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Solutions des équations d'Einstein, des applications d'onde et de l'équation d'onde semi-linéaire en régime de rayonnement

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Contents

Chapter 1

Résumé

Cette thèse traite de l'existence locale de solutions d'une certaine classe de systèmes hyperboliques dans des espaces fonctionnels à poids de type Hölder et Sobolev ainsi que de l'existence de solutions polyhomogènes. Les espaces fonctionnels utilisés, définis pour une variété à bord N au moyen d'une fonction régulière positive x caractérisant le bord (ie. $x|\partial N = 0$), permettent des singularités au bord.

Ce travail trouve son origine dans l'étude du comportement asympotique à l'infini isotrope du champs de gravitation. Dans les années soixantes, Bondi *et al.* [6] avec Sachs [38] et Penrose [37] ont proposé un ensemble de conditions asymptotiques approprié pour la description du régime de rayonnement du champs de gravitation. Une manière simplifiée d'introduire ces conditions est de supposer l'existence de coordonnées "asymptotiquement Minkowskiennes" $(z^\mu) = (t, x, y, z)$ dans lesquelles la métrique de l'espace-temps g prend la forme

$$\mathfrak{g}_{\mu\nu} - \eta_{\mu\nu} = \frac{\overset{1}{h}_{\mu\nu}\,(t-r,\theta,\varphi)}{r} + \frac{\overset{2}{h}_{\mu\nu}\,(t-r,\theta,\varphi)}{r^2} + \dots\ , \tag{1.0.1}$$

où $\eta_{\mu\nu}$ est la métrique de Minkowski diag$(-1, 1, 1, 1)$, avec $u = t-r$, r, θ, φ étant les coordonées sphériques standard de $\mathbb{R}^3$. L'expansion asymptotique ci-dessus devant être comprise à u fixé, r tendant vers l'infini. L'existence de classes de solutions des équations d'Einstein du vide satisfaisant les conditions précédentes est établie dans [4, 22]. Mais la question de la généralité des solutions ayant le comportement asymptotique (0.2) reste ouverte. En fait, les résultats de [4, 18] suggèrent fortement que le cadre approprié pour décrire les champs gravitationnels rayonnants est celui des expansions asymptotiques *polyhomogènes* :

$$\mathfrak{g}_{\mu\nu} - \eta_{\mu\nu} \in \mathscr{A}_{\rm phg}\ . \tag{1.0.2}$$

où dans notre contexte une fonction est dite polyhomogène — $f \in \mathscr{A}_{\rm phg}$ — si et seulement si

$$f \sim \sum_{i=0}^{\infty}\sum_{j=0}^{N_i} f_{ij}(u,\theta,\varphi)\frac{\ln^j r}{r^{n_i}}\ , \tag{1.0.3}$$

pour des suites n_i, N_i, avec $n_i \nearrow \infty$, où $\sim$ signifie "asymptotique à", et où les f_{ij} sont réguliers. La suggestion que les expansions (1.0.2) sont celles décrivant le champs de gravitation dans le régime de rayonnement vient du fait que des données initiales *génériques* en un sens bien précis, telles que celles construites dans [4, 18] sont polyhomogènes. Cela mène naturellement à se poser la question si des données initiales polyhomogènes sont préservées par évolution pour les équations du type équation d'onde.

Il s'avère que l'étape indispensable pour étudier ces questions est l'établissement de théorèmes d'existence locale dans des espaces de Sobolev et de Hölder à poids dans lesquels est inclus $\mathscr{A}_{\rm phg}$, théorèmes qui ont leur intérêt propre.

La première partie de la thèse, après quelques résultats généraux sur les espaces à poids considérés, traite du problème de Cauchy, dit "hyperboloïdal", pour l'équation d'onde scalaire linéaire ou semi-linéaire et de l'équation d'application d'onde dans l'espace-temps de Minkowski compactifié. Nous nous intéressons en particulier au comportement des solutions près du morceau du bord $\mathscr{I}^+$

de $\mathcal{M}$ qui représente l'infini isotrope futur. Les données initiales sont dans des espaces de Sobolev à poids sur une hypersurface compacte issue de la compactification d'un hyperboloïde, et peuvent présenter des singularités au bord, après compactification conforme. A l'aide d'une inégalité de type energie à poids que nous démontrons pour une certaine classe de systèmes hyperboliques linéaires, nous établissons diverses estimées à poids pour les équations d'onde étudiées. L'étape décisive pour obtenir les estimées consiste en une décomposition isotrope de $d\widetilde{f}$ en composante transverses et parallèles à $\mathscr{I}^+$. L'existence locale se déduit des estimations par des arguments standards. Nous montrons en outre, que si les données initiales sont polyhomogènes (ie. admettent une expansion asympotique à l'infini isotrope de la forme (0.1)) et satisfont certaines conditions de compatibilités, alors la solution est aussi polyhomogène. Les résultats précédents permettent des singularités plus générales que celles traitées dans les approches conformes classiques, qui, en particulier, ne peuvent permettre de traiter les cas de dimensions d'espace paires.

La deuxième partie a pour but d'établir des résultats similaires pour les équations d'Einstein du vide avec données initiales sur une hypersurface asymptotiquement hyperboloïdale. Pour appliquer les techniques mises en place dans la première partie, nous prenons une formulation par Friedrich du système conforme des équations d'Einstein avec un choix de jauge isotrope, combinée avec une décomposition des équations de type Newman-Penrose dans le formalisme de Christodoulou-Klainerman. En particulier nous avons dérivé une version plus générale des équations de Bianchi isotropes établies par ces derniers pour l'adapter à notre choix de jauge et en déduire des estimées à poids sur les composantes isotropes du tenseur de Weyl. Nous en déduisons des estimées dans les espaces de Sobolev à poids pour les différents champs du système et le théorème principal de cette thèse: existence locale de solutions du problème de Cauchy pour les équations d'Einstein avec données initiales dans ces espaces. Comme précédemment, notre approche permet un comportement singulier au voisinage du bord des données initiales, et en particulier autorise un comportement en $1/\Omega$ du tenseur $\Omega^{-1}W$, où W est le tenseur de Weyl. Il est important de noter que ce type de comportement est associée à une obstruction géométrique de la régularité de $\mathscr{I}$ et que notre théorème d'existence locale est en principe compatible avec les données initiales génériques construites dans [4].

Chapter 2

Introduction

This thesis deals with the local existence of solutions of some hyperbolic systems in weighted and polyhomogeneous spaces. By weighted spaces we mean Hölder and Sobolev weighted spaces on a Riemannian smooth manifold M with compact closure and nonempty boundary ∂M, where the weight is provided by powers of a positive regular function x defining ∂M. The motivation behind this work is as follows: In the sixties Bondi *et al.* [6] together with Sachs [38] and Penrose [37], building upon the pioneering work of Trautman [40, 41], have proposed a set of boundary conditions appropriate for the gravitational field in the radiation regime. A somewhat simplified way of introducing the Bondi-Penrose (BP) conditions is to assume existence of "asymptotically Minkowskian coordinates" $(x^\mu) = (t, x, y, z)$ in which the space-time metric $\mathfrak{g}$ takes the form

$$\mathfrak{g}_{\mu\nu} - \eta_{\mu\nu} = \frac{\overset{1}{h}_{\mu\nu}\,(t-r,\theta,\varphi)}{r} + \frac{\overset{2}{h}_{\mu\nu}\,(t-r,\theta,\varphi)}{r^2} + \dots\ , \tag{2.0.1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric diag$(-1, 1, 1, 1)$, u stands for $t-r$, with r, θ, φ being the standard spherical coordinates on $\mathbb{R}^3$. The expansion above has to hold at, say, fixed u, with r tending to infinity. Existence of classes of solutions of the vacuum Einstein equations satisfying the asymptotic conditions (2.0.1) follows from the work in [22] together with [3, 4, 19]. As of today it remains an open problem how general, within the class of radiating solutions of vacuum Einstein equations, are those solutions which display the behaviour (2.0.1). Indeed, the results in [1–4, 18] suggest strongly[1] that a more appropriate setup for such gravitational fields is that of *polyhomogeneous* asymptotic expansions:

$$\mathfrak{g}_{\mu\nu} - \eta_{\mu\nu} \in \mathscr{A}_{\rm phg}\ . \tag{2.0.2}$$

In the context of expansions in terms of a radial coordinate r tending to infinity, the space of polyhomogeneous functions is defined as the set of smooth functions which have an asymptotic expansion of the form

$$f \sim \sum_{i=0}^{\infty} \sum_{j=0}^{N_i} f_{ij}(u,\theta,\varphi) \frac{\ln^j r}{r^{n_i}}\ , \tag{2.0.3}$$

[1] *Cf.* [34] and references therein for some further related results.

for some sequences n_i, N_i, with $n_i \nearrow \infty$. Here the symbol $\sim$ stands for "being asymptotic to": if the right-hand-side is truncated at some finite i, the remainder term falls off appropriately faster. Further, the functions f_{ij} are supposed to be smooth, and the asymptotic expansions should be preserved under differentiation.[2]

The suggestion that the expansions (2.0.2) are *the ones* describing the gravitational field in the radiation regime arises from the fact that *generic*, in a well defined sense, initial data constructed in [1–4, 18] are polyhomogeneous. This leads naturally to the question, whether polyhomogeneity of initial data is preserved under evolution under wave equations. In the first part of this thesis (Chapter 3) we answer in the affirmative this question for semi-linear wave equations, and for the wave map equation, on Minkowski space-time. We develop a functional framework appropriate for the analysis of such questions. We prove preservation of polyhomogeneity for a large class of linear symmetric hyperbolic systems. We prove local in time existence of solutions of semi-linear wave equations, and for the wave map equation, on Minkowski space-time, with conormal and with polyhomogeneous initial data. We show that polyhomogeneity is preserved under evolution when appropriate (necessary) corner conditions are satisfied by the initial data. We note that the existing related results [7, 32, 35] do not answer the questions raised here.

Our main results in Chapter 3 are the existence and polyhomogeneity of solutions with appropriate polyhomogeneous initial data for the nonlinear scalar wave equation, and for the wave map equation. We achieve this in a few steps. First, we prove local existence of solutions of these equations in weighted Sobolev spaces, *cf.* Theorems 3.5.1 and 3.6.1. The next step is to obtain estimates on the time derivatives, *cf.* Theorems 3.5.4 and 3.6.4. Those estimates are uniform in time in a neighbourhood of the initial data surface if the initial data satisfy compatibility conditions. Somewhat surprisingly, we show that all initial data in weighted Sobolev spaces, not necessarily satisfying the compatibility conditions, evolve in such a way that the compatibility conditions will hold on all later time slices; this is done in Corollary 3.5.5 and Theorem 3.6.4. Finally, in Theorems 3.5.10 and 3.6.5 we prove polyhomogeneity of the solutions with polyhomogeneous initial data; this requires a hierarchy of compatibility conditions.

The restriction to Minkowski space-time in Theorem 3.6.5 is not necessary, and is only made for simplicity of presentation of the results; the same remark applies to Theorem 3.5.1. Similarly the choice of the initial data hypersurface as the standard unit hyperboloid is not necessary.

The second part of this work is concerned with the Einstein equations. The long term goal is to prove analogous theorems for general relativistic "hyperboloidal initial data sets"; this requires, first, setting up a framework to which

[2]The choice of the sequences n_i, N_i is not arbitrary, and is dictated by the equations at hand. For example, the analysis of $3+1$ dimensional Einstein equations in [18] suggests that consistent expansions can be obtained with $n_i = i$. On the other hand, Theorem 3.6.5 below gives actually $n_i = i/2$ for wave-maps on $2+1$ dimensional Minkowski space-time. We note that the $2+1$ dimensional wave map equation is related to the vacuum Einstein equations with cylindrical symmetry (*cf., e.g.*, [5, 15, 16]).

the techniques developed in the first part of the thesis apply. We use a mixture of the conformal Friedrich's form of the Einstein field equations with an appropriate null choice of gauge, together with a Newman-Penrose type decomposition of the equations; we actually work with the Christodoulou-Klainerman version of the Newman-Penrose formalism. We need those equations in a setting more general than the one already considered in the literature, which forces us to rederive the equations from scratch; this is done in Chapter 4. In Chapter 5 we apply the techniques developed in the first part of the thesis to prove estimates in weighted Sobolev spaces for solutions of those equations, which leads to a local existence theorem of space-times with "a piece of $\mathscr{I}^+$" to the future of a hyperboloidal hypersurface with initial data in weighted Sobolev spaces. This is the contents of Theorem 5.5.2, which is the main result of this work. As before, our framework allows initial data which are singular at the boundary, in particular initial data for which the conformally rescaled Weyl tensor has a $1/\Omega$ singularity are allowed in our theorems. This is precisely the behaviour associated with a geometric obstruction to smoothness of the conformal null infinity $\mathscr{I}$ [1, 2]. A rough inspection shows that our existence theorem is compatible with the generic initial data constructed in [3, 4]; a precise statement would, however involve a lengthy and tedious but otherwise straightforward analysis of the initial data which we have not carried out. It is clear at this stage that the methods developed in the first and second part of the thesis will lead — for polyhomogeneous initial data — to an existence theorem of space-times with "a piece of polyhomogeneous $\mathscr{I}$" to the future of the initial data hyperboloidal hypersurface; such a result, however, requires a further lengthy adaptation of the remaining methods of the first part of the thesis to the equations considered in the second part, a task which we are planning to finish in the near future.

Chapter 3

Nonlinear equations on Minkowski space-time

3.1 Conformal completions

Consider an $n+1$ dimensional space-time $(\mathscr{M}, \mathfrak{g})$ and let

$$\tilde{\mathfrak{g}} = \Omega^2 \mathfrak{g} \ . \tag{3.1.1}$$

Let $\Box_h$ denote the wave operator associated with a Lorentzian metric h,

$$\Box_h f = \frac{1}{\sqrt{|\det h_{\rho\sigma}|}} \partial_\mu \left(\sqrt{|\det h_{\alpha\beta}|} h^{\mu\nu} \partial_\nu f\right).$$

We recall that the scalar curvature $R = R(\mathfrak{g})$ of $\mathfrak{g}$ is related to the corresponding scalar curvature $\tilde{R} = \tilde{R}(\tilde{\mathfrak{g}})$ of $\tilde{\mathfrak{g}}$ by the formula

$$\tilde{R}\Omega^2 = R - 2n \left\{ \frac{1}{\Omega} \Box_{\mathfrak{g}} \Omega + \frac{n-3}{2} \frac{|\nabla \Omega|^2_{\mathfrak{g}}}{\Omega^2} \right\} \ . \tag{3.1.2}$$

It then follows from (3.1.2) that we have the identity

$$\Box_{\tilde{\mathfrak{g}}} (\Omega^{-\frac{n-1}{2}} f) = \Omega^{-\frac{n+3}{2}} \left(\Box_{\mathfrak{g}} f + \frac{n-1}{4n} (\tilde{R}\Omega^2 - R) f \right) \ . \tag{3.1.3}$$

It has been observed by Penrose [37] that the Minkowski space-time $(\mathscr{M}, \eta)$ can be conformally completed to a space-time with boundary $(\widetilde{\mathscr{M}}, \tilde{\eta})$, $\tilde{\eta} = \Omega^{-2}\eta$ on $\mathscr{M}$, by adding to $\mathscr{M}$ two null hypersurfaces, usually denoted by $\mathscr{I}^+$ and $\mathscr{I}^-$, which can be thought of as end points ($\mathscr{I}^+$) and initial points ($\mathscr{I}^-$) of inextendible null geodesics [36, 37, 42]. We will only be interested in "the future null infinity" $\mathscr{I}^+$; an explicit construction (of a subset of $\mathscr{I}^+$) which is convenient for our purposes proceeds as follows: for $(x^0)^2 < \sum_i (x^i)^2$ we define

$$y^\mu = \frac{x^\mu}{x^\alpha x_\alpha} \ ; \tag{3.1.4}$$

in the coordinate system $\{y^\mu\}$ the Minkowski metric $\eta \equiv -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$ takes the form

$$\eta = \frac{1}{\Omega^2} \eta_{\alpha\beta} dy^\alpha dy^\beta \ , \qquad \Omega = \eta_{\alpha\beta} y^\alpha y^\beta \ . \tag{3.1.5}$$

We note that under (3.1.4) the exterior of the light cone $C_0^{x^\mu} \equiv \{\eta_{\alpha\beta} x^\alpha x^\beta = 0\}$ emanating from the origin of the x^μ-coordinates is mapped to the exterior of the light cone $C_0^{y^\mu} = \{\eta_{\alpha\beta} y^\alpha y^\beta = 0\}$ emanating from the origin of the y^μ-coordinates. The conformal completion is obtained by adding $C_0^{y^\mu}$ to $\mathscr{M}$,

$$\widetilde{\mathscr{M}} = \mathscr{M} \cup (C_0^{y^\mu} \setminus \{0\}) \ ,$$

with the obvious differential structure arising from the coordinate system y^μ. We shall use the symbol $\mathscr{I}$ to denote $C_0^{y^\mu} \setminus \{0\}$, and $\mathscr{I}^+$ to denote $C_0^{y^\mu} \setminus \{0\} \cap \{y^0 > 0\}$. As already mentioned, $\mathscr{I}$ so defined is actually a subset of the usual $\mathscr{I}$, but this will be irrelevant for our purposes.

We note that (3.1.4) is singular at the light cone $C_0^{x^\mu}$. This is again irrelevant from our point of view because we are only interested in the behavior of the solutions near $\mathscr{I}^+$, and causality allows us to ignore this.

The above procedure can be adapted for several metrics of interest, such as the Schwarzschild, Kerr, or Robinson-Trautman metrics, to similarly yield conformal completions of space-time by the addition of null hypersurfaces $\mathscr{I}^+$. This observation was at the origin of Penrose's proposal to describe systems which are asymptotically flat in lightlike directions through the use of conformal completions.

It is noteworthy that the conformal technique allows one to reduce global-in-time existence problems to local ones; this has been exploited by various authors [8–13] for wave equations on a fixed background space-time. Further, Friedrich [24, 25, 30] has used this approach to obtain global existence result for Einstein equations to the future of a "hyperboloidal" Cauchy surface, with "small" smoothly compactifiable initial data, *cf.* also [23, 27].

On a more modest level, the identity (3.1.3) can be used as a starting point for the analysis of the asymptotic behavior of solutions of the scalar wave equation near $\mathscr{I}^+$, as it reduces the problem to a study of solutions near a null hypersurface. This is the approach used in this paper. There are associated identities for fields of any spin [37], which provide a convenient framework for similar questions for those fields.

3.2 Function spaces, embeddings, inequalities

Throughout this paper the letter C denotes a constant the exact value of which is irrelevant for the problem at hand, and which may vary from line to line.

Let M be a smooth manifold such that

$$\overline{M} \equiv M \cup \partial M$$

is a compact manifold with smooth boundary ∂M. We shall generally use the notations and conventions of [3]. Throughout this work the symbol x stands for a smooth defining function for ∂M, *i.e.*, a smooth function on $\overline{M}$ such that $\{x = 0\} = \partial M$, with dx nowhere vanishing on ∂M. It follows that there exists $x_0 > 0$ and a compact neighborhood $\mathscr{V}$ of ∂M on which x can be used as a coordinate, with $\mathscr{V}$ being diffeomorphic to $[0, x_0] \times \partial M$. For $0 \leq x_1 < x_2 \leq x_0$ we set

$$M_{x_1} = \{p \in M \mid x(p) < x_1\} \,, \tag{3.2.1a}$$

$$M_{x_1,x_2} = \{p \in M \mid x_1 < x(p) < x_2\} \,, \tag{3.2.1b}$$

$$\tilde{\partial} M_{x_1} = \{p \in M \mid x(p) = x_1\} \approx \partial M \,. \tag{3.2.1c}$$

In all that follows the symbol Ω denotes one of the sets M, M_{x_1}, or M_{x_1,x_2}. Any subset of M_{x_0} can be locally coordinatized by coordinates $y^i = (x, v^A)$, where the v^A's can be thought of as local coordinates on ∂M. We cover ∂M by a finite number of coordinate charts $\mathcal{O}_i$ so that the sets $\Omega_i \equiv [0, x_0] \times \mathcal{O}_i$ cover M_{x_0}. We use the usual multi-index notation for partial derivatives: for

$\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ we set $\partial^\beta = \partial_1^{\beta_1} \ldots \partial_n^{\beta_n}$. We will write ∂_v^β for derivatives of the form $\partial_2^{\beta_2} \ldots \partial_n^{\beta_n}$, which do not involve the $x^1 \equiv x$ variable.

For $\alpha \in \mathbb{R}$, $k \in \mathbb{N}$ and $\lambda \in (0,1]$, we define $\mathscr{C}_0^\alpha(\Omega_i)$ (respectively $\mathscr{C}_{0+\lambda}^\alpha(\Omega_i)$, $\mathscr{C}_k^\alpha(\Omega_i)$, $\mathscr{C}_{k+\lambda}^\alpha(\Omega_i)$) as the spaces of functions appropriately differentiable (on Ω_i for $\alpha < 0$, $\overline{\Omega_i}$ otherwise) such that the respective norms

$$\begin{aligned}
\|f\|_{\mathscr{C}_0^{-\alpha}(\Omega_i)} &\equiv \sup_{p\in\Omega_i} |x^\alpha f(p)| \,, \\
\|f\|_{\mathscr{C}_{0+\lambda}^\alpha(\Omega_i)} &\equiv \|f\|_{\mathscr{C}_0^\alpha(\Omega_i)} + \sup_{y\in\Omega_i} \sup_{y\neq y'\in B(y,\frac{x(y)}{2})\cap\Omega_i} \frac{x(y)^{-\alpha-\lambda}|f(y)-f(y')|}{|y-y'|^\lambda} \,, \\
\|f\|_{\mathscr{C}_k^\alpha(\Omega_i)} &\equiv \sum_{0\le|\beta|\le k} \|x^{\beta_1}\partial^\beta f\|_{\mathscr{C}_0^\alpha(\Omega_i)} \,, \\
\|f\|_{\mathscr{C}_{k+\lambda}^\alpha(\Omega_i)} &\equiv \|f\|_{\mathscr{C}_{k-1}^\alpha(\Omega_i)} + \sum_{|\beta|=k} \|x^{\beta_1}\partial^\beta f\|_{\mathscr{C}_{0+\lambda}^\alpha(\Omega_i)} \,,
\end{aligned} \tag{3.2.2}$$

are finite. Let Ω be an open subset of M, or a submanifold with boundary in M; for such sets we define:

$$\begin{aligned}
\|f\|_{\mathscr{C}_k^\alpha(\Omega)} &\equiv \sup_i \|f\|_{\mathscr{C}_k^\alpha(\Omega_i\cap\Omega)} + \|f\|_{C_k(\complement M_{x_0/2}\cap\Omega)} \,, \\
\|f\|_{\mathscr{C}_{k+\lambda}^\alpha(\Omega)} &\equiv \sup_i \|f\|_{\mathscr{C}_{k+\lambda}^\alpha(\Omega_i\cap\Omega)} + \|f\|_{C_{k+\lambda}(\complement M_{x_0/2}\cap\Omega)} \,.
\end{aligned} \tag{3.2.3}$$

Here $C_{k+\lambda}(\mathcal{U})$, for $\mathcal{U}$ being any of the set $\Omega_i, \Omega, \Omega_i \cap \Omega$ above, denotes the space of k-times continuously differentiable functions on $\mathcal{U}$ (differentiable up to boundary if $\mathcal{U}$ is a submanifold with boundary), with λ-Hölder continuous k'th derivatives, equipped with the usual norm. The associated function spaces are defined in the obvious way. We note that $f \in \mathscr{C}_{k+\lambda}^{\alpha+\sigma}(\Omega)$ if and only if $x^{-\sigma} f \in \mathscr{C}_{k+\lambda}^\alpha(\Omega)$.

We define the spaces $\mathscr{H}_k^\alpha(\Omega_i)$ as the spaces of those functions in $H_k^{\mathrm{loc}}(\Omega_i)$ for which the norms $\|\cdot\|_{\mathscr{H}_k^\alpha(\Omega_i)}$ are finite, where

$$\|f\|^2_{\mathscr{H}_k^\alpha(\Omega_i)} = \sum_{0\le|\beta|\le k} \int_{\Omega_i} (x^{-\alpha+\beta_1}\partial^\beta f)^2 \frac{dx}{x}\, d\nu \,, \tag{3.2.4}$$

where we identify $M_{a,b}$ and $[a,b] \times \partial M$ and $d\nu$ is, say, a measure on ∂M arising from some smooth Riemannian metric on ∂M. This is equivalent to

$$\sum_{0\le\beta_1+|\beta|\le k} \int_{\Omega_i} (x^{-\alpha}(x\partial_x)^{\beta_1}\partial_v^\beta f)^2 \frac{dx}{x}\, d\nu \,, \tag{3.2.5}$$

and it will sometimes be convenient to use (3.2.5) as the definition of $\|f\|^2_{\mathscr{H}_k^\alpha(\Omega_i)}$. For Ω's such that $\Omega_i \subset \Omega$ the spaces $\mathscr{H}_k^\alpha(\Omega)$ are defined as the spaces of those functions in $H_k^{\mathrm{loc}}(\Omega)$ for which the norm squared

$$\|f\|^2_{\mathscr{H}_k^\alpha(\Omega)} = \sum_i \|f\|^2_{\mathscr{H}_k^\alpha(\Omega_i)} + \|f\|^2_{H_k(\Omega\cap\complement M_{x_0/2})} \tag{3.2.6}$$

is finite. We note that

$$\|f\|_{H_0(\Omega)} \approx \|f\|_{\mathscr{H}_0^{-1/2}(\Omega)} ,$$

and that $\mathscr{H}_k^\alpha(M_{x_1,x_2}) = H_k(M_{x_1,x_2})$ for all α and k (in this last equality we are implicitly assuming that $x_1 > 0$); the norms are equivalent with the constants involved depending upon x_1 and x_2.

It is often awkward to work with coordinate charts, in order to avoid that one can proceed as follows: Choose a fixed smooth complete Riemannian metric b on $\overline{M}$. Let x be any smooth defining function for ∂M, we let X_1 be the gradient of x with respect to the metric b; rescaling b by a smooth function if necessary we may without loss of generality assume that X_1 has length one in the metric b in a neighbourhood of ∂M. We cover ∂M by a finite number of coordinate charts $\mathcal{O}_i$, with associated coordinates v^A; the v^A's are then propagated to a neighbourhood of ∂M by requiring

$$X_1(v^A) = 0 .$$

This leads to a covering of M_{x_0} of the kind already used, and one easily checks that

$$X_1 = \partial_x$$

in the resulting local coordinates. This gives then a globally defined vector ∂_x on M_{x_0}.

For $i = 2, \dots, r$ we let X_i be any smooth vector fields on ∂M satisfying the condition that at any $p \in \partial M$ the linear combinations of the X_i exhaust the tangent space $T_p\partial M$. (If ∂M is a sphere, a convenient choice is the collection of all Killing vectors of $(S^{n-1}, \mathring{h})$, where $\mathring{h}$ is the unit round metric on S^{n-1}.) Over the domain of a chart (v^A) of ∂M, one thus has

$$\partial_A = \sum_{i=2}^{r} f^i_A(v^B) X_i , \qquad (3.2.7a)$$

$$X_i = \sum_{A=2}^{n} X_i^A(v^B)\partial_A , \qquad (3.2.7b)$$

for some locally defined smooth functions f^i_A, X_i^A; clearly things can be arranged so that those functions are bounded, together with all their partial derivatives. We propagate the X_i's to M_{x_0} by requiring

$$[X_1, X_i] = 0 ,$$

equivalently

$$\partial_x X_i^A = 0 . \qquad (3.2.8)$$

It follows that (3.2.7) still holds with x-independent functions. For any multi-index $\beta = (\beta_1, \beta_2, \dots, \beta_r) \in \mathbb{N}^r$ we set, on M_{x_0},

$$\mathscr{D}^\beta f = X_1^{\beta_1} X_2^{\beta_2} \cdots X_r^{\beta_r} f = \partial_x^{\beta_1} X_2^{\beta_2} \cdots X_r^{\beta_r} f . \qquad (3.2.9)$$

It follows that we have

$$\|f\|_{\mathscr{C}_k^\alpha(M_{x_0})} \approx \sum_{0\le|\beta|\le k} \|x^{\beta_1}\mathscr{D}^\beta f\|_{\mathscr{C}_0^\alpha(M_{x_0})}\,,$$

$$\|f\|^2_{\mathscr{H}_k^\alpha(M_{x_0})} \approx \sum_{0\le|\beta|\le k} \int_{M_{x_0}} (x^{-\alpha+\beta_1}\mathscr{D}^\beta f)^2 \frac{dx}{x}\,d\nu\,,$$

where $\approx$ denotes the fact that the norms are equivalent, *etc.* Here, $|\beta| = \beta_1 + \ldots + \beta_r$. **Remark:** An equivalent norm can be obtained if we replace the volume element $dx\,d\nu$ by the volume element associated to any C_0- Riemannian metric defined on M_{x_0}. There is a useful way of rewriting $\|\cdot\|_{\mathscr{H}_k^\alpha(M_{x_0})}$ which proceeds as follows: for $f \in \mathscr{H}_k^\alpha(M_{x_0})$, $s \in [1,2]$, and $n \in \mathbb{N}$ we set

$$f_n(s,v) = f(x = x_0\frac{s}{2^n}, v)\;; \tag{3.2.10}$$

letting $\approx$ denote equivalence one then has, after a change of variables,

$$\begin{aligned}\|f\|^2_{\mathscr{H}_k^\alpha(M_{x_0})} &= \sum_{n\ge1}\sum_{0\le|\beta|\le k}\int_{[2^{-n}x_0,2^{1-n}x_0]\times\partial M} |x^{-\alpha+\beta_1}\mathscr{D}^\beta f(x,v)|^2\frac{dx}{x}\,d\nu\\ &\approx x_0^{-2\alpha}\sum_{n\ge1}\sum_{0\le|\beta|\le k} 2^{2n\alpha}\int_{[1,2]\times\partial M}|\mathscr{D}^\beta f_n(s,v)|^2 ds\;d\nu\\ &= x_0^{-2\alpha}\sum_{n\ge1}2^{2n\alpha}\|f_n\|^2_{H_k([1,2]\times\partial M)}\,.\end{aligned} \tag{3.2.11}$$

Note that above we use the notation $\mathscr{D}^\beta$ also for $\partial_s^{\beta_1}\partial^{\beta_1}X_2^{\beta_2}\ldots X_r^{\beta_r}$, (as for $\partial_x^{\beta_1}X_2^{\beta_2}\ldots X_r^{\beta_r}$). More precisely, we write $A \approx B$ if there exist constants $C_1, C_2 > 0$ such that $C_1 A \le B \le C_2 A$. In (3.2.11) the relevant constants depend only upon α and k. It turns out to be useful to have a formula similar to (3.2.11) for functions in M_{x_2,x_1}; this can be done for any x_1 and x_2, but in order to obtain uniform control of certain constants it is convenient to require $2x_2 \le x_1$. For such values of x_1 and x_2 we let $n_0(x_1,x_2) \in \mathbb{N}$ be such that $\frac{x_1}{2^{n_0+1}} \le x_2 \le \frac{x_1}{2^{n_0}}$. For $n \in \mathbb{N}$, $n \ge 1$, and for any $f : M_{x_2,x_1} \to \mathbb{R}^N$ we then define $f_n : [1,2]\times\partial M \to \mathbb{R}^N$ by

$$\begin{aligned} n \le n_0\,, &\qquad f_n(s,v) = f(x_1\frac{s}{2^n}, v)\,,\\ n = n_0+1\,, &\qquad f_n(s,v) = f(x_2\,s, v)\,,\\ n > n_0+1\,, &\qquad f_n = 0\,.\end{aligned} \tag{3.2.12}$$

(This coincides with the definition already given for M_{x_1}, when this set is thought of as being an "M_{x_2,x_1} with $x_2 = 0$", if we set $n_0 = +\infty$.) A calculation as in (3.2.11) shows that for any $2x_2 \le x_1 \le x_0$, there exist constants C_1 and c_1, independent of x_0, x_1 and x_2, such that for all $f \in \mathscr{H}_k^\alpha(M_{x_2,x_1})$,

$$\begin{aligned} c_1 x_1^{-2\alpha}\sum_n\{2^{n\alpha}\|f_n\|_{H_k([1,2]\times\partial M)}\}^2 &\le \|f\|^2_{\mathscr{H}_k^\alpha(M_{x_2,x_1})}\\ &\le C_1 x_1^{-2\alpha}\textstyle\sum_n\{2^{n\alpha}\|f_n\|_{H_k([1,2]\times\partial M)}\}^2\,.\end{aligned} \tag{3.2.13}$$

Equation (3.2.11) leads one to introduce spaces $\mathscr{B}^\alpha_{k+\lambda}$, that arise naturally from weighted Sobolev embeddings, *cf.* Equation (3.2.25) below: we define

$$\|f\|^2_{\mathscr{B}^\alpha_{k+\lambda}(M_{x_0})} \quad = \quad x_0^{-2\alpha}\sum_{n\geq 1} 2^{2n\alpha}\|f_n\|^2_{C_{k+\lambda}([1,2]\times\partial M)} \ , \tag{3.2.14}$$

f_n as in (3.2.10), and we set

$$\mathscr{B}^\alpha_{k+\lambda}(M_{x_0}) = \{f\in C_{k+\lambda}(\Omega) \mid \|f\|_{\mathcal{B}^\alpha_{k+\lambda}(M_{x_0})}<\infty\} \ .$$

Clearly

$$\mathcal{B}^\alpha_{k+\lambda}(M_{x_0})\subset \mathcal{C}^\alpha_{k+\lambda}(M_{x_0}) \ .$$

Since the general term f_N, as well as sums of the form $\Sigma_{n\geq N} f_n$, of a convergent series tend to zero as N tends to infinity, for $f\in\mathcal{B}^\alpha_{k+\lambda}(M_{x_0})$ we actually have

$$\lim_{x_1\to 0}\|f\|_{\mathscr{C}^\alpha_{k+\lambda}(M_{x_1})} = 0 \ . \tag{3.2.15}$$

We have the trivial inclusion,

$$\alpha' > \alpha \quad \Longrightarrow \quad \mathscr{C}^{\alpha'}_{k+\lambda}(M_{x_1})\subset \mathscr{H}^\alpha_k(M_{x_1}) \ . \tag{3.2.16}$$

The open inequality $\alpha'>\alpha$ in (3.2.16) has various unpleasant consequences, which are best avoided by introducing yet another space — the space $\mathcal{G}^\alpha_k$ of functions in $H^k_{\mathrm{loc}}(M_{x_0})$ for which the norm squared

$$\|f\|^2_{\mathcal{G}^\alpha_k(M_{x_0})} = \sup_{n\geq 1}\left\{\sum_{0\leq\beta\leq k}\int_{[2^{-n}x_0,2^{1-n}x_0]\times\partial M}|x^{-\alpha+\beta_1}\mathscr{D}^\beta f(x,v)|^2\frac{dx}{x}\,d\nu\right\} \tag{3.2.17}$$

is finite. We note that $\|f\|_{\mathcal{G}^\alpha_k(M_{x_0})}$ is equivalent to

$$x_0^{-\alpha}\sup_{n\geq 1}\left\{2^{n\alpha}\|f_n\|_{H_k([1,2]\times\partial M)}\right\} \ , \tag{3.2.18}$$

with $f_n(s,v)=f(\frac{x_0 s}{2^n},v)$, as in (3.2.10). To define the $\mathcal{G}^\alpha_k(M_{x_2,x_1})$'s, assuming again that $x_2\leq x_1/2$, we let $I_n(x_1,x_2)$ be defined as

$$\begin{aligned} n\leq n_0 \ , &\qquad I_n=[2^{-n}x_1,2^{1-n}x_1] \ , \\ n=n_0+1 \ , &\qquad I_{n_0+1}=[x_2,2x_2] \ , \\ n>n_0+1 \ , &\qquad I_n=\emptyset \ , \end{aligned} \tag{3.2.19}$$

where n_0 is as in (3.2.12). For all $f\in H^{\mathrm{loc}}_k(M_{x_2,x_1})$ we set

$$\|f\|^2_{\mathcal{G}^\alpha_k(M_{x_2,x_1})} = \sup_n\{\sum_i\sum_{0\leq|\beta|\leq k}\int_{\Omega_i\cap\{I_n\times\partial M\}}(x^{-\alpha+\beta_1}\mathscr{D}^\beta f)^2\frac{dx}{x}\,d\nu\ \} \tag{3.2.20}$$

Similarly to (3.2.13), there exist constants c_2 and C_2, which do *not* depend upon x_0, x_1, and x_2, such that for all $2x_2\leq x_1\leq x_0$,

$$c_2 x_1^{-\alpha}\sup_n\|f_n\|_{H_k([1,2]\times\partial M)}\leq \|f\|_{\mathcal{G}^\alpha_k(M_{x_2,x_1})}\leq C_2 x_1^{-\alpha}\sup_n\|f_n\|_{H_k([1,2]\times\partial M)} \ . \tag{3.2.21}$$

We have the obvious inequality

$$\|f\|_{\mathcal{G}_k^\alpha(\Omega)} \le \|f\|_{\mathscr{H}_k^\alpha(\Omega)} \ , \tag{3.2.22}$$

together with the modified version of (3.2.16),

$$\alpha' \ge \alpha \quad \Longrightarrow \quad \mathscr{C}_{k+\lambda}^{\alpha'} \subset \mathscr{G}_k^\alpha \ ; \tag{3.2.23}$$

in particular the function $(x, v) \to x^\alpha$ is in $\mathcal{G}_k^\alpha(M_{x_0})$.

If S_k denotes a space of functions, where $k \in \mathbb{N}$ is a differentiability index, we set

$$S_\infty \equiv \cap_{k\in\mathbb{N}} S_k \ ,$$

e.g., $\mathscr{G}_\infty^\alpha \equiv \cap_{k\in\mathbb{N}} \mathcal{G}_k^\alpha$, *etc.*

We note the following:

Proposition 3.2.1 Let $\Omega = M$, or $\Omega = M_{x_1}$, $0 < x_1 \le x_0$, or $\Omega = M_{x_2,x_1}$, $2x_2 < x_1 \le x_0$. For $0 < k + \lambda - n/2 \notin \mathbb{N}$ we have the continuous embeddings

$$\mathscr{H}_k^\alpha \subset \mathscr{B}_{k+\lambda-n/2}^\alpha \subset \mathscr{C}_{k+\lambda-n/2}^\alpha \ , \qquad \mathscr{H}_k^\alpha \subset \mathscr{G}_k^\alpha \subset \mathscr{C}_{k+\lambda-n/2}^\alpha \ , \tag{3.2.24}$$

and there exists an x_2-independent constant C such that we have

$$\forall f \in \mathscr{H}_k^\alpha \qquad \|f\|_{\mathscr{B}_{k+\lambda-n/2}^\alpha(\Omega)} \le C\|f\|_{\mathcal{H}_k^\alpha(\Omega)} \ , \tag{3.2.25}$$

$$\forall f \in \mathscr{G}_k^\alpha \qquad \|f\|_{\mathscr{C}_{k+\lambda-n/2}^\alpha(\Omega)} \le C\|f\|_{\mathcal{G}_k^\alpha(\Omega)} \ . \tag{3.2.26}$$

PROOF: (3.2.25)-(3.2.26) follow immediately from (3.2.11) and (3.2.13), together with the standard Sobolev embedding; the remaining inclusions in (3.2.24) are trivial. □

All other inequalities involving Sobolev spaces have their counterpart in the weighted setting; we shall in particular need various weighted versions of the Moser inequalities:

Proposition 3.2.2 Let $\Omega = M$, or $\Omega = M_{x_1}$, $0 < x_1 \le x_0$, or $\Omega = M_{x_2,x_1}$, $2x_2 < x_1 \le x_0$, and let $\mathscr{H}_k^\alpha = \mathscr{H}_k^\alpha(\Omega)$, *etc.*

1. There exists a constant $C = C(\alpha, \alpha', \beta, k, x_1)$ such that, for all $f \in \mathscr{H}_k^{\alpha'} \cap \mathscr{C}_0^\alpha$ and $g \in \mathscr{H}_k^\beta \cap \mathscr{C}_0^{\alpha+\beta-\alpha'}$, we have

$$\|fg\|_{\mathscr{H}_k^{\alpha+\beta}} \le C\left(\|f\|_{\mathscr{C}_0^\alpha}\|g\|_{\mathscr{H}_k^\beta} + \|f\|_{\mathscr{H}_k^{\alpha'}}\|g\|_{\mathscr{C}_0^{\alpha+\beta-\alpha'}}\right) \ . \tag{3.2.27}$$

   Further, $\forall\ |\gamma| \le k$,

$$\|x^{\gamma_1}\mathscr{D}^\gamma(fg) - (x^{\gamma_1}\mathscr{D}^\gamma f)g\|_{\mathscr{H}_0^{\alpha+\beta}} \le C\Bigg(\|f\|_{\mathscr{C}_0^\alpha}\|g\|_{\mathscr{H}_k^\beta} + \|f\|_{\mathscr{H}_{k-1}^{\alpha'}}\left(\|x\partial_x g\|_{\mathscr{C}_0^{\alpha+\beta-\alpha'}} + \sum_{i=2}^N \|X_i g\|_{\mathscr{C}_0^{\alpha+\beta-\alpha'}}\right)\Bigg) \ , \tag{3.2.28}$$

   where the vector fields X are defined in Equation (3.2.7).

2. Let $F \in C_k(M \times \mathbb{R}^N)$ be a function such that for all $p_0 \in \mathbb{R}^+$ there exists a constant $C_1 = C_1(p_0)$ so that, for all $p \in \mathbb{R}^N$, $|p| \le p_0$, we have

$$\|F(\cdot, p)\|_{\mathscr{C}^0_k(M_{x_0})} \le C_1 \ .$$

Then for all $\alpha < 0$, $\beta \in \mathbb{R}$, and $p_0 \in \mathbb{R}^+$ there exists a constant $C_2(p_0, k, \alpha, \beta, x_1)$ such that for all $\mathbb{R}^N$-valued functions $f \in \mathscr{H}^{\alpha-\beta}_k(\Omega)$ with $\|x^\beta f\|_{L^\infty(\Omega)} \le p_0$ we have

$$\left\|F(\cdot, x^\beta f)\right\|_{\mathscr{H}^\alpha_k} \le C_2(1 + \|f\|_{\mathscr{H}^{\alpha-\beta}_k}) \ . \tag{3.2.29}$$

Further, if F has a *uniform* zero of order $l > 0$, in the sense that there exists a constant $\hat{C}$ such that, for all $p \in \mathbb{R}^N$ and $0 \le i \le \min(k, l)$,

$$\left\|\frac{\partial^i F(\cdot, p)}{\partial p^i}\right\|_{\mathscr{C}^0_{k-i}} \le \hat{C}|p|^{l-i} \ , \tag{3.2.30}$$

then for all $\alpha \in \mathbb{R}$, $\beta \ge 0$, there exists a constant $C_3(\hat{C}, l, k, \alpha, \beta, p_0)$ such that, for all $f \in \mathscr{H}^{\alpha-l\beta}_k(\Omega)$ with $\|f\|_{L^\infty(\Omega)} \le p_0$, we have

$$\left\|F(\cdot, x^\beta f)\right\|_{\mathscr{H}^\alpha_k} \le C_3\|f\|_{\mathscr{H}^{\alpha-l\beta}_k} \ . \tag{3.2.31}$$

Remark: The hypothesis (3.2.30) will hold if F is *e.g.* a polynomial in p with coefficients of p^j vanishing for $j < l$, and being functions belonging to $\mathscr{C}^0_k$ for $j \ge l$.

PROOF: We shall give a detailed proof of (3.2.29) and (3.2.31), the inequalities (3.2.27)-(3.2.28) follow by an analogous argument using [39, Volume III, p. 10, Equations (3.21)-(3.22)], *cf.* the calculation of Proposition 3.2.3 below. Let, similarly to (3.2.10),

$$F_n(s, v) = F\left((x = \frac{x_0 s}{2^n}, v); (\frac{x_0 s}{2^n})^\beta f(x = \frac{x_0 s}{2^n}, v)\right) \ ;$$

from Equation (3.2.11) we have

$$\|F(\cdot, x^\beta f)\|^2_{\mathscr{H}^\alpha_k(M_{x_0})} \approx x_0^{2\alpha} \sum_{n \ge 1} 2^{2n\alpha} \|F_n\|^2_{H_k([1,2]\times\partial M)} \ . \tag{3.2.32}$$

We have the obvious bound

$$\sup_{[1,2]\times\partial M} \left|\left(\frac{x_0 s}{2^n}\right)^\beta f\left(\frac{x_0 s}{2^n}, v\right)\right| \le \|x^\beta f\|_{L^\infty(M_{x_0})} \le p_0 \ .$$

Further the partial derivatives of $(s, v) \to F_n(s, v, p)$ with respect to s and v at $p \in \mathbb{R}^N$ fixed, $|p| \le p_0$, can be bounded by a constant depending only upon

$$\sup_{|p| \le p_0} \|F(\cdot, p)\|_{\mathscr{C}^0_k(M_{x_0})} \ .$$

The usual Moser inequalities [39][Volume III, p. 11, Equation (3.30)] give

$$\|F_n\|^2_{H_k([1,2]\times\partial M)} \le C\left(1 + 2^{-2n\beta}\|f_n\|^2_{H_k([1,2]\times\partial M)}\right) \ ,$$

with f_n as in (3.2.10), and with a constant C depending upon k and p_0. Inserting this in (3.2.32) one obtains

$$\begin{aligned} \|F(\cdot,x^\beta f)\|^2_{\mathscr{H}^\alpha_k(M_{x_0})} &\leq C\sum_{n\leq 1} 2^{2n\alpha}(1+2^{-2n\beta}\|f_n\|^2_{H_k([1,2]\times\partial M)}) \\ &\leq C\left(1+\|f\|_{\mathscr{H}^{\alpha-\beta}_k(M_{x_0})}\right) . \end{aligned} \tag{3.2.33}$$

This establishes (3.2.29) for $\Omega=M_{x_0}$, and (3.2.29) with $\Omega=M$ readily follows. The remaining Ω's are handled in a similar way.

To establish (3.2.31), we note the inequality

$$\left|\frac{\partial^{|\gamma|+i}F_n(\cdot,p)}{\partial y^\gamma\partial p^i}\right|\leq C|p|^{\max(l-i,0)} ,$$

which follows from (3.2.30) when $|\gamma|+i\leq k$. Letting y stand for $(s,v)\in[1,2]\times\partial M$, it then follows that for $|\sigma|\leq k$ we have

$$\begin{aligned} |\partial^\sigma F_n| &= \Bigg|\sum_{|\gamma|+|\sigma_1|+\cdots+|\sigma_i|=|\sigma|} C(\sigma_1,\ldots,\sigma_i,\beta)\left(\frac{x_0}{2^n}\right)^{\beta(|\sigma_1|+\cdots+|\sigma_i|)} \\ &\qquad\times\frac{\partial^{|\gamma|+i}F_n}{\partial y^\gamma\partial p^i}\partial^{\sigma_1}(s^\beta f_n)\cdots\partial^{\sigma_i}(s^\beta f_n)\Bigg| \\ &\leq 2^{-l\beta n}C\sum_{|\sigma_1|+\cdots+|\sigma_i|\leq|\sigma|}|\partial^{\sigma_1}(s^\beta f_n)|\cdots|\partial^{\sigma_i}(s^\beta f_n)| . \end{aligned}$$

The usual inequalities [39, Volume III, Chapter 13, Section 3] give

$$\|F_n\|_{H_k([1,2]\times\partial M)}\leq C(k,p_0)2^{-l\beta n}\|f_n\|_{H_k([1,2]\times\partial M)} ,$$

for some constant $C(k,p_0)$, and one concludes from (3.2.32), as in (3.2.33). □

We have the following sharper version of (3.2.27)-(3.2.28):

Proposition 3.2.3 Let $\Omega=M$, or $\Omega=M_{x_1}$, $0<x_1\leq x_0$, or $\Omega=M_{x_2,x_1}$, $2x_2\leq x_1\leq x_0$, and let $\mathscr{H}^\alpha_k=\mathscr{H}^\alpha_k(\Omega)$, etc. There exists a constant $C_s=C_s(\alpha,\beta,k)$ such that, for all $f\in\mathscr{H}^\alpha_k\cap\mathcal{B}^\alpha_0$ and $g\in\mathcal{G}^\beta_k\cap\mathscr{C}^\beta_0$ we have

$$\|fg\|_{\mathscr{H}^{\alpha+\beta}_k}\leq C_s(\|f\|_{\mathcal{B}^\alpha_0}\|g\|_{\mathcal{G}^\beta_k}+\|f\|_{\mathscr{H}^\alpha_k}\|g\|_{\mathscr{C}^\beta_0}) , \tag{3.2.34}$$

$$\begin{aligned} &\forall|\gamma|\leq k\ ,\ \|x^{\gamma_1}\mathscr{D}^\gamma(fg)-(x^{\gamma_1}\mathscr{D}^\gamma f)g\|_{\mathscr{H}^{\alpha+\beta}_0} \\ &\quad\leq C\left(\|f\|_{\mathcal{B}^\alpha_0}\|g\|_{\mathcal{G}^\beta_k}+\|f\|_{\mathscr{H}^\alpha_{k-1}}\left(\|x\partial_x g\|_{\mathscr{C}^\beta_0}+\sum_{i=2}^r\|X_i g\|_{\mathscr{C}^\beta_0}\right)\right) \end{aligned} \tag{3.2.35}$$

where the vector fields X are defined in Equation (3.2.7).

Remark: A useful, though less elegant, inequality related to (3.2.34) is

$$\forall\, |\gamma+\sigma| \le k \qquad \|x^{\gamma_1}(\mathscr{D}^\gamma f)x^{\sigma_1}(\mathscr{D}^\sigma g)\|_{\mathscr{H}_0^{\alpha+\beta}} \le C_s(\|f\|_{\mathcal{B}_0^\alpha}\|g\|_{\mathcal{G}_k^\beta}+\|f\|_{\mathscr{H}_k^\alpha}\|g\|_{\mathscr{C}_0^\beta})\,. \tag{3.2.36}$$

PROOF: We will prove (3.2.35), the proof of (3.2.34) is essentially identical. When $\Omega = M_{x_0}$ we do the rescaling $f_n(s,v) = f(\frac{x_0 s}{2^n}, v)$, $g_n(s,v) = g(\frac{x_0 s}{2^n}, v)$, we then have, for all $|\gamma| \le k$,

$$\begin{aligned}
&\|x^{\gamma_1}\mathscr{D}^\gamma(fg) - (x^{\gamma_1}\mathscr{D}^\gamma f)g\|^2_{\mathscr{H}_0^{\alpha+\beta}} \\
&\approx\; x_0^{-2(\alpha+\beta)}\sum_n 2^{2n(\alpha+\beta)}\|\mathscr{D}^\gamma(f_n g_n) - (\mathscr{D}^\gamma f_n)g_n\|^2_{H_0([1,2]\times\partial M)} \\
&\le\; Cx_0^{-2(\alpha+\beta)}\sum_n 2^{2n(\alpha+\beta)}\left(\|f_n\|^2_{L^\infty}\|g_n\|^2_{H_k} + \|f_n\|^2_{H_{k-1}}\|\mathscr{D}g_n\|^2_{L^\infty}\right) \\
&\le\; Cx_0^{-2(\alpha+\beta)}\left(\left(\sum_n 2^{2n\alpha}\|f_n\|^2_{L^\infty}\right)\sup_n\left(2^{2n\beta}\|g_n\|^2_{H_k}\right)\right. \\
&\qquad \left. + \left(\sum_n 2^{2n\alpha}\|f_n\|^2_{H_{k-1}}\right)\sup_n\left(2^{2n\beta}\|\mathscr{D}g_n\|^2_{L^\infty}\right)\right) \\
&\approx\; C\left(\|f\|^2_{\mathcal{B}_0^\alpha}\|g\|^2_{\mathcal{G}_k^\beta} + \|f\|^2_{\mathscr{H}_{k-1}^\alpha}\|g\|^2_{\mathscr{C}_1^\beta}\right) \\
&\le\; C_s\left(\|f\|_{\mathcal{B}_0^\alpha}\|g\|_{\mathcal{G}_k^\beta} + \|f\|_{\mathscr{H}_{k-1}^\alpha}\|g\|_{\mathscr{C}_1^\beta}\right)^2\,.
\end{aligned} \tag{3.2.37}$$

(In the third line above we have used the inequality [39, Volume III, p. 10, Equation (3.22)].) The case $\Omega = M$ follows immediately from the above; the case $\Omega = M_{x_2x_1}$ is treated similarly using (3.2.12)-(3.2.13) and (3.2.19)-(3.2.21). □

Similar results can be proved in weighted Hölder spaces:

Lemma 3.2.4 Let $\Omega = M$, or $\Omega = M_{x_1}$, $0 < x_1 \le x_0$, or $\Omega = M_{x_2,x_1}$, $2x_2 \le x_1 \le x_0$, and let $\mathscr{C}_k^\alpha = \mathscr{C}_k^\alpha(\Omega)$. Let $f \in \mathscr{C}_k^\alpha \cap \mathscr{C}_0^\beta$ and $g \in \mathscr{C}_k^\gamma \cap \mathscr{C}_0^\delta$ with $\alpha+\delta = \gamma+\beta = \sigma$. Then we have $fg \in \mathscr{C}_k^\sigma$ and

$$\|fg\|_{\mathscr{C}_k^\sigma} \le C_i(\|f\|_{\mathscr{C}_0^\beta}\|g\|_{\mathscr{C}_k^\gamma} + \|g\|_{\mathscr{C}_0^\delta}\|f\|_{\mathscr{C}_k^\alpha})\,, \tag{3.2.38}$$

PROOF: The proof is very similar to that of Propositions 3.2.2 and 3.2.3. We use the same conventions as in (3.2.12), (3.2.19). We have $\|fg\|_{\mathscr{C}_k^\sigma} \approx \sup_n \|fg\|_{C_k(\omega)}$, where

$$\omega \equiv [1,2]\times\partial M\,, \tag{3.2.39}$$

similarly for f and g. The interpolation inequality [31, Appendix A] gives $\|f_n g_n\|_{C_k(\omega)} \le C(\|f_n\|_\infty\|g_n\|_{C_k(\omega)} + \|g_n\|_\infty\|f_n\|_{C_k(\omega)})$, which leads to the conclusion. □

We have the following $\mathscr{C}_k^\beta$ equivalent of the second part of Proposition 3.2.2, with a similar proof, based on Lemma 3.2.4:

Lemma 3.2.5 Let F be a function satisfying the hypotheses of point 2 of Proposition 3.2.2, with a uniform zero of order l in p in the sense of Equation (3.2.30). Then, for any $\epsilon > 0$, $\beta \in \mathbb{R}$ and $f \in \mathscr{C}_k^\beta \cap L^\infty$ we have $F(.,x^\epsilon f) \in \mathscr{C}_k^{\beta+l\epsilon}$, and there exists a constant C depending upon $\|f\|_{L^\infty}$ such that

$$\|F(.,x^\epsilon f)\|_{\mathscr{C}_k^{\beta+l\epsilon}} \leq C(\|f\|_\infty)\|f\|_{\mathscr{C}_k^\beta} \; . \tag{3.2.40}$$

The space of polyhomogeneous functions $\mathscr{A}_{\mathrm{phg}} = \mathscr{A}_{\mathrm{phg}}(M)$ is defined as the set of smooth functions on $\overline{M}$ which have an asymptotic expansion of the form

$$f \sim \sum_{i=0}^{\infty} \sum_{j=0}^{N_i} f_{ij} x^{n_i} \ln^j x \; , \tag{3.2.41}$$

for some sequences n_i, N_i, with $n_i \nearrow \infty$. The polyhomogeneous expansions of the introduction are of this form if r there is replaced by $1/x$; this corresponds to the conformal transformation of Section 3.1, which brings "null infinity" to a finite distance. We emphasize that we allow non-integer values of the n_i's; however, we shall mostly be interested in rational ones, as those arise naturally in the problem at hand. Here the symbol $\sim$ stands for "being asymptotic to": if the right-hand-side is truncated at some finite i, the remainder term falls off appropriately faster. Further, the functions f_{ij} are supposed to be smooth on $\overline{M}$, and the asymptotic expansions should be preserved under differentiation. It is easily checked that the space $\mathscr{A}_{\mathrm{phg}}$ is independent of the choice of the function x, within the class of defining functions of ∂M.

3.3 ODE's in weighted spaces

We begin with some *a priori* estimates in weighted spaces for ODE's. While the results are well-known in principle, and easy to prove, we present them in detail here because their precise form is useful for our arguments later in this work. For a vector w we denote by $\|w\|$ or by $|w|$ the usual Euclidean norm, while for a matrix b the symbol $\|b\|$ denotes its matrix norm.

3.3.1 Solutions of $\partial_\tau \varphi + b\varphi = c$ in weighted spaces

Let $\mathscr{O}$ be a subset of ∂M, which might be the whole of ∂M, or a coordinate patch of ∂M with coordinates v^A, whichever appropriate in the context; we set

$$\mathcal{U}_{x_2,x_1} \equiv]x_2,x_1] \times \mathscr{O} \times [0,T] \; , \tag{3.3.1}$$

$$\mathscr{S}_{x_2,x_1} \equiv]x_2,x_1] \times \mathscr{O} \; , \tag{3.3.2}$$

with $0 \leq x_2 < x_1$. We define $\mathscr{C}_k^\alpha(\mathcal{U}_{x_2,x_1})$ as in(3.2.3), with ∂_τ being considered as a tangential derivative like ∂_{v^A}.

Proposition 3.3.1 Let $\alpha \in \mathbb{R}$, $b \in \mathscr{C}_k^0(\mathcal{U}_{x_2,x_1}, End(\mathbb{R}^N))$, $c \in \mathscr{C}_k^\alpha(\mathcal{U}_{x_2,x_1}, \mathbb{R}^N)$, then the unique solution φ of the equation

$$\partial_\tau \varphi + b\varphi = c \; , \tag{3.3.3}$$

with initial data $\tilde\varphi \equiv \varphi|_{\tau=0} \in \mathscr{C}^\alpha_k(\mathscr{S}_{x_2,x_1},\mathbb{R}^N)$ is in $\mathscr{C}^\alpha_k(\mathcal{U}_{x_2,x_1},\mathbb{R}^N)$ with

$$\|\varphi\|_{\mathscr{C}^\alpha_k(\mathcal{U}_{x_2,x_1})} \le C\left(n,N,k,T,x_1,\|b\|_{\mathscr{C}^0_k(\mathcal{U}_{x_2,x_1})}\right)\left(\|\tilde\varphi\|_{\mathscr{C}^\alpha_k(\mathscr{S}_{x_2,x_1})}+\|c\|_{\mathscr{C}^\alpha_k(\mathcal{U}_{x_2,x_1})}\right). \tag{3.3.4}$$

We also have the estimates

$$\|\varphi(\tau)\|_{\mathscr{C}^\alpha_0(\mathscr{S}_{x_2,x_1})} \le Ce^{\|b\|_\infty\tau}\left(\|\varphi(0)\|_{\mathscr{C}^\alpha_0(\mathscr{S}_{x_2,x_1})}+\int_0^\tau e^{-\|b\|_\infty s}\|c(s)\|_{\mathscr{C}^\alpha_0(\mathscr{S}_{x_2,x_1})}\,ds\right). \tag{3.3.5}$$

$$\begin{aligned}\|\varphi(\tau)\|_{\mathscr{C}^\alpha_k(\mathscr{S}_{x_2,x_1})} \;\le\;& Ce^{C\|b\|_\infty\tau}\times\Bigg(\|\varphi(0)\|_{\mathscr{C}^\alpha_k(\mathscr{S}_{x_2,x_1})}+\int_0^\tau e^{-C|b|_\infty s}\|c(s)\|_{\mathscr{C}^\alpha_k(\mathscr{S}_{x_2,x_1})}\,ds\\ &+\int_0^\tau e^{(1-C)|b|_\infty\, s}\|b(s)\|_{\mathscr{C}^\alpha_k(\mathscr{S}_{x_2,x_1})}\Big(\|\varphi(0)\|_{\mathscr{C}^\alpha_0(\mathscr{S}_{x_2,x_1})}\\ &+\int_0^s e^{-|b|_\infty\, t}\|c(t)\|_{\mathscr{C}^\alpha_0(\mathscr{S}_{x_2,x_1})}\,dt\Big)\,ds\Bigg),\end{aligned} \tag{3.3.6}$$

for $\tau\in[0,T]$.

Remarks :

1. Analogous results in $\mathcal{B}^\alpha_k$ spaces can be proved by similar arguments.
2. An *a-priori* estimate in weighted Sobolev spaces for (3.3.3) follows from Proposition 3.4.1 below by choosing $e_- = \partial_\tau$ and $\psi\in\emptyset$, there.

PROOF: Let $k\in\mathbb{N}^*$, and let $\beta=(\beta_1,\beta_2,\dots\beta_n)$ be a multi-index with $|\beta|\le k$; $\partial^\beta\varphi$ verifies the equation

$$\partial_\tau\partial^\beta\varphi \;=\; -\partial^\beta(b\varphi)+\partial^\beta c\,. \tag{3.3.7}$$

Let $\epsilon>0$ and set

$$e(.,\tau,\epsilon)=\left(\epsilon+\sum_{|\beta|\le k}x^{2(\beta_1-\alpha)}\langle\partial^\beta\varphi,\ \partial^\beta\varphi\rangle\right)^{1/2},$$

$$E(\tau,\epsilon)=\|e(.,\tau,\epsilon)\|_{L^\infty(\mathscr{S}_{x_2,x_1})}\,.$$

When $k=0$ one easily finds

$$\partial_\tau e\le\|b\|e+|c|\,,$$

and (3.3.5) readily follows. For $k>0$ we have

$$\begin{aligned}\partial_\tau e \;&=\; \frac1e\sum_{|\beta|\le k}x^{2(\beta_1-\alpha)}\langle\partial_\tau\partial^\beta\varphi,\ \partial^\beta\varphi\rangle\,,\\ &\le\; \frac1e\sum_{|\beta|\le k}x^{2(\beta_1-\alpha)}|\partial^\beta(-b\varphi+c)|\;|\partial^\beta\varphi|\,,\\ &\le\; \frac{C(k,n)}{e}(\|b\varphi\|_{\mathscr{C}^\alpha_k(\mathscr{S}_{x_2,x_1})}+\|c\|_{\mathscr{C}^\alpha_k(\mathscr{S}_{x_2,x_1})})e\,,\\ &\le\; C(k,n)(\|b\varphi\|_{\mathscr{C}^\alpha_k(\mathscr{S}_{x_2,x_1})}+\|c\|_{\mathscr{C}^\alpha_k(\mathscr{S}_{x_2,x_1})})\,,\end{aligned}$$

where $C(k,n)$ is a constant depending upon k and the space dimension n, and which arises from the inequality $\sum_{i=1}^p |a_i| \le \sqrt{p}\sqrt{\sum_i |a_i|^2}$ for any real sequence (a_i). The weighted interpolation inequalities, Lemma 3.2.4, imply

$$\|b\varphi\|_{\mathscr{C}_k^\alpha(\mathscr{S}_{x_2,x_1})} \le C(\|b\|_{L^\infty(\mathscr{S}_{x_2,x_1})}\|\varphi\|_{\mathscr{C}_k^\alpha(\mathscr{S}_{x_2,x_1})} + \|b\|_{\mathscr{C}_k^0(\mathscr{S}_{x_2,x_1})}\|\varphi\|_{\mathscr{C}_0^\alpha(\mathscr{S}_{x_2,x_1})})\ ,$$

where C is a constant which depends upon k, N and n. It follows that

$$\begin{aligned}\partial_\tau e &\le C\left(\|b\|_{L^\infty(\mathscr{S}_{x_2,x_1})}\|\varphi\|_{\mathscr{C}_k^\alpha(\mathscr{S}_{x_2,x_1})} + \|b\|_{\mathscr{C}_k^0(\mathscr{S}_{x_2,x_1})}\|\varphi\|_{\mathscr{C}_0^\alpha(\mathscr{S}_{x_2,x_1})} + \|c\|_{\mathscr{C}_k^\alpha(\mathscr{S}_{x_2,x_1})}\right)\\ &\le C\left(\|b\|_\infty E(\epsilon,t) + \|b\|_{\mathscr{C}_k^0(\mathscr{S}_{x_2,x_1})}\|\varphi\|_{\mathscr{C}_0^\alpha(\mathscr{S}_{x_2,x_1})} + \|c\|_{\mathscr{C}_k^\alpha(\mathscr{S}_{x_2,x_1})}\right)\ ,\end{aligned}$$

with perhaps a different constant C. By integration we obtain

$$e(\tau) \le e(0) + C\int_0^\tau \left(\|b\|_\infty E(s,\epsilon) + \|b(s)\|_{\mathscr{C}_k^0(\mathscr{S}_{x_2,x_1})}\|\varphi(s)\|_{\mathscr{C}_0^\alpha(\mathscr{S}_{x_2,x_1})} + \|c(s)\|_{\mathscr{C}_k^\alpha(\mathscr{S}_{x_2,x_1})}\right) ds\ ,$$

from which we deduce

$$E(t,\epsilon) \le E(0,\epsilon) + C\int_0^\tau \left(\|b\|_\infty E(s,\epsilon) + \|b(s)\|_{\mathscr{C}_k^0(\mathscr{S}_{x_2,x_1})}\|\varphi(s)\|_{\mathscr{C}_0^\alpha(\mathscr{S}_{x_2,x_1})} + \|c(s)\|_{\mathscr{C}_k^\alpha(\mathscr{S}_{x_2,x_1})}\right) ds\ .$$

Using Gronwall's Lemma and letting $\epsilon \to 0$ one obtains

$$\begin{aligned}E(\tau,0) &\le e^{C\|b\|_\infty t}E(0,0)\\ &\quad + C\int_0^\tau e^{C\|b\|_\infty(\tau-s)}\left(\|b(s)\|_{\mathscr{C}_k^0(\mathscr{S}_{x_2,x_1})}\|\varphi(s)\|_{\mathscr{C}_0^\alpha(\mathscr{S}_{x_2,x_1})} + \|c(s)\|_{\mathscr{C}_k^\alpha(\mathscr{S}_{x_2,x_1})}\right)ds\ .\end{aligned}$$

The estimate (3.3.5) for $\|\varphi\|_{\mathscr{C}_0^\alpha(\mathscr{S}_{x_2,x_1})}$ inserted in the last inequality leads to Equation (3.3.6). The time-derivative estimates follow immediately from the above and from the equation satisfied by φ. □

3.3.2 Solutions of $\partial_x\phi + b\phi = c$ in weighted spaces

All the results in this section, as well as in Section 3.3.4 below, remain valid if we replace the set $\mathcal{U}_{x_2,x_1}$ defined in Equation (3.3.1) with $\mathscr{S}_{x_2,x_1}$ defined in (3.3.2) — the time dimension does not play a preferred role in the current problem. We start with the following elementary result; the point is to ensure that the relevant constants are x_2 independent:

Lemma 3.3.2 Let $g \in \mathscr{C}_k^\alpha(\mathcal{U}_{x_2,x_1}, \mathbb{R}^N)$, $0 \le x_2 < x_1$, then f defined for $\alpha > -1$ by

$$f(x, v^A, \tau) = \int_{x_2}^x g(s, v^A, \tau)\, ds$$

is in $\mathscr{C}_k^{\alpha+1}(\mathcal{U}_{x_2,x_1}, \mathbb{R}^N)$, with

$$\|f\|_{\mathscr{C}_k^{\alpha+1}(\mathcal{U}_{x_2,x_1})} \le \max\left\{1, \frac{1}{\alpha+1}\right\}\|g\|_{\mathscr{C}_k^\alpha(\mathcal{U}_{x_2,x_1})}\ .$$

Similarly f_2 defined by

$$f_2(x,v,\tau) = -\int_x^{x_1} g(s,v,\tau)\, ds$$

satisfies

$$(1+(\ln x)^2)^{-1/2} f_2 \in \mathscr{C}_k^0(\mathcal{U}_{x_2,x_1}) \text{ for } \alpha = -1 \ ,$$
$$f_2 \in \mathscr{C}_k^{\min\{\alpha+1,0\}}(\mathcal{U}_{x_2,x_1}) \text{ for } \alpha < 0 \text{ and } \alpha \neq -1 \ ,$$

with

$$\|f_2\|_{\mathscr{C}_k^{\min\{\alpha+1,0\}}(\mathcal{U}_{x_2,x_1})} \le \max\left\{1, \left|\frac{1}{1+\alpha}\right|, \left|\frac{x_1^{\alpha+1}}{1+\alpha}\right|\right\} \|g\|_{\mathscr{C}_k^\alpha(\mathcal{U}_{x_2,x_1})} \ .$$

PROOF: We have the trivial relations

$$\begin{aligned} \int_{x_2}^x s^\alpha \, ds &\le \frac{1}{\alpha+1} x^{\alpha+1} \quad \text{for } \alpha > -1 \ , \\ \int_x^{x_1} s^{-1}\, ds &= \ln x_1 - \ln x \ , \end{aligned}$$

as well as the commutation rules:

$$\begin{aligned} \partial_x \int_a^x f\, dx &= f(x) \ , \\ \partial_{v^A} \int_a^x f dx &= \int_a^x \partial_{v^A} f\, dx \ , \\ \partial_\tau \int_a^x f\, dx &= \int_a^x \partial_\tau f\, dx \ . \end{aligned}$$

Note that

$$\|f\|_{\mathscr{C}_k^{\alpha+1}(\mathcal{U}_{x_2,x_1})} = \|\partial_x f\|_{\mathscr{C}_{k-1}^\alpha(\mathcal{U}_{x_2,x_1})} + \sum_{0\le i+|\delta|\le k} \|\partial_\tau^i \partial_{v^A}^\delta f\|_{\mathscr{C}_0^{\alpha+1}(\mathcal{U}_{x_2,x_1})} \ , \tag{3.3.8}$$

with $\|\partial_x f\|_{\mathscr{C}_{k-1}^\alpha(\mathcal{U}_{x_2,x_1})} = \|g\|_{\mathscr{C}_{k-1}^\alpha(\mathcal{U}_{x_2,x_1})}$. To estimate $\partial_\tau^i \partial_{v^A}^\delta f$ one writes

$$\begin{aligned} |\partial_\tau^i \partial_v^\delta f| &\le \int_{x_2}^x |\partial_\tau^i \partial_v^\delta g|\, ds \ , \\ &\le \int_{x_2}^x \|\partial_\tau^i \partial_v^\delta g\|_{\mathscr{C}_0^\alpha} s^\alpha\, ds \ , \\ &\le \frac{1}{\alpha+1} x^{\alpha+1} \|\partial_\tau \partial_v^\delta g\|_{\mathscr{C}_0^\alpha} \ . \end{aligned}$$

The results for f_2 are established in a similar way. □

We shall use the following notation

$$\mathscr{I}_{x_2} = \{x = x_2\} \ , \tag{3.3.9}$$

with the range of the other variables being in principle clear from the context; this is the equivalent of the set $\tilde{\partial} M_{x_2}$ of Equation (3.2.1) when the set-up described there is assumed.

