CHAPTER 1

Black Holes – an Introduction

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This chapter is an introduction to the mathematical aspects of the theory of black holes, solutions of vacuum Einstein equations, possibly with a cosmological constant, in arbitrary dimensions.

1. Stationary black holes

Stationary solutions are of interest for a variety of reasons. As models for compact objects at rest, or in steady rotation, they play a key role in astrophysics. They are easier to study than non-stationary systems because stationary solutions are governed by elliptic rather than hyperbolic equations. Further, like in any field theory, one expects that large classes of dynamical solutions approach a stationary state in the final stages of their evolution. Last but not least, explicit stationary solutions are easier to come by than dynamical ones.

1.1. *Asymptotically flat examples*

The simplest stationary solutions describing compact isolated objects are the spherically symmetric ones. A theorem due to Birkhoff shows that in the vacuum region any spherically symmetric metric, even without assuming stationarity, belongs to the family of Schwarzschild metrics, parameterized by a positive mass parameter m:

$$g = -V^2dt^2 + V^{-2}dr^2 + r^2d\Omega^2 \ , \tag{1.1}$$

$$V^2 = 1 - \tfrac{2m}{r} \ , \quad t \in \mathbb{R} \ , \ r \in (2m, \infty) \ . \tag{1.2}$$

Here $d\Omega^2$ denotes the metric of the standard 2-sphere. Since the metric (1.1) seems to be singular as $r = 2m$ is approached, there arises the need to understand the geometry of the metric (1.1) there. The simplest way to do that, for metrics of the form (1.1) is to replace t by a new coordinate v defined as

$$v = t + f(r) \;, \quad f' = \frac{1}{V^2} \;, \tag{1.3}$$

leading to

$$v = t + r + 2m \ln(r - 2m) \;.$$

This brings g to the form

$$g = -(1 - \frac{2m}{r})dv^2 + 2dvdr + r^2 d\Omega^2 \;. \tag{1.4}$$

We have $\det g = -r^4 \sin^2\theta$, with all coefficients of g smooth, which shows that g is a well defined Lorentzian metric on the set

$$v \in \mathbb{R} \;, \quad r \in (0, \infty) \;. \tag{1.5}$$

More precisely, (1.4)-(1.5) is an analytic extension of the original space-time[a] (1.1).

It is easily seen that the region $\{r \le 2m\}$ for the metric (1.4) is a *black hole region*, in the sense that

observers, or signals, can enter this region, but can never leave it. (1.6)

In order to see that, recall that observers in general relativity always move on *future directed timelike curves*, that is, curves with timelike future directed tangent vector. For signals the curves are *causal future directed*, these are curves with timelike or null future directed tangent vector. Let, then, $\gamma(s) = (v(s), r(s), \theta(s), \varphi(s))$ be such a timelike curve, for the metric (1.4) the timelikeness condition $g(\dot\gamma, \dot\gamma) < 0$ reads

$$-(1 - \frac{2m}{r})\dot v^2 + 2\dot v \dot r + r^2(\dot\theta^2 + \sin^2\theta \dot\varphi^2) < 0 \;.$$

This implies

$$\dot v \Big(-(1 - \frac{2m}{r})\dot v + 2\dot r \Big) < 0 \;.$$

[a]The term *space–time* denotes a smooth, paracompact, connected, orientable and time–orientable Lorentzian manifold.

It follows that $\dot{v}$ does not change sign on a timelike curve. The usual choice of time orientation corresponds to $\dot{v} > 0$ on future directed curves, leading to

$$-(1-\frac{2m}{r})\dot{v} + 2\dot{r} < 0 \ .$$

For $r \le 2m$ the first term is non-negative, which enforces $\dot{r} < 0$ on all future directed timelike curves in that region. Thus, r is a strictly decreasing function along such curves, which implies that future directed timelike curves can cross the *event horizon* $\{r = 2m\}$ only if coming from the region $\{r > 2m\}$. The same conclusion applies for causal curves, by approximation.

Note that we could have chosen a time orientation in which future directed causal curves satisfy $\dot{v} < 0$. The resulting space-time is then called a *white hole* space-time, with $\{r = 2m\}$ being a *white hole event horizon*, which can only be crossed by those future directed causal curves which originate in the region $\{r < 2m\}$.

The transition from (1.1) to (1.4) is not the end of the story, as further extensions are possible. For the metric (1.1) a maximal analytic extension has been found independently by Kruskal, Szekeres, and Fronsdal, see Ref. 73 for details. This extension is visualised[b] in Figure 1. The region I there corresponds to the space-time (1.1), while the extension just constructed corresponds to the regions I and II.

A discussion of causal geodesics in the Schwarzschild geometry can be found in R. Price's contribution to this volume.

Higher dimensional counterparts of metrics (1.1) have been found by Tangherlini. In space-time dimension $n+1$, the metrics take the form (1.1) with

$$V^2 = 1 - \frac{2m}{r^{n-2}} \ , \tag{1.7}$$

and with $d\Omega^2$ — the unit round metric on S^{n-1}. The parameter m is the *Arnowitt-Deser-Misner mass* in space-time dimension four, and is proportional to that mass in higher dimensions. Assuming again $m > 0$, a maximal analytic extension can be constructed using a method of Walker[92] (which applies to all spherically symmetric space-times),[c] leading to a space-time with global structure identical to that of Figure 1 (except for the replacement $2M \to (2M)^{1/(n-2)}$ there). Global coordinate systems for the stan-

[b] I am grateful to J.-P. Nicolas for allowing me to use his electronic figure.[78]

[c] A generalisation of the Walker extension technique to arbitrary Killing horizons can be found in Ref. 85.

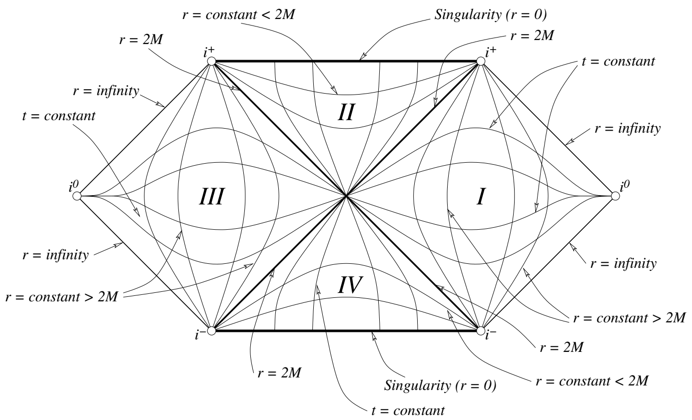

Fig. 1. The Carter-Penrose diagram for the Kruskal-Szekeres space-time with mass M. There are actually two asymptotically flat regions, with corresponding event horizons defined with respect to the second region. Each point in this diagram represents a two-dimensional sphere, and coordinates are chosen so that light-cones have slopes plus minus one.

dard maximal analytic extensions can be found in Ref. 67. The isometric embedding, into six-dimensional Euclidean space, of the $t = 0$ slice in a $(5+1)$–dimensional Tangherlini solution is visualised in Figure 2.

