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Polyhomogeneous solutions of wave equations in the radiation regime

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Abstract

While the physical properties of the gravitational field in the radiation regime are reasonably well understood, several mathematical questions remain unanswered. The question here is that of existence and properties of gravitational fields with asymptotic behavior compatible with existence of gravitational radiation. A framework to study those questions has been proposed by R. Penrose [41], and developed by H. Friedrich [25, 27, 28] using conformal completions techniques. In this conformal approach one has to 1) construct initial data, which satisfy the general relativistic constraint equations, with appropriate behavior near the conformal boundary, and 2) show a local (and perhaps also a global) existence theorem for the associated evolution problem. In this context solutions of the constraint equations can be found by solving a nonlinear elliptic system of equations, one of which resembles the Yamabe equation (and coincides with this equation in some cases), with the system degenerating near the conformal boundary. In the first part of the talk I (PTC) will describe the existence and boundary regularity results about this system obtained some years ago in collaboration with Helmut Friedrich and Lars Andersson. Some new applications of those techniques are also presented. In the second part of the talk I will describe some new results, obtained in collaboration with Olivier Lengard, concerning the evolution problem.

1. Introduction

Bondi *et al.* [9] together with Sachs [42] and Penrose [41], building upon the pioneering work of Trautman [43, 44], have proposed in the sixties a set of boundary conditions appropriate for the gravitational field in the radiation regime. A somewhat simplified way of introducing the Bondi–Penrose (BP) conditions is to assume

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existence of "asymptotically quasi-Minkowskian coordinates" $(x^\mu) = (t, x, y, z)$ in which the space-time metric $\mathfrak{g}$ takes the form

$$\mathfrak{g}_{\mu\nu} - \eta_{\mu\nu} = \frac{\overset{1}{h}_{\mu\nu}(t-r,\theta,\varphi)}{r} + \frac{\overset{2}{h}_{\mu\nu}(t-r,\theta,\varphi)}{r^2} + \dots \ , \tag{1.1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric diag$(-1, 1, 1, 1)$, u stands for $t-r$, with r, θ, φ being the standard spherical coordinates on $\mathbb{R}^3$. The expansion above has to hold at, say, fixed u, with r tending to infinity. Existence of classes of solutions of the vacuum Einstein equations satisfying the asymptotic conditions (1.1) follows from the work in [25] together with [2, 3, 24]. As of today it remains an open problem how general, within the class of radiating solutions of vacuum Einstein equations, are those solutions which display the behavior (1.1). Indeed, the results in [1, 2, 23] suggest strongly[1] that the correct setup for such gravitational fields is that of *polyhomogeneous* asymptotic expansions:

$$\mathfrak{g}_{\mu\nu} - \eta_{\mu\nu} \in \mathscr{A}_{\rm phg} \ . \tag{1.2}$$

In the context of of expansions in terms of a radial coordinate r tending to infinity, the space of polyhomogeneous functions is defined as the set of smooth functions which have an asymptotic expansion of the form

$$f \sim \sum_{i=0}^{\infty}\sum_{j=0}^{N_i} f_{ij}(u,\theta,\varphi)\frac{\ln^j r}{r^{n_i}} \ , \tag{1.3}$$

for some sequences n_i, N_i, with $n_i \nearrow \infty$. Here the symbol $\sim$ stands for "being asymptotic to": if the right-hand-side is truncated at some finite i, the remainder term falls off appropriately faster. Further, the functions f_{ij} are supposed to be smooth, and the asymptotic expansions should be preserved under differentiation. The choice of the sequences n_i, N_i is not arbitrary, and is dictated by the equations at hand. For example, the analysis of $3+1$ dimensional Einstein equations in [23] suggests that consistent expansions can be obtained with $n_i = i$. On the other hand, Theorem 3.2 below gives actually $n_i = i/2$ for wave-maps on $2+1$ dimensional Minkowski space-time. We note that the $2+1$ dimensional wave map equation is related to the vacuum Einstein equations with cylindrical symmetry (*cf., e.g.*, [8, 19, 20]).

To understand how the polyhomogeneous expansions (1.2) arise in general relativity, recall that one systematic method of constructing solutions of the vacuum Einstein equations,

$$R_{\mu\nu} = 0 \ , \tag{1.4}$$

proceeds by solving a Cauchy problem. In such a set-up one provides an initial data set (M, g_{ij}, K_{ij}), where M is a three dimensional manifold, g_{ij} is a Riemannian metric on M, and K_{ij} is a symmetric tensor field on M. Those data are not arbitrary, as they have to satisfy the general relativistic constraint equations:

$$D_i(K^{ij} - g^{kl}K_{kl}g_{ij}) = 0 \ , \tag{1.5}$$

$$R = K^{ij}K_{ij} - (g^{kl}K_{kl})^2 \ . \tag{1.6}$$

[1] *Cf.* [37] and references therein for some further related results.

One then constructs a space–time $(\mathscr{M}, \mathfrak{g})$ using (M, g_{ij}, K_{ij}) as Cauchy data for Equation (1.4). We shall start, in the next section, by a discussion of those aspects of Equations (1.5)-(1.6) which are relevant to the problem at hand.