Proposition 3.3.3 Let $0 \le x_2 < x_1$, suppose that $b \in \mathscr{C}_k^{-\epsilon}(\mathcal{U}_{x_2,x_1}, End(\mathbb{R}^N))$, $0 \le \epsilon < 1$, $c \in \mathscr{C}_k^{\alpha}(\mathcal{U}_{x_2,x_1}, \mathbb{R}^N)$, and let ϕ be a solution in $C_k(\mathcal{U}_{x_2,x_1})$ of the equation

$$\partial_x \phi + b\phi = c \; . \tag{3.3.10}$$

Then the following hold:

1. If $\alpha < -1$, then $\phi \in \mathscr{C}_k^{\alpha+1}(\mathcal{U}_{x_2,x_1})$ and we have, for $\alpha + 2 - \epsilon \neq 0$ and for $x_2 \le x_3 \le 1$ small enough so that $C(\|b\|_{\mathscr{C}_0^{-\epsilon}}, x_3) < 1$,

$$\|\phi\|_{\mathscr{C}_0^{\alpha+1}(\mathcal{U}_{x_2,x_3})} \le \frac{1}{1 - C(\|b\|_{\mathscr{C}_0^{-\epsilon}}, x_3)} (x_3^{-\alpha-1} \|\phi\|_{C_0(\mathscr{I}_{x_3})} + \frac{1}{|1+\alpha|} \|c\|_{\mathscr{C}_0^{\alpha}(\mathcal{U}_{x_2,x_3})}) \; , \tag{3.3.11}$$

where

$$C(\|b\|_{\mathscr{C}_0^{-\epsilon}}, x_3) = \frac{x_3^{1-\epsilon}}{|2+\alpha-\epsilon|} \|b\|_{\mathscr{C}_0^{-\epsilon}(\mathcal{U}_{x_2,x_3})} \; . \tag{3.3.12}$$

Moreover, if $x_2 \le x_3 \le 1$ is small enough so that $C_i\, C(\|b\|_{\mathscr{C}_0^{-\epsilon}}, x_3) < 1$, where C_i is the constant in the interpolation inequality (3.2.38), then

$$\begin{aligned} \|\phi\|_{\mathscr{C}_k^{\alpha+1}(\mathcal{U}_{x_2,x_3})} \;\; \le \;\; & C_\alpha(\|b\|_{\mathscr{C}_0^{-\epsilon}}, C_i, x_3) \Big(\|\phi(x_3)\|_{C_k(\mathscr{I}_{x_3})} + \|c\|_{\mathscr{C}_k^{\alpha}(\mathcal{U}_{x_2,x_3})} \\ & + \|b\|_{\mathscr{C}_k^{-\epsilon}(\mathcal{U}_{x_2,x_3})} (\|\phi(x_3)\|_{C_0(\mathscr{I}_{x_3})} + \|c\|_{\mathscr{C}_0^{\alpha}(\mathcal{U}_{x_2,x_3})}) \Big) \; , \end{aligned} \tag{3.3.13}$$

with $C_\alpha(\|b\|_{\mathscr{C}_0^{-\epsilon}}, C_i, x_3)$ an increasing function in the first and third variable.

2. If $\alpha = 1$, then $(1 + (\ln x)^2)^{-1/2} \phi \in \mathscr{C}_k^0(\mathcal{U}_{x_2,x_1})$.

3. If $\alpha > -1$, then $\phi_{x_2} \equiv \lim_{x \to x_2} \phi$ is in $C_k(\mathscr{I}_{x_2})$, with

$$\phi - \phi_{x_2} \in \mathscr{C}_k^{1-\epsilon}(\mathcal{U}_{x_2,x_1}) + \mathscr{C}_k^{\alpha+1}(\mathcal{U}_{x_2,x_1}) \; , \tag{3.3.14}$$

$\phi \in \mathscr{C}_k^{\alpha+1}(\mathcal{U}_{x_2,x_1})$ if $\phi_{x_2} = 0$, and

$$\|\phi\|_{L^\infty(\mathcal{U}_{x_2,x_3})} \le \frac{1}{1 - C'(\|b\|_{\mathscr{C}_0^{-\epsilon}}, x_3)} \left(\|\phi\|_{L^\infty(\mathscr{I}_{x_3})} + \frac{x_3^{1+\alpha}}{1+\alpha} \|c\|_{\mathscr{C}_0^{\alpha}(\mathcal{U}_{x_2,x_3})} \right) \tag{3.3.15}$$

for $x_2 \le x_3 \le 1$ small enough so that

$$C'(\|b\|_{\mathscr{C}_0^{-\epsilon}}, x_3) := \frac{x_3^{1-\epsilon}}{1-\epsilon} \|b\|_{\mathscr{C}_0^{-\epsilon}(\mathcal{U}_{x_2,x_3})} < 1 \; .$$

Moreover for x_3 small enough so that $C_i C'(\|b\|_{\mathscr{C}_0^{-\epsilon}}, x_3) < 1$ we also have

$$\begin{aligned} \|\phi\|_{\mathscr{C}_k^{0}(\mathcal{U}_{x_2,x_3})} \;\; \le \;\; & C'_\alpha(\|b\|_{\mathscr{C}_0^{-\epsilon}}, C_i, x_3) \Big(\|\phi(x_3)\|_{C_k(\mathscr{I}_{x_3})} + \|c\|_{\mathscr{C}_k^{\alpha}(\mathcal{U}_{x_2,x_3})} \\ & + \|b\|_{\mathscr{C}_k^{-\epsilon}(\mathcal{U}_{x_2,x_3})} (\|\phi(x_3)\|_{C_0(\mathscr{I}_{x_3})} + \|c\|_{\mathscr{C}_0^{\alpha}(\mathcal{U}_{x_2,x_3})}) \Big) \; , \end{aligned} \tag{3.3.16}$$

with C'_α an increasing function in its first and third argument.

Remarks : 1. The inequalities above are standard when $x_2 > 0$ and when the constants are allowed to depend upon x_2, regardless of whether or not x_3 can be made small. As already mentioned, the point here is to make sure that the constants do not blow up as x_2 gets small.

2. In case 2. log-weighted estimates are easily derived; they will, however, not be needed in what follows.

PROOF: 1. For simplicity, we will write $\mathscr{C}^\delta_k$ for $\mathscr{C}^\delta_k(\mathcal{U}_{x_3,x_2})$. Let ϕ be a (local) solution of (3.3.10), corresponding to initial data at $\{x = x_1\}$ in $C_k(\mathscr{I}_{x_1})$. For $a > 0$ set

$$e_a(x, v^A, \tau) := (a + \sum_{|\beta|\le k} x^{2\beta_1}(\partial^\beta\phi|\partial^\beta\phi))^{1/2} ,$$

and $e := e_0$. Let $x_3 \in]x_2, x_1[\cap]0, 1]$ be such that $\frac{x_3^{1-\epsilon}}{|2+\alpha-\epsilon|}\|b\|_{\mathscr{C}_0^{-\epsilon}} < 1$. We have for all $x_2 < x \le x_3$,

$$\begin{aligned} -\partial_x e_a &= -\frac{1}{e_a}\sum \beta_1 x^{2\beta_1-1}(\partial^\beta\phi|\partial^\beta\phi) \quad \mathrm{I} \\ &\quad -\frac{1}{e_a}\sum_{|\beta|\le k} x^{2\beta_1}(\partial^\beta\partial_x\phi|\partial^\beta\phi) \quad \mathrm{II} , \end{aligned} \tag{3.3.17}$$

Since β_1 is non-negative we have $-\partial_x e_a(x, v^A, \tau) \le$ II ; further

$$\begin{aligned} \mathrm{II} &= \frac{1}{e_a}\sum_{|\beta|\le k} x^{2\beta_1}(\partial^\beta(b\phi - c)|\partial^\beta\phi) , \\ &\le \frac{1}{e_a}\sum_{|\beta|\le k} (|x^{\beta_1}\partial^\beta c| + |x^{\beta_1}\partial^\beta(b\phi)|)\,|x^{\beta_1}\partial^\beta\phi| , \\ &\le \sum_{|\beta|\le k} |x^{\beta_1}\partial^\beta c| + |x^{\beta_1}\partial^\beta(b\phi)| . \end{aligned} \tag{3.3.18}$$

Clearly

$$\begin{aligned} \sum |x^{\beta_1}\partial^\beta c| &= x^\alpha \sum |x^{-\alpha+\beta_1}\partial^\beta c| , \\ &\le x^\alpha \|c\|_{\mathscr{C}^\alpha_k} , \\ \sum_{|\beta|\le k} |x^{\beta_1}\partial^\beta(b\phi)| &= x^{\alpha+1-\epsilon}\sum_{|\beta|\le k} |x^{-\alpha-1+\epsilon+\beta_1}\partial^\beta(b\phi)| , \\ &\le x^{\alpha+1-\epsilon}\|b\phi\|_{\mathscr{C}^{\alpha+1-\epsilon}_k} , \end{aligned}$$

which gives

$$-\partial_x e_a \le x^\alpha\|c\|_{\mathscr{C}^\alpha_k} + x^{\alpha+1-\epsilon}\|b\phi\|_{\mathscr{C}^{\alpha+1-\epsilon}_k} . \tag{3.3.19}$$

Consider, first, the case $k = 0$; in this case (3.3.19) reads

$$-\partial_x e_a \le x^\alpha\|c\|_{\mathscr{C}^\alpha_0} + x^{\alpha+1-\epsilon}\|b\|_{\mathscr{C}^{-\epsilon}_0}\|\phi\|_{\mathscr{C}^{\alpha+1}_0} ,$$

which, after integrating over $[x_3, x]$ and passing to the limit $a \to 0$, gives

$$\begin{aligned} e(x, v^A, \tau) &\leq e(x_3, v^A, \tau) + \left(-\frac{x^{\alpha+1}}{(1+\alpha)} + \frac{x_3^{\alpha+1}}{(1+\alpha)} \right) \|c\|_{\mathscr{C}_0^\alpha} \\ &\quad + (\frac{x_3^{\alpha+2-\epsilon}}{(2+\alpha-\epsilon)} - \frac{x^{\alpha+2-\epsilon}}{(2+\alpha-\epsilon)}) \|b\|_{\mathscr{C}_0^{-\epsilon}} \|\phi\|_{\mathscr{C}_0^{\alpha+1}} , \\ &\leq \|\phi\|_{C_0(\mathscr{I}_{x_3})} + \frac{x^{\alpha+1}}{|1+\alpha|} \|c\|_{\mathscr{C}_0^\alpha} \\ &\quad + (\frac{x_3^{\alpha+2-\epsilon}}{(2+\alpha-\epsilon)} - \frac{x^{\alpha+2-\epsilon}}{(2+\alpha-\epsilon)}) \|b\|_{\mathscr{C}_0^{-\epsilon}} \|\phi\|_{\mathscr{C}_0^{\alpha+1}} . \quad (3.3.20) \end{aligned}$$

Suppose for the moment that $\alpha + 2 - \epsilon < 0$; Equation (3.3.20) yields

$$e(x, v^A, \tau) \leq \|\phi\|_{C_0(\mathscr{I}_{x_3})} + \frac{x^{\alpha+1}}{|1+\alpha|} \|c\|_{\mathscr{C}_0^\alpha} + \frac{x^{\alpha+2-\epsilon}}{|2+\alpha-\epsilon|}) \|b\|_{\mathscr{C}_0^{-\epsilon}} \|\phi\|_{\mathscr{C}_0^{\alpha+1}} , \tag{3.3.21}$$

and since $x^{-1-\alpha} \leq x_3^{-1-\alpha} \leq 1$ we obtain

$$x^{-\alpha-1} e(x, v^A, \tau) \quad \leq \quad x_3^{-1-\alpha} \|\phi\|_{C_0(\mathscr{I}_{x_3})} + \frac{1}{|1+\alpha|} \|c\|_{\mathscr{C}_0^\alpha} + \frac{x_3^{1-\epsilon}}{|2+\alpha-\epsilon|} \|b\|_{\mathscr{C}_0^{-\epsilon}} \|\phi\|_{\mathscr{C}_0^{\alpha+1}} .$$

Suppose, further, that $\alpha + 2 - \epsilon > 0$; we then have

$$e(x, v^A, \tau) \quad \leq \quad \|\phi\|_{C_0(\mathscr{I}_{x_3})} + \frac{x^{\alpha+1}}{|1+\alpha|} \|c\|_{\mathscr{C}_0^\alpha} + \frac{x_3^{\alpha+2-\epsilon}}{(2+\alpha-\epsilon)}) \|b\|_{\mathscr{C}_0^{-\epsilon}} \|\phi\|_{\mathscr{C}_0^{\alpha+1}} ,$$

which gives

$$\begin{aligned} x^{-\alpha-1} e(x, v^A, \tau) &\leq x_3^{-1-\alpha} \|\phi\|_{C_0(\mathscr{I}_{x_3})} + \frac{1}{|1+\alpha|} \|c\|_{\mathscr{C}_0^\alpha} \\ &\quad + \frac{x_3^{1-\epsilon}}{|2+\alpha-\epsilon|} \|b\|_{\mathscr{C}_0^{-\epsilon}} \|\phi\|_{\mathscr{C}_0^{\alpha+1}} . \end{aligned}$$

The inequality $\|\phi\|_{\mathscr{C}_0^{\alpha+1}(\mathscr{U}_{x_2,x_3})} \leq \sup_{[x_2,x_3]} x^{-1-\alpha} e$ shows that in all cases we have

$$\|\phi\|_{\mathscr{C}_0^{\alpha+1}(\mathscr{U}_{x_2,x_3})} \leq \frac{1}{1 - C(\|b\|_{\mathscr{C}_0^{-\epsilon}}, x_3)} (x_3^{-1-\alpha} \|\phi\|_{C_0(\mathscr{I}_{x_3})} + \frac{1}{|1+\alpha|} \|c\|_{\mathscr{C}_0^\alpha}) ,$$

with the constant as in Equation (3.3.12). Consider, now, any $0 < k \in \mathbb{N}$; Equation (3.3.19) and the interpolation inequality (3.2.38) give

$$-\partial_x e_a \leq x^\alpha \|c\|_{\mathscr{C}_k^\alpha} + x^{\alpha+1-\epsilon} C_i (\|b\|_{\mathscr{C}_0^{-\epsilon}} \|\phi\|_{\mathscr{C}_k^{\alpha+1}} + \|b\|_{\mathscr{C}_k^{-\epsilon}} \|\phi\|_{\mathscr{C}_0^{\alpha+1}}) .$$

An argument identical to the one before, considering separately the cases $\alpha + 2 - \epsilon > 0$ or < 0, leads to

$$\|\phi\|_{\mathscr{C}_k^{\alpha+1}} \quad \leq \quad \frac{1}{1 - C_i\, C(\|b\|_{\mathscr{C}_0^{-\epsilon}}, x_3)} \left(x_3^{-1-\alpha} \|\phi\|_{C_k(\mathscr{I}_{x_3})} + \frac{1}{|1+\alpha|} \|c\|_{\mathscr{C}_k^\alpha} \right)$$

$$
\begin{aligned}
&+\frac{C_i}{1-C_i\,C(\|b\|_{\mathscr{C}_0^{-\epsilon}},x_3)}\frac{x_3^{1-\epsilon}}{|2+\alpha-\epsilon|}\|b\|_{\mathscr{C}_k^{-\epsilon}}\|\phi\|_{\mathscr{C}_0^{\alpha+1}}\ ,\\
\leq\ & \frac{C(x_3)}{1-C_i\,C(\|b\|_{\mathscr{C}_0^{-\epsilon}},x_3)}\Big(\|\phi\|_{C_k(\mathscr{I}_{x_3})}+\|c\|_{\mathscr{C}_k^{\alpha}}\\
&+\frac{C_i}{1-C\|b\|_{\mathscr{C}_0^{-\epsilon}},x_3)}\|b\|_{\mathscr{C}_k^{-\epsilon}}\Big(x_3^{-\alpha-1}\|\phi\|_{C_0(\mathscr{I}_{x_3})}+\frac{1}{|1+\alpha|}\|c\|_{\mathscr{C}_0^{\alpha}}\Big)\Big)\ ,
\end{aligned}
$$

which gives (3.3.13). We have thus shown that $\phi\in\mathscr{C}_k^{\alpha+1}(\mathcal{U}_{x_2,x_3})$; the property that $\phi\in\mathscr{C}_k^{\alpha+1}(\mathcal{U}_{x_2,x_1})$ immediately follows.

2. The proof is identical, except for a few obvious modifications in the calculations.

3. To obtain the L^∞ estimate, we start from (3.3.17)-(3.3.18) with $k=0$, which upon integration and passing to the limit $a\to 0$ gives

$$
e(x,v^A,\tau)\ \ \leq\ \ e(x_3,v^A,\tau)+\frac{x_3^{\alpha+1}}{1+\alpha}\|c\|_{\mathscr{C}_0^{\alpha}}+\frac{x_3^{1-\epsilon}}{1-\epsilon}\|b\|_{\mathscr{C}_0^{-\epsilon}}\|\phi\|_{\mathscr{C}_0^{0}}\ ,
$$

from which we deduce

$$
\|\phi\|_{L^\infty(\mathcal{U}_{x_2,x_3})}\leq\|\phi\|_{L^\infty(\mathscr{I}_{x_3})}+\frac{x_3^{\alpha+1}}{\alpha+1}\|c\|_{\mathscr{C}_0^{\alpha}}+\frac{x_3^{1-\epsilon}}{1-\epsilon}\|b\|_{\mathscr{C}_0^{-\epsilon}}\|\phi\|_{L^\infty(\mathcal{U}_{x_2,x_3})}\ ,
$$

and (3.3.15) follows. The proof of (3.3.16) is similar to that of the analogous statement in point 1. From what has been said it can be seen that $\phi_{x_2}\equiv\lim_{x\to x_2}\phi$ exists and is in $C_k(\mathscr{I}_{x_2})$. It remains to show that $\phi-\phi_{x_2}$ satisfies (3.3.14). Integrating (3.3.10) we have

$$
\phi(x,\cdot)=\phi_{x_2}(\cdot)e^{-\int_{x_2}^{x}b(s,\cdot)ds}+\int_{x_2}^{x}e^{\int_x^y b(s,\cdot)ds}c(y,\cdot)dy\ , \tag{3.3.22}
$$

from which the result easily follows. □

3.3.3 Polyhomogeneous solutions of $\partial_\tau\varphi+b\varphi=c$

We pass now to an analysis of ODE's with polyhomogeneous sources. The results here have an auxiliary character, and several of them are rather elementary; they will be needed to handle the real problem at hand, with partial differential operators. Let $\mathcal{O}$ be an open subset of ∂M, we set

$$
\mathcal{U}_{x_1}=]0,x_1]\times\mathcal{O}\times[0,T]\ . \tag{3.3.23}
$$

Integer space-dimensions force us to consider polyhomogeneous expansions with half-integer power of x; in order to account for that, we introduce an index

$$
\delta=\frac{1}{d}\ ,
$$

where d is a non-zero integer, $d\in\mathbb{N}^*$. We will mostly be interested in the case $d=1/2$ or $d=1$, however other values are also possible in the formalism here.

Results analogous to the ones below hold for the general polyhomogeneous expansions of Equation (3.2.41), which can be established by similar methods. We find it of interest that a consistent framework can be obtained in the setting considered below:

Proposition 3.3.4 Consider the system

$$\partial_\tau \varphi + b\varphi = c\ , \tag{3.3.24a}$$

$$\varphi|_{\{\tau=0\}}(x,v) \equiv \tilde\varphi(x,v) = x^\beta \sum_{i=0}^{p}\sum_{j=0}^{N_i} x^{i\delta} \ln^j x\ \tilde\varphi_{ij}(x,v)\ +\ \tilde\varphi_{p\delta+\beta+\epsilon}(x,v)\ , \tag{3.3.24b}$$

$$\tilde\varphi_{ij} \in C_\infty(\overline{\{\tau=0\}})\ , \qquad \tilde\varphi_{p\delta+\beta+\epsilon} \in \mathscr{C}^{p\delta+\beta+\epsilon}_\infty(\{\tau=0\})\ , \tag{3.3.24c}$$

with

$$b(x,v,\tau) = \sum_{i=0}^{p}\sum_{j=0}^{N_i'} x^{i\delta} \ln^j x\ b_{ij}(x,v,\tau)\ +\ b_{p\delta+\epsilon}(x,v,\tau)\ , \tag{3.3.25a}$$

$$b_{p\delta+\epsilon} \in \mathscr{C}^{p\delta+\epsilon}_\infty(\mathcal{U}_{x_1})\ , \qquad b_{ij} \in C_\infty(\overline{\mathcal{U}_{x_1}})\ , \tag{3.3.25b}$$

$$c(x,v,\tau) = x^\beta \sum_{i=0}^{p}\sum_{j=0}^{N_i''} x^{i\delta} \ln^j x\ c_{ij}(x,v,\tau)\ +\ c_{p\delta+\beta+\epsilon}(x,v,\tau)\ , \tag{3.3.25c}$$

$$c_{p\delta+\beta+\epsilon} \in \mathscr{C}^{p\delta+\beta+\epsilon}_\infty(\mathcal{U}_{x_1})\ , \qquad c_{ij} \in C_\infty(\overline{\mathcal{U}_{x_1}})\ , \tag{3.3.25d}$$

where $0 < \epsilon < \delta$, and $(N_i), (N_i'), (N_i'')$ are sequences with integer values, and with

$$b \in L^\infty(\mathcal{U}_{x_1})\ .$$

Then the solution φ takes the form

$$\varphi(x,v,\tau) = x^\beta \sum_{i=0}^{p}\sum_{j=0}^{M_i} x^{i\delta} \ln^j x\ \varphi_{ij}(x,v,\tau)\ +\ \varphi_{p\delta+\beta+\epsilon}(x,v,\tau)\ , \tag{3.3.26}$$

with $\varphi_{ij} \in C_\infty(\overline{\mathcal{U}_{x_1}})$, M_k is an integer sequence and $\varphi_{p\delta+\beta+\epsilon} \in \mathscr{C}^{p\delta+\beta+\epsilon}_\infty(\mathcal{U}_{x_1})$.

To prove the proposition we shall need the following lemma:

Lemma 3.3.5 Under the hypotheses of Proposition 3.3.4, suppose that in addition we have

$$\tilde\varphi_{p\delta+\beta+\epsilon} = b_{p\delta+\epsilon} = c_{p\delta+\beta+\epsilon} = 0\ .$$

Then for any $\epsilon \in]0,\delta[$ we have

$$\varphi = x^\beta \sum_{i=0}^{p}\sum_{j=0}^{M_i} x^{i\delta} \ln^j x\ \varphi_{ij}\ + \varphi_{p\delta+\beta+\epsilon}\ , \tag{3.3.27}$$

with $\varphi_{ij} \in C_\infty(\overline{\mathcal{U}_{x_1}})$, $\varphi_{p\delta+\beta+\epsilon} \in \mathscr{C}^{p\delta+\beta+\epsilon}_\infty(\mathcal{U}_{x_1})$, for some integer-valued sequence M_k.

PROOF: Inserting (3.3.27) in the equation (3.3.24a) and tracking the coefficients in front of $x^{i\delta}\ln^j x$ one finds the following set of equations

$$M_0 = \max\{N_0, N_0''\}\ , \qquad M_{i+1} = \max\{\max_{0\le k\le i} M_k + N'_{i-k},\ N''_{i+1},\ N_{i+1}\}\ ,$$

$$i \in [\![0,p]\!]\ ,\ j\in[\![0,M_i]\!]\ , \qquad \partial_\tau\varphi_{ij} + \sum_{k=0}^{i}\sum_{l=0}^{\min\{N'_k,j\}} b_{kl}\varphi_{i-k\,j-l} = c_{ij}\ ,$$

$$\partial_\tau\varphi_{p\delta+\beta+\epsilon} + b\varphi_{p\delta+\beta+\epsilon} = -\sum_{i=p+1}^{2p} x^\beta \sum_{j=0}^{M_i} x^{i\delta}\ln^j x\,\{\sum_{k=0}^{i}\sum_{l=0}^{\min\{N'_k,j\}} b_{kl}\varphi_{i-k\,j-l}\}\ .$$

Here $[\![a,b]\!] := [a,b]\cap\mathbb{N}$. This system is easily solved: one begins with $i=0$ and solves the equations for j running from 0 to M_0. This can then be repeated for $i=1$, *etc*, until $i=p$ is reached. This provides the functions φ_{ij}. Finally, one solves the last equation for the remainder term $\varphi_{p\delta+\beta+\epsilon}$, with initial value zero, noting that the right hand side of the resulting equation is in $\mathscr{C}^{p\delta+\beta+\epsilon}_\infty(\mathcal{U}_{x_1})$, and one concludes using Proposition 3.3.1. □.

PROOF OF PROPOSITION 3.3.4: With the notation of the proposition, we set $b_{\rm phg} = b - b_{p\delta+\epsilon}$, $c_{\rm phg} = c - c_{p\delta+\beta+\epsilon}$, $\tilde\varphi_{\rm phg} = \tilde\varphi - \tilde\varphi_{p\delta+\beta+\epsilon}$. We use the Lemma above to obtain a solution $\varphi_{\rm phg}$ of the problem

$$\partial_\tau\varphi + b_{\rm phg}\varphi = c_{\rm phg}\ , \tag{3.3.28}$$

$$\varphi|_\Sigma = \tilde\varphi = x^\beta\sum_{i=0}^{p}\sum_{j=0}^{N_i} x^{i\delta}\ln^j x\ \tilde\varphi_{ij}(x,v)\ . \tag{3.3.29}$$

Then we solve

$$\partial_\tau\varphi' + b\varphi' = c_{p\delta+\beta+\epsilon} - b_{p\delta+\epsilon}\varphi_{\rm phg}$$

with $\varphi'|_{\tau=0} = \tilde\varphi_{p\delta+\beta+\epsilon}$. According to Proposition 3.3.1 we have $\varphi'\in\mathscr{C}^{p\delta+\beta+\epsilon}_\infty(\mathcal{U}_{x_1})$. To conclude we set $\varphi = \varphi_{\rm phg}+\varphi'$ which is of the required form, and solves (3.3.24c). □

3.3.4 Polyhomogeneous solutions of $\partial_x\varphi + b\varphi = c$

Proposition 3.3.6 Let φ be a solution in $C_\infty(\mathcal{U}_{x_1})$ of

$$\partial_x\varphi + \frac{b}{x}\varphi = c\ , \tag{3.3.30}$$

and suppose that (3.3.25) holds with some $\epsilon\in]0,\delta[$, $\beta\in\mathbb{R}$, and with some integer-valued sequences $(N'_i), (N''_i)$. If

$$b = o(x)$$

(equivalently, $b_{0j}(0,v,\tau)=0$), then

$$\varphi = \sum_{i=0}^{p}\sum_{j=0}^{M_i} x^{i\delta}\ln^j x\ \widehat\varphi_{ij}\ + x^{\beta+1}\sum_{i=0}^{p}\sum_{j=0}^{M_i} x^{i\delta}\ln^j x\ \varphi_{ij}\ +\ \varphi_{p\delta+1+\beta+\epsilon}\ , \tag{3.3.31}$$

with

$$\widehat{\varphi}_{ij}\ ,\varphi_{ij} \in C_\infty(\overline{\mathcal{U}_{x_1}})\ , \qquad \varphi_{p\delta+1+\beta+\epsilon} \in \mathscr{C}_\infty^{p\delta+1+\beta+\epsilon}(\mathcal{U}_{x_1})\ ,$$

for some integer sequence (M_i).

PROOF: Proposition 3.3.3 shows that for $\beta > -1$ the limit

$$\varphi_0(\cdot) := \lim_{x\to 0} \varphi(x,\cdot)$$

exists and is a smooth function on $\mathcal{O} \times [0,T]$. If b is a multiple of the identity matrix the result is then obtained by a straightforward analysis of the formula

$$\varphi(x,\cdot) = \varphi_0(\cdot)e^{-\int_0^x b(s,\cdot)ds} + \int_0^x e^{\int_x^y b(s,\cdot)ds} c(y,\cdot)dy\ , \tag{3.3.32}$$

using the estimates of Lemma 3.3.2. For $\beta < -1$, and again for b — a multiple of the identity matrix — we use instead

$$\varphi(x,\cdot) = \varphi(x_1/2,\cdot)e^{-\int_{x_1/2}^x b(s,\cdot)ds} + \int_{x_1/2}^x e^{\int_x^y b(s,\cdot)ds} c(y,\cdot)dy\ . \tag{3.3.33}$$

In the general case, we first note that it follows from Proposition 3.3.3 that there exists $\lambda \in \mathbb{R}$ such that $\psi \in \mathscr{C}_\infty^\lambda$. We then write

$$\partial_x\psi - c = -\frac{b}{x}\psi \in \mathscr{C}_\infty^{\lambda+\delta-1}\ ; \tag{3.3.34}$$

integrating gives

$$\psi - \int_0^x c \in \mathscr{C}_\infty^{\lambda+\delta}\ .$$

Inserting this equation in the right-hand-side of (3.3.34) and integrating again one obtains a similar equation with a remainder term falling-off one power of δ faster. The result is proved by repeating this procedure a finite number of times. □

3.4 A class of linear symmetric hyperbolic systems

In this section we shall consider a class of linear symmetric hyperbolic first order systems on a set of the form $M_{x_0} \times I$, where I is an interval corresponding to the time variable, which will be denoted by τ. (We note that in some of our further applications the vector $\partial/\partial\tau$ will be lightlike, and not timelike as is usually the case. It should be pointed out that in our conventions the time variable is the last coordinate, allowing x to be the first variable, consistently with the conventions of the preceding sections.) We start by introducing some notation for the sets within the “space-time” $M_{x_0} \times I$, which will be relevant in what follows[1]:

$$t \geq 0, \quad 2(x_2+t) < x_1 \leq x_0, \quad \Sigma_{x_2,x_1,t} = \{\tau = t,\ x_2 < x < x_1 - 2t\}, \tag{3.4.1a}$$
$$T > 0, \quad 2(x_2+T) < x_1 \leq x_0, \qquad \Omega_{x_2,x_1,T} = \bigcup_{0<\tau<T} \Sigma_{x_2,x_1,\tau}, \tag{3.4.1b}$$
$$0 \leq 2t < x_1 \leq x_0\ , \qquad \Sigma_{x_1,t} = \{\tau = t, 0 < x < x_1 - 2t\}, \tag{3.4.1c}$$
$$0 < 2T < x_1, \qquad \Omega_{x_1,T} = \bigcup_{0<t<T} \Sigma_{x_1,t}\ . \tag{3.4.1d}$$

There is a natural identification between $\Sigma_{x_2,x_1,t}$ and M_{x_2,x_1-2t}, similarly between $\Sigma_{x_1,t}$ and M_{x_1-2t}, and we shall freely make use of such identifications throughout. We shall write $\|f(t)\|_{\mathscr{H}^\alpha_k}$ for $\|f|_{\Sigma_{x_2,x_1,t}}\|_{\mathscr{H}^\alpha_k(\Sigma_{x_2,x_1,t})}$, or for $\|f|_{\Sigma_{x_1,t}}\|_{\mathscr{H}^\alpha_k(\Sigma_{x_1,t})}$, *etc.*; the distinction should be clear from the context.

We shall be interested in symmetric hyperbolic first order systems which in local coordinates take the form

$$[A^\mu(z^\nu)\partial_\mu + A(z)]\, f = F, \tag{3.4.2}$$

where $z^\nu = (y^i, \tau)$ and $(y^i) = (x, v^A)$, with the following properties:

$\mathscr{C}1$) f and F are sections of a bundle which is a direct sum of two N_1 and N_2 dimensional Riemannian bundles over M obtained as some tensorial products of subspaces of TM; we will write

$$f = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \qquad F = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{3.4.3}$$

In local coordinates φ and a are thus $\mathbb{R}^{N_1}$ valued, while ψ and b are $\mathbb{R}^{N_2}$ valued. The respective scalar products will be denoted by $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$. We suppose there exists a smooth background Riemannian metric b on M_{x_0} (*cf.* p17) whose line element can be written

$$b = dx^2 + h_{AB}(x, v^C)dv^A dv^B\ ,$$

and we define the Lorentzian metric g on $M_{x_0} \times I$ by

$$g = 2dxd\tau + dx^2 + h_{AB}(x, v^C)dv^A dv^B\ ,$$

so that b is the metric induced on $M_{x_0} \times \{\tau\}$ (identified to M_{x_0}) by g. In this section ∇ will denote the Levi-Civita connection associated to g on $M \times I$. The form of the metric g leads that $\nabla_X Y \in T\ M \times \{\tau\}$ for $X, Y \in T\ M \times \{\tau\}$ and

[1] The motivation for the factors of 2, and the general form of the sets considered, arises as follows: The set $\partial M \times I$ should be thought of as a smooth null hypersurface in space-time; *e.g.*, in Minkowski space-time with Minkowskian coordinates y^μ, it can be the intersection of the half-space $\{y^0 \geq 1/2\}$ with the light cone emanating from the origin $y^\mu = 0$. Then τ is the Minkowski time, perhaps shifted by a constant, say $\tau = y^0 - 1/2$. The coordinate x is a coordinate which vanishes on $\partial M \times I$, in the current example *e.g.* $x = \sqrt{\sum (y^i)^2} - y^0$. Finally, in such a Minkowskian setup, the hypersurfaces $x = x_1 - 2\tau$, which determine one of the boundaries of the Σ's and Ω's defined in (3.4.1), correspond to the converging light cones $y^0 + \sqrt{\sum (y^i)^2} =$ const. The restrictions $2(x_2+t) < x_1 \leq x_0$ (in the definition of $\Sigma_{x_2,x_1,t}$) and $2(x_2+T) < x_1$ (in the definition of $\Omega_{x_2,x_1,T}$) are not necessary, and are only made for simplicity of discussion.

∇ is the Levi-Civita connection associated to b on $M \times \{\tau\}$[2]. We denote $\not\nabla$ some generic connection $\not\nabla$ defined on N_1 and N_2 compatible with the scalar products defined above, *e.g.*, if X is a vector field on $M_{x_0} \times I$, then

$$X(\langle \phi, \psi \rangle_1) = \langle \not\nabla_X \phi, \psi \rangle_1 + \langle \phi, \not\nabla_X \psi \rangle_1 \ , \tag{3.4.4}$$

for ϕ, ψ in N_1, similarly for $\langle \cdot, \cdot \rangle_2$. [3]

$\mathscr{C}2$) The left hand side of (3.4.2) can be written as

$$\begin{pmatrix} E^\mu_- \not\nabla_\mu \varphi & +L\psi \\ -L^\dagger \varphi & +E^\mu_+ \not\nabla_\mu \psi \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} , \tag{3.4.5}$$

where L is a first order differential operator. Here $L^\dagger$ denotes the formal adjoint of L, in the sense that if $\Omega = M$, or M_{x_1}, or M_{x_2,x_1}, and if φ, ψ are in $C_1(\overline{\Omega})$, then

$$\int_\Omega \langle \varphi, L\psi \rangle_1 \ d\mu = \int_\Omega \langle L^\dagger \varphi, \psi \rangle_2 \ d\mu \ , \tag{3.4.6}$$

where $d\mu$ is a measure on M which will, we hope, be obvious from the context. By density Equation (3.4.6) will still hold with $\Omega = M_{x_2,x_1}$ for all $\alpha, \beta \in \mathbb{R}$, all $\varphi \in \mathscr{H}_1^\alpha(M_{x_2,x_1})$ and all $\psi \in \mathscr{H}_1^\beta(M_{x_2,x_1})$. Equation (3.4.6) forces L not to contain any τ- or x- derivatives, where the letter x denotes a coordinate as defined in Section 3.2, thus

$$L = \ell^A(x, v, \tau)\partial_A + \ell(x, v, \tau) \ . \tag{3.4.7}$$

It follows that the principal part of the system (3.4.5) is of the form

$$\begin{pmatrix} E^\mu_- \partial_\mu & \ell^A \partial_A \\ (\ell^A)^t \partial_A & E^\mu_+ \partial_\mu \end{pmatrix} , \tag{3.4.8}$$

where A^t denotes the transpose of a matrix A. Equation (3.4.8) explicitly shows that (3.4.5) is symmetric hyperbolic when the $E^\mu_\pm$'s are symmetric with $E^\tau_\pm$ positive definite; the notions of "symmetric hyperbolic" and "symmetrizable hyperbolic" are identified throughout this work.

The hypotheses above will be assumed throughout this section.

3.4.1 Estimates on the space derivatives of the solutions

Let us pass now to the description of the hypotheses needed to derive weighted energy estimates for space derivatives of f. To obtain such estimates, we shall require the existence of a constant C_1 such that the (matrix-valued) coefficients ℓ^A and ℓ satisfy, in the relevant range of τ's,

$$\|\ell(\tau)\|_{\mathcal{G}^0_k(M_{x_1-2\tau})} + \sum_A \|\ell^A(\tau)\|_{\mathcal{G}^0_k(M_{x_1-2\tau})} \le C_1 \ . \tag{3.4.9}$$

[2]This hypothese will simplify the notation here, note this will no longer be satisfied in the Einstein analysis in the following chapter.

[3]In the setting of Minkowski space-time and wave equations, $\not\nabla$ will be the connection induced by ∇ on $TS_{x,\tau}$ and some of its tensor products (corresponding to N_1 and N_2), where $S_{x,\tau}$ is the sphere defined as the intersection of the level set of x and τ.

Similarly writing

$$L^\dagger = \ell^{\dagger A}(x,v,\tau)\partial_A + \ell^\dagger(x,v,\tau)\ , \tag{3.4.10}$$

we require

$$\|\ell^\dagger(\tau)\|_{\mathcal{G}^0_k(M_{x_1-2\tau})} + \sum_A \|\ell^{\dagger A}(\tau)\|_{\mathcal{G}^0_k(M_{x_1-2\tau})} \le C_1\ . \tag{3.4.11}$$

$\mathscr{C}3$) The matrices $E^\mu_\pm$ are symmetric and satisfy

$$E^\mu_\pm n_\mu \ge \varepsilon \mathrm{Id}\ , \qquad E^\mu_+\partial_\mu x \le -\varepsilon \mathrm{Id}\ , \qquad |E^\mu_-\partial_\mu x| \le C_1 x\ , \tag{3.4.12}$$

for some $\varepsilon > 0$. Here n_μ denotes the field of future directed (*i.e.*, $g(d\tau, n) < 0$) g-unit normals to the surfaces $\{\tau = \text{const}\}$. (Later on we will mainly be interested in the case of E^μ_+s of the form $E^\mu_\pm = e^\mu_\pm \otimes \mathrm{Id}$, for some vector fields $e^\mu_\pm$.) For simplicity we shall also assume

$$\partial_i E^\tau_\pm = 0\ ; \tag{3.4.13}$$

this is by no means necessary, but is sufficient for the purposes of this paper. We will further assume[4] that the E^μ_-'s satisfy a bound of the form:

$$\begin{aligned} &\|E^\mu_-(\tau)\|_{\mathscr{G}^0_k(M_{x_1-2\tau})} + \|\partial_x E^x_-(\tau)\|_{\mathscr{G}^0_{k-1}(M_{x_1-2\tau})} \\ &+\|\partial_A E^A_-\|_{\mathscr{G}^1_{k-1}(M_{x_1-2\tau})} + \|(D_\mu E^\mu_-)(\tau)\|_{L^\infty(M_{x_1-2\tau})} \quad \le \quad C_1\ , \end{aligned} \tag{3.4.14}$$

where we set

$$D_\mu E^\mu_\pm = \nabla\!\!\!/_\mu E^\mu_\pm + (\nabla_\mu \frac{\partial}{\partial \nu})^\mu E^\nu_\pm\ .$$

As far as the E^μ_+'s are concerned, we allow singular behavior which should, however, be somewhat less singular than $1/x$; to control that, we require existence of a function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$, satisfying $\lim_{x\to 0}\zeta(x) = 0$, such that for $0 < x \le x_1 - 2\tau$ we have

$$\begin{aligned} &\|E^\mu_+(\tau)\|_{\mathscr{G}^{-1}_k(M_x)} + \|\partial_x E^x_+(\tau)\|_{\mathscr{G}^{-1}_{k-1}(M_x)} \\ &+\|\partial_A E^x_+(\tau)\|_{\mathcal{G}^0_{k-1}(M_x)} + \|x D_\mu E^\mu_+(\tau)\|_{L^\infty(M_x)} \quad \le \quad \zeta(x)\ . \end{aligned} \tag{3.4.15}$$

When the operators $E^\mu_\pm \nabla\!\!\!/_\mu$ are written out explicitly as

$$E^\mu_\pm \nabla\!\!\!/_\mu = E^\mu_\pm \partial_\mu + B_\pm\ , \tag{3.4.16}$$

we require that

$$\|B_-(\tau)\|_{\mathcal{G}^0_k(M_{x_1-2\tau})} \le C_1\ , \quad \|B_+(\tau)\|_{\mathscr{G}^{-1}_k(M_x)} \le \zeta(x)\ , \quad 0 < x < x_1 - 2\tau\ . \tag{3.4.17}$$

$\mathscr{C}4$) The matrices B_{ab}, $a, b = 1, 2$, satisfy the bounds

$$\begin{aligned} &\|B_{11}(\tau)\|_{\mathscr{G}^0_k(M_{x_1-2\tau})} \le C_1\ , \\ &\|B_{12}(\tau)\|_{\mathscr{G}^{-1/2}_k(M_x)} + \|B_{21}(\tau)\|_{\mathscr{G}^{-1/2}_k(M_x)} + \|B_{22}(\tau)\|_{\mathscr{G}^{-1}_k(M_x)} \le \zeta(x)\ , \end{aligned} \tag{3.4.18}$$

[4] We use a convention in which the covariant derivatives $D_\mu E^\mu_\pm$ include terms associated with the vector density character of X^μ defined by (3.4.21); in particular this should be taken into account when verifying that the estimates (3.4.14)-(3.4.15) hold.

this last equation holding again for $0 < x < x_1 - 2\tau$.

Our final hypothesis concerns the "acausal" nature of the boundary of $\Omega_{x_2,x_1,T}$ (quotes used to avoid confusion with the Lorentzian definition of acausal):

$\mathscr{C}5$) $\partial\Omega_{x_2,x_1,T}$ is "non-timelike"[5] in the sense that for any covector n_μ, outwards-directed and $g-$normal to the differentiable part of $\partial\Omega_{x_2,x_1,T} \cap \{\tau > 0\}$, we have

$$E^\mu_\pm n_\mu \geq 0 \; . \tag{3.4.19}$$

(We note that (3.4.12) already guarantees that (3.4.19) holds on $\partial\Omega_{x_2,x_1,T} \cap \{\tau = T \text{ or } \tau = 0\}$.)

Weighted energy inequalities in $\mathscr{H}^\alpha_k$ spaces with arbitrary values of k may be proved under various hypotheses on the coefficients which appear in (3.4.2). We note one such result for systems satisfying $\mathscr{C}1$)-$\mathscr{C}5$), which lies in line with our remaining investigations. The restriction $\alpha \leq -1/2$ seems to be inherent to the problem at hand. We consider the case $\alpha < -1/2$; the case $\alpha = 1/2$ can be handled by the same methods, under somewhat more restrictive conditions on the coefficients.

Proposition 3.4.1 Suppose that $\alpha < -\frac{1}{2}$, $k > \frac{n}{2} + 1$, $k \in \mathbb{N}$, and set either $f(t) = f|_{\Sigma_{x_1,t}}$, $0 < x_1 \leq x_0$, $0 \leq t \leq t_{\max} \equiv x_1/2$, or $f(t) = f|_{\Sigma_{x_2,x_1,t}}$, $0 < 2x_2 < x_1 \leq x_0$, $0 \leq t < t_{\max} \equiv x_1 - 2x_2$. Under the hypotheses $\mathscr{C}1$)-$\mathscr{C}5$), there exists a constant C_2 depending upon x_1, C_1, n, N, k and α, as well as upon the "error function" ζ and the boundary manifold ∂M, such that for all f satisfying $f(0) \in H^{loc}_k$ and for all $0 < t \leq t_{\max}$ we have

$$\begin{aligned} \|f(t)\|^2_{\mathscr{H}^\alpha_k(M_{x_1-2t})} \quad \leq \quad & C_2 \Bigg(\|f(0)\|^2_{\mathscr{H}^\alpha_k(M_{x_1})} + e^{C_2 t} \int_0^t e^{C_2(t-s)} \Big(\|a(s)\|^2_{\mathscr{H}^\alpha_k(M_{x_1-2s})} \\ & + \|b(s)\|^2_{\mathscr{H}^{\alpha-1/2}_k(M_{x_1-2s})} \Big) ds \Bigg) \; . \end{aligned} \tag{3.4.20}$$

Remark: The condition $k > n/2+1$ is needed to derive a C_1 weighted control of the solution; there are no restrictions on k if we have at our disposal an *a priori* C_1 weighted bound for f. In such a case, for $k \leq n/2+1$, the inequality (3.4.20) should be modified by adding a term $\|f(s)\|^2_{\mathscr{B}^\alpha_1(M_{x_1-2s})}$ under the integral appearing in (3.4.20).

PROOF: We are mainly interested in small values of x_2, with eventually x_2 tending to zero, otherwise the estimate is standard. Keeping this in mind, let X^μ be the "energy-momentum vector density",

$$X^\mu = \sum_{0 \leq |\beta| \leq k} x^{-2\alpha-1+2\beta_1} \{ \langle \mathscr{D}^\beta \varphi, E^\mu_- \mathscr{D}^\beta \varphi \rangle_1 + \langle \mathscr{D}^\beta \psi, E^\mu_+ \mathscr{D}^\beta \psi \rangle_2 \}. \tag{3.4.21}$$

Suppose, first, that $f(0) \in H^{loc}_{k+1}$; standard results [39, Vol. III] show that $f(t) \in H^{loc}_{k+1}$, and we then have[4]

$$\nabla_\mu X^\mu = N_1 + D_1 + D_2 + E_1 + E_2 + E_3 \; , \tag{3.4.22}$$

[5]Note that $\partial\Omega_{x_2,x_1,T}$ do not need to be non-timelike in the Lorentzian metric g sense, however, both definitions of terms such that "non-timelike" or "causal" will coincide in our applications.

where

$$
\begin{aligned}
N_1 &= \sum_{0\le|\beta|\le k} (2\beta_1 - 2\alpha - 1) x^{-2\alpha-2+2\beta_1} \langle \mathscr{D}^\beta \psi, (E_+^\mu \partial_\mu x) \mathscr{D}^\beta \psi \rangle_2 \ , \\
D_1 &= 2 \sum_{0\le|\beta|\le k} x^{-2\alpha-1+2\beta_1} \langle \mathscr{D}^\beta \varphi, E_-^\mu \nabla_\mu \mathscr{D}^\beta \varphi \rangle_1 \ , \\
D_2 &= 2 \sum_{0\le|\beta|\le k} x^{-2\alpha-1+2\beta_1} \langle \mathscr{D}^\beta \psi, E_+^\mu \nabla_\mu \mathscr{D}^\beta \psi \rangle_2 \ , \\
E_1 &= \sum_{0\le|\beta|\le k} (2\beta_1 - 2\alpha - 1) x^{-2\alpha-1+2\beta_1} \langle \mathscr{D}^\beta \varphi \ , \frac{(E_-^\mu \partial_\mu x)}{x} \mathscr{D}^\beta \varphi \rangle_1 \ , \\
E_2 &= \sum_{0\le|\beta|\le k} x^{-2\alpha-1+2\beta_1} \langle \mathscr{D}^\beta \varphi, (D_\mu E_-^\mu) \mathscr{D}^\beta \varphi \rangle_1 \ , \\
E_3 &= \sum_{0\le|\beta|\le k} x^{-2\alpha-1+2\beta_1} \langle \mathscr{D}^\beta \psi, (D_\mu E_+^\mu) \mathscr{D}^\beta \psi \rangle_2 \ .
\end{aligned} \tag{3.4.23}
$$

Since $2\alpha + 1 < 0$, from (3.4.12) one finds that

$$\int_{\Sigma_{x_2,x_1,s}} N_1 \, dx \, d\nu \le -|2\alpha+1| \varepsilon \|\psi\|^2_{\mathscr{H}_k^{\alpha+\frac{1}{2}}} \tag{3.4.24}$$

which is strictly negative for $\psi \neq 0$, and can be used to control some of the error terms which occur at the right hand side of (3.4.22). (Here we have used the form (3.2.4) of $\|\psi\|^2_{\mathscr{H}_k^{\alpha+\frac{1}{2}}}$.) For example, to control E_3 we take any x_3 satisfying $2x_2 \le x_3 \le x_1 - 2s$ (we will make a more precise choice of x_3 later), and we write

$$
\begin{aligned}
\int_{\Sigma_{x_2,x_1,s}} E_3 \, dx \, d\nu &= E_{3,1} + E_{3,2} \ , \\
E_{3,1} &\equiv \int_{\Sigma_{x_2,x_1,s} \bigcap \{x \ge x_3\}} E_3 \, dx \, d\nu \ , \\
E_{3,2} &\equiv \int_{\Sigma_{x_2,x_1,s} \bigcap \{x \le x_3\}} E_3 \, dx \, d\nu \ .
\end{aligned}
$$

By (3.4.15), $E_{3,2}$ can be estimated as follows:

$$
\begin{aligned}
|E_{3,2}| &\le \sum_{0\le\beta\le k} \int_{\Sigma_{x_2,x_1,s} \bigcap \{x \le x_3\}} \zeta(x) x^{-2\alpha-2+2\beta_1} |\mathscr{D}^\beta \psi|^2 \, dx \, d\nu \\
&\le \frac{(2\alpha+1)\varepsilon}{10} \|\psi\|^2_{\mathscr{H}_k^{\alpha+\frac{1}{2}}} \ ,
\end{aligned}
$$

if x_3 is chosen small enough. Once this choice has been done, we can clearly estimate $E_{3,1}$ as

$$E_{3,1} \le C \|\psi\|^2_{\mathscr{H}_k^\alpha} \ ,$$

with some constant which is determined by x_3. The integrals of the error terms E_1 and E_2 are estimated in the obvious way, *cf.* (3.4.12) and (3.4.14):

$$\int_{\Sigma_{x_2,x_1,s}} (E_1 + E_2) \, dx \, d\nu \le C \|\varphi(s)\|^2_{\mathscr{H}_k^\alpha} \ .$$

To control the terms D_1 and D_2 we use the evolution equations (3.4.5):

$$\begin{aligned}
E_-^\mu \nabla_\mu \mathscr{D}^\beta \varphi &= \mathscr{D}^\beta (E_-^\mu \nabla_\mu \varphi) + [E_-^\mu \nabla_\mu, \mathscr{D}^\beta]\varphi \\
&= -\mathscr{D}^\beta (L\psi + B_{11}\varphi + B_{12}\psi - a) + [E_-^\mu \nabla_\mu, \mathscr{D}^\beta]\varphi \\
&= -L\mathscr{D}^\beta \psi + \mathscr{D}^\beta a + E_4^\beta \ , \qquad (3.4.25) \\
E_4^\beta &= -[\mathscr{D}^\beta, L]\psi + [E_-^\mu \nabla_\mu, \mathscr{D}^\beta]\varphi - \mathscr{D}^\beta (B_{11}\varphi + B_{12}\psi) \ , \\
E_+^\mu \nabla_\mu \mathscr{D}^\beta \psi &= L^\dagger \mathscr{D}^\beta \varphi + \mathscr{D}^\beta b + E_5^\beta , \qquad (3.4.26) \\
E_5^\beta &= [\mathscr{D}^\beta, L^\dagger]\varphi + [E_+^\mu \nabla_\mu, \mathscr{D}^\beta]\psi - \mathscr{D}^\beta (B_{21}\varphi + B_{22}\psi) \ .
\end{aligned}$$

Integrating $D_1 + D_2$ over $\Sigma_{x_2,x_1,s}$, one finds that the terms containing $L\mathscr{D}^\beta\psi$ and $-L^\dagger\mathscr{D}^\beta\varphi$ in (3.4.25) and (3.4.26) cancel out; the terms containing $\mathscr{D}^\beta a$ and $\mathscr{D}^\beta b$ are estimated as

$$\begin{aligned}
2 \sum_{0\le|\beta|\le k} \int_{\Sigma_{x_2,x_1,s}} & x^{-2\alpha-1+2\beta_1} \left(\langle \mathscr{D}^\beta \varphi, \mathscr{D}^\beta a\rangle_1 + \langle \mathscr{D}^\beta \psi, \mathscr{D}^\beta b\rangle_2 \right) dx\, d\nu \\
&\le \|\varphi\|^2_{\mathscr{H}_k^\alpha} + \|a\|^2_{\mathscr{H}_k^\alpha} + \frac{(2\alpha+1)\varepsilon}{10} \|\psi\|^2_{\mathscr{H}_k^{\alpha+\frac12}} + \frac{10}{(2\alpha+1)\varepsilon}\|b\|^2_{\mathscr{H}_k^{\alpha-\frac12}} \ .
\end{aligned}$$

The terms containing the commutators $[\mathscr{D}^\beta, L]\psi$ and $[\mathscr{D}^\beta, L^\dagger]\varphi$, can be estimated using the weighted commutator inequality (3.2.35), while the B_{11}, B_{12}, *etc.*, terms can be estimated using (3.2.34), by an expression of the form

$$CC_1 \left(\|\psi\|^2_{\mathscr{H}_k^\alpha} + \|\varphi\|^2_{\mathscr{H}_k^\alpha} + \frac{(2\alpha+1)\varepsilon}{10}\|\psi\|^2_{\mathscr{H}_k^{\alpha+\frac12}} \right) . \qquad (3.4.27)$$

To estimate the commutator terms arising from $E_\pm^\mu$, we note that for $|\beta| > 0$,

$$\begin{aligned}
x^{\beta_1}[E_\pm^\mu \partial_\mu, \mathcal{D}^\beta]\chi &= x^{\beta_1} E_\pm^A \partial_x^{\beta_1} [\partial_A, X_2^{\beta_2} \cdot \ldots X_r^{\beta_r}]\chi \\
&\quad - \sum_{\sigma+\delta=\beta} c(\sigma,\beta) x^{\sigma_1} (\mathcal{D}^\sigma E_\pm^\mu) x^{\delta_1} (\mathcal{D}^\delta \partial_\mu \chi) \\
&= E_6 + E_7 \ .
\end{aligned}$$

The hypothese (3.4.13) imply the terms in $\partial_\tau\chi$ vanish. Then the difficult term in E_7 is

$$E_7^x = - \sum_{\sigma+\delta=\beta} c(\sigma,\beta) x^{\sigma_1} (\mathcal{D}^\sigma E_\pm^x) x^{\delta_1} (\mathcal{D}^\delta \partial_x \chi) \ .$$

The terms arising from E_7^x in (3.4.25 - 3.4.26) can again be estimated as in (3.4.27) provided that $x\partial_x E_-^x$, $\partial_A E_-^x \in \mathscr{G}_{k-1}^1$, that $x\partial_x E_+^x$, $\partial_A E_+^x \in \mathscr{G}_{k-1}^0$, and that (3.4.15) holds. Summarizing, we have derived

$$\left| \int_{\Sigma_{x_2,x_1,s}} \nabla_\mu X^\mu d^n\mu \right| \le CC_1 \left(\|a(s)\|^2_{\mathscr{H}_k^\alpha} + \|b(s)\|^2_{\mathscr{H}_k^{\alpha-\frac12}} + \|\psi(s)\|^2_{\mathscr{H}_k^\alpha} + \|\varphi(s)\|^2_{\mathscr{H}_k^\alpha} \right) . \qquad (3.4.28)$$

Stokes theorem,

$$\int_{\Omega_{x_2,x_1,t}} D_\mu X^\mu \, d^n\mu \, d\tau \quad = \quad \int_{\partial\Omega_{x_1,x_2,t}} X^\mu dS_\mu \ ,$$

and our hypotheses on the geometry of the problem lead to

$$\|f(t)\|^2_{\mathscr{H}_k^\alpha} \le C \left(\|f(0)\|^2_{\mathscr{H}_k^\alpha} + C_1 \int_0^t \left(\|a(s)\|^2_{\mathscr{H}_k^\alpha} + \|b(s)\|^2_{\mathscr{H}_k^{\alpha-1/2}} + \|f(s)\|^2_{\mathscr{H}_k^\alpha} \right) ds \right) \ .$$

Gronwall's lemma establishes (3.4.20) on the family of hypersurfaces $\Sigma_{x_2,x_1,t}$ for $f(t) \in H^{loc}_{k+1}$. If $f(t) \in H^{loc}_k$, we approximate $f(0)$ by a sequence of functions $f_n(0)$, with $f_n(0) \in H^{loc}_{k+1}$ converging to $f(0)$ in $\mathscr{H}_k(\Sigma_{x_2,x_1,t})$, and we solve Equation (3.4.2) with initial data $f_n(0)$. The inequality (3.4.20) applied to the functions $f_n(t) - f_m(t)$ shows that $f_n(t)$ is Cauchy in $\mathscr{H}_k^\alpha$; passing to the limit $n \to \infty$ the desired result for f's such that $f(0) \in H^{loc}_k$ easily follows.

Since all the constants above are x_2 independent, an elementary argument using the the monotone convergence theorem shows that the inequality (3.4.20) for the $\Sigma_{x_1,t}$'s follows from the one for the $\Sigma_{x_2,x_1,t}$'s by passing to the limit $x_2 \to 0$. □

3.4.2 Estimates on the time derivatives of the solutions

The hypotheses done in the previous section ensure that we can algebraically solve Equation (3.4.2) for $\partial_\tau f$, and then recursively obtain formulae for $\partial^i_\tau f$. Under the hypotheses of Proposition 3.4.1, it is then straightforward to obtain estimates on the norms

$$\|((x\partial_\tau)^i f)(\tau)\|_{\mathscr{H}^\alpha_{k-i}(\Sigma_{x_1-2\tau})} \ , \qquad 0 \le i \le k \ ,$$

provided suitable weighted conditions are imposed on the τ derivatives of the coefficients of Equation (3.4.2). However, we would like to obtain derivative estimates without the x factors, uniformly in τ. Clearly a necessary condition for the existence of such estimates is that

$$\|(\partial^i_\tau f)(0)\|_{\mathscr{H}^\alpha_{k-i}(\Sigma_{x_1})} < \infty \ , \qquad 0 \le i \le k \ . \tag{3.4.29}$$

It turns out that (3.4.29) needs not to hold for arbitrary initial data $f(0) \in \mathscr{H}_k^\alpha$, and the requirement that it does leads to the *j-th order compatibility conditions*: by definition, these are the conditions on $f(0)$ which ensure that Equation (3.4.29) holds for $0 \le i \le j$. Since, for solutions of Equation (3.4.2), all the derivatives $\partial^i_\tau f(0)$ can be explicitly written as an i-th order differential operator acting on $f(0)$, the compatibility conditions are conditions on the behavior of the initial data $f(0)$ near the "corner" $x = 0$; we shall therefore sometimes refer to them as "*corner conditions*". We note that there can be corner conditions in weighted Sobolev spaces, or in weighted Hölder spaces; in this section we will be mainly interested in the latter, defined by Equation (3.4.33) below.

The following example is instructive in this context: For $0 \leq t < y$ let g be a solution of the two dimensional wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\right) g = 0 , \tag{3.4.30}$$

with initial condition

$$g\Big|_{t=0} = 2Cy^{\alpha+1} , \qquad \frac{\partial g}{\partial t}\Big|_{t=0} = 2(\alpha+1)y^{\alpha} ,$$

for some constants $C, \alpha \in \mathbb{R}$. From Equation (3.4.30) we can obtain a system of the form (3.4.5) by introducing $\tau = t$, $x = y - t$, $\varphi = (g, (\partial_\tau - 2\partial_x)g)$, $\psi = \partial_\tau g$, and setting $L = 0$, $E^\mu_- \partial_\mu = \partial_\tau \otimes \mathrm{id}$, $E^\mu_+ \partial_\mu = (\partial_\tau - 2\partial_x)$, so that we have

$$\begin{aligned} \partial_\tau \begin{pmatrix} g \\ (\partial_\tau - 2\partial_x)g \end{pmatrix} - \begin{pmatrix} \psi \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} , \\ (\partial_\tau - 2\partial_x)\psi &= 0 . \end{aligned}$$

The solution is

$$\begin{aligned} g &= (C+1)(y+t)^{\alpha+1} + (C-1)(y-t)^{\alpha+1} \\ &= (C+1)(2\tau + x)^{\alpha+1} + (C-1)x^{\alpha+1} . \end{aligned}$$

It follows that for each $0 \leq \tau \leq 1$, $k \in \mathbb{N}$, and $\beta < \alpha + 1$ we have $g(\tau, \cdot) \in \mathscr{H}^\beta_k((0, 10])$, consistently with Proposition (3.4.1). Somewhat surprisingly, for $\tau > 0$ and for all $i \in \mathbb{N}$ the functions $\partial^i_\tau g(\tau, \cdot)$ are smooth in x up to $x = 0$. However, the L^∞ for bound $\partial^i_\tau g(\tau, \cdot)$ blows up as τ tends to zero except in the case

$$C = -1 . \tag{3.4.31}$$

Condition (3.4.31) is precisely the corner condition needed for $\partial_\tau g(0, \cdot)$ to be better behaved than $\partial_x g(0, \cdot)$ at $x = 0$. In the example under consideration the fulfillment of the first order corner condition guarantees already that all the τ derivatives of g will be well behaved, but we do not expect this to be true in general.