One of the features of the metric (1.1) is its *stationarity*, with Killing vector field $X = \partial_t$. A Killing field, by definition, is a vector field the local flow of which preserves the metric. A space–time is called *stationary* if there exists a Killing vector field X which approaches ∂_t in the asymptotically flat region (where r goes to ∞, see below for precise definitions) *and* generates a one parameter groups of isometries. A space–time is called *static* if it is stationary and if the stationary Killing vector X is hypersurface-orthogonal, i.e. $X^\flat \wedge dX^\flat = 0$, where

$$X^\flat = X_\mu dx^\mu = g_{\mu\nu} X^\nu dx^\mu \ .$$

A space–time is called *axisymmetric* if there exists a Killing vector field Y, which generates a one parameter group of isometries, and which behaves like a *rotation*: this property is captured by requiring that all orbits 2π periodic, and that the set $\{Y = 0\}$, called the *axis of rotation*, is non-empty. Killing vector fields which are a non-trivial linear combination of a time translation and of a rotation in the asymptotically flat region are called *stationary-rotating*, or *helical*. Note that those definitions require completeness of orbits of all Killing vector fields (this means that the equation $\dot{x} = X$ has a global

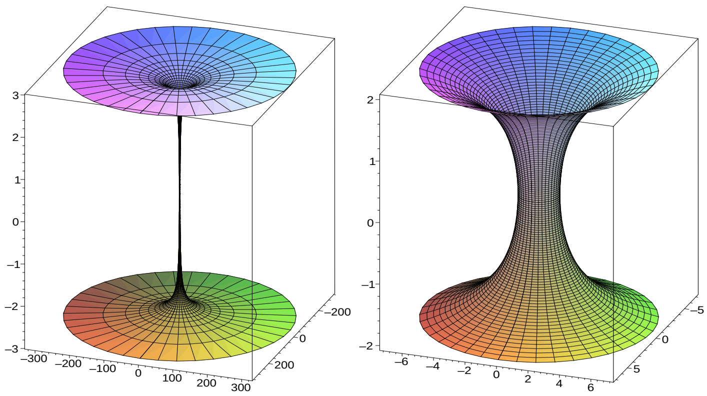

Fig. 2. Isometric embedding of the *space-geometry* of a $(5+1)$–dimensional Schwarzschild black hole into six-dimensional Euclidean space, near the throat of the Einstein-Rosen bridge $r = (2m)^{1/3}$, with $2m = 2$. The variable along the vertical axis asymptotes to $\approx \pm 3.06$ as r tends to infinity. The right picture is a zoom to the centre of the throat. The corresponding embedding in $(3+1)$–dimensions is known as the *Flamm paraboloid*.

solution for all initial values), see Refs. 22 and 51 for some results concerning this question.

In the extended Schwarzschild space-time the set $\{r = 2m\}$ is a null hypersurface $\mathscr{E}$, the Schwarzschild event horizon. The stationary Killing vector $X = \partial_t$ extends to a Killing vector $\hat{X}$ in the extended spacetime which becomes tangent to and null on $\mathscr{E}$, except at the "bifurcation sphere" right in the middle of Figure 1, where $\hat{X}$ vanishes. The global properties of the Kruskal–Szekeres extension of the exterior Schwarzschild[d] spacetime, make this space-time a natural model for a non-rotating black hole.

There is a rotating generalisation of the Schwarzschild metric, also discussed in the chapter by R. Price in this volume, namely the two parameter family of *exterior Kerr metrics*, which in Boyer-Lindquist coordinates take

[d]The exterior Schwarzschild space-time (1.1) admits an infinite number of non-isometric vacuum extensions, even in the class of maximal, analytic, simply connected ones. The Kruskal-Szekeres extension is singled out by the properties that it is maximal, vacuum, analytic, simply connected, with all maximally extended geodesics γ either complete, or with the curvature scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ diverging along γ in finite affine time.

the form

$$g = -\frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 - \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}dtd\varphi + \\ + \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta d\varphi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \ . \qquad (1.8)$$

Here

$$\Sigma = r^2 + a^2\cos^2\theta \ , \qquad \Delta = r^2 + a^2 - 2mr = (r - r_+)(r - r_-) \ ,$$

and $r_+ < r < \infty$, where

$$r_\pm = m \pm (m^2 - a^2)^{\frac{1}{2}} \ .$$

The metric satisfies the vacuum Einstein equations for any real values of the parameters a and m, but we will only discuss the range $0 \le a < m$. When $a = 0$, the Kerr metric reduces to the Schwarzschild metric. The Kerr metric is again a vacuum solution, and it is stationary with $X = \partial_t$ the asymptotic time translation, as well as axisymmetric with $Y = \partial_\varphi$ the generator of rotations. Similarly to the Schwarzschild case, it turns out that the metric can be smoothly extended across $r = r_+$, with $\{r = r_+\}$ being a smooth null hypersurface $\mathscr{E}$ in the extension. The simplest extension is obtained when t is replaced by a new coordinate

$$v = t + \int_{r_+}^{r} \frac{r^2+a^2}{\Delta}dr \ , \qquad (1.9)$$

with a further replacement of φ by

$$\phi = \varphi + \int_{r_+}^{r} \frac{a}{\Delta}dr \ . \qquad (1.10)$$

It is convenient to use the symbol $\hat{g}$ for the metric g in the new coordinate system, obtaining

$$\hat{g} = -\Big(1 - \frac{2mr}{\Sigma}\Big)dv^2 + 2drdv + \Sigma d\theta^2 - 2a\sin^2\theta d\phi dr \\ + \frac{(r^2+a^2)^2 - a^2\sin^2\theta\Delta}{\Sigma}\sin^2\theta d\phi^2 - \frac{4amr\sin^2\theta}{\Sigma}d\phi dv \ . \quad (1.11)$$

In order to see that (1.11) provides a smooth Lorentzian metric for $v \in \mathbb{R}$ and $r \in (0,\infty)$, note first that the coordinate transformation (1.9)-(1.10) has been tailored to remove the $1/\Delta$ singularity in (1.8), so that all coefficients are now analytic functions on $\mathbb{R} \times (0,\infty) \times S^2$. A direct calculation of the determinant of $\hat{g}$ is somewhat painful, a simpler way is to proceed as follows: first, the calculation of the determinant of the metric (1.8) reduces to that of a two-by-two determinant in the (t, φ) variables, leading to $\det g = -\sin^2\theta\Sigma^2$. Next, it is very easy to check that the determinant of the Jacobi matrix $\partial(v, r, \theta, \phi)/\partial(t, r, \theta, \varphi)$ is one. It follows that $\det \hat{g} = -\sin^2\theta\Sigma^2$ for $r > r_+$. Analyticity implies that this equation holds globally, which (since Σ has no zeros) establishes the Lorentzian signature of $\hat{g}$ for all positive r.

Let us show that the region $r < r_+$ is a black hole region, in the sense of (1.6). We start by noting that ∇r is a causal vector for $r_- \le r \le r_+$, where $r_- = m - \sqrt{m^2 + a^2}$. A direct calculation using (1.11) is again somewhat lengthy, instead we use (1.8) in the region $r > r_+$ to obtain there

$$\hat{g}(\nabla r, \nabla r) = g(\nabla r, \nabla r) = g^{rr} = \frac{1}{g_{rr}} = \frac{\Delta}{\Sigma} = \frac{(r - r_+)(r - r_-)}{r^2 + a^2\cos^2\theta} \ . \tag{1.12}$$

But the left-hand-side of this equation is an analytic function throughout the extended manifold $\mathbb{R}\times(0,\infty)\times S^2$, and uniqueness of analytic extensions implies that $\hat{g}(\nabla r, \nabla r)$ equals the expression at the extreme right of (1.12). (The intermediate equalities are of course valid only for $r > r_+$.) Thus ∇r is spacelike if $r < r_-$ or $r > r_+$, null on the "Killing horizons" $\{r = r_\pm\}$, and timelike in the region $\{r_- < r < r_+\}$. We choose a time orientation so that ∇r is future pointing there.

Consider, now, a future directed causal curve $\gamma(s)$. Along γ we have

$$\frac{dr}{ds} = \dot{\gamma}^i \nabla_i r = g_{ij}\dot{\gamma}^i \nabla^j r = g(\dot{\gamma}, \nabla r) < 0$$

in the region $\{r_- < r < r_+\}$, because the scalar product of two future directed causal vectors is always negative. This implies that r is strictly decreasing along future directed causal curves in the region $\{r_- < r < r_+\}$, so that such curves can only leave this region through the set $\{r = r_-\}$. In other words, no causal communication is possible from the region $\{r < r_+\}$ to the "exterior world" $\{r > r_+\}$.