2. The constraints

There is no systematic way known of constructing solutions of Equations (1.5)-(1.6), except in the special case

$$g^{kl}K_{kl} = \text{const} \tag{2.1}$$

(*cf.*, however, [6, 7, 15, 32, 33] for results under various restrictive assumptions). When (2.1) holds, solutions of (1.5)-(1.6) can be produced using the conformal method of Choquet–Bruhat — Lichnerowicz — York [17]. In this method one prescribes "seed fields" (h_{ij}, A_{ij}), where h_{ij} is a Riemannian metric on M and A_{ij} is a symmetric and traceless tensor field on M. In our context (M, h_{ij}) will be a "smoothly compactifiable" Riemannian manifold, to be defined below, and we can further use the normalisation $g^{kl}K_{kl} = 3$; in this case the fields (g_{ij}, K_{ij}) are obtained by setting

$$g_{ij} = \phi^4 h_{ij} \ , \qquad K^{ij} = \phi^{-10} B^{ij} + g^{ij} \ , \tag{2.2}$$

where

$$B^{ij} \equiv A^{ij} + D^i X^j + D^j X^i - \frac{2}{3} D_k X^k \ , \tag{2.3}$$

and where ϕ and X^i are solutions of the equations

$$D_i \left(D^i X^j + D^j X^i - \tfrac{2}{3} D_k X^k \right) = -D_i A^{ij} \ , \tag{2.4}$$

$$8\Delta_h \phi - R(h)\phi + h^{ij} h^{kl} B_{ij} B_{kl} \phi^{-7} - 6\phi^5 = 0 \ . \tag{2.5}$$

In Equations (2.3)–(2.4) the symbol D denotes the Levi-Civita derivative associated with the metric h_{ij}. Further, in (2.5), Δ_h is the Laplace-Beltrami operator of the metric h_{ij}, while $R(h)$ is the scalar curvature of that same metric.

In order to obtain initial data corresponding to the gravitational field in the radiating regime one has to impose suitable boundary conditions. A first guess as to what class of (h_{ij}, A_{ij})'s to consider, stems from the already mentioned proposal of Bondi, as geometrised by Penrose [41] using conformal methods: Penrose advocates the use of *conformally compactifiable* (M, h_{ij})'s. This means that M is the interior of a compact manifold with boundary $\overline{M} = M \cup \partial M$. Further, if x is a defining function for ∂M, then the metric h should be of the form

$$h_{ij} = x^{-2} \mathring{h}_{ij} \ ,$$

where $\mathring{h}_{ij}$ is a smooth Riemannian metric on $\overline{M}$. Finally the tensor field $x^{-3}A^{ij}$ should be a smooth tensor field on $\overline{M}$ (*cf., e.g.*, [2, Section 2.1] for a detailed discussion). By analogy with hyperboloids in Minkowski space-time, such initial data will be called *hyperboloidal initial data.* The first question that arises here is that of existence, and of properties, of solutions of Equations (2.4)-(2.5) under such conditions; we note that those equations constitute an elliptic system which degenerates uniformly at ∂M. In [2] a general framework has been developed to handle such systems, and in particular the following has been shown:

Theorem 2.1 *For any smooth* (M, h_{ij}, A^{ij}) *as above there exists a unique* polyhomogeneous *solution of (2.4)–(2.5). Further,*

1. *For given* M *and* x *there exists an open dense set (in the* $C^\infty(\overline{M})$ *topology) of* (h_{ij}, A^{ij})*'s for which the function* ϕ^{-2} *can be extended to a* C^2 *function on* $\overline{M}$*, but* not *to a* C^3 *function on* $\overline{M}$ *(the third derivatives of any extension of* ϕ *will blow up logarithmically as one approaches* ∂M*); in particular for generic (in the above sense) couples* (h_{ij}, A^{ij}) *the initial data set* (g_{ij}, K^{ij}) *will display asymptotic behaviour* incompatible *with the Bondi-Penrose asymptotic conditions.*

2. *There exists a "large set" of* non-generic (h_{ij}, A^{ij}) *for which* ϕ *and the* B^{ij}*'s are smooth on* $\overline{M}$*; for initial data resulting from such "seed fields" the Bondi-Penrose asymptotic conditions will be satisfied.*

In the theorem above *polyhomogeneity* should be understood in terms of asymptotic expansions in x at $x = 0$, analogous to (1.3) with r there replaced by $1/x$.

As another illustration of the results in [2], we shall consider here two toy equations which are often encountered in the literature on non-linear[2] elliptic equations. Let, in the remainder of this section, g be a smooth Riemannian metric on a smooth compact manifold with boundary $\overline{M}$, consider the problem

$$\Delta_g \varphi = \mid \varphi \mid^{\alpha-1} \varphi \ , \qquad \alpha > 1 \ , \tag{2.6}$$

$$\lim_{p \to \partial M} \varphi(p) = \infty \quad , \tag{2.7}$$

Δ_g — the Laplace-Beltrami operator on functions. It was shown in [4] that one then necessarily has (*cf.* also [45])

$$\lim_{p \to \partial M} \left[\varphi(p)\ x(p)^{\frac{2}{\alpha-1}} \right] = C_\alpha \equiv \left(\frac{2(\alpha+1)}{(\alpha-1)^2} \right)^{\frac{1}{\alpha-1}} \quad . \tag{2.8}$$

Here $x(p)$ denotes the Riemannian distance from p to ∂M. It follows from the results of [2] that the behavior of φ near ∂M can be described in a much more precise way. More precisely, we have the following ([21]; compare [5]):

Theorem 2.2 *Let* $(\overline{M}, g)$ *be a smooth compact Riemannian manifold with boundary, let* $\alpha > 1$*, set* $M = int\overline{M}$*. Let* $\varphi \in W_2^{p,loc}(M)$ *satisfy*

$$\Delta_g \varphi = \mid \varphi \mid^{\alpha-1} \varphi,$$

$$\lim_{p \to \partial M} \left[\varphi(p)\ x(p)^{\frac{2}{\alpha-1}} \right] = C_\alpha,$$

where $x(p) = dist(p, \partial M)$*. Define*

$$\mu_+ = \frac{2(\alpha+1)}{\alpha-1}.$$

[2] We wish to stress that while all the equations considered in this section are semi-linear, the techniques of [2] apply as well to fully non–linear equations, *cf.* [2, Section 5.1].