Let us pass to a derivation of the desired estimates. We shall use a method which avoids the use of weighted Sobolev spaces; the price one pays is the need to consider systems somewhat less general than (3.4.5), but still general enough for our purposes. More precisely, in this section we restrict our attention to systems of the form

$$\partial_\tau \varphi + B_{11}\varphi + B_{12}\psi = L_{11}\varphi + L_{12}\psi + a , \tag{3.4.32a}$$
$$e_+\psi + B_{21}\varphi + B_{22}\psi = L_{21}\varphi + L_{22}\psi + b , \tag{3.4.32b}$$

with

$$e_+\psi \equiv (\partial_\tau - 2\partial_x)\psi .$$

We assume that the L_{ab}'s, $a, b = 1, 2$ are first order differential operators of the form

$$L_{ab} = L^A_{ab}\partial_A + xL^\tau_{ab}\partial_\tau + xL^x_{ab}\partial_x , \tag{3.4.33}$$

with bounded coefficients L^μ_{ab}; no symmetry hypotheses are made. Clearly the intersection of systems of equations satisfying (3.4.32) with those of the form (3.4.5) is non-empty. (As we will see in Sections 3.5 and 3.6 below, non-linear waves equations on Minkowski space-time can be written in the form (3.4.32).) In particular Proposition 3.4.1 provides a large class of solutions of (3.4.32) such that

$$(\varphi,\psi)(\tau)\in \mathscr{H}^\alpha_k(M_{x_1-2\tau})\subset \mathscr{C}^\alpha_\ell(M_{x_1-2\tau})$$

for $\ell<k-n/2$. We shall therefore assume that a solution $f=(\varphi,\psi)$ satisfying $f(\tau)\in\mathscr{C}^\alpha_\ell(M_{x_1-2\tau})$ is given, and study its τ-differentiability properties. For this purpose it is convenient to introduce a space $\mathscr{C}^\alpha_{\ell|p}(\Omega)$ defined, for $p\le\ell$, as the space of functions f in $C^(_\ell\Omega)$ such that the norm

$$\|f\|_{\mathscr{C}^\alpha_{\ell|p}(\Omega)}\equiv\sup_\Omega \sum_{\substack{0\le i+j+k+|\gamma|\le\ell\\ 0\le k\le p}} x^{-\alpha}|(x\partial_x)^i(x\partial_\tau)^j\mathscr{D}^\gamma_v\partial^k_\tau f|$$

is finite. Similarly one defines $\mathscr{C}^{\alpha,\beta}_{\ell|p}(\Omega)$ using the norm

$$\|f\|_{\mathscr{C}^{\alpha,\beta}_{\ell|p}(\Omega)}\equiv\sup_\Omega \sum_{\substack{0\le i+j+k+|\gamma|\le\ell\\ 0\le k\le p}} (1+|\ln x|)^{-\beta}x^{-\alpha}|(x\partial_x)^i(x\partial_\tau)^j\mathscr{D}^\gamma_v\partial^k_\tau f|\ .$$

Clearly $\mathscr{C}^\alpha_{\ell|p}(\Omega)=\mathscr{C}^{\alpha,0}_{\ell|p}(\Omega)$. **Remark:** The spaces $\mathscr{C}^{\alpha,\beta}_{\ell|p}$ generalize the spaces $\mathscr{C}^{\alpha,\beta}_k$ which can be defined as the spaces of functions f such that $|1+\ln x|^{-\beta}f\in\mathscr{C}^\alpha_k$.

Proposition 3.4.2 Let $\alpha\le 0$, $\ell\in\mathbb{N}$, write Ω for $\Omega_{x_1,T}$, and suppose that $L^\mu_{ab}, B_{ab}\in\mathscr{C}^0_\ell(\Omega)$, $a\in\mathscr{C}^\alpha_{\ell-1}(\Omega)$, $b\in\mathscr{C}^{\alpha-1}_{\ell-1}(\Omega)$. Consider $f\equiv(\varphi,\psi)$ — a solution of (3.4.32) satisfying

$$\forall\tau\in[0,T]\qquad f(\tau)\in\mathscr{C}^\alpha_\ell(M_{x_1-2\tau})\ .$$

Then:

1. For all $\epsilon>0$ we have

$$(\varphi,\psi)\in\mathscr{C}^{\alpha,\beta}_{\lfloor\ell/2\rfloor}\left(\Omega\cap\{x+2\tau\ge\epsilon\}\right)\ ,$$

   in particular for any $\tau>0$ the compatibility conditions of order $p=\lfloor\ell/2\rfloor$ (the integer part of $\ell/2$) are satisfied by $(\varphi(\tau),\psi(\tau))$:

$$\forall\ 1\le i\le p\qquad \partial^i_\tau\varphi(\tau),\partial^i_\tau\psi(\tau)\in\mathscr{C}^{\alpha,\beta}_{\ell-i}(M_{x_1})\ ,\tag{3.4.34}$$

   Here $\beta=\lfloor\ell/2\rfloor$ if $\alpha=0$, and $\beta=0$ otherwise.

2. If there exists $1\le p\le\ell/2$, $p\in\mathbb{N}$, such that Equation (3.4.34) holds with $\beta=0$ at $\tau=0$, then

$$(\varphi,\psi)\in\mathscr{C}^{\alpha,\beta}_{\ell-p|p}(\Omega)\subset\mathscr{C}^{\alpha,\beta}_p(\Omega)\ ,\tag{3.4.35}$$

   with $\beta=p$ if $\alpha=0$, and $\beta=0$ otherwise.

Remark: The method of proof here gives a number of well controlled time derivatives smaller by a factor 2 than the number of space ones. This is, however, irrelevant, when $\ell = \infty$, which is the main point of interest in this work. We note that energy estimates as in the proof of Theorem 3.5.4 below provide an alternative, more complicated way of establishing a stronger statement, with more controlled time derivatives for large ℓ's.

PROOF: By rearranging terms and redefining the L_{ab}'s, the B_{ab}'s, and the source functions a and b we may without loss of generality assume that

$$L^\tau_{ab} \equiv 0 \ .$$

One can rewrite Equations (3.4.32) as $x\partial_\tau(\varphi, \psi) =$ a partial differential operator linear in $x\partial_x$ and ∂_v; by iteration this immediately yields $(\varphi, \psi) \in \mathscr{C}^\alpha_{\ell|0}$. Equation (3.4.32a) shows then that $\partial_\tau\varphi \in \mathscr{C}^\alpha_{\ell-1|0}$, hence $\varphi \in \mathscr{C}^\alpha_{\ell|1}$. On the other hand, Equation (3.4.32b) gives $e_+(\psi) \in \mathscr{C}^\alpha_{\ell-1|0} + \mathscr{C}^{\alpha-1}_{\ell-1}$, hence $\partial_\tau e_+(\psi) \in \mathscr{C}^{\alpha-1}_{\ell-2|0}$. Integrating Equation (3.4.32b) one finds

$$\psi(x, v^A, \tau) = \psi(x+2\tau, v^A, 0) + \int_{x/2}^{\tau+x/2} e_+(\psi)(2v, v^A, \tau - v + x/2)\, dv \ . \quad (3.4.36)$$

(We note that for each $\epsilon > 0$ the first term above is uniformly C_ℓ on the set $\Omega \cap \{x + 2\tau \geq \epsilon\} \cap \{x \leq x_0\}$.) Differentiating Equation (3.4.36) one obtains

$$\partial_\tau\psi(x, v^A, \tau) = \partial_\tau\psi(x + 2\tau, v^A, 0) + \int_{x/2}^{\tau+x/2} \partial_\tau e_+(\psi)(2v, v^A, \tau - v + x/2)\, dv \ ;$$

since $\alpha \leq 0$ and $\partial_\tau e_+(\psi) \in \mathscr{C}^{\alpha-1}_{\ell-2|0}$, straightforward estimations show that $\partial_\tau\psi \in \mathscr{C}^\alpha_{\ell-2|0}$, hence $\psi \in \mathscr{C}^\alpha_{\ell-1|1}$ if $\alpha \neq 0$, while $\psi \in \mathscr{C}^{0,1}_{\ell-1|1}$ when $\alpha = 0$.

Let $\beta_r = 0$ if $\alpha \neq 0$ and $\beta_r = r$ when $\alpha = 0$, and suppose that $\varphi \in \mathscr{C}^{\alpha,\beta_r}_{\ell+1-r|r}$ and $\psi \in \mathscr{C}^{\alpha,\beta_r}_{\ell-r|r}$ for some $1 \leq r \leq (\ell-1)/2$; we have already shown this to hold for $r = 1$. Equation (3.4.32a) gives

$$\partial_\tau\varphi \in \mathscr{C}^{\alpha,\beta_r}_{\ell-r-1|r} \quad \Longrightarrow \quad \varphi \in \mathscr{C}^{\alpha,\beta_r}_{\ell-r|r+1} \ .$$

It then follows from Equation (3.4.32b) that

$$e_+(\psi) \in \mathscr{C}^{\alpha,\beta_r}_{\ell-r-1|r} \quad \Longrightarrow \quad \partial^{r+1}_\tau e_+(\psi) \in \mathscr{C}^{\alpha-1,\beta_r}_{\ell-2r-2|0} \ .$$

Differentiating $r + 1$ times Equation (3.4.36) with respect to τ we obtain

$$\partial^{r+1}_\tau\psi(x, v^A, \tau) = \partial^{r+1}_\tau\psi(x+2\tau, v^A, 0) + \int_{x/2}^{\tau+x/2} \partial^{r+1}_\tau e_+(\psi)(2v, v^A, \tau-v+x/2)\, dv \ ,$$

which gives $\partial^{r+1}_\tau\psi \in \mathscr{C}^{\alpha,\beta_r}_{\ell-2r-2|0}$, hence $\psi \in \mathscr{C}^{\alpha,\beta_r}_{\ell-r-1|r+1}$, and the induction is completed. □

3.4.3 Polyhomogeneous solutions

Let $\Omega_{x_0,T}$ be defined by Equation (3.4.1d); we shall denote by $\mathscr{A}^\delta_k(\Omega_{x_0,T})$ the space of functions f defined on $\Omega_{x_1,T}$ which can be written in the form

$$\sum_{i=0}^{k}\sum_{j=0}^{N_i} x^{i\delta}\ln^j x\, f_{ij} + f_{\alpha+k\delta+\epsilon}\ ,$$

for some $\epsilon > 0$, some functions $f_{ij} \in C_\infty(\overline{\Omega_{x_0,T}})$, and some sequence (N_i) of non-negative integers. We also require that $f_{\alpha+k\delta+\epsilon} \in \mathscr{C}^{\alpha+k\delta+\epsilon}_\infty(\Omega_{x_0,T})$. We set

$$\mathscr{A}^\delta_\infty := \cap_{k\in\mathbb{N}}\mathscr{A}^\delta_k\ .$$

The following properties are useful in what follows:

- If $0 < x_1 < x_0 - T/2$, then a function $f \in C_\infty(\Omega_{x_0,T})$ is in $\mathscr{A}^\delta_k(\Omega_{x_0 T})$ if and only if for any coordinate patch $\mathscr{O}$ of ∂M we have $f \in \mathscr{A}^\delta_k(\mathcal{U}_{x_1})$, where $\mathcal{U}_{x_1} =]0, x_1[\times\mathscr{O}\times[0,T]$, and if $f \in C_\infty(\overline{\Omega_{int}})$, where $\Omega_{int} = \Omega_{x_0,T}\cap\{x \geq x_1\}$.
- For all $\epsilon > 0$ we have $\mathscr{C}^{\beta+p\delta+\epsilon}_\infty \subset x^\beta\mathscr{A}^\delta_p$; in particular $\mathscr{C}^\epsilon_\infty \subset \mathscr{A}^\delta_0$;
- It does not hold that $\mathscr{A}^\delta_k \subset \mathscr{C}^0_\infty$, however, for all $\epsilon > 0$ we have $\mathscr{A}^\delta_k \subset \mathscr{C}^{-\epsilon}_\infty$. More precisely, if $f \in \mathscr{A}^\delta_k$, then there exists $p \in \mathbb{N}$ such that $(1 + |\ln x|^2)^{-p/2} f \in \mathscr{C}^0_\infty$.
- As before we assume that $1/\delta \in \mathbb{N}$, which implies $x\mathscr{A}^\delta_k \subset \mathscr{A}^\delta_{k+1/\delta} \subset \mathscr{A}^\delta_{k+1}$.
- $\mathscr{A}^\delta_k$ is stable under multiplication: if $f, g \in \mathscr{A}^\delta_k$, then $fg \in \mathscr{A}^\delta_k$.
- $\mathscr{A}^\delta_k$ is stable under differentiation with respect to τ and to v, as well as under $x\partial_x$: if $f \in \mathscr{A}^\delta_k$, then $\partial_\tau f\ , X_i\cdot f\ (i \geq 2)$, $x\partial_x f \in \mathscr{A}^\delta_k$, with the vector fields X_i defined in Section 3.2, *cf.* Equation (3.2.7).

In this section we will consider systems of the form

$$\partial_\tau\varphi + B_{11}\varphi + B_{12}\psi = L_{11}\varphi + L_{12}\psi + a\ , \tag{3.4.37a}$$
$$\partial_x\psi + B_{21}\varphi + B_{22}\psi = L_{21}\varphi + L_{22}\psi + b\ , \tag{3.4.37b}$$

with the L_{ij}'s, $i, j = 1, 2$ of the form

$$L_{ij} = L^A_{ij}\partial_A + L^\tau_{ij}\partial_\tau + xL^x_{ij}\partial_x\ , \tag{3.4.38}$$

with

$$L^\mu_{11} \in x^\delta\mathscr{A}^\delta_{k-1}\ , \quad L^\mu_{21}\ , L^\mu_{12}\ , L^\mu_{22} \in \mathscr{A}^\delta_k\ . \tag{3.4.39}$$

No symmetry hypotheses are made on the matrices L^μ_{ij}. Conditions (3.4.37a)-(3.4.39) are easily seen to be compatible with those made elsewhere in this paper, *cf., e.g.,* the proof of Corollary 3.4.4 below. The reader is warned, however, that the operators L_{ij} here do *not* coincide with those in (3.4.32): to bring (3.4.32) into the form (3.4.37) one needs to multiply Equation (3.4.32b) by

$-1/2$, transfer the operator ∂_τ from the left- to the right-hand-side of (3.4.32), and appropriately redefine the L_{2j}'s.

We start with the following result, which assumes that the solutions have both space and time derivatives controlled, in the sense of weighted Sobolev spaces:

Theorem 3.4.3 Let $\beta,\ \beta' \in \mathbb{R},\ k \in \mathbb{N}\cup\{\infty\}$, and let (φ,ψ) be a solution of (3.4.37) in $\mathscr{C}^{\beta'}_\infty(\Omega_{x_0,T})$. Suppose that (3.4.39) holds, and that

$$B_{11} \in \left(\mathscr{A}^\delta_k \cap L^\infty\right)(\Omega_{x_0,T})\ , \qquad B_{12}, B_{22}, B_{21} \in \mathscr{A}^\delta_k(\Omega_{x_0,T})\ , \tag{3.4.40a}$$

$$a, b \in x^\beta \mathscr{A}^\delta_k(\Omega_{x_0,T})\ , \qquad \varphi(0) \in x^\beta \mathscr{A}^\delta_k(M_{x_0})\ . \tag{3.4.40b}$$

Then

$$\varphi \in \left(x^\beta \mathscr{A}^\delta_k + \mathscr{A}^\delta_k\right)(\Omega_{x_0,T})\ , \quad \psi \in \left(x^{\beta+1} \mathscr{A}^\delta_k + x\mathscr{A}^\delta_k\right)(\Omega_{x_0,T}) + C_\infty(\overline{\Omega_{x_0,T}})\ .$$

If one further assumes

$$L^\mu_{12}, B_{12}, a, \varphi(0) \in L^\infty(\Omega_{x_0,T})\ ,$$

then it also holds that

$$\varphi \in \left(x^\beta \mathscr{A}^\delta_k + \mathscr{A}^\delta_k \cap L^\infty\right)(\Omega_{x_0,T})\ .$$

PROOF: It is convenient to decompose B_{11} in the obvious way as

$$B_{11} = B^0_{11} + B^\delta_{11}\ ,$$

with $B^\delta_{11} \in x^\delta \mathscr{A}^\delta_{k-1}$ and $B^0_{11} \in C_\infty$. We rewrite (3.4.37) as

$$\partial_\tau \varphi + B^0_{11}\varphi = c_1\ , \tag{3.4.41a}$$

$$\partial_x \psi = c_2\ , \tag{3.4.41b}$$

where

$$c_1 := L_{11}\varphi + L_{12}\psi + a - B_{12}\psi - B^\delta_{11}\varphi\ , \tag{3.4.42a}$$

$$c_2 := L_{21}\varphi + L_{22}\psi + b - B_{21}\varphi - B_{22}\psi\ , \tag{3.4.42b}$$

In what follows we let $\epsilon > 0$ be a positive constant, which can be made as small as desired, and which may change from line to line. We note that c_2 is in $\mathscr{C}^{\beta'-\epsilon}_\infty + x^\beta \mathscr{A}^\delta_k$, and integration in x of (3.4.41b) gives

$$\psi = \psi_0 + \psi_{\beta'+1-\epsilon} + \psi_{\rm phg}\ ,$$

where

$$\psi_0(\cdot) = \begin{cases} \lim_{x\to 0}\psi(x,\cdot)\ , & \text{if } \beta'+1-\epsilon > 0, \\ 0\ , & \text{otherwise,} \end{cases}$$

with

$$\psi_0 \in C_\infty(\overline{\Omega_{x_0,T}})\ , \quad \psi_{\beta'+1-\epsilon} \in \mathscr{C}^{\beta'+1-\epsilon}_\infty(\Omega_{x_0,T})\ , \quad \psi_{\rm phg} \in x^{\beta+1}\mathscr{A}^\delta_k(\Omega_{x_0,T})\ ,$$

hence

$$\psi \in C_\infty + \mathscr{C}_\infty^{\beta'+1-\epsilon} + x^{\beta+1}\mathscr{A}_k^\delta \ .$$

Since $L_{11}\varphi \in \mathscr{C}_\infty^{\beta'+\delta-\epsilon}$ ($\partial_x\varphi \in \mathscr{C}_\infty^{\beta'-1}$ and $xL_{11}^x \in x\mathscr{A}_k^\delta \cap \mathscr{C}_0^\delta \subset \mathscr{C}_\infty^\delta$; similarly for the other derivatives), we find that

$$c_1 \in \mathscr{A}_k^\delta + x^\beta\mathscr{A}_k^\delta + \mathscr{C}_\infty^{\beta'+\delta-\epsilon} \ .$$

We can then apply Proposition 3.3.4 to (3.4.41a) to conclude that

$$\varphi \in \mathscr{A}_k^\delta + x^\beta\mathscr{A}_k^\delta + \mathscr{C}_\infty^{\beta'+p\delta-\epsilon} \ , \tag{3.4.43}$$

with $p = 1$. Coming back to c_2 we find now that $c_2 \in \mathscr{A}_k^\delta + x^\beta\mathscr{A}_k^\delta + \mathscr{C}_\infty^{\beta'+p\delta-\epsilon}$, and by Proposition 3.3.6 we obtain

$$\psi \in C_\infty + x\mathscr{A}_k^\delta + x^{\beta+1}\mathscr{A}_k^\delta + \mathscr{C}_\infty^{\beta'+p\delta+1-\epsilon} \ , \tag{3.4.44}$$

still with $p = 1$. To conclude, we proceed by induction; let $\beta' + p\delta \le \beta + k$ and suppose that Equations (3.4.43)-(3.4.44) hold; it follows that $c_1 \in \mathscr{A}_k^\delta + x^\beta\mathscr{A}_k^\delta + \mathscr{C}_\infty^{\beta'+(p+1)\delta-\epsilon}$. Applying Proposition 3.3.4 to (3.4.41a) gives (3.4.43) with p replaced by $p+1$. It follows that $c_2 \in \mathscr{A}_k^\delta + x^\beta\mathscr{A}_k^\delta + \mathscr{C}_\infty^{\beta'+(p+1)\delta-\epsilon}$; Proposition 3.3.6 applied to (3.4.37b) gives (3.4.44) with p replaced by $p+1$, and the result is established. □

As a straightforward consequence of Theorem 3.4.3 we obtain:

Corollary 3.4.4 Let $\beta' \in \mathbb{R}$, let $(\varphi,\psi) \in \mathscr{C}_\infty^{\beta'}(\Omega_{x_0,T})$ be a solution of the system (3.4.5), and suppose that

$$B_{ij}, E_\pm^\mu, B_\pm, \ell, \ell^\dagger, \ell^A, (\ell^A)^\dagger \in \mathscr{A}_k^\delta(\Omega_{x_0,T}) \ , \tag{3.4.45a}$$

$$E_-^\tau \text{ and } E_+^x \text{ — invertible, with } (E_-^\tau)^{-1}, (E_+^x)^{-1} \in \mathscr{A}_k^\delta(\Omega_{x_0,T}) \ , \tag{3.4.45b}$$

$$(E_-^\tau)^{-1}E_-^x \in x\left(\mathscr{A}_k^\delta \cap \mathscr{C}_0^\delta\right)(\Omega_{x_0,T}) \ , \quad (E_-^\tau)^{-1}E_-^A \in x^\delta\mathscr{A}_{k-1}^\delta(\Omega_{x_0,T}) \ , \tag{3.4.45c}$$

$$(E_-^\tau)^{-1}(B_{11} + B_-) \in L^\infty(\Omega_{x_0,T}) \ . \tag{3.4.45d}$$

If

$$a, b \in x^\beta\mathscr{A}_k^\delta(\Omega_{x_0,T}) \ , \quad \varphi(0) \in x^\beta\mathscr{A}_k^\delta(M_{x_0}) \ ,$$

with $\beta \in \mathbb{R}$, then

$$\varphi \in \left(x^\beta\mathscr{A}_k^\delta + \mathscr{A}_k^\delta\right)(\Omega_{x_0,T}) \ , \quad \psi \in \left(x^{\beta+1}\mathscr{A}_k^\delta + x\mathscr{A}_k^\delta\right)(\Omega_{x_0,T}) + C_\infty(\overline{\Omega_{x_0,T}}) \ .$$

In particular, if $k = \infty$ then the solution is polyhomogeneous.

Proof: : We write Equation (3.4.5) as

$$\begin{aligned} \partial_\tau\varphi + (E_-^\tau)^{-1}\left\{(B_{11} + B_-)\varphi + \ell\psi\right\} &= (E_-^\tau)^{-1}(E_-^i\partial_i\varphi - \ell^A\partial_A\psi + a) \\ \partial_x\psi - (E_+^x)^{-1}\left\{\ell^\dagger\varphi - (B_{22} + B_+)\psi\right\} &= (E_-^\tau)^{-1}((\ell^A)^\dagger\partial_A\varphi + E_+^\tau\partial_\tau\psi + E_+^A\partial_A\psi + b) \ , \end{aligned} \tag{3.4.46}$$

which is of the form (3.4.37), and we note that the hypotheses made on the coefficients of Equation (3.4.46) imply those of Theorem 3.4.3. □.

An unsatisfactory feature of results such as Theorem 3.4.3 is that uniform estimates both on space and time derivatives of the solutions are assumed. Recall that uniform time derivatives can be obtained only if corner conditions are satisfied, and the hypotheses of Theorem 3.4.3 require an infinite number of those to be fulfilled. The same techniques can be used to obtain various expansions of solutions when a finite number of time derivatives are controlled only, but the statements turn to be out somewhat less elegant. We give an example of such results when $\delta = 1$:

Theorem 3.4.5 Let $\beta \in \mathbb{R}$, $k \in \mathbb{N} \cup \{\infty\}$, and let (φ, ψ) be a solution of (3.4.37) in $\mathscr{C}^{\beta}_{\ell}(\Omega_{x_0,T})$ for some $\ell \geq 1$. If Equations (3.4.39)-(3.4.45) hold with $\delta = 1$, then for any $\lambda < 1$ we have

$$\varphi \in \left(x^{\beta}\mathscr{A}^1_k + \mathscr{A}^1_k + \cap_{\ell-2j-2\geq 0}\mathscr{C}^{\beta+j+\lambda}_{\ell-2j-2}\right)(\Omega_{x_0,T}) \ ,$$

$$\psi \in \left(x^{\beta+1}\mathscr{A}^1_k + x\mathscr{A}^1_k + \cap_{\ell-2j-1\geq 0}\mathscr{C}^{\beta+j+1+\lambda}_{\ell-2j-1}\right)(\Omega_{x_0,T}) + C_\infty(\overline{\Omega_{x_0,T}}) \tag{3.4.47}$$

If one further assumes

$$L^{\mu}_{12}, B_{12}, a, \varphi(0) \in L^{\infty}(\Omega_{x_0,T}) \ ,$$

then it also holds that

$$\varphi \in \left(x^{\beta}\mathscr{A}^1_k + \mathscr{A}^1_k \cap L^{\infty} + \cap_{\ell-2j-2\geq 0}\mathscr{C}^{\beta+j+\lambda}_{\ell-2j-2}\right)(\Omega_{x_0,T}) \ .$$

Proof: The result is obtained through a repetition of the proof of Theorem 3.4.3, keeping track of the differentiability of the remainder terms. □

We are ready now to prove polyhomogeneity of solutions of the Cauchy problem for Equation (3.4.5). We consider only the simplest case of equations satisfying the conditions (3.4.48) below, considerably more general statements can be proved using similar methods. The differentiability hypotheses below are clearly satisfied by equations with smooth bounded coefficients; however, they also allow for a wide class of equations with polyhomogeneous coefficients. We restrict ourselves to the case in which the corner conditions are satisfied to arbitrary order; if not, one obtains expansions as in (3.4.47), with a remainder in which a finite number only of time derivative are controlled; such results can be proved by identical arguments, compare the proof of Theorem 3.4.5.

Theorem 3.4.6 Consider a solution $(\varphi, \psi) \in C_\infty \times C_\infty$ of the system (3.4.5), suppose that in addition to (3.4.12), (3.4.13), (3.4.19), and (3.4.45a) we have

$$B_{11}, B_-, E^{\mu}_{\pm}, \ell, \ell^{\dagger} \in L^{\infty}(\Omega_{x_0,T}) \ , \tag{3.4.48a}$$

$$E^{\mu}_-\Big|_{x=0} = \partial_\tau \otimes \mathrm{id} \ , \qquad E^{\mu}_+\Big|_{x=0} = (\partial_\tau - 2\partial_x) \otimes \mathrm{id} \ , \tag{3.4.48b}$$

$$E^x_{\pm} - E^x_{\pm}\Big|_{x=0} , E^{\tau}_{\pm} - E^{\tau}_{\pm}\Big|_{x=0} \in x^{1+\delta}\mathscr{A}^{\delta}_{\infty}(\Omega_{x_0,T}) \ , \tag{3.4.48c}$$

$$E^A_- \in x\mathscr{A}^{\delta}_{\infty}(\Omega_{x_0,T}) \ . \tag{3.4.48d}$$

If

$$a, b \in x^\beta \mathscr{A}_k^\delta(\Omega_{x_0,T}) \ , \quad \varphi(0) \in x^\beta \mathscr{A}_k^\delta(M_{x_0}) \ ,$$

with $\beta \in \mathbb{R}$, and if the initial data satisfy *corner conditions to arbitrary order*, in the sense that

$$\forall\, i \in \mathbb{N} \qquad \partial_\tau^i \varphi(0),\ \partial_\tau^i \psi(0) \in \mathscr{C}_\infty^\lambda(M_{x_0}) \ , \tag{3.4.49}$$

for some (i-independent) $\lambda \in \mathbb{R}$, then

$$\varphi \in \left(x^\beta \mathscr{A}_k^\delta + \mathscr{A}_k^\delta\right)(\Omega_{x_0,T}) \ , \quad \psi \in \left(x^{\beta+1} \mathscr{A}_k^\delta + x\mathscr{A}_k^\delta\right)(\Omega_{x_0,T}) + C_\infty(\overline{\Omega_{x_0,T}}) \ .$$

In particular, if $k = \infty$ then the solution is polyhomogeneous.

Remark: The class of initial data satisfying corner conditions to arbitrary order is rather large; for example, if an initial data set $(\varphi(0), \psi(0))$ satisfies them, and if f, g are arbitrary functions smooth up to boundary on the initial data hypersurface, then $(\varphi(0) + f, \psi(0) + g)$ will also satisfy those conditions. More generally, large classes of such initial data can be constructed using a polyhomogeneous generalization of the Borel summation lemma.

PROOF: The hypothesis (3.4.49) with $i = 0$ and Proposition 3.4.1 show that for all $\tau \in [0, T]$ we have

$$\varphi(\tau),\ \psi(\tau) \in \mathscr{C}_\infty^\lambda(M_{x_0/2}) \ .$$

Proposition 3.4.2 shows then that the hypotheses of Corollary 3.4.4 are satisfied, and the result follows. □

3.5 The semi-linear scalar wave equation

Let f be a solution of the semi-linear wave equation

$$\Box_{\mathfrak{g}} f = H(x^\mu, f) \ , \tag{3.5.1}$$

here $\Box_{\mathfrak{g}}$ is the d' Alembertian associated with $\mathfrak{g}$. Set

$$\tilde f = \Omega^{-\frac{(n-1)}{2}} f \ ; \tag{3.5.2}$$

Letting $\tilde{\mathfrak{g}} = \Omega^2 \mathfrak{g}$ as in (3.1.1), from (3.1.3) we obtain

$$\Box_{\tilde{\mathfrak{g}}} \tilde f = \frac{n-1}{4n}(\tilde R - \frac{R}{\Omega^2})\tilde f + \Omega^{-\frac{n+3}{2}} H(x^\mu, \Omega^{\frac{n-1}{2}} \tilde f) \ . \tag{3.5.3}$$

Let $\mathfrak{g} = \eta$ be the Minkowski metric; under the conformal transformation (3.1.4) one obtains from (3.1.5) that $\tilde{\mathfrak{g}}$ is again the Minkowski metric, and (3.5.3) becomes

$$\Box_\eta \tilde f = \Omega^{-\frac{n+3}{2}} H(x^\mu, \Omega^{\frac{n-1}{2}} \tilde f) \ . \tag{3.5.4}$$

We shall assume that the initial data for f are given on a hypersurface $\Sigma \subset \mathscr{M}$, which, in a neighborhood $\mathscr{O}$ of $\mathscr{I}^+$ is given by the equation

$$\Sigma \cap \mathscr{O} = \{y^0 = \frac{1}{2}\} \ . \tag{3.5.5}$$

This correspond to a hyperboloid in $\mathscr{M}$ given by the equation $x^0+1=\sqrt{1+\vec{x}^2}$. It is convenient to introduce the following coordinate system (x,v,τ) in a $\widetilde{\mathscr{M}}$-neighborhood of $\mathscr{I}^+$:

$$\begin{aligned} \tau &= y^0-1/2\geq 0\ , \\ x &= (\sum (y^i)^2)^{\frac{1}{2}}-y^0\geq 0\ , \\ y^i &= (\sum (y^i)^2)^{\frac{1}{2}}n^i(v)\ , \end{aligned} \tag{3.5.6}$$

$n^i(v)\in S^{n-1}$, with $v=(v^A)$ denoting spherical coordinates on S^{n-1}. Equation (3.1.5) gives

$$\Omega = x(2\tau+x+1)\approx x\ . \tag{3.5.7}$$

If we let h denote the unit round metric on S^{n-1}, we then have

$$\eta = 2dxd\tau + dx^2 + (x+\tau+1/2)^2 h\ , \tag{3.5.8}$$

and

$$\begin{aligned} \Box_\eta \tilde f &= \frac{1}{(x+\tau+1/2)^{n-1}\sqrt{\det h}}\partial_\mu\left((x+\tau+1/2)^{n-1}\sqrt{\det h}\ \eta^{\mu\nu}\partial_\nu \tilde f\right) \\ &= \{-\partial_\tau(\partial_\tau-2\partial_x)+\frac{n-1}{x+\tau+1/2}\partial_x+\frac{\triangle_h}{(x+\tau+1/2)^2}\}\tilde f\ , \end{aligned} \tag{3.5.9}$$

where $\triangle_h$ is the Laplace-Beltrami operator of the metric h. We set

$$e_- = \partial_\tau\ ,\quad e_+=\partial_\tau-2\partial_x\ ,\quad e_A=\tfrac{1}{(x+\tau+1/2)}h_A\ , \tag{3.5.10}$$

$$\phi_- = e_-(\tilde f)\ ,\quad \phi_+=e_+(\tilde f)\ , \tag{3.5.11}$$

$$\phi_A=\psi_A=\tfrac{1}{(x+\tau+1/2)}h_A(\tilde f)\ , \tag{3.5.12}$$

where h_A denotes an h-orthonormal frame on S^{n-1}. We use the symbol D to denote the covariant derivative operator associated to the metric h. (The usefulness of introducing two different objects for $h_A(\tilde f)/(x+\tau+1/2)$ will become clear shortly.) Equation (3.5.4) implies the following set of equations:

$$\begin{aligned} e_-(\phi_+)\ -D_{e_A}\psi_A\ -\tfrac{n-1}{2(x+\tau+1/2)}\phi_+ &= -\tfrac{n-1}{2(x+\tau+1/2)}\phi_- + a_+\ , \\ -e_A(\phi_+)\ +e_+(\psi_A)\ -\tfrac{1}{(x+\tau+1/2)}\psi_A &= b_A\ , \end{aligned} \tag{3.5.13}$$

$$\begin{aligned} e_-(\phi_A)\ -e_A(\phi_-)\ +\tfrac{1}{(x+\tau+1/2)}\phi_A &= a_A\ , \\ -D_{e_A}\phi_A\ +e_+(\phi_-)\ +\tfrac{n-1}{2(x+\tau+1/2)}\phi_- &= \tfrac{n-1}{2(x+\tau+1/2)}\phi_+ + b_-\ , \end{aligned} \tag{3.5.14}$$

$$e_-(\tilde f)=\phi_-\ , \tag{3.5.15}$$

$$e_+(\tilde f)=\phi_+\ , \tag{3.5.16}$$

with $a_A=b_A=0$ and

$$a_+\equiv b_-\equiv -G\equiv -\Omega^{-\frac{n+3}{2}}H(x^\mu,\Omega^{\frac{n-1}{2}}\tilde f)\ . \tag{3.5.17}$$

3.5.1 Existence of solutions, space derivatives estimates

We note that the partial differential operator standing on the left-hand-side of (3.5.13) is symmetric hyperbolic; the same holds true for (3.5.14), or for the joint system (3.5.13)-(3.5.16). Now, part of our technique consists in deriving weighted energy estimates for symmetric hyperbolic systems having the structure above, *cf.* Section 3.4. Each such system comes with his own estimates, so that for the systems (3.5.13) and (3.5.14) we can obtain estimates with different weights. This allows us to handle a reasonably wide range of non-linearities, giving existence and blow-up control for initial data in weighted Sobolev spaces (with conormal-type blow-up at $\mathscr{I}^+$):

Theorem 3.5.1 Consider Equation (3.5.1) on $\mathbb{R}^{n,1}$ with initial data given on a hyperboloid $\mathscr{S} \supset \Sigma_{x_0,0}$ in Minkowski space-time, and satisfying

$$\widetilde{f}|_{\Sigma_{x_0,0}} \equiv \Omega^{-\frac{n-1}{2}} f|_{\Sigma_{x_0,0}} \in \mathscr{H}^{\alpha}_{k+1}(\Sigma_{x_0,0}) \,, \tag{3.5.18}$$

$$\partial_x(\Omega^{-\frac{n-1}{2}} f)|_{\Sigma_{x_0,0}} \in \mathscr{C}^{\alpha}_0(\Sigma_{x_0,0}) \cap \mathscr{H}^{\alpha-1/2}_k(\Sigma_{x_0,0}) \,, \tag{3.5.19}$$

$$\partial_\tau(\Omega^{-\frac{n-1}{2}} f)|_{\Sigma_{x_0,0}} \in \mathscr{H}^{\alpha}_k(\Sigma_{x_0,0}) \,, \tag{3.5.20}$$

with some $k > \frac{n}{2} + 1$, $-1 < \alpha < -1/2$. Suppose further that H has a uniform zero of order ℓ at $f = 0$, in the sense of (3.2.30), with

$$\ell \geq \begin{cases} 4 \,, & n = 2 \,, \\ 3 \,, & n = 3 \,, \\ 2 \,, & n \geq 4 \,. \end{cases} \tag{3.5.21}$$

Then:

1. There exists $0 < \tau_+ \leq T$ $(< x_0/2)$, depending only upon x_0 and a bound on the norms of the initial data in the spaces appearing in Equations (3.5.18)-(3.5.20), and a solution f of Equation (3.5.1), defined on a set containing Ω_{x_0,τ_+}, satisfying the given initial conditions, and satisfying

$$\|\widetilde{f}\|_{L^\infty(\Omega_{x_0,\tau_+})} < \infty \,.$$

2. Further, if τ_* is such that f exists on Ω_{x_0,τ_*} and satisfies $\|\widetilde{f}\|_{L^\infty(\Omega_{x_0,\tau_*})} < \infty$, then for $0 \leq \tau < \tau_*$ we have

$$\widetilde{f}|_{\Sigma_{x_0,\tau}} \in L^\infty(\Sigma_{x_0,\tau}) \cap \mathscr{H}^{\alpha}_{k+1}(\Sigma_{x_0,\tau}) \,,$$

$$\partial_\tau \widetilde{f}|_{\Sigma_{x_0,\tau}} \in \mathscr{H}^{\alpha}_k(\Sigma_{x_0,\tau}) \,, \qquad \partial_x \widetilde{f}|_{\Sigma_{x_0,\tau}} \in \mathscr{H}^{\alpha-\frac{1}{2}}_k(\Sigma_{x_0,\tau}) \cap \mathscr{C}^{\alpha}_0(\Sigma_{x_0,\tau}) \,.$$

Remarks :
1. Integration in x of condition (3.5.19) implies that $\widetilde{f} \in L^\infty(\Sigma_{x_0,0})$.
2. Some further information can be found in Theorem 3.5.3 below.

Proof: As before, we write $\|f(\tau)\|_{\mathscr{H}^{\alpha}_k}$ for $\|f|_{\Sigma_{x_0,\tau}}\|_{\mathscr{H}^{\alpha}_k(\Sigma_{x_0,\tau})}$, *etc.* Recall that the standard theory of hyperbolic systems (*cf., e.g.,* [39, Chapter 16, Vol. III][7]) shows that for any $0 < x_1 \leq x_0$ there exists $T(x_1) > 0$, satisfying $2x_1 + T \leq x_0$, and a solution $\widetilde{f}$ of (3.5.4), defined on $\Omega_{x_1,x_0,T}$, with initial data on Σ_{x_1,x_0} obtained from those on Σ_{x_0} by restriction. The idea of the proof is to derive x_1-independent, weighted *a priori* estimates for the solution. These estimates will guarantee that the existence time $T(x_1)$ does not shrink to zero as x_1 goes to zero; they will also guarantee that the weighted Sobolev regularity is preserved by evolution. We start with the following:

Lemma 3.5.2 Under the hypotheses of Theorem 3.5.1, consider on $\Omega_{x_1,x_0,T}$ the system (3.5.12)-(3.5.16), set

$$\begin{aligned} E_\alpha(t) &= \|\widetilde{f}(t)\|^2_{\mathscr{H}^\alpha_k} + \|\phi_-(t)\|^2_{\mathscr{H}^\alpha_k} \\ &\quad + \|\phi_+(t)\|^2_{\mathscr{H}^{\alpha-\frac{1}{2}}_k} + \sum_A \|\phi_A(t)\|^2_{\mathscr{H}^\alpha_k} . \end{aligned} \tag{3.5.22}$$

Then there exists a x_1-independent constant C such that

$$E_\alpha(t) \leq C\left\{E_\alpha(0) + \int_0^t e^{C(t-s)} S(s) ds\right\} , \tag{3.5.23}$$

where

$$\begin{aligned} S(s) &\equiv \sum_A \|a_A(s)\|^2_{\mathscr{H}^\alpha_k} + \|a_+(s)\|^2_{\mathscr{H}^{\alpha-1/2}_k} \\ &\quad + \|b_-(s)\|^2_{\mathscr{H}^{\alpha-1/2}_k} + \sum_A \|b_A(s)\|^2_{\mathscr{H}^{\alpha-1}_k} . \end{aligned} \tag{3.5.24}$$

Proof: We wish, first, to apply Proposition 3.4.1 to the system consisting of Equation (3.5.14) together with $e_-(\widetilde{f}) = \phi_-$; in order to do this we set

$$\varphi = \begin{pmatrix} \widetilde{f} \\ \phi_A \end{pmatrix} , \qquad \psi = \phi_- .$$

We choose $E^\mu_\pm \partial_\mu = e_\pm \otimes \mathrm{Id}$, we set

$$L\psi = \begin{pmatrix} 0 \\ -e_A(\psi) \end{pmatrix} , \tag{3.5.25}$$

and we define

$$\widetilde{E}_\alpha(t) = \|\widetilde{f}(t)\|^2_{\mathscr{H}^\alpha_k} + \|e_-(\widetilde{f})(t)\|^2_{\mathscr{H}^\alpha_k} + \sum_A \|e_A(\widetilde{f})(t)\|^2_{\mathscr{H}^\alpha_k} .$$

The hypotheses $\mathscr{C}1 - \mathscr{C}5$ of Proposition 3.4.1 are readily verified, and for any $\alpha < -\frac{1}{2}$ the inequality (3.4.20) gives

$$\begin{aligned} \widetilde{E}_\alpha(t) &\leq C\left\{\widetilde{E}_\alpha(0)e^{Ct} + \int_0^t e^{C(t-s)} \left(\|a_+(s)\|^2_{\mathscr{H}^\alpha}\right.\right. \\ &\quad \left.\left. + \|\phi_+(s)\|^2_{\mathscr{H}^{\alpha-1/2}} + \sum_A \|b_A(s)\|^2_{\mathscr{H}^{\alpha-1/2}}\right) ds\right\} . \end{aligned} \tag{3.5.26}$$

Next, we consider (3.5.13)-(3.5.15) as a whole; to apply Proposition 3.4.1 we take

$$\varphi = \begin{pmatrix} \widetilde{f} \\ \phi_+ \\ \phi_A \end{pmatrix} , \qquad \psi = \begin{pmatrix} \widetilde{f} \\ \psi_A \\ \phi_- \end{pmatrix} . \tag{3.5.27}$$

We set again $E^\mu_\pm \partial_\mu = e_\pm \otimes \mathrm{Id}$ and

$$L\psi = \begin{pmatrix} 0 \\ -D_{e_A}\psi_A \\ -e_A(\phi_-) \end{pmatrix} ,$$

hence

$$L^+\varphi = \begin{pmatrix} 0 \\ e_A(\phi_+) \\ D_{e_A}\phi_A \end{pmatrix} .$$

We define

$$\begin{aligned} \hat{E}_{\alpha'}(t) &= \|\widetilde{f}(t)\|^2_{\mathscr{H}^{\alpha'}_k} + \|e_-(\widetilde{f})(t)\|^2_{\mathscr{H}^{\alpha'}_k} \\ &\quad + \|e_+(\widetilde{f})(t)\|^2_{\mathscr{H}^{\alpha'}_k} + \sum_A \|e_A(\widetilde{f})(t)\|^2_{\mathscr{H}^{\alpha'}_k} , \end{aligned}$$

and write (3.5.13)-(3.5.15) in the form (3.4.5), with all the terms linear in ϕ_- and ϕ_+ in (3.5.13)-(3.5.14) transferred to the left-hand-side. The hypotheses of Proposition 3.4.1 hold again, and for any $\alpha' < -1/2$ it follows from (3.4.20) that

$$\begin{aligned} \hat{E}_{\alpha'}(t) &\le C\Bigg\{ \hat{E}_{\alpha'}(0)e^{Ct} + \int_0^t e^{C(t-s)} \Bigg(\sum_A \|a_A(s)\|^2_{\mathscr{H}^{\alpha'}_k} + \|b_-(s)\|^2_{\mathscr{H}^{\alpha'-1/2}_k} \\ &\quad + \|a_+(s)\|^2_{\mathscr{H}^{\alpha'}_k} + \sum_A \|b_A(s)\|^2_{\mathscr{H}^{\alpha'-1/2}_k} \Bigg) ds \Bigg\} . \end{aligned} \tag{3.5.28}$$

We set

$$E(t) = \widetilde{E}_\alpha(t) + \hat{E}_{\alpha-1/2}(t) .$$

It follows from (3.5.26) and (3.5.28) with $\alpha' = \alpha - 1/2$ that we have

$$E(t) \le C\left(E(0)e^{Ct} + \int_0^t e^{C(t-s)}(E(s) + S(s))ds \right) , \tag{3.5.29}$$

with $S(s)$ as in (3.5.24). Equation (3.5.23) with E_α replaced[6] by E follows now from Gronwall's Lemma. Since E_α is equivalent to E, our claims follow. □

Returning to the proof of Theorem 3.5.1, Lemma 3.5.2 applied to (3.5.13)-(3.5.15) gives

$$E_\alpha(t) \le C\left(E_\alpha(0)e^{Ct} + \int_0^t e^{C(t-s)} \|G(s)\|^2_{\mathscr{H}^{\alpha-1/2}_k} ds \right) . \tag{3.5.30}$$

[6] The constant C in Equation (3.5.23) does not necessarily coincide with that in (3.5.29).

By hypothesis the function H appearing in (3.5.1) has a uniform zero of order $\ell \geq 2$, in the sense of (3.2.30); we wish to use (3.2.31) to control the term containing $G(s)$ in (3.5.30). This requires an L^∞ bound on $\widetilde{f}$, which will be obtained next. As $k > n/2+1$, the Sobolev embedding (3.2.24) gives

$$\|e_-(\widetilde{f})(s)\|^2_{\mathscr{C}^\alpha_1} + \|e_+(\widetilde{f})(s)\|^2_{\mathscr{C}^{\alpha-1/2}_1} + \|e_A(\widetilde{f})(s)\|^2_{\mathscr{C}^\alpha_1} \leq CE_\alpha(s). \tag{3.5.31}$$

Now the conditions (3.5.21) on n and ℓ give

$$|G(\tau)| \leq C\|\widetilde{f}(\tau)\|^\ell_{L^\infty}\, x^{\ell\frac{n-1}{2}-\frac{n+3}{2}} \leq C\|\widetilde{f}(\tau)\|^\ell_{L^\infty}\, x^{-1/2}\,,$$

so that (recall that $\alpha < -1/2$)

$$\|G(\tau)\|_{\mathscr{C}^\alpha_0} \leq C\,\|\widetilde{f}(\tau)\|^\ell_{L^\infty}\,. \tag{3.5.32}$$

From (3.5.13) we have

$$\partial_\tau\phi_+ - \frac{n-1}{2(x+\tau+1/2)}\phi_+ = D_{e_A}\psi_A - \frac{n-1}{2(x+\tau+1/2)}\phi_- - G\,, \tag{3.5.33}$$

and (3.5.32) together with Proposition 3.3.1 yield

$$\begin{aligned}\|\phi_+(t)\|_{\mathscr{C}^\alpha_0} &\leq Ce^{Ct}\|\phi_+(0)\|_{\mathscr{C}^\alpha_0} \\ &\quad +C\int_0^t e^{C(t-s)}(\|D_{e_A}\psi_A(s)\|_{\mathscr{C}^\alpha_0} + \|\phi_-(s)\|_{\mathscr{C}^\alpha_0} + \|G(s)\|_{\mathscr{C}^\alpha_0})\,ds \\ &\leq Ce^{Ct}\|\phi_+(0)\|_{\mathscr{C}^\alpha_0} + \int_0^t e^{C(t-s)}C(E_\alpha(s),\|\widetilde{f}(s)\|_{L^\infty})ds\,,\end{aligned} \tag{3.5.34}$$

for some continuous function $C(E_\alpha(\cdot),\|f(\cdot)\|_{L^\infty})$. Integration of $\partial_x\widetilde{f} = \frac{1}{2}(\phi_- - \phi_+)$ over $[x, x_0-2\tau]$ gives

$$|\widetilde{f}(\tau,x)| \leq |\widetilde{f}(\tau,x_0-2\tau)| + \frac{1}{2}\|(\phi_- - \phi_+)(\tau)\|_{\mathscr{C}^\alpha_0}\int_x^{x_0-2\tau} s^\alpha ds\,.$$

For any $0 \leq \tau \leq \tau_* < x_0/2$ the $f(\tau, x_0-2\tau)$ term is estimated by a multiple of the initial energy in a standard way, which leads to the estimate

$$\begin{aligned}\|\widetilde{f}(\tau)\|_{L^\infty} &\leq CE_\alpha(\tau) + Ce^{C\tau}\|\phi_+(0)\|_{\mathscr{C}^\alpha_0} \\ &\quad + \int_0^\tau e^{C(\tau-s)}C(E_\alpha(s),\|\widetilde{f}(s)\|_{L^\infty})ds\,.\end{aligned} \tag{3.5.35}$$

Next,

$$\|G(s)\|_{\mathscr{H}^{\alpha-1/2}_k} \leq C\|H(s,\cdot,x^{\frac{n-1}{2}}\widetilde{f})\|_{\mathscr{H}^{\alpha-1/2+\frac{n+3}{2}}}\,,$$

and our hypothesis that H has a uniform zero of order ℓ together with (3.2.31) gives

$$\|G(s)\|_{\mathscr{H}^{\alpha-1/2}_k} \leq C\left(\|\widetilde{f}(s)\|_{L^\infty}\right)\|\widetilde{f}\|_{\mathscr{H}_k^{\alpha+\frac{n+2}{2}-l\frac{n-1}{2}}}\,.$$

In view of (3.5.35) this can be estimated by a function of $E_\alpha(s)$ and of $\|\widetilde{f}(s)\|_{L^\infty}$,

$$\begin{aligned}\|G(s)\|^2_{\mathscr{H}_k^{\alpha-1/2}} &\leq C\left(\|\widetilde{f}(s)\|_{L^\infty}\right)\|\widetilde{f}(s)\|^2_{\mathscr{H}_k^\alpha} \\ &\leq C\left(\|\widetilde{f}(s)\|_{L^\infty}\right)E_\alpha(s)\;, \qquad (3.5.36)\end{aligned}$$

provided that

$$l \geq \frac{n+2}{n-1} \qquad (3.5.37)$$

(which coincides again with (3.5.21)). If (3.5.37) holds, from (3.5.30) and (3.5.35) we obtain

$$\begin{aligned}\|\widetilde{f}(\tau)\|_{L^\infty} + E_\alpha(\tau) &\leq Ce^{C\tau}\left(E_\alpha(0) + \|\partial_x\widetilde{f}(0)\|_{\mathscr{C}_0^\alpha} + \|\partial_\tau\widetilde{f}(0)\|_{\mathscr{H}_k^\alpha}\right) \\ &\quad + \int_0^\tau \Phi\left(\tau, s, \|\widetilde{f}(s)\|_{L^\infty}, E_\alpha(s)\right)ds\;, \qquad (3.5.38)\end{aligned}$$

for some constant C, and for a function Φ which is bounded on bounded sets. It then easily follows that there exists a time τ_+ and a constant M, depending only upon x_0 and a bound on the norms of the initial data in the spaces appearing in Equations (3.5.18)-(3.5.20), such that $\|\widetilde{f}(\tau)\|_{L^\infty}$ and $E_\alpha(\tau)$ remain bounded by M for $0 \leq \tau \leq \tau_+$. Since all the objects above were x_1-independent, so is τ_+. By the usual continuation criterion (*cf., e.g.,* [39, Proposition 1.5, Chapter 16, Vol. III][7]) the solution exists on Ω_{x_1,x_0,τ_+} for all x_1; it thus follows that the maximally extended solution of the initial value problem considered here exists on a set which includes Ω_{x_0,τ_+}.

To establish point 2, suppose that a global a-priori L^∞ bound on $\widetilde{f}$ is known. Then (3.5.30) and (3.5.36) give a linear integral inequality on E_α, and Gronwall's Lemma gives a global bound on E_α. Arguments of the last part of the proof of point 1 yield the result. □

For the purpose of estimating time derivatives of the solutions we will need a generalization of Theorem 3.5.1. There are lots of ways to relax the hypotheses thereof; for simplicity we shall only make those generalizations which are strictly necessary for the arguments in the next section to go through. First, the fact that f is scalar valued plays no role in our considerations above; henceforth we assume that f has values in $\mathbb{R}^N$ for some $N \geq 1$. Next, the definitions (3.5.10) of $e_\pm$ and e_A will be kept, as well as those of ϕ_A and ψ_A given in (3.5.12). We will consider systems of the form

$$P\begin{pmatrix}\varphi\\ \psi\end{pmatrix} + \begin{pmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{pmatrix}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \begin{pmatrix}a\\ b\end{pmatrix} + G\;, \qquad (3.5.39a)$$

$$\varphi = \begin{pmatrix}\phi_+\\ \phi_A\end{pmatrix}\;,\quad \psi = \begin{pmatrix}\phi_-\\ \psi_A\end{pmatrix} \qquad (3.5.39b)$$

[7] In that reference symmetric hyperbolic systems on a torus are considered; however simple domain of dependence considerations show that the results there apply to the setup here.

together with

$$e_-(\tilde{f}) = B_0\phi_- + B_1\widetilde{f} \,, \qquad (3.5.40a)$$
$$e_+(\tilde{f}) = \phi_+ \,, \qquad (3.5.40b)$$

for some matrix valued functions B_0, B_1, with B_0 — invertible. Here

$$P = \begin{pmatrix} e_- & \ell^A D_A \\ (\ell^A)^t D_A & e_+ \end{pmatrix} \qquad (3.5.41)$$

is the (geometric) principal part of Equations (3.5.13)-(3.5.14). The nonlinear term $G = G(x^\mu, \widetilde{f})$ will be labelled as

$$G = \left(G_{e_+(\phi_-)}, G_{e_+(\psi_A)}, G_{e_-(\phi_A)}, G_{e_+(\phi_-)}\right) \,, \qquad (3.5.42)$$

with the order of the components following that of Equations (3.5.13)-(3.5.14). The B_{ab}'s will be labeled as B_{ϕ_-,ϕ_+}, B_{ϕ_-,ϕ_A}, *etc.*; for example, in this notation, the second of Equations (3.5.14) takes the form

$$\begin{aligned} e_+(\phi_-) &= D_{e_A}\phi_A - B_{\phi_-,\phi_-}\phi_- \\ &\quad - B_{\phi_-,\phi_+}\phi_+ - B_{\phi_-,\phi_A}\phi_A - B_{\phi_-,\psi_A}\psi_A + b_- + G_{e_+(\phi_-)} \,, \end{aligned} \qquad (3.5.43)$$

with actually $B_{\phi_-,\phi_A} = B_{\phi_-,\psi_A} = 0$.

Some effort will be needed to prove the information of point 3 of the theorem that follows; this is needed to be able to iteratively apply that theorem in the next section:

Theorem 3.5.3 Consider the system (3.5.39)-(3.5.40) with

$$\begin{aligned} &\|a(\tau)\|_{\mathscr{H}_k^\alpha} + \|b(\tau)\|_{\mathscr{H}_k^\alpha} + \sup_{a,b=1,2} \|B_{ab}(\tau)\|_{\mathscr{C}_k^0} \\ &\quad + \|B_0(\tau)\|_{\mathscr{C}_k^0} + \|B_0^{-1}(\tau)\|_{L^\infty} + \|B_1(\tau)\|_{\mathscr{C}_k^0} \le \tilde{C} \,, \end{aligned} \qquad (3.5.44)$$

for some constant $\tilde{C}$, and suppose that

$$G(x^\mu, \widetilde{f}) = \Omega^{-(n+3)/2} H(x^\mu, \Omega^{(n-1)/2}\widetilde{f}) \,, \qquad (3.5.45)$$

with $G_{e_-(\phi_A)} = 0$, and with H having a uniform zero of order ℓ in the sense of (3.2.30), with ℓ satisfying (3.5.21). If the initial data satisfy (3.5.18)-(3.5.20) with some $k > \frac{n}{2} + 1$, $-1 < \alpha < -1/2$, then:

1. The conclusions of point 1. of Theorem 3.5.1 hold with a time τ_+ depending only upon the constant $\tilde{C}$ in (3.5.44) and a bound on the norms of the initial data in the spaces appearing in Equations (3.5.18)-(3.5.20).

2. The conclusions of point 2. of Theorem 3.5.1 hold.

3. Under the hypotheses of point 2. of Theorem 3.5.1 we also have

$$\|(x + 2\tau)\partial_\tau \widetilde{f}\|_{L^\infty(\Omega_{x_0,\tau_*})} < \infty \,. \qquad (3.5.46)$$

Remark: The condition $G_{e_-(\phi_A)} = 0$ can be weakened to

$$G_{e_-(\phi_A)}(x^\mu, \widetilde{f}) = \Omega^{-(n+2)/2} H_{e_-(\phi_A)}(x^\mu, \Omega^{(n-1)/2}\widetilde{f}) \,, \tag{3.5.47}$$

for some function $H_{e_-(\phi_A)}$ with a uniform zero of order ℓ. Similarly it suffices to assume that

$$G_{e_+(\psi_A)}(x^\mu, \widetilde{f}) = \Omega^{-(n+4)/2} H_{e_+(\psi_A)}(x^\mu, \Omega^{(n-1)/2}\widetilde{f}) \,, \tag{3.5.48}$$

for some function $H_{e_+(\psi_A)}$ with a uniform zero of order ℓ.

PROOF: Let us start by remarking that, because $\psi_A = \phi_A$, in Equation (3.5.43) we can replace B_{ϕ_-,ϕ_A} by $B_{\phi_-,\phi_A}+B_{\phi_-,\psi_A}$ obtaining a system in which $B_{\phi_-,\psi_A} = 0$. Proceeding similarly with the other equations we may thus without loss of generality assume that

$$B_{\cdot,\psi_A} = 0 \,. \tag{3.5.49}$$

The proof of points 1 and 2 is then identical to that of Theorem 3.5.1, with the following minor changes: Equation (3.5.33) is replaced by the equation

$$\begin{aligned} &e_-(\phi_+) + B_{\phi_+,\phi_+}\phi_+ = \\ &\qquad D_{e_A}\phi_A - B_{\phi+-,\phi_-}\phi_- - B_{\phi_+,\phi_A}\phi_A + a_+ + G_{e_-(\phi_+)} \end{aligned} \tag{3.5.50}$$

to which Proposition 3.3.1 still applies, recovering (3.5.34). Further, the equation $\partial_x\widetilde{f} = (\phi_- - \phi_+)/2$ has to be replaced by

$$\partial_x\widetilde{f} + \frac{B_1}{2}\widetilde{f} = \frac{B_0\phi_- - \phi_+}{2} \,,$$

and the desired conclusion is obtained by Proposition 3.3.3. The remaining arguments do not require any modifications.

To prove point 3, from (3.5.43) we obtain

$$\begin{aligned} &e_+[(x+2\tau)\phi_-] = \\ &\quad (x+2\tau)\big(D_{e_A}\phi_A - B_{\phi_-,\phi_-}\phi_- \\ &\quad -B_{\phi_-,\phi_A}\phi_A - B_{\phi_-,\phi_+}\phi_+ + b_- + G_{e_+(\phi_-)}\big) \,, \end{aligned} \tag{3.5.51}$$

¿From Equations (3.5.32), (3.5.34), and (3.5.40a) together with

$$\phi_-, \phi_A \in \mathscr{H}^\alpha_k \subset \mathscr{C}^\alpha_0 \,, \qquad D_{e_A}\phi_A \in \mathscr{H}^\alpha_{k-1} \subset \mathscr{C}^\alpha_0 \,,$$

we obtain

$$e_+[(x+2\tau)\phi_-] \quad\le\quad \hat{C}x^{-\alpha} \,,$$

for some constant C depending only upon the initial data and $\|\widetilde{f}\|_{L^\infty(\Omega_{x_0,\tau_*})}$. Integrating as in the identity (3.4.36) we arrive at

$$\begin{aligned} &|B_0^{-1}\left\{(x+2\tau)(\partial_\tau\widetilde{f} - B_1\widetilde{f})(x,v,\tau)\right\}| \\ &\quad\le |B_0^{-1}\left\{(x+2\tau)\partial_\tau\widetilde{f}(x+2\tau,v,0)\right\}| + C\left(\|\widetilde{f}\|_{L^\infty(\Omega_{x_0,\tau_*})} + \hat{C}\right) \\ &\quad\le C\left(\|\partial_\tau\widetilde{f}\|_{\mathscr{C}_0^{-1}} + \|\widetilde{f}(0)\|_{L^\infty(\Omega_{x_0,\tau_*})} + \hat{C}\right) \end{aligned}$$

and Equation (3.5.46) follows.

□

3.5.2 Estimates on the time derivatives of the solutions

So far we have established existence of solutions with initial data in weighted Sobolev spaces, as well as weighted estimates on the space-derivatives of the solutions. The next step in proving polyhomogeneity is to establish estimates on time-derivatives. Similarly to the linear case, the question of corner conditions arises. In order to handle that, we introduce an index m, which corresponds to the number — perhaps zero — of corner conditions which are satisfied by the initial data. Next, the definition (3.2.30) of a uniform zero of order l has to be strengthened by adding conditions on time-derivatives: we shall require that there exists a constant $\hat{C}$ such that, for all $p \in \mathbb{R}^N$, $0 \le i \le \min(k,l)$ and $0 \le j \le m$ we have

$$\left\| \frac{\partial^{i+j} F(\tau,\cdot,p)}{\partial p^i \partial \tau^j} \right\|_{\mathscr{C}^0_{k+m-i-j}} \le \hat{C}|p|^{l-i} . \tag{3.5.52}$$

We start with the following:

Theorem 3.5.4 Let $\mathbb{N} \ni m \ge 0$, consider a solution $f : \Omega_{x_0,\tau_*} \to \mathbb{R}$ of Equation (3.5.1) satisfying

$$\|\widetilde{f}\|_{L^\infty(\Omega_{x_0,\tau_*})} < \infty ,$$

and suppose that

$$0 \le i \le m+1 \qquad \partial^i_\tau \widetilde{f}|_{\Sigma_{x_0,0}} \in \mathscr{H}^\alpha_{k+m+1-i}(\Sigma_{x_0,0}) , \tag{3.5.53}$$

$$0 \le i \le m \qquad \partial_x \partial^i_\tau \widetilde{f}|_{\Sigma_{x_0,0}} \in \mathscr{C}^\alpha_0(\Sigma_{x_0,0}) \cap \mathscr{H}^{\alpha-1/2}_{k+m-i}(\Sigma_{x_0,0}) , \tag{3.5.54}$$

with some $k > \frac{n}{2}+1$ and $-1 < \alpha < -1/2$. Suppose, further, that H is smooth in f and has a uniform zero of order ℓ at $f=0$, in the sense of (3.5.52), with ℓ as in Equation (3.5.21). Then for $0 \le \tau < \tau_*$ and for $0 \le i \le m$, $0 \le j+i < k+m-n/2$ we have

$$[(\tau+2x)\partial_\tau]^j \partial^i_\tau \widetilde{f}|_{\Sigma_{x_0,\tau}} \in L^\infty(\Sigma_{x_0,\tau}) \cap \mathscr{H}^\alpha_{k+m+1-i-j}(\Sigma_{x_0,\tau}) , \tag{3.5.55a}$$

$$\partial_x[(\tau+2x)\partial_\tau]^j \partial^i_\tau \widetilde{f}|_{\Sigma_{x_0,\tau}} \in \mathscr{H}^{\alpha-\frac{1}{2}}_{k+m-i-j}(\Sigma_{x_0,\tau}) \cap \mathscr{C}^\alpha_0(\Sigma_{x_0,\tau}) , \tag{3.5.55b}$$

and

$$0 \le p < k-n/2 \qquad [(\tau+2x)\partial_\tau]^p \partial^{m+1}_\tau \widetilde{f}|_{\Sigma_{x_0,\tau}} \in \mathscr{H}^\alpha_{k-p}(\Sigma_{x_0,\tau}) , \tag{3.5.56}$$

with τ-independent bounds on the norms.

The proof below actually proves the analogous result for the systems considered in Theorem 3.5.3; the same remark applies to Corollary 3.5.5 below. Before passing to that proof, we note that an important consequence of Theorem 3.5.4 is that corner conditions will hold at any time $\tau > 0$, regardless of whether or not they hold at $\tau = 0$:

Corollary 3.5.5 Under the conditions of point 2 of Theorem 3.5.1, for any $0 < \tau < \tau_*$ and for $0 \le i < k-1-n/2$ we have

$$\partial^i_\tau \widetilde{f}|_{\Sigma_{x_0,\tau}} \in L^\infty(\Sigma_{x_0,\tau}) \cap \mathscr{H}^\alpha_{k+1-i}(\Sigma_{x_0,\tau}) ,$$

$$\partial_\tau^{i+1}\widetilde{f}|_{\Sigma_{x_0,\tau}} \in \mathscr{H}^{\alpha}_{k-i}(\Sigma_{x_0,\tau})\ ,$$
$$\partial_x\partial_\tau^{i}\widetilde{f}|_{\Sigma_{x_0,\tau}} \in \mathscr{H}^{\alpha-\frac{1}{2}}_{k-i}(\Sigma_{x_0,\tau})\cap\mathscr{C}^{\alpha}_{0}(\Sigma_{x_0,\tau})\ .$$

We shall need the following simple Lemma:

Lemma 3.5.6 Let $F(x^\mu, p)$ be a function which is smooth in p at fixed x^μ and suppose that it has a uniform zero of order $\ell \geq 1$ in p. Then

1. For all $i \in \mathbb{N}$ the function $\partial_\tau^i\left(F(x^\mu, u(x^\mu)\right)$ has a uniform zero of order ℓ, when viewed as a function of $(u, \partial_\tau u, \dots, \partial_\tau^i u)$.
2. Let $H = \partial_p F$, then H has a uniform zero of order $\ell - 1$.