The Schwarzschild metric has the property that the set $g(X, X) = 0$, where X is the "static Killing vector" ∂_t, coincides with the event horizon $r = 2m$. This is not the case any more for the Kerr metric, where we have

$$g(\partial_t, \partial_t) = \hat{g}(\partial_v, \partial_v) = \hat{g}_{vv} = -\Big(1 - \frac{2mr}{r^2 + a^2\cos^2\theta}\Big) \ ,$$

and the equation $\hat{g}(\partial_v, \partial_v) = 0$ defines a set called the *ergosphere*:

$$\mathring{r}_\pm = m \pm \sqrt{m^2 - a^2\cos^2\theta} \ ,$$

see Figures 3 and 4. The ergosphere touches the horizons at the axes of

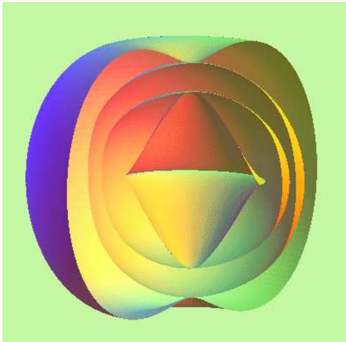

Fig. 3. A coordinate representation[81] of the outer ergosphere $r = \mathring{r}_+$, the event horizon $r = r_+$, the Cauchy horizon $r = r_-$, and the inner ergosphere $r = \mathring{r}_-$ with the singular ring in Kerr space-time. Computer graphics by Kayll Lake.[66]

symmetry $\cos\theta = \pm 1$. Note that $\partial\mathring{r}_\pm/\partial\theta \neq 0$ at those axes, so the ergosphere has a cusp there. The region bounded by the outermost horizon $r = r_+$ and the outermost ergosphere $r = \mathring{r}_+$ is called the *ergoregion*, with X spacelike in its interior. We refer the reader to Refs. 15 and 79 for an exhaustive analysis of the geometry of the Kerr space-time.

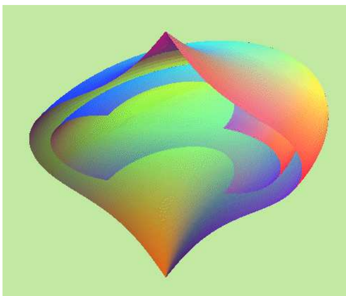

Fig. 4. Isometric embedding in Euclidean three space of the ergosphere (the outer hull), and part of the event horizon, for a rapidly rotating Kerr solution. The hole arises due to the fact that there is no global isometric embedding possible for the event horizon when $a/m > \sqrt{3}/2$.[81] Somewhat surprisingly, the embedding fails to represent accurately the fact that the cusps at the rotation axis are pointing inwards, and not outwards. Computer graphics by Kayll Lake.[66]

The hypersurfaces $\{r = r_\pm\}$ provide examples of *null acausal boundaries*. Causality theory shows that such hypersurfaces are threaded by a family of null geodesics, called *generators*. One checks that the stationary-rotating Killing field $X + \omega Y$, where $\omega = \frac{a}{2mr_+}$, is null on $\{r > r_+\}$, and hence *tangent* to the generators of the horizon. Thus, the generators are rotating with respect to the frame defined by the stationary Killing vector field X. This property is at the origin of the definition of ω as the *angular velocity* of the event horizon.

Higher dimensional generalisations of the Kerr metric have been constructed by Myers and Perry.[76]

In the examples discussed so far the black hole event horizon is a connected hypersurface in space-time. In fact,[13,25] there are no *static vacuum* solutions with several black holes, consistently with the intuition that gravity is an attractive force. However, static multi black holes become possible in presence of electric fields. The list of known examples is exhausted by the *Majumdar-Papapetrou* black holes, in which the metric g and the electromagnetic potential A take the form

$$g = -u^{-2}dt^2 + u^2(dx^2 + dy^2 + dz^2)\,, \tag{1.13}$$

$$A = u^{-1}dt\,, \tag{1.14}$$

with some nowhere vanishing function u. Einstein–Maxwell equations read then

$$\frac{\partial u}{\partial t} = 0\,, \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0\,. \tag{1.15}$$

Standard MP black holes are obtained if the coordinates x^μ of (1.13)–(1.14) cover the range $\mathbb{R} \times (\mathbb{R}^3 \setminus \{\vec{a}_i\})$ for a finite set of points $\vec{a}_i \in \mathbb{R}^3$, $i = 1, \ldots, I$, and if the function u has the form

$$u = 1 + \sum_{i=1}^{I} \frac{m_i}{|\vec{x} - \vec{a}_i|}\,, \tag{1.16}$$

for some positive constants m_i. It has been shown by Hartle and Hawking[54] that every standard MP space–time can be analytically extended to an electro–vacuum space–time with a non–empty black hole region. Higher-dimensional generalisations of the MP black holes, with very similar properties, have been found by Myers.[75]

1.2. $\Lambda \neq 0$

So far we have assumed a vanishing cosmological constant Λ. However, there is interest in solutions with $\Lambda \neq 0$: Indeed, there is strong evidence that we live in a universe with $\Lambda > 0$. On the other hand, space-times with a negative cosmological constant appear naturally in many models of theoretical physics, e.g. in string theory.

In space-time dimension four, examples are given by the *generalised Kottler* and the *generalised Nariai* solutions

$$ds^2 = -\Big(k - \frac{2m}{r} - \frac{\Lambda}{3}r^2\Big)dt^2 + \frac{dr^2}{k - \frac{2m}{r} - \frac{\Lambda}{3}r^2} + r^2 d\Omega_k^2 \ , \quad k = 0, \pm 1 \ , \tag{1.17}$$

$$ds^2 = -\Big(\lambda - \Lambda r^2\Big)dt^2 + \frac{dr^2}{\lambda - \Lambda r^2} + |\Lambda|^{-1} d\Omega_k^2 \ , \quad k = \pm 1 \ , \quad k\Lambda > 0 \ , \lambda \in \mathbb{R} \tag{1.18}$$

where $d\Omega_k^2$ denotes a metric of constant Gauss curvature k on a two-dimensional compact manifold ${}^2M$. These are static solutions of the vacuum Einstein equation with a cosmological constant Λ. The parameter $m \in \mathbb{R}$ is related to the *Gibbons-Hawking mass* of the foliation $t = \text{const}$, $r = \text{const}$.

As an example of the analysis in this context, consider the metrics (1.17) with $k = 0$ and $\Lambda = -3$:

$$ds^2 = -\Big(r^2 - \frac{2m}{r}\Big)dt^2 + \frac{dr^2}{r^2 - \frac{2m}{r}} + r^2(d\varphi^2 + d\psi^2) \ , \tag{1.19}$$

with φ and ψ parameterising S^1. If $m > 0$ there is a coordinate singularity at $r = (2m)^{1/3}$; an extension can be constructed as in (1.3) by replacing the coordinate t with

$$v = t + f(r) \ , \quad f' = \frac{1}{r^2 - \frac{2m}{r}} \ . \tag{1.20}$$

This leads to a smooth Lorentzian metric for all $r > 0$,

$$ds^2 = -\Big(r^2 - \frac{2m}{r}\Big)dv^2 + 2dvdr + r^2(d\varphi^2 + d\psi^2) \ . \tag{1.21}$$

We have now an exterior region $r > (2m)^{1/3}$, a black hole event horizon at $r = (2m)^{1/3}$, and a black hole region for $r < (2m)^{1/3}$.

Similarly when $\lambda\Lambda > 0$ the metrics (1.18) have an exterior region defined by the condition $r > \sqrt{\lambda/\Lambda}$. A procedure similar to the above leads to an extension across an event horizon $r = \sqrt{\lambda/\Lambda}$. Note that the asymptotic behavior of metrics (1.18) is rather different from that of metrics (1.17).

The Kottler examples can be generalised to higher dimensions as follows:[10] Let $M = \mathbb{R} \times (r_0, \infty) \times N^{n-1}$, with $N := N^{n-1}$ compact, and with metric of form:

$$g_m = -V dt^2 + V^{-1} dr^2 + r^2 g_N \ , \tag{1.22}$$

where g_N is any Einstein metric, $Ric_{g_N} = \lambda g_N$, with g_N scaled so that $\lambda = \pm(n-2)$ or 0. Then for $V = V(r)$ given by

$$V = c + r^2 - (2m)/r^{n-2} \,, \tag{1.23}$$

with $c = \pm 1$ or 0 respectively, g_m is a static solution of the vacuum Einstein equations, with $Ric_{g_m} = -n g_m$. When appropriately extended, the resulting space-times possess an event horizon at the largest positive root r_0 of $V(r)$.