Let $\overline{N}$ be a compact submanifold of $\overline{M}$ forming a neighborhood of ∂M such that $\varphi\,|_{\overline{N}}> 0$, and $x \in C^\infty(\overline{N})$. There exist functions $\varphi_i \in C^\infty(\overline{N}), i \in \mathbb{N}$, such that the following hold : For any $k \in \mathbb{N}$ let $I \in \mathbb{N}$ be such that $I\mu_+ \geq k$. Then

1. *if $\mu_+ \notin \mathbb{N}$, then*

$$x^{\frac{2}{\alpha-1}}\varphi - \varphi_0 - x^{\mu_+}\varphi_1 - \ldots - x^{I\mu_+}\varphi_I \in C^k(\overline{N}).$$

2. *If $\mu_+ \in \mathbb{N}$, then*

$$x^{\frac{2}{\alpha-1}}\varphi - \varphi_0 - x^{\mu_+}\ln x\varphi_1 - \ldots - x^{I\mu_+}\ln^I x\varphi_I \in C^k(\overline{N}).$$

If $\varphi_1\,|_{\partial M}= 0$, then (in both cases)

$$x^{\frac{2}{\alpha-1}}\varphi \in C^\infty(\overline{N}).$$

Remarks: 1. Similar results can be established using [2] when (M, g) is only of finite degree of differentiability, with some finite number of functions φ_i of finite degree of differentiability.
2. For $j \leq Int(\mu_+)$ and $j < \mu_+$ the functions $\partial_x^j\varphi_0|_{\partial M}$ are uniquely determined by $g_{ij}|_{\partial M}$ and $\partial_x^k g_{ij}|_{\partial M}$, for some finite number of transverse derivatives of g_{ij} at ∂M. Similarly when $\mu_+ \in \mathbb{N}$ the function $\varphi_1\,|_{\partial M}$ is uniquely determined by the boundary values of a finite number of derivatives of the metric in directions transverse to ∂M.

Proof: By elliptic regularity on $N = int\overline{N}$ we have $\varphi \in C^\infty(N)$. Set

$$u = \frac{\varphi x^{\frac{2}{\alpha-1}}}{C_\alpha} - 1.$$

$u \in C^\infty(N)$ and satisfies the equation

$$\begin{gathered} Lu = \omega + F(u)\ , \\ F(u) = C_\alpha^{\alpha-1}[(1+u)^\alpha - 1 - \alpha u]\ , \quad \omega = \tfrac{2x\Delta_g x}{\alpha-1}\ , \\ Lu = x^2\Delta_g u - \tfrac{4x}{\alpha-1}g(Du, Dx) - \tfrac{2}{\alpha-1}[\alpha + 1 + x\Delta x]u\ . \end{gathered}$$

L is of the form considered in [2] with indicial exponents

$$\mu_- = -1\ , \qquad \mu_+ = \frac{2(\alpha+1)}{\alpha-1}\ . \tag{2.9}$$

Let $\psi = \varphi\,|_{\partial N\cap M}$. By the arguments of proof of [2, Proposition 7.3.1] there exists a solution $\widehat{\varphi}$ of

$$\begin{gathered} \Delta\widehat{\varphi} = |\widehat{\varphi}|^{\alpha-1}\widehat{\varphi}\ , \\ \widehat{\varphi}|_{\partial N\cap M} = \psi\ , \\ \widehat{\varphi}x^{\alpha-1} - C_\alpha = O(x)\ . \end{gathered}$$

(The function u_0 required in [2, Proposition 7.3.1] can be taken as $u_0 \equiv 0$; *cf.* also Remark 2 following [2, Proposition 7.3.1]). By the asymptotic maximum principle (*cf., e.g.,* [31, Theorem 3.5]) we have $\varphi \equiv \widehat{\varphi}$ which implies

$$u = O(x) \ .$$

Theorem 7.4.1 of [2] gives

$$u \in \mathscr{A}_{\mathrm{phg}}(N) \cap C^0(\overline{N}) \ .$$

By a closer examination of the structure of Equation (2.6) one obtains the claimed results. □

Another example of equation to which the results of [2] apply is the equation

$$\Delta_g u = e^{\alpha u} \ , \qquad \alpha > 0 \ , \tag{2.10}$$

with the asymptotic condition

$$\lim_{p \to \partial M} u(p) = \infty \ . \tag{2.11}$$

It has been shown by Bandle and Marcus [M. Marcus, private communication] that u then necessarily satisfies, for $x \le \varepsilon < 1$, for some $\varepsilon > 0$,

$$|u - \frac{2}{\alpha} \ln \frac{1}{x} - \frac{1}{\alpha} \ln \frac{2}{\alpha}| \le Cx \ln \frac{1}{x} \ .$$

We have the following [21]:

Theorem 2.3 *Let $(\overline{M}, g)$ be a smooth Riemannian manifold with boundary, let $\alpha > 0$ and let $\varphi \in C^\infty(M)$ satisfy*

$$\Delta_g u = e^{\alpha u} \ .$$

Suppose that there exists a constant C such that

$$|u - \frac{2}{\alpha} \ln \frac{1}{x}| \le C \ .$$

Let $\overline{N}$ be a compact submanifold of $\overline{M}$ forming a neighborhood of ∂M such that $x \in C^\infty(\overline{M})$. There exist functions $\varphi_i \in C^\infty(\overline{N})$, $i \in \mathbb{N}$, such that, for all $k \in \mathbf{N}$,

$$u - \frac{2}{\alpha} \ln \frac{1}{x} - \frac{2}{\alpha} \ln \frac{2}{\alpha} - \varphi_1 x \ln x - \varphi_2 x^2 \ln^2 x - \ldots - \varphi_k x^k \ln^k x \in C^k(\overline{N}) \ .$$

