Stationary Black Holes

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Abstract

We review the theory of stationary black hole solutions of vacuum Einstein equations.
Keywords: black holes, event horizons, Schwarzschild metric, Kerr metric, no-hair theorems

1 Introduction

This article treats a specific class of stationary solutions to the Einstein field equations which read

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{1.1}$$

Here $R_{\mu\nu}$ and $R = g^{\mu\nu} R_{\mu\nu}$ are respectively the Ricci tensor and the Ricci scalar of the spacetime metric $g_{\mu\nu}$, G is the Newton constant and c the speed of light. The tensor $T_{\mu\nu}$ is the stress-energy tensor of matter. Spacetimes, or regions thereof, where $T_{\mu\nu} = 0$ are called vacuum.

Stationary solutions are of interest for a variety of reasons. As models for compact objects at rest, or in steady rotation, they play a key role in astrophysics. They are easier to study than non-stationary systems because stationary solutions are governed by elliptic rather than hyperbolic equations. Finally, like in any field theory, one expects that large classes of dynamical solutions approach ("settle down to") a stationary state in the final stages of their evolution.

The simplest stationary solutions describing compact isolated objects are the spherically symmetric ones. In the vacuum region these are all given by the Schwarzschild family. A theorem of Birkhoff shows that in the vacuum region any spherically symmetric metric, even without assuming stationarity, belongs to the family of Schwarzschild metrics, parameterized by a positive mass parameter m. Thus, regardless of possible motions of the matter, as long as they remain spherically symmetric, the exterior metric is the Schwarzschild one

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for some constant m. This has the following consequence for stellar dynamics: Imagine following the collapse of a cloud of pressureless fluid ("dust"). Within Newtonian gravity this dust cloud will, after finite time, contract to a point at which the density and the gravitational potential diverge. However, this result cannot be trusted as a sensible physical prediction because, even if one supposes that Newtonian gravity is still valid at very high densities, a matter model based on non-interacting point particles is certainly not. Consider, next, the same situation in the Einstein theory of gravity: Here a new question arises, related to the form of the Schwarzschild metric outside of the spherically symmetric body:

$$g = -V^2 dt^2 + V^{-2} dr^2 + r^2 d\Omega^2 \ , \quad V^2 = 1 - \frac{2Gm}{rc^2} \ , \quad t \in \mathbb{R} \ , \ r \in (\frac{2Gm}{c^2}, \infty) \tag{1.2}$$

Here $d\Omega^2$ is the line element of the standard 2-sphere. Since the metric (1.2) seems to be singular as $r = 2m$ is approached (from now on we use units in which $G = c = 1$) there arises the need to understand what happens at the surface of the star when the radius $r = 2m$ is reached. One thus faces the need of a careful study of the geometry of the metric (1.2) when $r = 2m$ is approached, and crossed.

The first key feature of the metric (1.2) is its stationarity of course with Killing vector field X given by $X = \partial_t$. A Killing field, by definition, is a vector field the local flow of which generates isometries. A space–time[1] is called *stationary* if there exists a Killing vector field X which approaches ∂_t in the asymptotically flat region (where r goes to ∞, see below for precise definitions) *and* generates a one parameter groups of isometries. A space–time is called *static* if it is stationary and if the stationary Killing vector X is hypersurface-orthogonal, i.e. $X^\flat \wedge dX^\flat = 0$, where $X^\flat = X_\mu dx^\mu = g_{\mu\nu} X^\nu dx^\mu$. A space–time is called *axisymmetric* if there exists a Killing vector field Y, which generates a one parameter group of isometries, and which behaves like a *rotation* in the asymptotically flat region, with all orbits 2π periodic. In asymptotically flat space-times this implies that there exists an axis of symmetry, that is, a set on which the Killing vector vanishes. Killing vector fields which are a non-trivial linear combination of a time translation and of a rotation in the asymptotically flat region are called *stationary-rotating*, or *helical*.

