, A, -\mathrm{i}\omega, \psi) \tag{4.28}$$

maps Lorentzian solutions to Riemannian ones.

As such, the first equation (4.27) does not make sense for periodic t's, but the second does. Note, however, that Dt and $|\mathrm{d}t|^2$ are singular at an axis of rotation of ∂_t, if there is one. This forces us to restrict ourselves to strictly stationary configurations, without black holes, when $\omega \neq 0$. As a consequence, in this section we merely reproduce the results already proved in [15] for rotating complex fields, albeit by a somewhat simpler argument.

Applying $\mathscr{L}_X$ to the second equation in (4.27) gives

$$\begin{aligned} 0 =& \Delta_{\mathfrak{g}}(\mathscr{L}_X\psi) + 2\omega\mathfrak{g}(Dt, D\mathscr{L}_X\psi) - \mathscr{L}_X\left(G_{\mathscr{V}}{}'(\psi\bar{\psi})\psi + G_W{}'(\psi\bar{\psi})\psi|F|^2\right) \\ &+ O(\psi\mathscr{L}_X\mathfrak{g}) + O(D\psi\mathscr{L}_X\mathfrak{g}) + O(DD\psi\mathscr{L}_X\mathfrak{g}) + O(\rho\mathscr{L}_X\psi), \end{aligned} \tag{4.29}$$

and therefore

$$(\Delta_{\mathring{\mathfrak{g}}} - G_{\mathscr{V}}{}'(0))\,(\mathscr{L}_X\psi) = q_3[\mathfrak{g}, \psi, U, A],$$

where each term in q_3 is at least linear in its arguments or their derivatives and contains an $\mathscr{L}_X$ derivative of $\mathfrak{g}$, ψ, U, or A.

We can now argue as before to obtain $\mathscr{L}_X(\mathfrak{g}, \psi, U, A) \equiv 0$ if the asymptotic data are invariant under $\mathscr{L}_X$.

The equation for ψ is then

$$V^{-1}D_i(Vg^{ij}\partial_j\psi) - \big(G_W{}'(\psi\bar{\psi})|F|^2 + G_{\mathscr{V}}{}'(\psi\bar{\psi})\big)\psi \\ + (V^{-2} - a^2\theta_k\theta^k)a^2\omega^2\psi + \mathrm{i}a^2\omega(\theta^j\partial_j\psi + V^{-1}D_j(V\theta^j\psi)) = 0. \tag{4.30}$$

All terms in this equation are well defined and, by the results of [15], we obtain a solution of the complete system of equations. As ψ is independent of t, no difficulties associated with the periodicity of the t coordinate arise. After transforming back via the inverse of (4.28) we obtain a time-periodic solution $\Phi(t, x) = \mathrm{e}^{\mathrm{i}\omega t}\psi(x)$ to the original equations.

5. Geometry of the solutions

We wish to show that the solutions constructed above with topology $\mathfrak{M} = \mathbb{R}^2 \times {}^{n-1}\mathcal{N}$ correspond to smooth black holes on the Lorentzian side. (In fact, the Lorentzian metric will be one-sided-analytic up-to-horizon [6] near the horizon, but this is irrelevant for the problem here.) For this, we recall some standard facts about isometries. Let us denote by

$$\mathscr{Z} := \{0\} \times {}^{n-1}\mathcal{N} \tag{5.1}$$

the codimension-two submanifold of $\mathfrak{M}$ which is the zero-set of the Killing vector X generating rotations of $\mathbb{R}^2$. Then $\mathscr{Z}$ is a totally-geodesic submanifold of $(\mathfrak{M}, \Re\mathfrak{g})$. In coordinates (x, y) normal for the metric $\Re\mathfrak{g}$, on each of the planes $\Re\mathfrak{g}$–orthogonal to $\mathscr{Z}$ the vector field X takes the standard Euclidean form

$$X = x\partial_y - y\partial_x.$$

This shows that in these coordinates a rotation R_π by an angle π, which is the map $(x, y) \mapsto (-x, -y)$, is an isometry of $\Re\mathfrak{g}$ which leaves invariant $\Im\mathfrak{g}$. Let us choose local coordinates (x^a) on $\mathscr{Z}$, and extend them to be constant along $\Re\mathfrak{g}$–geodesics $\Re\mathfrak{g}$–normal to $\mathscr{Z}$. We will denote by (x^A) the coordinates (x, y). One obtains

$$\Re\mathfrak{g}_{ab}(x, y, x^c) = \Re\mathfrak{g}_{ab}(-x, -y, x^c), \quad \Re\mathfrak{g}_{AB}(x, y, x^c) = \Re\mathfrak{g}_{AB}(-x, -y, x^c),$$

$$\Re\mathfrak{g}_{aA}(x, y, x^c) = -\Re\mathfrak{g}_{aA}(-x, -y, x^c).$$

In particular all odd-order derivatives of the metric functions $\Re\mathfrak{g}_{ab}$ and $\Re\mathfrak{g}_{AB}$ vanish on $\mathscr{Z}$.

An analogous argument applies to θ using $\mathscr{L}_X\theta = 0$.

Let us assume for definiteness that $a \in \mathbb{R}$, thus θ is purely real. It is then standard to derive the following form of the metric in coordinates (φ, ρ, x^a), where $(x, y) = (\rho\cos\varphi, \rho\sin\varphi)$ (compare [12, section 3] for detailed calculations in a closely related setting):

$$\Re\mathfrak{g} = u^2\mathrm{d}\varphi^2 + h_{jk}\mathrm{d}x^j\mathrm{d}x^k, \quad u = \rho(1 + \rho^2\psi), \tag{5.2}$$

$$\theta = \frac{\alpha\rho}{(1 + \rho^2\psi)^2}\mathrm{d}\rho + \gamma_a\mathrm{d}x^a, \tag{5.3}$$

$$h_{jk}\mathrm{d}x^j\mathrm{d}x^k = (1 + \rho^2\beta)\mathrm{d}\rho^2 + b_{ab}\mathrm{d}x^a\mathrm{d}x^b + 2\rho\lambda_a\mathrm{d}x^a\mathrm{d}\rho - u^2\theta_i\theta_j\mathrm{d}x^i\mathrm{d}x^j, \tag{5.4}$$

and where all the non-explicit functions are smooth functions of (ρ^2, x^a).

Passing to the Lorentzian regime, and replacing φ by a coordinate

$$\tau = \varphi + \log \rho, \tag{5.5}$$

one checks that the Lorentzian metric $\mathbf{g}$ smoothly extends to a Killing horizon at $\rho = 0$ after a final change of coordinates $\rho \to z = \frac{1}{2}\rho^2$. Indeed, the Lorentzian metric $\mathbf{g}$ is then given by

$$\begin{aligned} \mathbf{g} =& - u^2(\mathrm{d}\tau^2 - \frac{2}{\rho}\mathrm{d}\tau\mathrm{d}\rho) + \rho^2(\beta - 2\psi - \rho^2\psi^2)\mathrm{d}\rho^2 \\ &+ b_{ab}\mathrm{d}x^a\mathrm{d}x^b + 2\rho\lambda_a\mathrm{d}x^a\mathrm{d}\rho - u^2\theta_i\theta_j\mathrm{d}x^i\mathrm{d}x^j \\ =& - 2(1+\rho^2\psi)^2(z\,\mathrm{d}\tau^2 - \mathrm{d}\tau\mathrm{d}z) + (\beta - 2\psi - \rho^2\psi^2)\mathrm{d}z^2 \\ &+ b_{ab}\mathrm{d}x^a\mathrm{d}x^b + 2\lambda_a\mathrm{d}x^a\mathrm{d}z - 2z(1+\rho^2\psi)^2\theta_i\theta_j\mathrm{d}x^i\mathrm{d}x^j\,, \end{aligned} \tag{5.6}$$

after substituting $\mathrm{d}\Phi \mapsto \mathrm{i}\mathrm{d}\Phi$ in (5.2) and applying the coordinate transformation (5.5).

Indeed, (5.6) shows that that the Killing vector ∂_τ is null on the hypersurface $\{z = 0\}$, and that this hypersurface is null, hence a Killing horizon. This is a *non-rotating* horizon, in the sense that the Killing vector which is timelike at infinity is also tangent to the Killing horizon. (This explains why our solutions, which can have no further symmetries than stationarity, are compatible with the Hollands–Ishibashi–Wald [23] rigidity theorem, which provides at least one more symmetry for rotating horizons.) We also see from (5.6) that the surface gravity of the Killing horizon $\{z = 0\}$, calculated for the vector field ∂_τ, equals one. Rescaling τ to the scale of the original nearby seed Birmingham solution, the surface gravity of our solutions will coincide with that of the seed metric in those cases with matter sources where the gravitational free data at infinity have been chosen to coincide with the original ones; otherwise a nearby surface gravity will result when the asymptotic behaviour of the metric imposes a natural rescaling of the horizon Killing vector field.

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NON-DEGENERACY OF RIEMANNIAN SCHWARZSCHILD-ANTI DE SITTER METRICS: BIRKHOFF-TYPE RESULTS IN LINEARIZED GRAVITY

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ABSTRACT. We prove Birkhoff-type results showing that L^2 solutions of the linearized Einstein equations around Riemannian Kottler ("Schwarzschild-anti de Sitter") metrics in arbitrary dimension and horizon topology, which are not controlled by "master functions" are pure gauge. Together with earlier results this implies that the TT-gauge-fixed linearized Einstein operator for these metrics is non-degenerate for open ranges of the mass parameter.

1. INTRODUCTION

In a recent paper, together with Piotr Chruściel and Erwann Delay, we showed that the linearized Einstein operator at a subset of Riemannian Kottler metrics has no L^2 kernel[4]. This was motivated by [3] which gives, for each metric fulfilling this condition, a large class of new stationary black hole spacetimes.

Here we extend the results of [4] to a wider range of dimensions and horizon geometries. In fact the only thing we have to show is that all L^2 solutions of the linearized Einstein equations around Riemannian (generalized) Kottler metrics with negative cosmological constant, which are not controlled by the "master functions" of Kodama & Ishibashi [7], have to be pure gauge (except for the case of the critical mass value for spherical horizon geometry). This corresponds to showing that all solutions of the linearized Einstein equations with certain symmetry have to take a fixed form, i.e. a result similar to the Birkhoff theorem in full gravity (see Section 1.1 below).

Similar results are contained in [4, Appendices F–I] for spacetime dimension $n+2=4$ for $K \in \{1,-1\}$ and arbitrary dimension for $K=0$ (K is the (constant) sectional curvature of the horizon). Replacing these with the results proved below extends the conclusions of [4] to the stronger

THEOREM A. *Let us denote by P_L the linearization, at Riemannian Kottler metrics (2.2) with negative cosmological constant, of the TT-gauge-fixed Einstein operator. Then:*

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(1) *P_L has no L^2-kernel in spacetime dimension $n+2=4$ except for spherical black holes with mass parameter*

$$\mu = \mu_c := \frac{n}{n+1}\left(\ell\sqrt{\frac{n-1}{n+1}}\right)^{n-1} . \tag{1.1}$$

(2) *P_L has no L^2-kernel for open ranges of parameters $\mu \in (\mu_{min}(K), \mu(n))$ for $n > 2$, where $\mu(n) > \mu_{min}(K)$ solves a polynomial equation and*

$$\mu_{min}(K) := \begin{cases} 0 & K \in \{0,1\}, \\ -\frac{1}{n+1}\left(\frac{n+1}{\ell^2(n-1)}\right)^{\frac{1-n}{2}} & K = -1 . \end{cases} \tag{1.2}$$

(In contrast to the result of [4] we do not have to restrict to the case $K = 0$ for dimensions $n > 2$.)

In [4] it is conjectured that

Conjecture B. *P_L has no L^2-kernel except if $K = 1$ and μ is given by* (1.1).

With our results the only missing part to prove Conjecture B is a rigorous justification of the numerical arguments in [4, Section 3.2].

As mentioned above, the motivation to study the L^2 Kernel of P_L comes from [3]. Indeed, a trivial L^2 kernel of P_L for a Riemannian black hole metric $\mathring{\mathfrak{g}}$ implies the existence of infinite dimensional families of non-singular, stationary Lorentzian black hole solutions to the Einstein equations with negative cosmological constant, in vacuum or with various matter fields, and with conformal infinity close to that of a Lorentzian metric associated to $\mathring{\mathfrak{g}}$.

Theorem A thus implies the existence of such solutions in all spacetime dimensions and for flat, negatively, or positively curved conformal infinity.

1.1. **The Birkhoff theorem.** Our results can be understood as a linearized analogue to the Birkhoff theorem. Several different kinds of results have been referred to as "Birkhoff theorems" in the literature (see [14] for an overview). Here we will use the term to mean a classification result showing that under certain symmetry assumptions on a manifold the metric has to take a fixed form (which contains an additional Killing vector field). A classical result of this form is that spherically symmetric vacuum spacetimes are given by the Schwarzschild metric. As far as we are aware the most general such result is [1, Theorem 3.2]. This theorem applies to various kinds of Einstein-matter systems and, in fact, does not even require the full Einstein equations to be satisfied. Specializing to the case of solutions to the vacuum Einstein equations with cosmological constant it states

Theorem 1.1 (Birkhoff theorem for warped product vacuum spacetimes [1]). *Consider a warped product spacetime $(M = Q \times F, \bar{g} = g + r^2 h)$ satisfying the vacuum Einstein equations with cosmological constant Λ, where (Q, g) is a 2-dimensional manifold, (F, h) an $n \geq 2$ dimensional one and r is a function on Q. Then*

(1) *either $\bar{g}$ takes the standard Eddington-Finkelstein form*

$$\bar{g} = -\left(\frac{S^{[h]}}{n(n-1)} - \frac{2m}{r^{n-1}} - \frac{2\Lambda}{n(n+1)}r^2\right) du^2 \pm 2du dr + r^2 h\,,$$

where $S^{[h]} = const$ is the scalar curvature of h,

(2) *or $\Lambda = 0$, $R^{[h]}_{ij} = 0$, and*

$$\bar{g} = -dt^2 + dr^2 + (t \pm r)^2 h\,,$$

(3) *or r is constant, (Q, g) is maximally symmetric, (F, h) is Einstein, $S^{[h]} = 2r^2\Lambda$, and $S^{[g]} = 4\Lambda/n$.*

When (F, h) is $\mathbb{S}^n$ with the round metric this reduces to the classic Birkhoff theorem. In that case (2) does not apply, and (3) gives a limit case of (1) which cannot be described in the standard coordinates (see [13, Section 4]).

In Section 3 we consider perturbations of (Riemannian) Kottler metrics such that, in terms of the variables in Theorem 1.1, $\delta h \propto h$ and δg is constant on F. We conclude that the only such perturbations which satisfy the linearized Einstein equations are variations of the mass parameter, i.e. ones that (at the linear level) stay in the Kottler family. This is directly analogous to the Birkhoff theorem, with (F, h) being the spaces of constant sectional curvature which appear in the Kottler metrics.

In Section 4 we consider axially symmetric perturbations, and conclude that the only ones satisfying the linearized Einstein equations are variations of the angular momentum parameter in the (Riemannian) Kerr anti-de Sitter family. This result is of a similar type as the Birkhoff theorem but has no direct analogue in the nonlinear case.

2. Definitions & Background

We will consider the linearized Einstein equations on a Riemannian (generalized) Kottler [10] background (also referred to as "Schwarzschild Anti-de Sitter metrics" or "Birmingham metrics" [2]). These $n+2$ dimensional solutions of the Einstein equations are given by the manifold

$$M = S^1 \times [r_0, \infty) \times {}^n N_K \tag{2.1}$$

where $({}^n N_K, \gamma)$ is an n-dimensional space of constant sectional curvature $K \in \{-1, 0, 1\}$, together with the metric

$$\mathring{\mathfrak{g}} = \underbrace{\left(\frac{r^2}{\ell^2} + K - \frac{2\mu}{r^{n-1}}\right)}_{=:f(r)} dt^2 + \frac{dr^2}{\frac{r^2}{\ell^2} + K - \frac{2\mu}{r^{n-1}}} + r^2\gamma\,, \tag{2.2}$$

where t is a periodic coordinate on S^1 with period

$$T := \frac{f'(r_0)}{4\pi} > 0\,,$$

the parameter ℓ is related to the cosmological constant by

$$\ell = \sqrt{-\frac{n(n+1)}{2\Lambda}} > 0\,,$$

and $r_0 > 0$ is the largest zero of f. Note that $r = r_0$ is the axis of rotation for the "angular" coordinate t.

We use μ,ν,... for spacetime indices, a,b,... for indices on $S^1 \times [r_0,\infty)$ and i,j,... for those on ${}^nN_K$. We will denote by $\widehat{D}_i$, $\widehat{\Delta} := \gamma^{ij}\widehat{D}_i\widehat{D}_j$ the covariant derivative and Laplace-Beltrami operator on $({}^nN_K,\gamma)$ and by $\tilde{D}_a$, $\tilde{\Delta} := \tilde{D}^a\tilde{D}_a$ the corresponding operators on $(S^1\times[r_0,\infty), fdt^2+f^{-1}dr^2)$.