$\varphi_1 | \partial M$ is proportional to the mean extrinsic curvature of ∂M in $\overline{M}$. If that mean extrinsic curvature vanishes, then

$$u - \frac{2}{\alpha} \ln \frac{1}{x} \in C^\infty(\overline{N}).$$

PROOF: Write u in the form $u = \frac{1}{\alpha}[2 \ln \frac{1}{x} + \ln \frac{2}{\alpha} + \omega]$. The function ω solves the equation

$$x^2 \Delta_g \omega = 2\Big(e^\omega - 1\Big) + 2x \Delta_g x.$$

The proof follows now that of Theorem 2.2 (in the present case we have $\mu_- = 0$, $\mu_+ = 1$). □

3. The evolution problem

It is natural to enquire what are the asymptotic properties of those solutions of Einstein equations with *essentially* polyhomogeneous initial data as in Theorem 2.1, that is, initial data having some non–vanishing logarithmic coefficients in their asymptotic expansion. Recall that causality of hyperbolic PDE's allows one always to construct a solution of such equations in a neighborhood (perhaps very "small") of the initial data surface. However, standard theory applied to our problem at hand gives a neighbourhood which is smaller and smaller in time as one approaches ∂M. Supposing, *e.g.*, that ∂M corresponds to the set $u = 0$ in terms of the coordinates of Equation (1.1), it could occur that the range of u's for which the solution exists shrinks to $\{0\}$ as r tends to infinity. This would be quite unpleasant from a radiation point of view, because to describe gravitational radiation one usually requires some open interval of u's at "$r = \infty$". Inspired by Penrose's treatment [41], Friedrich [25] has developed a conformal framework in which the evolution problem can be reduced to one on a compact manifold. In particular, for initial data as described in point 2 of Theorem 2.1, Friedrich's theorems guarantee existence of a uniform interval of u's for which the solution exists. This is done by considering a system of equations for conformally rescaled fields. It turns out that the generic initial data of Theorem 2.1 are too singular at ∂M to be able to use the theorems of [25], so that some generalizations are needed. In work in progress [22, 38] we are currently analysing those issues, but no definitive results are available yet as far as the vacuum Einstein equations are concerned. We have, however, developed certain techniques to handle such asymptotic problems, and we wish to report here on the results which we have already obtained in some simpler cases. Before doing this, let us shortly recall the conformal method.

3.1. Conformal completions

Consider an $n+1$ dimensional space-time $(\mathscr{M}, \mathfrak{g})$ and let

$$\tilde{\mathfrak{g}} = \Omega^2 \mathfrak{g} \; . \tag{3.1}$$

Let $\Box_h$ denote the wave operator associated with a Lorentzian metric h,

$$\Box_h f = \frac{1}{\sqrt{|\det h_{\rho\sigma}|}} \partial_\mu (\sqrt{|\det h_{\alpha\beta}|} h^{\mu\nu} \partial_\nu f).$$

We recall that the scalar curvature $R = R(\mathfrak{g})$ of $\mathfrak{g}$ is related to the corresponding scalar curvature $\tilde{R} = \tilde{R}(\tilde{\mathfrak{g}})$ of $\tilde{\mathfrak{g}}$ by the formula

$$\tilde{R}\Omega^2 = R - 2n\{\frac{1}{\Omega}\Box_{\mathfrak{g}}\Omega + \frac{n-3}{2}\frac{|\nabla\Omega|^2_{\mathfrak{g}}}{\Omega^2}\} \; . \tag{3.2}$$

It then follows from (3.2) that we have the identity

$$\Box_{\tilde{\mathfrak{g}}}(\Omega^{-\frac{n-1}{2}} f) = \Omega^{-\frac{n+3}{2}} \left(\Box_{\mathfrak{g}} f + \frac{n-1}{4n}(\tilde{R}\Omega^2 - R) f \right) \; . \tag{3.3}$$

It has been observed by Penrose [41] that the Minkowski space-time $(\mathscr{M},\eta)$ can be conformally completed to a space-time with boundary $(\widetilde{\mathscr{M}},\tilde\eta)$, $\tilde\eta=\Omega^{-2}\eta$ on $\mathscr{M}$, by adding to $\mathscr{M}$ two null hypersurfaces, usually denoted by $\mathscr{I}^+$ and $\mathscr{I}^-$, which can be thought of as end points ($\mathscr{I}^+$) and initial points ($\mathscr{I}^-$) of inextendible null geodesics [40, 41, 46]. We will only be interested in "the future null infinity" $\mathscr{I}^+$; an explicit construction (of a subset of $\mathscr{I}^+$) which is convenient for our purposes proceeds as follows: for $(x^0)^2<\sum_i (x^i)^2$ we define

$$y^\mu=\frac{x^\mu}{x^\alpha x_\alpha}\ ; \tag{3.4}$$

in the coordinate system $\{y^\mu\}$ the Minkowski metric $\eta\equiv-(dx^0)^2+(dx^1)^2+(dx^2)^2+(dx^3)^2=\eta_{\alpha\beta}dx^\alpha dx^\beta$ takes the form

$$\eta=\frac{1}{\Omega^2}\eta_{\alpha\beta}dy^\alpha dy^\beta\ , \qquad \Omega=\eta_{\alpha\beta}y^\alpha y^\beta\ . \tag{3.5}$$