There exists a technique, due independently to Kruskal and Szekeres, of attaching together two regions $r > 2m$ and two regions $r < 2m$ of the Schwarzschild metric, in a way shown[2] in Figure 1, to obtain a manifold with a metric which is smooth at $r = 2m$. In the extended space-time the hypersurface $\{r = 2m\}$ is a null hypersurface $\mathscr{E}$, the Schwarzschild event horizon. The stationary Killing vector $X = \partial_t$ extends to a Killing vector in the extended spacetime which becomes tangent to and null on $\mathscr{E}$. The global properties of the Kruskal–Szekeres extension of the exterior Schwarzschild[3] spacetime, make

[1]The term *space–time* denotes a smooth, paracompact, connected, orientable and time–orientable Lorentzian manifold.

[2]We are grateful to J.-P. Nicolas for allowing us to use his electronic figures, based on those in *Dissertationes Math.* **408** (2002), 1–85.

[3]The exterior Schwarzschild space-time (1.2) admits an infinite number of non-isometric

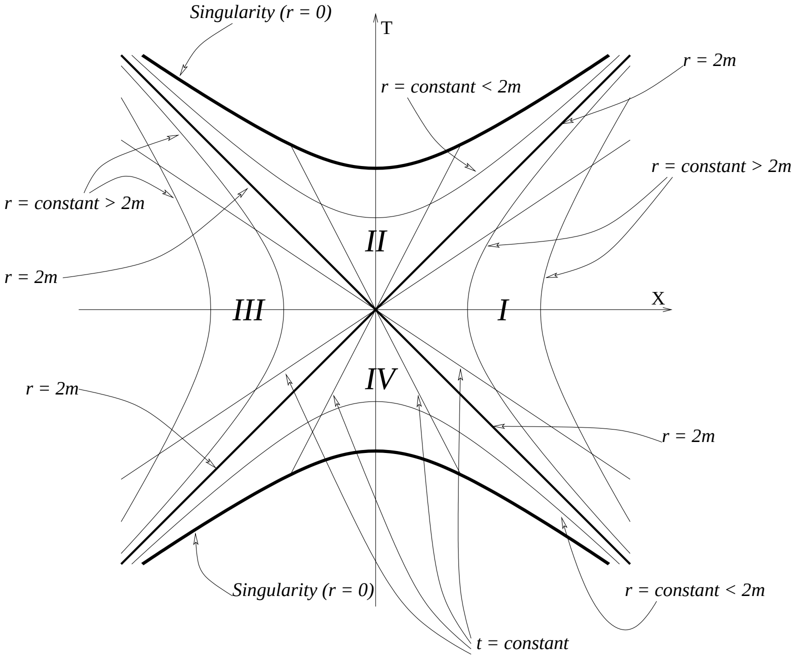

Figure 1: The Kruskal-Szekeres extension of the Schwarzschild solution.

this space-time a natural model for a non-rotating black hole.

We can now come back to the problem of the contracting dust cloud according to the Einstein theory. For simplicity we take the density of the dust to be uniform — the so-called Oppenheimer-Snyder solution. It then turns out that, in the course of collapse, the surface of the dust will eventually cross the Schwarzschild radius, leaving behind a Schwarzschild black hole. If one follows the dust cloud further, a singularity will eventually form, but will not be visible from the "outside region" (where $r > 2m$). For astronomical masses the Schwarzschild radius will be "astronomically large" (*e.g.*, $2m$ equals three kilometers when m is the mass of the sun), and standard phenomenological matter models such as that for dust can still be trusted, so the previous objection to the Newtonian scenario does not apply.

There is a rotating generalization of the Schwarzschild metric, namely the two parameter family of *exterior Kerr metrics*, which in Boyer-Lindquist coordinates take the form

$$g = -\frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 - \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}dtd\varphi + \\ + \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta d\varphi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \qquad (1.3)$$

with $0 \le a < m$. Here $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 + a^2 - 2mr$ and $r_+ < r < \infty$

vacuum extensions, even in the class of maximal, analytic, simply connected ones. The Kruskal-Szekeres extension is singled out by the properties that it is maximal, vacuum, analytic, simply connected, with all maximally extended geodesics either complete, or with the area r of the orbits of the isometry groups tending to zero along them.