PROOF: Let $u = (u^i)$; smoothness of F in p allows us to write

$$F(\vec{x}, \tau, u) = A_{i_1\dots i_\ell}u^{i_1}\dots u^{i_\ell}\ , \tag{3.5.57}$$

with some coefficients $A_{i_1\dots i_\ell} = A_{i_1\dots i_\ell}(\vec{x}, \tau, u)$ which are smooth in u, and totally symmetric in $i_1, \dots, i_\ell$; recall that the summation convention is used throughout. Point 2 immediately follows from (3.5.57). From that equation we also obtain

$$\begin{aligned}\partial_\tau F(\tau, \vec{x}, u) &= \left(\partial_\tau A_{i_1\dots i_\ell} + \partial_{u^i}A_{i_1\dots i_\ell}\partial_\tau u^i\right)u^{i_1}\dots u^{i_\ell}\\ &\quad +\ell A_{i_1\dots i_\ell}u^{i_1}\dots u^{i_{\ell-1}}\partial_\tau u^{i_\ell}\ ,\end{aligned}$$

which proves point 1 for $i = 1$. The result then follows by straightforward induction. □

We can pass now to the proof of Theorem 3.5.4:

PROOF: We assume that Equations (3.5.39)-(3.5.40) are satisfied; Theorem 3.5.3 shows that (3.5.55)-(3.5.56) hold with $i = j = p = 0$. Consider the vector-valued function

$$(\widetilde{f}, (x+2\tau)\partial_\tau\widetilde{f}, \varphi, (x+2\tau)\partial_\tau\varphi, \psi, (x+2\tau)\partial_\tau\psi)\ ;$$

we claim that it satisfies a set of equations of the form (3.5.39)-(3.5.40). Consider, for instance, Equation (3.5.40a); set

$$\hat{f} := (x+2\tau)\partial_\tau\widetilde{f}\ , \quad \hat{\phi}_- := (x+2\tau)\partial_\tau\phi_-\ ,$$

etc., we have

$$\begin{aligned}e_-(\hat{f}) &= \partial_\tau\Big((x+2\tau)(B_0\phi_- + B_1\widetilde{f})\Big)\\ &= B_0\hat{\phi}_- + (2B_0 + (x+2\tau)\partial_\tau B_0)\phi_-\\ &\quad +B_1\hat{f} + (2B_1 + (x+2\tau)\partial_\tau B_1)\widetilde{f}\ ,\end{aligned}$$

which is linear in $(\widetilde{f}, \hat{f}, \phi_-, \hat{\phi}_-)$. In fact

$$\begin{aligned}e_-\begin{pmatrix}\widetilde{f}\\ \hat{f}\end{pmatrix} &= \begin{pmatrix}B_0 & 0\\ 2B_0 + (x+2\tau)\partial_\tau B_0 & B_0\end{pmatrix}\begin{pmatrix}\phi_-\\ \hat{\phi}_-\end{pmatrix}\\ &\quad +\begin{pmatrix}B_1 & 0\\ 2B_1 + (x+2\tau)\partial_\tau B_1 & B_1\end{pmatrix}\begin{pmatrix}\widetilde{f}\\ \hat{f}\end{pmatrix}\ ,\end{aligned}$$

and the new matrix B_0 is again invertible, as desired. Next,

$$\begin{aligned}
e_-(\hat\phi_+) &= \partial_\tau\left((x+2\tau)\partial_\tau\phi_+\right) \\
&= \partial_\tau\Big((x+2\tau)\big(-D_A\phi_A - B_{\phi_+\phi_-}\phi_- \\
&\quad -B_{\phi_+\phi_A}\phi_A - B_{\phi_+\phi_+}\phi_+ + a_+ + G_{e_-(\phi_+)}\big)\Big) \\
&= -D_A\hat\phi_A - B_{\phi_+\phi_-}\hat\phi_- - B_{\phi_+\phi_A}\hat\phi_A - B_{\phi_+\phi_+}\hat\phi_+ \\
&\quad + \text{linear}(\varphi,\psi) + \widehat{a}_+ + G_{e_-(\hat\phi_+)}\ , \\
\widehat{a}_+ &= -2D_A\phi_A + \partial_\tau a_+ \ \in \mathscr{H}^\alpha_{k+m-1}\ , \\
G_{e_-(\hat\phi_+)} &= \partial_\tau\left(G_{e_-(\phi_+)}\right)\ ,
\end{aligned}$$

where "linear" denotes terms which are linear in the relevant variables. The equation for $e_-(\hat\phi_A)$ is handled in a similar way. The equations involving only e_+ or ∂_A are straightforward, since those operators commute with multiplication by $(x+2\tau)$. By Lemma 3.5.6 the new non-linearity has again a zero of order ℓ, when considered as a function of $(\widetilde f,(x+2\tau)\partial_\tau\widetilde f)$. In order to apply Theorem 3.5.3 we need to check whether the initial data are in the right spaces. Clearly

$$((x+2\tau)\partial_\tau\widetilde f)(0) = x\partial_\tau\widetilde f(0)\in\mathscr{H}^{\alpha+1}_{k+m}\subset\mathscr{H}^\alpha_{k+m}\cap L^\infty\ ,$$

$$\left(\partial_x((x+2\tau)\partial_\tau\widetilde f)\right)(0) = \left(\partial_\tau\widetilde f + x\partial_x\partial_\tau\widetilde f\right)(0)\in\mathscr{H}^\alpha_{k+m-1}\subset\mathscr{C}^\alpha_0\cap\mathscr{H}^{\alpha-1/2}_{k+m-1}\ .$$

Condition (3.5.20) requires some more work:

$$\begin{aligned}
\left(\partial_\tau((x+2\tau)\partial_\tau\widetilde f)\right)(0) &= \left(2\partial_\tau\widetilde f + x\partial^2_\tau\widetilde f\right)(0) \\
&= \left(2\partial_\tau\widetilde f + x(2\partial_x+e_+)\partial_\tau\widetilde f\right)(0) \\
&= \left(2\partial_\tau\widetilde f + 2x\partial_x\partial_\tau\widetilde f + xe_+\left(B_0\phi_- + B_1\widetilde f\right)\right)(0)\ .
\end{aligned}$$

The first two terms are obviously in $\mathscr{H}^\alpha_{k+m-1}$, and so is $xe_+(B_1\widetilde f) = x(\partial_\tau - 2\partial_x)(B_-\widetilde f)$. Equation (3.5.43) gives

$$\begin{aligned}
(xe_+(\phi_-))(0) &= x\big(D_{e_A}\phi_A - B_{\phi_-,\phi_-}\phi_- - B_{\phi_-,\phi_+}\phi_+ \\
&\quad -B_{\phi_-,\phi_A}\phi_A - B_{\phi_-,\psi_A}\psi_A + b_- + G_{e_+(\phi_-)}\big)(0)\ .
\end{aligned}$$

The desired property $(xe_+(B_0\phi_-))(0)\in\mathscr{H}^\alpha_{k+m-1}$ follows immediately; the only non-trivial term is $xG_{e_+(\phi_-)}$, the $\mathscr{H}^\alpha_{k+m+1}$ norm of which can be estimated by a function of $\|\widetilde f(0)\|_{L^\infty}$ and $\|\widetilde f(0)\|_{\mathscr{H}^\alpha_{k+m+1}}$, *cf.* Equation (3.5.36). Now, $(x+2\tau)\partial_\tau\widetilde f$ is uniformly bounded on Ω_{x_0,τ_*} by point 3 of Theorem 3.5.3, so that we can apply point 2 of Theorem 3.5.3 to conclude that Equations (3.5.55)-(3.5.56) hold with $j=p=1$ and $m=0$; straightforward induction establishes Theorem 3.5.4 for the remaining j's and p's.

Consider, now, $m=1$; the result already established with $m=0$ shows that $\partial_\tau\widetilde f(\tau)$ exists and satisfies (3.5.55) with $i=1$ for any $\tau>0$; similarly (3.5.56) holds with $m=1$ for any $\tau>0$. Now, a calculation similar (but simpler) to the one done above shows that $(\widetilde f,\partial_\tau\widetilde f)$ satisfies a system of equations of the form

(3.5.39)-(3.5.40) with initial data satisfying the conditions of Theorem 3.5.3 by hypothesis; the uniform bounds on some interval $[0,\tau_+)$ follow by point 1 of that theorem. We therefore have

$$\|(\widetilde{f},\partial_\tau\widetilde{f})\|_{L^\infty(\Omega_{x_0,\tau_*})}<\infty\ .$$

We can then apply the result already established for $m=0$ to the system of equations satisfied by $(\widetilde{f},\partial_\tau\widetilde{f})$ to obtain the conclusion of Theorem 3.5.4 with $m=1$. An induction upon m finishes the proof. □

3.5.3 Polyhomogeneous solutions

The aim of this section is to establish polyhomogeneity of solutions of a large class of semi-linear systems of the form

$$\begin{aligned}\partial_\tau\varphi+B_{11}\varphi+B_{12}\psi &= L_{11}\varphi+L_{12}\psi+a+G_\varphi\ , &\quad (3.5.58a)\\ \partial_x\psi+B_{21}\varphi+B_{22}\psi &= L_{21}\varphi+L_{22}\psi+b+G_\psi\ , &\quad (3.5.58b)\end{aligned}$$

with a nonlinearity

$$G=(G_\varphi,G_\psi)$$

of the form

$$G=x^{-p\delta}H(x^\mu,x^{q\delta}\psi_1,x^{q\delta+1}\psi_2,x^{q\delta+1}\varphi)\ . \tag{3.5.59}$$

Here we have decomposed ψ as

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\ ; \tag{3.5.60}$$

this is motivated by different *a priori* estimates we have at our disposal for various components of ψ in the applications we have in mind. Polyhomogeneity of solutions of (3.5.1) will follow as a special case, see Theorem 3.5.10 below. We will need to impose various restrictions on the function H, in order to do that some terminology will be needed. We shall say that a function $H(x^\mu,u)$ is *δ-polyhomogeneous in x with a uniform zero of order l in u* if H is smooth in $u\in\mathbb{R}^N$ at fixed x^μ, if H satisfies (3.2.30) for any $0\le i\le\min\{l,k\}$ and any $k\in\mathbb{N}$, if

$$\forall i\in\mathbb{N}\quad \partial_u^iH(\cdot,u)\in\mathscr{A}^\delta_\infty \tag{3.5.61}$$

at fixed constant u, and if we have the uniform estimate for constant u's

$$\forall\epsilon>0,M\ge0,i,k\in\mathbb{N}\ \exists\, C(\epsilon,M,i,k)\ \forall|u|\le M\quad \|\partial_u^iH(\cdot,u)\|_{\mathscr{C}_k^{-\epsilon}}\le C(\epsilon,M,i,k)\ . \tag{3.5.62}$$

The qualification "in u" in "uniform zero of order l in u" will often be omitted. The small parameter ϵ has been introduced above to take into account the possible logarithmic blow-up of functions in $\mathscr{A}^\delta_\infty$ at $x=0$; for the applications to the nonlinear scalar wave equation or to the wave map equation on Minkowski space-time, the alternative simpler requirement would actually suffice:

$$\forall\ M\ge0,i,k\in\mathbb{N}\ \ \exists\, C(M,i,k)\ \forall|u|\le M\quad \|\partial_u^iH(\cdot,u)\|_{\mathscr{C}^0_\infty}\le C(M,i,k)\ , \tag{3.5.63}$$

again for constant u's. Clearly functions which are jointly smooth in u and in x^μ satisfy the above conditions; Lemma 3.5.7 below provides another class of such functions. The following simple facts about functions in the above class will be useful:

Lemma 3.5.7 Let $m_1, m_2, k \in \mathbb{N}$, $m_1 \le m_2$, and let $P(x^\mu, u)$ be a polynomial in $u = (u^1, \dots u^N)$ of the form

$$P(x^\mu, u) = \sum_{m_1 \le j \le m_2} P_{i_1 \dots i_j}(x^\mu) u^{i_1} \dots u^{i_j} \ ,$$

with coefficients $P_{i_1 \dots i_j}(x^\mu) \in \mathscr{A}_k^\delta$. Then:

1. P is δ-polyhomogeneous in x with a uniform zero of order m_1.
2. If
$$f \in \mathscr{A}_k^\delta + \mathscr{C}_\infty^\lambda$$
for some $\lambda > 0$, then for any $\epsilon > 0$ we have
$$P(\cdot, x^{q\delta} f) \in x^{m_1 q\delta}(\mathscr{A}_k^\delta + \mathscr{C}_\infty^{\lambda - \epsilon}) \ .$$

The proof of Lemma 3.5.7 is elementary and will be left to the reader.

Lemma 3.5.8 Let $k, q \in \mathbb{N}$ and let $H(x^\mu, u)$ be δ-polyhomogeneous with respect to x with a zero of order m in u. If

$$f \in \begin{cases} \mathscr{A}_k^\delta \cap L^\infty + \mathscr{C}_\infty^\lambda \ , & q = 0 \ , \\ \mathscr{A}_k^\delta + \mathscr{C}_\infty^\lambda \ , & \text{otherwise,} \end{cases}$$

for some $\lambda > 0$, then for any $\epsilon > 0$

$$H(\cdot, x^{q\delta} f) \in x^{mq\delta}(\mathscr{A}_k^\delta + \mathscr{C}_\infty^{\lambda - \epsilon}) \ .$$

Proof: We Taylor-expand H in u to order r, where r is any number satisfying

$$rq\delta > mq\delta + \lambda \ .$$

We then have

$$H(x^\mu, x^{q\delta} f) = P(x^\mu, x^{q\delta} f) + R \ ,$$

where P is a polynomial and R is a remainder. We note that the coefficients of the expansion of P can be obtained by differentiating with respect to u and setting $u = 0$, and are therefore in $\mathscr{A}_\infty^\delta$ by (3.5.61). Further, the usual integral formula for the remainder in a Taylor expansion together with (3.5.62) shows that R has a uniform zero of order r, in the sense of Equation (3.2.30). The result follows from Lemma 3.5.7 and from Lemma 3.2.5. □

We are ready now to pass to the proof of the non-linear analogue of Theorem 3.4.3:

Theorem 3.5.9 Let $p \in \mathbb{Z}, q, 1/\delta \in \mathbb{N}, -1 < \beta' \in \mathbb{R}$, $k \in \mathbb{N} \cup \{\infty\}$, and let

$$(\varphi, \psi) \in \mathscr{C}^{\beta'}_{\infty}(\Omega_{x_0,T}) \times \mathscr{C}^{\beta'}_{\infty}(\Omega_{x_0,T}) \; , \quad \psi_1 \in L^\infty(\Omega_{x_0,T})$$

(ψ_1 as in Equation (3.5.60)), be a solution of (3.5.58) with G of the form (3.5.59), where H is δ-polyhomogeneous in x with a uniform zero of order

$$m > \frac{p - \frac{1}{\delta}}{q} \; . \tag{3.5.64}$$

Suppose that Equations (3.4.38)-(3.4.39) hold, and that

$$B_{11} \in \left(\mathscr{A}^\delta_k \cap L^\infty\right)(\Omega_{x_0,T}) \; , \qquad B_{12}, B_{22}, B_{21} \in \mathscr{A}^\delta_k(\Omega_{x_0,T}) \; , \tag{3.5.65a}$$

$$a, b \in \mathscr{A}^\delta_k(\Omega_{x_0,T}) \; , \qquad \varphi(0) \in \mathscr{A}^\delta_k(M_{x_0}) \; . \tag{3.5.65b}$$

Then

$$\varphi \in \left(x^{(mq-p)\delta}\mathscr{A}^\delta_k + \mathscr{A}^\delta_k\right)(\Omega_{x_0,T}) = x^{\min((mp-q)\delta,0)}\mathscr{A}^\delta_k(\Omega_{x_0,T}) \; ,$$

$$\psi \in x^{\min\{(mq-p)\delta+1,1\}}\mathscr{A}^\delta_k(\Omega_{x_0,T}) + C_\infty(\overline{\Omega_{x_0,T}}) \subset \left(\mathscr{A}^\delta_k \cap L^\infty\right)(\Omega_{x_0,T}) \; .$$

If one further assumes

$$L^\mu_{12}, B_{12}, a, \varphi(0), G_\varphi(\cdot, 0) \in L^\infty(\Omega_{x_0,T}) \; ,$$

then it also holds that

$$\varphi \in \left(x^{(mq-p)\delta}\mathscr{A}^\delta_k + \mathscr{A}^\delta_k \cap L^\infty\right)(\Omega_{x_0,T}) \; .$$

Remark: Obviously the theorem remains true if we replace G by a finite sum of nonlinearities satisfying the above hypotheses, with different p's and q's for each term of the sum.

PROOF: The result is established by a repetition of the proof of Theorem 3.4.3, using Lemma 3.2.5 and Lemma 3.5.8 to obtain the necessary estimates on the non-linear terms. We simply note that the condition on the order m of the non-linearity guarantees, using Lemma 3.2.5, that

$$\partial_x \psi = c_2 \in \mathscr{C}^{\lambda-\epsilon}_\infty \; ,$$

with

$$\lambda = \min\{\beta', mq\delta - p\delta\} > -1 \; ,$$

hence $\psi \in L^\infty$ by integration. Decreasing β' if necessary we may without loss of generality assume that $\beta' = \lambda$. When applying Lemma 3.5.8 it is convenient to view the function H as a function of the variable $f := (\psi_1, x\psi_2, x\varphi) \in L^\infty$. The remaining details are left to the reader. □

As a straightforward corollary of Theorem 3.5.9 one obtains:

Theorem 3.5.10 Let $\delta = 1$ in odd space dimensions, and let $\delta = 1/2$ in even space dimensions. Consider Equation (3.5.1) on $\mathbb{R}^{n,1}$, $n \geq 2$, with initial data

$$\widetilde{f}|_{\{\tau=0\}}\ , \quad \partial\widetilde{f}/\partial\tau|_{\{\tau=0\}} \in \left(\mathscr{A}^{\delta}_{\infty} \cap L^{\infty}\right)(M_{x_0})\ .$$

Suppose further that $H(x^\mu, f)$ is smooth in f at fixed x^μ, bounded and δ-polyhomogeneous in x^μ at constant f, and has a zero of order ℓ at $f = 0$, with ℓ as in (3.5.21). Then:

1. There exists $\tau_+ > 0$ such that f exists Ω_{x_0,τ_+}, with
$$\|\widetilde{f}\|_{L^\infty(\Omega_{x_0,\tau_+})}\ . \tag{3.5.66}$$

2. If the initial data are *compatible* polyhomogeneous in the sense that there exists $\lambda > -1$ such that
$$\forall i \in \mathbb{N} \qquad \partial^i_\tau\widetilde{f}(0) \in L^\infty(M_{x_0})\ , \quad \partial_x\partial^i_\tau\widetilde{f}(0) \in \mathscr{C}^{-\lambda}_{\infty}(M_{x_0})\ ,$$
then the solution is polyhomogeneous on each neighborhood Ω_{x_0,τ_*} of $\mathscr{I}^+$ on which f exists and satisfies (3.5.66) with τ_+ replaced by τ_*.

Proof: Point 1 is a Theorem 3.5.1 specialised to polyhomogeneous initial data. To prove point 2 we set

$$\psi = \begin{pmatrix} \psi_1 = \widetilde{f} \\ \psi_2 = \begin{pmatrix} \phi_- \\ \phi_A \end{pmatrix} \end{pmatrix}, \tag{3.5.67}$$

and

$$\varphi = \phi_+\ . \tag{3.5.68}$$

Then Equation (3.5.3) takes the form (3.5.58) with

$$G = -\Omega^{-\frac{n+3}{2}} H(x^\mu, \Omega^{\frac{n-1}{2}}\widetilde{f}) \equiv -\Omega^{-\frac{n+3}{2}} H(x^\mu, \Omega^{\frac{n-1}{2}}\psi_1)\ , \tag{3.5.69}$$

$$G_\varphi = -G \quad , \tag{3.5.70}$$

$$G_{\psi_1} = 0\ , \quad G_{\psi_2} = \begin{pmatrix} -G \\ 0 \end{pmatrix}\ . \tag{3.5.71}$$

For n even we take $\delta = 1/2$, $p = n + 3$, $q = n - 1$; the condition then (3.5.64) reads $m > \frac{n+1}{n-1}$, which coincides with (3.5.21). For n odd we take $\delta = 1$, $p = \frac{n+3}{2}$, $q = \frac{n-1}{2}$, and (3.5.21) guarantees again that (3.5.64) holds. □

3.6 Wave maps

Let $(\mathscr{N}, h)$ be a smooth Riemannian manifold, and let $f : (\mathscr{M}, \mathfrak{g}) \to (\mathscr{N}, h)$ solve the wave map equation. We will be interested in maps f which have the property that f approaches a constant map f_0 as r tends to infinity along lightlike directions, $f_0(x) = p_0 \in \mathscr{N}$ for all $x \in \mathscr{M}$. Introducing normal coordinates around p_0 we can write $f = (f^a)$, $a = 1, \dots, N = \dim \mathscr{N}$, with the functions f^a satisfying the set of equations

$$\Box_{\mathfrak{g}} f^a + \mathfrak{g}^{\mu\nu}\Gamma^a_{bc}(f)\frac{\partial f^b}{\partial x^\mu}\frac{\partial f^c}{\partial x^\nu} = 0 \,, \tag{3.6.1}$$

where the Γ^a_{bc}'s are the Christoffel symbols of the metric h. Setting as before $\widetilde{f}^a = \Omega^{-\frac{n-1}{2}} f^a$, $\tilde{\mathfrak{g}} = \Omega^2 \mathfrak{g}$, we then have from (3.1.3),

$$\Box_{\tilde{\mathfrak{g}}} \widetilde{f}^a = -\Omega^{-\frac{n-1}{2}} \tilde{\mathfrak{g}}^{\mu\nu}\Gamma^a_{bc}(\Omega^{\frac{n-1}{2}}\widetilde{f})\frac{\partial(\Omega^{\frac{n-1}{2}}\widetilde{f}^b)}{\partial x^\mu}\frac{\partial(\Omega^{\frac{n-1}{2}}\widetilde{f}^c)}{\partial x^\nu} + \frac{n-1}{4n}(\tilde{R} - R\Omega^{-2})\widetilde{f}^a \,. \tag{3.6.2}$$

In particular if $(\mathscr{M}, \mathfrak{g})$ is the Minkowski space-time (and if we use the same conformal transformation as in Section 3.1) we obtain a system of Equations (3.5.13)-(3.5.17) with $a_A = b_A = 0$, with the obvious replacements associated with $\widetilde{f} \to \widetilde{f}^a$, and with G in (3.5.17) replaced by

$$\begin{aligned} G^a \equiv & -\Gamma^a_{bc}(\Omega^{\frac{n-1}{2}}\widetilde{f})\Big\{\Omega^{\frac{n-1}{2}}(-\phi^b_+\phi^c_- + \phi^b_A\phi^c_A) \\ & - (n-1)\Omega^{\frac{n-3}{2}}\widetilde{f}^c\Big[\Big(x\phi^b_+ - (1+x+2\tau)\phi^b_-\Big) - (n-1)\widetilde{f}^b\Big]\Big\} \,. \end{aligned} \tag{3.6.3}$$

3.6.1 Existence of solutions, space derivatives estimates

As before, for even space-dimensions n the occurrence of non-integer powers of Ω above does not allow the use of the standard conformal method except for special target manifolds $(\mathscr{N}, h)$, *cf.* [11]. This can be handled in our approach, and we show:

Theorem 3.6.1 Consider Equation (3.6.1) on $\mathbb{R}^{n,1}$ with initial data given on a hyperboloid $\mathscr{S} \supset \Sigma_{x_0,0}$ in Minkowski space-time, and satisfying

$$\widetilde{f}^a|_{\Sigma_{x_0,0}} \equiv \Omega^{-\frac{n-1}{2}} f^a|_{\Sigma_{x_0,0}} \in \begin{cases} \left(\mathscr{H}^\alpha_{k+1} \cap L^\infty\right)(\Sigma_{x_0,0}) \,, & n \geq 3 \,, \\ \left(\mathscr{H}^\alpha_{k+1} \cap \mathscr{C}^0_1\right)(\Sigma_{x_0,0}) \,, & n = 2 \,, \end{cases} \tag{3.6.4}$$

$$\partial_x(\Omega^{-\frac{n-1}{2}} f^a)|_{\Sigma_{x_0,0}} \in \mathscr{H}^\alpha_k(\Sigma_{x_0,0}) \,, \tag{3.6.5}$$

$$\partial_\tau(\Omega^{-\frac{n-1}{2}} f^a)|_{\Sigma_{x_0,0}} \in \begin{cases} \mathscr{H}^\alpha_k(\Sigma_{x_0,0}) \,, & n \geq 3 \,, \\ \left(\mathscr{H}^\alpha_k \cap L^\infty\right)(\Sigma_{x_0,0}) \,, & n = 2 \,. \end{cases} \tag{3.6.6}$$

for some $k > \frac{n}{2} + 1$, $-1 < \alpha < -1/2$. Then:

1. There exists $\tau_+ > 0$ and a solution f^a of Equation (3.6.1), defined on a set containing Ω_{x_0,τ_+}, satisfying the given initial conditions, such that

$$\|\widetilde{f}^a\|_{\mathscr{C}^0_1(\Omega_{x_0,\tau_+})} < \infty \,, \qquad n = 2 \tag{3.6.7a}$$

$$\begin{aligned} & \|xe_+(\widetilde{f}^a)\|_{L^\infty(\Omega_{x_0,\tau_+})} + \sum_{i=1}^{r}\|xX_i\widetilde{f}^a\|_{L^\infty(\Omega_{x_0,\tau_+})} \\ & \quad + \|\widetilde{f}^a\|_{L^\infty(\Omega_{x_0,\tau_+})} + \|x\partial_\tau\widetilde{f}^a\|_{L^\infty(\Omega_{x_0,\tau_+})} < \infty \,, \quad n \geq 3 \,. \end{aligned} \tag{3.6.7b}$$

Here the X_i's are the vector fields defined in Section 3.2, *cf.* Equation (3.2.7).

2. Further, if τ_* is such that f^a exists on Ω_{x_0,τ_*} with (3.6.7) holding with $\tau_+ = \tau_*$, then for all $0 \le \tau < \tau_*$ we have
$$\widetilde{f}^a|_{\Sigma_{x_0,\tau}} \in L^\infty(\Sigma_{x_0,\tau}) \cap \mathscr{H}^\alpha_{k+1}(\Sigma_{x_0,\tau}) ,$$
$$\partial_\tau \widetilde{f}^a|_{\Sigma_{x_0,\tau}} \in \mathscr{H}^\alpha_k(\Sigma_{x_0,\tau}) , \qquad \partial_x \widetilde{f}^a|_{\Sigma_{x_0,\tau}} \in \mathscr{H}^\alpha_k(\Sigma_{x_0,\tau}) ,$$
uniformly in τ. If $n = 2$ we also have uniform bounds in the following spaces
$$\widetilde{f}^a|_{\Sigma_{x_0,\tau}} \in \left(\mathscr{C}^0_1 \cap \mathscr{H}^\alpha_{k+1}\right)(\Sigma_{x_0,\tau}) , \qquad \partial_\tau \widetilde{f}^a|_{\Sigma_{x_0,\tau}} \in \left(\mathscr{H}^\alpha_k \cap L^\infty\right)(\Sigma_{x_0,\tau}) .$$

Remark: Integration of condition (3.6.5) implies of course that $\widetilde{f} \in L^\infty(\Sigma_{x_0,0})$.

Proof: The proof is similar to that of Theorem 3.5.1, but simpler, because we do not need to gain a 1/2 in the decay rate, as done in Lemma 3.5.2. We write Equation (3.6.1) in the form (3.5.12)-(3.5.16), with $a_A = b_A = 0$ and with G in (3.5.17) replaced by G^a defined in (3.6.3). We write G^a as
$$G^a = A^a + B^a + C^a + D^a + E^a , \tag{3.6.8}$$
with the order of terms in (3.6.8) corresponding to that in (3.6.3). Since we are working in normal coordinates, Γ^a_{bc} has a uniform zero of order one in the sense of (3.2.30) at $f^a = 0$. We want to use Equation (3.4.20) to get an a-priori estimate for the solutions of (3.6.1); for this we shall need to estimate the $\mathscr{H}^\alpha_k$ norms of all the terms which occur in (3.6.8). The simplest such term is E^a:
$$\begin{aligned}\|E^a\|_{\mathscr{H}^\alpha_k} &\equiv (n-1)^2 \|\Gamma^a_{bc}(\Omega^{\frac{n-1}{2}}\widetilde{f})(\Omega^{\frac{n-1}{2}}\widetilde{f}^c)(\Omega^{\frac{n-1}{2}}\widetilde{f}^b)\Omega^{-1-\frac{n-1}{2}}\|_{\mathscr{H}^\alpha_k} \\ &\approx (n-1)^2 \|\Gamma^a_{bc}(\Omega^{\frac{n-1}{2}}\widetilde{f})(\Omega^{\frac{n-1}{2}}\widetilde{f}^c)(\Omega^{\frac{n-1}{2}}\widetilde{f}^b)\|_{\mathscr{H}^{\alpha+(n+1)/2}_k} ,\end{aligned}$$
where we have used the fact that Ω/x is a smooth, and therefore bounded, function. The function $\Gamma^a_{bc}(\Omega^{\frac{n-1}{2}}\widetilde{f})(\Omega^{\frac{n-1}{2}}\widetilde{f}^c)(\Omega^{\frac{n-1}{2}}\widetilde{f}^b)$ can be viewed as a smooth function F of $x^{\frac{n-1}{2}}\widetilde{f}^a$ with a uniform zero of order three. We can thus apply (3.2.31) with $l = 3$ to obtain
$$\begin{aligned}\|E(s)\|_{\mathscr{H}^\alpha_k} &\le C(\|\widetilde{f}(s)\|_{L^\infty})\|\widetilde{f}\|_{\mathscr{H}^{\alpha+2-n}_k} \\ &\le C(\|\widetilde{f}(s)\|_{L^\infty})\|\widetilde{f}\|_{\mathscr{H}^\alpha_k} ,\end{aligned} \tag{3.6.9}$$
since $n \ge 2$. We note that in dimensions larger than or equal to three we have at least one power of x "left unused" above, which will be made use of in estimating the remaining contributions to G^a. We proceed in a similar way with the other terms; in space dimension $n = 2$ we view $D^a \equiv (n-1)(1 + x + 2\tau)\Omega^{\frac{n-3}{2}}\Gamma^a_{bc}(\Omega^{\frac{n-1}{2}}\widetilde{f})\widetilde{f}^c\phi^b_-$ as a smooth function F with a uniform zero of order three of $(x^{\frac{n-1}{2}}\widetilde{f}^a, x^{\frac{n-1}{2}}\phi^a_-)$, which leads to the estimate
$$\|D(s)\|_{\mathscr{H}^\alpha_k} \le C(\|\widetilde{f}(s)\|_{L^\infty}, \|\phi_-(s)\|_{L^\infty})\left(\|\widetilde{f}\|_{\mathscr{H}^\alpha_k} + \|\phi_-(s)\|_{\mathscr{H}^\alpha_k}\right) \tag{3.6.10}$$
On the other hand, in dimension 3 or higher we can view D^a as a function F with a uniform zero of order three of $(x^{\frac{n-1}{2}}\widetilde{f}^a, x^{\frac{n-1}{2}}x\phi^a_-)$, which implies
$$\|D(s)\|_{\mathscr{H}^\alpha_k} \le C(\|\widetilde{f}(s)\|_{L^\infty}, \|x\phi_-(s)\|_{L^\infty})\left(\|\widetilde{f}\|_{\mathscr{H}^\alpha_k} + \|x\phi_-(s)\|_{\mathscr{H}^\alpha_k}\right) \tag{3.6.11}$$

Regardless of dimension we view $C^a \equiv (n-1)x\Omega^{\frac{n-3}{2}}\Gamma^a_{bc}(\Omega^{\frac{n-1}{2}}\widetilde{f})\widetilde{f}^c\phi^b_+$ as a smooth function with a uniform zero or order three of $(x^{\frac{n-1}{2}}\widetilde{f}^a, x^{\frac{n-1}{2}}x\phi^a_+)$, obtaining thus

$$\|C(s)\|_{\mathscr{H}^\alpha_k} \leq C(\|\widetilde{f}(s)\|_{L^\infty}, \|x\phi_+(s)\|_{L^\infty})\left(\|\widetilde{f}\|_{\mathscr{H}^\alpha_k} + \|x\phi_+(s)\|_{\mathscr{H}^\alpha_k}\right) \tag{3.6.12}$$

Viewing B^a as a function of $(x^{\frac{n-1}{2}}\widetilde{f}^a, x^{\frac{n-1}{2}}x\phi^a_A)$, and viewing A^a as a function of $(x^{\frac{n-1}{2}}\widetilde{f}^a, x^{\frac{n-1}{2}}x\phi^a_-, x^{\frac{n-1}{2}}x\phi^a_+)$, one similarly obtains for $n \geq 3$

$$\begin{aligned}\|A(s)\|_{\mathscr{H}^\alpha_k} &\leq C(\|\widetilde{f}(s)\|_{L^\infty}, \|x\phi_-(s)\|_{L^\infty}, \|x\phi_+(s)\|_{L^\infty}) \times \\ &\qquad \left(\|\widetilde{f}\|_{\mathscr{H}^\alpha_k} + \|x\phi_-(s)\|_{\mathscr{H}^\alpha_k} + \|x\phi_+(s)\|_{\mathscr{H}^\alpha_k}\right), \qquad (3.6.13)\\ \|B(s)\|_{\mathscr{H}^\alpha_k} &\leq C(\|\widetilde{f}(s)\|_{L^\infty}, \|x\phi_A(s)\|_{L^\infty})\left(\|\widetilde{f}\|_{\mathscr{H}^\alpha_k} + \|x\phi_A(s)\|_{\mathscr{H}^\alpha_k}\right), \qquad (3.6.14)\end{aligned}$$

while in dimension 2 it holds that

$$\begin{aligned}\|A(s)\|_{\mathscr{H}^\alpha_k} &\leq C(\|\widetilde{f}(s)\|_{L^\infty}, \|\phi_-(s)\|_{L^\infty}, \|x\phi_+(s)\|_{L^\infty}) \times \\ &\qquad \left(\|\widetilde{f}\|_{\mathscr{H}^\alpha_k} + \|\phi_-(s)\|_{\mathscr{H}^\alpha_k} + \|x\phi_+(s)\|_{\mathscr{H}^\alpha_k}\right). \qquad (3.6.15)\\ \|B(s)\|_{\mathscr{H}^\alpha_k} &\leq C(\|\widetilde{f}(s)\|_{L^\infty}, \|\phi_A(s)\|_{L^\infty})\left(\|\widetilde{f}\|_{\mathscr{H}^\alpha_k} + \|\phi_A(s)\|_{\mathscr{H}^\alpha_k}\right), \qquad (3.6.16)\end{aligned}$$

Summarizing, in space dimension two we have obtained

$$\begin{aligned}\|G(s)\|_{\mathscr{H}^\alpha_k} &\leq C(\|\widetilde{f}(s)\|_{L^\infty}, \|\phi_-(s)\|_{L^\infty}, \|\phi_A(s)\|_{L^\infty}, \|x\phi_+(s)\|_{L^\infty}) \times \\ &\qquad \left(\|\widetilde{f}\|_{\mathscr{H}^\alpha_k} + \|\phi_-(s)\|_{\mathscr{H}^\alpha_k} + \|x\phi_+(s)\|_{\mathscr{H}^\alpha_k} + \|\phi_A(s)\|_{\mathscr{H}^\alpha_k}\right) \\ &\leq C(\|\widetilde{f}(s)\|_{L^\infty}, \|\phi_-(s)\|_{L^\infty}, \|\phi_A(s)\|_{L^\infty}, \|x\phi_+(s)\|_{L^\infty}) \times \\ &\qquad \sqrt{E_\alpha(s)}, \qquad (3.6.17)\end{aligned}$$

where

$$\begin{aligned}E_\alpha(t) &= \|\widetilde{f}(t)\|^2_{\mathscr{H}^\alpha_k} + \|\phi_-(t)\|^2_{\mathscr{H}^\alpha_k} \\ &\quad + \|\phi_+(t)\|^2_{\mathscr{H}^\alpha_k} + \sum_A \|\phi_A(t)\|^2_{\mathscr{H}^\alpha_k}. \qquad (3.6.18)\end{aligned}$$

On the other hand in higher dimensions we can write

$$\begin{aligned}\|G(s)\|_{\mathscr{H}^\alpha_k} &\leq C(\|\widetilde{f}(s)\|_{L^\infty}, \|x\phi_A(s)\|_{L^\infty}, \|x\phi_-(s)\|_{L^\infty}, \|x\phi_+(s)\|_{L^\infty}) \times \\ &\qquad \sqrt{E_\alpha(s)}. \qquad (3.6.19)\end{aligned}$$

To obtain a closed inequality from Equations (3.4.20) and (3.6.17) or (3.6.19), we need to control all the L_∞ norms occuring there. Since $k > n/2+1$, from Equation (3.6.17) and the weighted Sobolev embeddings we obtain

$$\|G(s)\|_{\mathscr{C}^\alpha_1} \leq C(\|\widetilde{f}(s)\|_{L^\infty}, \|\phi_-(s)\|_{L^\infty}, \|\phi_A(s)\|_{L^\infty}, E_\alpha(s)), \tag{3.6.20}$$

in $n=2$, or — from (3.6.19) —

$$\|G(s)\|_{\mathscr{C}^\alpha_1} \leq C(\|\widetilde{f}(s)\|_{L^\infty}, E_\alpha(s)), \tag{3.6.21}$$

for $n \geq 3$. The identity

$$\widetilde{f}^a(\tau,x) = \widetilde{f}^a(\tau,x_0-2\tau) - \frac{1}{2}\int_x^{x_0-2\tau}(\phi_-^a+\phi_+^a)(\tau,s)\,ds \qquad (3.6.22)$$

yields

$$\begin{aligned}\|\widetilde{f}(s)\|_{L^\infty} &\leq C\left(\sqrt{E_\alpha(0)}+\|\phi_-(s)\|_{\mathscr{C}_0^\alpha}+\|\phi_+(s)\|_{\mathscr{C}_0^\alpha}\right)\\ &\leq C\left(\sqrt{E_\alpha(0)}+\sqrt{E_\alpha(s)}\right)\ . \qquad (3.6.23)\end{aligned}$$

for $n \geq 3$, while if $n = 2$ we use the estimate

$$\begin{aligned}\|\widetilde{f}(s)\|_{L^\infty}+\|\phi_A(s)\|_{L^\infty} &\leq C\left(\sqrt{E_\alpha(0)}+\|\phi_-(s)\|_{\mathscr{C}_1^\alpha}+\|\phi_+(s)\|_{\mathscr{C}_1^\alpha}\right)\\ &\leq C\left(\sqrt{E_\alpha(0)}+\sqrt{E_\alpha(s)}\right)\ . \qquad (3.6.24)\end{aligned}$$

In Equations (3.6.23)-(3.6.24) we have estimated $\widetilde{f}^a(\tau,x_0-2\tau)$ and its angular derivatives by a multiple of the initial energy $E_\alpha(0)$, which is justified for $\tau \leq \tau_* < x_0/2$. If $n \geq 3$ Equations (3.4.20), (3.6.21) and (3.6.23) give

$$E_\alpha(\tau) \leq CE_\alpha(0) + \int_0^\tau \Phi\left(E_\alpha(s)\right)\,ds\ , \qquad (3.6.25)$$

for some constant C, and for a function Φ which is bounded on bounded sets, and we conclude as in the proof of Theorem 3.5.1.

If $n = 2$, we note the identity

$$\phi_-(\tau,x) = \phi_-(0,x+2\tau) + \int_0^\tau e_+(\phi_-)(\sigma,2(\tau-\sigma)+x)\,d\sigma\ . \qquad (3.6.26)$$

From the second of Equations (3.5.14) we obtain

$$|e_+(\phi_-)(s,x)| \leq C\left(\|\phi_-(s)\|_{\mathscr{C}_0^\alpha}+\|\phi_A(s)\|_{\mathscr{C}_1^\alpha}+\|\phi_+(s)\|_{\mathscr{C}_0^\alpha}+\|G(s)\|_{\mathscr{C}_0^\alpha}\right)x^\alpha\ ,$$

so that

$$\begin{aligned}|\phi_-(\tau,x)| \leq\ & \|\phi_-(0)\|_{L^\infty} + C\int_0^\tau\left(\|\phi_-(\sigma)\|_{\mathscr{C}_0^\alpha}+\|\phi_A(\sigma)\|_{\mathscr{C}_1^\alpha}+\|\phi_+(\sigma)\|_{\mathscr{C}_0^\alpha}\right.\\ &\left.+\|G(\sigma)\|_{\mathscr{C}_0^\alpha}\right)(2(\tau-\sigma)+x)^\alpha\,d\sigma\ . \qquad (3.6.27)\end{aligned}$$

It follows that

$$\|\phi_-(\tau)\|_{L^\infty} \leq \|\phi_-(0)\|_{L^\infty} + C\int_0^\tau\left(\sqrt{E_\alpha(\sigma)}+\|G(\sigma)\|_{\mathscr{C}_0^\alpha}\right)(\tau-\sigma)^\alpha d\sigma \qquad (3.6.28)$$

Let

$$F(s) \equiv \|\widetilde{f}(s)\|_{L^\infty}+\|\phi_-(s)\|_{L^\infty}+\|\phi_A(s)\|_{L^\infty}+\sqrt{E_\alpha(s)}\ . \qquad (3.6.29)$$

It follows from (3.4.20), (3.6.24) and (3.6.28) that we have

$$F(\tau) \leq CF(0) + \int_0^\tau \Phi(F(s))\left(1+(\tau-\sigma)^\alpha\right)\,d\sigma\ , \qquad (3.6.30)$$

where Φ is a function bounded on bounded sets. We have the following:

Lemma 3.6.2 There exists a time τ_*, depending only upon C, $F(0)$, and the function Φ, such that any positive continuous function $F : [0, \tau_+) \to \mathbb{R}$ satisfying the inequality (3.6.30) with $\alpha > -1$ is bounded from above by $CF(0) + 1$ on $[0, \max(\tau_+, \tau_*))$.

PROOF: Let

$$M = \sup_{0 \le x \le CF(0)+1} |\Phi(x)| \ ;$$

if $M = 0$ the result is obviously true, so assume that $M \neq 0$. ¿From Equation (3.6.30) we obtain that on any interval $[0, \tau)$ on which $F \le CF(0) + 1$ we have

$$F(\tau) \le CF(0) + \int_0^\tau M\left(1 + (\tau - \sigma)^\alpha\right)\, d\sigma = CF(0) + M\left(\tau + \frac{\tau^{\alpha+1}}{\alpha+1}\right) .$$

(Equation (3.6.30) with $\tau = 0$ shows that $CF(0) \ge F(0)$, and continuity of F implies that the set of such intervals is non-empty.) The result is established by choosing

$$\tau_* = \min\left(\frac{1}{2M}, \left[\frac{\alpha+1}{2M}\right]^{1/(\alpha+1)}\right) .$$

□

Because the existence time τ_* in Theorem 3.6.1 does not depend upon x_1, Theorem 3.6.1 with $n = 2$ follows again by an argument identical to the one given at the end of Theorem 3.5.1. □

As in the case of the nonlinear wave equation (3.5.1), in order to obtain time derivative estimates we shall need a more general version of Theorem 3.6.1. Thus, we consider systems of the form (3.5.39)-(3.5.41) with a rather more general form of the non-linearity G appearing there. It should be clear from the proof of Theorem 3.6.1 that the case $n = 2$ needs separate treatment; in this thesis we will only consider dimensions $n \ge 3$; similar results hold in dimension $n = 2$ for equations with a nonlinearity of higher order:

Theorem 3.6.3 Let $n \ge 3$ and consider the system (3.5.39)-(3.5.40) with

$$\begin{aligned} &\|a(\tau)\|_{\mathscr{H}_k^\alpha} + \|b(\tau)\|_{\mathscr{H}_k^\alpha} + \sup_{a,b=1,2} \|B_{ab}(\tau)\|_{\mathscr{C}_k^0} \\ &\quad + \|B_0(\tau)\|_{\mathscr{C}_k^0} + \|B_0^{-1}(\tau)\|_{L^\infty} + \|B_1(\tau)\|_{\mathscr{C}_k^0} \le \tilde{C} \ , \end{aligned} \tag{3.6.31}$$

for some constant $\tilde{C}$, with the nonlinearity G in Equation (3.5.39a) of the form

$$G = x^{-(n+3)/2} H(x^\mu, x^{(n-1)/2}\widetilde{f}, x^{(n-1)/2} x\phi_A, x^{(n-1)/2} x\phi_+, x^{(n-1)/2} x\phi_-) \ , \tag{3.6.32}$$

with $G_{e_-(\phi_A)} = 0$ (*cf.* Equation (3.5.42)), and with H having a uniform zero of order $\ell \ge 3$ in the sense of (3.2.30). Suppose that the initial data satisfy

$$\widetilde{f}^a|_{\Sigma_{x_0,0}} \equiv \Omega^{-\frac{n-1}{2}} f^a|_{\Sigma_{x_0,0}} \ \in \ \left(\mathscr{H}_{k+1}^\alpha \cap L^\infty\right)(\Sigma_{x_0,0}) \ , \tag{3.6.33}$$

$$\partial_x \widetilde{f}^a|_{\Sigma_{x_0,0}} \ \in \ \mathscr{H}_k^\alpha(\Sigma_{x_0,0}) \ , \tag{3.6.34}$$

$$\partial_\tau \widetilde{f}^a|_{\Sigma_{x_0,0}} \ \in \ \mathscr{H}_k^\alpha(\Sigma_{x_0,0}) \ , \tag{3.6.35}$$

with some $k > \frac{n}{2} + 1$, $-1 < \alpha < -1/2$, then:

1. There exists $\tau_+ > 0$, depending only upon the constant $\tilde{C}$ in (3.6.31) and a bound on the norms of the initial data in the spaces appearing in Equations (3.6.33)-(3.6.35), and a solution f^a of Equations (3.5.39)-(3.5.40), defined on a set containing Ω_{x_0,τ_+}, satisfying the given initial conditions, such that

$$\|xe_+(\widetilde{f}^a)\|_{L^\infty(\Omega_{x_0,\tau_+})} + \sum_{i=1}^{r} \|xX_i\widetilde{f}^a\|_{L^\infty(\Omega_{x_0,\tau_+})} + \|\widetilde{f}^a\|_{L^\infty(\Omega_{x_0,\tau_+})} + \|x\partial_\tau \widetilde{f}^a\|_{L^\infty(\Omega_{x_0,\tau_+})} < \infty \,. \tag{3.6.36}$$

2. Further, if τ_* is such that f^a exists on Ω_{x_0,τ_*} with (3.6.36) holding with $\tau_+ = \tau_*$, then for all $0 \le \tau < \tau_*$ we have

$$\widetilde{f}^a|_{\Sigma_{x_0,\tau}} \in L^\infty(\Sigma_{x_0,\tau}) \cap \mathscr{H}^\alpha_{k+1}(\Sigma_{x_0,\tau}) \,, \tag{3.6.37a}$$

$$\partial_\tau \widetilde{f}^a|_{\Sigma_{x_0,\tau}} \in \mathscr{H}^\alpha_k(\Sigma_{x_0,\tau}) \,, \tag{3.6.37b}$$

$$\partial_x \widetilde{f}^a|_{\Sigma_{x_0,\tau}} \in \mathscr{H}^\alpha_k(\Sigma_{x_0,\tau}) \,, \tag{3.6.37c}$$

with uniform bounds in τ; this implies

$$\|x\partial_\tau\phi_+\|_{L^\infty(\Omega_{x_0,\tau_*})} + \|x\partial_\tau\phi_A\|_{L^\infty(\Omega_{x_0,\tau_*})} + \|(x+2\tau)\partial_\tau\widetilde{f}^a\|_{L^\infty(\Omega_{x_0,\tau_*})} < \infty \,. \tag{3.6.38}$$

If $k > n/2 + 2$ then we also have

$$\|x(x+2\tau)\partial_\tau\phi_-\|_{L^\infty(\Omega_{x_0,\tau_*})} < \infty \,. \tag{3.6.39}$$

Proof: The transition from Theorem 3.6.1 to Theorem 3.6.3 is rather similar to that from Theorem 3.5.1 to Theorem 3.5.3. We note that the estimates done in the course of the proof of Theorem 3.6.1, with $n \ge 3$ there, can be summed up in the inequality

$$\|x^{-(n+1)/2}H(x^\mu, x^{(n-1)/2}\hat{f})\|_{\mathscr{H}^\alpha_k} \le C(\|\hat{f}\|_{L^\infty})\|\hat{f}\|_{\mathscr{H}^\alpha_k} \,, \tag{3.6.40}$$

where

$$\hat{f} := (\widetilde{f}, x\phi_A, x\phi_+, x\phi_-) \,.$$

The minor modifications of the proof of Theorem 3.6.1 needed to obtain (3.6.37) and the estimate (3.6.38) on $(x+2\tau)\partial_\tau\widetilde{f}$ are identical to the ones described in the proof of Theorem 3.5.3. The estimate on $\|x\partial_\tau\phi_+\|_{L^\infty(\Omega_{x_0,\tau_*})}$ is obtained directly from Equation (3.5.50) and from (3.6.40). The estimate on $\|x\partial_\tau\phi_A\|_{L^\infty(\Omega_{x_0,\tau_*})}$ is obtained from the (3.5.39a)–equivalent of the first of Equations (3.5.14). Next, for $k > n/2 + 2$ Equations (3.5.43) and (3.6.40) give

$$e_+(\phi_-) \in \mathscr{H}^\alpha_{k-1} \subset \mathscr{C}^\alpha_1 \,, \tag{3.6.41}$$

Differentiating Equation (3.6.26) with respect to x gives

$$\partial_x\phi_-(\tau,x) = \partial_x\phi_-(0,x+2\tau) + \int_0^\tau \left(\partial_x e_+(\phi_-)\right)(\sigma, 2(\tau-\sigma)+x)\,d\sigma \,, \tag{3.6.42}$$

which together with (3.6.41) implies, by straightforward integration,

$$x(x+2\tau)|\partial_x\phi_-(\tau,x)| \le C \tag{3.6.43}$$

This, (3.6.41), and the identity

$$\partial_\tau\phi_- = (\partial_\tau - 2\partial_x + 2\partial_x)\phi_- = e_+(\phi_-) + 2\partial_x\phi_-$$

establish (3.6.39). □

3.6.2 Estimates on the time derivatives of the solutions

To control the time derivatives of the solutions, as in Section 3.5.2 we inroduce an index m which counts the number of corner conditions which are eventually satisfied by the initial data at the "corner" $\tau = x = 0$. As before we make a formal statement only for solutions of the wave-map equation (3.6.1), it should be clear from the proof that an analogous statement holds for solutions of (3.5.39)-(3.5.40) under appropriate conditions on the coefficients there.

Theorem 3.6.4 Let $\mathbb{N} \ni m \ge 0$, consider a solution $f : \Omega_{x_0,\tau_*} \to \mathbb{R}$ of Equation (3.6.1) satisfying

$$\begin{aligned}\|xe_+(\widetilde{f}^a)\|_{L^\infty(\Omega_{x_0,\tau_*})} &+ \sum_{i=1}^r \|xX_i\widetilde{f}^a\|_{L^\infty(\Omega_{x_0,\tau_*})} \\ &+\|\widetilde{f}^a\|_{L^\infty(\Omega_{x_0,\tau_*})} + \|x\partial_\tau\widetilde{f}^a\|_{L^\infty(\Omega_{x_0,\tau_*})} < \infty\ ,\end{aligned} \tag{3.6.44}$$

and suppose that

$$0 \le i \le m+1 \qquad \partial_\tau^i\widetilde{f}^a|_{\Sigma_{x_0,0}} \in \mathscr{H}^\alpha_{k+m+1-i}(\Sigma_{x_0,0})\ , \tag{3.6.45}$$

$$0 \le i \le m \qquad \partial_x\partial_\tau^i\widetilde{f}^a|_{\Sigma_{x_0,0}} \in \mathscr{H}^\alpha_{k+m-i}(\Sigma_{x_0,0})\ , \tag{3.6.46}$$

with some $k > \frac{n}{2}+2$, $-1 < \alpha < -1/2$. Then for $0 \le \tau < \tau_*$ and for $0 \le i \le m$, we have

$$\begin{aligned}&0 \le j+i < k+m-n/2 \\ &\qquad [(\tau+2x)\partial_\tau]^j\partial_\tau^i\widetilde{f}^a|_{\Sigma_{x_0,\tau}} \in L^\infty(\Sigma_{x_0,\tau}) \cap \mathscr{H}^\alpha_{k+m+1-i-j}(\Sigma_{x_0,\tau}),\end{aligned} \tag{3.6.47a}$$

$$\begin{aligned}&0 \le j+i < k+m-n/2-1 \\ &\qquad \partial_x[(\tau+2x)\partial_\tau]^j\partial_\tau^i\widetilde{f}^a|_{\Sigma_{x_0,\tau}} \in \mathscr{H}^\alpha_{k+m-i-j}(\Sigma_{x_0,\tau})\ ,\end{aligned} \tag{3.6.47b}$$

and

$$0 \le p < k-n/2 \qquad [(\tau+2x)\partial_\tau]^p\partial_\tau^{m+1}\widetilde{f}^a|_{\Sigma_{x_0,\tau}} \in \mathscr{H}^\alpha_{k-p}(\Sigma_{x_0,\tau})\ , \tag{3.6.48}$$

with τ-independent bounds on the norms.

Proof: The proof is an inductive application of Theorem 3.6.3, as in the proof of Theorem 3.5.4, and will be omitted. □

3.6.3 Polyhomogeneous solutions

Theorem 3.6.5 Let $\delta = 1$ in odd space dimensions, and let $\delta = 1/2$ in even space dimensions. Consider Equation (3.6.1) on $\mathbb{R}^{n,1}$, $n \geq 3$, with initial data

$$\widetilde{f}^a|_{\{\tau=0\}} \ , \quad \partial \widetilde{f}^a/\partial\tau|_{\{\tau=0\}} \in \mathscr{A}^\delta_\infty(M_{x_0}) \ .$$

Then:

1. There exists $\tau_+ > 0$ such that f^a exists on Ω_{x_0,τ_+}, with
$$\|\widetilde{f}\|_{L^\infty(\Omega_{x_0,\tau_+})} \ . \tag{3.6.49}$$

2. If the initial data are *compatible* polyhomogeneous in the sense that there exists $\lambda > -1$ such that
$$\forall i \in \mathbb{N} \qquad \partial^i_\tau \widetilde{f}^a(0) \in L^\infty(M_{x_0}) \ , \quad \partial_x \partial^i_\tau \widetilde{f}^a(0) \in \mathscr{C}^{-\lambda}_\infty(M_{x_0}) \ ,$$
then the solution is polyhomogeneous on each neighbourhood Ω_{x_0,τ_*} of $\mathscr{I}^+$ on which f exists and satisfies (3.6.49) with τ_+ replaced with $\tau*$.

PROOF: Existence of solutions follows from Theorem 3.6.1. Theorem 3.6.4 gives the time-derivative estimates which are necessary in Theorem 3.5.9. In order to apply that last theorem, we set

$$\varphi = \begin{pmatrix} \phi^c_+ \\ \phi^c_A \end{pmatrix} \ , \tag{3.6.50}$$

and

$$\psi_1 = (\widetilde{f}^c) \ , \quad \psi_2 = (\phi^c_-) \ . \tag{3.6.51}$$

Equation (3.6.2) takes then the form (3.5.58). As in Theorem 3.5.10, for n even we take $\delta = 1/2$, $p = n+3$, $q = n-1$; while for n odd we take $\delta = 1$, $p = \frac{n+3}{2}$, $q = \frac{n-1}{2}$. The non-linearity here is of order 3, which is compatible with the hypotheses of Theorem 3.5.9, and the result follows by that last theorem. □

Chapter 4

Einstein equations - the setup

4.1 Weyl connections in a doubly null frame

Consider any field of vectors e_i, $i = 1, \dots, 4$, such that

$$(g_{ij}) := (g(e_i, e_j)) = \begin{pmatrix} \delta^a_b & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix} , \tag{4.1.1}$$

where indices i, j *etc.* run from 1 to 4, while indices a, b *etc.* run from 1 to 2. One therefore has

$$(g^{ij}) := g(\theta^i, \theta^j) = \begin{pmatrix} \delta^a_b & 0 & 0 \\ 0 & 0 & -1/2 \\ 0 & -1/2 & 0 \end{pmatrix} ,$$

where θ^i is a basis of $T^*\mathscr{M}$ dual to e_i. If α_i, $i = 1, \cdots, 4$, is a usual Lorentzian orthonormal basis of $T\mathscr{M}$,

$$g(\alpha_i, \alpha_j) = \eta_{ij} = \mathrm{diag}(+1, +1, +1, -1) ,$$

then a basis e_i as above can be constructed by setting

$$e_a = \alpha_a , \quad e_3 = \alpha_3 + \alpha_4 , \quad e_4 = \alpha_4 - \alpha_3 .$$

Let Vol_g be the Lorentzian volume element of g, with the associated completely anti-symmetric tensor ϵ_{ijkl}:

$$\mathrm{Vol}_g = \beta^1 \wedge \beta^2 \wedge \beta^3 \wedge \beta^4 = \frac{1}{4!} \epsilon_{ijkl}\, \beta^i \wedge \beta^j \wedge \beta^k \wedge \beta^l ,$$

where β^i is a basis dual to α_j. We have $\theta^3 = (\beta^3 + \beta^4)/2$, $\theta^4 = (\beta^4 - \beta^3)/2$, $\beta^3 = \theta^3 - \theta^4$, $\beta^4 = \theta^3 + \theta^4$, hence

$$\mathrm{Vol}_g = 2\theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 = \frac{1}{4!} \epsilon_{ijkl}\, \theta^i \wedge \theta^j \wedge \theta^k \wedge \theta^l .$$

It follows that in the basis e_i the entries of the ϵ tensor are zeros and twos:

$$\epsilon_{1234} = 2 . \tag{4.1.2}$$

We let

$$\mathcal{S} = \mathrm{Vect}(\{e_1, e_2\}) ,$$

where $\mathrm{Vect}(X)$ denotes the vector space spanned by the elements of the set X. For any connection $\widehat{D}$ we define the connection coefficients $\widehat{\Gamma}_i{}^j{}_k$ by the formula

$$\widehat{\Gamma}_i{}^j{}_k := \theta^j(\widehat{D}_{e_i} e_k) ,$$

so that

$$\widehat{D}_{e_i} e_k = \widehat{\Gamma}_i{}^j{}_k e_j .$$

Let D be the Levi-Civita connection of the metric g, for any vector field b set

$$S(b)_i{}^k{}_\ell = \delta^k_i b_\ell + \delta^k_\ell b_i - g_{i\ell} g^{kj} b_j ,$$

and let the connection $\widehat{D}$ be defined as

$$\widehat{D} = D + S(f) \, , \tag{4.1.3}$$

for a vector field f which will be defined later. This equation has to be understood as follows: if $\Gamma_i{}^j{}_k$ are the connection coefficients of D, then

$$\widehat{\Gamma}_i{}^j{}_k = \Gamma_i{}^j{}_k + S(f)_i{}^j{}_k \, .$$

We note that f can be obtained from the $\widehat{\Gamma}_i{}^j{}_k$'s as follows:

$$f_i = \frac{1}{4}\widehat{\Gamma}_i{}^k{}_k \, .$$

The connection $\widehat{D}$ has no torsion,

$$\widehat{D}_{e_i}e_k - \widehat{D}_{e_k}e_i = [e_i, e_k] \, ,$$

however it is *not* metric compatible:

$$\widehat{D}_i g_{jk} \equiv (\widehat{D}_{e_i}g)(e_j, e_k) = -\widehat{\Gamma}_{ijk} - \widehat{\Gamma}_{ikj} = -2f_i g_{jk} \, . \tag{4.1.4}$$

Here and elsewhere,

$$\widehat{\Gamma}_{ijk} := g_{jm}\widehat{\Gamma}_i{}^m{}_k \, .$$

This shows that the usual anti-symmetry property $\widehat{\Gamma}_{ijk} = -\widehat{\Gamma}_{ikj}$ of the connection coefficients fails to hold for the following components of $\widehat{\Gamma}$:

$$\widehat{\Gamma}_{i34} = -\widehat{\Gamma}_{i43} - 4f_i \, , \quad \widehat{\Gamma}_{iab} = -\widehat{\Gamma}_{iba} + 2f_i\delta_{ab} \, . \tag{4.1.5}$$

In particular, the coefficients

$$\widehat{\Gamma}_{i11} = \widehat{\Gamma}_{i22} = f_i \tag{4.1.6}$$

do not vanish. However, both $\widehat{\Gamma}_{i33} = -2\widehat{\Gamma}_i{}^4{}_3$ and $\widehat{\Gamma}_{i44} = -2\widehat{\Gamma}_i{}^3{}_4$ vanish. For further use we note the identity

$$\widehat{D}_i g^{jk} = 2f_i g^{jk} \, , \tag{4.1.7}$$

which follows from (4.1.4).

The *null second fundamental forms* of a codimension two submanifold S are the two symmetric tensors on S defined as

$$\chi(X,Y) = g(D_X e_4, Y) \, , \qquad \underline{\chi}(X,Y) = g(D_X e_3, Y) \, , \tag{4.1.8}$$

where D is the Levi-Civita connection of $(\mathscr{M}, g)$, while X, Y are tangent to S. The *torsion* of S is a 1-form on S, defined for vector fields X tangent to S by the formula

$$\zeta(X) = -\frac{1}{2}g(D_X e_3, e_4) = \frac{1}{2}g(D_X e_4, e_3) \, . \tag{4.1.9}$$

In those definitions it is also assumed that e_3 and e_4 are normal to S, so that $\mathcal{S}$ coincides, over S, with the distribution TS of the planes tangent to

S. (Throughout the indices are raised and lowered with the metric g.) We shall need generalizations of those equations to the case when the connection is not metric compatible, and when $\mathcal{S}$ is not necessarily the tangent space to a submanifold. Thus, we define the hatted counterparts of the above objects as follows: for $X, Y \in \mathcal{S}$ we set

$$\hat\chi(X,Y) = g(\widehat{D}_X e_4, Y) \;, \qquad \underline{\hat\chi}(X,Y) = g(\widehat{D}_X e_3, Y) \;, \tag{4.1.10}$$

$$\hat\zeta(X) = -\frac{1}{2} g(\widehat{D}_X e_3, e_4) \;, \qquad \underline{\hat\zeta}(X) = -\frac{1}{2} g(\widehat{D}_X e_4, e_3) \;. \tag{4.1.11}$$

The fields $\chi, \underline{\chi}, \zeta, \underline{\zeta}$ are now defined in exactly the same way, with $\widehat{D}$ replaced by D. Since D is metric compatible we have $g(D_X e_3, e_4) = -g(D_X e_4, e_3)$ so that there is no real need of introducing an unhatted $\underline{\hat\zeta}$, but a) this antisymmetry property will not be true for general $\widehat{D}$; b) the unhatted equivalent of Equations (4.1.15) below looses its manifest symmetry under the exchange of e_3 with e_4, when $\underline{\zeta}$ is replaced by $-\zeta$. We stress that in Equations (4.1.10)-(4.1.11) we do not assume that $\mathcal{S}$ is integrable, so that the vector fields X and Y above are not necessarily tangent to some submanifold. While χ and $\underline{\chi}$ are symmetric (as tensor fields on S), $\hat\chi$ and $\underline{\hat\chi}$ are not:

$$\begin{aligned} \hat\chi_{ab} - \hat\chi_{ba} &= -g(e_4, [e_a, e_b]) \;, \\ \underline{\hat\chi}_{ab} - \underline{\hat\chi}_{ba} &= -g(e_3, [e_a, e_b]) \;, \end{aligned} \tag{4.1.12}$$

and when the distribution of planes $\mathcal{S}$ is not integrable the commutators $[e_a, e_b]$ will have non-zero e_3 or e_4 components.