We note that under (3.4) the exterior of the light cone $C_0^{x^\mu}\equiv\{\eta_{\alpha\beta}x^\alpha x^\beta=0\}$ emanating from the origin of the x^μ-coordinates is mapped to the exterior of the light cone $C_0^{y^\mu}=\{\eta_{\alpha\beta}y^\alpha y^\beta=0\}$ emanating from the origin of the y^μ-coordinates. The conformal completion is obtained by adding $C_0^{y^\mu}$ to $\mathscr{M}$,

$$\widetilde{\mathscr{M}}=\mathscr{M}\cup(C_0^{y^\mu}\setminus\{0\})\ ,$$

with the obvious differential structure arising from the coordinate system y^μ. We shall use the symbol $\mathscr{I}$ to denote $C_0^{y^\mu}\setminus\{0\}$, and $\mathscr{I}^+$ to denote $C_0^{y^\mu}\setminus\{0\}\cap\{y^0>0\}$. As already mentionted, $\mathscr{I}$ so defined is actually a subset of the usual $\mathscr{I}$, but this will be irrelevant for our purposes.

We note that (3.4) is singular at the light cone $C_0^{x^\mu}$. This is again irrelevant from our point of view because we are only interested in the behavior of the solutions near $\mathscr{I}^+$, and causality allows us to ignore this.

The above procedure can be adapted for several metrics of interest, such as the Schwarzschild, Kerr, or Robinson-Trautman metrics, to similarly yield conformal completions of space–time by the addition of null hypersurfaces $\mathscr{I}^+$. This observation was at the origin of Penrose's proposal to describe systems which are asymptotically flat in lightlike directions through the use of conformal completions.

It is noteworthy that the conformal technique allows one to reduce global-in-time existence problems to local ones; this has been exploited by various authors [11–14, 16, 18] for wave equations on a fixed background space-time. Further, Friedrich [27, 28, 30] has used this approach to obtain global existence result for Einstein equations to the future of a "hyperboloidal" Cauchy surface, with "small" smoothly compactifiable initial data, *cf.* also [26, 29].

On a more modest level, the identity (3.3) can be used as a starting point for the analysis of the asymptotic behavior of solutions of the scalar wave equation near $\mathscr{I}^+$, and we shall describe some such results in the next section. There are associated identities for fields of any spin [41], which provide a convenient framework for similar questions for those fields.

3.2. The semi-linear scalar wave equation

Let f be a solution of the semi-linear wave equation

$$\Box_{\mathfrak{g}} f = H(x^\mu, f) \;, \tag{3.6}$$

here $\Box_{\mathfrak{g}}$ is the d' Alembertian associated with $\mathfrak{g}$. Set

$$\tilde{f} = \Omega^{-\frac{(n-1)}{2}} f \;; \tag{3.7}$$

Letting $\tilde{\mathfrak{g}} = \Omega^2 \mathfrak{g}$ as in (3.1), from (3.3) we obtain

$$\Box_{\tilde{\mathfrak{g}}} \tilde{f} = \frac{n-1}{4n}(\tilde{R} - \frac{R}{\Omega^2})\tilde{f} + \Omega^{-\frac{n+3}{2}} H(x^\mu, \Omega^{\frac{n-1}{2}} \tilde{f}) \;. \tag{3.8}$$

Let $\mathfrak{g} = \eta$ be the Minkowski metric; under the conformal transformation (3.4) one obtains from (3.5) that $\tilde{\mathfrak{g}}$ is again the Minkowski metric, and (3.8) becomes

$$\Box_\eta \tilde{f} = \Omega^{-\frac{n+3}{2}} H(x^\mu, \Omega^{\frac{n-1}{2}} \tilde{f}) \;. \tag{3.9}$$

We shall assume that the initial data for f are given on a hypersurface $\Sigma \subset \mathscr{M}$, which, in a neighborhood $\mathscr{O}$ of $\mathscr{I}^+$ is given by the equation

$$\Sigma \cap \mathscr{O} = \{y^0 = \frac{1}{2}\} \;. \tag{3.10}$$

This correspond to a hyperboloid in $\mathscr{M}$ given by the equation $x^0 + 1 = \sqrt{1 + \vec{x}^2}$. It is convenient to introduce the following coordinate system (τ, x, v) in a $\widetilde{\mathscr{M}}$-neighborhood of $\mathscr{I}^+$:

$$\begin{aligned} \tau &= y^0 - 1/2 \geq 0 \;, \\ x &= (\sum (y^i)^2)^{\frac{1}{2}} - y^0 \geq 0 \;, \\ y^i &= (\sum (y^i)^2)^{\frac{1}{2}} n^i(v) \;, \end{aligned} \tag{3.11}$$