where $r_+ = m + (m^2 - a^2)^{\frac{1}{2}}$. When $a = 0$, the Kerr metric reduces to the Schwarzschild metric. The Kerr metric is again a vacuum solution, and it is stationary with $X = \partial_t$ the asymptotic time translation, as well as axisymmetric with $Y = \partial_\varphi$ the generator of rotations. Similarly to the Schwarzschild case, it turns out that the metric can be smoothly extended across $r = r_+$, with $\{r = r_+\}$ being a smooth null hypersurface $\mathscr{E}$ in the extension. The null generator K of $\mathscr{E}$ is the limit of the stationary-rotating Killing field $X + \omega Y$, where $\omega = \frac{a}{2mr_+}$. On the other hand, the Killing vector X is timelike only outside the hypersurface $\{r = m + (m^2 - a^2\cos^2\theta)^{\frac{1}{2}}\}$, on which X becomes null. In the region between r_+ and $r = m + (m^2 - a^2\cos^2\theta)^{\frac{1}{2}}$, which is called the *ergoregion*, X is spacelike. It is also spacelike on and tangent to $\mathscr{E}$, except where the axis of rotation meets $\mathscr{E}$, where X is null. By the above properties the Kerr family provides natural models for rotating black holes.

Unfortunately, as opposed to the spherically symmetric case, there are no known explicit collapsing solutions with rotating matter, in particular no known solutions having the Kerr metric as final state.

The aim of the theory outlined below is to understand the general geometrical features of stationary black holes, and to give a classification of models satisfying the field equations.

2 Model independent concepts

We now make precise some notions used informally in the introductory section. The mathematical notion of black hole is meant to capture the idea of a region of space-time which cannot be seen by "outside observers". Thus, at the outset, one assumes that there exists a family of physically preferred observers in the space-time under consideration. When considering isolated physical systems, it is natural to define the "exterior observers" as observers which are "very far" away from the system under consideration. The standard way of making this mathematically precise is by using conformal completions, discussed in more detail in the article about asymptotic structure in this encyclopedia: A pair $(\widetilde{\mathscr{M}}, \tilde{g})$ is called a *conformal completion at infinity,* or simply *conformal completion*, of $(\mathscr{M}, g)$ if $\widetilde{\mathscr{M}}$ is a manifold with boundary such that:

1. $\mathscr{M}$ is the interior of $\widetilde{\mathscr{M}}$,
2. there exists a function Ω, with the property that the metric $\tilde{g}$, defined as $\Omega^2 g$ on $\mathscr{M}$, extends by continuity to the boundary of $\widetilde{\mathscr{M}}$, with the extended metric remaining of Lorentzian signature,
3. Ω is positive on $\mathscr{M}$, differentiable on $\widetilde{\mathscr{M}}$, vanishes on the boundary

   $$\mathscr{I} := \widetilde{\mathscr{M}} \setminus \mathscr{M} \ ,$$

   with $d\Omega$ *nowhere vanishing* on $\mathscr{I}$.

The boundary $\mathscr{I}$ of $\widetilde{\mathscr{M}}$ is called Scri, a phonic shortcut for "script I". The idea here is the following: forcing Ω to vanish on $\mathscr{I}$ ensures that $\mathscr{I}$ lies infinitely far away from any physical object — a mathematical way of capturing the notion "very far away". The condition that $d\Omega$ does not vanish is a convenient technical condition which ensures that $\mathscr{I}$ is a smooth three dimensional hypersurface, instead of some, say, one or two dimensional object, or of a set with singularities here and there. Thus, $\mathscr{I}$ is an idealized description of a family of observers at infinity.

To distinguish between various points of $\mathscr{I}$ one sets

$$\mathscr{I}^+ = \{\text{points in } \mathscr{I} \text{ which are to the future of the physical space-time}\} .$$

$$\mathscr{I}^- = \{\text{points in } \mathscr{I} \text{ which are to the past of the physical space-time}\} .$$

(Recall that a point q is to the future, respectively to the past, of p if there exists a future directed, respectively past directed, causal curve from q to p. Causal curves are curves γ such that their tangent vector $\dot\gamma$ is causal everywhere, $g(\dot\gamma,\dot\gamma) \le 0$.) One then defines the black hole region $\mathscr{B}$ as

$$\begin{aligned}\mathscr{B} \quad := \quad & \{\text{the set of points in } \mathscr{M} \text{ from which} \\ & \qquad \text{no future directed causal curve in } \widetilde{\mathscr{M}} \text{ meets } \mathscr{I}^+\} . \end{aligned} \tag{2.1}$$