Following[1] Klainerman and Nicolò, we use the following labeling of the remaining Newman-Penrose coefficients associated with the frame fields e_i:

$$\hat\xi_a = \frac{1}{2} g(\widehat{D}_{e_4} e_4, e_a) \;, \tag{4.1.13a}$$

$$\underline{\hat\xi}_a = \frac{1}{2} g(\widehat{D}_{e_3} e_3, e_a) \;, \tag{4.1.13b}$$

$$\hat\eta_a = -\frac{1}{2} g(\widehat{D}_{e_3} e_a, e_4) = \frac{1}{2} g(\widehat{D}_{e_3} e_4, e_a) \;, \tag{4.1.13c}$$

$$\underline{\hat\eta}_a = -\frac{1}{2} g(\widehat{D}_{e_4} e_a, e_3) = \frac{1}{2} g(\widehat{D}_{e_4} e_3, e_a) \;, \tag{4.1.13d}$$

$$2\hat\omega = -\frac{1}{2} g(\widehat{D}_{e_4} e_3, e_4) \;, \tag{4.1.13e}$$

$$2\underline{\hat\omega} = -\frac{1}{2} g(\widehat{D}_{e_3} e_4, e_3) \;, \tag{4.1.13f}$$

$$2\hat\upsilon = -\frac{1}{2} g(\widehat{D}_{e_3} e_3, e_4) \;, \tag{4.1.13g}$$

$$2\underline{\hat\upsilon} = -\frac{1}{2} g(\widehat{D}_{e_4} e_4, e_3) \;, \tag{4.1.13h}$$

together with their unhatted counterparts, defined with $\widehat{D}$ replaced by D in the equations above. (The principle that determines which symbols are underlined,

[1] We are grateful to Klainerman and Nicolò for making their tex files available to us.

and which are not, should be clear from Equation (4.1.15) below: all the terms *at the right hand side* of that equation have a coefficient in front of e_4 which is underlined.) The above definitions, together with the properties of the $\widehat{D}$ connection coefficients $\widehat{\Gamma}_{ijk}$, imply the following:

$$
\begin{aligned}
\hat{\chi}_{ab} &= \widehat{\Gamma}_{ab4} = -\widehat{\Gamma}_{a4b} = 2\widehat{\Gamma}_a{}^3{}_b = -2\widehat{\Gamma}_{ab}{}^3 \,, && \text{(4.1.14a)}\\
\hat{\underline{\chi}}_{ab} &= \widehat{\Gamma}_{ab3} = -\widehat{\Gamma}_{a3b} = 2\widehat{\Gamma}_a{}^4{}_b = -2\widehat{\Gamma}_{ab}{}^4 \,, && \text{(4.1.14b)}\\
\hat{\zeta}_a &= \widehat{\Gamma}_a{}^3{}_3 = -\frac{1}{2}\widehat{\Gamma}_{a43} = \widehat{\Gamma}_{a4}{}^4 \,, && \text{(4.1.14c)}\\
\hat{\underline{\zeta}}_a &= \widehat{\Gamma}_a{}^4{}_4 = -\frac{1}{2}\widehat{\Gamma}_{a34} = -\widehat{\Gamma}_{a3}{}^3 \,, && \text{(4.1.14d)}\\
\hat{\xi}_a &= \widehat{\Gamma}_4{}^3{}_a = -\widehat{\Gamma}_{4a}{}^3 = \frac{1}{2}\widehat{\Gamma}_{4a4} = -\frac{1}{2}\widehat{\Gamma}_{44a} \,, && \text{(4.1.14e)}\\
\hat{\underline{\xi}}_a &= \widehat{\Gamma}_3{}^4{}_a = -\widehat{\Gamma}_{3a}{}^4 = \frac{1}{2}\widehat{\Gamma}_{3a3} = -\frac{1}{2}\widehat{\Gamma}_{33a} \,, && \text{(4.1.14f)}\\
\hat{\eta}_a &= \widehat{\Gamma}_3{}^3{}_a = -\frac{1}{2}\widehat{\Gamma}_{34a} = \frac{1}{2}\widehat{\Gamma}_{3a4} = -\widehat{\Gamma}_{3a}{}^3 \,, && \text{(4.1.14g)}\\
\hat{\underline{\eta}}_a &= \widehat{\Gamma}_4{}^4{}_a = -\frac{1}{2}\widehat{\Gamma}_{43a} = \frac{1}{2}\widehat{\Gamma}_{4a3} = -\widehat{\Gamma}_{4a}{}^4 \,, && \text{(4.1.14h)}\\
2\hat{\omega} &= \widehat{\Gamma}_4{}^3{}_3 = -\frac{1}{2}\widehat{\Gamma}_{443} = \widehat{\Gamma}_{44}{}^4 \,, && \text{(4.1.14i)}\\
2\hat{\underline{\omega}} &= \widehat{\Gamma}_3{}^4{}_4 = -\frac{1}{2}\widehat{\Gamma}_{334} = \widehat{\Gamma}_{33}{}^3 \,, && \text{(4.1.14j)}\\
2\hat{\upsilon} &= \widehat{\Gamma}_3{}^3{}_3 = -\frac{1}{2}\widehat{\Gamma}_{343} = \widehat{\Gamma}_{34}{}^4 \,, && \text{(4.1.14k)}\\
2\hat{\underline{\upsilon}} &= \widehat{\Gamma}_4{}^4{}_4 = -\frac{1}{2}\widehat{\Gamma}_{434} = \widehat{\Gamma}_{43}{}^3 \,. && \text{(4.1.14l)}
\end{aligned}
$$

This leads to

$$
\begin{aligned}
\widehat{D}_a e_b &= \widehat{\nabla\!\!\!/}_a e_b + \frac{1}{2}\hat{\chi}_{ab}e_3 + \frac{1}{2}\hat{\underline{\chi}}_{ab}e_4 \,, && \text{(4.1.15a)}\\
\widehat{D}_3 e_a &= \widehat{D\!\!\!/}_3 e_a + \hat{\eta}_a e_3 + \hat{\underline{\xi}}_a e_4 \,, && \text{(4.1.15b)}\\
\widehat{D}_4 e_a &= \widehat{D\!\!\!/}_4 e_a + \hat{\underline{\eta}}_a e_4 + \hat{\xi}_a e_3 \,, && \text{(4.1.15c)}\\
\widehat{D}_a e_3 &= \hat{\underline{\chi}}_a{}^b e_b + \hat{\zeta}_a e_3 \,, && \text{(4.1.15d)}\\
\widehat{D}_a e_4 &= \hat{\chi}_a{}^b e_b + \hat{\underline{\zeta}}_a e_4 \,, && \text{(4.1.15e)}\\
\widehat{D}_3 e_3 &= 2\hat{\underline{\xi}}^a e_a + 2\hat{\upsilon} e_3 \,, && \text{(4.1.15f)}\\
\widehat{D}_4 e_4 &= 2\hat{\xi}^a e_a + 2\hat{\underline{\upsilon}} e_4 \,, && \text{(4.1.15g)}\\
\widehat{D}_4 e_3 &= 2\hat{\underline{\eta}}_b e_b + 2\hat{\omega} e_3 \,, && \text{(4.1.15h)}\\
\widehat{D}_3 e_4 &= 2\hat{\eta}_b e_b + 2\hat{\underline{\omega}} e_4 \,. && \text{(4.1.15i)}
\end{aligned}
$$

We stress that Equations (4.1.15) are completely general — no simplifying assumptions have been made concerning the nature of the vector fields e_a, except for the orthonormality relations (4.1.1).

From Equations (4.1.5), (4.1.6), (4.1.14) and (4.1.15) one also has

$$
f_a = \frac{1}{2}(\hat{\zeta}_a + \hat{\underline{\zeta}}_a) = \widehat{\Gamma}_{a11} = \widehat{\Gamma}_{a22} \,,
$$

$$
\begin{aligned}
f_3 &= \hat{\upsilon} + \underline{\hat{\omega}} = \widehat{\Gamma}_{311} = \widehat{\Gamma}_{322} \,, \\
f_4 &= \underline{\hat{\upsilon}} + \hat{\omega} = \widehat{\Gamma}_{411} = \widehat{\Gamma}_{422} \,. \qquad (4.1.16)
\end{aligned}
$$

We have the following correspondence between the hatted and unhatted connection coefficients:

$$\hat{\chi}_{ab} = \chi_{ab} + h_{ab} f_4 \,, \tag{4.1.17a}$$
$$\underline{\hat{\chi}}_{ab} = \underline{\chi}_{ab} + h_{ab} f_3 \,, \tag{4.1.17b}$$
$$\hat{\zeta}_a = \zeta_a + f_a \,, \tag{4.1.17c}$$
$$\underline{\hat{\zeta}}_a = \underline{\zeta}_a + f_a = -\zeta_a + f_a \,, \tag{4.1.17d}$$
$$\hat{\xi}_a = \xi_a \,, \tag{4.1.17e}$$
$$\underline{\hat{\xi}}_a = \underline{\xi}_a \,, \tag{4.1.17f}$$
$$\hat{\eta}_a = \eta_a + f_a \,, \tag{4.1.17g}$$
$$\underline{\hat{\eta}}_a = \underline{\eta} + f_a \,, \tag{4.1.17h}$$
$$\hat{\omega} = \omega \,, \tag{4.1.17i}$$
$$\underline{\hat{\omega}} = \underline{\omega} \,, \tag{4.1.17j}$$
$$\hat{\upsilon} = \upsilon + f_3 = -\underline{\omega} + f_3 \,, \tag{4.1.17k}$$
$$\underline{\hat{\upsilon}} = \underline{\upsilon} + f_4 = -\omega + f_4 \,. \tag{4.1.17l}$$

4.2 Weyl-type tensors

4.2.1 The double-null decomposition of Weyl-type tensors

Let $d^i{}_{jkl}$ be any tensor field with the symmetries of the Weyl tensor,

$$d_{ijkl} = d_{klij} \,, \quad d_{ijkl} = -d_{jikl} \,, \quad g^{jk} d_{ijkl} = 0 \,, \quad d_{i[jkl]} = 0 \,; \tag{4.2.1}$$

we decompose $d^i{}_{jkl}$ into its null components, relative to the null pair $\{e_3, e_4\}$, as follows:

$$\underline{\alpha}(d)(X,Y) = d(X,e_3,Y,e_3) \,, \quad \alpha(d)(X,Y) = d(X,e_4,Y,e_4) \,, \tag{4.2.2a}$$
$$\underline{\beta}(d)(X) = \frac{1}{2} d(X,e_3,e_3,e_4) \,, \quad \beta(d)(X) = \frac{1}{2} d(X,e_4,e_3,e_4) \,, \tag{4.2.2b}$$
$$\rho(d) = \frac{1}{4} d(e_3,e_4,e_3,e_4) \,, \quad \sigma(d) := \rho({}^{\star_g} d) = \frac{1}{4} {}^{\star_g} d(e_3,e_4,e_3,e_4) \,, \tag{4.2.2c}$$

where X,Y are arbitrary vector fields tangent to the $S(x,\tau)$'s, and ^{*_g} denotes the space-time Hodge dual with respect to the first two indices of d_{ijkl}:

$$ {}^{\star_g} d_{ijkl} = \frac{1}{2} \epsilon_{ij}{}^{mn} d_{mnkl} \,. $$

α and $\underline{\alpha}$ are clearly symmetric, and they are also traceless:

$$
\begin{aligned}
0 = g^{ij} d(e_i,e_4,e_j,e_4) &= d(e_1,e_4,e_1,e_4) + d(e_2,e_4,e_2,e_4) \\
&\quad - \frac{1}{2}\big(\underbrace{d(e_4,e_4,e_3,e_4)}_{0} + \underbrace{d(e_3,e_4,e_4,e_4)}_{0}\big) \\
&= \alpha^a{}_a \,,
\end{aligned}
$$

similarly for $\underline{\alpha}$. From Equation (4.2.2) one finds

$$d_{a3b3} = \underline{\alpha}_{ab} \;, \qquad d_{a4b4} = \alpha_{ab} \;, \tag{4.2.3a}$$
$$d_{a334} = 2\underline{\beta}_a \;, \qquad d_{a434} = 2\beta_a \;, \tag{4.2.3b}$$
$$d_{3434} = 4\rho \;, \qquad d_{ab34} = 2\sigma\epsilon_{ab} \;, \tag{4.2.3c}$$
$$d_{abc3} = \epsilon_{ab}\,{}^\star\underline{\beta}_c \;, \quad d_{abc4} = -\epsilon_{ab}\,{}^\star\beta_c \;, \tag{4.2.3d}$$
$$d^a{}_{3b4} = -\rho\delta^a_b + \sigma\epsilon^a{}_b \;, \quad d_{abcd} = -\rho\epsilon_{ab}\epsilon_{cd} \;; \tag{4.2.3e}$$

where

$$\epsilon_{12} = -\epsilon_{21} = 1 \;, \qquad \epsilon_{11} = \epsilon_{22} = 0 \;. \tag{4.2.4}$$

Further, $\star$ denotes the Hodge dual on $\mathcal{S}$ with respect to the metric induced by g on $\mathcal{S}$:

$${}^\star\beta_a = \epsilon_a{}^b\beta_b \;. \tag{4.2.5}$$

The first three equations in (4.2.3) follow immediately from the definitions; we simply note that in the equation for d_{ab34} one uses

$$\sigma = \frac{1}{4}\epsilon^{ab}d_{ab34} \quad \Longleftrightarrow \quad d_{ab34} = 2\sigma\epsilon_{ab} \;, \tag{4.2.6}$$

with the factors appearing there justified by (4.1.2). To obtain the fourth equation in (4.2.3) one has, *e.g.*,

$$d_{1214} = -d_{2114} = \underbrace{-d_{2114} - \underbrace{d_{2224}}_{0} + \frac{1}{2}\underbrace{d_{2344}}_{0} + \frac{1}{2}d_{2434}}_{-g^{ij}d_{2ij4}=0} - \frac{1}{2}d_{2434} = -\beta_2 = -\epsilon_{12}\,{}^\star\beta_1 \;,$$

and the result follows by symmetry considerations. The last equation is obtained in a similar way.

4.2.2 Double-null decomposition of the Bianchi equations

We will need a double-null decomposition of the equations

$$\widehat{D}_i d^i{}_{jkl} \;=\; \hat{J}_{jkl} \;, \tag{4.2.7}$$
$$\widehat{D}_{[i}d_{jk]lm} \;=\; \hat{J}_{ijklm} \;, \tag{4.2.8}$$

where $\hat{J}_{jkl}$ and $\hat{J}_{ijklm}$ are source terms to be specified later; square brackets around a set of ℓ indices denote antisymmetrization, with a numerical factor $1/\ell!$. (Actually, Equation (4.2.7) will turn out to be sufficient for most of our purposes.) Equation (4.1.7) yields the following alternative form of (4.2.7):

$$\widehat{D}_i(g^{im}d_{mjkl}) = g^{im}\widehat{D}_i d_{mjkl} + 2f^m d_{mjkl} = \hat{J}_{jkl} \;. \tag{4.2.9}$$

Equation (4.2.9) with $k = 3$ and $k = 4$ gives

$$\widehat{D}_3 d_{43kl} \;=\; 2h^{ab}\widehat{D}_a d_{b3kl} + 4f^m d_{m3kl} - 2\hat{J}_{3kl} \;, \tag{4.2.10a}$$
$$\widehat{D}_4 d_{34kl} \;=\; 2h^{ab}\widehat{D}_a d_{b4kl} + 4f^m d_{m4kl} - 2\hat{J}_{4kl} \;, \tag{4.2.10b}$$

which will give equations for $\beta, \underline{\beta}, \sigma$ and ρ; we use the symbol h to denote the metric induced on $\mathcal{S}$ by g: for all $X, Y \in T\mathscr{M}$,

$$h(X,Y) = g(X,Y) + \frac{1}{2}g(e_3,X)g(e_4,Y) + \frac{1}{2}g(e_4,X)g(e_3,Y) \,. \tag{4.2.11}$$

To obtain the equations for α_{ab} and $\underline{\alpha}_{ab}$, we write

$$\begin{aligned} g^{ij}\widehat{D}_i d_{ja4b} &= -\frac{1}{2}(\widehat{D}_3 d_{4a4b} + \widehat{D}_4 d_{3a4b}) + h^{cd}\widehat{D}_c d_{da4b} \\ &= -\frac{1}{2}(\widehat{D}_3 d_{4a4b} + \underbrace{\widehat{D}_4 d_{3a4b} + \widehat{D}_3 d_{a44b} + \widehat{D}_a d_{434b}}_{3\hat{J}_{43a4b}} \\ &\quad - \underbrace{\widehat{D}_3 d_{a44b}}_{-\widehat{D}_3 d_{4a4b}} - \widehat{D}_a d_{434b}) + h^{cd}\widehat{D}_c d_{da4b} \\ &= -\widehat{D}_3 d_{4a4b} + \frac{1}{2}\widehat{D}_a d_{434b} + h^{cd}\widehat{D}_c d_{da4b} - \frac{3}{2}\hat{J}_{43a4b} \,, \end{aligned}$$

hence

$$\widehat{D}_3 d_{4a4b} = \frac{1}{2}\widehat{D}_a d_{434b} + h^{cd}\widehat{D}_c d_{da4b} + 2f^m d_{ma4b} - \frac{3}{2}\hat{J}_{43a4b} + \hat{J}_{a4b} \,, \tag{4.2.12}$$

with a similar equation in which the index 3 is interchanged with the index 4:

$$\widehat{D}_4 d_{3a3b} = \frac{1}{2}\widehat{D}_a d_{343b} + h^{cd}\widehat{D}_c d_{da3b} + 2f^m d_{ma3b} - \frac{3}{2}\hat{J}_{34a3b} + \hat{J}_{a3b} \,. \tag{4.2.13}$$

An equivalent, slightly more convenient, way of obtaining the null components version of Equations (4.2.12)-(4.2.13) proceeds as follows: Consider the "Bianchi" equation

$$\widehat{D}_i d^i{}_{ab4} = \hat{J}_{ab4} \,. \tag{4.2.14}$$

For any tensor field T_{ab} we denote by $\overline{T_{ab}}$ the symmetric traceless part of T_{ab}, and by tr$T$ its trace. Applying the "overline" operation to Equation (4.2.14) one has

$$\overline{\widehat{D}_i d^i{}_{(ab)4}} = \overline{\hat{J}_{(ab)4}} \,,$$

which we write in the form

$$2\overline{\widehat{D}_3 d^3{}_{(ab)4}} + 2\overline{\widehat{D}_4 d^4{}_{(ab)4}} + 2\overline{\widehat{D}_c d^c{}_{(ab)4}} \;=\; 2\overline{\hat{J}_{(ab)4}} \,. \tag{4.2.15}$$

Each of the terms on the left-hand-side of Equation (4.2.15) will be computed separately. First,

$$\begin{aligned} 2\overline{\widehat{D}_3 d^3{}_{(ab)4}} &= -\overline{\widehat{D}_3 d_{4(ab)4}} - 2f_3\overline{d_{4(ab)4}} \\ &= \widehat{D}_3 d_{a4b4} + 2f_3 d_{a4b4} \,, \end{aligned}$$

since $\widehat{D}_k d_{a4b4}$ is symmetric traceless. A calculation gives

$$2\overline{\widehat{D}_3 d^3{}_{(ab)4}} \;=\; \hat{\not{D}}_3\alpha_{ab} - 4\hat{\eta}_{(a}\beta_{b)} + 4{}^\star\hat{\eta}_{(a}{}^\star\beta_{b)} + (2\hat{\upsilon} - 2\underline{\hat{\omega}})\alpha_{ab} \,,$$

where

$$\hat{\not{D}}_3\alpha_{ab} := e_3(\alpha_{ab}) - \widehat{\Gamma}_3{}^c{}_a\alpha_{cb} - \widehat{\Gamma}_3{}^c{}_b\alpha_{ac} \,. \tag{4.2.16}$$

Noting that

$${}^\star\hat{\eta}_{(a}{}^\star\beta_{b)} = -\hat{\eta}_{(a}\beta_{b)} + g_{ab}\hat{\eta}\cdot\beta \,,$$

we finally obtain

$$2\overline{\widehat{D}_3 d^3{}_{(ab)4}} \;=\; \hat{\not{D}}_3\alpha - 4\hat{\eta}\overline{\otimes}_s\beta + (2\hat{\upsilon} - 2\hat{\underline{\omega}})\alpha \,, \tag{4.2.17}$$

where, following Christodoulou and Klainerman [14], $\hat{\eta}\overline{\otimes}_s\beta$ denotes *twice* the trace-free symmetric tensor product,

$$(X\overline{\otimes}_s Y)^{ab} = X^aY^b + X^bY^a - g^{ab}X_cY^c \,, \tag{4.2.18}$$

similarly for covectors. Next, we claim that

$$\overline{\widehat{D}_4 d^4{}_{(ab)4}} = 0 \,, \tag{4.2.19}$$

which can be seen either by a direct calculation, or as follows: Equation (4.2.3) shows that $d^4{}_{(ab)4} = -\frac{1}{2}d_{3(ab)4}$ is proportional to the metric h. Now any covariant differentiation preserves symmetries of tensors; further one easily checks that $\widehat{D}$-covariant diffentiation preserves the property of being proportional to the metric, and the result follows.

For the last term, note that $\widehat{D}_c d$ also has all the symmetries (4.2.1), and can therefore be decomposed as in (4.2.2), with $\alpha(d)$ there replaced by $\alpha(\widehat{D}_c d)$, $\beta_a(d)$ there replaced by $\beta_a(\widehat{D}_c d)$, *etc.*; one then has identities analogous to (4.2.3). It then follows that

$$\begin{aligned} -2\overline{\widehat{D}_c d^c{}_{(ab)4}} &= 2\varepsilon^c{}_{(a}\varepsilon_{b)d}\beta^d(\widehat{D}_c d) \\ &= \beta_a(\widehat{D}_b d) + \beta_b(\widehat{D}_a d) - g_{ab}\beta^c(\widehat{D}_c d) \\ &= 2\overline{\beta_{(a}(\widehat{D}_{b)})} \,. \end{aligned}$$

Further,

$$\begin{aligned} 2\beta_d(\widehat{D}_c d) &= g_{de}\widehat{D}_c d^e{}_{434} \\ &= 2g_{de}\widehat{\not{\nabla}}_c\beta^e - 3\hat{\chi}_{cd}\rho - 3{}^\star\hat{\chi}^t_{dc}\sigma - \underline{\hat{\chi}}_c{}^e\alpha_{de} - (4\hat{\underline{\zeta}}_c + 2\hat{\zeta}_c)\beta_d \\ &= 2\widehat{\not{\nabla}}_c\beta_d - 2\hat{\underline{\zeta}}_c\beta_d - 3\hat{\chi}_{cd}\rho - 3{}^\star\hat{\chi}^t_{dc}\sigma - \underline{\hat{\chi}}_c{}^e\alpha_{de} \,, \end{aligned} \tag{4.2.20}$$

where $\widehat{\not{\nabla}}$ is the orthogonal projection on $\mathcal{S}$ of the relevant covariant derivatives in directions tangent to $\mathcal{S}$, *e.g.*

$$\widehat{\not{\nabla}}_a e_b = \widehat{\Gamma}_a{}^c{}_b e_c \,. \tag{4.2.21}$$

and we have used $g_{de}\widehat{\not{\nabla}}_c\beta^e = \widehat{\not{\nabla}}_c\beta_d + 2f_c\beta_d$. Then, taking the symetric traceless part of Equation (4.2.20) one is led to

$$\begin{aligned} -2\overline{\widehat{D}_c d^c{}_{(ab)4}} &= 2\overline{\widehat{\not{\nabla}}_{(b}\beta_{a)}} - 3(\overline{\hat{\chi}_{(ab)}}\rho + \overline{{}^\star\hat{\chi}_{(ab)}}\sigma) \\ &\quad -2\overline{\hat{\underline{\zeta}}_{(a}\beta_{b)}} - \frac{1}{2}(\mathrm{tr}\underline{\hat{\chi}}\alpha_{ab} + a(\underline{\hat{\chi}})^\star\alpha_{ab}) \,, \end{aligned}$$

where ${}^\star\alpha_{ab} = \epsilon_a{}^c\alpha_{cb}$; further, we have used

$$\underline{\hat\chi}_a{}^e\alpha_{be} + \underline{\hat\chi}_b{}^e\alpha_{ae} - g_{ab}\underline{\hat\chi}^{cd}\alpha_{cd} = \mathrm{tr}\underline{\hat\chi}\alpha_{ab} + a(\underline{\hat\chi})^\star\alpha_{ab} \ ,$$

with

$$a(\underline{\hat\chi}) := \epsilon^{ab}\underline{\chi}_{ab} = \underline{\hat\chi}_{12} - \underline{\hat\chi}_{21} \ .$$

Summing up, one obtains

$$\begin{aligned}\hat{\not{D}}_3\alpha + \frac{1}{2}\mathrm{tr}\underline{\hat\chi}\alpha &= \widehat{\not\nabla}\widehat{\otimes}_s\beta + (2\underline{\hat\omega} - 2\hat\upsilon)\alpha - \frac{1}{2}a(\underline{\hat\chi})^\star\alpha \\ &\quad -3(\overline{\underline{\hat\chi}}\rho + {}^\star\overline{\underline{\hat\chi}}\sigma) + (4\hat\eta - \underline{\hat\zeta})\widehat{\otimes}_s\beta + 2\overline{\hat{J}(\cdot,\cdot,e_4)} \ .\end{aligned}$$

Consider, next, the equation for $e_3(\beta)$; its doubly-null decomposition can be obtained as follows: We start with the equation

$$\widehat{D}_i d^i{}_{34a} = \hat{J}_{34a} \ , \tag{4.2.22}$$

which gives

$$-\frac{1}{2}\widehat{D}_3 d_{a434} - f_3 d_{a434} + \widehat{D}_c d^c{}_{34a} = \hat{J}_{34a} \tag{4.2.23}$$

so that

$$\frac{1}{2}\widehat{D}_3 d_{a434} = -\widehat{D}_c d^c{}_{3a4} - f_3 d_{a434} - \hat{J}_{34a} \ . \tag{4.2.24}$$

We have the identity

$$-\widehat\Gamma_c{}^c{}_d d^d{}_{3a4} + \widehat\Gamma_c{}^d{}_a d^c{}_{3d4} = 0 \ , \tag{4.2.25}$$

which can be seen as follows: from the null decomposition of d we have

$$\begin{aligned}-\widehat\Gamma_c{}^c{}_d d^d{}_{3a4} + \widehat\Gamma_c{}^d{}_a d^c{}_{3d4} &= -\widehat\Gamma_c{}^c{}_d(-\rho\delta^d_a + \sigma\varepsilon^d{}_a) + \widehat\Gamma_c{}^d{}_a(-\rho\delta^c_d + \sigma\varepsilon^c{}_d) \\ &= (-\widehat\Gamma_c{}^{cd}\varepsilon_{da} + \widehat\Gamma_c{}^d{}_a\varepsilon^c{}_d)\sigma \ .\end{aligned}$$

Setting $a = 1$ one finds

$$-\widehat\Gamma_c{}^{cd}\varepsilon_{da} + \widehat\Gamma_c{}^d{}_a\varepsilon^c{}_d = \widehat\Gamma_1{}^{12} + \widehat\Gamma_2{}^{22} + \widehat\Gamma_1{}^{21} - \widehat\Gamma_2{}^{11} \ ,$$

and one concludes using $\widehat\Gamma_2{}^{22} = \widehat\Gamma_2{}^{11}$ and $\widehat\Gamma_1{}^{12} + \widehat\Gamma_1{}^{21} = 0$.

Using Equation (4.2.25) with some work one obtains

$$\begin{aligned}-\widehat{D}_c d^c{}_{3a4} &= \widehat{\not\nabla}_a\rho + ({}^\star\widehat{\not\nabla})_a\sigma - \mathrm{tr}\underline{\hat\chi}\beta_a - \underline{\hat\chi}_{cb}\varepsilon^{cb\star}\beta_a \\ &\quad +\hat\chi_{ba}\underline{\beta}^b + \hat\chi_c{}^d\varepsilon_{ad}{}^\star\underline{\beta}^c - \rho(\hat\zeta_a + \underline{\hat\zeta}_a) - \sigma({}^\star\hat\zeta_a + {}^\star\underline{\hat\zeta}_a) \ .\end{aligned} \tag{4.2.26}$$

On the other hand

$$\frac{1}{2}\widehat{D}_3 d_{a434} = \hat{\not{D}}_3\beta - (4\underline{\hat\omega} + 2\hat\upsilon)\beta - \underline{\hat\xi}\cdot\alpha - 3\hat\eta\rho - 3{}^\star\hat\eta\sigma \ , \tag{4.2.27}$$

where

$$\hat{\not{D}}_3\beta_a = e_3(\beta_a) - \widehat\Gamma_3{}^b{}_a\beta_b \ . \tag{4.2.28}$$

Using the identity

$$\hat\chi_c{}^d\varepsilon_{ad}{}^\star\underline{\beta}^c = \hat\chi_a{}^c\underline{\beta}_c - \mathrm{tr}\hat\chi\underline{\beta}_a$$

and the above intermediate calculations we finally obtain the equation

$$\begin{aligned}\beta_3 \quad := \quad & \hat{\not{D}}_3\beta + \mathrm{tr}\underline{\hat\chi}\beta = \widehat{\not\nabla}\rho + {}^\star\widehat{\not\nabla}\sigma + (\overline{\hat\chi} + \overline{\hat\chi}^t)\cdot\underline{\beta} + 2\underline{\hat\omega}\beta + 3(\hat\eta\rho + {}^\star\hat\eta\sigma) \\ & -((\hat\zeta + \underline{\hat\zeta})\rho + ({}^\star\hat\zeta + {}^\star\underline{\hat\zeta})\sigma) + \underline{\hat\xi}\cdot\alpha - \underline{\hat\chi}_{ab}\varepsilon^{ab\star}\beta - \hat J_{34a}\ . \qquad (4.2.29)\end{aligned}$$

Similar, tedious but otherwise straightforward, calculations allow one to obtain the remaining equations, listed out as Equation (4.2.33) below. A useful symmetry principle, which allows to reduce the number of calculations by half, is to note that under the interchange of e_3 with e_4 the underlined rotation coefficients (4.1.14) are exchanged with the non-underlined ones. On the other hand, the null components of the tensor d transform as follows:

$$\begin{aligned}\alpha \leftrightarrow \underline{\alpha}\ , \qquad & \rho \leftrightarrow \rho\ , \\ \beta \leftrightarrow -\underline{\beta}\ , \qquad & \sigma \leftrightarrow -\sigma\ . \qquad (4.2.30)\end{aligned}$$

A convenient identity in the relevant manipulations is

$$\widehat{\not\nabla}_c\epsilon_{ab} = -2f_c\epsilon_{ab} = -(\hat\zeta_c + \underline{\hat\zeta}_c)\epsilon_{ab}\ , \qquad (4.2.31)$$

as well as

$$\widehat{\not\nabla}_c\epsilon_a{}^b = 0\ . \qquad (4.2.32)$$

The full set of equations which are obtained by the doubly-null decomposition of Equation (4.2.7) reads[2]

$$\begin{aligned}
\hat{\not{D}}_4\underline{\alpha} \quad = \quad & -\frac{1}{2}\mathrm{tr}\hat\chi\underline{\alpha} - \widehat{\not\nabla}\overline{\otimes}_s\underline{\beta} + (2\hat\omega - 2\underline{\hat\upsilon})\underline{\alpha} - \frac{1}{2}a(\hat\chi)^\star\underline{\alpha} \\
& -3(\overline{\underline{\hat\chi}}\rho - {}^\star\overline{\underline{\hat\chi}}\sigma) - (4\underline{\hat\eta} - \hat\zeta)\overline{\otimes}_s\underline{\beta} + 2\overline{\hat J(\cdot,\cdot,e_3)}\ , \qquad (4.2.33a) \\
\hat{\not{D}}_3\underline{\beta} \quad = \quad & -2\mathrm{tr}\underline{\hat\chi\beta} - \widehat{\not{\mathrm{d}}\mathrm{iv}}\,\underline{\alpha} + 2\hat\upsilon\underline{\beta} - \underline{\alpha}\cdot(\hat\eta - 2\hat\zeta) + 2a(\underline{\hat\chi})^\star\underline{\beta} + 3(-\underline{\hat\xi}\rho + {}^\star\underline{\hat\xi}\sigma) \\
& -\hat J(e_3,\cdot,e_3)\ , \qquad (4.2.33b) \\
\hat{\not{D}}_4\underline{\beta} \quad = \quad & -\mathrm{tr}\hat\chi\underline{\beta} - \widehat{\not\nabla}\rho + {}^\star\widehat{\not\nabla}\sigma + 2\overline{\underline{\hat\chi}}\cdot\beta + 2\hat\omega\underline{\beta} + 3(-\underline{\hat\eta}\rho + {}^\star\underline{\hat\eta}\sigma) \\
& +(\hat\zeta + \underline{\hat\zeta})\rho - ({}^\star\hat\zeta + {}^\star\underline{\hat\zeta})\sigma - \hat\xi\cdot\underline{\alpha} - a(\hat\chi)^\star\underline{\beta} + \hat J(e_4,e_3,\cdot)\ , \qquad (4.2.33c) \\
\widehat{D}_3\rho \quad = \quad & -\frac{3}{2}\mathrm{tr}\underline{\hat\chi}\rho - \widehat{\not{\mathrm{d}}\mathrm{iv}}\,\underline{\beta} - \frac{1}{2}\overline{\hat\chi}\cdot\underline{\alpha} + (2\hat\zeta + \underline{\hat\zeta} - 2\hat\eta)\cdot\underline{\beta} \\
& +2\underline{\hat\xi}\cdot\beta + \frac{3}{2}a(\underline{\hat\chi})\sigma + 4(\hat\upsilon + \underline{\hat\omega})\rho + \frac{1}{2}\hat J_{334}\ , \qquad (4.2.33d) \\
\widehat{D}_4\rho \quad = \quad & -\frac{3}{2}\mathrm{tr}\hat\chi\rho + \widehat{\not{\mathrm{d}}\mathrm{iv}}\,\beta - \frac{1}{2}\overline{\underline{\hat\chi}}\cdot\alpha - (2\underline{\hat\zeta} + \hat\zeta - 2\underline{\hat\eta})\cdot\beta \\
& -2\hat\xi\cdot\underline{\beta} - \frac{3}{2}a(\hat\chi)\sigma + 4(\underline{\hat\upsilon} + \hat\omega)\rho + \frac{1}{2}\hat J_{443}\ , \qquad (4.2.33e)
\end{aligned}$$

[2]Equations (4.2.33) are essentially a subset of the Newman-Penrose equations written out in a tensor formalism. The equations in [14] or in [33] can be obtained from (4.2.33) by specialisation, and straightforward changes of notation. We note some (inessential) misprints in the equations in [33].

$$\begin{aligned}
\widehat{D}_3\sigma &= -\frac{3}{2}\mathrm{tr}\underline{\hat\chi}\sigma - \widehat{\mathrm{d\!/\!iv}}\,{}^\star\underline{\beta} + 2(\underline{\hat\omega} + \hat\upsilon)\sigma - \frac{1}{2}{}^t\hat\chi\cdot{}^\star\underline{\alpha} - 2\underline{\hat\xi}\cdot{}^\star\beta \\
&\quad +(\underline{\hat\zeta} + 2\hat\zeta - 2\hat\eta)\cdot{}^\star\underline{\mathring{\beta}} - \frac{3}{2}\rho a(\underline{\hat\chi}) - \frac{1}{2}a(\hat J(e_3,\cdot,\cdot))\,, \qquad (4.2.33\mathrm{f})\\
\widehat{D}_4\sigma &= -\frac{3}{2}\mathrm{tr}\hat\chi\sigma - \widehat{\mathrm{d\!/\!iv}}\,{}^\star\beta + 2(\hat\omega + \underline{\hat\upsilon})\sigma + \frac{1}{2}{}^t\underline{\hat\chi}\cdot{}^\star\alpha - 2\hat\xi\cdot{}^\star\underline{\beta} \\
&\quad +(\hat\zeta + 2\underline{\hat\zeta} - 2\underline{\hat\eta})\cdot{}^\star\beta + \frac{3}{2}\rho a(\hat\chi) - \frac{1}{2}a(\hat J(e_4,\cdot,\cdot))\,, \qquad (4.2.33\mathrm{g})\\
\hat{D\!\!\!/}_3\beta &= -\mathrm{tr}\underline{\hat\chi}\beta - \hat{\nabla\!\!\!/}\rho + {}^\star\hat{\nabla\!\!\!/}\sigma + 2\overline{\hat\chi}\cdot\underline{\beta} + 2\underline{\hat\omega}\beta + 3(\hat\eta\rho + {}^\star\hat\eta\sigma) \\
&\quad -(\hat\zeta + \underline{\hat\zeta})\rho - ({}^\star\hat\zeta + {}^\star\underline{\hat\zeta})\sigma + \underline{\hat\xi}\cdot\alpha - a(\underline{\hat\chi})^\star\beta - \hat J(e_3,e_4,\cdot)\,, \qquad (4.2.33\mathrm{h})\\
\hat{D\!\!\!/}_4\beta &= -2\mathrm{tr}\hat\chi\beta + \widehat{\mathrm{d\!/\!iv}}\,\alpha + 2\underline{\hat\upsilon}\beta + \alpha\cdot(\underline{\hat\eta} - 2\underline{\hat\zeta}) + 2a(\hat\chi)^\star\beta + 3(\hat\xi\rho + {}^\star\hat\xi\sigma) \\
&\quad -\hat J(e_4,\cdot,e_4)\,, \qquad (4.2.33\mathrm{i})\\
\hat{D\!\!\!/}_3\alpha &= -\frac{1}{2}\mathrm{tr}\underline{\hat\chi}\alpha + \hat{\nabla\!\!\!/}\overline{\otimes}_s\beta + (2\underline{\hat\omega} - 2\hat\upsilon)\alpha - \frac{1}{2}a(\underline{\hat\chi})^\star\alpha \\
&\quad -3(\overline{\underline{\hat\chi}}\rho + {}^\star\overline{\underline{\hat\chi}}\sigma) + (4\hat\eta - \underline{\hat\zeta})\overline{\otimes}_s\beta + 2\overline{\hat J(\cdot,\cdot,e_4)}\,. \qquad (4.2.33\mathrm{j})
\end{aligned}$$

For the convenience of the reader it is appropriate to give a summary of the notations used: The operators $\hat{D\!\!\!/}_4$ and $\hat{D\!\!\!/}_3$ are defined as the orthogonal projections on $\mathcal{S}$ of the $\widehat{D}$-covariant derivatives along the null directions e_3 and e_4:

$$\hat{D\!\!\!/}_3 e_a = \widehat{\Gamma}_3{}^b{}_a e_b\,, \qquad \hat{D\!\!\!/}_4 e_a = \widehat{\Gamma}_4{}^b{}_a e_b\,.$$

In particular

$$\hat{D\!\!\!/}_3\rho = \widehat{D}_3\rho = e_3(\rho)\,, \qquad \hat{D\!\!\!/}_3\sigma = \widehat{D}_3\sigma = e_3(\sigma)\,,$$

etc., with $\hat{D\!\!\!/}_3\beta$ and $\hat{D\!\!\!/}_3\alpha_{ab}$ written out explicitly in Equation (4.2.28) and Equation (4.2.16). Next, $\nabla\!\!\!/$ and $\hat{\nabla\!\!\!/}$ are differential operators in directions tangent to $\mathcal{S}$ defined as the orthogonal projection on $\mathcal{S}$ of the relevant covariant derivatives in directions tangent to $\mathcal{S}$, *cf.* Equation (4.2.21). We use the symbols $\mathrm{d\!/\!iv}$ and $\widehat{\mathrm{d\!/\!iv}}$ to denote the associated "divergence" operators: if $X = X^a e_a$ and $Y = Y^{ab} e_a \otimes e_b$ then

$$\mathrm{d\!/\!iv}\,X = \hat{\nabla\!\!\!/}_a X^a\,, \qquad \widehat{\mathrm{d\!/\!iv}}\,Y = \hat{\nabla\!\!\!/}_a Y^{ab}\,,$$

This gives

$$\widehat{\mathrm{d\!/\!iv}}\,\underline{\beta} := \hat{\nabla\!\!\!/}_a(h^{ab}\underline{\beta}_b) = h^{ab}\hat{\nabla\!\!\!/}_a\underline{\beta}_b + 2f^c\underline{\beta}_c = h^{ab}\hat{\nabla\!\!\!/}_a\underline{\beta}_b + (\hat\zeta + \underline{\hat\zeta})^c\underline{\beta}_c\,,$$

similarly

$$(\widehat{\mathrm{d\!/\!iv}}\,\alpha)_b := \hat{\nabla\!\!\!/}_a(h^{ac}\alpha_{cb})\,.$$

We have also set

$${}^t\hat\chi^{ab} = \hat\chi^{ba}\,.$$

Recall that a bar over a valence-two tensor denotes its *symmetric traceless part*, *e.g.*

$$\overline{\underline{\hat\chi}}_{ab} = \frac{1}{2}\left\{\underline{\hat\chi}_{ab} + \underline{\hat\chi}_{ba} - h^{cd}\underline{\hat\chi}_{cd}h_{ab}\right\}\,,$$

while

$$a(\chi) = \varepsilon^{ab}\chi_{ab} \ .$$

To avoid ambiguities, we emphasize that in Equations (4.2.33) the free slot in $\hat{J}$, whenever occurring, refers to vectors in $\mathcal{S}$, in particular

$$a(\hat{J}(e_4, \cdot, \cdot)) := \epsilon^{ab}\hat{J}_{4ab} \ , \quad a(\hat{J}(e_3, \cdot, \cdot)) := \epsilon^{ab}\hat{J}_{3ab} \ .$$

Finally the symbol $\overline{\otimes}_s$ has been defined in Equation (4.2.18).

4.3 Adaptation of Friedrich's form of the Einstein field equations

4.3.1 The conformal field equations

Let $\tilde{g}$ be the physical space-time metric, let Ω be a function and let $g = \Omega^2 \tilde{g}$ be the unphysical conformally rescaled counterpart of $\tilde{g}$. (To make easier reference to [20, 21, 26, 29], throughout this section the symbol g denotes the *unphysical* metric; this is opposite to the conventions used elsewhere in this work.) Consider any frame field $e_k = e^\mu{}_k \, \partial_{x^\mu}$ such that the $g(e_i, e_k) \equiv g_{ik}$'s are constants. Using the Einstein vacuum field equations, Friedrich [20, 21] has derived a set of equations for the fields[3]

$$e^\mu{}_k, \quad \Gamma_i{}^j{}_k, \quad d^i{}_{jkl} = \Omega^{-1} C^i{}_{jkl}, \quad L_{ij} = \frac{1}{2} R_{ij} - \frac{1}{12} R \, g_{ij},$$

$$\Omega, \quad \bar{s} = \frac{1}{4} D_i D^i \Omega + \frac{1}{24} R \, \Omega,$$

where $\Gamma_i{}^j{}_k$ denotes the Levi-Civita connection coefficients in the frame e_k while $C^i{}_{jkl}$, R_{ij}, and R stand, respectively, for the conformal Weyl tensor, the Ricci tensor, and the Ricci scalar of g. Friedrich's "conformal field equations" read

$$[e_p, e_q] = (\Gamma_p{}^l{}_q - \Gamma_q{}^l{}_p) \, e_l, \tag{4.3.1a}$$

$$\begin{aligned} &e_p(\Gamma_q{}^i{}_j) - e_q(\Gamma_p{}^i{}_j) - 2\,\Gamma_k{}^i{}_j \, \Gamma_{[p}{}^k{}_{q]} + 2\,\Gamma_{[p}{}^i{}_{|k|} \Gamma_{q]}{}^k{}_j \\ &\quad = 2\, g^i{}_{[p} \, L_{q]j} - 2\, g^{ik} \, g_{j[p} \, L_{q]k} + \Omega \, d^i{}_{jpq}, \end{aligned} \tag{4.3.1b}$$

$$D_i d^i{}_{jkl} = 0, \tag{4.3.1c}$$

$$D_i L_{jk} - D_j L_{ik} = D_l \Omega \, d^l{}_{kij}, \tag{4.3.1d}$$

$$D_i \, D_j \Omega = -\Omega \, L_{ij} + \bar{s} g_{ij}, \tag{4.3.1e}$$

$$D_i \bar{s} = -L_{ij} D^j \Omega, \tag{4.3.1f}$$

$$6 \Omega \, \bar{s} - 3 \, D_j \Omega \, D^j \Omega = 0. \tag{4.3.1g}$$

The first equation expresses the fact that the Levi-Civita connection is torsion free; the second is the definition of the Riemann tensor; the third is the Bianchi identity assuming that $\tilde{g}$ is Ricci flat. The remaining equations are obtained by algebraic manipulations from the vacuum Einstein equations, using the conformal transformation laws for the various objects at hand. In regions where $\Omega > 0$ the system is equivalent to the vacuum Einstein equations [20, 21].

4.3.2 The conformal equations in terms of conformal connections

In [26] H. Friedrich has presented a reformulation of his original conformal field equations, presented above, in terms of objects better adapted to the conformal approach used. The key idea is to replace the equations for the *Levi-Civita connection* by equations involving *Weyl connections* – for our purposes

[3] We are grateful to Helmut Friedrich for allowing us to use his tex files.

we define those as connections of the form (4.1.3). The "generalized conformal Einstein equations" derived in [26] form a system of equations for the unknown

$$u = \left(e^\mu{}_k,\, \hat\Gamma_i{}^j{}_k,\, \hat L_{jk} = \frac{1}{2}\,\hat R_{(jk)} - \frac{1}{12}\,\hat R\, g_{jk} - \frac{1}{4}\,\hat R_{[jk]},\, d^i{}_{jkl} = \Theta^{-1}\, C^i{}_{jkl}\right), \tag{4.3.2}$$

where $\hat\Gamma_i{}^j{}_k$ are the connections coefficients of the Weyl connection $\widehat D$ in the frame e_k, $\hat R_{jk}$ is the Ricci tensor of $\widehat D$, $C^i{}_{jkl}$ is its Weyl tensor, and $\hat R = g^{jk}\,\hat R_{jk}$; as elsewhere, round brackets denote symmetrization while square ones denote antisymmetrization. As before, one introduces an unphysical metric g via the formula

$$\tilde g \to g = \Omega^2\,\tilde g\ ,$$

where Ω is a conformal factor which will be determined later. The system of equations derived in [26] reads

$$\begin{aligned}
[e_p, e_q] &= (\hat\Gamma_p{}^l{}_q - \hat\Gamma_q{}^l{}_p)\, e_l\ , && (4.3.3a)\\
e_p(\hat\Gamma_q{}^i{}_j) - e_q(\hat\Gamma_p{}^i{}_j) &= 2\,\hat\Gamma_k{}^i{}_j\,\hat\Gamma_{[p}{}^k{}_{q]} - 2\,\hat\Gamma_{[p}{}^i{}_{|k|}\,\hat\Gamma_{q]}{}^k{}_j + 2\,\delta^i_{[p}\,\hat L_{q]j} &&\\
&\quad -2\,g^{ik}\,g_{j[p}\,\hat L_{q]k} - 2\,\delta^i_j\,\hat L_{[pq]} + \Omega\, d^i{}_{jpq}\ , && (4.3.3b)\\
\widehat D_p\,\hat L_{qj} - \widehat D_q\,\hat L_{pj} &= \Omega b_i\, d^i{}_{jpq}\ , && (4.3.3c)\\
D_i\, d^i{}_{jkl} &= 0\ . && (4.3.3d)
\end{aligned}$$

Here b is the field of one-forms appearing in Equation (4.1.3). To obtain an evolution system using the equations above, suppose first that a vacuum space-time $(\widetilde{\mathscr M}, \tilde g)$ "with a piece of $\mathscr I^+$" is given, with conformal completion $(\mathscr M, \mathring g)$. For simplicity we shall suppose that all the objects involved are smooth on $\mathscr M$. Let $\mathscr S$ be a *hyperboloidal hypersurface*; by definition, this is a smooth spacelike hypersurface in $\widetilde{\mathscr M}$ the closure $\overline{\mathscr S}$ in $\mathscr M$ of which intersects smoothly $\mathscr I^+$; further it is assumed that $\overline{\mathscr S}$ is uniformly spacelike up-to-boundary for the metric $\mathring g$. Recall that a *conformal geodesic* [43] is a space-time curve $x(\tau)$, together with a 1-form $b(\tau)$ along it, which satisfy the system of ODE's

$$(\tilde D_{\dot x}\dot x)^\mu + S(b)_\nu{}^\mu{}_\rho\,\dot x^\nu\,\dot x^\rho = 0, \tag{4.3.4a}$$

$$(\tilde D_{\dot x} b)_\nu - \frac{1}{2} b_\mu\, S(b)_\nu{}^\mu{}_\rho\,\dot x^\rho - \tilde L_{\nu\mu}\,\dot x^\mu = 0, \tag{4.3.4b}$$

where $\tilde L_{\nu\mu} = \frac{1}{2}\,(\tilde R_{\nu\mu} - \frac{1}{6}\,\tilde g_{\nu\mu}\,\tilde R)$ is defined in terms of the Ricci tensor and the Ricci scalar of $\tilde g$.

Because $(\widetilde{\mathscr M}, \tilde g)$ is vacuum, Equations (4.3.4a)-(4.3.4b) for a conformal geodesic $s(\tau)$ read

$$\begin{aligned}
\widetilde D_{\dot s}\dot s &= -2b(\dot s)\dot s + g(\dot s, \dot s)\tilde g^\sharp(b,\cdot)\ , && (4.3.5a)\\
\widetilde D_{\dot s} b &= b(\dot s) b - \frac{1}{2}\tilde g^\sharp(b,b)\tilde g(\dot s,\cdot)\ . && (4.3.5b)
\end{aligned}$$

From now on we will always require

$$\tilde g(\dot s, \dot s)|_{\mathscr S} = 0\ .$$

From (4.3.5) we have

$$\partial_\tau \tilde g(\dot s,\dot s) \;=\; -2b(\dot s)\tilde g(\dot s,\dot s)\ , \tag{4.3.6}$$

so that $\tilde g(\dot s,\dot s)$ remains zero along the conformal geodesics. It follows that the conformal geodesics considered here are null geodesics for $\tilde g$, with a parametrization which will not be the affine one in general. One easily verifies that $b(\dot s)$ and $\tilde g^\sharp(b,b)$ satisfy

$$\partial_\tau b(\dot s) \;=\; -b(\dot s)^2\ , \tag{4.3.7a}$$
$$\partial_\tau \tilde g^\sharp(b,b) \;=\; b(\dot s)\tilde g^\sharp(b,b)\ . \tag{4.3.7b}$$

In particular if $b(\dot s)|_{\mathscr S}=0$, then $b(\dot s)$ is zero along the geodesics and the parameter is the affine parameter of the $\tilde g-$ geodesic, with $\tilde g^\sharp(b,b)$ being constant along the geodesic. Further, it follows from Equation (4.3.7a) that

$$b(\dot s)(\tau)=\frac{b(\dot s)(0)}{b(\dot s)(0)\tau+1}\ , \tag{4.3.8}$$

as long as the conformal geodesic exists.

Choose any smooth field e_4 of future pointing null vector fields, and any field of forms b defined along $\mathscr S$, and use them as initial values for Equations (4.3.7a)-(4.3.7b). Solving those equations we obtain a field b defined on a neighbourhood of $\mathscr S$. We then set

$$\widehat D=\widetilde D+S(b)\ ,$$

which provides a conformal connection on this neighbourhood of $\mathscr S$.

Let us return to the conformal completion $(\mathscr M,\mathring g)$ of $(\widetilde{\mathscr M},\tilde g)$, so that

$$\mathring g=\mathring\Omega^2\tilde g\ ,$$

with $\mathring\Omega$ being a defining function for $\mathscr I^+:=\partial\mathscr M$. Let $(\overline{\mathscr S},\mathring h)$ be the associated conformal completion of $(\mathscr S,\tilde h)$, with $\mathring h=\mathring\Omega_0^2\tilde h$, where $\tilde h:=\tilde g|_{\mathscr S}$ is the metric induced on $\mathscr S$ by $\tilde g$, and $\mathring\Omega_0:=\mathring\Omega|_{\overline{\mathscr S}}$ is a defining function for $\partial\mathscr S$. The restriction of the metric $\mathring g$ to $\overline{\mathscr S}$ is defined to be $\mathring h$, so that $\mathring g|_{\overline{\mathscr S}}=\mathring\Omega_0^2\tilde g|_{\overline{\mathscr S}}$. Let n be a future-directed $\mathring g$-unit normal n to $\overline{\mathscr S}$. There exists a unique smooth strictly positive function a on $\mathscr S$ such that

$$e_4=a(n-c_3)\ ,$$

where c_3 is a $\mathring h$-unit vector field tangent to $\mathscr S$. Let $\{c_i\}_{1\le i\le 3}$ be any $\mathring h$-orthonormal frame along $\overline{\mathscr S}$, we define a half-null tetrad $\{e_i\}_{1\le i\le 4}$ there by setting

$$e_a\Big|_{\mathscr S}:=c_a\ ,\ a=1,2\ ,\quad e_3\Big|_{\mathscr S}=a^{-1}(n+c_3)\ . \tag{4.3.9}$$

Let $\dot s$ be the field of tangents to the above family of conformal geodesics, set

$$e_4:=\dot s\ .$$

We propagate the remaining e_i's by $\widehat{D}$-parallel transport,

$$\widehat{D}_{\dot{s}} e_i = 0 \;;$$

it follows from the definition of e_4 that this equation holds for $i = 4$ as well. In terms of $\widetilde{D}$ this gives

$$\widetilde{D}_{\dot{s}} e_i \;=\; -b(e_i)\dot{s} - b(\dot{s})e_i + \tilde{g}(\dot{s}, e_i)\tilde{g}^\sharp(b, \cdot) \;, \tag{4.3.10}$$

which implies

$$\frac{d}{d\tau}\tilde{g}(e_i, e_j) \;=\; -2b(\dot{s})\tilde{g}(e_i, e_j) \;,$$

so that

$$\theta^2 \tilde{g}(e_i, e_j)(\tau) \;=\; \tilde{g}(e_i, e_j)(0) \;, \tag{4.3.11}$$

where θ is a solution of

$$\partial_\tau(\theta^{-2}) \;=\; -2b(\dot{s})\theta^{-2} \;, \tag{4.3.12}$$

with initial value $\theta(0) = 1$. It follows from Equation (4.3.8) that

$$\theta(\tau) = 1 + b(\dot{s})(0)\tau \;. \tag{4.3.13}$$

We define a new metric g by requiring that *the tetrad e_i be a half-null tetrad for g* in the sense of Equation (4.1.1); equivalently, if θ^i is a frame dual to e_i, then

$$g = \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2 - 2\theta^3 \otimes \theta^4 - 2\theta^4 \otimes \theta^3 \;.$$

It follows from Equation (4.3.9) that $g = \mathring{g}$ along $\mathscr{S}$, and Equation (4.3.11) implies that there exists a function Ω such that

$$g = \Omega^{-2}\tilde{g} \;.$$

More precisely, we have

$$\Omega(s(\tau)) = \theta(\tau)\mathring{\Omega}(s(0)) \;, \tag{4.3.14}$$

thus Ω is determined by the value of $\mathring{\Omega}$ at the point where the conformal geodesic $s(\cdot)$ intersects $\mathscr{S}$. Similarly we have

$$g = (\Omega/\mathring{\Omega})^2 \mathring{g} \;.$$

If $b(\dot{s})(0)$ is a smooth strictly positive function on $\overline{\mathscr{S}}$, then it follows from Equations (4.3.13)-(4.3.14) that $\Omega/\mathring{\Omega}$ is a smooth, strictly positive function, bounded away from zero, for $\tau > 0$ in a neighbourhood of $\mathscr{S}$, and (replacing $\mathscr{M}$ and $\widetilde{\mathscr{M}}$ by subsets thereof if necessary) the part of $(\mathscr{M}, g)$ lying to the future of $\mathscr{S}$ is a smooth conformal completion of $(\widetilde{\mathscr{M}}, \tilde{g})$.

Let $b_i := b(e_i)$; from Equations (4.3.5) and (4.3.10) one obtains

$$\frac{db_i}{d\tau} = -b(\dot{s})b_i + \frac{1}{2}\tilde{g}^\sharp(b, b)\tilde{g}(e_4, e_i) \;. \tag{4.3.15}$$

This can be integrated to read

$$b_a(\tau) = \frac{b_a(0)}{1 + b(\dot{s})(0)\tau} \;, \quad a = 1, 2 \;. \tag{4.3.16}$$

Equations (4.3.7b) and (4.3.8) also give

$$\tilde{g}^\sharp(b, b)(\tau) = \frac{\tilde{g}^\sharp(b, b)(0)}{b(\dot{s})(0)\tau + 1} \;, \tag{4.3.17}$$

from which one obtains

$$b_3(\tau) = \frac{b_3(0)}{1 + b(\dot{s})(0)\tau} + \frac{\tau}{2(1 + b(\dot{s})(0)\tau)}\tilde{g}^\sharp(b, b)(0)\tilde{g}(e_3, e_4)(0) \;. \tag{4.3.18}$$

In the above construction the initial value of the field e_4 along $\mathscr{S}$ was completely arbitrary, it is convenient to restrict that freedom as follows: Let x be any defining function for $\partial\mathscr{S}$ on $\mathscr{S}$ and let x_0 be such that the level sets $\mathscr{S}_x$ of x are smooth two dimensional submanifolds for $0 \le x \le x_0$. As above n is the field of $\mathring{g}$-unit normals to $\mathscr{S}$ defined along $\mathscr{S}$, let m be the field of $\mathring{h}$-unit vectors which are tangent to $\mathscr{S}$, normal to the $\mathscr{S}_x$'s, pointing away from $\mathscr{S}$ on $\partial\mathscr{S}$ (thus $m \cdot x < 0$). Let a be any strictly positive function on $\mathscr{S}$ and set

$$e_4 \equiv e_- := a(n - m) \;, \quad e_+ \equiv e_3 := a^{-1}(n + m) \;, \tag{4.3.19}$$

and e_a, $a = 1, 2$ — any (locally defined) field of orthonormal vector field tangent to the $\mathscr{S}_x$'s. Letting b_0 be any field of one forms defined along $\mathscr{S}$, the hypersurfaces $\mathscr{I}^+_x$ are defined by shooting conformal geodesics from the $\mathscr{S}_x$'s, with initial velocity $\dot{x}(0) = e_-$, and with $b(0) = b_0$. Without loss of generality the time-parameter range can be assumed to be $\tau \in [0, \tau_0]$, with a τ_0 small enough. Further decreasing τ_0 if necessary, the $\mathscr{I}^+_x$'s will form a smooth foliation, with leaves diffeomorphic to $\mathscr{S}_x \times [0, \tau_0]$.

As already pointed out, the integral curves of e_- are null geodesics, and by construction they are normal to the two-dimensional surfaces $\mathscr{S}_x$. It is a standard fact in Lorentzian geometry that the resulting hypersurfaces $\mathscr{I}^+_x$ are null. The field $e_- = e_4$ is thus a $\widehat{D}$-auto-parallel, hypersurface-orthogonal, null vector field defined on

$$\mathscr{M}_{[0,x_0,\tau_0]} := \cup_{x\in[0,x_0]}\mathscr{I}^+_x \approx [0, x_0] \times \partial\mathscr{S} \times [0, \tau_0] \;. \tag{4.3.20}$$

We define a coordinate system on $\mathscr{M}_{[0,x_0,\tau_0]}$ by setting $\tau = 0$ on $\mathscr{S}$. Let v^a be any local coordinates on $\partial\mathscr{S}$, we propagate them to a neighbourhood of $\partial\mathscr{S}$ in $\mathscr{S}$ in any way to obtain a local coordinate system (x, v^a) near $\partial\mathscr{S}$. We then Lie-drag x and the local coordinates v^a along the null conformal geodesics $s(\tau)$, that is x and the v^a's are defined on $\mathscr{M}_{[0,x_0,\tau_0]}$ as those solutions of the equations

$$e_-(v^a) = 0 \;, \quad e_-(x) = 0$$

which assume the obvious initial values on $\mathscr{S}$. Clearly e_- must be proportional to $\partial_\tau = \frac{\partial}{\partial\tau}$, and one easily checks that in fact

$$e_- = \partial_\tau \;. \tag{4.3.21}$$

Since e_- is a field of null vectors tangent to the null hypersurfaces $\mathscr{I}^+_x$, every vector orthogonal to e_- is also tangent to $\mathscr{I}^+_x$. In particular we obtain that the e_a's, $a = 1, 2$ are tangent to $\mathscr{I}^+_x$; equivalently, if in the current coordinate system the e_i's are written as $e^\mu_i \partial_\mu$, we obtain

$$e^x_a = 0 \; . \tag{4.3.22}$$

The fact that the e_i's are $\widehat{D}$-parallel implies that

$$\widehat{\Gamma}_4{}^i{}_j = 0 \; . \tag{4.3.23}$$

Recall, now, that the conformal geodesics $s(\tau)$ and the associated connections $\widehat{D}$ are invariants of the conformal structure of $\tilde{g}$, in the following sense: if g is rescaled by a conformal factor Θ^{-2}, then a conformal geodesic of g remains a conformal geodesic for $\Theta^{-2}g$, with the field b replaced by $b + \Theta^{-1}d\Theta$. This follows immediately from the formulae of Appendix A.1. It follows from that the tensor field $\hat{L}_{jk}$ defined in Equation (4.3.2) satisfies

$$\hat{L}_{4j} = 0 \; . \tag{4.3.24}$$

Equations (4.3.21), (4.3.23) and (4.3.24) allow us to obtain from Equations (4.3.3a)-(4.3.3c) the following set of ODE's:

$$\begin{aligned} \frac{\partial e_q}{\partial \tau} &= -\hat{\Gamma}_q{}^l{}_4 \, e_l \; , && (4.3.25a)\\ \frac{\partial \hat{\Gamma}_q{}^i{}_j}{\partial \tau} &= -\hat{\Gamma}_k{}^i{}_j \, \hat{\Gamma}_q{}^k{}_4 + \delta^i_4 \, \hat{L}_{qj} \\ &\quad - g^{ik} \, g_{j4} \, \hat{L}_{qk} + \delta^i_j \, \hat{L}_{q4} + \Omega \, d^i{}_{j4q} \; , && (4.3.25b)\\ \frac{\partial \hat{L}_{qj}}{\partial \tau} &= \Omega b_i \, d^i{}_{j4q}. && (4.3.25c) \end{aligned}$$

The above have to be supplemented by an evolution equation for $d^i{}_{j4q}$. This last equation will be obtained by considering the null decomposition (4.2.2) of this tensor. Before passing to a detailed analysis of this issue, let us make some comments on our strategy here.

First, Equations (4.3.25) together with Equation (4.3.3d) for $d^i{}_{jpq}$, with Equations (4.3.8), (4.3.16)-(4.3.18) for b_i and with (4.3.14) for Ω *do* form a closed system of equations. Our aim in what follows is to use those equations to obtain *a priori* estimates for solutions of Einstein equations. Existence for a sufficiently long time — so that $\widetilde{\mathscr{M}}$ includes a whole "piece of $\mathscr{I}^+$" — will then follow by usual continuation-of-solutions arguments, presented in detail in Section 5.5 below.

Next, the above equations hold as well when

$$\widehat{D} = D \qquad \Longleftrightarrow \qquad \Omega = \mathring{\Omega}_0 \; , \quad b = \Omega^{-1} d\Omega \; . \tag{4.3.26}$$

The whole formalism is somewhat simpler in this case; in particular no hatted connection coefficients are needed, the raising and the lowering of indices commutes with covariant differentiation, *etc.* However, a formulation in which

b's other than in (4.3.26) are allowed has the esthetic advantage that it reflects the inherent conformal freedom existing in the conformal formulation of the problem. In particular any issues related to that freedom will be much easier to analyse in a setting in which general b's are allowed. For this reason we have decided not to restrict ourselves to the case (4.3.26) in most our calculations, for further reference. However we will soon concentrate on (4.3.26) in our main analytic results. We note that while the use of a covariant derivative operator $\widehat{D} \neq D$ plays a critical role in [29], in our problem at hand it seems only to play an esthetic one. We also note that when (4.3.26) holds with $\mathring{\Omega}_0 = x$, then the gradient $d\Omega$ of the conformal factor Ω is null, and Equation (4.3.24) easily follows from Equations (4.3.1).

Let us pass now to a convenient null reformulation of Equation (4.3.3d). Consider, first, the null coefficients (4.1.13) of $\widehat{\Gamma}$; it follows from Equation (4.3.23) and (4.1.13) that we have

$$\hat{\xi}_a = \underline{\hat{\eta}}_a = \hat{\omega} = \underline{\hat{\upsilon}} = 0 \ . \tag{4.3.27}$$

Next, let α, β, *etc*, be the null components of d, and for reasons which will become apparent below introduce

$$\mathring{\beta} := \beta \ , \quad \underline{\mathring{\beta}} := \underline{\beta} \ , \tag{4.3.28a}$$

$$\mathring{\sigma} := \sigma \ , \quad \mathring{\rho} := \rho \ . \tag{4.3.28b}$$

The doubly-null form of Equation (4.3.3d) is obtained from Equation (4.2.33) by obvious specialisation: we insert (4.3.27) in (4.2.33), and rewrite the resulting equations as follows

$$\not{D}_4\underline{\alpha} + \frac{1}{2}\mathrm{tr}\hat{\chi}\underline{\alpha} = -\not{\nabla}\overline{\otimes}_s\underline{\beta} - \frac{1}{2}a(\hat{\chi})^{\star}\underline{\alpha} - 3(\overline{\underline{\hat{\chi}}}\rho - {}^{\star}\overline{\underline{\hat{\chi}}}\sigma) + \hat{\zeta}\overline{\otimes}_s\underline{\beta} \ , \tag{4.3.29a}$$

$$\begin{aligned} \not{D}_3\underline{\beta} + 2\mathrm{tr}\underline{\hat{\chi}\beta} &= -\not{\mathrm{d}}\mathrm{iv}\,\underline{\alpha} + 2\hat{\upsilon}\underline{\beta} - \underline{\alpha}\cdot(\hat{\eta} - 2\hat{\zeta}) \\ &\quad + 2a(\underline{\hat{\chi}})^{\star}\underline{\beta} + 3(-\underline{\hat{\xi}}\rho + {}^{\star}\underline{\hat{\xi}}\sigma) \ , \end{aligned} \tag{4.3.29b}$$

$$\not{D}_4\underline{\mathring{\beta}} + \mathrm{tr}\hat{\chi}\underline{\mathring{\beta}} = -\not{\nabla}\mathring{\rho} + {}^{\star}\not{\nabla}\mathring{\sigma} + 2\overline{\underline{\hat{\chi}}}\cdot\beta - a(\hat{\chi})^{\star}\underline{\beta} \ , \tag{4.3.30a}$$

$$\begin{aligned} D_3\mathring{\sigma} + \frac{3}{2}\mathrm{tr}\underline{\hat{\chi}}\mathring{\sigma} &= -\not{\mathrm{d}}\mathrm{iv}\,{}^{\star}\underline{\mathring{\beta}} - \frac{1}{2}{}^{t}\hat{\chi}\cdot{}^{\star}\underline{\alpha} - 2\underline{\hat{\xi}}\cdot{}^{\star}\beta + (\hat{\zeta} - 2\hat{\eta})\cdot{}^{\star}\underline{\mathring{\beta}} \\ &\quad - \frac{3}{2}\rho a(\underline{\hat{\chi}}) \ , \end{aligned} \tag{4.3.30b}$$

$$\begin{aligned} D_3\mathring{\rho} + \frac{3}{2}\mathrm{tr}\underline{\hat{\chi}}\mathring{\rho} &= -\not{\mathrm{d}}\mathrm{iv}\,\underline{\mathring{\beta}} - \frac{1}{2}\overline{\hat{\chi}}\cdot\underline{\alpha} + (\hat{\zeta} - 2\hat{\eta})\cdot\underline{\beta} + 2\underline{\hat{\xi}}\cdot\beta \\ &\quad + \frac{3}{2}a(\underline{\hat{\chi}})\sigma \ , \end{aligned} \tag{4.3.30c}$$

$$D_4\rho + \frac{3}{2}\mathrm{tr}\hat{\chi}\rho = \not{\mathrm{d}}\mathrm{iv}\,\beta - \frac{1}{2}\overline{\underline{\hat{\chi}}}\cdot\alpha - \underline{\hat{\zeta}}\cdot\beta - \frac{3}{2}a(\hat{\chi})\sigma \ , \tag{4.3.31a}$$

$$D_4\sigma + \frac{3}{2}\mathrm{tr}\hat{\chi}\sigma = -\not{\mathrm{d}}\mathrm{iv}\,{}^{\star}\beta + \frac{1}{2}{}^{t}\underline{\hat{\chi}}\cdot{}^{\star}\alpha + \underline{\hat{\zeta}}\cdot{}^{\star}\beta + \frac{3}{2}\rho a(\hat{\chi}) \ , \tag{4.3.31b}$$

$$\begin{aligned} \not{D}_3\beta + \mathrm{tr}\underline{\hat{\chi}}\beta &= \not{\nabla}\rho + {}^{\star}\not{\nabla}\sigma + 2\overline{\hat{\chi}}\cdot\underline{\beta} + 2\underline{\hat{\omega}}\beta + 3(\hat{\eta}\rho + {}^{\star}\hat{\eta}\sigma) + \underline{\hat{\xi}}\cdot\alpha \\ &\quad - a(\underline{\hat{\chi}})^{\star}\beta \ , \end{aligned} \tag{4.3.31c}$$

$$\not{D}_4\mathring{\beta} + 2\mathrm{tr}\hat{\chi}\mathring{\beta} = \not{\mathrm{d}}\mathrm{iv}\,\alpha - 2\alpha\cdot\underline{\hat{\zeta}} + 2a(\hat{\chi})^{\star}\beta \ , \tag{4.3.32a}$$

$$\begin{aligned}\not{D}_3\alpha + \frac{1}{2}\mathrm{tr}\underline{\hat{\chi}}\alpha &= \not{\nabla}\otimes_s\mathring{\beta} + (2\underline{\hat{\omega}} - 2\hat{\upsilon})\alpha - \frac{1}{2}a(\underline{\hat{\chi}})^\star\alpha \\ &\quad -3(\overline{\hat{\chi}}\rho + {}^\star\overline{\hat{\chi}}\sigma) + (4\hat{\eta} - \underline{\hat{\zeta}})\overline{\otimes}_s\beta\ . \qquad (4.3.32b)\end{aligned}$$

In these equations the unhatted Christoffel coefficients should be expressed in terms of the hatted ones using Equations (4.1.17).[4]

4.3.3 Choices of gauge

Let $(\mathscr{M},\widetilde{g})$ be a maximal globally hyperbolic space-time solution of the vacuum Einstein equations with initial data on a spacelike hypersurface Σ which is supposed to admit a conformal compactification: denoting $\widetilde{h}$ the metric induced by $\widetilde{g}$ on Σ and $\overline{\Sigma} = \Sigma\cup\partial\Sigma$, with $\partial\Sigma$ being a two dimensional manifold, we suppose there exists a smooth function $\mathring{\Omega}_0$ such that

- $\mathring{\Omega}_0|_{\partial\Sigma} = 0$ and $d\mathring{\Omega}_0|_{T\Sigma}(p)\neq 0$ for any $p\in\partial\Sigma$,
- $(\overline{\Sigma},\mathring{h}_0)$ is a compact Riemannian manifold with $\mathring{h}_0 = \mathring{\Omega}_0^2\widetilde{h}$.