It turns out that the collection of static vacuum black holes with a negative cosmological constant is much richer than the one with $\Lambda = 0$. This is due to rather different asymptotic behavior of the solutions. An elegant way of capturing this asymptotic behavior, due to Penrose,[82] proceeds as follows (for notational simplicity we assume that $\Lambda < 0$ has been scaled as in (1.23)): Replacing in (1.22) the coordinate r by $x = 1/r$ one obtains $g_m = x^{-2}\tilde{g}_m$, where

$$\tilde{g}_m = -(1 + cx^2 - 2mx^n)dt^2 + \frac{dx^2}{1 + cx^2 - 2mx^n} + g_N \,. \tag{1.24}$$

We are interested in the metric $\tilde{g}_m$ for $r \geq r_0$ with some large r_0, this corresponds to x small, $0 < x \leq x_0 := 1/r_0$. The surprising fact is that

> $\tilde{g}_m$ *extends by continuity to a smooth Lorentzian metric on the set* $x \in [0, x_0]$.

It is then natural to look for static vacuum metrics of the form $x^{-2}\tilde{g}$, with $\tilde{g}$ smoothly extending to the conformal boundary at infinity $\{x = 0\}$. Such metrics will be called *conformally compactifiable.* In Refs. 2 and 3 the following is shown: write $\tilde{g}|_{x=0}$ as $-\alpha^2 dt^2 + g_N$, where g_N is a Riemannian metric on N, with $\partial_t \alpha = \partial_t g_N = 0$. Then:

(1) Let $\mathring{g}_N$ be a Riemannian metric, with sectional curvatures equal to minus one, on the compact manifold N. Then for all t-independent (α, g_N) close enough to $(1, \mathring{g}_N)$ there exists an associated static, vacuum, conformally compactifiable black hole metric.
(2) In space-time dimension $n+1 = 4$, for all compact N the set of (α, g_N) corresponding to conformally compactifiable static vacuum black holes contains an infinite dimensional manifold.

All metrics presented so far in this section were static. A family of rotating stationary solutions, generalising the Myers-Perry solutions to $\Lambda \neq 0$, can be found in Ref. 53.

Rather surprisingly, when $\Lambda < 0$ there exist static vacuum black holes in space-time dimension three,[e] discovered by Bañados, Teitelboim and Zanelli.[5] The static, circularly symmetric, vacuum solutions take the form

$$ds^2 = -\Big(\frac{r^2}{\ell^2} - m\Big)dt^2 + \Big(\frac{r^2}{\ell^2} - m\Big)^{-1}dr^2 + r^2 d\phi^2 \;, \qquad (1.25)$$

where m is related to the total mass and $\ell^2 = -1/\Lambda$. For $m > 0$, this can be extended, as in (1.3) with $V^2 = r^2/\ell^2 - m$, to a black hole space-time with event horizon located at $r_H = \ell\sqrt{m}$. There also exist rotating counterparts of (1.25), discussed in the reference just given.

1.3. *Black strings and branes*

Consider any vacuum black hole solution $(\mathscr{M}, g)$, and let (N, h) be a Riemannian manifold with a Ricci flat metric, $\mathrm{Ric}(h) = 0$. Then the space-time $(\mathscr{M} \times N, g \oplus h)$ is again a vacuum space-time, containing a black hole region in the sense used so far. (Similarly if $\mathrm{Ric}(g) = \sigma g$ and $\mathrm{Ric}(h) = \sigma h$ then $\mathrm{Ric}(g \oplus h) = \sigma g \oplus h$.) Objects of this type are called *black strings* when $\dim N = 1$, and *black branes* in general. Due to lack of space they will not be discussed here, see Refs. 70, 80 and references therein.

2. Model independent concepts

We now describe a general framework for the notions used in the previous sections. The mathematical notion of black hole is meant to capture the idea of a region of space-time which *cannot be seen by "outside observers"*. Thus, at the outset, one assumes that there exists a family of physically preferred observers in the space-time under consideration. When considering isolated physical systems, it is natural to define the "exterior observers" as observers which are "very far" away from the system under consideration. The standard way of making this mathematically precise is by using conformal completions, already mentioned above: A pair $(\widetilde{\mathscr{M}}, \tilde{g})$ is called a *conformal completion at infinity,* or simply *conformal completion*, of $(\mathscr{M}, g)$ if $\widetilde{\mathscr{M}}$ is a manifold with boundary such that:

(1) $\mathscr{M}$ is the interior of $\widetilde{\mathscr{M}}$,
(2) there exists a function Ω, with the property that the metric $\tilde{g}$, defined as $\Omega^2 g$ on $\mathscr{M}$, extends by continuity to the boundary of $\widetilde{\mathscr{M}}$, with the extended metric remaining of Lorentzian signature,

[e] There are no such vacuum black holes with $\Lambda > 0$, or with $\Lambda = 0$ and degenerate horizons.[58]

(3) Ω is positive on $\mathscr{M}$, differentiable on $\widetilde{\mathscr{M}}$, vanishes on the boundary

$$\mathscr{I} := \widetilde{\mathscr{M}} \setminus \mathscr{M} \ ,$$

with $d\Omega$ *nowhere vanishing* on $\mathscr{I}$.

(In the example (1.24) we have $\Omega = x$, and $\mathscr{I} = \{x = 0\}$.) The boundary $\mathscr{I}$ of $\widetilde{\mathscr{M}}$ is called Scri, a phonic shortcut for "script I". The idea here is the following: forcing Ω to vanish on $\mathscr{I}$ ensures that $\mathscr{I}$ lies infinitely far away from any physical object — a mathematical way of capturing the notion "very far away". The condition that $d\Omega$ does not vanish is a convenient technical condition which ensures that $\mathscr{I}$ is a smooth three-dimensional hypersurface, instead of some, say, one- or two-dimensional object, or of a set with singularities here and there. Thus, $\mathscr{I}$ is an idealised description of a family of observers at infinity.[f]

To distinguish between various points of $\mathscr{I}$ one sets

$$\mathscr{I}^+ = \{\text{points in } \mathscr{I} \text{ which are to the future of the physical space-time}\} \ ,$$
$$\mathscr{I}^- = \{\text{points in } \mathscr{I} \text{ which are to the past of the physical space-time}\} \ .$$

(Recall that a point p is to the future, respectively to the past, of q if there exists a future directed, respectively past directed, causal curve from q to p. Causal curves are curves γ such that their tangent vector $\dot\gamma$ is causal everywhere, $g(\dot\gamma, \dot\gamma) \le 0$.) One then defines the black hole region $\mathscr{B}$ as

$$\mathscr{B} := \{\text{the set of points in } \mathscr{M} \text{ from which no future directed causal curve in } \widetilde{\mathscr{M}} \text{ meets } \mathscr{I}^+\} \ . \quad (2.1)$$

By definition, points in the black hole region cannot thus send information to $\mathscr{I}^+$; equivalently, observers on $\mathscr{I}^+$ cannot see points in $\mathscr{B}$. The *white hole* region $\mathscr{W}$ is defined by changing the time orientation in (2.1).

In order to obtain a meaningful definition of black hole, one needs to assume further that $\mathscr{I}^+$ satisfies a few regularity conditions. For example, if we consider the standard conformal completion of Minkowski space-time, then of course $\mathscr{B}$ will be empty. However, one can remove points from that completion, obtaining sometimes a new completion with a non-empty black hole region. (Think of a family of observers who stop to exist at time $t = 0$, they will never be able to see any event with $t > 0$, leading to a black hole region with respect to this family.) We shall return to this question shortly.

[f] We note that the behavior of the metric in the asymptotic region for the black strings and branes of Section 1.3 is *not* compatible with this framework.

A key notion related to the concept of a black hole is that of *future* ($\mathscr{E}^+$) and *past* ($\mathscr{E}^-$) *event horizons*,

$$\mathscr{E}^+ := \partial\mathscr{B} \,, \qquad \mathscr{E}^- := \partial\mathscr{W} \,. \tag{2.2}$$

Under mild assumptions, event horizons in stationary space-times with matter satisfying the *null energy condition*,

$$T_{\mu\nu}\ell^\mu\ell^\nu \geq 0 \quad \text{for all null vectors } \ell^\mu, \tag{2.3}$$

are smooth null hypersurfaces, analytic if the metric is analytic.[28] This is, however, not the case in the non-stationary case: roughly speaking, event horizons are non-differentiable at end points of their generators. In Ref. 29 a horizon has been constructed which is non-differentiable on a dense set. The best one can say in general is that event horizons are Lipschitz,[83] semi-convex[28] topological hypersurfaces.