By definition, points in the black hole region cannot thus send information to $\mathscr{I}^+$; equivalently, observers on $\mathscr{I}^+$ cannot see points in $\mathscr{B}$. The *white hole* region $\mathscr{W}$ is defined by changing the time orientation in (2.1). A key notion related to the concept of a black hole is that of *future* ($\mathscr{E}^+$) and *past ($\mathscr{E}^-$) event horizons*,

$$\mathscr{E}^+ := \partial\mathscr{B} , \qquad \mathscr{E}^- := \partial\mathscr{W} . \tag{2.2}$$

Under mild assumptions, event horizons in stationary space-times with matter satisfying the *null energy condition*,

$$T_{\mu\nu}\ell^\mu\ell^\nu \ge 0 \quad \text{ for all null vectors } \ell^\mu, \tag{2.3}$$

are smooth null hypersurfaces, analytic if the metric is analytic.

In order to develop a reasonable theory one also needs a regularity condition for the interior of space-time. This has to be a condition which does not exclude singularities (otherwise the Schwarzschild and Kerr black holes would be excluded), but which nevertheless guarantees a well-behaved exterior region. One such condition, assumed in all the results described below, is the existence in $\mathscr{M}$ of an asymptotically flat space-like hypersurface $\mathscr{S}$ with compact interior. Further, either $\mathscr{S}$ has no boundary, or the boundary of $\mathscr{S}$ lies on $\overline{\mathscr{E}^+ \cup \mathscr{E}^-}$. To make things precise, for any spacelike hypersurface let g_{ij} be the induced metric, and let K_{ij} denote its extrinsic curvature. A space–like hypersurface $\mathscr{S}_{\rm ext}$ diffeomorphic to $\mathbb{R}^3$ minus a ball will be called *asymptotically flat* if the fields (g_{ij}, K_{ij}) satisfy the fall–off conditions

$$|g_{ij}-\delta_{ij}|+r|\partial_\ell g_{ij}|+\cdots+r^k|\partial_{\ell_1\cdots\ell_k}g_{ij}|+r|K_{ij}|+\cdots+r^k|\partial_{\ell_1\cdots\ell_{k-1}}K_{ij}| \le Cr^{-1} , \tag{2.4}$$

for some constants C, $k \ge 1$. A hypersurface $\mathscr{S}$ (with or without boundary) will be said to be *asymptotically flat with compact interior* if $\mathscr{S}$ is of the form $\mathscr{S}_{int} \cup \mathscr{S}_{\rm ext}$, with $\mathscr{S}_{int}$ compact and $\mathscr{S}_{ext}$ asymptotically flat.

There exists a canonical way of constructing a conformal completion with good global properties for stationary space-times which are asymptotically flat in the sense of (2.4), and which are vacuum sufficiently far out in the asymptotic region. This conformal completion is referred to as the *standard completion* and will be assumed from now on.

Returning to the event horizon $\mathscr{E} = \mathscr{E}^+ \cup \mathscr{E}^-$, it is not very difficult to show that every Killing vector field X is necessarily tangent to $\mathscr{E}$. Since the latter set is a null Lipschitz hypersurface, it follows that X is either null or spacelike on $\mathscr{E}$. This leads to a preferred class of event horizons, called *Killing horizons.* By definition, a Killing horizon associated with a Killing vector K is a *null hypersurface* which coincides with a connected component of the set

$$\mathcal{H}(K) := \{p \in \mathscr{M} : \ g(K,K)(p) = 0 \ , \ K(p) \neq 0\} \ . \tag{2.5}$$

A simple example is provided by the "boost Killing vector field" $K = z\partial_t + t\partial_z$ in Minkowski space-time: $\mathcal{H}(K)$ has four connected components

$$\mathcal{H}_{\epsilon\delta} := \{t = \epsilon z \ , \delta t > 0\} \ , \quad \epsilon, \delta \in \{\pm 1\} \ .$$

The closure $\overline{\mathcal{H}}$ of $\mathcal{H}$ is the set $\{|t| = |z|\}$, which is not a manifold, because of the crossing of the null hyperplanes $\{t = \pm z\}$ at $t = z = 0$. Horizons of this type are referred to as *bifurcate Killing horizons*, with the set $\{K(p) = 0\}$ being called the *bifurcation surface* of $\mathcal{H}(K)$. The bifurcate horizon structure in the Kruszkal-Szekeres-Schwarzschild space-time can be clearly seen on Figures 1 and 2.