(A hyperboloidal initial data set for the Einstein Equations satisfies these hypotheses.) Let us note that we do not suppose at this stage that $(\mathscr{M},\widetilde{g})$ admit a conformal completion neither that it contains a piece of $\mathscr{I}^+$.

However, we can repeat the construction of the previous section with $\Sigma = \mathscr{S}$ from (4.3.4a) to (4.3.22) except that $\mathring{\Omega}_0$ is defined only on Σ (and so on for $g = \mathring{\Omega}_0\tilde{g}$) and we do not have a set $\mathscr{M}_{[0,x_0]}$ with the relation (4.3.20). Now suppose the physical metric is locally smooth enough ($C^2(\Sigma)$ is enough), the compactness of $\overline{\Sigma}_{x_2,x_0} := \{p\in\Sigma\mid x_2\le x(p)\le x_0\}$ for $x_0 > x_2 > 0$, implies the existence of $\tau* > 0$ such that the geodesics $s(\tau)$ are defined for $\tau\in[0,\tau*[$, provided the initial fields $b(0)$ and $\dot{s}$ are in $L^\infty(\Sigma_{x_2,x_0})$.

Therefore, we have the following substitute for (4.3.20)

$$\mathscr{M}_{[x_2,x_0,\tau_0]} := \cup_{x\in[x_2,x_0]}\mathscr{I}^+_x \approx [x_2,x_0]\times\partial\mathscr{S}\times[0,\tau_0]\ , \qquad (4.3.33)$$

for $\tau_0 < \tau*$.

Now, we will make gauge choices which will simplify some expressions.

Before going further, let us note that dx is null since by construction $e_4\cdot x = e_a\cdot x = 0$.

First, we fix $b(0)$ by

$$x|_\Sigma = \mathring{\Omega}_0\ , \qquad (4.3.34)$$

$$b(0) = x^{-1}dx|_{\Sigma_{x_2,x_0}}\ . \qquad (4.3.35)$$

This implies that $b(0)$ is null and $b(0)(\dot{s}) = 0$, which gives with (4.3.7a-4.3.7b) that $b(\tau)$is null and $b(\tau)(\dot{s}) = 0$ for $\tau\in[0,\tau_0]$. Then (4.3.15) gives that

$$b = x^{-1}dx\ , \qquad (4.3.36)$$

[4] There is a certain amount of freedom which symbols at the right should be decorated with "o"'s; we shall not be consistent in this respect and change β's to $\mathring{\beta}$'s, *etc.*, according to the context.

on $\mathscr{M}_{[x_2,x_0,\tau_0]}$, while Equations 4.3.5 become

$$\widetilde{D}_{\dot{s}}\dot{s} = 0 \ , \tag{4.3.37}$$
$$\widetilde{D}_{\dot{s}}b = 0 \ . \tag{4.3.38}$$

Remark: The conclusion is that our choice of gauge gives us a conformal connection which is the Levi-Civita connection of the metric $g = x^{-2}\tilde{g}$, and for the results obtained in this thesis, we could avoid the use of conformal connections. The point is that we can obtain ordinary differential equations for L_{ij} with an adapted "null" conformal factor, without introducing the conformal connections.

Now, the next choices concern the initial data for x, that is to say $x|_\Sigma = \mathring{\Omega}_0$. We suppose the initial conformal factor can be chosen as a Gaussian function, which is explicited in the following lemma :

Lemma 4.3.1 Suppose $\mathring{h}_0 \in C_\infty(\Sigma)$, and suppose that the associated physical initial data $(\Sigma, \widetilde{h}_{ij}, \widetilde{K}_{ij})$ satisfy

$$\widetilde{h}^{ij}\widetilde{K}_{ij}\Big|_{\partial M} = 3 \ .$$

Then there exists x, a smooth function on $\overline{\Sigma}$ positive on Σ, such that

- x is defined on a neighbourhood of $\partial\Sigma$ and $\Sigma_{x_1} = \{p \in \Sigma \mid x(p) \leq x_1\}$ is diffeomorphic to $[0, x_1] \times \partial\Sigma$ for some $x_1 > 0$ small enough;
- $x|_{\partial\Sigma} = 0$;
- $h_0 = x^2\widetilde{h}$ is a Riemannian metric on $\overline{\Sigma}$;
- $h_0^\sharp(dx|_\Sigma, dx|_\Sigma) = 1$, where $h_0^\sharp$ is the dual metric of h_0 and $dx|_{T\Sigma}$ is the differential of x on Σ,[5]
- $\nabla_{dx|^\flat_{T\Sigma}} dx^\flat_{T\Sigma} = 0$, where ∇ is the Levi-Civita connection associated to h_0 and $dx^\flat$ is the h_0-gradiant of x.

So far, this result is justified for initial data metric which is sufficiently differentiable, say smooth. There is little doubt that this construction can be repeated for polyhomogeneous initial data, or initial data in a finite weighted Sobolev differentiability class. We have not examined this question in detail, and we are planning to remove this choice of gauge soon.

PROOF: See [4, Lemma 2.1]. □

From now we have on $\mathscr{M}_{x_2,x_0,\tau*}$,

$$b = x^{-1}dx \ , \qquad \Omega = x \ . \tag{4.3.39}$$

There remains some freedom in the choice of the coordinate system. The natural choice is to consider Gaussian coordinate systems (x, v^A) on Σ_{x_0}, where

[5] There is no use in the notation $dx|_{T\Sigma}$ within the lemma since x in defined on Σ, but later we will use dx for x defined on a piece of the space-time $\mathscr{M}$.

$dx|^\flat_{T\Sigma} \cdot v^A = 0$. The coordinate systems (x, v^A, τ) where x, v^A are Lie dragged along the geodesics $s(\tau)$, will be called x, τ-adapted coordinates. We will denote

$$\partial_x = \frac{\partial}{\partial x} . \qquad (4.3.40)$$

Further, our condition on $x|_\Sigma$ gives on Σ :

$$m = -\partial_x , \qquad (4.3.41)$$
$$n = \partial_\tau - \partial_x . \qquad (4.3.42)$$

We set

$$e_4(0) = \partial_\tau , \qquad (4.3.43)$$
$$e_3(0) = \partial_\tau - 2\partial_x , \qquad (4.3.44)$$

which correspond to the choice $a = 1$ in (4.3.19). We will need the precise form of some coefficients of e_i^μ. First, we note

$$e_4^\mu \partial_\mu = \partial_\tau , \qquad (4.3.45a)$$
$$e_a^x = 0 , \qquad (4.3.45b)$$
$$\partial_\tau e_3^x = 0 , \qquad (4.3.45c)$$

where the last equation is a consequence of $\partial_\tau e_3^x = -{\Gamma_3}^3{}_4 e_3^x$. At $\{\tau = 0\}$ we set

$$e_3^x = -2 , \qquad (4.3.46)$$

and it follows from Equation (4.3.45c) that Equation (4.3.46) will hold throughout. Moreover we have the evolution equations,

$$\partial_\tau e_3^\tau = -2\underline{\omega} - 2\eta^a e_a^\tau , \qquad (4.3.47a)$$
$$\partial_\tau e_3^A = -2\eta^a e_a^A , \qquad (4.3.47b)$$
$$\partial_\tau e_a^\tau = \zeta_a - {\chi_a}^b e_b^\tau , \qquad (4.3.47c)$$
$$\partial_\tau e_a^A = -{\chi_a}^b e_b^A . \qquad (4.3.47d)$$

In (4.3.47c) the equation $\zeta = -\underline{\zeta}$ has been used.

Let us turn our attention now to the connection coefficients; recall that in the current gauge Equation (4.3.27) holds, as well as

$$\underline{\zeta} = -\zeta , \qquad \upsilon = -\underline{\omega} . \qquad (4.3.48)$$

The vectors e_1, e_2 are tangent to the hypersurfaces $\{x = r\}$, and $g(e_4, [e_a, e_b]) = 0$, so summarising we obtain

$$\chi_{ab} = \chi_{ba} , \qquad (4.3.49a)$$
$$\xi_a = 0 , \qquad (4.3.49b)$$
$$\underline{\eta}_a = 0 , \qquad (4.3.49c)$$
$$\omega = 0 , \qquad (4.3.49d)$$
$$\upsilon = -\underline{\omega} , \qquad (4.3.49e)$$
$$\underline{\upsilon} = 0 , \qquad (4.3.49f)$$

Further

$$L_{i4} = L_{4i} = 0 \ , \tag{4.3.50}$$

which follows from our choice of gauge, *cf.* Equation (4.3.24), and from the symmetry of L_{ij}; indeed, metricity of D implies that $R_{[ij]} = 0$ in Equation (A.1.4). This shows that the L_{q4} term in Equation (4.3.25b) vanishes, and a double-null decomposition of the evolution equations (4.3.25b) for the NP coefficients gives

$$\begin{aligned}
\partial_\tau \chi_a{}^b &= -\chi_{ac}\cdot\chi^{cb} - x\alpha_a{}^b \ , & (4.3.51a)\\
\partial_\tau \underline{\chi}_a{}^b &= -\chi_a{}^c\underline{\chi}_c{}^b + x\rho\delta_a^b + x\sigma\varepsilon_a{}^b + 2L_a{}^b \ , & (4.3.51b)\\
\partial_\tau \zeta_a &= -\chi_a{}^c\zeta_c - x\beta_a \ , & (4.3.51c)\\
\partial_\tau \underline{\xi}_a &= -\eta^c\underline{\chi}_{ca} + L_{3a} - x\underline{\beta}_a \ , & (4.3.51d)\\
\partial_\tau \eta_a &= -\eta^c\chi_{ca} - x\beta_a \ , & (4.3.51e)\\
\partial_\tau \underline{\omega} &= \eta^c\zeta_c + x\rho \ , & (4.3.51f)\\
\partial_\tau \Gamma_a{}^b{}_c &= -\chi_a{}^d\Gamma_d{}^b{}_c + x\varepsilon_{bc}{}^\star\beta_a \ , & (4.3.51g)\\
\partial_\tau \Gamma_3{}^a{}_b &= -2\Gamma_c{}^a{}_b\eta^c - 2x\sigma\varepsilon^a{}_b \ . & (4.3.51h)
\end{aligned}$$

It follows that we also have

$$\begin{aligned}
\partial_\tau \mathrm{tr}\chi &= -\chi\cdot\chi \ , & (4.3.52a)\\
\partial_\tau \overline{\chi}_{ab} &= -\mathrm{tr}\chi\overline{\chi}_{ab} - x\alpha_{ab} \ , & (4.3.52b)\\
\partial_\tau \mathrm{tr}\underline{\chi} &= -\chi\cdot\underline{\chi} + 2x\rho + 2L_a{}^a \ , & (4.3.52c)\\
\partial_\tau \underline{\chi}_{[ab]} &= \underline{\chi}_{c[a}\chi^c{}_{b]} + x\sigma\varepsilon_{ab} \ . & (4.3.52d)
\end{aligned}$$

Consider, next, L_{ij}; Equation (4.3.50) shows that the only possibly non-zero components thereof are L_{ab}, L_{3a} and L_{33}. If follows from Equations (4.3.45)-(4.3.46) that

$$b_i := \langle b, e_i\rangle = -2x^{-1}\delta_i^3 \ ,$$

and Equation (4.3.25c) gives the following evolution equation for the null component L_{33}:

$$\partial_\tau L_{33} = 4\rho \ . \tag{4.3.53}$$

From Equation (A.1.7) (recall that $\tilde{L}_{ij} = 0$) one obtains the following explicit expressions for the remaining L_{ij}'s:

$$\begin{aligned}
L_{ab} &= x^{-1}\chi_{ab} \ , & (4.3.54a)\\
L_{3a} &= 2x^{-1}\eta_a \ . & (4.3.54b)
\end{aligned}$$

This leads to the following form of (4.3.51b) and (4.3.51d):

$$\begin{aligned}
\partial_\tau \underline{\chi}_{ab} &= -\chi_a{}^c\underline{\chi}_{cb} + x\rho g_{ab} + x\sigma\varepsilon_{ab} + 2x^{-1}\chi_{ab} \ , & (4.3.55a)\\
\partial_\tau \underline{\xi}_a &= -\eta_c\underline{\chi}_a{}^c + 2x^{-1}\eta_a - x\underline{\beta}_a \ . & (4.3.55b)
\end{aligned}$$

In this choice of gauge, the Bianchi equations become

$$\underline{\alpha}_4 \ := \ \not{D}_4\underline{\alpha} + \frac{1}{2}\mathrm{tr}\chi\underline{\alpha}$$

$$
\begin{aligned}
&= -\not\nabla\overline{\otimes}_s\underline{\beta} - 3(\underline{\overline{\chi}}\rho - {}^\star\underline{\overline{\chi}}\sigma) + \zeta\overline{\otimes}_s\mathring{\underline{\beta}}\ , && (4.3.56a)\\
\underline{\beta}_3 &:= \not{D}_3\underline{\beta} + 2\mathrm{tr}\underline{\chi}\mathring{\underline{\beta}}\\
&= -\not{\mathrm{d}}\mathrm{iv}\,\underline{\alpha} - 2\underline{\omega}\mathring{\underline{\beta}} - \underline{\alpha}\cdot(\eta - 2\zeta) + 2a(\underline{\chi})^\star\mathring{\underline{\beta}} + 3(-\underline{\xi}\rho + {}^\star\underline{\xi}\sigma)\ , && (4.3.56b)\\
\mathring{\underline{\beta}}_4 &:= \not{D}_4\mathring{\underline{\beta}} + \mathrm{tr}\chi\mathring{\underline{\beta}}\\
&= -\not\nabla\mathring{\rho} + {}^\star\not\nabla\mathring{\sigma} + 2\underline{\overline{\chi}}\cdot\beta\ , && (4.3.57a)\\
\mathring{\sigma}_3 &:= D_3\mathring{\sigma} + \frac{3}{2}\mathrm{tr}\underline{\chi}\mathring{\sigma}\\
&= -\not{\mathrm{d}}\mathrm{iv}\,{}^\star\mathring{\underline{\beta}} - \frac{1}{2}\overline{\chi}\cdot{}^\star\underline{\alpha} - 2\underline{\xi}\cdot{}^\star\beta + (\zeta - 2\eta)\cdot{}^\star\mathring{\underline{\beta}} - \frac{3}{2}\rho a(\underline{\chi})\ , && (4.3.57b)\\
\mathring{\rho}_3 &:= D_3\mathring{\rho} + \frac{3}{2}\mathrm{tr}\underline{\chi}\mathring{\rho}\\
&= -\not{\mathrm{d}}\mathrm{iv}\,\mathring{\underline{\beta}} - \frac{1}{2}\overline{\chi}\cdot\underline{\alpha} + (\zeta - 2\eta)\cdot\underline{\beta} + 2\underline{\xi}\cdot\beta + \frac{3}{2}a(\underline{\chi})\sigma\ , && (4.3.57c)\\
\rho_4 &:= D_4\rho + \frac{3}{2}\mathrm{tr}\chi\rho\\
&= \not{\mathrm{d}}\mathrm{iv}\,\beta - \frac{1}{2}\underline{\overline{\chi}}\cdot\alpha + \zeta\cdot\beta\ , && (4.3.58a)\\
\sigma_4 &:= D_4\sigma + \frac{3}{2}\mathrm{tr}\chi\sigma\\
&= -\not{\mathrm{d}}\mathrm{iv}\,{}^\star\beta + \frac{1}{2}\underline{\overline{\chi}}\cdot{}^\star\alpha - \zeta\cdot{}^\star\beta\ , && (4.3.58b)\\
\beta_3 &:= \not{D}_3\beta + \mathrm{tr}\underline{\chi}\beta\\
&= \not\nabla\rho + {}^\star\not\nabla\sigma + 2\overline{\chi}\cdot\underline{\beta} + 2\underline{\omega}\beta + 3(\eta\rho + {}^\star\eta\sigma) + \underline{\xi}\cdot\alpha - a(\underline{\chi})^\star\beta\ , && (4.3.58c)\\
\mathring{\beta}_4 &:= \not{D}_4\mathring{\beta} + 2\mathrm{tr}\chi\mathring{\beta}\\
&= \not{\mathrm{d}}\mathrm{iv}\,\alpha + 2\alpha\cdot\zeta + 2a(\chi)^\star\beta\ , && (4.3.59a)\\
\alpha_3 &:= \not{D}_3\alpha + \frac{1}{2}\mathrm{tr}\underline{\chi}\alpha\\
&= \not\nabla\overline{\otimes}_s\mathring{\beta} + (4\underline{\omega})\alpha - \frac{1}{2}a(\underline{\chi})^\star\alpha\\
&\quad -3(\overline{\chi}\rho + {}^\star\overline{\chi}\sigma) + (4\eta + \zeta)\overline{\otimes}_s\beta\ . && (4.3.59b)
\end{aligned}
$$

4.4 Bianchi equations and symmetric hyperbolic systems

We shall say that a system of first order PDE's for unknowns f, sections of a Riemannian bundle with scalar product $\langle\cdot,\cdot\rangle$, is symmetric hyperbolic if in local coordinates its principal part can be written in the form

$$A^\mu f_{,\mu}$$

with A^μ — symmetric for the scalar product $\langle\cdot,\cdot\rangle$:

$$\langle f, A^\mu g\rangle = \langle A^\mu f, g\rangle\ .$$

One further assumes that the set of covectors $X_\mu \in T^*\mathscr{M}$ such that $A^\mu X_\mu$ is a strictly positif endomorphism of the bundle of the f's is non-empty at each point $p \in \mathscr{M}$. Such covectors are said to be A^μ-timelike future directed. An exteriorally oriented hypersurface is said to be A^μ*-spacelike*, or simply spacelike, if its field of oriented co-normals n_μ is timelike future directed. An exteriorally oriented hypersurface is said to be locally A^μ*-acausal* if $A^\mu n_\mu$ is non-negative. To every symmetric hyperbolic system there is associated an energy-momentum vector

$$\mathscr{E}^\mu(f) := \langle f, A^\mu f\rangle \;, \tag{4.4.1}$$

which is used to derive energy inequalities, *cf.* Section 5.3. Let us show that the principal part of each of the systems (4.3.29)-(4.3.32) is symmetric hyperbolic when the scalar products are appropriately chosen.

1. **The $(\underline{\alpha},\underline{\beta})$ equations (4.3.29):** We have $\underline{\alpha}_{12}=\underline{\alpha}_{21}$, $\underline{\alpha}_{11}=-\underline{\alpha}_{22}$ hence the pair $(\underline{\alpha},\underline{\beta})$ can be parametrized by $f=(\underline{\alpha}_{11},\underline{\alpha}_{12},\underline{\beta}_1,\underline{\beta}_2)$. Equation (4.3.29) can be rewritten as

$$A^\mu\partial_\mu f + Af = F\;, \tag{4.4.2}$$

with

$$A^\mu\partial_\mu = \begin{pmatrix} e_4 & 0 & e_1 & -e_2\\ 0 & e_4 & e_2 & e_1\\ e_1 & e_2 & e_3 & 0\\ -e_2 & e_1 & 0 & e_3\end{pmatrix}\;, \tag{4.4.3}$$

which is obviously symmetric with respect to the scalar product

$$\begin{aligned}\langle f,f\rangle &= \underline{\alpha}_{11}^2+\underline{\alpha}_{12}^2+\underline{\beta}_1^2+\underline{\beta}_2^2 &\text{(4.4.4a)}\\ &= \frac{1}{2}h^{ac}h^{bd}\underline{\alpha}_{ab}\underline{\alpha}_{cd}+h^{ab}\underline{\beta}_a\underline{\beta}_b\;. &\text{(4.4.4b)}\end{aligned}$$

The associated energy-momentum vector is

$$\mathscr{E}^\mu(\underline{\alpha},\underline{\beta}) = \frac{1}{2}h^{ac}h^{bd}\underline{\alpha}_{ab}\underline{\alpha}_{cd}e_4^\mu + 2h^{ab}h^{cd}\underline{\beta}_a\underline{\alpha}_{bc}e_d^\mu + h^{ab}\underline{\beta}_a\underline{\beta}_b e_3^\mu\;. \tag{4.4.5}$$

Straightforward algebra shows that a hypersurface is A^μ-spacelike if and only if it is spacelike with respect to the space-time metric g; it is A^μ-locally acausal if and only if it is non-timelike with respect to the space-time metric g. In fact, for any covector $n=n_i\theta^i$ which is non-spacelike and satisfies $n_3>0$, $n_4>0$, and for any $\lambda>0$ we have

$$\begin{aligned}\mathscr{E}^\mu(\underline{\alpha},\underline{\beta})n_\mu &= \frac{\frac{1}{2}n_3n_4-\lambda^2}{n_3}h^{ac}h^{bd}\underline{\alpha}_{ab}\underline{\alpha}_{cd}\\ &\quad+\Big(\frac{\lambda}{\sqrt{n_3}}\underline{\alpha}^{ab}+\frac{\sqrt{n_3}}{\lambda}\underline{\beta}^a n^b\Big)\Big(\frac{\lambda}{\sqrt{n_3}}\underline{\alpha}_{ab}+\frac{\sqrt{n_3}}{\lambda}\underline{\beta}_a n_b\Big)\\ &\quad+\frac{n_3}{\lambda^2}(\lambda^2-n^an_a)h^{ab}\underline{\beta}_a\underline{\beta}_b\;,\end{aligned} \tag{4.4.6}$$

which explicitly shows that $\mathscr{E}^\mu$ can be used to control the L^2 norm of $\underline{\alpha}$ and $\underline{\beta}$ on any uniformly spacelike hypersurface, by choosing λ so that

$$n_an^a+\epsilon \le \lambda^2 \le -\epsilon+\frac{n_3n_4}{2}\;,\quad \epsilon>0. \tag{4.4.7}$$

Once the symmetric hyperbolic character of (4.3.29) has been established, it is convenient to drop the identification of f with $(\underline{\alpha}_{11}, \underline{\alpha}_{12}, \underline{\beta}_1, \underline{\beta}_2)$, and to consider f as the couple $(\underline{\alpha}, \underline{\beta})$, where $\underline{\alpha}$ and $\underline{\beta}$ have their usual tensorial representation; in that case the scalar product is given by (4.4.4b).

Now, we will need evaluate the expression $\partial_\mu(\sqrt{|\det g|}\mathscr{E}^\mu)/\sqrt{|\det g|})$ which appears in the energy identity considered in Section 5.3. For this purpose it is convenient to rewrite (4.4.2) as

$$A^i \nabla_i f + Bf = F \,, \tag{4.4.8}$$

using a covariant derivative ∇ which is compatible with the scalar product $\langle \cdot, \cdot \rangle$ *and* with the density character of the (odd) form $\sqrt{|\det g|}\mathscr{E}^\mu \partial_\mu \rfloor dx^1 \wedge \cdots dx^{n+1}$, in a sense made clear by the following:

$$\partial_\mu(\langle f, A^\mu f\rangle \sqrt{|\det g|}) = 2\langle f, A^i \nabla_i f)\rangle \sqrt{|\det g|} \,. \tag{4.4.9}$$

A straightforward calculation (*cf., e.g.,* the calculation in Equation (4.4.14) below) shows that the following choice will satisfy this requirement:

$$\begin{aligned} A^4 \nabla_{e_4} \begin{pmatrix} \underline{\alpha} \\ \underline{\beta} \end{pmatrix} &= \begin{pmatrix} (\not{D}_4 + \underline{\upsilon} + \frac{1}{2}\mathrm{tr}\chi)\underline{\alpha} \\ 0 \end{pmatrix} \,, \\ A^3 \nabla_{e_3} \begin{pmatrix} \underline{\alpha} \\ \underline{\beta} \end{pmatrix} &= \begin{pmatrix} 0 \\ (\not{D}_3 + \upsilon + \frac{1}{2}\mathrm{tr}\underline{\chi})\underline{\beta} + \frac{1}{2}\underline{\alpha}\cdot(\eta + \underline{\eta}) \end{pmatrix} \,, \\ \nabla_{e_a} \begin{pmatrix} \underline{\alpha} \\ \underline{\beta} \end{pmatrix} &= \begin{pmatrix} \not{\nabla}_a \underline{\alpha} \\ \not{\nabla}_a \underline{\beta} \end{pmatrix} \,. \end{aligned} \tag{4.4.10}$$

2. **The $(\mathring{\underline{\beta}}, (\mathring{\sigma}, \mathring{\rho}))$ equations (4.3.30):** The analysis of (4.3.30) is obtained by obvious renamings and permutations from that of (4.3.31):

3. **The $((\rho,\sigma),\beta)$ equations (4.3.31):** We set $f = ((\rho,\sigma),\beta) = (\rho,\sigma,\beta_1,\beta_2)$. Equation (4.3.31) can be rewritten in the form (4.4.2) with

$$A^\mu \partial_\mu = \begin{pmatrix} e_4 & 0 & -e_1 & -e_2 \\ 0 & e_4 & -e_2 & e_1 \\ -e_1 & -e_2 & e_3 & 0 \\ -e_2 & e_1 & 0 & e_3 \end{pmatrix} \,, \tag{4.4.11}$$

which is obviously symmetric with respect to the scalar product

$$\begin{aligned} \langle f, f \rangle &= \rho^2 + \sigma^2 + \beta_1^2 + \beta_2^2 \\ &= \rho^2 + \sigma^2 + h^{ab}\beta_a\beta_b \,. \end{aligned}$$

The associated energy-momentum vector is

$$\mathscr{E}^\mu((\rho,\sigma),\beta) = (\rho^2+\sigma^2)e_4^\mu - 2\rho h^{ab}\beta_a e_b^\mu + 2\sigma h^{ab\star}\beta_a e_b^\mu + h^{ab}\beta_a\beta_b e_3^\mu \,. \tag{4.4.12}$$

We can rewrite (4.4.12) in a form analogous to (4.4.6) with any $\lambda \in \mathbb{R}$ and any n_μ timelike future directed (so that $n_3 > 0$, $n_4 > 0$, $-\frac{1}{2}n_3 n_4 + n_a n^a < 0$):

$$\begin{aligned}
\mathscr{E}^\mu n_\mu &= (\rho^2+\sigma^2)\frac{n_3 n_4 - 2\lambda^2}{n_3} \\
&+2\left(\lambda\frac{\rho}{\sqrt{n_3}}h^{ab} - \frac{\sqrt{n_3}}{2\lambda}\beta^a n^b\right)\left(\lambda\frac{\rho}{\sqrt{n_3}}h_{ab} - \frac{\sqrt{n_3}}{2\lambda}\beta_a n_b\right) \\
&+2\left(\lambda\frac{\sigma}{\sqrt{n_3}}h^{ab} + \frac{\sqrt{n_3}}{2\lambda}{}^\star\beta^a n^b\right)\left(\lambda\frac{\sigma}{\sqrt{n_3}}h_{ab} + \frac{\sqrt{n_3}}{2\lambda}{}^\star\beta_a n_b\right) \\
&+\frac{n_3}{\lambda^2}(\lambda^2 - n_b n^b)\beta_a\beta^a \ . \qquad (4.4.13)
\end{aligned}$$

It follows as before that we can control the norm L^2 of ρ, σ and β over any uniformly spacelike hypersurface by choosing λ as in Equation (4.4.7). One can also check that a hypersurface is A^μ-spacelike for the system associated to (4.4.11) if and only if it is spacelike for the metric g.

Let us evaluate the expression $\partial_\mu(\mathscr{E}^\mu\sqrt{|\det g|}) = D_i\mathscr{E}^i\sqrt{|\det g|}$:

$$\begin{aligned}
D_i\mathscr{E}^i &= e_i.\mathscr{E}^i + \Gamma_i{}^i{}_k\mathscr{E}^k \\
&= e_4.(\rho^2+\sigma^2) + e_3.(h^{ab}\beta_a\beta_b) - 2e_a.(\rho h^{ab}\beta_b - \sigma h^{ab\star}\beta_b) \\
&\quad +\Gamma_4{}^4{}_4(\rho^2+\sigma^2) - \Gamma_4{}^4{}_a(2\rho h^{ab}\beta_b - 2\sigma h^{ab\star}\beta_b) \\
&\quad +\Gamma_3{}^3{}_3(h^{ab}\beta_a\beta_b) - \Gamma_3{}^3{}_a(2\rho h^{ab}\beta_b - 2\sigma h^{ab\star}\beta_b) \\
&\quad +\Gamma_a{}^a{}_4(\rho^2+\sigma^2) + \Gamma_a{}^a{}_3 h^{cd}\beta_c\beta_d - 2\Gamma_a{}^a{}_c(\rho h^{bc}\beta_c - \sigma h^{bc\star}\beta_c) \\
&= 2\langle(\rho,\sigma), e_4.(\rho,\sigma) + \frac{1}{2}(\Gamma_4{}^4{}_4 + \Gamma_a{}^a{}_4)(\rho,\sigma)\rangle \\
&\quad +2h^{cd}(\not{D}_3\beta_c + \frac{1}{2}(\Gamma_a{}^a{}_3 + \Gamma_3{}^3{}_3)\beta_c)\beta_d \\
&\quad -2(e_a.\rho)h^{ab}\beta_b + 2(e_a.\sigma)h^{ab\star}\beta_b \\
&\quad -2(e_a.\beta^a + \Gamma_a{}^a{}_c\beta^c + (\Gamma_4{}^4{}_a + \Gamma_3{}^3{}_a)\beta^a)\rho \\
&\quad +2(e_a.{}^\star\beta^a + \Gamma_a{}^a{}_c{}^\star\beta^c + (\Gamma_4{}^4{}_a + \Gamma_3{}^3{}_a){}^\star\beta^a)\sigma \\
&= 2\langle A^i\nabla_i U, U\rangle \ , \qquad (4.4.14)
\end{aligned}$$

with

$$\begin{aligned}
A^4\nabla_{e_4}\begin{pmatrix}(\rho,\sigma)\\ \beta\end{pmatrix} &= \begin{pmatrix}(\not{D}_4 + \underline{\upsilon} + \frac{1}{2}\mathrm{tr}\chi)(\rho,\sigma)\\ 0\end{pmatrix} , \\
A^3\nabla_{e_3}\begin{pmatrix}(\rho,\sigma)\\ \beta\end{pmatrix} &= \begin{pmatrix}0\\ (\not{D}_3 + \upsilon + \frac{1}{2}\mathrm{tr}\underline{\chi})\beta\end{pmatrix} , \\
\nabla_{e_a}\begin{pmatrix}(\rho,\sigma)\\ \beta\end{pmatrix} &= \begin{pmatrix}e_a(\rho,\sigma)\\ \not{\nabla}_a\beta + (\eta+\underline{\eta})\beta_a\end{pmatrix} . \qquad (4.4.15)
\end{aligned}$$

4. **The $(\mathring{\beta},\alpha)$ equations (4.3.32):** The analysis of (4.3.32) is obtained by obvious renamings and permutations from that of (4.3.29), done above.

Chapter 5

Einstein equations - the analysis

5.1 Construction of the functional spaces on space-time

Let $\mathscr{M}$ be a spacetime such that those described in 4.3.3. In particular, all the conditions and gauge choices of 4.3.3 are satisfied. There exists $t > 0$ such that $\mathscr{M}_{[x_2,x_0,t]} \subset \mathscr{M}$ for any $0 < x_2 < x_0$. We recall

$$g^{xx} \equiv 0 \ , \tag{5.1.1}$$

We define the following subset of $\mathscr{M}_{[x_2,x_0,t]}$ for any $x_1 \in]x_2, x_0[$:

$$\mathscr{M}_{x_2,x_1,t} = \{p \in \mathscr{M} \mid x_2 \le x \le x_1 \ , \ x(p) + 3\tau(p) \le x_1 \ , \ 0 \le \tau \le t\} \ . \tag{5.1.2}$$

The (completely arbitrary) choice of the factor 3 appearing in the equations above is motivated as follows: the coordinates here should be thought of as an approximation of the corresponding coordinates in Minkowski space-time of Chapter 3. We will be using the Stokes theorem on $\mathscr{M}_{x_1,t}$, and the causal character of its boundary will determine the sign of the various terms which will result. In the Minkowskian case the sets $\{x + 2\tau = \text{const}\}$ were null hypersurfaces, but in our case this does not need to be true anymore. On the other hand, the hypersurfaces $\{x + 3\tau = \text{const}\}$ are space-like in Minkowski space-time, and will turn out to be spacelike in our case as well; this is sufficient for our purposes.

We have the natural foliation

$$\mathscr{M}_{x_2,x_1,t} = \bigcup_{0 \le \tau \le t} M_{x_2,x_1-3\tau} \times \{\tau\} \ , \tag{5.1.3}$$

where $M_{x_2,x_1-3\tau}$ is a subset of Σ such that

$$x_2 \le x \le x_1 - 3\tau \ . \tag{5.1.4}$$

We denote $h_\tau \equiv h(\tau) \equiv g|_{M_{x_2,x_1-3\tau}\times\{\tau\}}$, $d^n\mu_\tau$ the element volume associated, and for any function f over $\mathscr{M}_{x_2,x_1,t}$,

$$f(\tau) = f|_{M_{x_2,x_1-3\tau}\times\{\tau\}} \ . \tag{5.1.5}$$

As in Section 4.3.3 we consider a finite number of "Gaussian" coordinates (x, v^A) the domains of which cover $M_{x_1} \times \{0\}$; here M_{x_1} is as in (3.2.1a). Let $(\Omega_i, (x, v^A))$ be such a coordinate system on the initial hypersurface, with $\Omega_i \subset M_{x_1} \times \{0\}$. We set

$$M^i_{x_2,x_1-3\tau} = \Omega_i \cap \{x_2 \le x \le x_2 - 3\tau\} \ , \tag{5.1.6a}$$

$$\mathscr{M}^i_{x_2,x_1,t} = \bigcup_{\tau\in[0,t]} M^i_{x_2,x_1-3\tau} \times \{\tau\} \ . \tag{5.1.6b}$$

Then we have the x,τ-adapted coordinate system $(\mathscr{M}^i_{x_2,x_1,t}, (x, v^A, \tau))$ (with $\partial_\tau v^A = 0$) and we have a covering of $\mathscr{M}_{x_2,x_1,t}$ by a finite $x,\tau-$adapted coordinates class.

Remark: We will denote $(z^\mu) = (x, v^A, \tau)$ and $(x^\delta) = (x, v^A)$, the respective coordinates systems on $\mathscr{M}_{x_2,x_1,t}$ and $M_{x_2,x_1-3\tau}$.
Remark: We can construct a partition of unity φ_i associated to the $\mathscr{M}^i_{x_2,x_1,t}$ well adapted to our coordinates : First we take a partition of unity $\mathring{\varphi}_i$ on $\partial\Sigma$ associated with a covering

$$\mathcal{O}_i := \Omega_i \cap \partial\Sigma \tag{5.1.7}$$

of $\partial\Sigma$. Then we propagate to $M_{x_1} \times \{0\}$ and to $\mathscr{M}_{x_2,x_1,t}$ with

$$\begin{aligned} \varphi_i|_{\mathscr{I}^+\cap\Omega_i\times\{0\}} &= \mathring{\varphi}_i\ , && (5.1.8)\\ \partial_x\varphi_i &= 0\ , && (5.1.9)\\ \partial_\tau\varphi_i &= 0\ . && (5.1.10)\end{aligned}$$

Using the φ_i's we define the operators $\mathcal{D}^\beta$ as the collection of the following objects:

$$\begin{aligned} \mathcal{D}^\emptyset &= \varphi_i\ , && (5.1.11)\\ \mathcal{D}^\beta &= \varphi_i\partial_x^{\beta_1}\partial_{v^2}^{\beta_2}\partial_{v^3}^{\beta_3}\ , && (5.1.12)\end{aligned}$$

where (x, v^2, v^3) is the coordinate system of $\Omega_i^\tau = M^i_{x_2,x_1-3\tau}$. The reader is warned that the operators $\mathcal{D}$ here are *not* the same as those used in the previous chapter as defined after Equation (3.2.8). In every sum over $\mathcal{D}^\beta$ below an implicit sum over the i's is understood. Strictly speaking, we should include an index i on the $\mathcal{D}^\beta$'s, but we shall not to do that in order not to overburden notation.

Then, we have over any coordinate patch

$$\partial^\beta = \sum_i \sum_{\gamma_2+\gamma_3=\beta_2+\beta_3} c(\gamma_2,\gamma_3,v^A)\varphi_i\partial_x^{\beta_1}\partial_{v^2}^{\gamma_2}\partial_{v^3}^{\gamma_3}\ , \tag{5.1.13}$$

so that

$$\partial^\beta = \sum_{|\gamma|=|\beta|,\gamma_1=\beta_1} c(\gamma,\beta,v^A)\mathcal{D}^\gamma\ , \tag{5.1.14}$$

where c is a smooth in v^A.

For any vector field Y on $\mathscr{M}_{x_2,x_1,t}$, we define $\|Y(\tau)\|_{\mathscr{H}_k^\beta(M_{x_2,x_1-3\tau})}$ by

$$\|Y(\tau)\|^2_{\mathscr{H}_k^\beta(M_{x_2,x_1-3\tau})} = \tag{5.1.15}$$

$$\sum_i \int_{M^i_{x_2,x_1-3\tau}} \sum_{|\beta|\le k} x^{-2\alpha+2\beta_1-1}\Big((\mathcal{D}^\beta Y^\tau(\tau))^2 + (\mathcal{D}^\beta Y^x(\tau))^2 + \sum_{A=1}^n (\mathcal{D}^\beta Y^A(\tau))^2\Big)\, d^n\mu_0 , \tag{5.1.16}$$

where we sum over all the coordinate patches $(M^i_{x_2,x_1-3\tau})$ of the $x,\tau-$adapted coordinates class, with $d^n\mu_0$ being the volume element associated to h_0 and ϕ_i

$(0 \leq \phi_i \leq 1)$ being a partition of unity of the Ω_i (and so of the $M^i_{x_2,x_1-3\tau}$), and with $Y = Y^x\partial_x + Y^A\partial_A + Y^\tau\partial_\tau$. We can define similarly $\|Y(\tau)\|_{\mathcal{G}^\alpha_k(M_{x_2,x_1-3\tau})}$:

$$\|Y(\tau)\|^2_{\mathcal{G}^\beta_k(M_{x_2,x_1-3\tau})} = \sup_n \sum_i \int_{M^i_{x_2,x_1-3\tau}\cap(I_n\times\mathcal{O}_i)} \sum_{|\beta|\leq k} x^{-2\alpha+2\beta_1-1}\Big((\mathcal{D}^\beta Y^\tau(\tau))^2 + (\mathcal{D}^\beta Y^x(\tau))^2 + \sum_{A=1}^n (\mathcal{D}^\beta Y^A(\tau))^2\Big)\, d^n\mu_0, \tag{5.1.17}$$

where I_n has been defined in Equation (3.2.19), and $\mathcal{O}_i$ in Equation (5.1.7).

We set

$$\|Y(\tau)\|_{\mathscr{C}^\alpha_k(M_{x_2,x_1-3\tau})} = \sup_i \|Y^\mu(\tau)\|_{\mathscr{C}^\alpha_k(M^i_{x_2,x_1-3\tau})}, \tag{5.1.18}$$

$$\|Y(\tau)\|_{\mathcal{B}^\alpha_k(M_{x_2,x_1-3\tau})} = \sum_n \sup_i \|Y^\mu(\tau)\|_{\mathscr{C}^\alpha_k(M^i_{x_2,x_1-3\tau}\cap I_n)}, \tag{5.1.19}$$

and so on for the L^∞ norm.

In the same way, we define $\|Y\|_{\mathscr{H}^\beta_k(\mathscr{M}_{x_2,x_1,t})}$ replacing $M^i_{x_2,x_1-3\tau}$ by $\mathscr{M}^i_{x_2,x_1,t}$ and so on for the $\mathscr{C}^\beta_k$ and $\mathcal{G}^\beta_k$ spaces. Note we will define similar norms in a more geometrical way for some tensors fields over Riemannian bundles of $\mathscr{M}$.

5.2 The boot-strap hypotheses on the tetrad fields, and some consequences

We will require some properties for the space-time. We will be considering null tetrads as in Section 4.3.3, on which some precise functional requirements will be imposed. Our goal is to show that a certain set of conditions on the tetrads is compatible with the evolution of the Weyl tensor via the equations derived in Chapter 4. We will then use the continuity method to obtain existence of the solutions of the vacuum Einstein equations with initial data in weighted Sobolev spaces, with conformal singularities at $\mathscr{I}$.

Throughout this section k will denote an integer, while ϵ is a real number in $]0, 1[$.

$\mathscr{C}1$) We suppose that there exists constants $t_1 > 0$, $C_{\overline{e}_+}$ such that $e_- = e_4 = \partial_\tau$ and $e_+ = e_3$ (defined in Section 4.3.3) can be written $e_+ = \overline{e}_+ + (\partial_\tau - 2\partial_x)$, with for all $\tau \in [0, t_1]$

$$\|\overline{e}_+(\tau)\|_{\mathscr{H}^\epsilon_k(M_{x_2,x_1-3\tau})} + \|\overline{e}_+(\tau)\|_{\mathscr{C}^1_0(M_{x_2,x_1-3\tau})} \leq C_{\overline{e}_+}, \tag{5.2.1}$$

$$\overline{e}_+|_{\tau=0} = 0, \tag{5.2.2}$$

$$\overline{e}^x_+ = 0, \tag{5.2.3}$$

$\mathscr{C}2$) The (e_a) which complete e_3, e_4 in a half null tetrad is such that we can write

$$e_a = \mathring{e}_a + \overline{e}_a, \tag{5.2.4}$$

with the following conditions:

- First, we require that

$$\mathring{e}_a(x, v^A, \tau) = e_a(x, v^A, 0) \,, \tag{5.2.5}$$

(we reminds $(\partial_x, \mathring{e}_a)$ is an orthonormal field of $(M_{x_1} \times \{0\}, h_0)$ and the e_a are tangent to the level set of x), with $\mathring{e}_a(0) \in \mathcal{G}^0_k(M_{x_1})$. Let us note that we have by hypothesis

$$\mathring{e}_a = e_a^A \partial_{v^A} \,. \tag{5.2.6}$$

Further we assume the existence of a constant $C_{\mathring{e}_a}$ such that

$$\|\mathring{e}_a(0)\|_{\mathcal{G}^0_k} + \|\mathring{e}_a(0)\|_{L^\infty} + \|[\mathring{e}_a^A(0)]^{-1}\|_{L^\infty} \leq C_{\mathring{e}_a} \,, \tag{5.2.7}$$

If $k \geq 2$ the second term above would be controlled by the first, however, we do not make any restrictions on k at this stage.

- There exists $C_{\overline{e}_a}$ and $t_2 > 0$ such that for any $\tau \in [0, t_2]$,

$$\begin{aligned} \|\overline{e}_a(\tau)\|_{\mathscr{H}^\epsilon_k(M_{x_2,x_1-3\tau})} + \|\overline{e}_a(\tau)\|_{\mathscr{C}^1_0} \;\; &\leq \;\; C_{\overline{e}_a} \,, & (5.2.8)\\ \overline{e}_a(0) &= 0 \,. & (5.2.9) \end{aligned}$$

Let $t*$ be defined by

$$t* = \min(t_1, t_2) \,. \tag{5.2.10}$$

For $k \geq 3$, the conditions $\mathscr{C}1 - \mathscr{C}2$) imply there exists $C_{\overline{e}}(C_{\overline{e}_a}, C_{\overline{e}_+}, C_s)$ such that

$$\sum_{i=1}^{3} \sup_{0 \leq \tau \leq t} \left\{ \|\overline{e}_i(\tau)\|_{\mathscr{C}^1_0} + \|\overline{e}_i(\tau)\|_{\mathscr{C}^\epsilon_1} + \|\overline{e}_i(\tau)\|_{\mathscr{H}^\epsilon_k} \right\} \;\; \leq \;\; C_{\overline{e}} \,, \tag{5.2.11}$$

for all $\tau \in [0, t*]$, and

$$\overline{e}_i(0) = 0 \,. \tag{5.2.12}$$

Here, and throughout this chapter, we use the generic symbol C_s to denote a constant which arises out of the functional inequalities in weighted spaces of Section 3.2 such as Sobolev inequalities, or Moser inequalities.

Such a tetrad will be called a x, τ- compatible null tetrad. To simplify some wording, we will denote

$$\begin{aligned} \mathring{e}_4 &= e_4 = \partial_\tau \,, & (5.2.13)\\ \overline{e}_4 &= 0 \,, & (5.2.14)\\ \mathring{e}_3 &= \partial_\tau - 2\partial_x \,. & (5.2.15) \end{aligned}$$

Lemma 5.2.1 Under $\mathscr{C}1$)-$\mathscr{C}2$) we have on $M_{x_1} \times \{0\}$,

$$g^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & h^{22}_{0,x} & h^{23}_{0,x} & 0 \\ 0 & h^{23}_{0,x} & h^{33}_{0,x} & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} , \tag{5.2.16}$$

with $h_{0,x} = g|_{TS_{0,x}}$, where $S_{0,x} = \{p \in M_{x_1} \times \{0\} \,|x(p) = x\}$ ($h^{AB}_{0,x} = e^A_a e^B_b g^{ab}$), so that

$$\|g^{\mu\nu}\|_{L^\infty(M_{x_1}\times\{0\})} \leq C(C_{\mathring{e}_a}) , \tag{5.2.17}$$

Further, there exists $c_h > 0$ such that $[h^{AB}_{0,x}] \geq c_h \mathrm{Id}_2$; $g^{\mu\nu}(0)$ on $M_{x_1} \times \{0\}$ is of the form

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & h_{0,x22} & h_{0,x23} & 0 \\ 0 & h_{0,x23} & h_{0,x33} & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} , \tag{5.2.18}$$

and

$$h_{\delta\gamma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & h_{0,x22} & h_{0,x23} \\ 0 & h_{0,x23} & h_{0,x33} \end{pmatrix} , \tag{5.2.19}$$

and we have the estimates

$$\|g_{\mu\nu}\|_{L^\infty(M_{x_1}\times\{0\})} \leq C(C_{\mathring{e}_a}) , \tag{5.2.20}$$

$$\|h_{\delta\gamma}\|_{L^\infty(M_{x_1}\times\{0\})} \leq C(C_{\mathring{e}_a}) . \tag{5.2.21}$$

(Recall that $(x^\delta) = (x, v^A)$, with the indices A, B running from two to three and the indices δ, γ running from one to three.) If we denote by $h^{\delta\gamma}$ the matrix inverse to $h_{\delta\gamma}$, then it also holds that

$$\|h^{\delta\gamma}\|_{L^\infty(M_{x_1}\times\{0\})} \leq C(C_{\mathring{e}_a}) . \tag{5.2.22}$$

Besides, denoting $\mathcal{V}_0 = \sqrt{\det h(0)_{\delta\gamma}}$ (*i.e.*, $\mathcal{V}_0 dx dv^2 dv^3$ is the volume element of $M_{x_1} \times \{0\}^i$), we have

$$C_1(C_{\mathring{e}}) \leq \|\mathcal{V}_0\|_{L^\infty} \leq C(C_{\mathring{e}}) . \tag{5.2.23}$$

PROOF: This follows immediately from the hypotheses made using $g^{\mu\nu} = g^{ij} e^\mu{}_i e^\nu{}_j$, with g^{ij} as at the beginning of Section **??**. □

We denote

$$C_{\mathring{e}} = \sum_i \|\mathring{e}_i\|_{\mathcal{G}^0_k(M_{x_1}\times\{0\})} + \|\mathring{e}_i\|_{L^\infty(M_{x_1}\times\{0\})} + \|[\mathring{e}^\mu_i]^{-1}\|_{L^\infty(M_{x_1}\times\{0\})} . \tag{5.2.24}$$

$\mathscr{C}3$) We suppose there exists a constant C^τ_e such that, for all $0 \leq \tau \leq t*$,

$$\|\partial_\tau \overline{e}_i(\tau)\|_{L^\infty} + \|\partial_\tau \overline{e}_i(\tau)\|_{\mathscr{C}^1_0} + \|\partial_\tau \overline{e}_i(\tau)\|_{\mathscr{H}^\epsilon_k} \leq C^\tau_e , \tag{5.2.25}$$

Remark: Since $\|\partial_\tau \overline{e}^\tau_i\|_{L^\infty} \leq x_1 \|\partial_\tau \overline{e}_i\|_{\mathscr{C}^1_0}$, the first term in (5.2.25) is controlled by the second one, so that (5.2.25) has some redundancy; we have added this term there in order to avoid the appearance of complicated constants later.

Lemma 5.2.2 There exists $C(C_{\overline{e}}, C_{\mathring{e}})$ bounded on bounded sets of variable, such that for all $\tau \in [0, t*]$,

$$\|g^{\mu\nu}(\tau)\|_{L^\infty} + \|h^{\delta\gamma}(\tau)\|_{L^\infty} \leq C(C_{\overline{e}}, C_{\mathring{e}}) \,, \tag{5.2.26}$$

$$\|\partial_\tau g^{\mu\nu}(\tau)\|_{L^\infty} + \|\partial_\tau h^{\delta\gamma}(\tau)\|_{L^\infty} \leq C_e^\tau C(C_{\overline{e}}, C_{\mathring{e}}) \,, \tag{5.2.27}$$

where $(x^\mu) = (x, v^A, \tau)$ and $(x^\delta) = (x, v^A)$. Further there exists $T_0'(C_{\mathring{e}}, C_{\overline{e}}, C_e^\tau)$ such that for all $0 \leq \tau \leq \min(t*, T_0')$, $[e_i^\mu(\tau)]$, $g^{\mu\nu}(\tau)$ and $h^{\delta\gamma}(\tau)$ are invertible with the estimates

$$\|\theta^i_\mu\|_{L^\infty} \leq C(C_{\overline{e}}, C_{\mathring{e}}) \,, \tag{5.2.28}$$

where $[\theta^i_\mu] = [e_i^\mu]^{-1}$,

$$\|h^{\delta\gamma}(\tau)\|_{L^\infty} + \|g^{\mu\nu}(\tau)\|_{L^\infty} + \|g_{\mu\nu}(\tau)\|_{L^\infty} + \|h_{\delta\gamma}(\tau)\|_{L^\infty} \leq C(C_{\overline{e}}, C_{\mathring{e}}) \,, \tag{5.2.29}$$

with $\delta, \gamma = 1, 2, 3$ and C being a constant bounded over bounded sets of variable.

Proof: We write

$$g^{\mu\nu} = e_i^\mu e_j^\nu g^{ij} \,, \tag{5.2.30}$$

$$g_{\mu\nu} = \theta^i_\mu \theta^j_\nu g_{ij} \,, \tag{5.2.31}$$

where θ^i_μ is the inverse matrix of e_i^μ. The first equation gives the estimates (5.2.26). The matrix $[e_i^\mu](0)+h$ is clearly invertible for $\|h\|$ small enough. Thus, the hypothese (5.2.25) ensures that there exists $T_0'(C_{\overline{e}}, C_{\mathring{e}}, C_e^\tau)$ such $[e_i^\mu](\tau)$ is invertible for $\tau \in [0, \min(t*, T_0')]$. The equation (5.2.31) gives the remaining estimates. □

Using the equations of the proof of the last lemma together with the weighted Moser inequalities one obtains, no details will be given:

Lemma 5.2.3 There exists a constant, $C_e(C_{\mathring{e}}, C_{\overline{e}}, x_1, \epsilon)$, such that for any $0 \leq \tau \leq \min(t*, T_0')$,

$$\sum_i \|e_i(\tau)\|_{L^\infty} + \|e_i(\tau)\|_{\mathcal{G}^0_k(M_{x_2, x_1 - 3\tau})} \leq C_e \,. \tag{5.2.32}$$

Further, we have the estimates

$$\|g_{\mu\nu}(\tau)\|_{\mathcal{G}^0_k} + \|g_{\mu\nu}(\tau)\|_{L^\infty} \leq C(C_e) \,, \tag{5.2.33}$$

$$\|\partial_\tau g_{\mu\nu}(\tau)\|_{\mathscr{H}^\epsilon_k} + \|\partial_\tau g_{\mu\nu}(\tau)\|_{\mathscr{C}^1_0} \leq C(C_e, C_e^\tau) \,, \tag{5.2.34}$$

$$\|\theta^i_\mu(\tau)\|_{\mathcal{G}^0_k} + \|\theta^i_\mu(\tau)\|_{L^\infty} \leq C(C_e) \,, \tag{5.2.35}$$

$$\|\partial_\tau \theta^i_\mu(\tau)\|_{\mathscr{H}^\epsilon_k} + \|\partial_\tau \theta^i_\mu(\tau)\|_{\mathscr{C}^1_0} \leq C(C_e, C_e^\tau) \,, \tag{5.2.36}$$

and so on for $h(\tau)$.

In the energy estimates the volume element associated to the metric g appears, the next result allows us to relate the resulting energies with the Sobolev spaces constructed in Section 5.1; the proof is a straightforward consequence of (5.2.29) and is left to the reader:

Lemma 5.2.4 For any $\tau \in [0, \min(t*, T_0')]$, we have

$$c_1(C_{\mathring{e}}, C_{\overline{e}}) \le \mathcal{V}_\tau \le c_2(C_{\mathring{e}}, C_{\overline{e}}) \; , \tag{5.2.37}$$

with

$$\mathcal{V}_\tau = \sqrt{\det |h_{\mu\nu}(\tau)|} \; , \tag{5.2.38}$$

and $0 < c_1 < c_2$. Besides, for any $f \in \mathscr{H}_k^\alpha$ we have

$$\begin{aligned} & c_2^{-1}(C_{\mathring{e}}, C_{\overline{e}}) \int_{M_{x_2,x_1-3\tau}\times\{\tau\}} \sum_{|\beta|\le k} x^{-2\alpha+2\beta_1-1} (\mathcal{D}^\beta f)^2 \; d^n\mu_\tau \\ \le \quad & \|f(\tau)\|_{\mathscr{H}_k^\alpha(M_{x_2,x_1-3\tau})} \\ & \le c_1^{-1}(C_{\mathring{e}}, C_{\overline{e}}) \int_{M_{x_2,x_1-3\tau}\times\{\tau\}} \sum_{|\beta|\le k} x^{-2\alpha+2\beta_1-1} (\mathcal{D}^\beta f)^2 \; d^n\mu_\tau \; , \end{aligned} \tag{5.2.39}$$

where $d\mu_\tau := \mathcal{V}_\tau dx dv^2 dv^3$. An obvious analogue of (5.2.39) holds for $f \in \mathcal{G}_k^\alpha$.

Lemma 5.2.5 Let us define

$$n^\mu = \frac{-g^{\mu\tau}}{\sqrt{-g^{\tau\tau}}} \; , \tag{5.2.40}$$

$$N_\tau = \frac{1}{\sqrt{-g^{\tau\tau}}} \; , \tag{5.2.41}$$

with $g^{\tau\tau} = g(d\tau, d\tau)$. Then, with the conditions $\mathscr{C}0$–$\mathscr{C}3$), there exists $T_0(C_{\overline{e}}, C_e^\tau) \le T_0'$ (with T_0' given by Lemma 5.2.4) such that, for all $0 \le \tau \le \min(t*, T_0)$,

- $|g^{\tau\tau} + 1| \le \frac{1}{4}$,
- $1/2 \le N_\tau \le 2$,
- $\frac{4}{3} h_{\delta\gamma}(0) \ge h_{\delta\gamma}(\tau) \ge \frac{1}{2} h_{\delta,\gamma}(0)$,
- the hypersurfaces

$$\{p \in \mathscr{M}_{x_2,x_1,t} \mid \tau(p) = \tau_0\} \; , \text{ and}$$
$$\{p \in \mathscr{M}_{x_2,x_1,t} \mid x(p) + 3\tau(p) = c \; , \; 0 \le \tau \le \min(t*, T_0) \; , 0 \le x \le x_1\} \; , \tag{5.2.42}$$

  are spacelike for all $0 \le \tau_0 \le \min(t*, T_0)$, and for all $c \in [0, x_1]$ (so that n^μ is the unit future pointing normal to $\{\tau = const.\}$),

and there exists a constant $C(C_{\overline{e}}, C_e^\tau)$ such that for any $\tau \in [0, \min(t*, T_0)]$,

$$\|n^i + \frac{1}{2}(\delta_3^i + \delta_4^i)\|_{L^\infty(M_{x_2,x_1-3\tau})} \le C(C_{\overline{e}}, C_e^\tau)\tau \; , \tag{5.2.43}$$

$$|N_\tau - 1| \le C(C_{\overline{e}}, C_e^\tau)\tau \; . \tag{5.2.44}$$

(Recall that we use the convention in which latin lower-case indices i, j, *etc.*, are tetrad indices, so that the index i in (5.2.43) is a tetrad one.) Besides, n^μ satisfies the estimates

$$\|n^\mu\|_{\mathcal{G}_k^0} + \|\partial_\tau n^\mu\|_{\mathcal{G}_k^0} \le C(C_{\mathring{e}}, C_{\overline{e}}, C_e^\tau) \; . \tag{5.2.45}$$

PROOF: We have

$$\begin{aligned} g(d\tau, d\tau) &= g^{ij} d_i\tau\, d_j\tau \;, \\ &= -(e_- \cdot \tau)(e_+ \cdot \tau) + \sum_a (e_a \cdot \tau)^2 \;, \\ &= -1 - \overline{e}^\tau_+ + \sum_a (\overline{e}^\tau_a)^2 \;. \end{aligned}$$

From (5.2.25) we obtain

$$\begin{aligned} \partial_\tau g(d\tau, d\tau) &= -\partial_\tau \overline{e}^\tau_+ + 2\sum_a (\partial_\tau \overline{e}^\tau_a)\overline{e}^\tau_a \;, \\ |\partial_\tau g(d\tau, d\tau)| &\le C^\tau_e + 2C^\tau_e C_{\overline{e}} \;, \\ |\partial_\tau N_\tau| &\le \frac{1}{2}(C^\tau_e + 2C^\tau_e C_{\overline{e}})N^3_\tau \;. \end{aligned}$$

We have

$$g(d\tau, d\tau)(\tau = 0) = -1 \;,$$

so that integrating in time for

$$0 \le \tau \le T_0 := \frac{1}{2(C^\tau_e + 2C^\tau_e C_{\overline{e}}})$$

we are led to

$$\begin{aligned} -1 - T_0(C^\tau_e + 2C^\tau_e C_{\overline{e}}) &\le g(d\tau, d\tau) \le \quad -1 + T_0(C^\tau_e + 2C^\tau_e C_{\overline{e}}) \;, \\ 1/4 &\le -g(d\tau, d\tau) \le \quad 3/4 \;, \end{aligned} \tag{5.2.46}$$

and

$$\sqrt{\frac{2}{3}} \le N_\tau \le \sqrt{2} \;.$$

Spacelikeness of the level sets of τ follows from (5.2.46). The estimate on $h_{\delta\gamma}$ is derived in a similar way. Next, for $\tau \le T_0$,

$$|\partial_\tau N_\tau| \le (C^\tau_e + 2C^\tau_e C_{\overline{e}})\sqrt{2} \;. \tag{5.2.47}$$

On the other hand writing $n^i = g^{ij}\tau_{,j}N_\tau$, we find

$$\begin{aligned} n^3 &= -\frac{1}{2}N_\tau \;, \\ n^4 &= -\frac{1}{2}(1 + \overline{e}^\tau_3)N_\tau \;, \\ n^a &= \overline{e}^\tau_a N_\tau \;, \end{aligned}$$

with the indices here being *tetrad* indices. This gives with (5.2.47) for any $\tau \in [0, T_0]$, there exists a constant $C(C_{\overline{e}}, C^\tau_e)$ such that

$$\begin{aligned} |\partial_\tau n^i| &\le C(C_{\overline{e}}, C^\tau_e) \;, \\ |\partial_\tau N_\tau| &\le C(C_{\overline{e}}, C^\tau_e) \;. \end{aligned}$$

Integrating in time gives (5.2.43). Equation (5.2.45) follows from Equations (5.2.40)-(5.2.41) and Lemma 5.2.3.

For the hypersurface $\{p \in \mathscr{M}_{x_2,x_1,t*} \mid x(p) + 3\tau(p) = x_1\}$, we compute the norm of the gradient

$$\begin{aligned} g(dx + 3d\tau, dx + 3d\tau) &= g(dx, dx) + 9g(d\tau, d\tau) + 6g(dx, d\tau) \\ &= 0 + 9g(d\tau, d\tau) - \frac{1}{2} 6 e_3^x e_4^\tau \\ &= 9g(d\tau, d\tau) + 6 < -3/4 \end{aligned}$$

(recall that $g(dx, dx) = 0$ by (5.1.1); we have also used (5.2.3), (5.2.13), (5.2.15) and (5.2.46)), and the result is established. □.

5.3 Energy estimates for a class of hyperbolic systems in $\mathscr{M}_{x_2,x_1,t}$

The aim of this section is to derive an energy inequality similar to that of Proposition 3.4.1, under hypotheses which are compatible with the various systems extracted out of the vacuum Einstein equations in Chapter 4 — the main point is to relax the hypothesis (3.4.13) of Chapter 3. While that hypothesis is relaxed, we impose some other hypotheses here that are more stringent that those of Section 3.4; this is not necessary, but it simplifies some estimations and is sufficient for our purposes here.

Let $\mathcal{N}_0$ be the space orthogonal to the bundle generated by e_+, e_-, (so that, with the notations of the previous section, (e_a) is a field of frames on $\mathcal{N}_0$), and $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$ with $\mathcal{N}_1$ and $\mathcal{N}_2$ obtained through some cartesian and tensorial products of $\mathcal{N}_0$ and its dual.

$$A^\mu \partial_\mu f + Af = F \ , \tag{5.3.1}$$

a first order system, with f a section of $\mathcal{N}$. We will denote

$$f = \begin{pmatrix} \phi \\ \psi \end{pmatrix} \ , \tag{5.3.2}$$

with $\phi \in \mathcal{N}_1$ and $\psi \in \mathcal{N}_2$. The Lorentzian metric g induces a Riemannian metric on $\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2$ and $\mathcal{N}$ (we take the metric product of the metrics of $\mathcal{N}_1$ and $\mathcal{N}_2$ so that these spaces are orthogonal). These metrics will be denoted h in both cases. We will write $\mathcal{N} = \mathcal{N}_1 + \mathcal{N}_2$ with the decomposition being orthogonal.

Therefore we can define

$$\langle f, f \rangle = h(f, f) \ . \tag{5.3.3}$$

Lemma 5.3.1 For a tensor field f in $\mathcal{N}$, the norms

$$\|f(\tau)\|_{\mathscr{H}_k^\alpha} = \left(\sum_{|\beta| \le k} \sum_i \int_{M^i_{x_2,x_1-3\tau}} x^{-2\alpha+2\beta_1-1} \sum_{a_1 \dots a_N} (\mathcal{D}^\beta f_{a_1\dots a_N}(\tau))^2 d^n\mu_0 \right)^{1/2} \tag{5.3.4}$$

$$\left(\sum_{|\beta|\leq k}\sum_i\int_{M^i_{x_2,x_1-3\tau}} x^{-2\alpha+2\beta_1-1}\sum_{a_1\ldots a_N}(\mathcal{D}^\beta f_{a_1\ldots a_N}(\tau))^2\, d^n\mu_\tau\right)^{1/2} \tag{5.3.5}$$

and

$$\left(\sum_{|\beta|\leq k}\sum_i\int_{M^i_{x_2,x_1-3\tau}} x^{-2\alpha+2\beta_1-1}\sum_{a_1\ldots a_N}(\partial^\beta f_{a_1\ldots a_N}(\tau))^2 d^n\mu_0\right)^{1/2}, \tag{5.3.6}$$

are equivalent for all $\tau\in[0,\min(t*,T_0')]$, with $f_{a_1\ldots a_N}$ being the expression of f in the tetrad field e_a $x,\tau-$adapted.