A symmetric 2-covariant tensor h on M can be split into "scalar", "vector", and "tensor" parts according to their behavior under diffeomorphisms acting on the n-dimensional submanifold ${}^nN_K$[9]:

$$h = h^S + h^V + h^T\,. \tag{2.3}$$

The three parts in (2.3) can be expanded into modes as [7, Sections 2.1, 5.1 and 5.2]

$$h^S_{ab} = \sum_I f^S_{ab,I}\mathbb{S}^I\,,\quad h^S_{ai} = \sum_I rf^S_{a,I}\mathbb{S}^I_i\,,\quad h^S_{ij} = \sum_I 2r^2(H^S_{L,I}\gamma_{ij}\mathbb{S}^I + H^S_{T,I}\mathbb{S}^I_{ij})\,, \tag{2.4}$$

$$h^V_{ab} = 0\,,\qquad h^V_{ai} = \sum_I rf^V_{a,I}\mathbb{V}^I_i\,,\quad h^V_{ij} = \sum_I 2r^2H^V_{T,I}\mathbb{V}^I_{ij}\,, \tag{2.5}$$

$$h^T_{ab} = 0\,,\qquad h^T_{ai} = 0\,,\qquad h^T_{ij} = \sum_I 2r^2H^T_{T,I}\mathbb{T}^I_{ij}\,, \tag{2.6}$$

where the $\mathbb{S}^I$, $\mathbb{V}^I_i$, $\mathbb{T}^I_{ij}$ are scalar, vector, and (symmetric, transverse, and traceless) tensor harmonics, i.e.

$$(\widehat{\Delta}_n + k^2)\mathbb{S}^I = 0\,,\quad (\widehat{\Delta}_n + k_V^2)\mathbb{V}^I_i = 0\,,\quad (\widehat{\Delta}_n + k_T^2)\mathbb{T}^I_{ij} = 0\,, \tag{2.7}$$

$$\mathbb{T}^I_{ij} = \mathbb{T}^I_{ji}\,,\qquad \gamma^{ij}\widehat{D}_i\mathbb{T}^I_{jk} = 0\,,\qquad \gamma^{ij}\mathbb{T}^I_{ij} = 0\,, \tag{2.8}$$

with eigenvalues k^2, k_V^2, k_T^2 and

$$\mathbb{S}^I_i = -\frac{1}{k}\widehat{D}_i\mathbb{S}^I\,,\quad k \neq 0\,, \tag{2.9}$$

$$\mathbb{S}^I_{ij} = \frac{1}{k^2}\widehat{D}_i\widehat{D}_j\mathbb{S}^I + \frac{1}{n}\gamma_{ij}\mathbb{S}^I\,,\quad k \neq 0\,, \tag{2.10}$$

$$\mathbb{V}^I_{ij} = -\frac{1}{2k_V}(\widehat{D}_i\mathbb{V}^I_j + \widehat{D}_j\mathbb{V}^I_i) = -\frac{1}{2k_V}\mathcal{L}_{\mathbb{V}^I}\gamma_{ij}\,,\quad k_V \neq 0\,, \tag{2.11}$$

with the corresponding quantities vanishing if $k = 0$ or $k_V = 0$. For the case $K = 1$ the eigenvalues are [12]

$$k^2 = l(l+n-1)\,,\qquad l = 0,1,2,\ldots\,, \tag{2.12}$$

$$k_V^2 = l(l+n-1) - 1\,,\qquad l = 1,2,3\ldots\,, \tag{2.13}$$

$$k_T^2 = l(l+n-1) - 2\,,\qquad l = 2,3,4,\ldots\,,\quad n > 2\,. \tag{2.14}$$

By [9, Appendix B], using the fact that $({}^nN_K,\gamma)$ is a space of constant curvature, the scalar, vector, and tensor parts of a solution to the linearized Einstein equations separately satisfy the equations.

Kodama and Ishibashi [7] introduced master functions, scalar functions $\Phi_{i,I}$ on the t,r space, satisfying

$$\tilde{\Delta}\Phi_{i,I} - V_{i,I}\Phi_{i,I} = 0\,,\quad i \in \{S,V,T\}\,, \tag{2.15}$$

where the $V_{i,I}(r)$ are some complicated potentials given in [7, p. 8, 13, 14]. These master functions control the behavior of perturbations for all modes for which they are defined. In [4, Section 3 & 4] it is shown that whenever the master functions are defined they can be used to prove that there are no L^2 solutions of the linearized Einstein equations.

The remaining cases, which have to be treated separately, are

(1) the $l = 0$ scalar and vector modes, i.e. those where $k = 0$ or $k_V = 0$,
(2) the $l = 1$ scalar and vector modes for $K = 1$.

We show in the following that L^2 perturbations of this form are purely gauge. The first case will be treated in Section 3 and the second one in Section 4.

For further reference we note that gauge transformation $h_{\mu\nu} \to h_{\mu\nu} + \mathcal{L}_Y \mathring{\mathfrak{g}}_{\mu\nu}$, of perturbations h, with (small) gauge vector Y, take the form

$$h_{tt} \to h_{tt} + Y^r \partial_r f + 2f\partial_t Y^t \,, \tag{2.16}$$

$$h_{tr} \to h_{tr} + f^{-1}\partial_t Y^r + f\partial_r Y^t \,, \tag{2.17}$$

$$h_{rr} \to h_{rr} + Y^r \partial_r f^{-1} + 2f^{-1}\partial_r Y^r \,, \tag{2.18}$$

$$h_{ti} \to h_{ti} + f\partial_i Y^t + r^2\gamma_{ki}\partial_t Y^k \,, \tag{2.19}$$

$$h_{ri} \to h_{ri} + r^2\gamma_{ik}\partial_r Y^k + f^{-1}\partial_i Y^r \,, \tag{2.20}$$

$$h_{ij} \to h_{ij} + 2rY^r\gamma_{ij} + r^2(\widehat{D}_i(\gamma_{jk}Y^k) + \widehat{D}_j(\gamma_{ik}Y^k)) \,. \tag{2.21}$$

By [11, Proposition 6.5 and Proposition E] elements of the L^2 kernel of $P_L = \Delta_L + 2(n+1)$ (see [4, Section 2]) behave as $|h|_{\mathring{\mathfrak{g}}} = O(r^{-n-1})$ for $r \to \infty$ which gives for the components $h_{\mu\nu}$

$$\begin{aligned} h_{tt} &= O(r^{1-n})\,, \quad h_{tr} = O(r^{-1-n})\,, \quad h_{rr} = O(r^{-3-n})\,, \\ h_{tj} &= O(r^{1-n})\,, \quad h_{rj} = O(r^{-1-n})\,, \quad h_{jk} = O(r^{1-n})\,. \end{aligned} \tag{2.22}$$

3. The $l = 0$ modes for $K \in \{-1, 0, 1\}$

In this section we show that L^2 solutions of the linearized Einstein equations consisting only of $l = 0$ modes have to be pure gauge.

For the cases $K = 1$ and $K = -1$ we only have to consider the scalar part: The tensor part is always controlled by the master functions and there are no (non-zero) harmonic vectors (i.e. vectors with $k_V = 0$) for $K \in \{1, -1\}$. For $K = 1$ this can be read of directly from (2.13). For $K = -1$ we consider the Hodge Laplacian

$$\widehat{\Delta}_H \mathbb{V}_i := (dd^*\mathbb{V} + d^\star d\mathbb{V})_i = -\widehat{\Delta}\mathbb{V}_i + {}^nR_i{}^j\mathbb{V}_j \,,$$

(see e.g. [6]). Using the fact that $\widehat{\Delta}_H$ is non-negative and that ${}^nR_{ij}X^iX^j = (n-1)K$ for all unit vectors X (as $({}^nN_K, \gamma)$ has constant curvature) we obtain, for $K = -1$, $k_V^2 \geq n - 1 > 0$.

3.1. Scalar perturbations.

We consider the scalar part of a $l = 0$ linearized solution $h_{\mu\nu}$ of the Einstein equations, i.e.

$$h_{ab} = h_{ab}(t, r)\,, \quad h_{ia} \equiv 0\,, \quad h_{ij} = \psi(t, r)\mathring{\mathfrak{g}}_{ij} = \psi(t, r)r^2\gamma_{ij}\,, \tag{3.1}$$

and assume that $h \in L^2$.

The angular part of the perturbation can be gauged away by defining a gauge vector Y as

$$Y^r = r\psi/2 = O(r^{-n}), \quad Y^i \equiv 0, \tag{3.2}$$

which implies

$$h_{ij} = \mathscr{L}_Y \mathring{\mathfrak{g}}_{ij}.$$

The remaining component Y^t of the gauge vector allows us to do the same for h_{tr}, by integrating (2.17) in r. However, it is not a priori clear that the resulting gauge vector is smooth at $r = r_0$. We circumvent this problem by cutting off at a finite distance ε from r_0, i.e. by defining a gauge vector Y_ε as $Y^r_\varepsilon = Y^r$, and

$$Y^t_\varepsilon = -\chi_\varepsilon(r) \int_r^\infty f^{-1}(h_{tr} - \partial_t Y^r f^{-1}) dr = O(r^{-n-2}), \tag{3.3}$$

where χ_ε is a smooth function such that $\chi_\varepsilon \equiv 1$ for $r > r_0 + \varepsilon$ and $\chi_\varepsilon \equiv 0$ for $r < r_0 + \varepsilon/2$. With this definition we have, for $r > r_0 + \varepsilon$,

$$h_{tr} = \mathscr{L}_{Y_\varepsilon} \mathring{\mathfrak{g}}_{tr}.$$

We set

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \mathscr{L}_{Y_\varepsilon} \mathring{\mathfrak{g}}_{\mu\nu}, \tag{3.4}$$

thus $\bar{h}_{\mu\nu}$ is a solution of the linearized Einstein equations with, for $r > r_0+\varepsilon$, all components vanishing except possibly $\bar{h}_{tt}$ and $\bar{h}_{rr}$.

We now define new functions Z_r and Z_t as

$$Z_r := r^{n-1} f^2 \bar{h}_{rr}, \quad Z_t := r^{n-1}(\bar{h}_{tt} + f^2 \bar{h}_{rr}), \tag{3.5}$$

chosen such that a variation of the mass in the coordinates of (2.2), which takes the form

$$2\frac{\delta\mu}{r^{n-1}}(-dt^2 + f^{-2} dr^2), \tag{3.6}$$

is captured purely by Z_r.

Using [8, Appendix B] we can write the linearized Einstein equations for our perturbation in terms of Z_r and Z_t.

For the t, r equation we find, for $r > r_0 + \varepsilon$,

$$G'_{tr}[h] = \frac{n \partial_t Z_r(t, r)}{2(-2\mu r + r^n + r^{2+n})}, \tag{3.7}$$

thus Z_r depends at most upon r. One can now eliminate the second radial derivative of Z_t between the G_{tt} and G_{rr} equations, obtaining, again for $r > r_0 + \varepsilon$,

$$\partial_r \left(\frac{Z_t}{r^{n-1} f} \right) = 0. \tag{3.8}$$

Hence, for $r > r_0 + \varepsilon$,

$$Z_t = C(t) r^{n-1} f. \tag{3.9}$$

for some function C depending only upon t. Inserting all this into the $G_{ij} = 0$ equations gives, for r as before, $\partial_r Z_r = 0$, and thus Z_r is a constant, say $2\delta\mu$ there.

In terms of $\bar{h}_{rr}$ and $\bar{h}_{tt}$ we now have, for $r > r_0 + \varepsilon$,

$$\bar{h}_{tt} = fC(t) - \frac{2\delta\mu}{r}\,, \quad \bar{h}_{rr} = \frac{2\delta\mu}{rf^2}\,. \tag{3.10}$$

As f behaves asymptotically like r^2, C has to vanish for this to be in L^2. We find that the only scalar $l = 0$ perturbations which satisfy the linearized Einstein equations are, up to gauge, variations of the mass.

For $K \in \{0, -1\}$ the tensor field $\bar{h}_{\mu\nu}$ is in L^2 if and only if $\delta\mu = 0$, while for $K = 1$ this holds with the exception of the case $\mu = \mu_c$, with the critical mass μ_c defined in (1.1). (See e.g. [4, Section 2] for a derivation of the critical mass.)

Hence, for these cases, $\bar{h}_{\mu\nu} \equiv 0$, i.e. $h_{\mu\nu} = \mathscr{L}_{Y_\varepsilon}\mathring{\mathfrak{g}}_{\mu\nu}$, for $r > r_0 + \varepsilon$. As $\varepsilon > 0$ is arbitrary and $Y_\delta \equiv Y_\varepsilon$ for $r > r_0 + \delta$, $\delta > \varepsilon$ this applies for all $r > r_0$ with Y_0.

The tensors h and $\mathring{\mathfrak{g}}$ are smooth by assumption, so we can conclude from $h_{\mu\nu} = \mathscr{L}_{Y_0}\mathring{\mathfrak{g}}_{\mu\nu}$ that the integrand in (3.3) is smooth and bounded, implying that $Y := Y_0$ is in fact smooth for all r, including the rotation axis $r = r_0$.

We find that, except for the case of critical mass,

$$h_{\mu\nu} = \mathscr{L}_Y\mathring{\mathfrak{g}}_{\mu\nu}\,, \qquad |Y|_{\mathring{\mathfrak{g}}} = O(r^{-n-1})\,, \tag{3.11}$$

i.e. h is pure gauge.

3.2. **Vector perturbations.** For the case $K = 0$ there are (constant) harmonic vectors with $k_V = 0$. Perturbations associated with these take the form

$$h_{ab} = 0\,, \quad h_{ai} = rf_a^V\mathbb{V}_i\,, \quad h_{ij} = 0\,, \tag{3.12}$$

where the $\mathbb{V}_i$ are constants and the f_a^V are functions of t and r. Defining $\bar{h}^V$ by $h^V_{\mu\nu} = \bar{h}^V_{\mu\nu} + \mathcal{L}_Y\mathring{\mathfrak{g}}$ with a gauge vector Y chosen as $Y^a \equiv 0$ and

$$Y^i = \gamma^{ij}\mathbb{V}_i\int\frac{f_r^V}{r}dr = O(r^{-n-2})\,, \tag{3.13}$$

we obtain $\bar{h}_{ri} = 0$, i.e. $f_r^V = 0$. The removed gauge part behaves asymptotically as

$$|\mathcal{L}_Y\mathring{\mathfrak{g}}|^2_{\mathring{\mathfrak{g}}} = O(r^{-2n-2})\,.$$

We find from (4.2) that Y^i is regular at r_0, and therefore the term $f^{-1}\partial_tY^i$ which occurs in $|\mathcal{L}_Y\mathring{\mathfrak{g}}|_{\mathring{\mathfrak{g}}}$ is as well (because of the behavior of g_{tt} there). This implies that $|\mathcal{L}_Y\mathring{\mathfrak{g}}|^2_{\mathring{\mathfrak{g}}}$ is in L^2.

Inserting $\bar{h}$ into the r, i component of the linearized Einstein equations gives

$$\frac{r^n\left(r\partial_t\partial_rf_t^V - \partial_tf_t^V\right)}{2\left(r^{n+2} - 2r\mu\right)} = 0\,,$$

and therefore $f_t^V = f_t^V(r)$ by the periodicity of t. Inserting back into the t, i equation we obtain

$$r^2\partial_r^2f_t^V + rn\partial_rf_t^V - nf_t^V = 0\,,$$

which gives, after integrating,

$$f_t^V = rC_1 + r^{-n}C_2\,.$$

Here C_1 has to vanish for the perturbation to be in L^2 and C_2 has to vanish as the tensors $dtdx^i$ are not smooth at the axis of rotation $r = r_0$.

3.3. **Tensor perturbations.** Additionally, for the case $K = 0$ there are (constant) harmonic tensors with $k_T = 0$. These are actually controlled by the master functions, but for completeness we show directly that they must vanish.

The associated tensor perturbations take the form

$$h_{ab} = 0\,, \quad h_{ai} = 0\,, \quad h_{ij} = 2r^2 H_T^T \mathbb{T}_{ij}\,, \tag{3.14}$$

where $\mathbb{T}_{ij}$ is a constant tensor satisfying $\mathbb{T}_{ij}\gamma^{ij} = \mathbb{T}_{ij}\delta^{ij} = 0$ and H_T^T is a function of t and r only.

The only nontrivial linearized Einstein equation is

$$\tilde{\Delta} H_T^T + \frac{nf}{r}\partial_r H_T^T = 0\,. \tag{3.15}$$

This gives $H_T^T = 0$ by the maximum principle, as $H_T^T = O(r^{-1-n})$ from (2.22).

4. The $l = 1$ modes for $K = 1$

For $K \in \{0, -1\}$ the $l = 0$ scalar and vector modes are the only ones not controlled by the master functions of Kodama & Ishibashi. For $K = 1$ however the $l = 1$ scalar and vector modes also need to be treated separately. In this section we therefore analyze these $l = 1$ modes when $({}^nN, \gamma_{ij})$ is an n-dimensional round unit sphere. We use the equations of [8, Appendix B] and our argument is similar to that of [5] in the 2 dimensional case.

4.1. **Vector perturbations.** The $l = 1$ vector perturbations take the form

$$h^V_{\mu\nu} = \sum_{m=1}^{n(n+1)/2} \begin{pmatrix} 0 & r f^V_{a,m} J^m_i \\ r f^V_{a,m} J^m_i & 0 \end{pmatrix}\,, \tag{4.1}$$

where the $f^V_{a,m}$ are functions of t and r and $\gamma^{ij} J^m_i \partial_j$ form a basis of Killing vector fields on S^n .

Gauge transformations defined by a gauge vector Y of the form

$$Y^a = 0\,, \quad Y^i = \sum_m Y_m(t,r)\gamma^{ij} J^m_j\,,$$

preserve the form (4.1) of the perturbations. The effect of such a gauge transformation on the perturbation is given by

$$h^V_{ai} \to \sum_m (r f^V_{a,m} + r^2 \partial_a Y_m) J^m_i\,,$$

with all other components unaffected.