In order to develop a reasonable theory one also needs a regularity condition for the interior of space-time. This has to be a condition which does not exclude singularities (otherwise the Schwarzschild and Kerr black holes would be excluded), but which nevertheless guarantees a well-behaved exterior region. One such condition, assumed in all the results described below, is the existence in $\mathscr{M}$ of an asymptotically flat space-like hypersurface $\mathscr{S}$ with compact interior region. This means that $\mathscr{S}$ is the union of a finite number[g] of *asymptotically flat ends* $\mathscr{S}_{\rm ext}$, each diffeomorphic to $\mathbb{R}^n \setminus B(0,R)$, and of a compact region $\mathscr{S}_{int}$. Further, either $\mathscr{S}$ has no boundary, or the boundary of $\mathscr{S}$ lies on $\mathscr{E}^+ \cup \mathscr{E}^-$. To make things precise, for any spacelike hypersurface let g_{ij} be the induced metric, and let K_{ij} denote its extrinsic curvature. A space–like hypersurface $\mathscr{S}_{\rm ext}$ diffeomorphic to $\mathbb{R}^n$ minus a ball will be called an α-*asymptotically flat* end, for some $\alpha > 0$, if the fields (g_{ij}, K_{ij}) satisfy the fall–off conditions

$$|g_{ij}-\delta_{ij}|+\cdots+r^k|\partial_{\ell_1\cdots\ell_k}g_{ij}|+r|K_{ij}|+\cdots+r^k|\partial_{\ell_1\cdots\ell_{k-1}}K_{ij}| \leq Cr^{-\alpha} \,, \tag{2.4}$$

for some constants C, $k \geq 1$. The fall-off rate is typically determined either by requiring that the leading deviations from flatness are identical to those in the Tangherlini solution (1.1) with V given by (1.7), or that the fall-off rate be the same as in (1.7) (which leads to $\alpha = n-2$), or by requiring a well-defined ADM mass (which leads to $\alpha > (n-2)/2$).

[g] There is no loss of generality in assuming that there is only one such region, if $\mathscr{S}$ is allowed to have a trapped or marginally trapped boundary. However, it is often more convenient to work with hypersurfaces without boundary.

In dimension $3 + 1$ there exists a canonical way of constructing a conformal completion with good global properties for stationary space-times which are asymptotically flat in the sense of (2.4) for some $\alpha > 0$, and which are vacuum sufficiently far out in the asymptotic region, as follows: Equation (2.4) and the stationary Einstein equations can be used[9] to prove a complete asymptotic expansion of the metric in terms of powers of $1/r$.[h] The analysis in Refs. 37 and 40 shows then the existence of a smooth conformal completion at null infinity. This conformal completion is referred to as the *standard completion* and will be assumed from now on. It coincides with the completion constructed in the last section for the metrics (1.22).

As already pointed out, an analysis along the lines of Beig and Simon[9] has only been performed so far in dimension $3 + 1$, and it is not clear what happens in general, because the proofs use an identity which is wrong in other dimensions. On one hand there sometimes exist smooth conformal completions — we have just constructed some in the previous section. On the other hand, it is known that the hypothesis of smoothness of the conformal completion is overly restrictive in odd space-time dimensions in general[i], though it could conceivably be justifiable for stationary solutions. Whatever the case, we shall follow the $\mathscr{I}$ approach here, and we refer the reader to Ref. 19 for a discussion of further drawbacks of this approach, and for alternative proposals.

Returning to the event horizon $\mathscr{E} = \mathscr{E}^+ \cup \mathscr{E}^-$, it is not very difficult to show that every Killing vector field X is necessarily tangent to $\mathscr{E}$: indeed, since $\mathscr{M}$ is invariant under the flow of X, so is $\mathscr{I}^+$, and therefore also $I^-(\mathscr{I}^+)$, and therefore also its boundary $\mathscr{E}^+ = \partial I^-(\mathscr{I}^+)$. Similarly for $\mathscr{E}^-$. Hence X is tangent to $\mathscr{E}$. the Since both $\mathscr{E}^\pm$ are null hypersurfaces, it follows that X is either null or spacelike on $\mathscr{E}$. This leads to a preferred class of event horizons, called *Killing horizons.* By definition, a Killing horizon associated with a Killing vector K is a *null hypersurface* which coincides with a connected component of the set

$$\mathcal{H}(K) := \{p \in \mathscr{M} : \; g(K,K)(p) = 0 \; , \; K(p) \neq 0\} \; . \tag{2.5}$$

---

[h]In higher dimensions it is straightforward to prove an asymptotic expansion of stationary vacuum solutions in terms of $\ln^j r/r^i$.

[i]In even space-time dimension smoothness of $\mathscr{I}$ might fail because of logarithmic terms in the expansion.[31,65] In odd space-time dimensions the situation is (seemingly) even worse, because of half-integer powers of $1/r$[57]

A simple example is provided by the "boost Killing vector field" $K = z\partial_t + t\partial_z$ in Minkowski space-time: $\mathcal{H}(K)$ has four connected components

$$\mathcal{H}_{\epsilon\delta} := \{t = \epsilon z \ , \delta t > 0\} \ , \quad \epsilon, \delta \in \{\pm 1\} \ .$$

The closure $\overline{\mathcal{H}}$ of $\mathcal{H}$ is the set $\{|t| = |z|\}$, which is not a manifold, because of the crossing of the null hyperplanes $\{t = \pm z\}$ at $t = z = 0$. Horizons of this type are referred to as *bifurcate Killing horizons*, with the set $\{K(p) = 0\}$ being called the *bifurcation surface* of $\mathcal{H}(K)$. The bifurcate horizon structure in the Kruskal-Szekeres-Schwarzschild space-time can be clearly seen in Figure 1.

The Vishveshwara-Carter lemma[16, 90] shows that if a Killing vector K in an $(n+1)$–dimensional space-time is hypersurface-orthogonal, $K^\flat \wedge dK^\flat = 0$, then the set $\mathcal{H}(K)$ defined in (2.5) is a union of smooth null hypersurfaces, with K being tangent to the null geodesics threading $\mathcal{H}$, and so is indeed a union of Killing horizons. It has been shown by Carter[16] that the same conclusion can be reached in asymptotically flat, vacuum, four-dimensional space-times if the hypothesis of hypersurface-orthogonality is replaced by that of existence of two linearly independent Killing vector fields. The proof proceeds via an analysis of the orbits of the isometry group in four-dimensional asymptotically flat manifolds, together with Papapetrou's orthogonal-transitivity theorem, and does not generalise to higher dimensions without further hypotheses.

In stationary-axisymmetric space-times a Killing vector K *tangent to the generators* of a Killing horizon $\mathcal{H}$ can be normalised so that $K = X + \omega Y$, where X is the Killing vector field which asymptotes to a time translation in the asymptotic region, and Y is the Killing vector field which generates rotations in the asymptotic region. The constant ω is called the *angular velocity of the Killing horizon* $\mathcal{H}$.

On a Killing horizon $\mathcal{H}(K)$ one necessarily has

$$\nabla^\mu (K^\nu K_\nu) = -2\kappa K^\mu. \tag{2.6}$$

Assuming that the horizon is bifurcate (Ref. 61, p. 59), or that the so-called *dominant energy condition* holds (this means that $T_{\mu\nu} X^\mu X^\nu \geq 0$ for all timelike vector fields X) (Ref. 56, Theorem 7.1), it can be shown that κ is constant (recall that Killing horizons are always connected in our terminology), it is called *the surface gravity of* $\mathcal{H}$. A Killing horizon is called *degenerate* when $\kappa = 0$, and non–degenerate otherwise; by an abuse of terminology one similarly talks of degenerate black holes, *etc.* In Kerr space-times we have $\kappa = 0$ if and only if $m = a$. All horizons in the multi-black hole Majumdar-Papapetrou solutions (1.13)-(1.16) are degenerate.