PROOF: The equivalence of (5.3.4) and (5.3.5) follows from the volume estimate of Lemma 5.2.4; the second equivalence follows from Equation (5.1.14). □

We will make the following assumptions:
$H0$) We have

$$A^\mu Y_\mu \geq 0\ , \tag{5.3.7}$$

for any future causal vector Y^μ.
$H1$) We can write $A^\mu\partial_\mu = A^i e_i$, with A^i constant, and (e_i) being a null tetrad satisfying the hypotheses made in the previous section.
$H2$) We can write

$$A^a f = \begin{pmatrix} A'^a\psi \\ {}^tA'^a\phi\end{pmatrix}, \tag{5.3.8a}$$

$$A^3 f = \begin{pmatrix} 0 \\ \psi\end{pmatrix}, \tag{5.3.8b}$$

$$A^4 f = \begin{pmatrix} \phi \\ 0\end{pmatrix}, \tag{5.3.8c}$$

$$Af = \begin{pmatrix} B_{11}\phi+B_{12}\psi \\ B_{21}\phi+B_{22}\psi\end{pmatrix}, \tag{5.3.8d}$$

where we write ${}^tC$ for the transpose of a matrix C to leave room for some indices on C. We denote by

$$C^a = \|A^a\|\ , \tag{5.3.9}$$

with $\|A^a\|^2 = \sup_{\langle U,U\rangle=1}\langle A^aU, AU\rangle$.
$H3$) There exists a covariant derivative ∇ on $\mathcal{N}$ such that

$$D_i\langle A^iU,U\rangle = 2\langle A^i\nabla_i U, U\rangle\ , \tag{5.3.10}$$

and there exist $\Gamma_3,\Gamma_3',\Gamma_4,\Gamma,\Gamma'$ such that

$$A^3\nabla_3 f = \begin{pmatrix} 0 \\ e_3^\mu\partial_\mu\psi+\Gamma_3\psi+\Gamma_3'\phi\end{pmatrix}, \tag{5.3.11a}$$

$$A^4\nabla_4 f = \begin{pmatrix} e_4^\mu\partial_\mu\phi+\Gamma_4\phi+\Gamma_4'\psi \\ 0\end{pmatrix}, \tag{5.3.11b}$$

$$A^a\nabla_a f = A^a e_a^\mu\partial_\mu f + \begin{pmatrix} \Gamma\psi \\ \Gamma'\phi\end{pmatrix}. \tag{5.3.11c}$$

Remark: These hypotheses are satisfied in systems such as (4.4.10). In our application the matrix A^i will be symmetric.
$H4$) There exist constants C_A, $C_{\mathring{\Gamma}}$ and $C_{\overline{\Gamma}}$ such that for all $\tau \in [0, t*]$,

$$\|A(\tau)\|_{\mathcal{G}^0_k} + \|A(\tau)\|_{\mathscr{C}^0_0} + \|\partial_\tau A(\tau)\|_{\mathscr{C}^0_0} \le C_A \ , \tag{5.3.12}$$

and

$$\begin{aligned} &\|\Gamma_3(\tau)\|_{L^\infty} + \|\Gamma'_3(\tau)\|_{L^\infty} + \|\Gamma_4(\tau)\|_{L^\infty} \\ &+\|\Gamma'_4(\tau)\|_{L^\infty} + \|\Gamma(\tau)\|_{L^\infty} + \|\Gamma'(\tau)\|_{L^\infty} \quad \le \quad C_\Gamma \ . \end{aligned} \tag{5.3.13}$$

With our hypotheses we have:

Lemma 5.3.2 There exist constants $C_1(C_e, C^a)$, $C_2(C^\tau_e, C^a)$, $C_3(C_{\overline{e}}, C^a)$, and $C_4(C_{\overline{e}}, C^\tau_e, C^a)$ such that for all $\tau \in [0, t*]$,

$$\begin{aligned} &\|A^3 e^x_3(\tau)\|_{\mathcal{G}^0_k} + \|A^4 e^\tau_4(\tau)\|_{\mathcal{G}^0_k} + \|A^3 e^\tau_3(\tau)\|_{\mathcal{G}^0_k} \\ &+\|A^3 e^A_3(\tau)\|_{\mathcal{G}^0_k} + \|A^a e^A_a(\tau)\|_{\mathcal{G}^0_k} \le C_1 \ , \end{aligned} \tag{5.3.14}$$

$$\|A^x(\tau)\|_{L^\infty} + \|A^\tau(\tau)\|_{L^\infty} + \|A^A(\tau)\|_{L^\infty} \quad \le \quad C_1 , \tag{5.3.15}$$

$$\|\partial_\tau A^\tau(\tau)\|_{\mathcal{G}^0_k} + \|\partial_\tau A^A(\tau)\|_{\mathcal{G}^0_k} \quad \le \quad C_2 , \tag{5.3.16}$$

$$\|A^a e^\tau_a(\tau)\|_{\mathscr{H}^\epsilon_k} + \|A^a e^\tau_a(\tau)\|_{\mathscr{C}^1_0} + \|\overline{e}^\tau_+(\tau)\|_{\mathscr{H}^\epsilon_k} + \|\overline{e}^\tau_+(\tau)\|_{\mathscr{C}^1_0} \quad \le \quad C_3 , \tag{5.3.17}$$

$$\begin{aligned} &\|A^a \partial_\tau e^\tau_a(\tau)\|_{\mathscr{H}^\epsilon_k} + \|A^a \partial_\tau e^\tau_a(\tau)\|_{\mathscr{C}^1_0} + \|\partial_\tau e^\tau_+(\tau)\|_{\mathscr{H}^\epsilon_k} \\ &+\|\partial_\tau \overline{e}^\tau_+(\tau)\|_{\mathscr{C}^1_0} \quad \le \quad C_4 . \end{aligned} \tag{5.3.18}$$

Remark: Let us note that $A^x = A^3 e^x_3$ is constant by Equations (5.2.3) and (5.2.15).
Remark: In the problem at hand — the Einstein equations or wave equations — the matrix A^a will be constants with norm equal to 1 or 2. Therefore, the dependance in C^a will be omitted in various estimates. Besides, we will use the letter C to denote various irrelevant constants, indicating whenever necessary the dependencies, and the letter C_s to denote the constants from the Sobolev and Moser-type inequalities.
PROOF: The various inequalities are direct consequences of the hypotheses (5.2.11), (5.2.25) and (5.2.32) on the e_i's and their time derivatives, together with (5.2.13)-(5.2.15). □

We define the energy associated to the system 5.3.1 :

$$\mathcal{E}^\alpha_k(\tau) \equiv \sum_{|\beta| \le k} \int_{M_{x_2, x_1 - 3\tau}} x^{-2\alpha - 1 + 2\beta_1} \langle A^i \mathcal{D}^\beta f(\tau), \mathcal{D}^\beta f(\tau) \rangle n_i \ d^n\mu_\tau \ . \tag{5.3.19}$$

We will need the following lemma to establish the equivalence between the norm associated to the energy above and the weighted Sobolev norms (5.3.4).

Lemma 5.3.3 Suppose $\mathscr{C}1)$-$\mathscr{C}3)$ and $H1)$-$H3)$ are satisfied. Then, there exists $T_1(C_{\overline{e}}, C_e^\tau) \leq T_0$ (T_0 defined in Lemma 5.2.5), such that, for any $0 \leq \tau \leq \min(t*, T_1)$,

$$\begin{aligned} 1/4 &\leq n_3 \leq \frac{3}{4} \ , \\ 1/4 &\leq n_4 \leq \frac{3}{4} \ , \\ -1/8 &\leq \sum_{a=1,2} C^a n_a \leq 1/8 \ . \end{aligned} \tag{5.3.20}$$

Further, for any $\tau \in [0, \min(t*, T_1)]$, we have

$$\frac{1}{8} c_1 \|f\|^2_{\mathscr{H}_k^\alpha(M_{x_2,x_1-3\tau})} \leq \mathcal{E}_k^\alpha(\tau) \leq c_2 \|f\|^2_{\mathscr{H}_k^\alpha(M_{x_2,x_1-3\tau})} \ , \tag{5.3.21}$$

where c_1, c_2 are the constants of Lemma 5.2.4.

Proof: For any $0 \leq \tau \leq \min(t*, T_0)$, we have from Equation (5.2.43), Lemma 5.2.5,

$$\|n^i + \frac{1}{2}(\delta_3^i + \delta_4^i)\|_{L^\infty(M_{x_2,x_1-3\tau})} \leq C(C_{\overline{e}}, C_e^\tau)\tau \ . \tag{5.3.22}$$

Then, we set

$$T_1 = \min\left(\frac{1}{4C}, \frac{1}{8(C^1 + C^2)C}, T_0\right) \ ,$$

and we easily check (5.3.20). With the estimates (5.3.20) we obtain (note $A^3 + A^4 = \mathrm{Id}$) Equation (5.3.21). □.

Lemma 5.3.4 Suppose that $k \geq 3$, and let f be a solution in $H_k^{\mathrm{loc}}(\mathscr{M}_{x_2,x_1,T_0})$ of (5.3.1). There exists $T_1'(C^a, C_e^\tau, C_{\overline{e}}) \leq T_1$, where T_1 is given by Lemma 5.3.3, such that if $f(\tau) \in \mathscr{H}_k^\alpha(M_{x_2,x_1-3\tau})$ for some $\tau \in [0, \min(t*, T_1)]$, then there exists a constant $C_5(C_{\overline{e}}, C_e, C_s, C_A, C^a, x_1, \alpha, \epsilon)$ such that

$$\begin{aligned} \|\partial_\tau \phi(\tau)\|_{\mathscr{H}_k^\alpha} &\leq C_5 \Big(\|\psi(\tau)\|_{L^\infty} + \|f(\tau)\|_{\mathscr{H}_k^\alpha} + \|a(\tau)\|_{\mathscr{H}_k^\alpha} \\ &\qquad + \|b(\tau)\|_{\mathscr{H}_k^{\alpha-1/2}} + \|b(\tau)\|_{\mathscr{C}_0^\alpha} \Big) \ , \end{aligned} \tag{5.3.23}$$

$$\|\partial_\tau \psi(\tau)\|_{\mathscr{H}_k^{\alpha-1}} \leq C_5 \left(\|f(\tau)\|_{\mathscr{H}_k^\alpha} + \|a(\tau)\|_{\mathscr{H}_k^\alpha} + \|b\|_{\mathscr{H}_k^{\alpha-1/2}} \right) \tag{5.3.24}$$

Remarks : 1) The proof below actually establishes Equations (5.3.23)-(5.3.24) with an $\mathscr{H}_k^{\alpha-1}$ norm on b, but the above is sufficient for our purposes.

2) We have a weaker estimate which do not require an L^∞ bound on ψ

$$\|\partial_\tau f(\tau)\|_{\mathscr{H}_k^{\alpha-1}} \leq C_5'(\|f(\tau)\|_{\mathscr{H}_k^\alpha} + \|F(\tau)\|_{\mathscr{H}_k^{\alpha-1}}) \ , \tag{5.3.25}$$

with C_5' depending upon $C_e, C_A, C^a, x_1, \alpha$, bounded on bounded sets of variable. This estimate is a straightforward consequence of (5.3.28).

Proof: We have

$$A^\tau \partial_\tau f = -A^x \partial_x f - A^A \partial_A f - Af + F \ , \tag{5.3.26}$$

with

$$A^\tau = A^3(1+\overline{e}_3^\tau) + A^4 + A^a e_a^\tau = \mathrm{Id} + A^3\overline{e}_3^\tau + A^a\overline{e}_a^\tau \ . \tag{5.3.27}$$

Because

$$\tau \le T_1' := \min\left(T_1, \frac{1}{4(1+C^1+C^2)C_{\overline{e}}^\tau}\right)$$

we have

$$\begin{aligned} \|\mathrm{Id} + A^3\overline{e}_3^\tau + A^a\overline{e}_a^\tau\| &\ge 1 - \|A^a\overline{e}_a^\tau\| \ , \\ &\ge 1-(1+C^1+C^2)C_{\overline{e}}^\tau T_1' \ , \\ &\ge 3/4 \ , \end{aligned}$$

which implies that A^τ is invertible. Then we write over any coordinate patch $M^i_{x_2,x_1-3\tau}$,

$$\partial_\tau f \ = \ -(A^\tau)^{-1}\left(A^x\partial_x f + A^A\partial_A f + Af - F\right) \ , \tag{5.3.28}$$

with $A^x = -2A^3\partial_x$. By Lemma 5.3.2 and by weighted Moser inequalities similar to Proposition 3.2.2 we have

$$\|(A^\tau)^{-1}(\tau)\|_{\mathcal{G}^0_k(M_{x_2,x_1-3\tau})} \le C(C_e) \ , \tag{5.3.29}$$

for some increasing function C (dependance upon the number of patches implicit), and for any $\tau \in [0, \min(t*, T_1)]$. More precisely, using the structure of (5.3.27), we have

$$(A^\tau)^{-1} \ = \ \mathrm{Id} + \overline{A} \ , \tag{5.3.30}$$

with

$$\|\overline{A}\|_{\mathscr{H}^\epsilon_k} + \|\overline{A}\|_{\mathscr{C}^1_0} \le C(C_{\overline{e}}, C^a) \ . \tag{5.3.31}$$

Denoting

$$\overline{A} = \begin{pmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{pmatrix} \ , \tag{5.3.32}$$

we have

$$\begin{aligned} \begin{pmatrix} \partial_\tau\phi \\ \partial_\tau\psi \end{pmatrix} \ &= \ \begin{pmatrix} -\overline{A}_{12}\partial_x\psi \\ (2-\overline{A}_{22})\partial_x\psi \end{pmatrix} + \begin{pmatrix} a + \overline{A}_{11}a + \overline{A}_{12}b \\ b + \overline{A}_{21}a + \overline{A}_{22}b \end{pmatrix} \\ &\quad -(A^\tau)^{-1}\left\{A^A\partial_A\begin{pmatrix}\phi\\ \psi\end{pmatrix} + A\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\} \ , \end{aligned} \tag{5.3.33}$$

which can be written

$$\begin{aligned} \begin{pmatrix} \partial_\tau\phi \\ \partial_\tau\psi \end{pmatrix} \ &= \ \begin{pmatrix} -\overline{A}_{12}\partial_x\psi \\ (2-\overline{A}_{22})\partial_x\psi \end{pmatrix} + \begin{pmatrix} a + \overline{A}_{11}a + \overline{A}_{12}b \\ b + \overline{A}_{21}a + \overline{A}_{22}b \end{pmatrix} \\ &\quad + \begin{pmatrix} L_1^A\partial_A f \\ L_2^A\partial_A f \end{pmatrix} + \begin{pmatrix} \ell_1 f \\ \ell_2 f \end{pmatrix} \ , \end{aligned} \tag{5.3.34}$$

with

$$\begin{aligned} &\|L_1(\tau)\|_{\mathcal{G}^0_k} + \|L_2(\tau)\|_{\mathcal{G}^0_k} + \|\ell_1(\tau)\|_{\mathcal{G}^0_k} + \|\ell_2(\tau)\|_{\mathcal{G}^0_k} \\ &+\|L_1(\tau)\|_{L^\infty} + \|L_2(\tau)\|_{L^\infty} + \|\ell_1(\tau)\|_{L^\infty} + \|\ell_2(\tau)\|_{L^\infty} \quad \leq \quad C'(C_e, C^a, C_A) \,. \end{aligned} \tag{5.3.35}$$

Then, with the weighted Moser inequalities of Proposition 3.2.2 we compute

$$\begin{aligned} \|\overline{A}_{12}\partial_x\psi\|_{\mathscr{H}^\alpha_{k-1}} &\leq \|\overline{A}_{12}\psi\|_{\mathscr{H}^{\alpha+1}_k} \\ &\leq C_s\left(\|\overline{A}_{12}\|_{\mathscr{C}^1_0}\|\psi\|_{\mathscr{H}^\alpha_k} + \|\overline{A}_{12}\|_{\mathscr{H}^{1+\alpha}_k}\|\psi\|_{L^\infty}\right) \\ &\leq C(C_{\overline{e}}, C^a, x_1, \epsilon, \alpha)(\|\psi\|_{L^\infty} + \|\psi\|_{\mathscr{H}^\alpha_k}) \,, \\ \|(2-\overline{A_{22}})\partial_x\psi\|_{\mathscr{H}^{\alpha-1}_k} &\leq (2\|\psi\|_{\mathscr{H}^\alpha_k} + C_s\|\overline{A_{22}}\|_{\mathscr{H}^0_k}\|\psi\|_{\mathscr{H}^\alpha_k}) \\ &\leq C(C_{\overline{e}}, C^a, C_e, x_1, \epsilon)\|\psi\|_{\mathscr{H}^\alpha_k} \,. \end{aligned}$$

Similarly

$$\begin{aligned} \|\overline{A}_{12}b\|_{\mathscr{H}^\alpha_k} &\leq C_s(\|\overline{A}_{12}\|_{\mathscr{C}^1_0}\|b\|_{\mathscr{H}^{\alpha-1}_k} + \|\overline{A}_{12}\|_{\mathscr{H}^0_k}\|b\|_{\mathscr{C}^\alpha_0}) \\ &\leq C(C_{\overline{e}}, C^a, C_s, x_1, \alpha, \epsilon)(\|b\|_{\mathscr{H}^{\alpha-1/2}_k} + \|b\|_{\mathscr{C}^\alpha_0}) \,. \end{aligned}$$

The estimates of remaining terms are straightforward, which gives (5.3.23). Equation (5.3.24) is starightforward. □

Now, we will derive various inequalities on ψ and time derivatives of f in Hölder spaces.

Lemma 5.3.5 Let f be a solution in $C^{\rm loc}_1(\mathscr{M}_{x_2,x_1,t*})$ of (5.3.1) with $-1<\alpha<0$, then, for $0\leq\tau\leq\min(t*,T_1')$, where T_1' is given by Lemma 5.3.4, there exists $C_6(C_{\overline{e}}, C_e, C_A, C^a, x_1, \alpha)$ such that

$$\|\partial_\tau\phi(\tau)\|_{\mathscr{C}^\alpha_0} \leq C_6\left(\|f(\tau)\|_{\mathscr{C}^\alpha_1} + \|a(\tau)\|_{\mathscr{C}^\alpha_0} + \|b(\tau)\|_{\mathscr{C}^{\alpha-1/2}_0}\right) \,, \tag{5.3.36}$$

$$\|\partial_\tau\psi(\tau)\|_{\mathscr{C}^{\alpha-1}_0} \leq C_6\left(\|f(\tau)\|_{\mathscr{C}^\alpha_1} + \|a(\tau)\|_{\mathscr{C}^\alpha_0} + \|b(\tau)\|_{\mathscr{C}^{\alpha-1/2}_0}\right) \,, \tag{5.3.37}$$

and

$$\begin{aligned} &\|\psi(\tau)\|^2_{L^\infty} - \|\psi(0)\|^2_{L^\infty} \\ &\quad\leq \int_0^\tau C_6(\tau-s)^\alpha\left(\|f(s)\|^2_{\mathscr{C}^\alpha_1} + \|F(s)\|^2_{\mathscr{C}^\alpha_0} + \|\psi(s)\|^2_{L^\infty}\right) ds \,. \end{aligned} \tag{5.3.38}$$

Further, suppose that $\epsilon > 1+\alpha$ and $k\geq 3$, where ϵ, k are the constants from (5.2.1) and (5.2.8), then

$$\begin{aligned} \|x\partial_x\psi(\tau)\|^2_{L^\infty} &\leq \|x\partial_x\psi(0)\|^2_{L^\infty} \\ &\quad + \int_0^\tau C_6'(\tau-s)^\alpha(\|f(s)\|^2_{\mathscr{C}^\alpha_2} + \|F(s)\|^2_{\mathscr{C}^\alpha_1} + \|x\partial_x\psi(s)\|^2_{L^\infty})\, ds \,, \end{aligned} \tag{5.3.39}$$

for $0\leq\tau\leq T_1'$, with C_6' depending upon $(C_{\overline{e}}, C_{\mathring{e}}, C^a, C_A, x_1, \epsilon, \alpha)$, bounded on bounded sets of variable.

PROOF: The estimates (5.3.36)-(5.3.37) can be proven from (5.3.33) in a straightforward manner. To prove (5.3.38), one writes

$$(\partial_\tau - 2\partial_x)\psi = -\overline{e}_+^\tau \partial_\tau\psi - \overline{e}_+^A\partial_A\psi - A^a(\overline{e}_a^\tau\partial_\tau\phi + e_a^A\partial_A\phi) - B_{21}\phi - B_{22}\psi + b \,. \tag{5.3.40}$$

Then, from (5.2.1), (5.2.8) and (5.3.36)-(5.3.37), one finds

$$\begin{aligned}\|(\partial_\tau - 2\partial_x)\psi(\tau)\|_{\mathscr{C}_0^\alpha} &\leq C(C_e, C_{\overline{e}}, C_A, C^a, x_1, \alpha)\left(\|f(\tau)\|_{\mathscr{C}_1^\alpha}\right. \\ &\qquad \left. + \|F(\tau)\|_{\mathscr{C}_0^\alpha}\right) \,, \end{aligned} \tag{5.3.41}$$

Setting $\|\psi\|^2 = \langle\psi,\psi\rangle$, the formula

$$\begin{aligned} g(x, v^A, \tau) &= g(x+2\tau, v^A, 0) + \int_{x/2}^{x/2+\tau} (\partial_\tau - 2\partial_x) g(2v, v^A, \tau - v + x/2) dv \\ &= g(x+2\tau, v^A, 0) + \int_0^\tau (\partial_\tau - 2\partial_x) g(2\tau - 2s + x, v^A, s) ds \,, \end{aligned} \tag{5.3.42}$$

valid for any function $g \in \mathscr{C}_1^{\mathrm{loc}}$, $0 < x < x_1 - 2\tau$, leads to

$$\|\psi\|^2(x, v^A, \tau) = \|\psi\|^2(x+2\tau, v^A, 0) + \int_0^\tau (\partial_\tau - 2\partial_x)\|\psi\|^2(2\tau - 2s + x, v^A, s) ds \,. \tag{5.3.43}$$

Further,

$$\begin{aligned} (\partial_\tau - 2\partial x)\|\psi\|^2(x, v^A, \tau) &= 2\langle(\partial_\tau - 2\partial_x)\psi, \psi\rangle(x, v^A, \tau) \\ &\leq 2\|(\partial_\tau - 2\partial_x)\psi(x, v^A, \tau)\| \; \|\psi(x, v^A, \tau)\| \\ &\leq 2x^\alpha \|\psi(\tau)\|_{L^\infty}\|(\partial_\tau - 2\partial_x)\psi(\tau)\|_{\mathscr{C}_0^\alpha} \,, \end{aligned}$$

where we have written

$$\|(\partial_\tau - 2\partial_x)\psi\|(x, v, \tau) \leq x^\alpha \|(\partial_\tau - 2\partial_x)\psi\|_{\mathscr{C}_0^\alpha} \,. \tag{5.3.44}$$

The last estimate with (5.3.41) gives

$$\begin{aligned} \|\psi\|^2(x, v^A, \tau) &\leq \|\psi\|^2(x+2\tau, v^A, 0) + \int_0^\tau (x + 2(\tau - s))^\alpha C(C_e, C_{\overline{e}}, C_A, C^a, x_1, \alpha)\|\psi(s)\|_{L^\infty} \times \\ &\qquad \left(\|f(s)\|_{\mathscr{C}_1^\alpha} + \|F(s)\|_{\mathscr{C}_0^\alpha}\right) \, ds \\ &\leq \|\psi\|^2(x+2\tau, v^A, 0) + \int_0^\tau (\tau - s)^\alpha C\left(\|\psi(s)\|^2_{L^\infty} + \|f(s)\|^2_{\mathscr{C}_1^\alpha} + \|F(s)\|^2_{\mathscr{C}_0^\alpha}\right) ds \,, \end{aligned}$$

which gives (5.3.38) taking the sup norm over x, v^A.

For the last estimate, one computes from (5.3.34),

$$\begin{aligned} (\partial_\tau - 2\partial_x)(x\partial_x\psi) &= -\overline{A}_{22}x\partial_x\partial_x\psi + L_2^A\partial_A(x\partial_x f) + l_2(x\partial_x f) + x\partial_x(b + \overline{A}_{21}a + \overline{A}_{22}b) \\ &\quad -(\partial_x\overline{A}_{22})x\partial_x\psi + (x\partial_x L_2^A)\partial_A f + (x\partial_x \ell_2) f \,. \end{aligned}$$

One writes

$$(\partial_\tau - 2\partial_x)(x\partial_x\psi) = -(\partial_x\overline{A}_{22})x\partial_x\psi + R\ . \tag{5.3.45}$$

By (5.3.31) and (5.3.35), we have

$$\|R(\tau)\|_{\mathscr{C}_0^\alpha} \leq C(C_{\overline{e}}, C_{\mathring{e}}, C_A, C^a, x_1, \alpha, \epsilon)(\|f(\tau)\|_{\mathscr{C}_2^\alpha} + \|F(\tau)\|_{\mathscr{C}_1^\alpha})\ .$$

For instance the first term is estimated as follows

$$\begin{aligned} \|-\overline{A}_{22}x\partial_x\partial_x\psi\|_{\mathscr{C}_0^\alpha} &\leq C_s\|\overline{A}\|_{\mathscr{C}_0^1}\|\psi(\tau)\|_{\mathscr{C}_2^\alpha} \\ &\leq C(C_{\overline{e}}, C_{\mathring{e}}, C^a)\|f(\tau)\|_{\mathscr{C}_2^\alpha}\ , \end{aligned}$$

and the term $(x\partial_x L_2^A)\partial_A f$:

$$\begin{aligned} \|(x\partial_x L_2^A)\partial_A f(\tau)\|_{\mathscr{C}_0^\alpha} &\leq \|L_2^A(\tau)\|_{\mathscr{C}_1^0}\|f_1(\tau)\|_{\mathscr{C}_1^\alpha} \\ &\leq C(C_e, C^a, \epsilon)\|f(\tau)\|_{\mathscr{C}_1^\alpha}\ . \end{aligned}$$

On the other hand, from (5.3.31) and $\epsilon \geq 1+\alpha$,

$$\begin{aligned} \|(\partial_x\overline{A}_{22})(x\partial_x\psi)\|_{\mathscr{C}_0^\alpha} &\leq \|\overline{A}_{22}\|_{\mathscr{C}_1^{1+\alpha}}\|x\partial_x\psi\|_{L^\infty} \\ &\leq C(C_{\overline{e}}, C_{\mathring{e}}, C^a, x_1, \alpha, \epsilon)\|x\partial_x\psi\|_{L^\infty}\ . \end{aligned}$$

Therefore

$$\begin{aligned} &(\partial_\tau - 2\partial_x)\|x\partial_x\psi\|^2(x, v^A, \tau) \leq \\ &\quad C(C_{\overline{e}}, C_{\mathring{e}}, C^a, C_A, x_1, \alpha, \epsilon)x^\alpha(\|x\partial_x\psi(\tau)\|_{L^\infty} + \|f(\tau)\|_{\mathscr{C}_2^\alpha} + \|F(\tau)\|_{\mathscr{C}_1^\alpha})\|x\partial_x\psi\|(x, v^A, \tau) \end{aligned}$$

which gives (5.3.39) using (5.3.42).

□

Proposition 5.3.6 Let $-1 < \alpha < -1/2$ in $\mathbb{R}$. We suppose the parameter k appearing in (5.2.1) and (5.2.8) is such that $k \geq 3$. Let x_2, x_1, t be such that $0 \leq 2x_2 < x_1 - t/2$. Let f be a solution of (5.3.1) with $f(0) \in H_k^{\mathrm{loc}}$, to which we associate the energies

$$\begin{aligned} \tilde{\mathcal{E}}_k^\alpha(\tau) &= \mathcal{E}_k^\alpha(\tau) + \|\psi(\tau)\|_{L^\infty}^2\ , &(5.3.46)\\ {\mathcal{E}'}_k^\alpha(\tau) &= \mathcal{E}_k^\alpha(\tau) + \|\psi(\tau)\|_{L^\infty}^2 + \|x\partial_x\psi(\tau)\|_{L^\infty}^2\ , &(5.3.47) \end{aligned}$$

where $\mathcal{E}_k^\alpha(\tau)$ is defined in (5.3.19). Under the hypotheses $\mathscr{C}1$)-$\mathscr{C}3$), $H1$)-$H4$), there exists $C_8(C_{\mathring{e}}, C_{\overline{e}}, C_e^\tau, C_\Gamma, C_A, C^a, C_s, x_1, \alpha, \epsilon)$, bounded over bounded sets of variables, such that for all $\tau \in [0, \min(t*, T_1')]$, where T_1' is given by Lemma 5.3.4 and $t*$ by (5.2.10),

$$\begin{aligned} &\mathcal{E}_k^\alpha(\tau) - \mathcal{E}_k^\alpha(0) \leq \\ &\quad \int_0^\tau C_8\left(\mathcal{E}_k^\alpha(s) + \|\partial_\tau f(s)\|_{\mathscr{C}_0^\alpha}^2 + \|a\|_{\mathscr{H}_k^\alpha}^2 + \|b\|_{\mathscr{H}_k^{\alpha-1/2}}^2\right) ds\ , \quad (5.3.48)\\ &\tilde{\mathcal{E}}_k^\alpha(\tau) - \tilde{\mathcal{E}}_k^\alpha(0) \leq \end{aligned}$$

$$\int_0^\tau C_8(1+(\tau-s)^\alpha)\Big(\tilde{\mathcal{E}}^\alpha_k(s)+\|\partial_\tau f(s)\|^2_{\mathscr{C}^\alpha_0} + \|a(s)\|^2_{\mathscr{H}^\alpha_k}+\|b(s)\|^2_{\mathscr{H}^{\alpha-1/2}_k}+\|b(s)\|^2_{\mathscr{C}^\alpha_0}\Big)\,ds\ , \qquad (5.3.49)$$

$$\mathcal{E}'^\alpha_k(\tau)-\mathcal{E}'^\alpha_k(0)\le \int_0^\tau C_8(1+(\tau-s)^\alpha)\Big(\mathcal{E}'^\alpha_k(s) + \|a(s)\|^2_{\mathscr{H}^\alpha_k}+\|b(s)\|^2_{\mathscr{H}^{\alpha-1/2}_k}+\|b(s)\|^2_{\mathscr{C}^\alpha_1}\Big)\,ds\ , \qquad (5.3.50)$$

where the last estimate holds only for $k\ge 4$ and $\epsilon\ge 1+\alpha$.

Remarks : 1) The various energy inequalities and their proofs are still valid for $\alpha=-1/2$, provided we replace $\|b\|_{\mathscr{H}^{\alpha-1/2}_k}$ by $\|b\|_{\mathscr{H}^\alpha_k}$ there.

2) The condition $k\ge 4$ for the last equation, a consequence of the need for an estimate of f in $\mathscr{C}^\alpha_1$ in Lemma 5.3.5 to obtain an estimate on $\|\partial_x\psi\|_{\mathscr{C}^{-1}_0}$, is somewhat artificial. It can be avoided under further initial condition by estimating $\|\partial_x\psi\|_{\mathscr{H}^\alpha_k}$.

Proof: Since our hypotheses imply that (5.3.1) is a symmetric hyperbolic system, there exists a solution with $f(0)\in H^{\rm loc}_k$. First we suppose $f(0)\in H^{\rm loc}_{k+1}(M_{x_2,x_1})$ and $x_2>0$. Then $f(\tau)\in H^{\rm loc}_{k+1}(M_{x_2,x_1-3\tau})$ and $\partial_\tau f(\tau)\in H^{\rm loc}_k(M_{x_2,x_1-3\tau})$ for any $\tau\in[0,\min(t*,T_1')]$. We will obtain weighted estimates for f in a region $\mathscr{M}_{x_2,x_1,t*}$ of the space time. We set

$$X^\mu=\sum_{|\beta|\le k}x^{-2\alpha-1+2\beta_1}\langle\mathcal{D}^\beta f,A^\mu\mathcal{D}^\beta f\rangle\ .$$

Now let us apply Stokes theorem to the piece of space-time $\mathcal{M}_{x_2,x_1,\tau}$. Noting that the hypersurfaces $\{x=cst\}$ and $\{x=3\tau\}$ are spacelike or null, one finds with $H0$),

$$0\le\int_{\{x=x_2\}\cap\mathcal{M}_{x_2,x_1,\tau}}X^\mu dS_\mu\ , \qquad (5.3.51)$$

with dS_μ being the volume element induced on $\{x=x_2\}\cap\mathcal{M}_{x_2,x_1,\tau}$, and similarly for the hypersurface $\{x=3\tau\}$. Therefore the Stokes theorem gives

$$\mathcal{E}^\alpha_k(\tau)\le\mathcal{E}^\alpha_k(0)+\int_{s=0}^\tau\int_{M_{x_2,x_1-3s}}N_\tau D_\mu X^\mu\ d^n\mu_\tau\ ds\ . \qquad (5.3.52)$$

With our hypotheses we have

$$D_\mu X^\mu = \sum_{|\beta|\le k}(-2\alpha-1+2\beta_1)(e_3.x)x^{-2\alpha-2+2\beta_1}\langle\mathcal{D}^\beta f,A^3\mathcal{D}^\beta f\rangle +2\sum_{|\beta|\le k}x^{-2\alpha-1+2\beta_1}\langle\mathcal{D}^\beta f,A^i\nabla_i\mathcal{D}^\beta f\rangle \qquad (5.3.53)$$

(*cf.* Equation (5.3.10)). Then we write

$$A^i\nabla_i\mathcal{D}^\beta f = A^ie_i^\mu\partial_\mu\mathcal{D}^\beta f + \begin{pmatrix} \Gamma_4\mathcal{D}^\beta\phi+\Gamma\mathcal{D}^\beta\psi \\ \Gamma_3\mathcal{D}^\beta\psi+\Gamma_3'\mathcal{D}^\beta\phi+\Gamma'\mathcal{D}^\beta\phi \end{pmatrix}, \quad (5.3.54)$$

with

$$A^ie_i^\mu\partial_\mu\mathcal{D}^\beta f = \mathcal{D}^\beta(A^ie_i^\mu\partial_\mu f) - \sum_{(0,\dots,0)<\gamma\le\beta} c(\gamma,\beta)A^i(\mathcal{D}^\gamma e_i^\mu)(\mathcal{D}^{\beta-\gamma}\partial_\mu f) - A^ie_i^\mu[\mathcal{D}^\beta,\partial_\mu]f\ . \quad (5.3.55)$$

We have using (5.1.14)

$$[\mathcal{D}^\beta,\partial_x] = 0\ , \quad (5.3.56a)$$
$$[\mathcal{D}^\beta,\partial_\tau] = 0\ , \quad (5.3.56b)$$
$$[\mathcal{D}^\beta,\partial_A] = -\sum_i(\partial_A\varphi_i)\partial^\beta = \sum_{|\gamma|=|\beta|,\gamma_1=\beta_1} d(\beta,\gamma,v^A)\mathcal{D}^\gamma\ , \quad (5.3.56c)$$

with d smooth in v^A. Hence,

$$A^ie_i^\mu[\mathcal{D}^\beta,\partial_\mu]f = \sum_{|\gamma|=|\beta|\,,\gamma_1=\beta_1} c_A(\beta,\gamma,v^2,v^3)A^ae_a^A\mathcal{D}^\gamma f\ , \quad (5.3.57)$$

so that there exist smooth functions $c^{\beta,\gamma}{}_A$ depending upon angular variables such that

$$\sum_{|\beta|\le k} x^{-2\alpha-2+2\beta_1}\langle A^ie_i^\mu[\mathcal{D}^\beta,\partial_\mu]f,\mathcal{D}^\beta f\rangle = \sum_{\beta_1=\gamma_1,|\beta|=|\gamma|\le k} x^{-2\alpha-2+2\beta_1}c^{\beta,\gamma}{}_A\langle A^ae_a^A\mathcal{D}^\gamma f,\mathcal{D}^\beta f\rangle\ .$$

Writing

$$\begin{aligned} D_\mu X^\mu = & \sum_{|\beta|\le k}(2\alpha+1-2\beta_1)x^{-2\alpha-2+2\beta_1}\langle\mathcal{D}^\beta f,A^3\mathcal{D}^\beta f\rangle \\ & +2\sum_{|\beta|\le k}x^{-2\alpha+2\beta_1-1}\langle\mathcal{D}^\beta f,A^ie_i^\mu\partial_\mu\mathcal{D}^\beta f\rangle \\ & +2\sum_{|\beta|\le k}x^{-2\alpha+2\beta_1-1}\langle\mathcal{D}^\beta\phi,\Gamma_4\mathcal{D}^\beta\phi+\Gamma_4'\mathcal{D}^\beta\psi+\Gamma\mathcal{D}^\beta\psi\rangle \\ & +2\sum_{|\beta|\le k}x^{-2\alpha+2\beta_1-1}\langle\mathcal{D}^\beta\psi,\Gamma_3\mathcal{D}^\beta\psi+\Gamma_3'\mathcal{D}^\beta\phi+\Gamma'\mathcal{D}^\beta\phi\rangle\ , \end{aligned}$$

and using what precedes one finds, after some rearrangements,

$$D_\mu X^\mu = N_1+S+G+T_x+T_A+T_\tau+R\ , \quad (5.3.58a)$$

with the splitting motivated as follows: N_1 contains the negative terms from Equation (5.3.53), which will help us to estimate some of the error terms; S contains the first term from Equation (5.3.48), which will be worked upon using the field equations; G contains the terms from Equation (5.3.54) involving the

Γ's, and which can be directly estimated; the T_μ's arise from the second and third term in Equation (5.3.48) and contain all the terms in $\partial_\mu f$ from there. More precisely:

$$N_1 = -2\sum_{|\beta|\le k}(2\alpha-2\beta_1+1)x^{-2\alpha+2\beta_1-2}\langle\mathcal{D}^\beta\psi,\mathcal{D}^\beta\psi\rangle\ , \tag{5.3.59a}$$

$$S = \sum_{|\beta|\le k}x^{-2\alpha+2\beta_1-1}\langle\mathcal{D}^\beta f,\mathcal{D}^\beta(A^i e_i^\mu\partial_\mu f)\rangle\ , \tag{5.3.59b}$$

$$G = \sum_{|\beta|\le k}x^{-2\alpha+2\beta_1-1}\Big(\langle\mathcal{D}^\beta\phi,\Gamma_4\mathcal{D}^\beta\phi\rangle+\langle\mathcal{D}^\beta\psi,\Gamma_3\mathcal{D}^\beta\psi\rangle \tag{5.3.59c}$$

$$+\langle\mathcal{D}^\beta\phi,(\Gamma+\Gamma_4')\mathcal{D}^\beta\psi\rangle+\langle\mathcal{D}^\beta\psi,(\Gamma_3'+\Gamma')\mathcal{D}^\beta\phi\rangle\Big) \tag{5.3.59d}$$

The T_μ terms require some work. Consider, first, the terms in Equation (5.3.48) which explicitly contain $\partial_x f$; writing β as $\delta+\gamma$ and using $e_i^x = -2\delta_i^3$ we obtain

$$\begin{aligned} T_x &= -\sum_{|\delta|\le k-1}\ \sum_{0<|\gamma|\le k-|\delta|}x^{-2\alpha+2\delta_1+2\gamma_1-1}\langle\mathcal{D}^{\delta+\gamma}f,A^i(\mathcal{D}^\gamma e_i^x)(\mathcal{D}^\delta\partial_x f)\rangle\ ,\\ &= 0\ . \end{aligned}$$

The remaining T_μ's read

$$\begin{aligned} T_A &= -\sum_{|\delta|\le k-1}\ \sum_{0<|\gamma|\le k-|\delta|}x^{-2\alpha+2\delta_1+2\gamma_1-1}\langle\mathcal{D}^{\delta+\gamma}f,A^a(\mathcal{D}^\gamma e_a^A)(\mathcal{D}^\delta\partial_A f)\rangle\\ &\quad -\sum_{|\delta|\le k-1}\ \sum_{0<|\gamma|\le k-|\delta|}x^{-2\alpha+2\delta_1+2\gamma_1-1}\langle\mathcal{D}^{\delta+\gamma}f,A^3(\mathcal{D}^\gamma e_3^A)(\mathcal{D}^\delta\partial_A f)\rangle\ ,\\ &\quad -\sum_{\beta_1=\gamma_1,|\beta|=|\gamma|\le k}c^{\beta,\gamma}{}_A x^{-2\alpha+\beta+\gamma-1}\langle A^a e_a^A\mathcal{D}^\gamma f,\mathcal{D}^\beta f\rangle\ , \end{aligned} \tag{5.3.60a}$$

$$\begin{aligned} T_\tau &= -\sum_{|\delta|\le k-1}\ \sum_{0<|\gamma|\le k-|\delta|}x^{-2\alpha+2\delta_1+2\gamma_1-1}\Big(\langle\mathcal{D}^{\delta+\gamma}\psi,(\mathcal{D}^\gamma e_3^\tau)(\mathcal{D}^\delta\partial_\tau\psi)\rangle\\ &\quad +\langle\mathcal{D}^{\gamma+\delta}f,A^a(\mathcal{D}^\gamma e_a^\tau)(\mathcal{D}^\delta\partial_\tau f)\rangle\Big)\ , \end{aligned} \tag{5.3.60b}$$

We will use Gronwall's Lemma to extract information out of Equation (5.3.52); for this we have to estimate

$$\begin{aligned} \int_{\mathscr{M}_{x_2,x_1,\tau}}\partial_\mu X^\mu d^{n+1}\mu &= \int_0^\tau\int_{M_{x_2,x_1-3\tau}\times\{\tau\}}\partial_\mu X^\mu N_\tau\ d^n\mu_\tau\ ds\\ &\le 2\int_0^\tau\int_{M_{x_2,x_1-3\tau}}\partial_\mu X^\mu(\tau)\ d^n\mu_\tau\ ds\ . \end{aligned} \tag{5.3.61}$$

More precisely, we will estimate

$$\int_{M_{x_2,x_1-3\tau}}\partial_\mu X^\mu(\tau)\ d^n\mu_\tau\ . \tag{5.3.62}$$

Whenever no confusion is possible, we will write $\mathscr{H}_k^\gamma$ and $\mathcal{G}_k^\gamma$ for $\mathscr{H}_k^\gamma(M_{x_2,x_1-3\tau})$ and $\mathcal{G}_k^\gamma(M_{x_2,x_1-3\tau})$. The most delicate term to estimate is $\int_{M_{x_2,x_1-3\tau}} T_\tau \, d^n\mu_\tau$. We will estimate it in two ways, the first will give (5.3.48), the second (5.3.49). We start with the second term in Equation (5.3.60b), we denote $I_{\delta,\gamma}$ the generic term in the sum. We have (recall that $\overline{e}_a^\tau = e_a^\tau$):

$$\begin{aligned}
I_{\delta,\gamma} := \int_{M^i_{x_2,x_1-3\tau}} & x^{-2\alpha-1+2\delta_1+2\gamma_1} \langle \mathcal{D}^{\gamma+\delta} f(\tau), A^a(\mathcal{D}^\gamma \overline{e}_a^\tau(\tau))(\mathcal{D}^\delta \partial_\tau f(\tau))\rangle \, d^n\mu_\tau \\
\leq \;& c_2 \int_{M^i_{x_2,x_1-3\tau}} x^{-2\alpha-1+2\delta_1+2\gamma_1} \langle \mathcal{D}^{\gamma+\delta} f(\tau), A^a(\mathcal{D}^\gamma \overline{e}_a^\tau(\tau))(\mathcal{D}^\delta \partial_\tau f(\tau))\rangle d^n\mu_0 \\
\leq \;& c_2 C_s \|f(\tau)\|_{\mathscr{H}_k^\alpha(M^i_{x_2,x_1-3\tau})} \Big(\|\overline{e}_a(\tau)\|_{\mathscr{C}_0^1(M^i_{x_2,x_1-3\tau})} \|\partial_\tau f(\tau)\|_{\mathscr{H}_{k-1}^{\alpha-1}(M^i_{x_2,x_1-3\tau})} \\
& \qquad + \|\overline{e}_a(\tau)\|_{\mathscr{H}_k^0(M^i_{x_2,x_1-3\tau})} \|\partial_\tau f(\tau)\|_{\mathscr{C}_0^\alpha(M^i_{x_2,x_1-3\tau})} \Big) \\
\leq \;& C(C_{\mathring{e}}, C_{\overline{e}}, C_s, x_1, \alpha, \epsilon) \|f(\tau)\|_{\mathscr{H}_k^\alpha(M_{x_2,x_1-3\tau})} \Big(\|\partial_\tau f(\tau)\|_{\mathscr{H}_{k-1}^{\alpha-1}(M_{x_2,x_1-3\tau})} \\
& \qquad + \|\partial_\tau f(\tau)\|_{\mathscr{C}_0^\alpha(M_{x_2,x_1-3\tau})} \Big) .
\end{aligned}$$

We have made these estimates first on $M^i_{x_2,x_1-3\tau}$ because the e_i are not defined globally in general; summing over i we obtain an estimate on $M_{x_2,x_1-3\tau}$,

$$\begin{aligned}
\int_{M_{x_2,x_1-3\tau}} & x^{-2\alpha-1+2\delta_1+2\gamma_1} \langle \mathcal{D}^{\gamma+\delta} f(\tau), A^a(\mathcal{D}^\gamma \overline{e}_a^\tau(\tau))(\mathcal{D}^\delta \partial_\tau f(\tau))\rangle \, d^n\mu_\tau \\
\leq \;& C(C_{\mathring{e}}, C_{\overline{e}}, C_s, x_1, \alpha, \epsilon) \|f(\tau)\|_{\mathscr{H}_k^\alpha} (\|\partial_\tau f(\tau)\|_{\mathscr{H}_{k-1}^{\alpha-1}} + \|\partial_\tau f(\tau)\|_{\mathscr{C}_0^\alpha}) \\
\leq \;& C(C_{\mathring{e}}, C_{\overline{e}}, C_A, C^a, C_s, x_1, \alpha, \epsilon) \|f(\tau)\|_{\mathscr{H}_k^\alpha} \Big(\|f(\tau)\|_{\mathscr{H}_k^\alpha} + \|\partial_\tau f(\tau)\|_{\mathscr{C}_0^\alpha} + \|F(\tau)\|_{\mathscr{H}_k^{\alpha-1/2}} \Big) \\
\leq \;& C(C_{\mathring{e}}, C_{\overline{e}}, C_A, C^a, C_s, x_1, \alpha, \epsilon) \Big(\|f(\tau)\|^2_{\mathscr{H}_k^\alpha} + \|\partial_\tau f(\tau)\|^2_{\mathscr{C}_0^\alpha} + \|F(\tau)\|^2_{\mathscr{H}_k^{\alpha-1/2}} \Big) \; ;
\end{aligned}$$

in the second inequality (5.3.25) has been used. The first term in T_τ can be estimated similarly leading to

$$\begin{aligned}
\int_{M_{x_2,x_1-3\tau}} & T_\tau(\tau) \, d^n\mu_\tau \\
\leq \;& C(C_{\mathring{e}}, C_{\overline{e}}, C_e^\tau C_A, C^a, C_s, x_1, \alpha, \epsilon) \Big(\|f(\tau)\|^2_{\mathscr{H}_k^\alpha} + \|\partial_\tau f(\tau)\|^2_{\mathscr{C}_0^\alpha} \\
& \qquad + \|a(\tau)\|^2_{\mathscr{H}_k^\alpha} + \|b(\tau)\|^2_{\mathscr{H}_k^{\alpha-1/2}} \Big) .
\end{aligned} \tag{5.3.63}$$

Let us now give an other estimate of T_τ. We write

$$I_{\delta,\gamma} = I_\phi + I_\psi \, ,$$

with

$$I_\phi = \int_{M^i_{x_2,x_1-3\tau}} x^{-2\alpha-1+2\delta_1+2\gamma_1} \langle \mathcal{D}^{\gamma+\delta} f(\tau), A^a(\mathcal{D}^\gamma \overline{e}_a^\tau(\tau))(\mathcal{D}^\delta \partial_\tau \phi(\tau))\rangle \, d^n\mu_\tau \, ,$$

$$I_\psi = \int_{M^i_{x_2,x_1-3\tau}} x^{-2\alpha-1+2\delta_1+2\gamma_1} \langle \mathcal{D}^{\gamma+\delta} f(\tau), A^a(\mathcal{D}^\gamma \overline{e}_a^\tau(\tau))(\mathcal{D}^\delta \partial_\tau \psi(\tau))\rangle \, d^n\mu_\tau \, .$$

I_ϕ can be estimated as follows

$$\begin{aligned} I_\phi &\le C(C_s)(\|A^a\overline{e}_a^\tau\|_{\mathscr{C}^1_0}\|\partial_\tau\phi\|_{\mathscr{H}^{\alpha-1}_k} + \|A^a\overline{e}_a^\tau\|_{\mathscr{H}^{1+\alpha}_k}\|\partial_\tau\phi\|_{\mathscr{C}^{-1}_0})\|f\|_{\mathscr{H}^\alpha_k} \\ &\le C(C_{\overline{e}}, C_{\mathring{e}}, C_A, C^a, C_s, x_1, \alpha, \epsilon)(\|f\|^2_{\mathscr{H}^\alpha_k} + \|a\|^2_{\mathscr{H}^\alpha_k} + \|b\|^2_{\mathscr{H}^{\alpha-1/2}_k}) \end{aligned}$$

where we have used

$$\|\partial_\tau\phi\|_{\mathscr{C}^{-1}_0} \le \|\partial_\tau\phi\|_{\mathscr{C}^\alpha_0} \, , \tag{5.3.64}$$

with $\|\partial_\tau\phi\|_{\mathscr{C}^\alpha_0}$ estimated through Lemma 5.3.5. For I_ψ, one writes similarly

$$I_\psi \le C(C_s)\|f\|_{\mathscr{H}^\alpha_k}(\|A^a\overline{e}_a^\tau\|_{\mathscr{C}^1_0}\|\partial_\tau\psi\|_{\mathscr{H}^{\alpha-1}_k} + \|A^a\overline{e}_a^\tau\|_{\mathscr{H}^{1+\alpha}_k}\|\partial_\tau\psi\|_{\mathscr{C}^{-1}_0}) \, .$$

To estimate $\partial_\tau\psi$, ones uses (5.3.34) to obtain

$$\|\partial_\tau\psi\|_{\mathscr{C}^{-1}_0} \le C(C_{\overline{e}}, C_{\mathring{e}}, C^a, C_A, x_1, \alpha)(\|\partial_x\psi\|_{\mathscr{C}^{-1}_0} + \|f\|_{\mathscr{C}^\alpha_1} + \|F\|_{\mathscr{C}^\alpha_0}) \, ,$$

which gives using (5.3.25)

$$\begin{aligned} I_\psi \le\; & C(C_{\overline{e}}, C_{\mathring{e}}, C^a, C_A, C_s, x_1, \alpha, \epsilon) \times \\ & (\|f\|^2_{\mathscr{H}^\alpha_k} + \|x\partial_x\psi\|^2_{L^\infty} + \|a\|^2_{\mathscr{H}^\alpha_k} + \|b\|^2_{\mathscr{H}^{\alpha-1/2}_k}) \, . \end{aligned}$$

To sum up, one has obtained

$$I \le C(C_{\overline{e}}, C_{\mathring{e}}, C^a, C_A, C_s, x_1, \alpha, \epsilon)\left(\mathcal{E}'^\alpha_k + \|a\|^2_{\mathscr{H}^\alpha_k} + \|b\|^2_{\mathscr{H}^{\alpha-1/2}_k}\right) \, .$$

Finally, estimating similarly the others terms in T_τ, one obtains

$$T_\tau \le C(C_{\overline{e}}, C_{\mathring{e}}, C^a, C_A, C_s, x_1, \alpha, \epsilon)\left(\mathcal{E}'^\alpha_k + \|a\|^2_{\mathscr{H}^\alpha_k} + \|b\|^2_{\mathscr{H}^{\alpha-1/2}_k}\right) \, . \tag{5.3.65}$$

Next,

$$\int_{M_{x_2,x_1-3\tau}} N_1(\tau) \, d^n\mu_\tau \le -2c_1|2\alpha+1|\|\psi(\tau)\|^2_{\mathscr{H}^{\alpha+1/2}_k} \, . \tag{5.3.66}$$

To estimate S, we use the field equation and write

$$\begin{aligned} &\int_{M^i_{x_2,x_1-3\tau}} S \;\; d^n\mu_\tau \\ &\le \int_{M^i_{x_2,x_1-3\tau}} \sum_{|\beta|\le k} x^{-2\alpha+2\beta_1-1}\langle \mathcal{D}^\beta f, \mathcal{D}^\beta(-Af+F)\rangle \, d^n\mu_\tau \\ &\le \int_{M^i_{x_2,x_1-3\tau}} \sum_{|\beta|\le k} x^{-2\alpha+2\beta_1-1}\Big(\langle \mathcal{D}^\beta\phi, \mathcal{D}^\beta a\rangle + \langle \mathcal{D}^\beta\psi, \mathcal{D}^\beta b\rangle \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad - \langle \mathcal{D}^\beta f, \mathcal{D}^\beta (Af)\rangle\Big)\ d^n\mu_\tau \\
\le\ & \int_{M^i_{x_2,x_1-3\tau}} \sum_{|\beta|\le k} \left(\frac{|2\alpha+1|c_1}{2c_2} x^{-2\alpha-2+2\beta_1} \langle \mathcal{D}^\beta\psi, \mathcal{D}^\beta\psi\rangle \right.\\
&\qquad\qquad \left. + \frac{2c_2}{|2\alpha+1|c_1} x^{-2\alpha+2\beta_1} \langle \mathcal{D}^\beta b, \mathcal{D}^\beta b\rangle \right)\ d^n\mu_\tau \\
&+ \int_{M^i_{x_2,x_1-3\tau}} \sum_{|\beta|\le k} x^{-2\alpha+2\beta_1-1} \Big(\langle \mathcal{D}^\beta \phi, \mathcal{D}^\beta\phi\rangle + \langle \mathcal{D}^\beta a, \mathcal{D}^\beta a\rangle \Big)\ d^n\mu_\tau \\
&+ c_2 C(k) \|f\|_{\mathscr{H}^\alpha_k(M^i_{x_2,x_1-3\tau})} \|Af\|_{\mathscr{H}^\alpha_k(M^i_{x_2,x_1-3\tau})} \\
\le\ & \frac{|2\alpha+1|c_1}{2} \|\psi\|^2_{\mathscr{H}^{\alpha+1/2}_k(M_{x_2,x_1-3\tau})} + \frac{2c_2^2}{|2\alpha+1|c_1} \|b\|^2_{\mathscr{H}^{\alpha-1/2}_k(M_{x_2,x_1-3\tau})} + c_2 \|\phi\|^2_{\mathscr{H}^\alpha_k(M_{x_2,x_1-3\tau})} \\
&+ c_2 \|a\|^2_{\mathscr{H}^\alpha_k(M_{x_2,x_1-3\tau})} + C(C_{\mathring{e}}, C_{\overline{e}}, C_s, k) \|A\|_{\mathcal{G}^0_k(M_{x_2,x_1-3\tau})} \|f\|^2_{\mathscr{H}^\alpha_k(M_{x_2,x_1-3\tau})}\ ,
\end{aligned}
$$

which gives

$$
\begin{aligned}
\int_{M_{x_2,x_1-3\tau}} S(\tau)\ d^n\mu_\tau \ \le\ & C(C_{\mathring{e}}, C_{\overline{e}}, C_s, k) \|A(\tau)\|_{\mathcal{G}^0_k(M_{x_2,x_1-3\tau})} \|f(\tau)\|^2_{\mathscr{H}^\alpha_k(M_{x_2,x_1-3\tau})} \\
& + |2\alpha+1|c_1 \|\psi(\tau)\|^2_{\mathscr{H}^{\alpha+1/2}_k} + C(C_{\mathring{e}}, C_{\overline{e}}, C_s, \alpha) \|b(\tau)\|^2_{\mathscr{H}^{\alpha-1/2}_k} \\
& + C(C_{\mathring{e}}, C_{\overline{e}}, C_s)(\|a(\tau)\|^2_{\mathscr{H}^\alpha_k} + \|\phi(\tau)\|^2_{\mathscr{H}^\alpha_k})\ .
\end{aligned}
$$

Similarly one derives

$$
\int_{M_{x_2,x_1-3\tau}} T_A(\tau)\ d^n\mu_\tau \ \le\ c_2 C(k, C_s) C_e \|f(\tau)\|^2_{\mathscr{H}^\alpha_k(M_{x_2,x_1-3\tau})}\ .
$$

The remaining estimate of the term in G is straightforward since all the Γ's are in L^∞ (5.3.13):

$$
\int_{M_{x_2,x_1-3\tau}} G(\tau)\ d^n\mu_\tau \le c_2 C(k) C_\Gamma \|f(\tau)\|^2_{\mathscr{H}^\alpha_k}\ . \tag{5.3.67}
$$

Therefore, using Equation (5.3.66) to get rid of the term $\|\psi\|_{\mathscr{H}^{\alpha+1/2}_k(M_{x_2,x_1-3\tau})}$ present in the estimation of $\int_{M_{x_2,x_1-3\tau}} S(\tau)\ d^n\mu_\tau$ one obtains from (5.3.63)

$$
\begin{aligned}
&\int_{\mathscr{M}_{x_2,x_1,\tau}} \partial_\mu X^\mu\ d^n\mu_\tau \le \\
&\quad \int_0^\tau C(C_{\mathring{e}}, C_{\overline{e}}, C^\tau_e, C_\Gamma, C_A, C^a, C_s, x_1, \alpha, k) \\
&\qquad \times \Big(\|f(s)\|^2_{\mathscr{H}^\alpha_k} + \|\partial_\tau f(s)\|^2_{\mathscr{C}^\alpha_0} \\
&\qquad + \|a(s)\|^2_{\mathscr{H}^\alpha_k} + \|b(s)\|^2_{\mathscr{H}^{\alpha-1/2}_k} \Big)\, ds\ .
\end{aligned}
$$

Equation (5.3.52) implies (5.3.48). The inequality (5.3.49) follows from (5.3.48) and (5.3.38). Then (5.3.50) follows from the above using estimate (5.3.65) and Lemma 5.3.5, *cf.* (5.3.39).

To extend the result to $f(0) \in H_k^{\rm loc}$, consider any sequence $f_n(0) \in H_{k+1}^{\rm loc}$ converging to $f(0)$ in $\mathscr{H}_k^\alpha$. The estimate (5.3.48) applies to the f_n's; Gronwall's Lemma applied to this estimate shows that all the objects appearing there remain finite when passing to the limit $n \to \infty$, and that (5.3.48) applies to f. A similar argument works for (5.3.49)-(5.3.50) — here a straightforward generalisation of Lemma 3.6.2 should be used instead of Gronwall's Lemma. □

5.4 Energy inequality for the Einstein Equations.

The aim of this section is to provide weighted energy estimates for the conformal system derived from the Einstein equations in Section 4.3.3.

We start with the space-time constructed in Section 4.3.3. The hypotheses (5.2.2),(5.2.5), (5.2.6), (5.2.12), (5.2.13)-(5.2.15) are satisfied by our choices of Gauge. We will address the boost-strap hypotheses on the weighted norms of the tetrad fields such as (5.2.1) and (5.2.8) in the next paragraph. Let us define $T(x_2)$ by

$$T(x_2) = \max\{t \mid \mathscr{M}_{x_2,x_1,t} \subset \mathscr{M}\} \tag{5.4.1}$$

where $\mathscr{M}_{x_2,x_1,t}$ is understood as the set of the null geodesics $s(\tau)$ starting from Σ_{x_2,x_1}, $T(x_2)$ being determined by the time of existence of the geodesics $s(\tau)$. The Friedrich conformal system equivalent to the Einstein equations takes the form explicited in Section 4.3.3 on $\mathscr{M}_{x_2,x_1,t}$ for $0 \le t < T(x_2)$, which will allows us to obtain the weighted estimates on various fields. We will use the notations of Chapter 4, with $g = x^2\tilde{g}$, $\Gamma_i{}^j{}_k$ being the connection coefficient of D with the decomposition (4.1.14), (without hat), $\alpha, \beta, \sigma, \rho, \underline{\alpha}, \underline{\beta}, \underline{\alpha}$ being the null components of the rescaled Weyl tensor $d_{ijkl} = x^{-1}W_{ijkl}$, where W_{ijkl} is the Weyl tensor associated to $\widetilde{g}$; the indices correspond to the half-null tetrad constructed above.

Then we set

$$\begin{aligned}
\mathring{\Gamma}_a{}^b{}_c(x, v^A, \tau) &= \Gamma_a{}^b{}_c(x, v^A, 0) \,, & (5.4.2)\\
\mathring{\Gamma}_3{}^b{}_c(x, v^A, \tau) &= \Gamma_3{}^b{}_c(x, v^A, 0) \,, & (5.4.3)\\
\overline{\Gamma}_a{}^b{}_c &= \Gamma_a{}^b{}_c - \mathring{\Gamma}_a{}^b{}_c \,, & (5.4.4)\\
\overline{\Gamma}_3{}^b{}_c &= \Gamma_a{}^b{}_c - \mathring{\Gamma}_3{}^b{}_c \,. & (5.4.5)
\end{aligned}$$

We will consider the following sets of variables (we recall that indices a, b, c run from 1 to 2):

$$\begin{aligned}
f_1 &= x^{-1}\left(\overline{e}_3^\tau, \overline{e}_3^A, \overline{e}_a^\tau, \overline{e}_a^A\right) \,, & (5.4.6a)\\
f_2 &= \left(x^{-1}\underline{\omega}, x^{-1}\eta^a, x^{-1}\zeta_a, x^{-1}\chi_a{}^b, x^{-1}\overline{\Gamma}_a{}^b{}_c, x^{-1}\overline{\Gamma}_3{}^b{}_c, \underline{\chi}_a{}^b, \underline{\zeta}_a, \underline{\xi}_a\right) \,, & (5.4.6b)\\
f_3 &= \left(\alpha_{ab}, \beta_a, \rho, \sigma, \underline{\beta}_a\right) \,, & (5.4.6c)\\
f_3' &= (\alpha_{ab}, \beta_a, \rho, \sigma) \,, & (5.4.6d)\\
f_4 &= x(\underline{\alpha}_{ab}, \underline{\beta}_a) \,, & (5.4.6e)\\
f &= (f_1, f_2, f_3, f_4) \,, & (5.4.6f)\\
f' &= (f_1, f_2, f_3', f_4) \,, & (5.4.6g)\\
\mathring{f} &= (\mathring{e}_i, \mathring{\Gamma}_a{}^b{}_c, \mathring{\Gamma}_3{}^b{}_c) \,. & (5.4.6h)
\end{aligned}$$

Remark: The component $\underline{\beta}$ is present both in f_5, f_6 because it will be estimated in $\mathscr{H}_k^\alpha$ and in $\mathscr{H}_k^{\alpha-1}$. In f_3' we have put the components of the rescaled Weyl tensor which can be estimated in L^∞, so that f' contains all the fields of f which will be bounded in our boot-strap setting.

The ordinary differential equations of Section 4.3.3 are rewritten in a form adapted to (5.4.6). For (4.3.47) one writes

$$\begin{aligned}
\partial_\tau(x^{-1}\overline{e}_3^\tau) &= -2(x^{-1}\underline{\omega}) - 2x(x^{-1}\eta^a)(x^{-1}e_a^\tau)\ , &(5.4.7a)\\
\partial_\tau(x^{-1}\overline{e}_3^A) &= -2(x^{-1}\eta^a)(\mathring{e}_a^A + x(x^{-1}\overline{e}_a^A))\ , &(5.4.7b)\\
\partial_\tau(x^{-1}\overline{e}_a^\tau) &= (x^{-1}\zeta_a) - x(x^{-1}\chi_a{}^b)(x^{-1}e_b^\tau)\ , &(5.4.7c)\\
\partial_\tau(x^{-1}\overline{e}_a^A) &= -(x^{-1}\chi_a{}^b)(\mathring{e}_b^A + x(x^{-1}\overline{e}_b^A))\ . &(5.4.7d)
\end{aligned}$$

The system (4.3.51) is rewritten as

$$\begin{aligned}
\partial_\tau(x^{-1}\chi_a{}^b) &= -x(x^{-1}\chi_{ac}).(x^{-1}\chi^{cb}) - \alpha_a{}^b\ , &(5.4.8a)\\
\partial_\tau\underline{\chi}_a{}^b &= -x(x^{-1}\chi_a{}^c)\underline{\chi}_c{}^b + x\rho\delta_a^b + x\sigma\varepsilon_a{}^b + 2(x^{-1}\chi_a{}^b)\ , &(5.4.8b)\\
\partial_\tau(x^{-1}\zeta_a) &= -x(x^{-1}\chi_a{}^c)(x^{-1}\zeta_c) - \beta_a\ , &(5.4.8c)\\
\partial_\tau\underline{\xi}_a &= -x(x^{-1}\eta^c)\underline{\chi}_{ca} + 2x^{-1}\eta_a - x\underline{\beta}_a\ , &(5.4.8d)\\
\partial_\tau(x^{-1}\eta_a) &= -x(x^{-1}\eta^c)(x^{-1}\chi_{ca}) - \beta_a\ , &(5.4.8e)\\
\partial_\tau(x^{-1}\underline{\omega}) &= x(x^{-1}\eta^c)(x^{-1}\zeta_c) + \rho\ , &(5.4.8f)\\
\partial_\tau(x^{-1}\overline{\Gamma}_a{}^b{}_c) &= -(x^{-1}\chi_a{}^d)\mathring{\Gamma}_d{}^b{}_c - x(x^{-1}\chi_a{}^d)(x^{-1}\overline{\Gamma}_d{}^b{}_c) + \varepsilon_{bc}{}^\star\beta_a\ , &(5.4.8g)\\
\partial_\tau(x^{-1}\overline{\Gamma}_3{}^a{}_b) &= -2\mathring{\Gamma}_c{}^a{}_b(x^{-1}\eta^c) - 2x(x^{-1}\overline{\Gamma}_c{}^a{}_b)(x^{-1}\eta^c) - 2\sigma\varepsilon^a{}_b\ . &(5.4.8h)\\
& &(5.4.8i)
\end{aligned}$$

For f_1, f_2, our conformal system can be summed up in an evolution equation of the kind

$$\partial_\tau(f_1, f_2) \;=\; Q_0((f_1,f_2),\mathring{f}) + xQ_1(f',f') + L_0f'\ , \qquad (5.4.9)$$

with Q_0, Q_1 bilinear forms in $\mathbb{R}^N$ with constant coefficients, and L_0 — a matrix with constant coefficients.