$n^i(v) \in S^{n-1}$, with $v = (v^A)$ denoting spherical coordinates on S^{n-1}. Equation (3.5) gives

$$\Omega = x(2\tau + x + 1) \approx x \;. \tag{3.12}$$

If we let h denote the unit round metric on S^{n-1}, we then have

$$\eta = 2dxd\tau + dx^2 + (x + \tau + 1/2)^2 h \;, \tag{3.13}$$

and

$$\begin{aligned} \Box_\eta \tilde{f} &= \frac{1}{(x+\tau+1/2)^{n-1}\sqrt{\det h}} \partial_\mu \left((x+\tau+1/2)^{n-1} \sqrt{\det h}\, \eta^{\mu\nu} \partial_\nu \tilde{f} \right) \\ &= \{ -\partial_\tau(\partial_\tau - 2\partial_x) + \frac{n-1}{x+\tau+1/2}\partial_x + \frac{\triangle_h}{(x+\tau+1/2)^2} \} \tilde{f} \;, \end{aligned} \tag{3.14}$$

where $\triangle_h$ is the Laplace-Beltrami operator of the metric h. We set

$$
\begin{aligned}
e_- = \partial_\tau \ , \quad e_+ = \partial_\tau - 2\partial_x \ , \quad e_A = \tfrac{1}{(x+\tau+1/2)} h_A \ , \qquad (3.15)\\
\phi_- = e_-(\tilde f) \ , \quad \phi_+ = e_+(\tilde f) \ , \qquad (3.16)\\
\phi_A = \psi_A = \tfrac{1}{(x+\tau+1/2)} h_A(\tilde f) \ , \qquad (3.17)
\end{aligned}
$$

where h_A denotes an h-orthonormal frame on S^{n-1}. (The usefulness of introducing two different objects for $h_A(\tilde f)/(x+\tau+1/2)$ will become clear shortly.) Equation (3.9) implies the following set of equations:

$$
\left.\begin{array}{rrrcr}
e_-(\phi_+) & -\mathscr{D}_{e_A}\psi_A & -\frac{n-1}{2(x+\tau+1/2)}\phi_+ & = & -\frac{n-1}{2(x+\tau+1/2)}\phi_- - G \ , \\
-e_A(\phi_+) & +e_+(\psi_A) & -\frac{1}{(x+\tau+1/2)}\psi_A & = & 0 \ ,
\end{array}\right. \qquad (3.18)
$$

$$
\left.\begin{array}{rrrcr}
e_-(\phi_A) & -e_A(\phi_-) & +\frac{1}{(x+\tau+1/2)}\phi_A & = & 0 \ , \\
-\mathscr{D}_{e_A}\phi_A & +e_+(\phi_-) & +\frac{n-1}{2(x+\tau+1/2)}\phi_- & = & \frac{n-1}{2(x+\tau+1/2)}\phi_+ - G \ ,
\end{array}\right. \qquad (3.19)
$$

$$
\begin{aligned}
e_-(\tilde f) = \phi_- \ , \qquad (3.20)\\
e_+(\tilde f) = \phi_+ \ , \qquad (3.21)\\
G = \Omega^{-\frac{n+3}{2}} H(x^\mu, \Omega^{\frac{n-1}{2}}\tilde f) \ . \qquad (3.22)
\end{aligned}
$$

We note that the partial differential operator standing on the left-hand-side of (3.18) is symmetric hyperbolic; the same holds true for (3.19), or for the joint system (3.18)-(3.21). Now, part of our technique consists in deriving weighted energy estimates for symmetric hyperbolic systems having the structure above; this is described in more detail in Section 3.4. Each such system comes with his own estimates, so that for the systems (3.18) and (3.19) we can obtain estimates with different weights. This allows us to handle a reasonably wide range of non-linearities, giving existence and blow-up control for initial data in weighted Sobolev spaces (with conormal-type blow-up at $\mathscr{I}^+$). Before proceeding further, some terminology is needed. As is well known, the Cauchy data for the wave equation consist of the values $f|_M$ of f on the initial data hypersurface M and of a transverse derivative there — *e.g.*, $(\partial f/\partial\tau)|_M$. Given such data all the derivatives $(\partial^i f/\partial\tau^i)|_M$ of a solution can be calculated on M in terms of $f|_M$, $(\partial f/\partial\tau)|_M$, and of space-derivatives thereof. For example, from Equation (3.14) we have

$$
\left.\frac{\partial^2 \tilde f}{\partial^2\tau}\right|_M = 2\partial_x(\partial_\tau \tilde f|_M) + \left\{\frac{n-1}{x+\tau+1/2}\partial_x + \frac{\triangle_h}{(x+\tau+1/2)^2}\right\}\tilde f|_M \ . \qquad (3.23)
$$

For *general* polyhomogeneous $f|_M$ and $(\partial f/\partial\tau)|_M$ the resulting $(\partial^i f/\partial\tau^i)_M$ will be polyhomogeneous at ∂M but *not* necessarily bounded; *e.g.*, the occurrence of $x\ln x$ terms in $\tilde f|_M$ or in $(\partial\tilde f/\partial\tau)|_M$ will cause — unless cancellations occur — a logaritmic singularity of $(\partial^2 f/\partial\tau^2)|_M$, with a further $1/x$ singularity in $(\partial^3 f/\partial\tau^3)|_M$, *etc.* We say that polyhomogeneous initial data are *k-compatible* if $(\partial^i f/\partial\tau^i)|_M$ are bounded by some power of $\ln x$ for $0 \le i \le k$. We say that the initial data are *compatible polyhomogeneous* if $(\partial^i f/\partial\tau^i)|_M$ are bounded by some (perhaps i–dependent) powers of $\ln x$ for all i. This condition is clearly necessary for propagation of polyhomogeneity — it follows from Theorem 3.1 below that it is also sufficient. It should be clear from Equation (3.23) that the condition of k–compatibility imposes a finite number of algebraic conditions on the coefficients which appear in the polyhomogeneous expansions (1.3) of and $(\partial\tilde f/\partial\tau)|_M$, and that those conditions are easy to satisfy. Similarly, the condition of compatibility for all k imposes an infinite number of conditions relating those coefficients, with a large set of initial data satisfying those conditions. (*E.g.*, if $\tilde f|_M$ and $(\partial\tilde f/\partial\tau)|_M$ are compatible polyhomogeneous, then for any functions $g,h \in C^\infty(\overline{M})$ the new initial data $\tilde f|_M \to \tilde f|_M + g$ and $(\partial\tilde f/\partial\tau)|_M \to (\partial\tilde f/\partial\tau)|_M + h$ will also be compatible polyhomogeneous.) In [22] we prove the following:

Theorem 3.1 *Consider Equation (3.6) on $\mathbb{R}^{n,1}$, $n \geq 2$, with polyhomogeneous bounded initial data $\tilde f|_{\{\tau=0\}}$, $\partial\tilde f/\partial\tau|_{\{\tau=0\}}$. Suppose further that $H(x^\mu, f)$ is bounded and polyhomogeneous in x^μ at constant f, and has a zero of order ℓ at $f=0$, with*

$$\ell \geq \begin{cases} 4\,, & n=2\,, \\ 3\,, & n=3\,, \\ 2\,, & n\geq 4\,. \end{cases}$$

Then:

1. *There exists $\tau_+ > 0$ such that f exists on a neighbourhood of $\mathscr{I}^+$ containing $\{\tau\in[0,\tau_+), x\in[0,x_0)\}$.*

2. *The solution has the property that D-derivatives (as defined in Equation (3.29) below) of arbitrary order of $\tilde f(\cdot,\tau)$ are continuous up-to-boundary on each neighbourhood $\mathscr{U}$ of $\mathscr{I}^+$ of the form $\mathscr{U} = \{\tau\in[0,\tau_*), x\in[0,x_1)\}$ on which $\tilde f$ exists.*

3. *If the initial data are* compatible *polyhomogeneous, then the solution is polyhomogeneous on each neighbourhood $\mathscr{U}$ of $\mathscr{I}^+$ on which f exists, where $\mathscr{U}$ is as in point 2. above.*

We note that the results of Joshi [35] (in the special case of H — smooth in both variables) on propagation of polyhomogeneity do not apply to the problem at hand when n is even, due to the occurence of non–integer powers of Ω in (3.22) in such a case.

3.3. Wave maps

Let $(\mathscr{N},h)$ be a smooth Riemannian manifold, and let $f : (\mathscr{M},\mathfrak{g}) \to (\mathscr{N},h)$ solve the wave map equation. We will be interested in maps f which have the property that f approaches a constant map f_0 as r tends to infinity along lightlike directions, $f_0(x) = p_0 \in \mathscr{N}$ for all $x\in\mathscr{M}$. Introducing normal coordinates around p_0 we can write $f=(f^a)$, $a=1,\ldots,N=\dim\mathscr{N}$, with the functions f^a satisfying the set of equations

$$\Box_{\mathfrak{g}} f^a + \mathfrak{g}^{\mu\nu}\Gamma^a_{bc}(f)\frac{\partial f^b}{\partial x^\mu}\frac{\partial f^c}{\partial x^\nu} = 0\,, \tag{3.24}$$

where the Γ^a_{bc}'s are the Christoffel symbols of the metric h. Setting as before $\tilde{f}^a = \Omega^{-\frac{n-1}{2}} f^a$, $\tilde{\mathfrak{g}} = \Omega^2 \mathfrak{g}$, we then have from (3.3),

$$\Box_{\tilde{\mathfrak{g}}} \tilde{f}^a = -\Omega^{-\frac{n-1}{2}} \tilde{\mathfrak{g}}^{\mu\nu} \Gamma^a_{bc}(\Omega^{\frac{n-1}{2}} \tilde{f}) \frac{\partial(\Omega^{\frac{n-1}{2}} \tilde{f}^b)}{\partial x^\mu} \frac{\partial(\Omega^{\frac{n-1}{2}} \tilde{f}^c)}{\partial x^\nu} + \frac{n-1}{4n}(\tilde{R} - R\Omega^{-2}) \tilde{f}^a \ . \quad (3.25)$$

In particular if $(\mathscr{M}, \mathfrak{g})$ is the Minkowski space-time (and if we use the same conformal transformation as in Section 3.1) we obtain a system of Equations (3.19)-(3.21) with the obvious replacements associated with $\tilde{f} \to \tilde{f}^a$, and with G there replaced by

$$\begin{aligned} G^a \equiv & -\Gamma^a_{bc}(\Omega^{\frac{n-1}{2}} \tilde{f}) \Big\{ \Omega^{\frac{n-1}{2}} (-\phi^b_+ \phi^c_- + \phi^b_A \phi^c_A) \\ & - (n-1)\Omega^{\frac{n-3}{2}} \tilde{f}^c \Big[\big(x\phi^b_+ - (1 + x + 2\tau)\phi^b_- \big) - (n-1)\tilde{f}^b \Big] \Big\} \ . \end{aligned} \quad (3.26)$$

As before, for even space-dimensions n the occurence of non–integer powers of Ω above does not allow the use of the standard conformal method except for special target manifolds $(\mathscr{N}, h)$, *cf.* [14]. This can be handled in our approach, and in [22] we show:

Theorem 3.2 *The conclusions of Theorem 3.1 remain true for Equation (3.24) on $\mathbb{R}^{n,1}$, $n \geq 2$, with arbitrary polyhomogeneous bounded initial data $\tilde{f}^a|_{\{\tau=0\}}$, $\partial \tilde{f}^a / \partial\tau|_{\{\tau=0\}}$ if $n \geq 3$, and with 2-compatible polyhomogeneous initial data if $n = 2$.*

The restriction to Minkowski space–time in Theorem 3.2 is not necessary, and is only made for simplicity of presentation of the results; the same remark applies to Theorem 3.1. Similarly the choice of the initial data hypersurface as the standard unit hyperboloid is not necessary.

