Initial data engineering

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Abstract

We present a local gluing construction for general relativistic initial data sets. The method applies to *generic* initial data, in a sense which is made precise. In particular the trace of the extrinsic curvature is not assumed to be constant near the gluing points, which was the case for previous such constructions. No global conditions on the initial data sets such as compactness, completeness, or asymptotic conditions are imposed. As an application, we prove existence of spatially compact, maximal globally hyperbolic, vacuum space-times without any closed constant mean curvature spacelike hypersurface.

1 Introduction

Let (M_a, γ_a, K_a), $a = 1, 2$, be two (arbitrary dimensional) general relativistic initial data sets; by this we mean that each γ_a is a Riemannian metric on the n dimensional manifold M_a, while each K_a is a symmetric two-covariant tensor field on M_a. Such a data set is called vacuum data if it satisfies the vacuum Einstein constraint equations

$$R(\gamma) - (2\Lambda + |K|^2_\gamma - (\mathrm{tr}_\gamma K)^2) = 0 \tag{1.1}$$

$$D_i(K^{ij} - \mathrm{tr}_\gamma K \gamma^{ij}) = 0 \tag{1.2}$$

where $R(\gamma)$ is the scalar curvature (Ricci scalar) of the metric γ, and Λ is the cosmological constant. The *vacuum local gluing problem* can be formulated as follows:

> Let $p_a \in M_a$ be two points, and let M be the manifold obtained by removing from M_a geodesic balls of radius ϵ around the p_a, and gluing in a neck $\mathbb{S}^{n-1} \times I$, where I is an interval. Can one find vacuum initial data (γ, K) on M which coincide with the original vacuum data away from a small neighborhood of the neck?

It is natural to pose the same question in field theoretical models with matter. A formulation that avoids the issue of specifying the precise nature of the matter fields is obtained if we represent these fields in the initial data set by the matter energy density function ρ and the matter energy-momentum vector J, requiring that they satisfy the dominant energy condition[1]

$$\rho \geq |J| \,. \tag{1.3}$$

The Einstein-matter constraints then relate ρ and J to the gravitational fields via the following

$$16\pi\rho = R(\gamma) - (2\Lambda + |K|_\gamma^2 - (\mathrm{tr}_\gamma K)^2) \,, \tag{1.4}$$

$$16\pi J^j = 2D_i(K^{ij} - \mathrm{tr}_\gamma K \gamma^{ij}) \,. \tag{1.5}$$

As a variation of the local gluing problem, one has the *wormhole creation problem*, or the *vacuum wormhole creation* problem: one starts with a single initial data set $(\tilde{M}, \tilde{\gamma}, \tilde{K})$, and chooses a pair of points $p_a \in \tilde{M}$. As before one forms the new manifold M by replacing small geodesic balls around these points by a neck $S^{n-1} \times I$, and one asks for the existence of initial data on M which satisfy either the vacuum or the Einstein-matter constraints, and which coincide with the original data away from the neck region.

It is easily seen that for certain special sets of initial data, such constructions are not possible: consider, for example, the flat initial data set $(\mathbb{R}^3, \delta, 0)$ associated with Minkowski space-time. It follows from the positive energy theorem that this set of data cannot be glued to any data on a compact manifold without globally perturbing the metric (so that the mass is nonzero).

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[1]Recall that (1.3) might fail when quantum phenomena are taken into account. We note that the local gluing problem is trivial if no energy restrictions, or matter content restrictions, are imposed, as then both the metric and the extrinsic curvature can be glued together in many different ways.

The object of this work is to show that the above gluings can be performed for *generic* initial data sets. To make our notion of genericity precise, some terminology is needed. Let P denote the linearisation of the map which takes a set of data (g, K) to the constraint functions appearing in (1.1)-(1.2), and let P^* be its formal adjoint. Recall that a *Killing Initial Data* (KID) is defined as a solution (N, Y) of the set of equations $P^*(N, Y) = 0$. These equations are given explicitly by

$$0 = \begin{pmatrix} 2(\nabla_{(i}Y_{j)} - \nabla^l Y_l g_{ij} - K_{ij}N + \mathrm{tr}K\, Ng_{ij}) \\ \\ \nabla^l Y_l K_{ij} - 2K^l{}_{(i}\nabla_{j)}Y_l + K^q{}_l \nabla_q Y^l g_{ij} - \Delta N g_{ij} + \nabla_i \nabla_j N \\ +(\nabla^p K_{lp}g_{ij} - \nabla_l K_{ij})Y^l - N\mathrm{Ric}\,(g)_{ij} \\ +2N K^l{}_i K_{jl} - 2N(\mathrm{tr}\,K)K_{ij} \end{pmatrix} . \quad (1.6)$$

We shall denote by $\mathscr{K}(\Omega)$ the set of KIDs defined on an open set Ω (note that we impose no boundary conditions on (N, Y).) In a vacuum space-time $(\mathscr{M}, g)$ (possibly with non-zero cosmological constant) the KIDs on a spacelike hypersurface Ω are in one-to-one correspondence with the Killing vectors of g on the domain of dependence of Ω [21]. A similar statement, with an appropriately modified equation for the KIDs, holds in electro-vacuum for appropriately invariant initial data for the gravitational and electromagnetic fields. (The reader is referred to [6] for comments about such data for general matter fields.)