Defining $\bar{h}^V$ by $h^V_{\mu\nu} = \bar{h}^V_{\mu\nu} + \mathcal{L}_Y \mathring{\mathfrak{g}}$ with a gauge vector Y given by $Y^a = 0$ and

$$Y^i = \sum_m \gamma^{ij} J^m_j \int_{r_0}^{r} \frac{f^V_{r,m}(t,r')}{r'} dr' = O(r^{-n-2})\,, \tag{4.2}$$

we find that the components $\bar{h}^V_{ri}$ vanish, leaving only $\bar{h}^V_{ti}$. The norm of the gauge part is found to be

$$|\mathcal{L}_Y \mathring{\mathfrak{g}}|^2_{\mathring{\mathfrak{g}}} = O(r^{-2n-2}),$$

as before, and, as it is regular at r_0, $\mathcal{L}_Y \mathring{\mathfrak{g}} \in L^2$.

Inserting $\bar{h}$ into the r,i component of the linearized Einstein equations gives

$$\frac{r^n\left(r\partial_t\partial_r f^V_t - \partial_t f^V_t\right)}{2\left(-2\mu r + r^{n+2} + r^n\right)} = 0\,.$$

Integrating twice and using the periodicity of t we obtain

$$f^V_t = f^V_t(r)\,.$$

Inserting into the t,i equation gives

$$-nf^V_t + nr\partial_r f^V_t + r^2\partial_r^2 f^V_t = 0\,,$$

and therefore

$$f^V_t = rC_1 + r^{-n}C_2\,,$$

for constants C_1 and C_2. As the tensors $dtdx^i$ are not smooth at the axis of rotation $r = r_0$ we require $C_1 r_0 + C_2 r_0^{-n} = 0$, i.e.

$$f^V_t = C_1\frac{r^{n+1} - r_0^{n+1}}{r^n}\,.$$

Perturbations of this form are exactly variations of the angular momentum parameter a in the Riemannian Kerr anti-de Sitter family (cf. [4, Appendix J]).

As they are not in L^2 we have $\bar{h} = 0$ and

$$h^V = \mathscr{L}_Y \mathring{\mathfrak{g}}\,, \quad |Y|_{\mathring{\mathfrak{g}}} = O(r^{-n-1})\,. \tag{4.3}$$

4.2. **Scalar perturbations.** Scalar $l = 1$ solutions of the linearized Einstein equations take the form

$$(h^S_{\alpha\beta}) = \sum_m \begin{pmatrix} f^S_{ab,m}\mathbb{S}^m & rf^S_{a,m}\mathbb{S}^m_i \\ rf^S_{a,m}\mathbb{S}^m_i & 2r^2\mathbb{S}^m H^S_{L,m}\gamma_{ij}\end{pmatrix}, \tag{4.4}$$

where $\mathbb{S}^m_i = -k^{-1}\widehat{D}_i\mathbb{S}^m = -n^{-1/2}\widehat{D}_i\mathbb{S}^m$ and the $\mathbb{S}^m$ are the $l = 1$ scalar harmonics on S^n.

Under gauge transformations with gauge-vector Y of the form

$$(Y_\alpha) = (Y_a, r^2\,\widehat{D}_i X) = \sum_m(\tilde{Y}_{a,m}\mathbb{S}^m, r^2\tilde{X}_m\widehat{D}_i\mathbb{S}^m)\,, \tag{4.5}$$

where $\tilde{Y}_{a,m}$ and $\tilde{X}_m$ are functions of t and r only, $(h^S_{\alpha\beta})$ transforms to $(\bar{h}^S_{\alpha\beta})$ given by

(4.6)

$$\sum_m \begin{pmatrix} (f^S_{ab,m} + \tilde{D}_a\tilde{Y}_{b,m} + \tilde{D}_b\tilde{Y}_{a,m})\mathbb{S}^m & (rf^S_{a,m} - \sqrt{n}(\tilde{Y}_{a,m} + r^2\partial_a\tilde{X}_m))\mathbb{S}^m_i \\ (rf^S_{a,m} - \sqrt{n}(\tilde{Y}_{a,m} + r^2\partial_a\tilde{X}_m))\mathbb{S}^m_i & (2r^2H^S_L + 2fr\tilde{Y}_{r,m} - 2r^2\tilde{X}_m)\mathbb{S}^m\gamma_{ij}\end{pmatrix}.$$

We can use the gauge freedom to set $\bar{h}^S_{ai} = 0$ and $\mathring{\mathfrak{g}}^{ab}\bar{h}^S_{ab} = 0$ by choosing (X, Y_a) such that they solve the following system of equations:

$$\sqrt{n}(\tilde{Y}_{a,m} + r^2\partial_a\tilde{X}_m) = rf^S_{a,m} = O(r^{-n+1})\,, \tag{4.7}$$

$$\tilde{D}^b\tilde{Y}_{b,m} = -\tfrac{1}{2}\mathring{\mathfrak{g}}^{ab}f^S_{ab,m} = O(r^{-n-1})\,. \tag{4.8}$$

With this choice, $\bar{h}^S$ satisfies

$$\bar{h}^S_{ai} = 0\,, \qquad \mathring{\mathfrak{g}}^{ab}\bar{h}^S_{ab} = 0\,. \tag{4.9}$$

Note that (4.7)-(4.8) imply

$$\tilde{D}^b(r^2\tilde{D}_b\tilde{X}_m) = \frac{1}{2}\mathring{\mathfrak{g}}^{ab}f^S_{ab,m} + \frac{1}{\sqrt{n}}\tilde{D}^b(rf^S_{b,m})\,. \tag{4.10}$$

The homogeneous version of the equation (4.10) for $\tilde{X}_m$ has no non-trivial solutions tending to zero at infinity by the maximum principle. The operator at the left-hand side of (4.10) has indicial exponents in $\{0, -3\}$, and therefore (4.10) has a unique solution $\tilde{X}_m = O(r^{-3})$.

The conditions (4.9) do not fix the gauge uniquely: an additional gauge transformation satisfying

$$\tilde{Y}_{a,m} + r^2\partial_a\tilde{X}_m = 0\,, \quad \tilde{D}^b\tilde{Y}_{b,m} = 0\,, \tag{4.11}$$

preserves the form of $\bar{h}^S$.

We define new variables $Z_{a,m}$ as

$$\begin{aligned} f^S_{tr,m} = \frac{1}{f}\Big[&Z_{t,m} + 2r^{1-n}\left(r^n - \mu(n+1)r\right)\partial_t H^S_{L,m} \\ &- 2r^{4-2n}\mu(n+1)\left(-2\mu + r^{n+1} + r^{n-1}\right)\partial_t\partial_r Z_{r,m}\Big]\,, \end{aligned} \tag{4.12}$$

$$f^S_{rr,m} = \frac{1}{f}\Big[2r\partial_r H^S_{L,m} + 2\mu(n+1)r^{2-n}\partial_r Z_{r,m}\Big]\,. \tag{4.13}$$

Note that this defines Z_r only up to a term which depends on t alone.

The t, r linearized Einstein equation directly gives $Z_{t,m} = 0$. Eliminating third order derivatives from the remaining equations we obtain

$$\begin{aligned} &r^n\partial_r H^S_{L,m} + r\left(-2\mu r + r^{n+2} + r^n\right)\partial_r^2 Z_{r,m} \\ &\quad + \left(2\mu(n-2)r + \left(3r^2 + 2\right)r^n\right)\partial_r Z_{r,m} = 0\,. \end{aligned} \tag{4.14}$$

Differentiating the Einstein equations by r and using (4.14) to express derivatives of $H^S_{L,m}$ by $Z^{(1)}_r$ gives two fifth order and two fourth order equation for $Z^{(1)}_r$. Eliminating higher derivatives we finally obtain a third order equation for $Z^{(1)}_r$

$$\partial_r\tilde{D}^a(r^2\tilde{D}_a Z_{r,m}) - 2\frac{r^{n-1} - \mu(n+1)}{r^{n+2} + r^n - 2\mu r}\tilde{D}^a(r^2\tilde{D}_a Z_{r,m}) = 0\,. \tag{4.15}$$

This implies

$$\tilde{D}^a(r^2\tilde{D}_a Z_{r,m}) = \frac{Cr^2}{f}\,, \tag{4.16}$$

with a constant C which has to vanish for $Z_{r,m}$ to be regular at r_0.

We now consider the remaining gauge freedom. We see from (4.11) that for any X satisfying

$$\tilde{D}^a(r^2\tilde{D}_a\tilde{X}_m) = 0 \tag{4.17}$$

there exists an associated Y_a giving a gauge transformation which preserves (4.9).

Inserting the definition of our new variables into (4.6) we find that under a gauge transformation satisfying (4.11) $Z_{r,m}$ and $H^S_{L,m}$ transform as

$$\partial_r Z_{r,m} \mapsto \partial_r(Z_{r,m} + \tilde{X}_m)\,, \tag{4.18}$$

$$H^S_{L,m} \mapsto H^S_{L,m} - \tilde{X}_m + \frac{f}{r}\tilde{Y}_r\,. \tag{4.19}$$

As, by (4.16), $Z_{r,m}$ satisfies (4.17) we can set $\partial_r Z_{r,m} \equiv 0$ using a gauge transformation with $\partial_r\tilde{X}_m = -\partial_r Z_{r,m}$, which preserves (4.9).

Inserting this into (4.14) we see that $H^S_{L,m}$ can only depend on t. From the remaining equations $\partial_t^2 H^S_{L,m} = 0$, i.e. $H^S_{L,m}$ is constant by periodicity.

We can exploit the remaining freedom in X to set

$$\tilde{X}_m = H^S_{L,m}\,, \qquad Y_a = 0\,, \tag{4.20}$$

obtaining $H^S_{L,m} \equiv 0$. This gives $Z_{r,m} = \text{const}$ and therefore $f^S_{tt,m} \equiv f^S_{rr,m} \equiv f^S_{tr,m} \equiv 0$.

We arrive at $h^S = \mathcal{L}_{\bar{Y}}\mathring{\mathfrak{g}}$ where $\bar{Y}$ is the combined gauge vector consisting of the part defined by (4.7)–(4.8), that given by (4.18) and that given by (4.20). From the asymptotics (2.22) of h and from (2.16)–(2.20), with the right-hand sides set to zero, we conclude that

$$h^S = \mathscr{L}_{\bar{Y}}\mathring{\mathfrak{g}}\,, \quad |\bar{Y}|_{\mathring{\mathfrak{g}}} = O(r^{-n+1})\,. \tag{4.21}$$

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6 TIMELIKE SINGULARITIES AND HAMILTONIAN COSMOLOGICAL BILLIARDS

6.1 IWASAWA COORDINATE SINGULARITY

In the following paper we construct timelike AVTD solutions (see Section 3.3.2) of the vacuum Einstein equations without symmetries. The solutions are parametrized by asymptotic data at the singularity. In general, vacuum solutions without symmetries are expected to exhibit chaotic instead of AVTD-type behavior. Here, we impose constraints on the asymptotic data which simplify the dynamics and allow us to use the Fuchs theorem to conclude.

In terms of the cosmological billiards formalism [39] these constraints correspond to an "asymptotic suppression" of some of the potential walls. This allows the abstract "particle" to continue along a straight line trajectory indefinitely, if it is pointed in the right direction.

Separately from this construction we analyze the structure of the equations in the timelike case, including all the potential walls. As two of the wall coefficients change signs compared to the standard case of spacelike singularities, the corresponding wall terms become potential wells instead. In the limit of $\tau \to \infty$ (i.e. as the singularity is approached) these wells appear to become infinitely deep. This contradicts some of the assumptions in [39] and appears to render their (heuristic) arguments for chaotic behavior invalid.

Shortly after the publication of the following paper Shaghoulian and Wang [98] pointed out that this is, in fact, an artefact of the Iwasawa coordinates used in our analysis. These coordinates become singular at the location of the well, which can be seen as follows:

Let us consider the metric in a Kasner coframe $\{\mathrm{d}t, l, m, r\}$,

$$g = N^2\mathrm{d}t^2 + (-a(t,x)^2 l_i l_j + b(t,x) m_i m_j + c(t,x)^2 r_i r_j)\mathrm{d}x^i \mathrm{d}x^j\,, \tag{24}$$

where ∂t is now a spacelike vector. In terms of the Iwasawa variables β^a and $\mathcal{N}_i^a$ the metric takes the form

$$g = N^2\mathrm{d}t^2 + \sum_a (1 - 2\delta_{a1}) e^{-2\beta^a} \mathcal{N}_i^a \mathcal{N}_j^a \mathrm{d}x^i \mathrm{d}x^j\,. \tag{25}$$

Comparing coefficients gives

$$e^{-2\beta^1} = -a(t,x)^2 l_1^2 + b(t,x) m_1^2 + c(t,x)^2 r_1^2\,. \tag{26}$$

In contrast to the case of spacelike singularity (c.f. [39, Section 4.2]) this expression is not positive definite. The zero crossing corresponds to $\beta^1 \to \infty$ which is exactly what happens when the "particle" falls into one of the potential wells (similar behavior occurs for the other well).

Shaghoulian and Wang show in [98, Appendix B] that, at least for spatially homogeneous Bianchi IX solutions with a timelike singularity, the affected walls turn into finite potential wells when using appropriate non-singular variables. These finite wells do not prevent the particle from passing through, meaning the "gravitational wall terms" lying behind them become relevant. The arguments of [39] would then again suggest the existence of chaotic behavior, although in a larger allowed region of the parameters. See Figure 1 and the corresponding Figure 5 in the following paper.

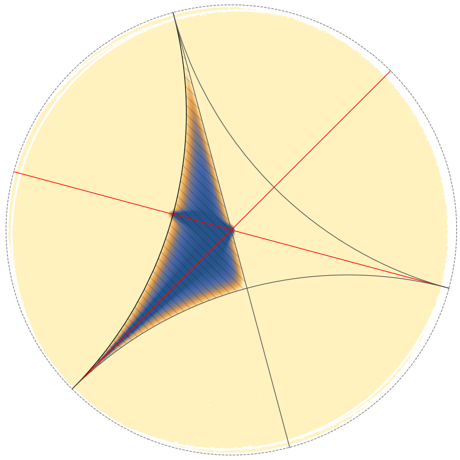

Figure 1: Qualitative picture of the potential including finite well terms for the case of timelike x^2 coordinate, shown in the hyperbolic space of the γ^a, projected onto the Poincaré disk. Thick black lines mark the dominant walls, thick gray lines the subdominant walls and red lines the walls with negative coefficients. Compare with Figure 5 in the following paper. The (hatched) allowed region is bigger than in the standard case of spacelike singularity but still has finite volume.

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Note

Timelike singularities and Hamiltonian cosmological billiards*

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Abstract

We construct a large class of vacuum solutions of the Einstein equations without any symmetries and with controlled asymptotics near a timelike singularity. The solutions are obtained by a Fuchs analysis of the equations which evolve the metric in a spacelike direction. We further observe that the change of sign of some of the terms (walls) in the associated Hamiltonian invalidate the 'cosmological billards' heuristic arguments for the existence of singularities of the mixmaster type in the current context.

Keywords: cosmological billiards, timelike singularity, BKL conjecture

(Some figures may appear in colour only in the online journal)

1. Introduction

An important issue in general relativity is the nature of singularities. While it is widely believed that the strong cosmic censorship conjecture holds, which can be loosely stated as the expectation that timelike singularities do not form by evolution from generic spatially compact or asymptotically flat initial data sets [9], the issue is wide open. From this perspective it is of interest to consider timelike singularities and therefore the ways in which cosmic censorship could be violated.

There are numerous exact solutions with timelike singularities (e.g. [4]). Such solutions are typically obtained when searching for solutions with symmetries. This naturally leads to the question of whether solutions with timelike singularities and without symmetries exist. We prove in this work that this is indeed the case: We construct an analog to the class of the non-chaotic solutions without symmetries and with controlled asymptotics of [1, 6], by changing the time parameter τ from a timelike to a spacelike coordinate. As the Hamiltonian differs from the one of [1, 6] only by sign changes, which do not affect the analysis in the

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analytic case, we obtain a family of solutions with the same free functions and asymptotics (in terms of the now spacelike τ coordinate), but with a timelike instead of a spacelike singularity.

The construction of the solutions is based on the cosmological billiard formalism using the Iwasawa decomposition of the metric. This method was introduced by Damour, Henneaux and Nicolai in [3] to give a heuristic argument for the chaotic picture of spacelike singularities provided by the Belinsky–Khalatnikov–Lifshitz (BKL) conjecture, and later used by Damour and DeBuyl in [2] to provide a precise statement of the conjecture.

We also show that the change of the time parameter τ from a timelike to a spacelike coordinate, i.e. considering timelike instead of spacelike singularities, switches the signs of some of the terms (walls) in the Hamiltonian considered. These changes violate the property of the spacelike case that the coefficients of the dominant wall terms are positive, thus rendering the arguments of Damour *et al* in [3] inapplicable. The affected terms become attractive rather than repulsive, allowing subdominant walls lying behind the dominant ones to become relevant. This does not affect the class of solutions we construct here, as these are non-generic and use an ansatz that suppresses the wall terms asymptotically.

2. Derivation of the Hamiltonian for the spacelike 'time'-variable

We follow the derivation of the Hamiltonian formalism by Wald [11, appendix E.2]. The spacetime metric is denoted by $\bar{g}_{\alpha\beta}$ while the induced Lorentzian metric on the timelike hypersurfaces of constant τ is denoted by g_{ij}. We choose a zero shift gauge, i.e. the metric takes the form

$$\mathrm{d}s^2 = N^2\mathrm{d}\tau^2 + g_{ij}\mathrm{d}x^i\mathrm{d}x^j. \tag{2.1}$$

As the hypersurfaces of constant τ are timelike, their normal vector is spacelike. This means that the Gauss equation takes the form

$$R_{abc}{}^{d} = g_a{}^{f}g_b{}^{g}g_c{}^{k}g^{d}{}_{j}\bar{R}_{fgk}{}^{j} + K_{ac}K_b{}^{d} - K_{bc}K_a{}^{d} \tag{2.2}$$

where $R_{abc}{}^{d}$ and $\bar{R}_{abc}{}^{d}$ are the Riemann curvature tensors of the induced and full metric respectively and K_{ab} is the second fundamental form of the hypersurface. Compared to the case of spacelike hypersurfaces, the signs of the KK terms are interchanged.