The Vishveshwara-Carter lemma shows that if a Killing vector K is hypersurface-orthogonal, $K^\flat \wedge dK^\flat = 0$, then the set $\mathcal{H}(K)$ defined in (2.5) is a union of smooth null hypersurfaces, with K being tangent to the null geodesics threading $\mathcal{H}$ ("$\mathcal{H}$ is generated by K"), and so is indeed a Killing horizon. It has been shown by Carter that the same conclusion can be reached if the hypothesis of hypersurface-orthogonality is replaced by that of existence of two linearly independent Killing vector fields.

In stationary-axisymmetric space-times a Killing vector K *tangent to the generators* of a Killing horizon $\mathcal{H}$ can be normalised so that $K = X + \omega Y$, where X is the Killing vector field which asymptotes to a time translation in the asymptotic region, and Y is the Killing vector field which generates rotations in the asymptotic region. The constant ω is called the *angular velocity of the Killing horizon* $\mathcal{H}$.

On a Killing horizon $\mathcal{H}(K)$ one necessarily has

$$\nabla^\mu (K^\nu K_\nu) = -2\kappa K^\mu. \tag{2.6}$$

Assuming the so-called dominant energy condition on $T_{\mu\nu}$,[4] it can be shown that κ is constant (recall that Killing horizons are always connected in our terminology), it is called *the surface gravity of* $\mathcal{H}$. A Killing horizon is called

[4]See the article by Bray on "Positive Energy Theorem and other inequalities in General Relativity"

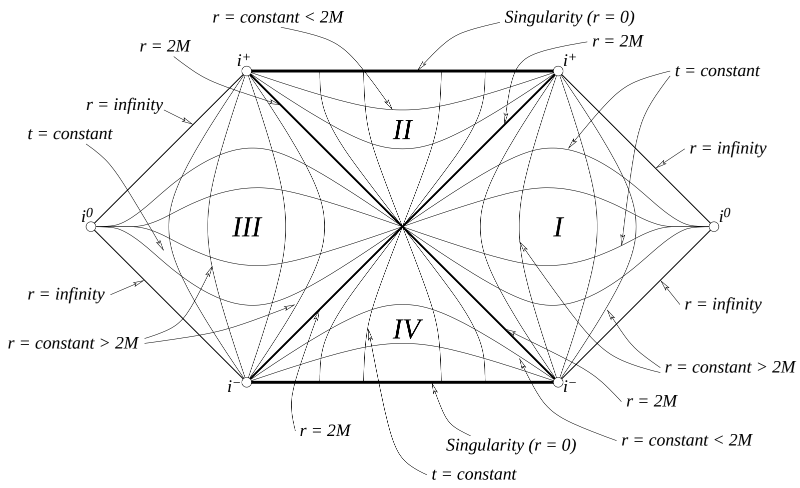

Figure 2: The Carter-Penrose diagram for the Kruskal-Szekeres space-time. There are actually two asymptotically flat regions, with corresponding $\mathscr{I}^{\pm}$ and $\mathscr{E}^{\pm}$ defined with respect to the second region, but not indicated on this diagram. Each point in this diagram represents a two-dimensional sphere, and coordinates are chosen so that light-cones have slopes plus minus one. Regions are numbered as in Figure 1.

degenerate when $\kappa = 0$, and non–degenerate otherwise; by an abuse of terminology one similarly talks of degenerate black holes, *etc.* In Kerr space-times we have $\kappa = 0$ if and only if $m = a$. A fundamental theorem of Boyer shows that degenerate horizons are closed. This implies that a horizon $\mathcal{H}(K)$ such that K has zeros in $\overline{\mathcal{H}}$ is non-degenerate, and is of bifurcate type, as described above. Further, a *non-degenerate* Killing horizon with *complete* geodesic generators always contains zeros of K in its closure. However, it is not true that existence of a non-degenerate horizon implies that of zeros of K: take the Killing vector field $z\partial_t + t\partial_z$ in Minkowski space-time from which the 2-plane $\{z = t = 0\}$ has been removed. The universal cover of that last space-time provides a space-time in which one cannot restore the points which have been artificially removed, without violating the manifold property.