A fundamental theorem of Boyer shows that degenerate horizons are closed. This implies that a horizon $\mathcal{H}(K)$ such that K has zeros in $\overline{\mathcal{H}}$ is non-degenerate, and is of bifurcate type, as described above. Further, a *non-degenerate* Killing horizon with *complete* geodesic generators always contains zeros of K in its closure. However, it is not true that existence of a non-degenerate horizon implies that of zeros of K: take the Killing vector field $z\partial_t + t\partial_z$ in Minkowski space-time from which the 2-plane $\{z = t = 0\}$ has been removed. The universal cover of that last space-time provides a space-time in which one cannot restore the points which have been artificially removed, without violating the manifold property.

The *domain of outer communications* (d.o.c.) of a black hole space-time is defined as

$$\langle\langle \mathscr{M} \rangle\rangle := \mathscr{M} \setminus \{\mathscr{B} \cup \mathscr{W}\}\ . \tag{2.7}$$

Thus, $\langle\langle \mathscr{M} \rangle\rangle$ is the region lying outside of the white hole region and outside of the black hole region; it is the region which can both be seen by the outside observers and influenced by those.

The subset of $\langle\langle \mathscr{M} \rangle\rangle$ where X is spacelike is called the *ergoregion*. In the Schwarzschild space-time $\omega = 0$ and the ergoregion is empty, but neither of these is true in Kerr with $a \neq 0$.

A very convenient method for visualising the global structure of space-times is provided by the *Carter-Penrose diagrams*. An example of such a diagram is presented in Figure 1.

A corollary of the *topological censorship theorem* of Friedman, Schleich and Witt[43,46,47] is that d.o.c.'s of regular black hole space-times satisfying the dominant energy condition are simply connected.[45,50] This implies that connected components of event horizons in stationary, asymptotically flat, four-dimensional space-times have $\mathbb{R} \times S^2$ topology.[12,35] The restrictions in higher dimension are less stringent,[14,48] in particular in space-time dimension five an $\mathbb{R} \times S^2 \times S^1$ topology is allowed. A vacuum solution with this horizon topology has been indeed found by Emparan and Reall.[42]

Space-times with good causality properties can be sliced by families of spacelike surfaces $\mathscr{S}_t$, this provides an associated slicing $\mathscr{E}_t = \mathscr{S}_t \cap \mathscr{E}$ of the event horizon. It can be shown that the area of the $\mathscr{E}_t$'s is well defined,[28] this is not a completely trivial statement in view of the poor differentiability properties of $\mathscr{E}$. A key theorem of Hawking[55] (compare Ref. 28) shows that, in suitably regular asymptotically flat space-times, the area of $\mathscr{E}_t$'s is a monotonous function of t. This property carries over to black-hole regions associated to *null-convex* families of observers, as in Ref. 19.

Vacuum or electrovacuum regions with a *timelike* Killing vector can be endowed with an analytic chart in which the metric is analytic. This result has often been misinterpreted as holding up-to-the horizon. However, rather mild global conditions forbid timelike Killing vectors on event horizons. The Curzon metric, studied by Scott and Szekeres[88] provides an example of failure of analyticity at degenerate horizons. One-sided analyticity at *static non-degenerate* vacuum horizons has been proved recently.[26] It is expected that the result remains true for stationary Killing horizons, but the proof does not generalise in any obvious way.

3. Classification of asymptotically flat stationary black holes ("No hair theorems")

We confine attention to the "outside region" of black holes, the domain of outer communications (2.7). For reasons of space we only consider vacuum solutions; there is a similar theory for electro-vacuum black holes.[17,18,23,24,95] There also exists a somewhat less developed theory for black hole spacetimes in the presence of nonabelian gauge fields.[91]

Based on the facts below, it is expected that the d.o.c.'s of appropriately regular, stationary, asymptotically flat four-dimensional vacuum black holes are isometrically diffeomorphic to those of Kerr black holes.

(1) The *rigidity theorem* (Hawking[44,55]): event horizons in regular, *non–degenerate*, stationary, *analytic*, four-dimensional vacuum black holes are either *Killing horizons* for X, or there exists a second Killing vector in $\langle\langle \mathscr{M} \rangle\rangle$. The proof does not seem to generalise to higher dimensions without further assumptions.
(2) The *Killing horizons theorem* (Sudarsky-Wald[89]): *non–degenerate* stationary vacuum black holes such that the *event horizon is the union of Killing horizons of X* are *static*. Both the proof in Ref. 89, and that of existence of maximal hypersurfaces needed there,[34] are valid in any space dimensions $n \geq 3$.
(3) The Schwarzschild black holes exhaust the family of *static* regular vacuum black holes (Israel,[60] Bunting – Masood-ul-Alam,[13] Chruściel[25]). The proof in Ref. 25 carries over immediately to all space dimensions $n \geq 3$ (compare Refs. 52, 87), with the proviso of validity of the rigidity part of the Riemannian positive energy theorem.[j]

[j]The proofs of this last theorem, known at the time of writing of this work, require the existence of a spin structure in space dimensions larger than eleven,[41] though the result

(4) The Kerr black holes satisfying

$$m^2 > a^2 \tag{3.1}$$

exhaust the family of *non–degenerate*, *stationary–axisymmetric,* vacuum, *connected, four-dimensional* black holes. Here m is the total ADM mass, while the product am is the total ADM angular momentum. The framework for the proof has been set-up by Carter, and the statement above is due to Robinson.[86] The Emparan-Reall metrics[42] show that there is no uniqueness in higher dimensions, even if three commuting Killing vectors are assumed; see, however, Ref. 74.

The above results are collectively known under the name of *no hair theorems*, and they have *not* provided the final answer to the problem so far even in four dimensions: First, there are no *a priori* reasons known for the analyticity hypothesis in the rigidity theorem. Next, degenerate horizons have been completely understood in the static case only.

In all results above it has been assumed that the metric approaches the Minkowski one in the asymptotic region. Anderson[1] has shown that, under natural regularity hypothesis, the only alternative concerning the asymptotic behavior for *static* $(3+1)$–dimensional vacuum black holes are "small ends", as defined in his work. Solutions with this last behavior have been constructed by Korotkin and Nicolai,[63] and it would be of interest to prove that there are no others. In higher dimension other asymptotic behaviors are possible, examples are given by the metrics (1.22) with $V = c-(2m)/r^{n-2}$, and g_N as described there.

Yet another key open question is that of existence of *non-connected* regular stationary-axisymmetric vacuum black holes. The following result is due to Weinstein:[93] Let $\partial\mathscr{S}_a$, $a=1,\ldots,N$ be the connected components of $\partial\mathscr{S}$. Let $X^\flat = g_{\mu\nu}X^\mu dx^\nu$, where X^μ is the Killing vector field which asymptotically approaches the unit normal to $\mathscr{S}_{ext}$. Similarly set $Y^\flat = g_{\mu\nu}Y^\mu dx^\nu$, Y^μ being the Killing vector field associated with rotations. On each $\partial\mathscr{S}_a$ there exists a constant ω_a such that the vector $X+\omega_a Y$ is tangent to the generators of the Killing horizon intersecting $\partial\mathscr{S}_a$. The constant ω_a is called the angular velocity of the associated Killing horizon. Define

$$m_a = -\frac{1}{8\pi}\int_{\partial\mathscr{S}_a} *dX^\flat \ , \tag{3.2}$$

$$L_a = -\frac{1}{4\pi}\int_{\partial\mathscr{S}_a} *dY^\flat \ . \tag{3.3}$$

---

is expected to hold without any restrictions.