Using $g_{\alpha\beta} = \bar{g}_{\alpha\beta} - n_\alpha n_\beta$ with n^α the unit normal vector of the hypersurface ($n_\alpha n^\alpha = 1$) gives

$$\bar{R}_{\alpha\beta\gamma\delta}g^{\alpha\gamma}g^{\beta\delta} = -2\bar{G}_{\alpha\beta}n^\alpha n^\beta. \tag{2.3}$$

This leads to a change of sign in the constraint equation:

$$0 = \bar{G}_{\mu\nu}n^\mu n^\nu = -\frac{1}{2}R - (K^\mu{}_\mu)^2 + K_{\mu\nu}K^{\mu\nu}. \tag{2.4}$$

Contracting the Einstein tensor twice with the normal vector n^a gives an expression for the scalar curvature:

$$\bar{R} = -2n^\alpha n^\beta(\bar{G}_{\alpha\beta} - \bar{R}_{\alpha\beta}). \tag{2.5}$$

The definition of the Riemann tensor gives for the last term

$$\begin{aligned}\bar{R}_{\alpha\beta}n^\alpha n^\beta = \bar{R}_{\alpha\gamma\beta}{}^\gamma n^\alpha n^\beta &= -n^\alpha(\nabla_\alpha\nabla_\gamma - \nabla_\gamma\nabla_\alpha)n^\gamma \\ &= (\nabla_\alpha n^\alpha)(\nabla_\gamma n^\gamma) - (\nabla_\gamma n^\alpha)(\nabla_\alpha n^\gamma) \\ &\quad - \nabla_\alpha(n^\alpha\nabla_\gamma n^\gamma) + \nabla_\gamma(n^\alpha\nabla_\alpha n^\gamma) \\ &= (K^\alpha_\alpha)^2 - K_{\alpha\gamma}K^{\alpha\gamma} - \nabla_\alpha(n^\alpha\nabla_\gamma n^\gamma) + \nabla_\gamma(n^\alpha\nabla_\alpha n^\gamma)\end{aligned} \tag{2.6}$$

where the last two terms are divergences, which will be discarded in the Lagrangian.

Using (2.4)–(2.6) and $\sqrt{-\bar{g}} = N\sqrt{-g}$ to express the Einstein–Hilbert action gives

$$\mathcal{L} = \sqrt{-\bar{g}}\bar{R} = -\sqrt{-g}N(R - K_{ab}K^{ab} + (K^a{}_a)^2). \tag{2.7}$$

The canonically conjugate momenta to the metric components g_{ij} are given by

$$\pi^{ij} = \frac{\partial\mathcal{L}}{\partial\dot{g}_{ij}} = \sqrt{-g}(K^{ij} - K^k{}_k g^{ij}) = N^{-1}\sqrt{-g}\frac{1}{2}(\dot{g}_{kl}g^{ki}g^{lj} - \dot{g}_{kl}g^{kl}g^{ij}), \tag{2.8}$$

which is unchanged from the standard case.

The Hamitonian, expressed in terms of the canonical coordinates g_{ab} and momenta π^{ab}, is finally

$$\mathcal{H} = \pi^{ab}\dot{g}_{ab} - \mathcal{L} = (-g)^{-1/2}N\left(\pi^{ab}\pi_{ab} - \frac{1}{2}(\pi^a{}_a)^2\right) + RN\sqrt{-g} \tag{2.9}$$

i.e. the standard one with the sign of the curvature term changed.

3. Iwasawa variable Hamiltonian

Here, we will describe the changes to the derivation of the Iwasawa variable Hamiltonian, as given in appendix A of [6].

Since the level sets of τ are timelike, we need to decide which frame vector is the timelike one. As the Iwasawa ansatz breaks the symmetry between the frame vectors, different choices will lead to different dynamical systems. We will use an index $J \in \{1, 2, 3\}$ to distinguish between those cases: x^J will denote the timelike coordinate.

The Lorentzian metric g_{ij} on the $\tau =$ const hypersurfaces is split in Iwasawa variables as

$$g_{ij} = \sum_a m_a^J e^{-2\beta^a}\mathcal{N}^a{}_i\mathcal{N}^a{}_j \tag{3.1}$$

where $m_a^J = 1 - 2\delta_{Ja}$, i.e. -1 for $a = J$ and 1 otherwise.

We set the lapse function N equal to $\sqrt{-g}$ where g is the determinant of the metric g_{ij}. The (timelike) singularity will be approached as $\tau \to \infty$.

The conjugate momenta π_a to the β^a and $P^i{}_a$ to the $\mathcal{N}^a{}_i$ are given by

$$\pi_a = \frac{\partial\mathcal{L}}{\partial\dot{\beta}^a} = \frac{\partial\mathcal{L}}{\partial\dot{g}_{ij}}\frac{\partial\dot{g}_{ij}}{\partial\dot{\beta}^a} = -2\pi^{ij}m_a^J e^{-2\beta^a}\mathcal{N}^a{}_i\mathcal{N}^a{}_j \tag{3.2}$$

and

$$P^i{}_a = 2m_a^J\pi^{ij}\mathcal{N}^a{}_j e^{-2\beta^a}, \tag{3.3}$$

i.e. the same as in the spacelike case except for the additional factor m_a^J.

The non-curvature terms of the Hamiltonian (2.9), with $N = \sqrt{-g}$ inserted, are

$$\pi^{ab}\pi_{ab} - \frac{1}{2}(\pi^a{}_a)^2. \tag{3.4}$$

The first term can be split into

$$\frac{1}{4}\sum_a \pi_a^2 + \frac{1}{2}\sum_{a<b} m_a^J m_b^J e^{-2(\beta^b-\beta^a)} (P^j{}_a \mathcal{N}^b{}_j)^2. \tag{3.5}$$

The first term of (3.5) together with the second term of (3.4) give the kinetic term $G^{ab}\pi_a\pi_b$ of the Hamiltonian, unchanged from the spacelike case. The second term of (3.5) is the symmetry wall term, with the addition of m_a^J and m_b^J. These cause a sign change for two of the symmetry walls.

The Iwasawa form of the curvature term in the Hamiltonian (2.9), which gives the gravitational potential walls, is calculated in the Iwasawa frame, where the metric takes the form

$$\gamma_{ab} = \delta_{ab} m_a^J e^{-2\beta^a} = \delta_{ab} m_a^J A_a^2 \tag{3.6}$$

with $A_a^2 := \exp(-2\beta^a)$ i.e. with an additional factor m_a^J compared to the spacelike case.

This is the only change in the derivation of the curvature term, as the Cartan formulas remain unchanged. The terms corresponding to the dominant gravitational walls are

$$\frac{1}{4}\sum_{a\neq b\neq c\neq a} (C^a{}_{bc})^2 \frac{A_a^2}{A_c^2 A_b^2}\frac{m_a^J}{m_b^J m_c^J}. \tag{3.7}$$

In $3+1$ dimensions exactly one of a, b, c is equal to J, which adds an additional minus in front of this term. This cancels the change of sign in the Hamiltonian.

In addition, there are sign changes in the subdominant gravitational terms, but as they have an indeterminate sign even for the spacelike case, this does not affect the analysis.

In conclusion, for $3+1$ dimensions, the prefactors of two of the symmetry walls change sign. In the case $J = 1$ this involves the $\beta^2 - \beta^1$ and $\beta^3 - \beta^1$ walls, for $J = 2$ the $\beta^3 - \beta^2$ and $\beta^2 - \beta^1$ walls and for $J = 3$ the $\beta^3 - \beta^2$ and $\beta^3 - \beta^1$ walls. In all cases, the sign of at least one dominant wall term changes.

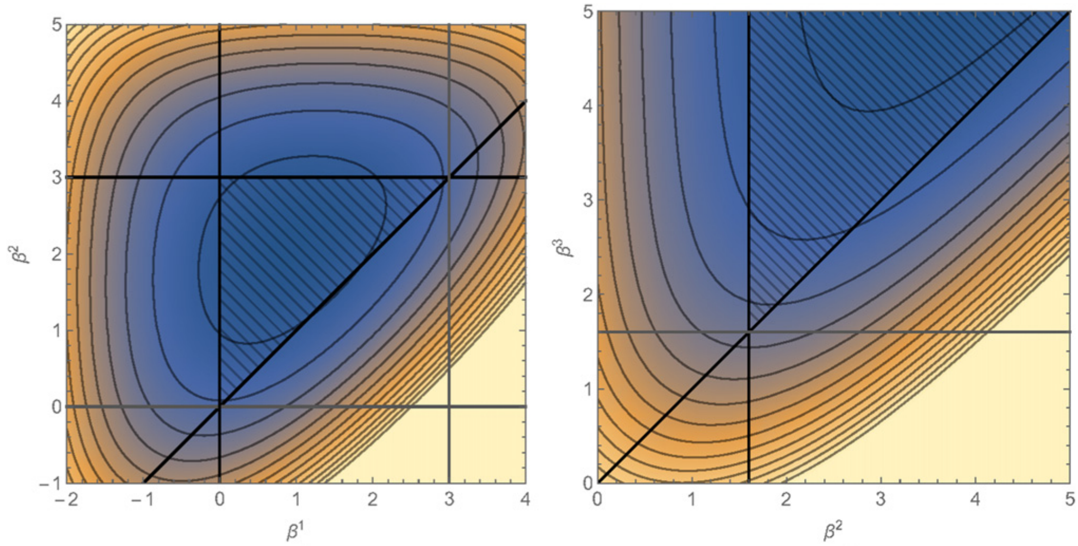
The potential (i.e. the Hamiltonian without the terms containing π^a) for the spacelike case and the three choices of J are sketched in figures 1 to 4.

4. Consequences for cosmological billiard

The arguments for the asymptotic billiard picture depend on the positive sign of the prefactors of the dominant wall terms: In hyperbolic coordinates ρ and γ^a, such that $G_{ab}\gamma^a\gamma^b = -1$ (G_{ab} is the constant matrix $G_{ab} = -\sum_{c\neq d}\delta_a^c\delta_b^d$) and $\beta^a = \rho\gamma^a$ (such a splitting is possible, assuming the solution is close to a Kasner state asymptotically [3]) the Hamiltonian takes the form

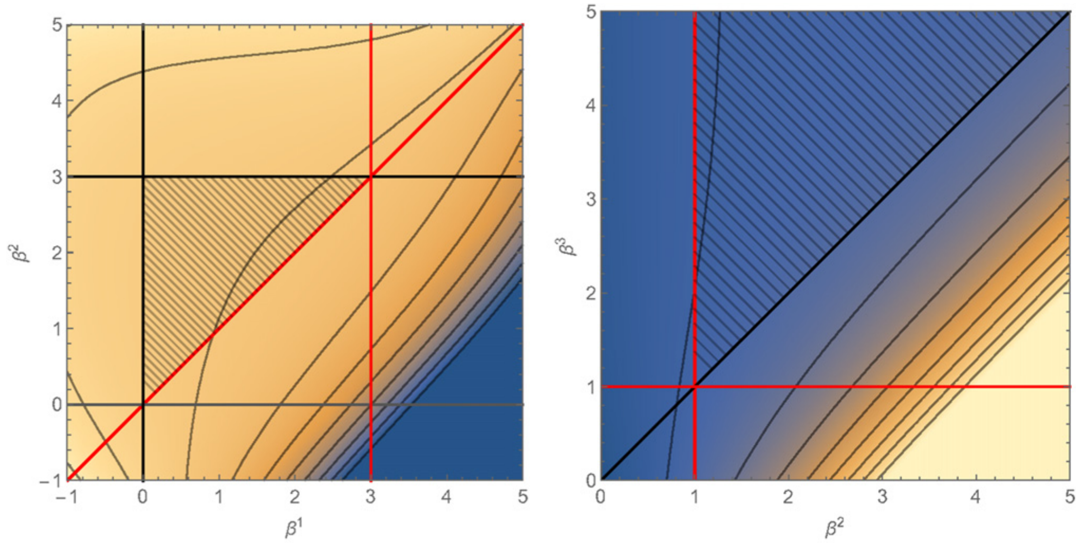
$$H = \frac{1}{4}(-\rho^2\pi_\rho^2 + \pi_\gamma^2) + \rho^2\sum_A c_A e^{-2\rho\omega_A(\gamma)}. \tag{4.1}$$

If the prefactors c_A for the dominant walls are positive this approaches the 'sharp wall Hamiltonian'



(a) τ timelike, $\beta^3 = 3$ slice (b) τ timelike, $\beta^1 = 1$ slice

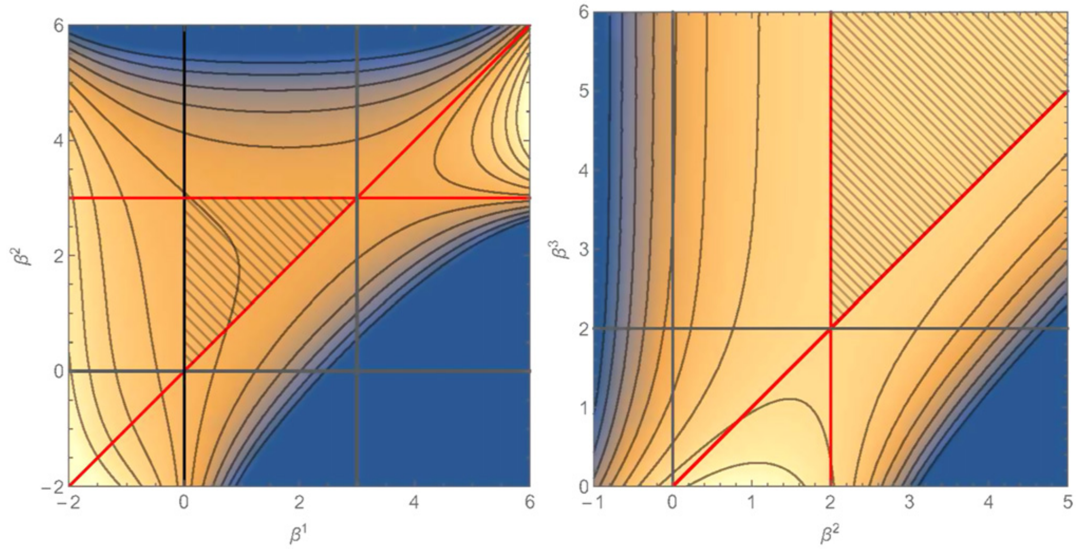
Figure 1. Sketch of the potential for the case of timelike τ (i.e. with spacelike singularity). Only the exponential terms are plotted, the coefficients are set to 1. The potential increases from dark blue through orange to light yellow. Black lines mark the dominant walls, gray lines the subdominant walls and the allowed region (i.e. the 'billiard table') is hatched.



(a) x^1 timelike, $\beta^3 = 3$ slice (b) x^1 timelike, $\beta^1 = 1$ slice

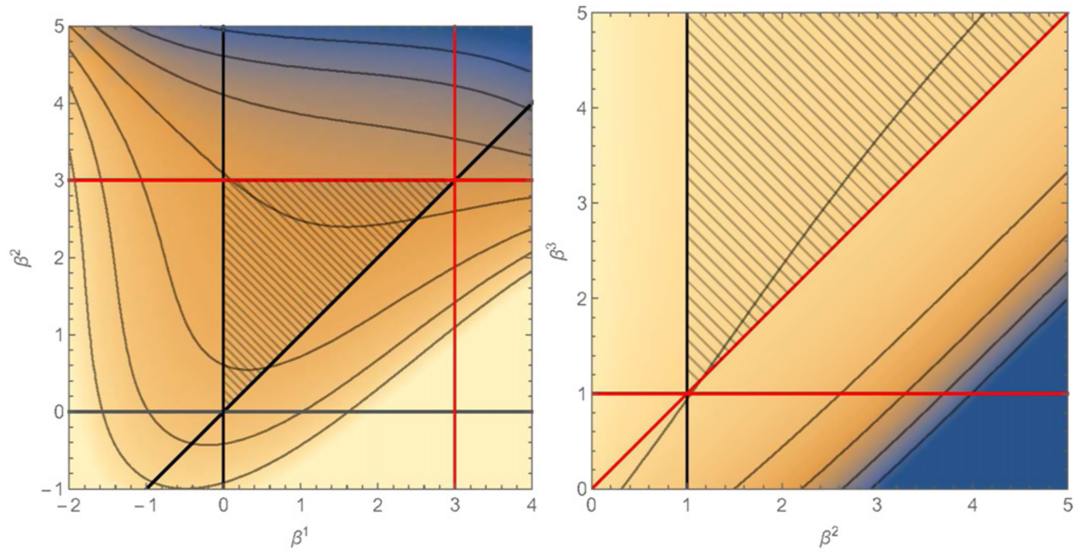
Figure 2. $J = 1$, i.e. x^1 timelike case. The walls with negative coefficients (set to -1 in the plots) are marked by red lines. The hatched region is the same as in figure 1 but no longer corresponds to an allowed region, as the potential approaches $-\infty$ outside it.

$$H = \frac{1}{4}(-\rho^2\pi_\rho^2 + \pi_\gamma^2) + \sum_{A'}\Theta_\infty(-\omega_{A'}(\gamma)) \tag{4.2}$$



(a) x^2 timelike, $\beta^3 = 3$ slice (b) x^2 timelike, $\beta^1 = 2$ slice

Figure 3. $J = 2$ i.e. x^2 timelike case.