The *domain of outer communications* (d.o.c.) of a black hole space-time is defined as

$$\langle\langle \mathscr{M} \rangle\rangle := \mathscr{M} \setminus \{\mathscr{B} \cup \mathscr{W}\} \,. \tag{2.7}$$

Thus, $\langle\langle \mathscr{M} \rangle\rangle$ is the region lying outside of the white hole region and outside of the black hole region; it is the region which can both be seen by the outside observers and influenced by those.

The subset of $\langle\langle \mathscr{M} \rangle\rangle$ where X is spacelike is called the *ergoregion.* In the Schwarzschild space-time $\omega = 0$ and the ergoregion is empty, but neither of these is true in Kerr with $a \neq 0$.

A very convenient method for visualising the global structure of space-times is provided by the *Carter-Penrose diagrams.* An example of such a diagram is given[2] in Figure 2.

A corollary of the *topological censorship theorem* of Friedman, Schleich and Witt is that d.o.c.'s of regular black hole space-times satisfying the dominant energy condition are simply connected. This implies that connected components of event horizons in stationary space-times have $\mathbb{R} \times S^2$ topology.

We end our review of the concepts associated with stationary black hole spacetimes by summarising the properties of the Schwarzschild and Kerr geometries: The extended Kerr spacetime with $m > a$ is a black hole spacetime with the hypersurface $\{r = r_+\}$ forming a non-degenerate, bifurcate Killing horizon generated by the vector field $X + \omega Y$ and surface gravity given by $\kappa = \frac{(m^2-a^2)^{1/2}}{2m[m+(m^2-a^2)^{1/2}]}$. In the case $a = 0$, where the angular velocity ω vanishes, X is hypersurface-orthogonal and becomes the generator of $\mathcal{H}$. The bifurcation surface in this case is the totally geodesic 2-sphere, along which the four regions in Figure 1 are joined.

3 Classification of stationary solutions ("No hair theorems")

We confine attention to the "outside region" of black holes, the domain of outer communications (d.o.c.).[5] For reasons of space we only consider vacuum solutions; there exists a similar theory for electro-vacuum black holes. (There is a somewhat less developed theory for black hole spacetimes in the presence of nonabelian gauge fields, see the review by Gal'tsov and Volkov.) In connection with a collapse scenario the vacuum condition begs the question: collapse of what? The answer is twofold: First there are large classes of solutions of Einstein equations describing pure gravitational waves. It is believed that sufficiently strong such solutions will form black holes. (Whether or not they will do that is related to the *cosmic censorship conjecture*, discussed in the article on Spacetime Topology, Global Structure and Singularities in this encyclopedia.) Consider, next, a dynamical situation in which matter is initially present. The conditions imposed in this section correspond then to a final state in which matter has either been radiated away to infinity, or has been swallowed by the black hole (as in the spherically symmetric Oppenheimer-Snyder collapse described above).

Based on the facts below, it is expected that the d.o.c.'s of appropriately regular, stationary, vacuum black holes are isometrically diffeomorphic to those of Kerr black holes:

1. The *rigidity theorem* (Hawking): event horizons in regular, *non–degenerate*, stationary, *analytic* vacuum black holes are either *Killing horizons* for X, or there exists a second Killing vector in $\langle\langle \mathscr{M} \rangle\rangle$.

2. The *Killing horizons theorem* (Sudarsky-Wald): *non–degenerate* stationary vacuum black holes such that the *event horizon is the union of Killing horizons of* X are *static*.