Such integrals are called *Komar integrals*. One usually thinks of L_a as the angular momentum of each connected component of the black hole. Set

$$\mu_a = m_a - 2\omega_a L_a \ . \tag{3.4}$$

Weinstein shows that one necessarily has $\mu_a > 0$. The problem at hand can be reduced to a *harmonic map* equation, also known as the *Ernst equation*, involving a singular map from $\mathbb{R}^3$ with Euclidean metric δ to the two-dimensional hyperbolic space. Let $r_a > 0$, $a = 1, \ldots, N-1$, be the distance in $\mathbb{R}^3$ along the axis between neighboring black holes as measured with respect to the (unphysical) metric δ. Weinstein proves that for *non-degenerate* regular black holes the inequality (3.1) holds, and that the metric on $\langle\langle \mathscr{M} \rangle\rangle$ is determined up to isometry by the $3N-1$ parameters

$$(\mu_1, \ldots, \mu_N, L_1, \ldots, L_N, r_1, \ldots, r_{N-1}) \tag{3.5}$$

just described, with $r_a, \mu_a > 0$. These results by Weinstein contain the no-hair theorem of Carter and Robinson as a special case. Weinstein also shows that for every $N \geq 2$ and for every set of parameters (3.5) with $\mu_a, r_a > 0$, there exists a solution of the problem at hand. It is known that for some sets of parameters (3.5) the solutions will have "strut singularities" between some pairs of neighboring black holes,[69,71,77,94] but the existence of the "struts" for all sets of parameters as above is not known, and is one of the main open problems in our understanding of stationary–axisymmetric electro–vacuum black holes. The existence and uniqueness results of Weinstein remain valid when strut singularities are allowed in the metric at the outset, though such solutions do not fall into the category of regular black holes discussed so far.

Some of the results above have been generalised to $\Lambda \neq 0$.[4,11,33,49,84]

4. Dynamical black holes: Robinson-Trautman metrics

The only known family of vacuum, singularity-free (in the sense described in the previous section), dynamical black holes, with exhaustive understanding of the global structure to the future of a Cauchy surface, is provided by the *Robinson-Trautman* (RT) metrics.

By definition, the Robinson–Trautman space–times can be foliated by a null, hypersurface orthogonal, shear free, expanding geodesic congruence. It has been shown by Robinson and Trautman that in such a space–time there always exists a coordinate system in which the metric takes the form

$$ds^2 = -\Phi\, du^2 - 2du\, dr + r^2 e^{2\lambda} \mathring{g}_{ab}\, dx^a\, dx^b, \quad x^a \in {}^2M, \quad \lambda = \lambda(u, x^a), \tag{4.1}$$

$$\mathring{g}_{ab} = \mathring{g}_{ab}(x^a), \quad \Phi = \frac{R}{2} + \frac{r}{12m}\Delta_g R - \frac{2m}{r}, \quad R = R(g_{ab}) \equiv R(e^{2\lambda}\mathring{g}_{ab}),$$

m is a constant which is related to the total Bondi mass of the metric, R is the Ricci scalar of the metric $g_{ab} \equiv e^{2\lambda}\mathring{g}_{ab}$, and $({}^2M, \mathring{g}_{ab})$ is a smooth Riemannian manifold which we shall assume to be a two-dimensional sphere (other topologies are considered in Ref. 21).

For metrics of the form (4.1), the Einstein vacuum equations reduce to a single parabolic evolution equation for the two-dimensional metric $g = g_{ab}dx^a dx^b$:

$$\partial_u g = \frac{\Delta R}{12m} g \ . \tag{4.2}$$

This is equivalent to a non-linear fourth order parabolic equation for the conformal factor λ. The Schwarzschild metric provides an example of a time-independent solution.

The Cauchy data for an RT metric consist of $\lambda_0(x^a) \equiv \lambda(u = u_0, x^a)$. Equivalently, one prescribes a metric $g_{\mu\nu}$ of the form (4.1) on the null hypersurface $\{u = u_0, x^a \in {}^2M, \ r \in (0,\infty)\}$. Note that this hypersurface extends up to a curvature singularity at $r = 0$, where the scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ diverges as r^{-6}. This is a 'white hole singularity", familiar from all known black hole spaces-times.

It is proved in Ref. 20 that, for $m > 0$, every such initial λ_0 leads to a black hole space-time. More precisely, one has the following: For any $\lambda_0 \in C^\infty(S^2)$ there exists a Robinson–Trautman space-time $(\mathscr{M}, g)$ with a "half-complete" $\mathcal{I}^+$, the global structure of which is shown in Figure 5. Moreover, there exist an infinite number of non-isometric vacuum

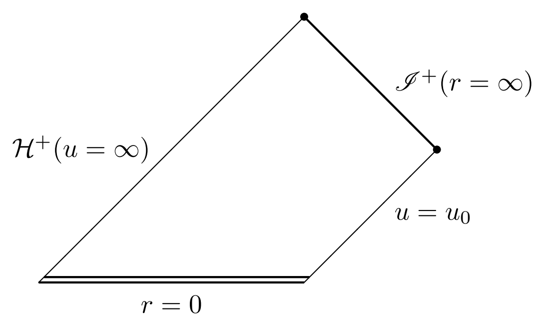

Fig. 5. The global structure of RT space-times with $m > 0$ and spherical topology.

Robinson–Trautman C^5 extensions[k] of $(\mathscr{M}, g)$ through $\mathscr{H}^+$, which are obtained by gluing to $(\mathscr{M}, g)$ any other Robinson–Trautman spacetime with the same mass parameter m, as shown in Figure 6. Each such extension leads to a black-hole space-time, in which $\mathscr{H}^+$ becomes a black hole event horizon. (There also exist an infinite number of C^{117} vacuum RT extensions of $(\mathscr{M}, g)$ through $\mathscr{H}^+$ — one such extension can be obtained by gluing a copy of $(\mathscr{M}, g)$ to itself. Somewhat surprisingly, no extensions of C^{123} differentiability class exist in general.)

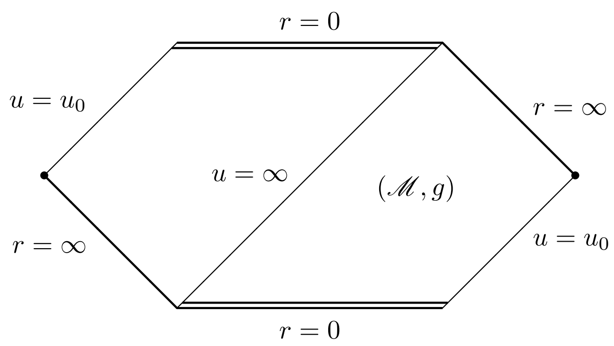

Fig. 6. Vacuum RT extensions beyond $\mathcal{H}^+$

5. Initial data sets containing trapped, or marginally trapped, surfaces

Let $\mathscr{T}$ be a *compact, $(n-1)$–dimensional, spacelike* submanifold in a $(n+1)$–dimensional space-time $(\mathscr{M}, g)$. We assume that there is a continuous choice ℓ of a field of future directed null normals to $\mathscr{T}$, which will be referred to as the *outer one*. Let e_i, $i = 1, \cdots, n-1$ be a local ON frame on $\mathscr{T}$, one sets

$$\theta_+ = \sum_{i=1}^{n-1} g(\nabla_{e_i}\ell, e_i) \ .$$

Then $\mathscr{T}$ will be called *future outer trapped* if $\theta_+ > 0$, and *marginally future outer trapped* if $\theta_+ = 0$. A marginally trapped surface lying within a spacelike hypersurface is often referred to as an *apparent horizon*.

[k]By this we mean that the metric can be C^5 extended beyond $\mathscr{H}^+$; the extension can actually be chosen to be of $C^{5,\alpha}$ differentiability class, for any $\alpha < 1$.

It is a folklore theorem in general relativity that, under appropriate global conditions, existence of a future outer trapped or marginally trapped surface implies that of a non-empty black hole region. So one strategy in constructing black hole space-times is to find initial data which will contain trapped, or marginally trapped, surfaces[6,8,38,39,68,72]

It is useful to recall how apparent horizons are detected using initial data: let $(\mathscr{S}, g, K)$ be an initial data set, and let $S \subset \mathscr{S}$ be a compact embedded two-dimensional two-sided submanifold in $\mathscr{S}$. If n^i is the field of outer normals to S and H is the outer mean extrinsic curvature[l] of S within $\mathscr{S}$ then, in a convenient normalisation, the divergence θ_+ of future directed null geodesics normal to S is given by

$$\theta_+ = H + K_{ij}(g^{ij} - n^i n^j) \ . \tag{5.1}$$

In the time-symmetric case θ_+ reduces thus to H, and S is trapped if and only if $H < 0$, marginally trapped if and only if $H = 0$. Thus, in this case apparent horizons correspond to compact minimal surfaces within $\mathscr{S}$.