(a) x^3 timelike, $\beta^3 = 3$ slice (b) x^3 timelike, $\beta^1 = 2$ slice

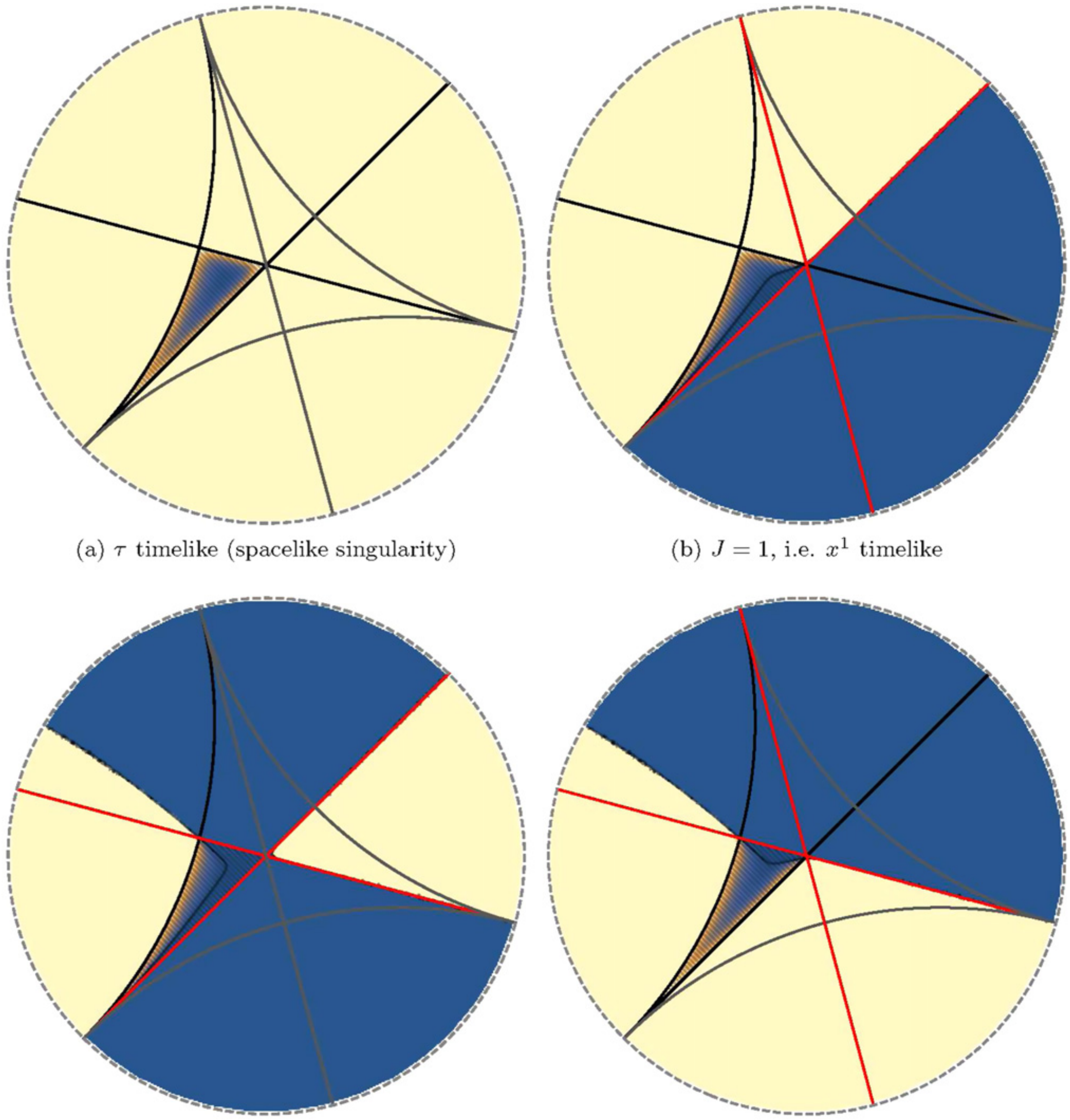
Figure 4. $J = 3$ i.e. x^3 timelike case.

where the sum over A' only covers the dominant walls and

$$\Theta_\infty(x) = \begin{cases} 0 & x < 0, \\ \infty & x > 0. \end{cases} \tag{4.3}$$

If, however, some of the prefactors are negative the corresponding terms are potential wells instead of walls. In the timelike case in 3 + 1 dimensions, this affects at least one of the dominant symmetry walls.

Figure 5 shows the potentials in the hyperbolic space of the γ^a, projected onto the Poincaré disk.



(a) τ timelike (spacelike singularity)

(b) $J = 1$, i.e. x^1 timelike

(c) $J = 2$, i.e. x^2 timelike

(d) $J = 3$, i.e. x^3 timelike

Figure 5. Sketch of the potentials in the hyperbolic space of the γ^a, projected onto the Poincaré disk. As before, the allowed region for the case of spacelike singularity is hatched, thick black lines mark the dominant walls, thick gray lines the subdominant walls and red lines the walls with negative coefficients.

5. Consequences for solutions constructed in [1, 6]

The class of solutions constructed in [1, 6] for the case of timelike τ also exists for spacelike τ. The sign changes in the Hamiltonian have no effect on the arguments concerning the evolution equations in the context of the analytic Fuchs theorem, as the decay of the exponential terms is unchanged.

An additional factor m_b^J appears in the term $\tilde{\pi}^b{}_a$, which enters in the Iwasawa variable momentum constraint:

$$\tilde{\pi}^b{}_a = \frac{1}{2}\begin{cases} -\pi_b & \text{for } a = b, \\ \mathcal{N}^b{}_i P^i{}_a & \text{for } b > a, \\ m_b^J e^{-2(\beta^a - \beta^b)} \mathcal{N}^a{}_i P^i{}_b & \text{for } a > b. \end{cases} \tag{5.1}$$

As the factor m_b^J is only present in the asymptotically decaying case $a > b$, which is discarded in the asymptotic constraints, this leaves the conditions on the free functions unchanged.

Similarly, there are sign changes in the derivation of the evolution equations for the constraints, given in appendix D of [6], which cancel out in the final equations.

As in the case of a spacelike singularity, the presence of a cosmological constant does not affect the result (see appendix F of [6]).

This leads to the following theorem, in close analogy with the results of [1, 6]:

Theorem 1. *For any choice of $J \in \{1, 2, 3\}$ and analytic functions $\beta_\circ^2$, $\beta_\circ^3$ and $P_\circ{}^2{}_1$ depending on coordinates x^i, $i \in \{1, 2, 3\}$, and for any two analytic functions, $p_\circ^2$ and $p_\circ^3$ depending on x^i, which satisfy the inequalities*

$$0 < p_\circ^2 < (\sqrt{2} - 1)p_\circ^3 \tag{5.2}$$

we obtain a solution of the vacuum Einstein equations with an arbitrary cosmological constant given by the metric

$$g = e^{-2\sum_{a=1}^{3}\beta^a}\mathrm{d}\tau^2 + \sum_{a=1}^{3} m_a^J e^{-2\beta^a}\mathcal{N}^a{}_i\mathcal{N}^a{}_j \mathrm{d}x^i\mathrm{d}x^j. \tag{5.3}$$

Here $m_a^J = 1 - 2\delta_{Ja}$, β^a and $\mathcal{N}^a{}_i$, $i, a \in \{1, 2, 3\}$, depend on all coordinates τ, x^i and behave asymptotically as

$$\beta^a = \beta_\circ^a + \tau p_\circ^a + O(e^{-\tau\nu}) \quad \text{and} \quad \mathcal{N}^a{}_i =: \delta_i^a + \mathcal{N}_s{}^a{}_i = \delta_i^a + O(e^{-\tau\nu}), \tag{5.4}$$

where ν is a positive constant, the $\beta_\circ{}^a$'s and $p_\circ{}^a$'s depend only upon x^i and $\mathcal{N}_s{}^a{}_i = 0$ for $a \geqslant i$ with the non-vanishing terms given by

$$\mathcal{N}_s{}^1{}_2 = -\frac{P_\circ{}^2{}_1 e^{-2(\beta_\circ^2 - \beta_\circ^1)}}{2(p_\circ^2 - p_\circ^1)} e^{-\tau(2p_\circ^2 - 2p_\circ^1)} + O(e^{-\tau(2p_\circ^2 - 2p_\circ^1 + \nu)}), \tag{5.5}$$

$$\mathcal{N}_s{}^2{}_3 = -\frac{P_\circ{}^3{}_2 e^{-2(\beta_\circ^3 - \beta_\circ^2)}}{2(p_\circ^3 - p_\circ^2)} e^{-\tau(2p_\circ^3 - 2p_\circ^2)} + O(e^{-\tau(2p_\circ^3 - 2p_\circ^2 + \nu)}), \tag{5.6}$$

$$\mathcal{N}_s{}^1{}_3 = e^{-2(\beta_\circ^3 - \beta_\circ^1)}\left(P_\circ{}^3{}_1 - \frac{P_\circ{}^2{}_1 P_\circ{}^3{}_2}{2p_\circ^3 - 2p_\circ^2}\right)\frac{1}{2p_\circ^3 - 2p_\circ^1} e^{-\tau(2p_\circ^3 - 2p_\circ^1)} + O(e^{-\tau(2p_\circ^3 - 2p_\circ^1 + \nu)}), \tag{5.7}$$

where the functions $\{P_\circ{}^i{}_a\}_{1 \leqslant a < i \leqslant 3}$ depend only on x^i.

The remaining functions $p_\circ^1$, $\beta_\circ^1$, $P_\circ{}^3{}_2$ and $P_\circ{}^3{}_1$ are then determined from the asymptotic constraint equations:

$$p_\circ^1 = -\frac{p_\circ^2 p_\circ^3}{p_\circ^2 + p_\circ^3}, \tag{5.8}$$

$$\beta_{\circ,3}^1 = -(p_\circ^2 + p_\circ^3)^{-1}(p_{\circ,3}^2 + p_{\circ,3}^1 + \beta_{\circ,3}^2(p_\circ^1 + p_\circ^3) + \beta_{\circ,3}^3(p_\circ^1 + p_\circ^2)), \tag{5.9}$$

$$P_\circ{}^3{}_{2,3} = 2(G_{2c}p^c_{\circ,2} + \beta^d_{\circ,2}p^f_\circ G_{df}), \tag{5.10}$$

$$P_\circ{}^3{}_{1,3} = -P_\circ{}^2{}_{1,2} + 2(G_{1c}p^c_{\circ,1} + \beta^d_{\circ,1}p^f_\circ G_{df}). \tag{5.11}$$

Here G_{ab} denotes the constant matrix $G_{ab} = -\sum_{c\neq d}\delta^c_a\delta^d_b$.

Finally, the Kretschmann scalar behaves as

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \left(\frac{16e^{4(\beta^1_\circ+\beta^2_\circ+\beta^3_\circ)}(p^2_\circ p^3_\circ)^2}{(p^2_\circ + p^3_\circ)^2}((p^2_\circ)^2 + p^2_\circ p^3_\circ + (p^3_\circ)^2) + O(e^{-\nu\tau})\right)e^{\tau 4(p^1_\circ+p^2_\circ+p^3_\circ)},$$

and therefore, since $p^2_\circ p^3_\circ > 0$, the curvature diverges as $\tau \to \infty$. Along curves $\gamma(\tau) = (\tau, \gamma^i(\tau))$, $\tau \in [\tau_0, \infty)$, fulfilling $|\gamma'^i(\tau)| = O(e^{(p^i_\circ(\gamma^j(\tau))-\epsilon)\tau})$ for some $\epsilon > 0$ and for $i = 1, 2, 3$, the curvature diverges in finite proper time/length.

6. Conclusion

We have constructed a large class of vacuum spacetimes containing a timelike singularity. The solutions asymptotically approach a timelike Kasner metric at each point (x^i), which can be interpreted as the field of an infinitely extended thin rod, with positive mass for $J \neq 1$ and negative mass for $J = 1$ [5]. As the Kasner exponents now depend upon the coordinates x^i the solutions might represent the field of more complicated, non-symmetric and non-static, one-dimensional sources.

We have also noted that the cosmological billiards arguments of Damour, Henneaux, and Nicolai [3] are not directly applicable to this case, because of the transformation of asymptotically infinite potential walls into infinite wells. One should keep in mind the results of Parnovsky [7, 8], who applied the original procedure used by BKL to the timelike case, and concluded that the heuristic construction of chaotic singularities remains applicable. It would be of interest to resolve this apparent contradiction.

In [10] the authors argue, using a model Bianchi IX spacetime, that the change of sign of some of the wall terms is an artifact of the Iwasawa decomposition and that the affected walls vanish in a different gauge. It is not clear to us whether their arguments apply to the general inhomogeneous case.

Acknowledgments

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The annoying null boundaries

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Abstract. We consider a class of globally hyperbolic space-times with "expanding singularities". Under suitable assumptions we show that no C^0-extensions across a compact boundary exist, while the boundary must be null wherever differentiable (which is almost everywhere) in the non-compact case.

1. Introduction

One of the major open questions in mathematical general relativity is the behavior of globally hyperbolic space-times when singular boundaries are approached. In particular the question of extendibility of the metric across such boundaries lies at the heart of the "cosmic censorship conjecture" [1], compare [2, 3].

In a recent important paper, Sbierski [4] has established C^0-inextendibility of the Kruskal-Szekeres extension of the Schwarzschild metric. His proof makes uses of the $SO(2)\times\mathbb{R}$ symmetry of the metric, which renders the argument unsuitable in situations where no isometries exist. (See also [5].)

It is of interest to enquire whether Sbierski's analysis can be adapted to more general spacetimes of interest, without isometries. The object of this note is to point out a class of space-times where C^0-extendibility can only happen across null boundaries, if at all. Indeed, let us consider a globally hyperbolic space-time with a differentiable Cauchy time function t covering $(0,\infty)$; we will be interested in possible extensions of M towards the past, see Definition 2.3 below. The function t determines a topological splitting $M=(0,\infty)\times\mathscr{S}$, where the slices $\{\tau\}\times\mathscr{S}$ are the level sets of t; here one travels from a slice $\{t_1\}\times\mathscr{S}$ to $\{t_2\}\times\mathscr{S}$ by following the integral curves of ∇t.

Definition 1.1. We shall say that a globally hyperbolic space-time (M,g), with a time function t as just described, contains a *globally expanding singularity towards the past* if for every open set $A\subset\mathscr{S}$ there exists a sequence t_i decreasing to zero such that the (Riemannian) diameter of $\{t_i\}\times A$ within $\{t_i\}\times\mathscr{S}$ tends to infinity as $t_i\to 0$.

As discussed in more detail in Section 6 below, the space-times constructed in [6, 7], as well as Gowdy, and various other "Asymptotically Velocity Term Dominated" (AVTD) spacetimes obtained via Fuchsian methods are of this type, and the theorems below apply.

All the boundaries that we consider in this work will be achronal, hence differentiable almost everywhere by standard arguments (cf. the beginning of Section 4).

We have:

THEOREM 1.2. *Suppose that* (M, g) *contains a globally expanding singularity towards the past. Then, in every continuous past extension of* M*, the boundary* $\partial\iota(M)$ *of the image of* M *in the extension is null at all its differentiability points.*

Theorem 1.2 is an immediate consequence of Theorem 3.1 below, which is proved under a condition weaker, but somewhat more involved, than that of Definition 1.1.

As such, it also holds:

PROPOSITION 1.3. *Under the hypotheses of Theorem 3.1, every spacelike hypersurface in the extension which intersects the boundary* $\partial\iota(M)$ *also intersects* $\iota(M)$ *itself.*

Some further results in the same spirit concerning general extensions are also established, see Theorems 3.3 and 3.5 below.

We suspect that null extensions cannot occur either under the hypotheses above, and thus such space-times are inextendible, but we have not been able to establish this in general. However we have:

PROPOSITION 1.4. *Under the hypotheses of Theorem 1.2,* (M, g) *has no past extensions with a continuous metric and a compact boundary.*

The proof of Proposition 1.4 can be found in Section 5.

Recall that electrovacuum space-times with compact Cauchy horizons have been studied in [8, 9], and that, in space-times in which the metric is C^3-extendible [10, 11], for such metrics compactness of the horizon implies its differentiability. We emphasise that our arguments do not need such results.

It should be recognised that our definition of expanding space-times is tied to the choice of a time-function t, and large deformations of a good time function, if one exists, will not preserve the condition. In particular it might be very difficult to determine whether or not a given space-time, presented in a coordinate system where the conditions of Definition 1.1 are *not* met, admits a time-function which will satisfy the conditions. But our results here give some geometric meaning to the notion: space-times extendible through a compact Cauchy horizon, or past-extendible through a spacelike boundary, will not be expanding in the sense of Definition 1.1 no matter what time function is used.

2. Conventions and definitions

We use the standard definition of the Riemannian diameter of a set A in the set $B \supseteq A$, where both A and B are subsets of a Riemannian manifold (M, g)

$$\operatorname{diam}(A, B) := \sup_{x,y\in A} \inf_{\substack{\gamma:[0,1]\to B\\ \gamma(0)=x\,,\,\gamma(1)=y}} \int_0^1 |\dot\gamma(s)|_g \mathrm{d}s\,. \tag{2.1}$$

Our remaining definitions follow Sbierski [4], in particular:

DEFINITION 2.1. A C^0-*extension* of a spacetime (M, g), where M is a smooth manifold and g a Lorentzian metric, is a spacetime $(\tilde M, \tilde g)$ of the same dimension, with $\tilde M$ again a smooth manifold and $\tilde g$ a continuous Lorentzian metric, together with a smooth isometric embedding $\iota : M \to \tilde M$ such that $\iota(M)$ is a proper subset of $\tilde M$.

The timelike futures and pasts $I^\pm$ will be defined using piecewise smooth timelike curves in both M and $\tilde M$.