On the other hand the Bianchi equations (4.3.56)-(4.3.59) can be written[4] as

$$\begin{aligned}
& e_4\cdot(x\underline{\alpha}_{ab}) + e_a\cdot(x\underline{\beta}_b) + e_b\cdot(x\underline{\beta}_a) - g^{ab}e_c\cdot(x\underline{\beta}^c)\\
&\qquad -(\mathring{\Gamma}_a{}^c{}_b + \mathring{\Gamma}_b{}^c{}_a - \mathring{\Gamma}_d{}^{dc})(x\underline{\beta}_c)\\
&= x\,x^{-1}(\overline{\Gamma}_a{}^c{}_b + \overline{\Gamma}_b{}^c{}_a - \overline{\Gamma}_d{}^{dc})(x\underline{\beta}_c) - \frac{1}{2}x\mathrm{tr}(x^{-1}\chi)(x\underline{\alpha}_{ab})\\
&\qquad -3x(\underline{\chi}_{ab}\rho - {}^\star\underline{\chi}_{ab}\sigma)\\
&\qquad +x(x^{-1}\zeta_a)(x\underline{\beta}_b) + x(x^{-1}\zeta_b)(x\underline{\beta}_a) - xg_{ab}(x^{-1}\zeta_c)(x\underline{\beta}^c)\ , &(5.4.10a)\\
& e_3\cdot(x\underline{\beta}_a) + e_c\cdot(x\underline{\alpha}^c{}_a) + (\mathring{\Gamma}_c{}^c{}_a - \mathring{\Gamma}_c{}^d{}_a)(x\underline{\alpha}^c{}_d) - \mathring{\Gamma}_3{}^c{}_a(x\underline{\beta}_c)\\
&= -x\,x^{-1}(\overline{\Gamma}_c{}^c{}_a - \overline{\Gamma}_c{}^d{}_a)x\underline{\alpha}^c{}_d + x\,x^{-1}\overline{\Gamma}_3{}^c{}_a x\underline{\beta}_c\\
&\qquad -2\mathrm{tr}\underline{\chi}(x\underline{\beta}_a) - 2x(x^{-1}\underline{\omega})(x\underline{\beta}_a) - x(x\underline{\alpha}_{ac})(x^{-1}\eta^c - 2x^{-1}\zeta^c)\\
&\qquad +2xa(\underline{\chi})^\star\underline{\beta}_a + 3x(-\underline{\xi}_a\rho + {}^\star\underline{\xi}_a\sigma) + 2\underline{\beta}_a\ , &(5.4.10b)\\
& e_4\cdot\underline{\beta}_a + e_a\cdot\rho - \epsilon^{ab}e_b\cdot\sigma
\end{aligned}$$

$$
\begin{aligned}
&= 2\overline{\chi}\cdot\beta - x^{-1}\mathrm{tr}\chi x\underline{\beta}\ , && (5.4.11a)\\
&e_3\cdot\sigma + \epsilon^{ab}e_a\cdot\underline{\beta}_b + \epsilon^{cb}\mathring{\Gamma}_a{}^a{}_c\underline{\beta}_b \\
&= -x\epsilon^{cb}(x^{-1}\overline{\Gamma}_a{}^a{}_c)\underline{\beta}_b - \frac{1}{2}(x^{-1}\overline{\chi})\cdot{}^\star(x\underline{\alpha}) - \frac{3}{2}\mathrm{tr}\underline{\chi}\sigma \\
&\quad -2\underline{\xi}\cdot{}^\star\beta + x(x^{-1}\zeta - 2x^{-1}\eta)\cdot{}^\star\underline{\beta} - \frac{3}{2}\rho a(\underline{\chi})\ , && (5.4.11b)\\
&e_3\cdot\rho + e_a\cdot\underline{\beta}^a + \mathring{\Gamma}_a{}^a{}_c\underline{\beta}^c \\
&= -x(x^{-1}\overline{\Gamma}_a{}^a{}_c)\underline{\beta}^c - \frac{3}{2}\mathrm{tr}\underline{\chi}\rho - \frac{1}{2}(x^{-1}\overline{\chi})\cdot(x\underline{\alpha}) \\
&\quad +x(x^{-1}\zeta - 2x^{-1}\eta)\cdot\underline{\beta} + 2\underline{\xi}\cdot\beta + \frac{3}{2}a(\underline{\chi})\sigma\ , && (5.4.11c)\\
&e_4\cdot\rho - e_a\cdot\beta^a - \mathring{\Gamma}_a{}^a{}_c\beta^c \\
&= x(x^{-1}\overline{\Gamma}_a{}^a{}_c)\beta^c - \frac{3}{2}x\mathrm{tr}(x^{-1}\chi)\rho - \frac{1}{2}\underline{\overline{\chi}}\cdot\alpha + x^{-1}\zeta\cdot(x\beta)\ , && (5.4.12a)\\
&e_4\sigma + e_a\cdot{}^\star\beta^a + \mathring{\Gamma}_a{}^a{}_c{}^\star\beta^c \\
&= -x^{-1}\overline{\Gamma}_a{}^a{}_c(x^\star\beta^c) - \frac{3}{2}x\mathrm{tr}(x^{-1}\chi)\sigma + \frac{1}{2}\underline{\overline{\chi}}\cdot{}^\star\alpha - x^{-1}\zeta\cdot(x^\star\beta)\ , && (5.4.12b)\\
&e_3\cdot\beta_a - e_a\cdot\rho - \epsilon_a^b e_b\cdot\sigma - \mathring{\Gamma}_3{}^c{}_a\beta_c \\
&= x(x^{-1}\overline{\Gamma}_3{}^c{}_a)\beta_c - \mathrm{tr}\underline{\chi}\beta + 2x(x^{-1}\overline{\chi})\cdot\underline{\beta} + 2x^{-1}\underline{\omega}(x\beta) \\
&\quad +3x(x^{-1}\eta\rho + x^{-1\star}\eta\sigma) + \underline{\xi}\cdot\alpha - a(\underline{\chi})^\star\beta\ , && (5.4.12c)\\
&e_4\cdot\beta_a - e_b\cdot\alpha^b{}_a - \mathring{\Gamma}_b{}^b{}_c\alpha^c{}_a + \mathring{\Gamma}_b{}^c{}_a\alpha^b{}_c \\
&= x(x^{-1}\overline{\Gamma}_b{}^b{}_c\alpha^c{}_a - x^{-1}\overline{\Gamma}_b{}^c{}_a\alpha^b{}_c) - 2\mathrm{tr}\chi\beta \\
&\quad +2x\alpha_{ab}(x^{-1}\zeta^b) + 2xa(x^{-1}\chi)^\star\beta_a\ , && (5.4.13a)\\
&e_3\alpha_{ab} - e_a\cdot\beta_b - e_b\cdot\beta_a + g^{ab}e_c\cdot(\beta^c) \\
&\quad -\mathring{\Gamma}_3{}^c{}_b\alpha_{ac} - \mathring{\Gamma}_3{}^c{}_a\alpha_{cb} + (\mathring{\Gamma}_a{}^c{}_b + \mathring{\Gamma}_b{}^c{}_a - \mathring{\Gamma}_d{}^{dc})\beta_c \\
&= x\, x^{-1}(\overline{\Gamma}_d{}^{dc} - \overline{\Gamma}_a{}^c{}_b - \overline{\Gamma}_b{}^c{}_a)\beta_c + x(x^{-1}\overline{\Gamma}_3{}^c{}_b)\alpha_{ac} - x(x^{-1}\overline{\Gamma}_3{}^c{}_a)\alpha_{cb} \\
&\quad -\frac{1}{2}\mathrm{tr}\underline{\chi}\alpha_{ab} + 4x(x^{-1}\underline{\omega})\alpha_{ab} - \frac{1}{2}a(\underline{\chi})^\star\alpha_{ab} \\
&\quad -3x(x^{-1}\overline{\chi}_{ab}\rho + x^{-1\star}\overline{\chi}_{ab}\sigma) + x(4x^{-1}\eta_a + x^{-1}\zeta_a)\overline{\otimes}_s\beta_b\ . && (5.4.13b)
\end{aligned}
$$

Let us note that we have written $e_+\cdot(x\underline{\beta}) = xe_+\cdot\underline{\beta} - 2\underline{\beta}$ to obtain the last equation of (5.4.11). Each of the subsystems (5.4.10),(5.4.11),(5.4.12) can be put in the form

$$
\begin{aligned}
\partial_\tau\phi + A'^a e_a\cdot\psi + B_{11}\phi + B_{12}\psi &= \langle Q_2 f', f'\rangle + x\langle Q_3 f', f\rangle\ , && (5.4.14)\\
e_3\cdot\psi + {}^tA'^a e_a\cdot\psi + B_{21}\phi + B_{22}\psi &= \langle Q_4 f', f'\rangle + x\langle Q_5 f', f\rangle + L_1 f\ , && (5.4.15)
\end{aligned}
$$

with the B_{ij} arising from those terms in (5.4.10)-(5.4.13) that contains the $\mathring{\Gamma}$'s, and can be written as

$$
B_{ij} = B'_{ij}(\mathring{f})\ . \qquad (5.4.16)
$$

The matrices Q_2, Q_3, Q_4, Q_5 have constant coefficients; B'_{ij} is linear in $\mathring{f}$, again with constant coefficients; all the coefficients take values in

$$
\{1, 2, 3, 4, -1, -2, -3, -4, 1/2, -1/2, 3/2, -3/2, 0\}\ .
$$

Now let us give some details how (5.4.10)-(5.4.13) can be put in the form (5.4.14)-(5.4.15) and satisfy the hypotheses $H0$)-$H3$), p. 111. For (5.4.10) one sets

$$\phi = \begin{pmatrix} x\underline{\alpha}_{11} \\ x\underline{\alpha}_{12} \end{pmatrix} = (x\underline{\alpha}_{ab}) \;, \tag{5.4.17}$$

$$\psi = \begin{pmatrix} x\underline{\beta}_1 \\ x\underline{\beta}_2 \end{pmatrix} . \tag{5.4.18}$$

Then, in the notation of (5.3.8) the A^i are given by (4.4.3) so that $H1$)-$H2$) hold. Condition $H0$) holds by (4.4.6). Equation (4.4.10) shows that $H3$) holds. Using that last equation together with (4.3.49) one can read off the coefficients appearing in (5.3.11):

$$\begin{aligned}
(\Gamma_3\psi)_a &= (\Gamma_3 x\underline{\beta})_a \\
&= (-\underline{\omega} + \frac{1}{2}\mathrm{tr}\underline{\chi})x\underline{\beta}_a - {\Gamma_3}^a{}_b\underline{\beta}_b \;, && (5.4.19a) \\
(\Gamma'_3\phi)_a &= (\Gamma_3 x\underline{\beta})_a \\
&= \frac{1}{2}\eta^b x\underline{\alpha}_{ab} \;, && (5.4.19b) \\
(\Gamma_4\phi)_{ab} &= (\Gamma_4 x\underline{\alpha})_{ab} \\
&= \frac{1}{2}\mathrm{tr}\chi x\underline{\alpha}_{ab} \;, && (5.4.19c) \\
(\Gamma'_4\psi)_a &= (\Gamma'_4 x\underline{\beta})_a \\
&= 0 \;, && (5.4.19d) \\
(\Gamma\psi)_{ab} &= (\Gamma x\underline{\beta})_{ab} \\
&= -{A'^e}_{ab}{}^d{\Gamma_e}^c{}_d x\underline{\beta}_c \;, && (5.4.19e) \\
(\Gamma'\phi)_a &= (\Gamma' x\underline{\alpha})_a \\
&= {A'^g}_{cd}{}^i h^{ce}h^{df}h_{ai}({\Gamma_g}^h{}_e x\underline{\alpha}_{hf} + {\Gamma_g}^h{}_f x\underline{\alpha}_{he}) \;. && (5.4.19f)
\end{aligned}$$

(recall that $h_{ab} = \delta_a^b$ denotes the metric induced by g on Vect$\{e_a\}$; the matrices $A'^a = ({A'^a}_{bc}{}^e)$ have been defined in Equation (5.3.8a), and can be read off from Equation (4.4.3) — they have constant coefficients in $\{1, -1\}$).

For (5.4.13) one sets

$$\phi = \begin{pmatrix} \beta_1 \\ \beta_1 \end{pmatrix} , \tag{5.4.20}$$

$$\psi = \begin{pmatrix} \alpha_{11} \\ \alpha_{12} \end{pmatrix} = (\alpha_{ab}) \;. \tag{5.4.21}$$

In the notation of (5.3.8) the A^i are given by obvious renamings and permutations of (4.4.3). The (α, β) equivalent of 4.4.6 gives $H0$). From the (α, β) version of (4.4.10) and from (4.3.49), one can read off the coefficients appearing in (5.3.11):

$$(\Gamma_4\phi)_a = (\Gamma_4\beta)_a$$

$$
\begin{aligned}
&= \frac{1}{2}\mathrm{tr}\chi\beta_a \, , && (5.4.22a)\\
(\Gamma'_4\phi)_a &= (\Gamma'_4\beta)_a \\
&= \frac{1}{2}\eta^b\alpha_{ab} \, , && (5.4.22b)\\
(\Gamma_3\psi)_a &= (\Gamma_3\alpha)_{ab} \\
&= (\frac{1}{2}\mathrm{tr}\underline{\chi} - \underline{\omega})\alpha_{ab} \, , && (5.4.22c)\\
(\Gamma'_3\phi)_{ab} &= (\Gamma'_3\beta)_{ab} \\
&= 0 \, , && (5.4.22d)\\
(\Gamma'\phi)_{ab} &= (\Gamma'\beta)_{ab} \\
&= -A'^e{}_{ab}{}^d\Gamma_e{}^c{}_d\beta_c \, , && (5.4.22e)\\
(\Gamma\psi)_a &= (\Gamma\alpha)_a \\
&= A'^g{}_{cd}{}^i h^{ce}h^{df}h_{ai}(\Gamma_g{}^h{}_e\alpha_{hf} + \Gamma_g{}^h{}_f\alpha_{he}) \, . && (5.4.22f)
\end{aligned}
$$

Here the matrices $A'^a = (A'^a{}_{bc}{}^e)$ are the same as in Equation (5.4.19f), and thus again have constant coefficients in $\{1, -1\}$.

For (5.4.12), one sets

$$\phi = \begin{pmatrix} \rho \\ \sigma \end{pmatrix} \, , \quad \psi = (\beta_a) \, . \tag{5.4.23}$$

Then the A_i are given by (4.4.11). The identity (4.4.13) ensures that $H0)$ is satisfied. The gamma's can be read off from (4.4.15):

$$
\begin{aligned}
(\Gamma_3\psi)_a &= (\Gamma_3\beta)_a \\
&= -\Gamma_3{}^a{}_b\beta^b + (\frac{1}{2}\mathrm{tr}\underline{\chi} - \underline{\omega})\beta_a \, , && (5.4.24a)\\
(\Gamma'_3\phi) &= \left(\Gamma'_3\begin{pmatrix} \rho \\ \sigma \end{pmatrix}\right) \\
&= 0 \, , && (5.4.24b)\\
(\Gamma_4\phi) &= \left(\Gamma_4\begin{pmatrix} \rho \\ \sigma \end{pmatrix}\right) \\
&= \frac{1}{2}\mathrm{tr}\chi\begin{pmatrix} \rho \\ \sigma \end{pmatrix} \, , \\
(\Gamma'_4\psi)_a &= \left(\Gamma'_4\beta\right)_a \\
&= 0 \, , && (5.4.24c)\\
(\Gamma\psi) &= (\Gamma\beta) \\
&= \begin{pmatrix} -\Gamma_a{}^a{}_c\,\beta^c - \eta_c\beta^c \\ \Gamma_a{}^a{}_c\,\epsilon^{cd}\beta_d + \epsilon^{cd}\eta_c\beta_d \end{pmatrix} \, , && (5.4.24d)\\
(\Gamma'\phi)_a &= \left(\Gamma'\begin{pmatrix} \rho \\ \sigma \end{pmatrix}\right)_a \\
&= 0 \, . && (5.4.24e)
\end{aligned}
$$

For (5.4.11), one sets

$$\phi = (\underline{\beta}_a) \tag{5.4.25}$$

$$\psi = \begin{pmatrix} \rho \\ \sigma \end{pmatrix} . \tag{5.4.26}$$

Again the A_i are given by permutations of (4.4.11) and the gammas by the $((\rho,\sigma),\underline{\beta})$ version of (4.4.15) :

$$\begin{aligned}
(\Gamma_4\phi)_a &= \left(\Gamma_4\underline{\beta}\right)_a \\
&= \frac{1}{2}\mathrm{tr}\chi\underline{\beta}_a , && (5.4.27a) \\
(\Gamma_4'\psi) &= \left(\Gamma_4'\begin{pmatrix} \rho \\ \sigma \end{pmatrix}\right) \\
&= 0 , && (5.4.27b) \\
(\Gamma_3\phi) &= \left(\Gamma_3\begin{pmatrix} \rho \\ \sigma \end{pmatrix}\right) \\
&= (\frac{1}{2}\mathrm{tr}\underline{\chi} - \underline{\omega})\begin{pmatrix} \rho \\ \sigma \end{pmatrix} , \\
(\Gamma_3'\phi)_a &= \left(\Gamma_3'\underline{\beta}\right)_a \\
&= 0 , && (5.4.27c) \\
(\Gamma'\phi) &= (\Gamma'\underline{\beta}) && (5.4.27d) \\
&= \begin{pmatrix} -\Gamma_a{}^a{}_c\,\underline{\beta}^c - \eta_c\underline{\beta}^c \\ \Gamma_a{}^a{}_c\,\epsilon^{cd}\underline{\beta}_d + \epsilon^{cd}\eta_c\underline{\beta}_d \end{pmatrix} , && (5.4.27e) \\
(\Gamma\psi)_a &= \left(\Gamma\begin{pmatrix} \rho \\ \sigma \end{pmatrix}\right))_a \\
&= 0 . && (5.4.27f)
\end{aligned}$$

Proposition 5.4.1 Consider a field f, defined by Equation (5.4.6), arising from a metric which solves the vacuum Einstein equation, such that $f(\tau) \in H_k^{\mathrm{loc}}(M_{x_2,x_1-3\tau})$ for some $k \geq 4$, and such that the conclusions of Lemma 4.3.1 hold. Let $T(x_2)$ be given by (5.4.1), and let us define, for $0 \leq \tau < T(x_2)$ and $-1 < \alpha < -1/2$,

$$\begin{aligned}
E_k^\alpha(x_2,x_1,\tau) := {}& \|(f_1,f_2)(\tau)\|_{\mathscr{H}_k^\alpha(M_{x_2,x_1-3\tau})} + \|(f_1,f_2)(\tau)\|_{L^\infty(M_{x_2,x_1-3\tau})} \\
&+ \|(f_3,f_4)(\tau)\|^2_{\mathscr{H}_k^\alpha(M_{x_2,x_1-3\tau})} + \|(f_3',f_4)(\tau)\|^2_{L^\infty(M_{x_2,x_1-3\tau})} \\
&+ \|x\partial_x\alpha_{ab}\|^2_{L^\infty(M_{x_2,x_1-3\tau})} .
\end{aligned} \tag{5.4.28}$$

Suppose that

$$M_1(x_2,x_1) := 2E_k^\alpha(x_2,x_1,0) + 2\|\mathring{f}\|_{\mathcal{G}_k^0} + 1 < \infty , \tag{5.4.29}$$

and let $T^*(M_1,x_2,x_1,\alpha,k,T(x_2)) > 0$ be defined as

$$T^* := \sup\{0 \leq \tau < T(x_2)\ |\forall s \in [0,\tau]\ E_k^\alpha(x_2,x_1,s) \leq M_1\} . \tag{5.4.30}$$

Then there exists $C_1(M_1,x_1,\alpha,k)$ bounded over bounded sets of variables such that, for $0 \leq \tau < T^*$,

$$\begin{aligned}
\|\partial_\tau(f_1,f_2)(\tau)\|_{L^\infty} &\leq C_1(M_1,x_1) , && (5.4.31) \\
\|\partial_\tau(f_1,f_2)(\tau)\|_{\mathscr{H}_k^\alpha} &\leq C_1(M_1,x_1) , && (5.4.32) \\
\|(f_1,f_2)(\tau)\|_{\mathscr{H}_k^\alpha} &\leq \|(f_1,f_2)(0)\|_{\mathscr{H}_k^\alpha} + \tau C_1(M_1,x_1) , && (5.4.33) \\
\|(f_1,f_2)(\tau)\|_{L^\infty} &\leq \|(f_1,f_2)(0)\|_{L^\infty} + \tau C_1(M_1,x_1) . && (5.4.34)
\end{aligned}$$

Further there exist constants $T_3(M_1, x_1, \alpha, k) > 0$, $C_2(M_1, x_1, \alpha, k)$ such that, for all $0 \le \tau < \min(T^*, T_3)$,

$$\begin{aligned}
&\|(f_3, f_4)(\tau)\|^2_{\mathscr{H}^\alpha_k} + \|(\alpha_{ab}, \beta_a, \rho, \sigma, x\underline{\alpha}_{ab})(\tau)\|^2_{L^\infty} + \|x\partial_x \alpha_{ab}(\tau)\|^2_{L^\infty} \\
\le\ & 2\|(f_3, f_4)(0)\|^2_{\mathscr{H}^\alpha_k} + \|(\alpha_{ab}, \beta_a, \rho, \sigma, x\underline{\alpha}_{ab})(0)\|^2_{L^\infty} + \|x\partial_x \alpha_{ab}(0)\|^2_{L^\infty} \\
& + C_2(M_1, x_1, \alpha, k)(\tau + \tau^{\alpha+1}) \ . \qquad (5.4.35)
\end{aligned}$$

PROOF: Inequality (5.4.31) follows immediately from (5.4.9), (5.4.32) is a direct consequence of Equation (5.4.9) using the Moser inequality

$$\|fg\|_{\mathscr{H}^{\alpha+\beta}_k} \le C_s \|f_1\|_{\mathscr{H}^\alpha_k} \|g\|_{\mathcal{G}^\beta_k} \le C_s \|f_1\|_{\mathscr{H}^\alpha_k} \|g\|_{\mathscr{H}^\beta_k} \ , \qquad (5.4.36)$$

which is a straightforward consequence of (3.2.35). We estimate the right hand side terms as

$$\begin{aligned}
\|Q_0((f_1, f_2), \mathring{f})\|_{\mathscr{H}^\alpha_k} &\le C(Q_0) C_s \|(f_1, f_2)\|_{\mathscr{H}^\alpha_k} \|\mathring{f}\|_{\mathcal{G}^0_k} \\
&\le C(C_s) \left(\|(f_1, f_2)\|^2_{\mathscr{H}^\alpha_k} + \|\mathring{f}\|^2_{\mathcal{G}^0_k} \right) \\
&\le C(C_s) M_1^2 \ , \\
\|xQ_1(f', f')\|_{\mathscr{H}^\alpha_k} &\le C(x_1, \alpha, C_s) \|f'\|^2_{\mathscr{H}^\alpha_k} \\
&\le C(x_1, \alpha, C_s) M_1 \ ,
\end{aligned}$$

and similarly for $L_0 f'$. Inequalities (5.4.33)-(5.4.34) follow from (5.4.31)-(5.4.32) using the next lemma with $b = 0$ and $c = \partial_\tau(f_1, f_2)$:

Lemma 5.4.2 Let $U : M_{x_2,x_3} \times [0, T] \to \mathbb{R}^N$ for some $T > 0$ and $0 < x_2 < x_3$, satisfy the equation

$$\partial_\tau U = bU + c \ , \qquad (5.4.37)$$

with $b : M_{x_2,x_3} \times [0, T] \to End(\mathbb{R}^N)$, $c : M_{x_2,x_3} \to \mathbb{R}^N$. Suppose $U(0) \in H^{\rm loc}_k(M_{x_2,x_3})$ and $b(\tau), c(\tau) \in H^{\rm loc}_k(M_{x_2,x_3})$ for $\tau \in [0, T]$. If $k > n/2$, then U satisfies the inequality

$$\|U(\tau)\|_{\mathscr{H}^\alpha_k} \le \|U(0)\|_{\mathscr{H}^\alpha_k} + C(C_s) \int_0^\tau \|b(s)\|_{\mathcal{G}^0_k} \|U(s)\|_{\mathscr{H}^\alpha_k} + \|c(s)\|_{\mathscr{H}^\alpha_k} ds \ , \qquad (5.4.38)$$

(recall that we use the symbol C_s for constants arising from Sobolev embedding and the likes) and

$$\|U(\tau)\|_{L^\infty} \le \|U(0)\|_{L^\infty} + \int_0^\tau \|b(s)\|_{L^\infty} \|U(s)\|_{L^\infty} + \|c(s)\|_{L^\infty} ds, \qquad (5.4.39)$$

for all $\tau \in [0, t]$.

PROOF: Let $\varepsilon > 0$, we note that $U \in C^1([0, T], H^{\rm loc}_k(M_{x_2, x_1 - 3\tau}))$ (*cf.* Remark 2 after 3.3.1). We have

$$\partial_\tau (\|U(\tau)\|^2_{\mathscr{H}^\alpha_k(M_{x_2,x_3})} + \varepsilon)^{1/2} \le \|\partial_\tau U(\tau)\|_{\mathscr{H}^\alpha_k(M_{x_2,x_3})} \ , \qquad (5.4.40)$$

similarly for the L^∞ norm. Then, we deduce

$$\|U(\tau)\|_{\mathscr{H}_k^\alpha(M_{x_2,x_3})} \leq \|U(0)\|_{\mathscr{H}_k^\alpha(M_{x_2,x_3})} + \int_0^T \|\partial_\tau U(s)\|_{\mathscr{H}_k^\alpha(M_{x_2,x_3})} ds \ , \quad (5.4.41)$$

Therefore, using Equation (5.4.37) and the Moser inequality (5.4.36),

$$\|U(\tau)\|_{\mathscr{H}_k^\alpha} \leq \|U(0)\|_{\mathscr{H}_k^\alpha} + \int_0^\tau \|(bU+c)(s)\|_{\mathscr{H}_k^\alpha} ds \ , \quad (5.4.42)$$

$$\leq \|U(0)\|_{\mathscr{H}_k^\alpha} + \int_0^\tau C_s\|b(s)\|_{\mathcal{G}_k^0}\|U\|_{\mathscr{H}_k^\alpha} + \|c(s)\|_{\mathscr{H}_k^\alpha} ds \quad (5.4.43)$$

The L^∞ inequality is a corollary of Lemma 3.3.1. □

Let us turn our attention to the proof of inequality (5.4.35). We wish to apply Proposition 5.3.6 to the systems (5.4.10)-(5.4.13), in order to do that we need to verify the relevant hypotheses. We have already shown that (5.4.10)-(5.4.13) can be rewritten in the from (5.4.14)-(5.4.15), with the hypotheses $H0$)-$H3$), pp. 111-112, being satisfied. We need, next, to verify that conditions $\mathscr{C}1$)-$\mathscr{C}3$), p. 104-106, hold on $\mathscr{M}_{x_2,x_1,t}$ for $t < T(x_2)$ (taking $t_1 = t_2 = t* = t$). The definitions (5.4.28) and (5.4.30) show that Equation (5.2.1) holds with

$$\epsilon = 1 + \alpha > 0 \ ,$$

and

$$C_{\overline{e}_+} = M_1 \ .$$

Equation (5.2.2) and Eq4.2.3 hold by hypothesis, *cf.* Lemma 4.3.1. From the definition (5.4.29) of M_1 we have

$$\|\mathring{e}_a\|_{\mathcal{G}_k^0} \leq M_1 \ , \quad (5.4.44)$$

which gives (5.2.7), using the weighted Sobolev embedding (since $k \geq 2$) and the fact that the $\mathring{e}_a(0)$'s are defined on a compact set. Equation (5.2.8) follows immediately from (5.4.29) with

$$C_{\overline{e}_a} = M_1 \ , \quad (5.4.45)$$

while (5.2.9) holds by construction. The estimates (5.4.31)-(5.4.32) give the existence of $C_e^\tau(M_1, x_1, \alpha)$ such that (5.2.25) is satisfied with $\epsilon = 1 + \alpha$. We have thus shown that conditions $\mathscr{C}1$)-$\mathscr{C}3$) are satisfied.

Consider, next, Condition $H4$), p. 112. The map A of Equation (5.3.1) corresponds to the matrices B_{ij} in the system (5.4.14)-(5.4.15); those depend linearly upon $\mathring{f}$ (*cf.* Equation (5.4.16)), which gives the existence of a constant $C_A(M_1)$ such that the inequality (5.3.12) is satisfied. Further the maps $\Gamma, \Gamma', \Gamma_3, \Gamma_4$ given by (5.4.19), (5.4.22), (5.4.24) and (5.4.27) are linear combinations of $\mathring{\Gamma}_a{}^b{}_c$, $\mathring{\Gamma}_3{}^a{}_b$, $\overline{\Gamma}_a{}^b{}_c$, $\overline{\Gamma}_3{}^a{}_b$, and $\mathrm{tr}\chi, \mathrm{tr}\underline{\chi}, \underline{\omega}$, which gives

$$\begin{aligned}\|(\Gamma,\Gamma',\Gamma_3,\Gamma_4)\|_{L^\infty} &\leq C(C_s,x_1)(\|\mathring{f}\|_{\mathcal{G}_k^0} + \|f_2\|_{L^\infty}) \\ &\leq C(C_s,x_1)M_1 \ ,\end{aligned}$$

hence (5.3.13) is satisfied. It follows that all the hypotheses $H1$)-$H4$) are satisfied as well, and Proposition 5.3.6 applies — the source terms a and b there are now the non-linear terms appearing in the right-hand side of (5.4.14)-(5.4.15). Letting Q stand for one of the Q_2, Q_3, Q_4, Q_5, we estimate the corresponding contribution as

$$\begin{aligned} \|Q(f',f')(\tau)\|_{\mathscr{H}_k^\alpha} &\leq c(C_s)(\|f'(\tau)\|_{L^\infty}\|f'(\tau)\|_{\mathscr{H}_k^\alpha}) \\ &\leq C(C_s,M_1)\;, \qquad (5.4.46a) \\ \|xQ(f',f)(\tau)\|_{\mathscr{H}_k^\alpha} &\leq c(C_s)\|f'(\tau)\|_{\mathscr{H}_k^\alpha}\|xf(\tau)\|_{\mathscr{H}_k^0} \\ &\leq C(x_1,\alpha)\|f'(\tau)\|_{\mathscr{H}_k^\alpha}\|f(\tau)\|_{\mathscr{H}_k^\alpha} \\ &\leq C(M_1,x_1,\alpha)\;. \qquad (5.4.46b) \end{aligned}$$

The estimate 5.3.36 of Lemma 5.3.5 applied to each of the systems (5.4.10)-(5.4.13) written in the form (5.4.14)-(5.4.15), gives

$$\|\partial_\tau(\beta_a,(\rho,\sigma),\underline{\beta}_a,x\underline{\alpha}_{ab})(\tau)\|_{\mathscr{C}_0^\alpha} \leq C(M_1,x_1,\alpha,k)\;. \tag{5.4.47}$$

Proposition 5.3.6 applied to (5.4.10) gives the existence of a time $T_1>0$ depending only upon M_1, such that the estimate (5.3.49) holds for $0\leq\tau\leq\min(T^*,T_1)$:

$$\begin{aligned} &c_1(\|x\underline{\alpha}(\tau)\|^2_{\mathscr{H}_k^\alpha}+\|x\underline{\beta}(\tau)\|^2_{\mathscr{H}_k^\alpha})+\|x\underline{\beta}(\tau)\|^2_{L^\infty} \\ &\quad\leq\; c_2(\|x\underline{\alpha}(0)\|^2_{\mathscr{H}_k^\alpha}+\|x\underline{\beta}(0)\|^2_{\mathscr{H}_k^\alpha})+\|x\underline{\beta}(0)\|^2_{L^\infty} \\ &\qquad+\int_0^\tau C(M_1,x_1,\alpha)\,(1+(\tau-s)^\alpha)\;ds\;. \end{aligned}$$

We have also used Lemma 5.3.3 and Equations (5.4.46); further, Equation (5.4.47) has been taken into account to control the τ-derivative terms appearing at the right-hand-side of (5.3.49). The inequality (5.3.49) applied to (5.4.11) and (5.4.12) leads to a similar inequality involving $\|\underline{\beta}\|_{\mathscr{H}_k^\alpha}$, $\|(\rho,\sigma)\|_{\mathscr{H}_k^\alpha}$, $\|(\rho,\sigma)\|_{L^\infty}$, $\|\beta\|_{\mathscr{H}_k^\alpha}$ and $\|\beta\|_{L^\infty}$. Note that (5.3.49) is not of any use for (5.4.13), because we have no estimate on $\partial_\tau\alpha$. We use Equation (5.3.50) instead, which gives

$$\begin{aligned} &\|\alpha_{ab}(\tau)\|^2_{\mathscr{H}_k^\alpha}+\|\alpha_{ab}(\tau)\|^2_{L^\infty}+\|x\partial_x\alpha_{ab}(\tau)\|^2_{L^\infty}+\|\beta_a(\tau)\|^2_{\mathscr{H}_k^\alpha} \\ &\quad\leq\;\|\alpha_{ab}(0)\|^2_{\mathscr{H}_k^\alpha}+\|\alpha_{ab}(0)\|^2_{L^\infty}+\|x\partial_x\alpha_{ab}(0)\|^2_{L^\infty}+\|\beta_a(0)\|^2_{\mathscr{H}_k^\alpha} \\ &\qquad+C(M_1,x_1,\alpha,\epsilon)\int_0^\tau(1+(\tau-s)^\alpha)\;ds\;. \end{aligned}$$

The term involving $\|b\|_{\mathscr{C}_1^\alpha}$ at the right-hand-side of Equation (5.3.50) has been estimated using (5.4.46) and the weighted Sobolev embedding.

To finish the proof of (5.4.35) we need an inequality involving $\|x\underline{\alpha}\|_{L^\infty}$. In order to obtain an estimate for this quantity we rewrite Equation (5.4.10a) as

$$\begin{aligned} \partial_\tau\underline{\alpha}_{ab} \;=\; &-\{e_a\cdot(\underline{\beta}_b)+e_b\cdot(\underline{\beta}_a)-g^{ab}e_c\cdot(\underline{\beta}^c)-(\mathring{\Gamma}_a{}^c{}_b+\mathring{\Gamma}_b{}^c{}_a-\mathring{\Gamma}_d{}^{dc})(\underline{\beta}_c)\} \\ &+(\overline{\Gamma}_a{}^c{}_b+\overline{\Gamma}_b{}^c{}_a-\overline{\Gamma}_d{}^{dc})\underline{\beta}_c-\frac{1}{2}\mathrm{tr}(\chi)\underline{\alpha}_{ab} \\ &-3(\underline{\chi}_{ab}\rho-{}^\star\underline{\chi}_{ab}\sigma)+\zeta_a\underline{\beta}_b+\zeta_b\underline{\beta}_a-g_{ab}\zeta_c\underline{\beta}^c\;. \end{aligned} \tag{5.4.48}$$

Terms such as $e_a \cdot \beta_b$ are estimated as follows:

$$\begin{aligned} \|e_a \cdot \underline{\beta}_b\|_{\mathscr{C}_0^\alpha} &\leq \|e_a^A \partial_A \underline{\beta}_b\|_{\mathscr{C}_0^\alpha} + \|e_a^\tau \partial_\tau \underline{\beta}_b\|_{\mathscr{C}_0^\alpha} \\ &\leq C(C_e)\|\underline{\beta}\|_{\mathscr{C}_1^\alpha} + C_{\overline{e}}\|\partial_\tau \underline{\beta}\|_{\mathscr{C}_0^{\alpha-1}} , \end{aligned}$$

where the last term can be estimated with Lemma 5.3.5 applied to (5.4.11). The terms containing ρ and σ are estimated using the fact that

$$\|\rho(\tau)\|_{L^\infty} \leq \sqrt{M_1} ,$$

similarly for σ. Handling the remaining terms in a similar way one is led to

$$\|\partial_\tau \underline{\alpha}_{ab}\|_{\mathscr{C}_0^\alpha} \leq C(M_1, x_1, \alpha) , \tag{5.4.49}$$

which obviously implies

$$\|\partial_\tau (x\underline{\alpha}_{ab})\|_{L^\infty} \leq C(M_1, x_1, \alpha) . \tag{5.4.50}$$

Integrating in τ one thus has the desired inequality

$$\|x\underline{\alpha}_{ab}(\tau)\|^2_{L^\infty} \leq \|x\underline{\alpha}_{ab}(0)\|^2_{L^\infty} + \tau C(M_1, x_1, \alpha) . \tag{5.4.51}$$

Summing all the estimates gives (5.4.35). □

Choosing T_4 sufficiently small so that all the expressions at the left-hand-sides of the inequalities of Proposition 5.4.1 are smaller than some multiple of their initial values, one obtains the main result of this section:

Theorem 5.4.3 Let $0 < x_2 < x_1/2$ and $k \geq 4$. Let $f_1, f_2, f_3, f_4, \mathring{f}$ be defined on $\mathscr{M}_{x_2,x_1,t}$ for any $0 \leq t < T(x_2)$ as in (5.4.6), and satisfy Equations (5.4.7)-(5.4.8) and (5.4.10)-(5.4.13) there. Suppose that the conclusions of Lemma 4.3.1 hold, and that there exists $-1 < \alpha < -1/2$ such that

$$\begin{aligned} M_0 \;:=\; & \|f(0)\|_{\mathscr{H}_k^\alpha(\Sigma_{x_1})} + \|\mathring{f}(0)\|_{\mathcal{G}_k^0(\Sigma_{x_1})} + \|(f'(0)\|_{L^\infty(\Sigma_{x_1})} \\ & + \|x\partial_x \alpha_{ab}(0)\|_{L^\infty(\Sigma_1)} < +\infty . \end{aligned} \tag{5.4.52}$$

Then there exists $T_4(M_0, x_1, k, \alpha)$ and $C(M_0, x_1, \alpha, k)$, independent of $T(x_2)$ (where $T(x_2)$ is defined before Equation (5.4.1)), such that for any $0 \leq \tau < \min(T(x_2), T_4)$, we have for any $x_2 > 0$,

$$\begin{aligned} & \|f(\tau)\|_{\mathscr{H}_k^\alpha(M_{x_2,x_1-3\tau})} + \|f'(\tau)\|_{L^\infty(M_{x_2,x_1-3\tau})} + \|x\partial_x \alpha_{ab}(\tau)\|_{L^\infty(M_{x_2,x_1-3\tau})} \\ \leq\; & C(M_0, x_1, \alpha, k) \end{aligned} \tag{5.4.53}$$

5.5 Local existence theorem for the Einstein equations in weighted spaces.

We want to show that there exists a development of initial data on Σ which contains $\mathscr{M}_{0,x_1,T}$ for some T; this claim follows essentially from the estimate of Theorem 5.4.3. The result is, however, not completely obvious, because that estimate applies only to the system of equations considered there, which has been derived as a subset of the set of Equations (4.3.1); while those equations are equivalent to the vacuum Einstein equations, Equations (5.4.7)-(5.4.8) and (5.4.10)-(5.4.13) are not. Now, it is standard to show that the estimate (5.4.53) implies existence of solutions of the reduced set (5.4.7)-(5.4.8) and (5.4.10)-(5.4.13) on a set containing $\mathscr{M}_{0,x_1,T_*}$ for some τ_*, however it could turn out that the solution so obtained does not satisfy the vacuum Einstein equations everywhere. The fact that this does not happen would follow if one showed, for appropriate initial data, that all the equations (4.3.1) hold when the reduced ones do. A proof along those lines would involve quite heavy calculations. We shall instead present a general abstract argument which avoids those.

Before proceeding further it is useful to recall a convenient form of the local existence theorem for Einstein's equations. By definition, *vacuum initial data* are defined as the triple $(N, \widetilde{h}_{ij}, \widetilde{K}_{ij})$, where $\widetilde{h}_{ij}$ is a Riemannian metric on a manifold N, $\widetilde{K}_{ij}$ is a symmetric tensor field on N, and $(\widetilde{h}_{ij}, \widetilde{K}_{ij})$ satisfy the vacuum constraint equations:

$$\tilde{D}_i \left(\widetilde{K}^{ij} - \widetilde{h}^{ij}\widetilde{h}^{kl}\widetilde{K}_{kl} \right) = 0 \; ,$$
$$R(\widetilde{h}) = \widetilde{h}^{ij}\widetilde{h}^{kl}(\widetilde{K}_{ik}\widetilde{K}_{jl} - \widetilde{K}_{ij}\widetilde{K}_{kl}) \; .$$

Here $\tilde{D}$ is the covariant derivative of $\widetilde{h}$, and $R(\widetilde{h})$ is the curvature scalar thereof. An imbedding i of N into a space-time $(\mathscr{M}, \widetilde{g})$ is said to be compatible with the initial data $(\widetilde{h}_{ij}, \widetilde{K}_{ij})$ on N if the pull-back $i^*\widetilde{g}$ of the space-time metric $\widetilde{g}$ on $\mathscr{M}$ coincides with $\widetilde{h}_{ij}$, while $\widetilde{K}_{ij}$ is the pull-back of the extrinsic curvature tensor of $i(N)$.

We use the symbol $\mathring{h}$ to denote some arbitrarily chosen smooth background Riemannian metric, which is introduced for notational convenience only; $\mathring{D}$ denotes the covariant derivative of the metric $\mathring{h}$. $L^2(N, d\mu_{\mathring{h}})$ is the L^2 space defined with respect to the canonical measure $d\mu_{\mathring{h}}$ of the metric $\mathring{h}$. An n dimensional manifold M with topological boundary ∂M is said to be a *smooth manifold with boundary with corner at* S if ∂M is the union of smooth $n-1$ dimensional manifolds with boundaries which intersect transversally at a smooth $n-2$ dimensional manifold S. A *generator* of a null hypersurface $\mathscr{H}$ is a null geodesic segment contained in $\mathscr{H}$.

Theorem 5.5.1 Let N be a three dimensional compact manifold with boundary with a smooth metric $\mathring{h}$, let $C_0 > 0$ and suppose that the vacuum initial data $(\widetilde{h}_{ij}, \widetilde{K}_{ij})$ satisfy

$$\sum_{0\leq i\leq \ell+1} \|\mathring{D}^i\widetilde{h}_{mn}\|_{L^2(N,d\mu_{\mathring{h}})} + \sum_{0\leq i\leq \ell} \|\mathring{D}^i\widetilde{K}_{mn}\|_{L^2(N,d\mu_{\mathring{h}})} \leq C_0 \; , \qquad (5.5.1)$$

with some $\ell > 5/2$. Then:

1. There exists a constant $\epsilon > 0$ depending only upon N, $\mathring{h}$, and C_0, and a vacuum globally hyperbolic Lorentzian development $(\mathscr{M}_{(N,\tilde{h},\tilde{K})}, \tilde{g})$ of the initial data with smooth null boundary $\partial\mathscr{M}_{(N,\tilde{h},\tilde{K})}$ with corner at ∂N, and with the following properties:

    (a) Every future directed timelike geodesic $\gamma(s)$ normal to N at $s = 0$ can be defined for proper time parameter s ranging over $[0, \epsilon]$, except when γ meets $\partial\mathscr{M}_{(N,\tilde{h},\tilde{K})}$ at some s smaller than ϵ; similarly for past directed timelike geodesics.

    (b) Every generator $\gamma(s)$ of $\partial\mathscr{M}_{(N,\tilde{h},\tilde{K})}$ with $\gamma(0) \in \partial N$ is defined for affine parameter s ranging over $[0, \epsilon]$; here s is normalised so that $|\tilde{g}(\dot{\gamma}(0), n)| = 1$, where n is the field of unit normals to N.

2. Let $(\mathscr{M}, \tilde{g})$ be a C^4 maximal globally hyperbolic vacuum space-time and suppose that there exists an embedding $i : N \to \mathscr{M}$, compatible with the initial data, with $i(N)$ — achronal. Then there exists an isometric embedding of the interior of $\mathscr{M}_{(N,\tilde{h},\tilde{K})}$ into $\mathscr{M}$ extending i.

Remark: The hypothesis that $(\mathscr{M}, \tilde{g})$ is C^4 in point 2 can be considerably relaxed, and is made only for simplicity of presentation.

PROOF: Point 1 is established by solving on $\mathbb{R} \times N$ the equations obtained by reducing the vacuum Einstein equations using the background metric $\hat{h}$:

$$\Box_{\tilde{g}} x^\mu = \Box_{\hat{h}} x^\mu \ , \qquad \hat{h} = -dt^2 + \mathring{h} \ , \tag{5.5.2}$$

where $\Box_k$ denotes the d'Alembertian of a metric k. Under (5.5.2) the Einstein equations become a set of hyperbolic wave equations for the metric coefficients

$$\tilde{g}^{\mu\nu} \equiv \tilde{g}(dx^\mu, dx^\nu) \ .$$

The initial conditions at $t = 0$ are derived from the initial data $(\tilde{h}_{ij}, \tilde{K}_{ij})$ in the usual way, and one further impose the boundary conditions

$$\tilde{g}_{\mu\nu}(t, x) = \mathring{g}_{\mu\nu}(t, x) \quad \text{on } \mathbb{R} \times \partial N \ .$$

Here $\mathring{g}_{\mu\nu}(t, x)$ on $\mathbb{R} \times \partial N$ is any Lorentzian metric chosen so that the corner conditions on $\{0\} \times \partial N$ are satisfied. Standard theory of hyperbolic PDE's provides a solution $\tilde{g}$ of that problem defined on $(-T, T) \times N$, for some T which depends only upon N, $\mathring{h}$, and the constant C_0 of (5.5.1). The metric $\tilde{g}$ will not be vacuum on $(-T, T) \times N$ in general; however, the obstruction for $\tilde{g}$ to be vacuum is governed by a vector field which satisfies a wave equation, the characteristics of which are the light cones of $\tilde{g}$; this implies that $\tilde{g}$ will be vacuum in the domain of dependence $\mathscr{D}(N)$ of N in $(-T, T) \times N$. $\mathscr{D}(N)$ with the metric obtained from that on $(-T, T) \times N$ by restriction provides the required vacuum manifold $(\mathscr{M}_{(N,\tilde{h},\tilde{K})}, \tilde{g})$ (with a boundary which has a corner at ∂N). Point 2 is Proposition 2.4 of [17]. □

We are ready now to pass to the proof of our main theorem:

Theorem 5.5.2 Consider vacuum hyperboloidal initial data $(M, \widetilde{h}_{ij}, \widetilde{K}_{ij})$ with

$$\widetilde{h}^{ij}\widetilde{K}_{ij}\Big|_{\partial M} = 3 \ ,$$

and suppose that the conclusions of Lemma 4.3.1 hold. Let $(\mathscr{M}, \widetilde{g})$ be the maximal globally hyperbolic vacuum development thereof, let Σ_{x_1} be the subset of $\Sigma \equiv M$ defined in Lemma 4.3.1, and assume that the fields (5.4.6) satisfy along Σ_{x_1}

$$\|f(0)\|_{\mathscr{H}^\alpha_k(\Sigma_{x_1})} + \|\mathring{f}(0)\|_{\mathcal{G}^0_k(\Sigma_{x_1})} + \|\partial_x\alpha_{ab}(0)\|_{\mathscr{C}^\alpha_0(\Sigma_1)} < +\infty \ , \quad k \geq 6. \quad (5.5.3)$$

Then there exists $T_* > 0$ and an isometric embedding of $\mathscr{M}_{0,x_1,T_*}$ into $\mathscr{M}$. In particular there exists a conformal completion of $\mathscr{M}$ with

$$\mathscr{I}^+ \supset [0, T_*] \times S^2 \ .$$

Remark: The differentiability condition $k \geq 6$ arises from the requirements of point 2 of Theorem 5.5.1. The analytical considerations in this work lead to the restriction $k \geq 4$, and we believe that this restriction should be sufficient for our arguments to go through. This requires a reexamination of Theorem 5.5.1, which we plan to do in a near future.

PROOF: Local existence theorem such that 5.5.1 with the construction of the beginning of Section 5.4 ensures that, for any $0 < x_2 < x_1/2$, there exists some $t > 0$, a vacuum metric on $\mathscr{M}_{x_2,x_1,t}$, and an isometric embedding $i_{x_2,x_1,t}$ of $\mathscr{M}_{x_2,x_1,t}$ into $\mathscr{M}$ which is compatible with the initial data and we can identify $\mathscr{M}_{x_2,x_1,t}$ with a subset of $\mathscr{M}$. In what follows we will always use this identification; $i_{x_2,x_1,t}$ is then the identity map. As in Section 5.4, let us define

$$T(x_2) = \sup\{t \mid \mathscr{M}_{x_2,x_1,t} \subset \mathscr{M}\} \ ,$$

from what has been said we have $T(x_2) > 0$ for all $x_2 > 0$. In order to prove Theorem 5.5.2 we will show that

$$T(x_2) \geq T_* := T_4 \ , \quad (5.5.4)$$

where the time T_4 is given by Theorem 5.4.3. We shall need the following results:

Lemma 5.5.3 In $x, \tau-$adapted coordinates we have, for all $\tau \in [0, t(x_2))$,

$$\|\widetilde{\Gamma}_\mu{}^\nu{}_\eta(\tau)\|_{\mathcal{G}^{-1}_k} \leq C(x_1, \alpha, \|\mathring{f}\|_{\mathcal{G}^0_k}, E^\alpha_k(x_2, x_1, \tau)) \ , \quad (5.5.5)$$

$$\|\widetilde{h}_{\delta\gamma}(\tau)\|_{\mathcal{G}^{-2}_k} \leq C(x_1, \alpha, \|\mathring{f}\|_{\mathcal{G}^0_k}, E^\alpha_k(x_2, x_1, \tau)) \ , \quad (5.5.6)$$

$$\|\widetilde{n}_\mu(\tau)\|_{\mathcal{G}^{-1}_k} + \|\partial_\tau\widetilde{n}_\mu\|_{\mathcal{G}^{-1}_k} \leq C(x_1, \alpha, \|\mathring{f}\|_{\mathcal{G}^0_k}, E^\alpha_k(x_2, x_1, \tau)) \ , \quad (5.5.7)$$

for some constants depending upon the variables listed, with E^α_k defined in (5.4.28), $\widetilde{n}_\mu = x^{-1}n_\mu$ — the $\widetilde{g}$-unit normal to $i_{x_2,x_1,\tau}(M)$, $\widetilde{\Gamma}_\mu{}^\nu{}_\eta$ — the Christoffel symbols of $\widetilde{g}$. Further there exist constants $0 < c_1 \leq c_2$ depending upon x_1, α, the initial data and $E^\alpha_k(x_2, x_1, \tau)$ such that

$$c_1(x_1)\widetilde{h}_{\delta\gamma}(0) \leq \widetilde{h}_{\delta\gamma}(\tau) \leq c_2(x_1)\widetilde{h}_{\delta\gamma}(0) \ . \quad (5.5.8)$$

Proof: Because of the relation

$$\widetilde{\Gamma}_\mu{}^\nu{}_\eta = \Gamma_\mu{}^\nu{}_\eta + S(x^{-1}dx)_\mu{}^\nu{}_\eta \ , \tag{5.5.9}$$

in order to establish (5.5.5) it suffices to show that

$$\|\Gamma_\mu{}^\nu{}_\eta\|_{\mathcal{G}^{-1}_k} \le C(E^\alpha_k(x_2,x_1,\tau),x_1) \ . \tag{5.5.10}$$

The definitions $\Gamma_\mu{}^\nu{}_\eta = dx^\nu(\nabla_{\partial_\mu}\partial_\eta)$ and $\Gamma_i{}^j{}_k = \theta^j(\nabla_{e_i}e_k)$ lead to

$$\Gamma_\mu{}^\nu{}_\eta = \Gamma_i{}^j{}_k\theta^i_\mu\theta^k_\eta e^\nu_j + (\partial_\mu\theta^j_\eta)e^\nu_j \ . \tag{5.5.11}$$

From Lemma 5.2.3 we obtain

$$\|\theta^i_\mu(\tau)\|_{\mathcal{G}^0_k} \le C(E^\alpha_k(x_2,x_1,\tau)) \ , \tag{5.5.12}$$

similarly for $\partial_\tau\theta^i_\mu$, which gives an estimate in $\mathcal{G}^{-1}_k$ for $\partial_x\theta^i_\mu$. The estimate (5.5.10) immediately follows. In order to prove (5.5.8) one writes

$$\widetilde{n}_\mu = x^{-1}g_{\mu\nu}n^\nu \ , \tag{5.5.13}$$

and the result follows by Lemma 5.2.5, *cf.* Equation (5.2.45). □

Corollary 5.5.4 In a x,τ compatible coordinates system $(x^\delta) = (x, v^A)$, The extrinsinc curvature form $\widetilde{K}_{\delta\gamma}$ of the level sets of τ in $(\mathscr{M}_{x_2,x_1,t},\widetilde{g})$ satisfies

$$\|\widetilde{K}_{\delta\gamma}\|_{\mathcal{G}^{-2}_k(M_{x_2,x_1-3\tau})} \ \le \ C(x_2,E^\alpha_k(x_2,x_1,\tau)) \ . \tag{5.5.14}$$

Proof: Let

$$\widetilde{K}_{\mu\nu} = \tilde{\nabla}_\mu\widetilde{n}_\nu \tag{5.5.15}$$

$$= \partial_\mu\widetilde{n}_\nu - \widetilde{\Gamma}_\mu{}^\eta{}_\nu\widetilde{n}_\eta \ . \tag{5.5.16}$$

In an x,τ compatible coordinate system the extrinsic curvature tensor $\widetilde{K}_{\delta\gamma}$ is a submatrix of $\widetilde{K}_{\mu\nu}$ (recall that we use a convention in which $(x^\delta) = (x, v^A)$ and $(z^\mu) = (x, v^A, \tau)$). The result follows from Lemma 5.5.3. □

Lemma 5.5.5 There exists a constant C_{x_2} such that for every $t < \min(T(x_2), T_*)$ we have on $\mathscr{M}_{x_2,x_1,t}$

$$-C_{x_2} < \widetilde{g}^{\tau\tau} < -1/C_{x_2} \ , \tag{5.5.17}$$

$$\forall\ X \in T\mathscr{M}_{x_2,x_1,t} \text{ satisfying } d\tau(X) = 0 \text{ we have}$$
$$\mathring{h}(X,X)/C_{x_2} < \widetilde{g}(X,X) < C_{x_2}\mathring{h}(X,X) \ , \tag{5.5.18}$$

$$\sum_{0\le i\le k}\|\mathring{D}^i\widetilde{h}_{mn}(\tau)\|_{L^2(N)} + \sum_{0\le i\le k}\|\mathring{D}^i\widetilde{K}_{mn}(\tau)\|_{L^2(N,d\mu_{\mathring{h}})} \le C_{x_2} \ , \tag{5.5.19}$$

where $\widetilde{h}_{mn}(\tau)$ and $\widetilde{K}_{mn}(\tau)$ are the metric and the extrinsic curvature of the level sets of τ in $\mathscr{M}_{x_2,x_1,t}$, with $N = M_{x_2,x_1-3t}$.

Remark: Unlike most of the constants in our work, the constant C_{x_2} depends upon x_2; however, the key point of Lemma 5.5.5 is that the bounds in (5.5.17) and (5.5.19) are t-independent in the specified range of t's.

PROOF: This is a direct consequence of Lemma 5.5.3 and Corollary 5.5.4. □

Let us return to the proof of Theorem 5.5.2. Let

$$0 \le s < T(x_2) \ , \tag{5.5.20}$$

by definition of $T(x_2)$ we have

$$\{\tau = s, x_2 \le x \le x_1 - 3s\} \subset \mathscr{M} \ ,$$

and compactness of the set $\{\tau = s, x = x_2\}$ implies that there exists $\delta(x_2) > 0$ such that

$$\{\tau = s, x_2 - \delta(x_2) \le x \le x_1 - 3s\} \subset \mathscr{M} \ .$$

Replacing $\delta(x_2)$ by $x_2/2$ if necessary we may without loss of generality assume that

$$0 < \delta(x_2) \le x_2/2 \ .$$

Global hyperbolicity of $\mathscr{M}$ implies then that

$$\mathscr{M}_{x_2-\delta(x_2),x_1,t} \subset \mathscr{M} \ .$$

Consider the family of initial data $(\widetilde{h}_{ij}(s), \widetilde{K}_{ij}(s))$ induced by the metric $\widetilde{g}$ on $\mathscr{M}$ on the family of hypersurfaces

$$\{\tau = s, x_2 - \delta(x_2) \le x \le x_1 - 3\tau\} \subset \mathscr{M}$$

parameterized by $0 < s < T(x_2)$; by part 1 of Theorem 5.5.1 there exists a metric, vacuum development of the initial data, defined on a manifold

$$\mathscr{M}(s) \equiv \mathscr{M}_{\left(\{\tau=s,x_2-\delta(x_2)\le x\le x_1-3s\},\widetilde{h}_{ij}(s),\widetilde{K}_{ij}(s)\right)}$$

with the properties spelled-out there. Point 2 of Theorem 5.5.2 shows that $\mathscr{M}(s)$ can be isometrically embedded in $\mathscr{M}$. If s satisfies (5.5.20) and if

$$T(x_2) < T_* \ ,$$

it follows from Lemma 5.5.5 that the null generators of the hypersurfaces $x =$ const starting from any point $p \in \{\tau = s, x_2 - \delta(x_2) \le x_2\}$ can be extended a uniform (s-independent) affine time ϵ to the future within $\mathscr{M}(s)$, where the affine parameter is normalized as in the statement of Theorem 5.5.1. This implies, for s close enough to $T(x_2)$, that the hypersurface

$$\{\tau = T(x_2), x_2 \le x \le x_1 - 3T(x_2)\}$$

will be included in $\mathscr{M}(s)$, hence in $\mathscr{M}$. This is compatible with maximality of $T(x_2)$ only if Equation (5.5.4) holds, and the theorem is established. □

Chapter 6

Concluding remarks

We close this work with some concluding remarks.

First, all the results in Chapter 3, concerning non-linear wave equations, have been formulated on a Minkowski background. It is clear that these results generalise without any difficulties to a large class of space-times with a smooth or polyhomogeneous Scri. The proof of such a fact would require checking that all the arguments used go through in a conveniently chosen coordinate system in a neighbourhood of $\mathscr{I}^+$. We are confident that this can be done and leave such results to interested readers.

Next, so far we have established polyhomogeneity of solutions of the wave map equation only in dimensions larger than or equal to three. We are planning to extend this result to dimension two in a near future; we believe that there is no difficulty in doing that if corner conditions are imposed on all orders. However, we hope to obtain a result where at most a finite number of such conditions would be needed.

We note that the polyhomogeneity results presented in Chapter 3 require an infinite number of corner conditions. This is not necessary, this result will be proved elsewhere.

Our results concerning Einstein equations are unsatisfactory in several respects. First, we had to assume that the conclusions of Lemma 4.3.1 hold. We have justified this fact only for initial data with a three-dimensional metric with sufficiently high degree of differentiability, together with a restriction on the trace of the extrinsic curvature. There is little doubt that those restrictions are not necessary, and we expect to be able to remove them soon. We note, however, that those conditions still allow initial data which are not covered by Friedrich's results, because they do not exclude the possibility that the conformally rescaled Weyl tensor is non-zero on the conformal boundary, *cf.* [1, 2].

Next, a full understanding of the gravitational problem requires checking the compatibility of the hypotheses of Theorem 5.5.2 with the properties of the initial data which can be constructed by various methods, *e.g.* the conformal method. This should be straightforward using the results and methods of [3], but requires lengthy and tedious calculations which have not been carried out so far. This question clearly deserves further investigations.

Finally, one would like to show that polyhomogeneous initial data sets for vacuum Einstein equations lead to space-time with a polyhomogeneous conformal completion. We believe that the methods of Chapter 3 can be adapted to prove such a result, but a detailed analysis of this question remains to be done.

Appendix A

Miscellaneous

A.1 Conformal connections.

We consider a space-time $(\mathscr{M}, \tilde{g})$ with the Levi-Civita connection $\widetilde{D}$. Now, we consider for a conformal factor Ω the metric $g = \Omega^2\tilde{g}$ and its Levi Civita connection D. Let b a one form over $(\mathscr{M}, \tilde{g})$ and $\widehat{D}$ the weyl connection associated by $\widehat{D} = \widetilde{D} + S(b)$.

We have

$$\begin{aligned} D &= \widetilde{D} + S(\Omega^{-1}d\Omega) \, , & \text{(A.1.1)} \\ \widehat{D} &= \widetilde{D} + S(b) \, , & \text{(A.1.2)} \\ \widehat{D} &= D + S(f) \, , & \text{(A.1.3)} \end{aligned}$$

with $f = b - \Omega^{-1}d\Omega$.

Let us denote respectively $R_{ij}, R^i{}_{jkl}$, $\tilde{R}_{ij}, \tilde{R}^i{}_{jkl}$, $\hat{R}_{ij}, \hat{R}^i{}_{jkl}$ the Ricci and Riemann tensor associated to $D, \widetilde{D}, \widehat{D}$. Following Friedrich we define

$$\hat{L}_{ij} = \frac{1}{2}\hat{R}_{(ij)} - \frac{1}{12}g^{lk}R_{kl}g_{ij} - \frac{1}{4}\hat{R}_{[ij]} \, , \qquad \text{(A.1.4)}$$

[1] and so on for $\tilde{L}$ and L. Note that since the product $g^\sharp \otimes g$ only depends upon the conformal class of the metric, the definition in (A.1.4) could have been done with replacing $g^{lk}R_{kl}g_{ij}$ by $\tilde{g}^{lk}R_{kl}\tilde{g}_{ij}$.

We have the following relations

$$\begin{aligned} \hat{L}_{jk} &= \tilde{L}_{jk} - \widetilde{D}_j b_k + \frac{1}{2} b_i S(b)_j{}^i{}_k \, , & \text{(A.1.5)} \\ L_{jk} &= \tilde{L}_{jk} - \widetilde{D}_j(\Omega^{-1}d_k\Omega) + \frac{1}{2}\Omega^{-1}d_i\Omega S(\Omega^{-1}d\Omega)_j{}^i{}_k \, , & \text{(A.1.6)} \\ \hat{L}_{jk} &= L_{jk} - D_j f_k + \frac{1}{2} f_i S(f)_j{}^i{}_k \, . & \text{(A.1.7)} \end{aligned}$$

On the other hand, the Weyl tensors W of the different connections are the same:

$$\hat{W}^i{}_{jkl} = W^i{}_{jkl} = \tilde{W}^i{}_{jkl} \, .$$

PROOF: We will prove the first equation of (A.1.7), for the others are equivalent, replacing b by f or by $\Omega^{-1}d\Omega$. To simplify, we denote S for $S(b)$, so that $\widehat{D}_X Y = \widetilde{D}_X Y + S(X, Y)$ for any vectorfield X, Y. Let us note that S is symmetric. We write

$$\begin{aligned} \hat{R}(X,Y)Z &= \widehat{D}_X\widehat{D}_Y Z - \widehat{D}_Y\widehat{D}_X Z - \widehat{D}_{[X,Y]}Z \, , \\ &= \widehat{D}_X(\widetilde{D}_Y Z + S(Y,Z)) - \widehat{D}_Y(\widetilde{D}_X Z + S(X,Z)) - \widetilde{D}_{[X,Y]}Z \\ &\quad -S([X,Y],Z) \, , \\ &= \widetilde{D}_X\widetilde{D}_Y Z + S(X, \widetilde{D}_Y Z) + \widetilde{D}_X(S(Y,Z)) + S(X, S(Y,Z)) \\ &\quad -\widetilde{D}_Y\widetilde{D}_X Z - S(Y, \widetilde{D}_X Z) - (\widetilde{D}_Y S(X,Z)) - S(Y, S(X,Z)) \\ &\quad -\widetilde{D}_{[X,Y]}Z - S([X,Y],Z) \, , \end{aligned}$$

[1] The tensor $\hat{L}_{ij}$ defined in Equation (A.1.4) coincides with that in [28], and with the tensor A_{ij} of [27, p. 138]; it equals A_{ji} of [29, Eq. (2.34), p. 96]

$$\begin{aligned}
&= \tilde{R}(X,Y)Z + (\widetilde{D}_X S)(Y,Z) + S(\widetilde{D}_X Y, Z) + S(X, S(Y,Z)) \\
&\quad -(\tilde{\nabla}_Y S)(X,Z) - S(\tilde{\nabla}_Y X, Z) - S(Y, S(X,Z)) - S([X,Y],Z) \, , \\
&= \tilde{R}(X,Y)Z + (\widetilde{D}_X S)(Y,Z) - (\widetilde{D}_Y S)(X,Z) \\
&\quad +S(X,S(Y,Z)) - S(Y,S(X,Z)) \, ,
\end{aligned}$$

where we have used

$$\begin{aligned}
S(\widetilde{D}_X Y, Z) - S(\widetilde{D}_Y X, Z) - S([X,Y],Z) &= S(\widetilde{D}_X Y - \widetilde{D}_Y X - [X,Y], Z) \, . \\
&= 0
\end{aligned}$$

With the indices convention

$$\hat{R}^l{}_{kij}\partial_l = \hat{R}(\partial_i, \partial_j)\partial_k \, , \tag{A.1.8}$$

we deduce

$$\hat{R}^l{}_{kij} = \tilde{R}^l{}_{kij} + 2\left(\widetilde{D}_{[i} S_{j]}{}^l{}_k + S_m{}^l{}_{[i} S_{j]}{}^m{}_k\right) \, , \tag{A.1.9}$$

$$\hat{R}_{kj} = \tilde{R}_{jk} + \widetilde{D}_k b_j - 3\widetilde{D}_j b_k - \tilde{g}_{jk}\widetilde{D}_m b^m + 2b_k b_j - 2g_{jk} b_m b^m \, , \tag{A.1.10}$$

$$\hat{R}^k{}_k = \tilde{R}^k{}_k - 6(\widetilde{D}_m b^m + b_m b^m) \, , \tag{A.1.11}$$

where the indices are moved with $\tilde{g}$. Therefore

$$\hat{L}_{jk} - \tilde{L}_{jk} = -\widetilde{D}_k b_j + \frac{1}{2} S(b)_k{}^l{}_j b_l \, . \tag{A.1.12}$$

□

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