3.4. A weighted energy inequality for a class of symmetric hyperbolic systems

The first step of the proof of existence of solutions, and propagation of polyhomogeneity, consists in deriving weighted energy inequalities. We do this in [22] for a class of symmetric hyperbolic systems which can be written in the form[3]

$$\begin{pmatrix} E^\mu_- \nabla_\mu \varphi & +L\psi \\ -L^\dagger \varphi & +E^\mu_+ \nabla_\mu \psi \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} , \quad (3.27)$$

where φ and ψ are sections of N_1 and N_2 dimensional Riemannian bundles; L is a first order differential operator involving only derivatives tangential to the spheres $\tau =$const, $x =$const, and $L^\dagger$ denotes the formal adjoint of L on those spheres; we use the generic symbol ∇ to denote a covariant derivative on the relevant bundle. The matrices $E^\mu_\pm$ are assumed to satisfy

$$E^\tau_\pm \geq \varepsilon \mathrm{Id} \ , \qquad E^x_+ \leq -\varepsilon \mathrm{Id} \ , \qquad |E^x_-| \leq C_1 x \ , \quad (3.28)$$

[3]Some somewhat more general systems are also considered in [38].

for some $\varepsilon > 0$. We note that Equations (3.18)–(3.19) (which follow from the scalar wave equation) are of this form, with $E^\mu_\pm = e^\mu_\pm \otimes \mathrm{Id}$. Moreover, several other equations of interest can be written in this form, including the Maxwell or Yang-Mills equations, as well as the Bianchi identities for the Weyl tensor. (This allows us to derive energy inequalities, as well as propagation of polyhomogeneity, for the Maxwell equations, or for the Weyl equations on Minkowski space-time. See also [36] for some results in the Maxwell case.)

We use spaces $\mathscr{H}^\alpha_k$ of conormal-type distributions, defined as follows: Let $\overline{M}$ be the (conformal completion of) the initial data hypersurface $\tau = 0$. We set $y^1 = x$, a defining function for ∂M. We cover ∂M by a finite number of coordinate charts $\mathscr{O}_i$ with coordinates v^A, and in each of the coordinate charts $\Omega_i \equiv [0, x_0] \times \mathscr{O}_i$ the operators D_i are defined as

$$D_1 \equiv x\frac{\partial}{\partial y^1} = x\frac{\partial}{\partial x}, \qquad D_A \equiv \frac{\partial}{\partial y^A} = \frac{\partial}{\partial v^A}\ . \tag{3.29}$$

We define the spaces $\mathscr{H}^\alpha_k(\Omega_i)$ as the spaces of those functions in $H^{\mathrm{loc}}_k(\Omega_i)$ for which the norms $\|\cdot\|_{\mathscr{H}^\alpha_k(\Omega_i)}$ are finite, where

$$\|f\|^2_{\mathscr{H}^\alpha_k(\Omega_i)} = \sum_{0\le|\beta|\le k} \int_{\Omega_i} (x^{-\alpha} D^\beta f)^2 \frac{dx}{x} d^{n-1}\mu\ . \tag{3.30}$$

The spaces $\mathscr{H}^\alpha_k(M)$ are defined as the spaces of those fonctions in $H^{\mathrm{loc}}_k(M)$ for which the norm squared

$$\|f\|^2_{\mathscr{H}^\alpha_k(M)} = \sum_i \|f\|^2_{\mathscr{H}^\alpha_k(\Omega_i)} + \|f\|^2_{H_k(M\setminus\{x\le x_0/2\})} \tag{3.31}$$

is finite. Weighted energy inequalities in $\mathscr{H}^\alpha_k$ spaces with arbitrary values of k may be proved under various hypotheses on the objects which appear in (3.27); assuming for simplicity that all the coefficients are bounded and polyhomogeneous[4], in [22] we show, for $\alpha < -\frac{1}{2}$, an inequality[5] of the form:

$$\|f(t)\|^2_{\mathscr{H}^\alpha_k} \le C\left(\|f(0)\|^2_{\mathscr{H}^\alpha_k} e^{Ct} + \int_0^t e^{C(t-s)}(\|a(s)\|^2_{\mathscr{H}^\alpha_k} + \|b(s)\|^2_{\mathscr{H}^{\alpha-\frac{1}{2}}_k}) ds\right)\ . \tag{3.32}$$

Here $f(t)$ stands for f restricted to the hypersurface $\{\tau = t\}$, *etc.* For linear systems (3.27) the existence of solutions on domains of dependence with initial data in $\mathscr{H}^\alpha_k$ spaces follows immediately from the standard theorems using causality arguments. The inequality (3.32) shows then preservation of the $\mathscr{H}^\alpha_k$ character of solutions of Equations (3.27) when $a \in \mathscr{H}^\alpha_k$ and $b \in \mathscr{H}^{\alpha-\frac{1}{2}}_k$. Similarly, the inequality (3.32) together with appropriate weighted generalizations of Moser-type non-linear inequalities in Sobolev spaces, allows one to derive existence, as well as preservation of the $\mathscr{H}^\alpha_k$ character, of solutions of Equations (3.6) or (3.24). (Compare [10, 39].) To obtain polyhomogeneity of solutions, for compatible polyhomogeneous initial

[4] More general results can be found in [22].

[5] A similar inequality for the scalar wave equation has been independently derived in [34].

data, for systems of the form (3.27) we need the further restriction (satisfied by (3.18)-(3.21)) that

$$E_{-}^{x} = O(x^2)\ , \qquad E_{-}^{A} = O(x)\ .$$

The result follows then by a rather simple boot-strap argument, based on a more careful examination of the structure of Equations (3.27).

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