We define the future/past boundary of a spacetime as in [12], namely:

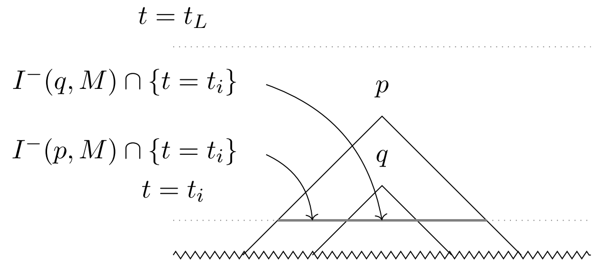

Figure 2.1: An illustration of the definition of an *expanding singularity*.

Definition 2.2. The *future, respectively past, boundary of* M, denoted $\partial^+\iota(M)$, respectively $\partial^-\iota(M)$, is the set of points $p \in \partial\iota(M)$ such that there exists a future/past directed timelike curve $\gamma : [0,1] \to \tilde{M}$ with $\gamma([0,1)) \subset \iota(M)$ and $\gamma(1) = p$.

Definition 2.3. An extension will be called *past*, respectively *future*, if $\partial^+\iota(M) = \emptyset$, respectively $\partial^-\iota(M) = \emptyset$.

We will also need a definition closely related to, but somewhat weaker than Definition 1.1:

Definition 2.4. We shall say that a globally hyperbolic space-time (M, g) contains an *expanding singularity towards the past* if there exists a time function t ranging over $(0, \infty)$ and a real number $t_L > 0$ such that for all pairs of points $p \in \{t < t_L\}$ and $q \in I^-(p, M)$ there exists a sequence t_i decreasing to zero such that the diameter of $I^-(q, M) \cap \{t = t_i\}$ in $I^-(p, M) \cap \{t = t_i\}$ tends to infinity as $t_i \to 0$.

This is illustrated in Figure 2.1.

Remark 2.5. We note that a globally expanding singularity in the sense of Definition 1.1 is expanding in the sense of Definition 2.4. For this, let us identify the spacetime M with $\mathbb{R} \times \mathscr{S}$ by flowing along the gradient ∇t of t. For a subset Ω of $\mathscr{S}$ set $\Omega(t) = \{t\} \times \Omega$, and let

$$\Omega_i(t_i) := I^-(q, M) \cap \{t = t_i\} .$$

We then have, for $j > i$,

$$\Omega_i(t_j) \subseteq I^-(q, M) \cap \{t = t_j\} \equiv \Omega_j(t_j) .$$

By its definition, given in (2.1), the diameter appearing in Definition 2.4, namely

$$\operatorname{diam}(\Omega_i(t_i), I^-(p, M) \cap \{t = t_i\}) ,$$

is bounded from below by the diameter of $\Omega_i(t_i)$ in the whole level set, $\operatorname{diam}(\Omega_i(t_i), \{t = t_i\})$. By Definition 1.1 of a globally expanding singularity the diameter of $\Omega_0(t_i)$ in $\{t = t_i\}$ diverges as t_i tends to zero, and the inclusion $\Omega_0(t_i) \subset \Omega_i(t_i)$ proves the claim.

3. Nonexistence of spacelike boundaries

We have the following result, which immediately implies Theorem 1.2:

Theorem 3.1. *Suppose that* (M, g) *contains an expanding singularity towards the past. Then in every continuous past extension of* M *the boundary* ∂M *is null wherever differentiable.*

The following proposition gives some more information about extensions in the current context:

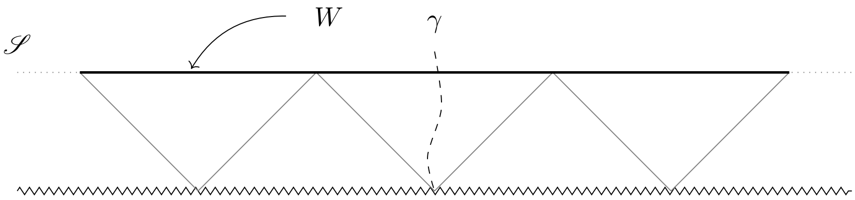

Figure 3.1: The set W appearing in the second condition of Theorem 3.5.

Proposition 3.2. *Under the hypotheses of Theorem 3.1, every spacelike hypersurface in the extension which intersects $\partial^- \iota(M)$ also intersects $\iota(\{t < t_L\}) \subseteq \iota(M)$.*

For a general extension (i.e. where $\partial^+ \iota(M)$ might be non-empty) we have:

Theorem 3.3. *Suppose that (M, g) contains an expanding singularity towards the past. If the Cauchy hypersurface $\mathscr{S}$ is compact then the past boundary $\partial^- \iota(M)$ of M in every extension is null wherever differentiable.*

Proposition 3.4. *Under the hypotheses of Theorem 3.3, every spacelike hypersurface in the extension which intersects $\partial^- \iota(M)$ also intersects $\iota(\{t < t_L\}) \subseteq \iota(M)$.*

Under weaker conditions on $\mathscr{S}$ we obtain a similar but more involved result:

Theorem 3.5. *Suppose that (M, g) contains an expanding singularity towards the past. If either*

- *there exist constants $t_C > 0, C > 0$ such that for all timelike curves γ in M the intersection $I^+(\gamma, M) \cap \{t = t_C\}$ has diameter less than C,*
- *or for all timelike curves $\gamma \subset M$ the intersection*
$$W := I^+(I^-(I^+(\gamma, I^-(\mathscr{S}, M)), M), M) \cap \mathscr{S}$$
(compare Figure 3.1) is precompact.

Then:

- *every spacelike hypersurface in the extension which intersects $\partial^- \iota(M)$ also intersects $\iota(\{t < t_L\}) \subseteq \iota(M)$*
- *and for every point $p \in \partial^- \iota(M)$ there exists a neighborhood $\tilde{O} \subseteq \tilde{M}$ of p such that the hypersurface*
$$\{q \in \partial^- \iota(M) \cap \tilde{O} \,|\, I^+(q, \tilde{O}) \cap \partial^- \iota(M) = \emptyset\}, \tag{3.1}$$
which will be referred to as the futuremost part *of $\partial^- \iota(M)$ in $\tilde{O}$, is null wherever it is differentiable.*

Remark 3.6. The first condition in Theorem 3.5 implies the second one.

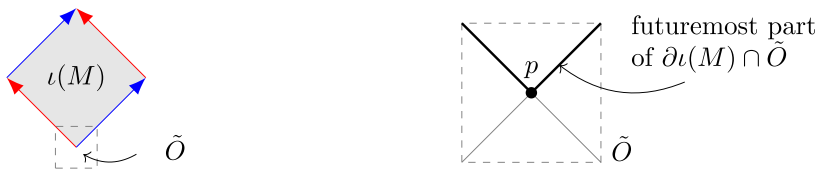

Figure 3.2: An example of a case where the futuremost part of $\partial^- \iota(M)$ in $\tilde{O}$ is not the same as $\partial^- \iota(M) \cap \tilde{O}$. The picture on the left shows the whole spacetime, with the extension given by gluing the edges according to the arrows. The picture on the right shows a neighborhood $\tilde{O}$ of the point $p \in \partial^-(M)$. Here $\partial^- \iota(M) = \partial^+ \iota(M) = \partial\iota(M)$ and there is no neighborhood $\tilde{O}$ of p such that $\partial\iota(M) \cap \tilde{O}$ is achronal. The futuremost part, however, is achronal.

Figure 3.2 illustrates the difference between the *futuremost part* of $\partial^- \iota(M)$, which appears in the second consequence of Theorem 3.5, and $\partial^- \iota(M)$ itself.

4. Achronality of boundaries

In order to prove the results above, we start by showing that the relevant sets are achronal. As such, achronal sets are locally Lipschitz continuous (for every point we can find a coordinate neighborhood on which the metric is close to Minkowski, which provides a Lipschitz bound) and therefore differentiable almost everywhere by Rademacher's theorem. By achronality they are null or spacelike wherever differentiable. To finish the proofs we will need to rule out spacelike tangent planes; this will be done in the next section.

We have:

LEMMA 4.1. *The boundary of a future or past extension is an achronal topological hypersurface.*

PROOF. We assume a past extension. For a future extension, replace $I^\pm$ with $I^\mp$ in the following.

Achronality of $\partial\iota(M)$ is equivalent to $I^+(\partial\iota(M), \tilde{M}) \cap \partial\iota(M) = \emptyset$. We first show that

$$I^+(\partial\iota(M), \tilde{M}) \subseteq I^+(\iota(M), \tilde{M}) .$$

Indeed, for every point $p \in I^+(\partial\iota(M), \tilde{M})$ there is $q \in \partial\iota(M)$ such that $p \in I^+(q, \tilde{M})$. Therefore $q \in I^-(p, \tilde{M})$ and, because I^- is open and q lies on the boundary of $\iota(M)$, there is a point $\iota(M) \ni r \in I^-(p, \tilde{M})$. This implies $p \in I^+(r, \tilde{M})$, which proves the claim.

As $\tilde{M}$ is a past extension, no future directed timelike curve leaves M, i.e. $I^+(r, \tilde{M}) \subseteq \iota(M)\ \forall r \in \iota(M)$, and therefore

$$I^+(\partial\iota(M), \tilde{M}) \subseteq \iota(M) \subseteq \tilde{M} \setminus \partial\iota(M)$$

which implies that $\partial\iota(M)$ is achronal.

To show that it is a topological hypersurface we need to show $\partial\iota(M) \cap \mathrm{edge}(\partial\iota(M)) = \emptyset$ (see [13, Section 14, Lemma 25], and note that the proof there does not use differentiability of the metric). We consider a point $p \in \partial\iota(M)$. By the above, $I^+(p, \tilde{M}) \subseteq \iota(M)$ and, as $\partial^+(\iota(M)) = \emptyset$, $I^-(p, \tilde{M}) \subset \tilde{M} \setminus \iota(M)$. Therefore every (past directed) timelike curve from $I^+(p, \tilde{M})$ to $I^-(p, \tilde{M})$ has to cross $\partial^-(M) \subseteq \partial\iota(M)$ and so $p \notin \mathrm{edge}(\partial\iota(M))$. □

REMARK 4.2. A very similar result was shown by Galloway and Ling [12, Theorem 2.6].

Next we show that every point on an achronal boundary is the limit of a timelike curve in $\iota(M)$. For a general boundary this is only true for at least one point [4, Lemma 2.17].

LEMMA 4.3. *Given a point $p \in \partial\iota(M)$ and a neighborhood $p \in \tilde{O} \subset \tilde{M}$ such that $\partial\iota(M) \cap \tilde{O}$ is achronal in $\tilde{O}$ then p is the end point of a differentiable timelike curve in M, i.e. there exists a differentiable timelike curve $\sigma : [-1, 0] \to \tilde{M}$ such that $\sigma([-1, 0)) \subset \iota(M)$ and $\sigma(0) = p$.*

PROOF. The curve σ might be future- or past-directed, we consider both cases here. We choose coordinates $(x^\alpha) \in (-\delta, \delta) \times (-\epsilon, \epsilon)^d$ on $\tilde{O}$ such that $p = (0, \dots, 0)$, the metric is close to Minkowski and the sets $\{\pm\frac{19}{20}\delta\} \times (-\epsilon, \epsilon)^d$ are contained in $I^\pm(p, \tilde{O})$ (by choosing $\tilde{O}$ smaller if necessary).

For every $\underline{x} \in (-\epsilon, \epsilon)^d$ there is at most one $t \in (-\delta, \delta)$ such that $(t, \underline{x}) \in \partial\iota(M)$: If there were more than one then they could be connected by a (vertical) timelike curve, contradicting achronality.

As p is a boundary point there is a point $\tilde{O} \ni q \in \iota(M)$. It can be connected by a vertical curve which does not cross $\partial\iota(M)$ to one of $\{\pm\frac{19}{20}\delta\} \times (-\epsilon, \epsilon)^d$. Therefore the differentiable timelike curve $\sigma : (0, \frac{19}{20}\delta) \to \tilde{O}, \sigma(s) = (\pm s, 0, \dots, 0)$ lies in $\iota(M)$, as it can be connected to q by a curve which does not cross the boundary. □

REMARK 4.4. Lemma 4.1 and Lemma 4.3 together imply that if $\partial^+\iota(M) = \emptyset$ then $\partial\iota(M) = \partial^-\iota(M)$, with a similar statement obtained by reversing time-orientation.

In order to prove Theorems 3.3 and 3.5 we will need the following two Lemmas.

LEMMA 4.5. *We consider a globally hyperbolic spacetime (M, g) and a C^0-extension $(\tilde{M}, \tilde{g})$ with embedding $\iota : M \to \tilde{M}$. If there exists a spacelike Cauchy hypersurface $\mathscr{S} \subset M$ such that for all timelike curves $\gamma \subset M$ the set*

$$W := I^+(I^-(I^+(\gamma, I^-(\mathscr{S}, M)), M), M) \cap \mathscr{S}$$

is precompact in M, then for any $p \in \partial^-\iota(M)$ there exists a neighborhood $\tilde{O}$ of p such that the futuremost part of $\partial^-\iota(M)$ in $\tilde{O}$, $\{q \in \partial^-\iota(M) \cap \tilde{O} \,|\, I^+(q, \tilde{O}) \cap \partial^-\iota(M) = \emptyset\}$, is a non-empty achronal topological hypersurface.

Recall that the set W is shown in Figure 3.1.

REMARK 4.6. A simple condition that gives a precompact W is an upper bound for the diameter of $I^+(\gamma, M) \cap \mathscr{S}$ which is uniform with respect to γ.

PROOF. Let $p \in \partial^-\iota(M)$ be a point on the past boundary, γ the timelike curve approaching it, and $\mathscr{S}$ the spacelike Cauchy hypersurface as in the Lemma. As $\bar{W}$ (closure in M) is compact we can choose a neighborhood $\tilde{O}$ of p such that $\tilde{O} \cap \iota(\bar{W}) = \emptyset$. Choosing a smaller neighborhood if necessary we introduce coordinates $(x^\alpha) \in (-\delta, \delta) \times (-\epsilon, \epsilon)^d$ such that the metric is close to Minkowski, $\dot{\gamma} = \partial_{x^0}$ and $\{\pm\frac{19}{20}\delta\} \times (-\epsilon, \epsilon)^d \subseteq I^\pm(p, \tilde{O})$.

We consider the straight (in coordinates) timelike curves $\sigma_{\underline{x}} : (-\delta, \frac{19}{20}\delta] \to \tilde{O}$, $s \mapsto (s, \underline{x})$ where $\underline{x} \in (-\epsilon, \epsilon)^d$. These end in $\iota(M)$ by the construction of the coordinates. We define

$$b(\underline{x}) = \inf\{s \in (-\delta, \frac{19}{20}\delta] \mid \forall s' > s, \sigma_{\underline{x}}(s') \subset \iota(M)\}$$

and show that $b(\underline{x}) > -\delta$, i.e. that the $\sigma_{\underline{x}}$ intersect $\partial^-\iota(M)$ at least once, at $\sigma_{\underline{x}}(b(\underline{x}))$, and that

$$\sigma_{\underline{x}}(b(\underline{x})) \in \tilde{O} \setminus (I^+(p, \tilde{O}) \cup I^-(p, \tilde{O}))\,.$$

As $\mathscr{S}$ is a Cauchy hypersurface and by the definition of W, the future in $\tilde{O}$ of any point in $I_\gamma \cap \tilde{O}$, where

$$I_\gamma := \iota(I^-(I^+(\gamma, M) \cap I^-(\mathscr{S}, M), M))\,,$$

has to be contained in $\iota(M)$. In particular, as the future of a point is the future of any past directed curve ending at that point and $\gamma \cap \tilde{O} \subset I_\gamma$, $I^+(p, \tilde{O}) = I^+(\gamma, \tilde{O}) \subset \iota(M)$ and therefore

$$\sigma_{\underline{x}}(b(\underline{x})) \notin I^+(p, \tilde{O})\,.$$

We now have to show that $b(\underline{x}) > -\delta$ and $\sigma_{\underline{x}}(b(\underline{x})) \notin I^-(p, \tilde{O})$. If we assume this is false then there exists $s^- \in (b(\underline{x}), \frac{19}{20}\delta)$ such that $\sigma_{\underline{x}}(s^-) \in I^-(p, \tilde{O})$. But we also have $\sigma_{\underline{x}}(s^-) \subset I_\gamma$: The end point $\sigma_{\underline{x}}(\frac{19}{20}\delta)$ of $\sigma_{\underline{x}}$ is contained in $\{\frac{19}{20}\delta\} \times (-\epsilon, \epsilon)^d \subseteq I^+(p, \tilde{O}) = I^+(\gamma, \tilde{O})$, and therefore $\sigma_{\underline{x}}((b(\underline{x}), \frac{19}{20}\delta]) \subset I_\gamma$. Now, as $\sigma_{\underline{x}}(s^-) \subset I_\gamma$ and $p \in I^+(\sigma_{\underline{x}}(s^-), \tilde{O})$, we have $p \in \iota(M)$ by the argument in the previous paragraph, which is a contradiction.

Repeating the argument with p replaced by $\sigma_{\underline{x}}(b(\underline{x}))$ for all $\underline{x} \in (-\epsilon, \epsilon)^d$ shows that the set $\{\sigma_{\underline{x}}(b(\underline{x})) \mid \underline{x} \in (-\epsilon, \epsilon)^d\} = \{q \in \partial^-\iota(M) \mid I^+(q, \tilde{O}) \cap \partial^-\iota(M) = \emptyset\}$ is achronal in $\tilde{O}$. It is a topological hypersurface as p can't be an edge point by the properties of $b(\underline{x})$. $\square$

Under the stronger assumption that the Cauchy hypersurface $\mathscr{S}$ is compact, we obtain the following simpler result.