It should be emphasised that the existence of disconnected apparent horizons within an initial data set does not guarantee, as of the time of writing this work, a multi-black-hole spacetime, because our understanding of the long time behavior of solutions of Einstein equations is way too poor. Some very partial results concerning such questions can be found in Ref. 32.

5.1. *Brill-Lindquist initial data*

Probably the simplest examples are the time-symmetric initial data of Brill and Lindquist. Here the space-metric at time $t = 0$ takes the form

$$g = \psi^{4/(n-2)}\Big((dx^1)^2 + \ldots + (dx^n)^2\Big) \ , \tag{5.2}$$

with

$$\psi = 1 + \sum_{i=1}^{I} \frac{m_i}{2|\vec{x} - \vec{x}_i|^{n-2}} \ .$$

The positions of the poles $\vec{x}_i \in \mathbb{R}^n$ and the values of the mass parameters $m_i \in \mathbb{R}$ are arbitrary. If all the m_i are positive and sufficiently small, then for each i there exists a small minimal surface with the topology of a sphere which encloses $\vec{x}_i$.[32] From Ref. 62, in dimension 3+1 the associated maximal

[l]We use the definition that gives $H = 2/r$ for round spheres of radius r in three-dimensional Euclidean space.

globally hyperbolic development possesses a $\mathscr{I}^+$ which is complete to the past. However $\mathscr{I}^+$ cannot be smooth,[64] and it is not known how large it is to the future. One expects that the intersection of the event horizon with the initial data surface will have more than one connected component for sufficiently small values of $m_i/|\vec{x}_k - \vec{x}_j|$, but this is not known.

5.2. *The "many Schwarzschild" initial data*

There is a well-known special case of (5.2), which is the space-part of the Schwarzschild metric centred at $\vec{x}_0$ with mass m :

$$g = \left(1 + \frac{m}{2|\vec{x} - \vec{x}_0|^{n-2}}\right)^{4/(n-2)} \delta \ , \tag{5.3}$$

where δ is the Euclidean metric. Abusing terminology in a standard way, we call (5.3) simply the Schwarzschild metric. The sphere $|\vec{x}-\vec{x}_0| = m/2$ is minimal, and the region $|\vec{x} - \vec{x}_0| < m/2$ corresponds to the second asymptotic region. This feature of the geometry, as connecting two asymptotic regions, is sometimes referred to as the *Einstein-Rosen bridge*, see Figure 2.

Now fix the radii $0 \le 4R_1 < R_2 < \infty$. Denoting by $B(\vec{a}, R)$ the open coordinate ball centred at $\vec{a}$ with radius R, choose points

$$\vec{x}_i \in \Gamma_0(4R_1, R_2) := \begin{cases} B(0, R_2) \setminus \overline{B(0, 4R_1)} \ , & R_1 > 0 \\ B(0, R_2) \ , & R_1 = 0 \ , \end{cases}$$

and radii r_i, $i = 1, \ldots, 2N$, so that the closed balls $\overline{B(\vec{x}_i, 4r_i)}$ are all contained in $\Gamma_0(4R_1, R_2)$ and are pairwise disjoint. Set

$$\Omega := \Gamma_0(R_1, R_2) \setminus \left(\cup_i \overline{B(\vec{x}_i, r_i)}\right) \ . \tag{5.4}$$

We assume that the $\vec{x}_i$ and r_i are chosen so that Ω is invariant with respect to the reflection $\vec{x} \to -\vec{x}$. Now consider a collection of nonnegative mass parameters, arranged into a vector as

$$\vec{M} = (m, m_0, m_1, \ldots, m_{2N}),$$

where $0 < 2m_i < r_i$, $i \ge 1$, and in addition with $2m_0 < R_1$ if $R_1 > 0$ but $m_0 = 0$ if $R_1 = 0$. We assume that the mass parameters associated to the points $\vec{x}_i$ and $-\vec{x}_i$ are the same. The remaining entry m is explained below.

Given this data, it follows from the work in Refs.[27,36] that there exists a $\delta > 0$ such that if

$$\sum_{i=0}^{2N} |m_i| \le \delta \ , \tag{5.5}$$

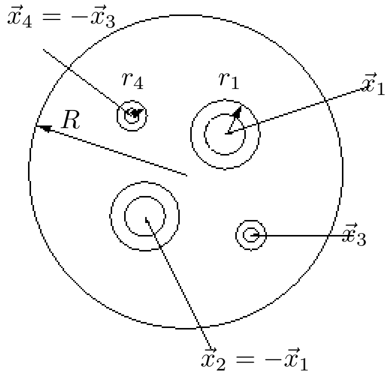

Fig. 7. "Many Schwarzschild" initial data with four black holes. The initial data are *exactly* Schwarzschild within the four innermost circles and outside the outermost one. The free parameters are R, $(\vec{x}_1, r_1, m_1)$, and $(\vec{x}_3, r_3, m_3)$, with sufficiently small m_a's. We impose $m_2 = m_1$, $r_2 = r_1$, $m_4 = m_3$ and $r_4 = r_3$.

then there exists a number

$$m = \sum_{i=0}^{2N} m_i + O(\delta^2)$$

and a C^∞ metric $\hat{g}_{\vec{M}}$ which is a solution of the time-symmetric vacuum constraint equation

$$R(\hat{g}_{\vec{M}}) = 0 \ ,$$

such that:

(1) On the punctured balls $B(\vec{x}_i, 2r_i) \setminus \{\vec{x}_i\}$, $i \geq 1$, $\hat{g}_{\vec{M}}$ is the Schwarzschild metric, centred at $\vec{x}_i$, with mass m_i;
(2) On $\mathbb{R}^n \setminus \overline{B(0, 2R_2)}$, $\hat{g}_{\vec{M}}$ agrees with the Schwarzschild metric centred at 0, with mass m;
(3) If $R_1 > 0$, then $\hat{g}_{\vec{M}}$ agrees on $B(0, 2R_1) \setminus \{0\}$ with the Schwarzschild metric centred at 0, with mass m_0.

By point (1) above each of the spheres $|\vec{x} - \vec{x}_i| = m_i/2$ is an apparent horizon.

A key feature of those initial data is that we have complete control of the *space-time metric* within the domains of dependence of $B(\vec{x}_i, 2r_i) \setminus \{\vec{x}_i\}$ and of $\mathbb{R}^n \setminus \overline{B(0, 2R_2)}$, where the space-time metric is a Schwarzschild metric.

Because of the high symmetry, one expects that "all black holes will eventually merge", so that the event horizon will be a connected hypersurface in space-time.

5.3. *Black holes and gluing methods*

A recent alternate technique for gluing initial data sets is given in Refs. 59. In this approach, general initial data sets on compact manifolds or with asymptotically Euclidean or hyperboloidal ends are glued together to produce solutions of the constraint equations on the connected sum manifolds. Only very mild restrictions on the original initial data are needed. The neck regions produced by this construction are again of Schwarzschild type. The overall strategy of the construction is similar to that used in many previous gluing constructions in geometry. Namely, one takes a family of approximate solutions to the constraint equations and then attempts to perturb the members of this family to exact solutions. There is a parameter η which measures the size of the neck, or gluing region; the main difficulty is caused by the tension between the competing demands that the approximate solutions become more nearly exact as $\eta \to 0$ while the underlying geometry and analysis become more singular. In this approach, the conformal method of solving the constraints is used, and the solution involves a conformal factor which is exponentially close to one (as a function of η) away from the neck region. It has been shown[30] that the deformation can actually be localised near the neck in generic situations.

Consider, now, an asymptotically flat time-symmetric initial data set, to which several other time-symmetric initial data sets have been glued by this method. If the gluing regions are made small enough, the existence of a non-trivial minimal surface, hence of an apparent horizon, follows by standard results. This implies the existence of a black hole region in the maximal globally hyperbolic development of the data.

It is shown in Ref. 32 that the intersection of the event horizon with the initial data hypersurface will have more than one connected component for several families of glued initial data sets.

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