[5]Except for the so-called degenerate case discussed later, the "inside"(black hole) region is not stationary, so that this restriction already follows from the requirement of stationarity.

3. The Schwarzschild black holes exhaust the family of *static* regular vacuum black holes (Israel, Bunting – Massood-ul-Alam, Chruściel).

4. The non–degenerate Kerr black holes satisfying

$$m^2 > a^2 \tag{3.1}$$

exhaust the family of *non–degenerate*, *stationary–axisymmetric,* vacuum, *connected* black holes. Here m is the total ADM mass (see the article by Bray in this encyclopedia), while the product am is the total ADM angular momentum. (Of course these quantities generalize the constants a and m appearing in the Kerr metric.) The framework for the proof has been set-up by Carter, and the statement above is due to Robinson.

The above results are collectively known under the name of *no hair theorems*, and they have not provided the final answer to the problem so far. There are no *a priori* reasons known for the analyticity hypothesis in the rigidity theorem. Further, degenerate horizons have been completely understood in the static case only.

Yet another key open question is that of existence of *non-connected* regular stationary-axisymmetric vacuum black holes. The following result is due to Weinstein: Let $\partial\mathscr{S}_a$, $a = 1, \ldots, N$ be the connected components of $\partial\mathscr{S}$. Let $X^\flat = g_{\mu\nu}X^\mu dx^\nu$, where X^μ is the Killing vector field which asymptotically approaches the unit normal to $\mathscr{S}_{ext}$. Similarly set $Y^\flat = g_{\mu\nu}Y^\mu dx^\nu$, Y^μ being the Killing vector field associated with rotations. On each $\partial\mathscr{S}_a$ there exists a constant ω_a such that the vector $X + \omega_a Y$ is tangent to the generators of the Killing horizon intersecting $\partial\mathscr{S}_a$. The constant ω_a is called the angular velocity of the associated Killing horizon. Define

$$m_a = -\tfrac{1}{8\pi}\int_{\partial\mathscr{S}_a} *dX^\flat \;, \tag{3.2}$$

$$L_a = -\tfrac{1}{4\pi}\int_{\partial\mathscr{S}_a} *dY^\flat \;. \tag{3.3}$$

Such integrals are called *Komar integrals.* One usually thinks of L_a as the angular momentum of each connected component of the black hole. Set

$$\mu_a = m_a - 2\omega_a L_a \;. \tag{3.4}$$

Weinstein shows that one necessarily has $\mu_a > 0$. The problem at hand can be reduced to a *harmonic map* equation, also known as the *Ernst equation*, involving a singular map from $\mathbb{R}3$ with Euclidean metric δ to the two-dimensional hyperbolic space. Let $r_a > 0$, $a = 1, \ldots, N-1$, be the distance in $\mathbb{R}3$ along the axis between neighboring black holes as measured with respect to the (unphysical) metric δ. Weinstein proved that for *non-degenerate* regular black holes the inequality (3.1) holds, and that the metric on $\langle\langle\mathscr{M}\rangle\rangle$ is determined up to isometry by the $3N-1$ parameters

$$(\mu_1, \ldots, \mu_N, L_1, \ldots, L_N, r_1, \ldots, r_{N-1}) \tag{3.5}$$

just described, with $r_a, \mu_a > 0$. These results by Weinstein contain the no-hair theorem of Carter and Robinson as a special case. Weinstein also shows that for every $N \geq 2$ and for every set of parameters (3.5) with $\mu_a, r_a > 0$, there exists a solution of the problem at hand. It is known that for some sets of parameters (3.5) the solutions will have "strut singularities" between some pairs of neighboring black holes, but the existence of the "struts" for all sets of parameters as above is not known, and is one of the main open problems in our understanding of stationary–axisymmetric electro–vacuum black holes. The existence and uniqueness results of Weinstein remain valid when strut singularities are allowed in the metric at the outset, though such solutions do not fall into the category of regular black holes discussed here.

See also: Asymptotic Structure and Conformal Infinity. Black Hole Thermodynamics. Initial Value problem for Einstein Equations. Positive energy Theorem and other inequalities in General Relativity. Spacetime Topology, Causal Structure and Singularities.

Suggestions for Further Reading: [1–8]

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