LEMMA 4.7. *We consider a globally hyperbolic spacetime (M,g) and a C^0-extension $(\tilde{M},\tilde{g})$ with embedding $\iota : M \to \tilde{M}$. If there exists a compact spacelike Cauchy hypersurface of M then $\partial^-\iota(M)$ is a locally achronal topological hypersurface.*

PROOF. Let $\mathscr{S}$ be the compact spacelike Cauchy hypersurface. As any subset of a compact set is precompact by definition, Lemma 4.5 applies. What remains to be shown is that the set $\{q \in \partial^-\iota(M) \cap \tilde{O} \,|\, I^+(q,\tilde{O}) \cap \partial^-\iota(M) = \emptyset\}$ is actually the full $\partial^-\iota(M) \cap \tilde{O}$, i.e. that there are no additional points in $\partial^-\iota(M)$ below it.

We choose a neighborhood $\tilde{O}$ and coordinates as in the proof of Lemma 4.5, but with $\tilde{O} \cap \mathscr{S} = \emptyset$. Now the future in $\tilde{O}$ of any point $q \in \partial^-\iota(M) \cap \tilde{O}$ lies in $\iota(M)$.

If there was a point $q \in \partial^-\iota(M) \cap \tilde{O}$ such that $\partial^-\iota(M) \ni p \in I^+(q,\tilde{O}) \cap \partial^-\iota(M)$ then p would lie in $\iota(M)$. $\square$

5. Proofs of the main theorems

The proofs depend on the following Lemma, which is a slight variation of a result of Sbierski [14]. We use his notation for the sets

$$C_a^- := \{0 \neq X \in \mathbb{R}^{d+1} | \frac{< X, e_0 >_{\mathbb{R}^{d+1}}}{|X|_{\mathbb{R}^{d+1}}} < -a\},$$

where $0 < a < 1$, $< .,. >_{\mathbb{R}^{d+1}}$ is the Euclidean scalar product in $\mathbb{R}^{d+1}$ and $|.|_{\mathbb{R}^{d+1}}$ is the Euclidean norm. The C_a^- are cones of vectors with angle less than $\cos^{-1}(a)$ to the x^0 axis with the tip of the cone pointing up.

LEMMA 5.1. *We consider a spacetime (M,g) with extension $(\tilde{M},\tilde{g})$. Given a neighborhood $\tilde{O}$ of a point on the boundary of $\iota(M)$, a point $p \in \tilde{O} \cap \iota(M)$ and a chart $\tilde{\psi} : \tilde{O} \to (-\delta,\delta) \times (-\epsilon,\epsilon)^d$ such that*

- *(i)* ∂_{x_0} *is timelike,*
- *(ii)* $|\tilde{g}_{\alpha\beta} - \eta_{\alpha\beta}| < \nu$ *where* η *is the Minkowski metric and* $1/2 > \nu > 0$ *a constant such that* $\forall a \in (-\delta,\delta) \times (-\epsilon,\epsilon)^d$, $\tilde{\psi}^{-1}(a + C_{5/6}^-) \subseteq I^-(\tilde{\psi}^{-1}(a),\tilde{O}) \subseteq \tilde{\psi}^{-1}(a + C_{5/8}^-)$,
- *(iii)* $\tilde{\psi}^{-1}(\{\tilde{x}^0 < -\frac{1}{10}\delta\}) \subseteq \tilde{M} \setminus I^-(\tilde{\psi}^{-1}(\{\frac{19}{20}\delta\} \times (-\epsilon,\epsilon)^d), \iota(M))$,
- *(iv)* $I^-(p,\iota(M)) \subset \tilde{O}$,
- *(v)* $\tilde{\psi}^{-1}(\{\frac{19}{20}\delta\} \times (-\epsilon,\epsilon)^d) \subseteq \iota(I^+(p,\tilde{O}))$,

there exist $q \in I^-(p,\iota(M)) \cap \tilde{O}$ and a constant $0 < C_d < \infty$ such that for all Cauchy hypersurfaces $\mathscr{S}$ of M the distance in $I^-(p,\iota(M)) \cap \iota(\mathscr{S})$ of any two points in $I^-(q,\iota(M)) \cap \iota(\mathscr{S})$ is bounded above by C_d.

PROOF. We choose $q \in I^-(p,\tilde{O}) \cap \iota(M)$ such that

$$(\tilde{\psi}(q) + C_{5/8}^-) \cap \{\tilde{x}^0 > -\delta/10\} \subset (\tilde{\psi}(p) + C_{5/6}^-) \cap \{\tilde{x}^0 > -\delta/10\},$$

i.e. such that the past of q in M lies completely inside a (Euclidean) cone contained in the past of p.

We assume $q \in \iota(I^+(\mathscr{S},M))$, as otherwise $I^-(\iota^{-1}(q),M) \cap \mathscr{S} = \emptyset$ and there is nothing to show. By property (iii) of the chart $\tilde{\psi}$ there exists a function $(-\epsilon,\epsilon)^d \to (-\delta,\delta)$, $\underline{x} \mapsto L_{\underline{x}}$ such that the timelike curves $\sigma_{\underline{x}} : (L_{\underline{x}},\delta) \to M, \sigma_{\underline{x}}(s) = \iota^{-1}(\tilde{\psi}^{-1}(s,\underline{x}))$ are past inextendible in M. As q lies in the past of p, property (v) holds for q as well. This implies that the curves $\sigma_{\underline{x}}$ intersect $I^+(\mathscr{S},M)$ and therefore intersect $\mathscr{S}$ exactly once, say at $s = f(\underline{x})$. We take this as the definition of $f : (-\epsilon,\epsilon)^d \to (-\delta,\delta)$.

We first show that f is smooth: As $\tilde{\psi}(\iota(\mathscr{S}) \cap \tilde{O})$ is a smooth submanifold, there exists for every point $(f(\underline{x}_0), \underline{x}_0)$ a neighborhood W and a smooth submersion $g : W \to \mathbb{R}$ such that $\tilde{\psi}(\iota(\mathscr{S}) \cap \tilde{O}) \cap W = \{g = 0\}$. As $\mathscr{S}$ is a Cauchy hypersurface no timelike vector can be tangent to it. Therefore $\partial_0 g|_{(f(\underline{x}_0), \underline{x}_0)} \neq 0$ and by the implicit function theorem there exists a smooth function $h : (-\epsilon, \epsilon)^d \supseteq V \to (-\delta, \delta)$, where V is a neighborhood of $\underline{x}_0$, such that $g(h(\underline{x}), \underline{x}) = 0$. Thus $f|_V = h$ and therefore f is smooth.

The next step is to show that $|\partial_i f|$ is bounded by a positive constant in $\tilde{O}$ for all i. As vectors tangent to $\mathscr{S}$ cannot be timelike we obtain the inequality

$$0 \leq \tilde{g}((\partial_i f)\partial_0 + \partial_i, (\partial_i f)\partial_0 + \partial_i) = (\partial_i f)^2 \tilde{g}_{00} + 2(\partial_i f)\tilde{g}_{0i} + \tilde{g}_{ii} \,. \tag{5.1}$$

By property (ii), $\tilde{g}_{00} < -1/2$, $\tilde{g}_{ii} > 1/2$, and $\tilde{g}_{0i} < 1/2$ and therefore this inequality is only satisfied for $(\partial_i f)_- \leq (\partial_i f) \leq (\partial_i f)_+$ where $(\partial_i f)_\pm$ are the values where equality holds in (5.1), i.e.

$$(\partial_i f)_\pm = \frac{-\tilde{g}_{0i} \mp \sqrt{(\tilde{g}_{0i})^2 - \tilde{g}_{ii}\tilde{g}_{00}}}{\tilde{g}_{00}} \,.$$

Again using property (ii), we see that $(\partial_i f)_\pm$, and therefore also $(\partial_i f)$, are bounded by a constant independent of f.

We define $\omega : (-\epsilon, \epsilon)^d \to (-\delta, \delta) \times (-\epsilon, \epsilon)^d$, $\omega(\underline{x}) = (f(\underline{x}), \underline{x})$. This parameterizes a smooth submanifold $\tilde{S}$ which is isometric to an open subset of $\mathscr{S}$ in M by $\iota^{-1} \circ \tilde{\psi}^{-1}$. We denote by $\bar{g}$ the metric induced on $\tilde{S}$ by $\tilde{g}$. The components of $\bar{g}$ are

$$\bar{g}_{ij} = \tilde{g}_{\mu\nu} \frac{\partial \omega^\mu}{\partial x_i} \frac{\partial \omega^\nu}{\partial x_j} = \tilde{g}_{00} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} + \tilde{g}_{0j} \frac{\partial f}{\partial x_i} + \tilde{g}_{i0} \frac{\partial f}{\partial x_j} + \tilde{g}_{ij} \,.$$

As $|\tilde{g}_{\mu\nu}|$ and $\partial_i f$ are bounded by the above, we have $|\bar{g}_{ij}| < C_{\bar{g}}$ for a positive constant $C_{\bar{g}}$.

We now consider two points $r, s \in I^-(q, \iota(M)) \cap \iota(M)$ as in the lemma. As $I^-(q, M) \subset \tilde{O}$ there exist $\underline{x}, \underline{y} \in (-\epsilon, \epsilon)^d$ such that $\omega(\underline{x}) = \tilde{\psi}(\iota(r))$ and $\omega(\underline{y}) = \tilde{\psi}(\iota(s))$. The length of the straight line $\sigma : [0, 1] \to (-\epsilon, \epsilon)^d$, $\sigma(\ell) = \underline{x} + \ell(\underline{y} - \underline{x})$ is given by

$$\begin{aligned}
L(\sigma) &= \int_0^1 \sqrt{\bar{g}(\dot{\sigma}(\ell), \dot{\sigma}(\ell))}\, \mathrm{d}\ell \\
&= \int_0^1 \sqrt{\sum_{i,j=1}^{d} (\underline{y}_i - \underline{x}_i)\bar{g}_{ij}(\sigma(\ell))(\underline{y}_j - \underline{x}_j)}\, \mathrm{d}\ell \\
&\leq \int_0^1 \sqrt{\sum_{i,j=1}^{d} 2\epsilon \cdot C_{\bar{g}} \cdot 2\epsilon}\, \mathrm{d}\ell \\
&= 2\epsilon d \sqrt{C_{\bar{g}}} \,.
\end{aligned}$$

As $I^-(q, \iota(M)) \cap \tilde{O} \subset \tilde{\psi}^{-1}(\tilde{\psi}(p) + C^-_{5/6})$ the curve σ is contained in $I^-(p, M) \cap \mathscr{S}$. As the distance in $I^-(p, M) \cap \mathscr{S}$ between r and s is defined as the infimum over the length of all piecewise smooth curves connecting them, and $\iota^{-1} \circ \tilde{\psi}^{-1} \circ \sigma$ is one such curve, this completes the proof. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. By Lemma 4.1 the boundary $\partial\iota(M)$ of the extension is achronal, hence differentiable almost everywhere, and by Lemma 4.3 we have for every $p_\partial \in \partial\iota(M)$ a timelike curve $\gamma : [0, 1] \to \tilde{M}$ such that $\gamma([0, 1)) \subset \iota(M)$ and $\gamma(1) = p_\partial$.

We assume that $\partial\iota(M)$ is spacelike at a point p_∂ at which it is differentiable and establish a contradiction. We choose a neighborhood $\tilde{O}$ of p_∂ such that $t < t_L$ in $\tilde{O} \cap \iota(M)$, where t_L is the constant appearing in the Definition 2.4 of expanding singularity. This is possible as $\partial\iota(M)$ is achronal: In any neighborhood of p_∂, either $t < t_L$ or $\{t = t_L\}$ is some achronal set lying above $\partial\iota(M)$ and we can find a smaller neighborhood of p_∂ which doesn't intersect it.

We choose coordinates $\tilde{x}^i$ on $\tilde{O}$ such that $\tilde{x}^i(p_\partial) = 0$, $\tilde{g}(p_\partial) = \eta$ (where η is the Minkowski metric) and $|\tilde{g}(x)_{\alpha\beta} - \eta_{\alpha\beta}| < \nu$ for all $x \in \tilde{O}$ and ν such that condition 2 of Lemma 5.1 is satisfied.

We perform a Lorentz boost to transform the normal vector of $\partial\iota(M)$ at p_∂ to ∂_{x^0} and, by choosing a smaller neighborhood $\tilde{O}$ if necessary, ensure that $\partial\iota(M)$ is almost horizontal, i.e. $\partial\iota(M) \cap \tilde{O} \subset \{-\delta/10 < \tilde{x}^0 < +\delta/10\}$ and therefore condition 3 is satisfied.

We choose a point $p \in \tilde{O} \cap \iota(M)$ such that $I^-(p, \iota(M)) \subset \tilde{O}$, satisfying condition 4. Finally, by choosing a smaller neighborhood, we can satisfy the remaining condition 5.

Applying Lemma 5.1 we obtain, for a point $q \in I^-(p, \iota(M))$ and for all Cauchy hypersurfaces $\mathscr{S}$, an upper bound for the distance in $I^-(p, \iota(M)) \cap \mathscr{S}$ of any two points $r, s \in I^-(q, M) \cap \mathscr{S}$. This is a contradiction to Definition 2.4 of expanding singularities. $\square$

Proof of Proposition 3.2. To prove that there exists no spacelike hypersurface $\Sigma_{p_\partial} \subset \tilde{M} \setminus \iota(M)$ such that $p_\partial \in \Sigma_{p_\partial}$ we use the same argument, but choose coordinates in $\tilde{O}$ such that Σ_{p_∂} instead of $\partial\iota(M)$ is almost horizontal. As Σ_{p_∂} is spacelike, the part of $I^-(p, \tilde{M})$ lying above Σ_{p_∂} is entirely contained in $\tilde{O}$, we denote it by $I_{\Sigma_{p_\partial}, p}$. As any past directed curve from p has to cross Σ_{p_∂} before leaving $\tilde{O}$ we have $I^-(p, \iota(M)) \subseteq I_{\Sigma_{p_\partial}, p}$ and therefore we obtain a contradiction as before. $\square$

The proofs of Theorems 3.3 and 3.5 proceed in a very similar way:

Proof of Theorems 3.3 and 3.5. By Lemma 4.7 the conditions of Theorem 3.3 imply that $\partial^-\iota(M)$ is a locally achronal topological hypersurface. Similarly, under the conditions of Theorem 3.5, Lemma 4.5 implies that the "locally futuremost part" of the past boundary $\partial^-\iota(M)$ is achronal, i.e. for every $p \in \partial^-\iota(M)$ there exists a neighborhood U of p such that

$$\{z \in \partial^-\iota(M) \cap U \mid I^+(z, U) \cap \partial^-\iota(M) = \emptyset\}$$

is an achronal topological hypersurface in U.

By the definition of $\partial^-\iota(M)$ we have for every $p \in \partial^-\iota(M)$ a timelike curve γ ending at p. The proof now proceeds analogously to that of Theorem 3.1 with $\partial\iota(M)$ replaced by $\partial^-\iota(M)$ in the case of Theorem 3.3 and with the "locally futuremost part" of $\partial^-\iota(M)$ in the case of Theorem 3.5.

The argument excluding the existence of a spacelike hypersurface $\Sigma_{p_\partial} \subset \tilde{M} \setminus \iota(M)$ such that $p_\partial \in \Sigma_{p_\partial}$ follows the proof of Proposition 3.2, except that to guarantee that $I^-(p, \iota(M))$ lies above Σ_{p_∂} we only need $\Sigma_{p_\partial} \cap \iota(\{t < t_L\}) = \emptyset$ as $t < t_L$ in $I^+(\partial^-\iota(M), \tilde{O})$. $\square$

We pass now to the

Proof of Proposition 1.4. In contrast to the proofs above, if the boundary is null we cannot use the pasts of points in M to identify subsets of Cauchy hypersurfaces in neighborhoods in $\tilde{M}$ with those in M. Instead we consider the total diameter of the (compact) Cauchy hypersurface. We first need to ensure that neighborhoods of the boundary contain the whole (embedded) Cauchy hypersurface $\mathscr{S}_s := \{t = s\}$, for s sufficiently small.

By the proof of Theorem 3.1 in [15] there exists a constant $\delta > 0$ such that for all $0 < s \leq \delta$ there is a map $\psi_s : \tilde{M} \to \tilde{M}$, the flow of a continuous timelike vector field on $\tilde{M}$, such that

$\psi_s(\partial\iota(M)) = \iota(\mathscr{S}_s)$ and these are compact. In [15] it is assumed that the extended manifold is at least C^3, but this can be relaxed in our case: The results of [15, Lemma 3.2] hold by our Lemma 4.1. Note that [15, Lemma 2.1] and [15, Lemma 3.3] require only differentiabilty of the manifold, not the metric. The rest of the proof takes place in (M, g) which is smooth in any case. In addition, [15, Theorem 3.1] assumes that the boundary itself is C^1 but this can similarly be relaxed as long as the Cauchy hypersurfaces $\mathscr{S}_t$ are C^1.

By Lemma 4.1 and Lemma 4.3 the boundary $\partial\iota(M) = \partial^-\iota(M)$ of the extension is achronal. We choose for each point $p \in \partial\iota(M)$ an open neighborhood in $\tilde{M}$ and coordinates such that the metric is close to Minkowski. As $\partial\iota(M)$ is compact we can find a finite subcover $\{O_i\}$. For each i there exists a constant δ_i such that for each $p \in \bar{O}_i \cap \partial\iota(M)$ and all $0 < s \leq \delta_i$, $\psi_s(p) \in \bar{O}_i$. As there are only finitely many i we can set $\delta_{min} = \min\{\delta_i\}$ and obtain $\mathscr{S}_s = \psi_s(\partial\iota(M)) \subset \bigcup_i \bar{O}_i$ for all $0 < s \leq \delta_{min}$.

The distance in $\mathscr{S}$ between any two points of a Cauchy hypersurface $\mathscr{S}$ is bounded above in each O_i by a constant independent of $\mathscr{S}$ by a similar argument as in the proof of Lemma 5.1. As $I^+(\partial\iota(M), \tilde{M}) \subset \iota(M)$ and the Cauchy hypersurfaces $\mathscr{S}_t$ are achronal in M they can intersect O_i only once (i.e. $\iota(\mathscr{S}_t) \cap O_i$ is connected), and therefore the total diameter of $\mathscr{S}_t$ is bounded. This contradicts the Definition 1.1 of a globally expanding singularity. $\square$

6. Examples

We will use the following Lemma to show that the examples below contain an expanding singularity towards the past, as required by Theorem 3.1:

LEMMA 6.1. *We consider a globally hyperbolic spacetime (M, g) of dimension $n+1$ with a Cauchy time function $t : M \to (0, \infty)$ such that $M = (0, \infty) \times \mathscr{S}$. Suppose that there exists $t_L > 0$ such that the subset $\{t < t_L\} \subset M$ can be covered by charts of the form $(0, t_L) \times U$, for some open subset $U \subseteq \mathscr{S}$, in which the metric takes the form*

$$g = g_{00}(t, x)\mathrm{d}t^2 + g_{ij}(t, x)\mathrm{d}x^i\mathrm{d}x^j\,, \quad g_{00} < 0\,,$$

and satisfies

$$g_{11}(t, x) \xrightarrow{t\to 0} \infty\,, \quad g_{ij}(t, x) \xrightarrow{t\to 0} 0 \quad \text{for} \quad (i, j) \neq (1, 1)\,, \tag{6.1}$$

$$\text{and either} \quad g_{1i} = 0 \quad \text{for} \quad i \neq 1 \quad \text{or} \quad \frac{g_{mi}g_{mj}}{g_{mm}g_{ij}} \xrightarrow{t\to 0} 0 \quad \text{for} \quad i, j < m \tag{6.2}$$

uniformly on compact subsets in x. Assume moreover that either

- *$\mathscr{S}$ is compact and $\{t < t_L\}$ is covered by a single chart as above,*
- *or for every $p \in \{t < t_L\} \subset M$ there exists a chart as above which contains $I^-(p, M)$ and a compact set $K_p \subset \mathscr{S}$ such that $I^-(p, M) \subset (0, t_\sigma) \times K_p$.*

Then (M, g) contains an expanding singularity towards the past.

PROOF. We need to show that for every $p \in \{t < t_L\}$ and $q \in I^-(p, M)$ there exist a sequence t_i decreasing to zero such that the diameter of $I^-(q, M) \cap \{t = t_i\}$ in $I^-(p, M) \cap \{t = t_i\}$ tends to infinity as $t_i \to 0$.

If the second condition in (6.2) holds, we start by defining an orthonormal frame $\{\theta^m\}$, $g_{ij}dx^idx^j = \delta_{ij}\theta^i\theta^j$ on each slice by

$$\theta^m = \sqrt{h^m_{mm}}(dx^m + \sum_{i=1}^{m-1}\frac{h^m_{mi}}{h^m_{mm}}dx^i)\,,$$

where
$$h^{m-1} = h^m - (\theta^m)^2 \quad \text{and} \quad h^n_{ij} := g_{ij}\,.$$
The tensor field h^{m-1} is positive definite on the subspace spanned by $\{\partial_1,\dots,\partial_{m-1}\}$: Indeed, if there were a vector $X = X^1\partial_1 + \dots X^{m-1}\partial_{m-1}$ such that $h^{m-1}(X,X) \le 0$ then we could choose X^m so that $\theta^m(X + X^m\partial_m) = 0$. Setting $0 \neq Y := X + X^m\partial_m$, this gives $h^{m-1}(Y,Y) = h^{m-1}(X,X) \le 0$ and $h^m(Y,Y) = h^{m-1}(X,X) \le 0$, giving a contradiction to the positive definiteness of g.

Writing h^{m-1} in terms of h^m we obtain
$$h^{m-1} = \sum_{i,j=1}^{m-1} \left(h^m_{ij} - \frac{h^m_{mi}h^m_{mj}}{h^m_{mm}} \right) dx^i dx^j\,.$$
One checks that $h^{m-1}_{ij}/h^m_{ij} \to 1$ follows from (6.2), which further implies $h^m_{ij}/g_{ij} \to 1$ for $i,j \le m$. By construction we have $\theta^1 = \alpha(t,x)dx^1$, with $\alpha(t,x)/\sqrt{g_{11}(t,x)} \to 1$, and with all convergences uniform on compact subsets.

We now choose $k_0 \in \mathbb{N}$ and $\epsilon > 0$ such that the sequences of points
$$y^k_1 = (1/k, x^1(q) - \epsilon, x^2(q), \dots) \;\text{ and }\; y^k_2 = (1/k, x^1(q) + \epsilon, x^2(q), \dots)$$
fulfill $y^k_1, y^k_2 \in I^-(q,M)$ for all $k \in \{k_0, k_0+1, \dots\}$. The distance between y^k_1 and y^k_2, and therefore the diameter of $I^-(q,M) \cap \{t = 1/k\}$, in $I^-(p,M) \cap \{t = 1/k\}$ is bounded from below by
$$d_{\{1/k\}\times K_p}(y^k_1, y^k_2) = \inf_\sigma \int_{-\epsilon}^{\epsilon} \sqrt{\sum_{i=1}^{n} (\theta^i(\dot\sigma))^2 \mathrm{d}s} \ge \inf_\sigma \int_{-\epsilon}^{\epsilon} \sqrt{(\theta^1(\dot\sigma))^2}\mathrm{d}s = \inf_\sigma \int_{-\epsilon}^{\epsilon} |\alpha(t,x)\dot\sigma^1|\mathrm{d}s\,,$$
where the infimum is taken over curves $\sigma : [-\epsilon,\epsilon] \to \{1/k\} \times K_p$ (with $K_p := \mathscr{S}$ in the case where $\mathscr{S}$ is compact), such that $\sigma(-\epsilon) = y^k_1$ and $\sigma(\epsilon) = y^k_2$. If the first condition holds in (6.2) we obtain the same expression with $\alpha := g_{11}$.

We have
$$\begin{aligned}
\lim_{k\to\infty} d_{\{1/k\}\times K_p}(y^k_1, y^k_2) &= \lim_{k\to\infty} \inf_\sigma \int_{-\epsilon}^{+\epsilon} |\alpha(\sigma(s))\dot\sigma^1(s)|\mathrm{d}s \\
&> \lim_{k\to\infty} \left(\min_{x\in K_p} \alpha(1/k, x) \right) \inf_\sigma \int_{-\epsilon}^{+\epsilon} |\dot\sigma^1(s)|\mathrm{d}s\,,
\end{aligned}$$
where the minimum diverges by the uniform divergence of α on K_p and the integral is independent of k and positive as $\int_{-\epsilon}^{+\epsilon} \dot\sigma^1(s)\mathrm{d}s = 2\epsilon$. □

REMARK 6.2. Lemma 6.1 directly generalizes to the case where the expanding direction rotates within the $t = \text{const}$ hypersurfaces. Indeed, let $\{t_k\}$ be a sequence of times and $\{X_k\}$ a sequence of associated expanding directions (constant in x unit vectors w.r.t. the Euclidean metric δ_{ij}) in each hypersurface $\mathscr{S}_k := \{t = t_k\}$. A spatial rotation for each t_k gives $X_k = \partial_{x^1}$ and the conditions on the metric are as before in these adapted coordinates. This is relevant for the case of Bianchi IX and BKL singularities discussed in Sections 6.4 and 6.5 below.

6.1. AVTD metrics without symmetries

These solutions, constructed in [6, 7], take the form

$$\mathrm{d}s^2 = -e^{-2\sum_{a=1}^{3}\beta^a}\mathrm{d}\tau^2 + \sum_{a=1}^{3} e^{-2\beta^a}\mathcal{N}^a{}_i\mathcal{N}^a{}_j\,\mathrm{d}x^i\mathrm{d}x^j\,, \tag{6.3}$$

with β^a and $\mathcal{N}^a{}_i$, $i, a \in \{1,2,3\}$, depending on all coordinates τ, x^i and behaving asymptotically as

$$\beta^a = \beta^a_\circ + \tau p^a_\circ + O(e^{-\tau\nu}) \quad \text{and} \quad \mathcal{N}^a{}_i = \delta^a_i + O(e^{-2\tau(p^i_\circ - p^a_\circ)})\,, \tag{6.4}$$

with $\nu > 0$. They are parameterized by freely prescribable analytic functions $\beta^2_\circ, \beta^3_\circ$ and $P_\circ{}^2{}_1$ depending on all space coordinates and two analytic functions $p^2_\circ$ and $p^3_\circ$ depending on all space coordinates which are free except for the inequalities

$$0 < p^2_\circ < (\sqrt{2}-1)p^3_\circ\,. \tag{6.5}$$

The function $P_\circ{}^2{}_1$ does not appear in the asymptotic expansion (6.4) but influences lower order terms in the expansion of $\mathcal{N}^a{}_i$.

The remaining exponent $p^1_\circ$ is given by

$$p^1_\circ = -\frac{p^2_\circ p^3_\circ}{p^2_\circ + p^3_\circ} < 0\,, \tag{6.6}$$

i.e. x^1 is the expanding direction.

The solutions approach a curvature singularity as $\tau \to \infty$.

The construction of these spacetimes in [7] was done in a purely local manner, regardless of the topology of the solutions. Here we will assume that the spatial topology is compact.

We replace the time coordinate τ with $t = -\log\tau$, giving, with the asymptotic expansion (6.4) inserted,

$$\mathrm{d}s^2 = -t^{2\sigma_{p_\circ}-2}e^{-2\sigma_{\beta_\circ}}(1+O^\nu)\mathrm{d}t^2+\sum_{a=1}^{3} t^{2p^a_\circ}e^{-2\beta^a_\circ}(1+O^\nu)(\delta^a_i+O(t^{2(p^i_\circ-p^a_\circ)}))(\delta^a_j+O(t^{2(p^j_\circ-p^a_\circ)}))\mathrm{d}x^i\mathrm{d}x^j \tag{6.7}$$

where $O^\nu = O(t^\nu)$, $\sigma_{p_\circ} = p^1_\circ + p^2_\circ + p^3_\circ$, and $\sigma_{\beta_\circ} = \beta^1_\circ + \beta^2_\circ + \beta^3_\circ$. We see directly that $g_{11} \to \infty$ and $g_{ij} \to 0$, satisfying (6.1). From (6.7) we find that the metric component g_{ij} is of the same order as the faster decaying one of g_{ii} and g_{jj} and that $g_{ii}/g_{jj} \to 0$ for $i > j$. This implies that (6.2) is also satisfied, i.e. the solutions are of the form required in Lemma 6.1.

6.2. T^3 Gowdy

The T^3 Gowdy spacetimes have metrics of the form

$$g = e^{(\tau-\lambda)/2}(-e^{-2\tau}\mathrm{d}\tau^2 + \mathrm{d}\theta^2) + e^{-\tau}(e^P\mathrm{d}\sigma^2 + 2e^PQ\mathrm{d}\sigma\mathrm{d}\delta + (e^PQ^2 + e^{-P})\mathrm{d}\delta^2)\,,$$

where λ, P and Q are functions of τ and θ and the singularity is approached as $\tau \to \infty$. The $\tau = \text{const}$ slices are toroidal and therefore compact. Ringström showed [16] that generic Gowdy spacetimes asymptotically behave as follows:

$$P(\tau,\theta) = v_a(\theta)\tau + \phi(\theta) + o(1), \qquad Q(t,\theta) = q(\theta) + o(1), \qquad \lambda = v_a(\theta)^2\tau + o(\tau)$$

where $0 < v_a(\theta) < 1$ and the lower order terms converge uniformly. Therefore the $g_{\theta\theta}$ component of the metric diverges towards the singularity while all other space components converge to zero. By redefining the time coordinate as $t := -\ln\tau$ the metric can be brought to the form required in Lemma 6.1: since the off-diagonal components $g_{\theta i}$ vanish, so (6.2) is satisfied.

6.3. Further AVTD spacetimes

Using Fuchsian methods, asymptotically Kasner-like spacetimes without symmetries have been constructed in the presence of various matter fields and in vacuum (either for spacetime dimension higher than 10, or the ones described in section 6.1 which exist for a restricted set of asymptotic data). The constructions generally start by defining a reduced evolution system, the "velocity term dominated" (VTD) or "Kasner-like" system, which does not include spatial derivatives, and then using Fuchs-type theorems to show that solutions of the full Einstein equations which approach these exist. These theorems guarantee a convergence which is uniform on compact subsets, as required by Lemma 6.1. Assuming that the spatial manifold is compact, the only things left to verify are the conditions (6.1), (6.2)

In the case of stiff fluid or scalar field matter there is no expanding direction and Lemma 6.1 does not apply [17, 18].

In the case of ≥ 10 dimensional vacuum, there is at least one expanding direction [18]. These solutions can be constructed using the same approach as those in section 6.1, leading to the same behavior of the $\mathcal{N}^a_{\ i}$ and therefore also satisfy (6.2). Lemma 6.1 is applicable if the spatial manifold is compact.

There are various results on general (non-Gowdy) T^2 symmetric spacetimes, the most general of which assumes the so-called "half-polarization" condition [19, 20]. These have one expanding direction, satisfy (6.2), and are spatially compact.

6.4. "Mixmaster" Bianchi IX spacetimes

The Bianchi models are homogeneous, but generically anisotropic, spacetimes, which are divided into types according to the structure constants of their Killing vector fields.

As shown by Ringström [21], generic solutions of type IX have at least three α-limit points on the "Kasner circle", i.e. they approach at least three different Kasner metrics arbitrarily closely as the singularity is approached. It is conjectured that generic α-limit sets contain an infinite number of points on the Kasner circle, and that the dynamics approaches that of the discrete "Kasner map", which shows chaotic behavior [22]. One would naively expect to be able to choose a sequence of times and directions as in Remark 6.2. However, it has been pointed out to us by Hans Ringström (private communication) that this expectation is incorrect, and that with some work one can infer from [21] that the space-diameter of the surfaces of homogeneity in all Bianchi IX vacuum models approaches zero as the singularity is approached.

6.5. Generic spacelike singularities in the context of the BKL conjecture

The BKL conjecture states, roughly speaking, that generic spacelike singularities behave at each spatial point as a "Mixmaster" Bianchi IX solution.

There are various heuristic arguments supporting this conjecture. Using the so-called "cosmological billiards formalism", as described in [23], the metric takes the form (6.3), with the same behavior of the $\mathcal{N}^a_{\ i}$ as in (6.4), and with the $\beta^a(\tau, x)$ now *not* showing linear behavior in τ, but rather a sequence of approximately linear phases (so-called Kasner epochs) connected by "bounces" off increasingly sharp potential walls. The β^a are expected to be unbounded towards the singularity ($\tau \to \infty$), but they might well be bounded from below, and therefore it is not clear (and perhaps unlikely, given the Bianchi IX result mentioned above) whether one can choose a sequence of times and directions as described in Remark 6.2.

It is expected that such solutions will develop particle horizons (this is sometimes referred to as "asymptotic silence") [24]. This would imply that the second option in Lemma 6.1 is fulfilled, i.e. $I^-(p, M) \subset (0, t_\sigma) \times K_p$ for all points p sufficiently close to the singularity and some compact set K_p. Whether or not this can be used to infer that these spacetimes are C^0 inextendible remains to be seen.

It should be emphasized that the arguments of [23] are heuristic.

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8 CONCLUSIONS

In this thesis we have investigated several asymptotic problems in non-symmetric spacetimes. We have constructed stationary solutions of the Einstein equations with a negative cosmological constant coupled to various matter fields. These put recent numerical constructions on a rigorous footing and are of independent physical interest, e.g. because of their relation to the AdS/CFT conjecture. They exemplify the much wider variety of possible stationary solutions compared to the $\Lambda = 0$ case.

We investigated the asymptotic behavior of singularities, first within the framework of BKL: We proved existence of a large class of non-symmetric AVTD spacetimes containing a timelike instead of spacelike singularity, and we provided a new analysis of the only known class of inhomogeneous chaotic BKL solutions. Inspired by the question of strong cosmic censorship we analyzed C^0-extensions across a newly defined category of "expanding singularities" and showed that such extensions are only possible through non-compact boundaries which are null almost everywhere.

8.1 OPEN QUESTIONS

There are a number of open questions and opportunities for further research arising from our results:

- From our numerical investigations in Section 3.2 of [25] it seems clear that the non-degeneracy results of Chapter 5, and therefore the construction of stationary black hole solutions in Chapter 4, should work for all values of the mass parameter (except for the critical one). Making these numerical results rigorous would require proving global existence of solutions for a scalar Riccati equation with a complicated potential term. There has been some recent work in this area, which unfortunately does not apply directly to our case [57].

- Our use of a periodic t coordinate in Chapter 4 means that we cannot construct boson star solutions with a black hole. Such solutions are of physical interest, so it would be useful to remove this restriction.

- In Chapter 7 we can rule out spacelike, but not null, C^0-extensions across expanding singularities. It would be of interest to either prove C^0-inextendibility, by ruling out the null case as well, or to provide an example of a spacetime that shows this behavior.

Any such example would necessarily have to be spatially inhomogeneous, as otherwise the arguments of Sbierski [96] show its inextendibility.

- Our results in Chapter 7 apply to the spacetimes constructed in [69] but not to their close analogues with timelike singularities in Chapter 6. In general, C^0-extendibility across timelike boundaries, such as the Kerr ring singularity, has not yet been studied in detail. This would require the development of new methods, as futures and pasts of points cannot be used to identify subsets between the original and extended manifold.

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