We note that the gluing problem is in fact a special case of the wormhole creation problem if one allows $\tilde{M}$ to be a non connected manifold. Hence from now on we shall assume that $\tilde{M}$ has either one or two components, with $p_a \in \tilde{M}$, $a = 1, 2$. The first main result of our paper concerns vacuum initial data:

THEOREM 1.1 *Let $(\tilde{M}, \tilde{\gamma}, \tilde{K})$ be a smooth vacuum initial data set, and consider two open sets $\Omega_a \subset \tilde{M}$ with compact closure and smooth boundary such that*

$$\text{the set of KIDs, } \mathscr{K}(\Omega_a), \text{ is trivial.}$$

Then for all $p_a \in \Omega_a$, $\epsilon > 0$ and $k \in \mathbb{N}$ there exists a smooth vacuum initial data set $(M, \gamma(\epsilon), K(\epsilon))$ on M such that $(\gamma(\epsilon), K(\epsilon))$ is ϵ-close to $(\tilde{\gamma}, \tilde{K})$ in a $C^k \times C^k$ topology away from $B(p_1, \epsilon) \cup B(p_2, \epsilon)$. Moreover $(\gamma(\epsilon), K(\epsilon))$ coincides with $(\tilde{\gamma}, \tilde{K})$ away from $\Omega_1 \cup \Omega_2$.

The hypothesis of smoothness has been made for simplicity. Similar results, with perhaps some finite loss in differentiability, can be obtained for initial data sets with finite Hölder or Sobolev differentiability.

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Some comments about the no-local-KIDs condition $\mathscr{K}(\Omega_a) = \{0\}$ are in order. As noted above, this is equivalent to the condition that there are no Killing vectors defined on the domain of dependence of the regions Ω_a in the associated vacuum space-time. First, the result is sharp in the following sense: as discussed above, initial data for Minkowski space-time cannot locally be glued to anything which is non-singular and vacuum. This meshes with the fact that for Minkowskian initial data, we have $\mathscr{K}(\Omega) \neq \{0\}$ for any open set Ω. Next, it is intuitively clear that for generic space-times there will be no locally defined Killing vectors, and several precise statements to this effect have been proved in [7]. Thus, our result can be interpreted as the statement that for generic vacuum initial data sets the local gluing can be performed around arbitrarily chosen points p_a. In particular it follows from the results in [7] that the collection of initial data with generic regions Ω_a satisfying the hypotheses of Theorem 1.1 is not empty. Further, it follows from the results here together with those in [18] and in [7] that the following initial data sets can always be glued together, near arbitrary points, after a (perhaps global) perturbation which is ϵ-small away from the gluing region:

- initial data containing an asymptotically flat region

- initial data containing a conformally compactifiable CMC region

- CMC initial data on a compact boundaryless manifold.

Let us denote by $\mathscr{N}(\Omega)$ the set of functions N satisfying, on Ω, the second of equations (1.6) with $K \equiv 0$. Theorem 1.1 has the following purely Riemannian "time-symmetric" counterpart:

THEOREM 1.2 *Let* $(\tilde{M}, \tilde{\gamma})$ *be a smooth Riemannian manifold with non-positive constant scalar curvature* ν, *and consider two open sets* $\Omega_a \subset \tilde{M}$ *with compact closure and smooth boundary such that*

$$\mathscr{N}(\Omega_a) = \{0\} \ .$$

Then for all $p_a \in \Omega_a$, $\epsilon > 0$ *and* $k \in \mathbb{N}$ *there exists a Riemannian manifold* $(M, \gamma(\epsilon))$ *with scalar curvature* ν *such that* $\gamma(\epsilon)$ *is* ϵ-close *to* $\tilde{\gamma}$ *in a* C^k *topology away from* $B(p_1, \epsilon) \cup B(p_2, \epsilon)$. *Moreover* $\gamma(\epsilon)$ *coincides with* $\tilde{\gamma}$ *away from* $\Omega_1 \cup \Omega_2$.

The proof is a simplified version of that of Theorem 1.1; we leave the details to the reader. This result is the local counterpart of the gluing theorem of Joyce [20] for $\nu < 0$, and appears to be completely new in the case $\nu = 0$.

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A noteworthy consequence of Theorem 1.1, proved in Section 5.1, is the following:

COROLLARY 1.3 *There exist vacuum maximal globally hyperbolic space-times with compact Cauchy surfaces which contain no compact boundaryless space-like hypersurfaces with constant mean curvature.*

It is clear that there exist equivalents of Theorem 1.1 in non-vacuum field theoretical models. However, proofs for such models require a case-by-case analysis of the corresponding gluing and KID equations. It is therefore noteworthy that one can make a general statement assuming only the dominant energy condition, which we use in its $(n+1)$-dimensional formulation:

$$T_{\mu\nu}X^\mu Y^\nu \geq 0 \quad \text{for all timelike future directed vectors } X^\mu \text{ and } Y^\mu . \quad (1.7)$$

The second main result of this paper reads:

THEOREM 1.4 *Consider a smooth solution $(\mathscr{M}, g)$ of the Einstein field equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$, with one or two connected components, and with matter fields satisfying the dominant energy condition (1.7). Let $\tilde{M}$ be a spacelike hypersurface in $\mathscr{M}$ with induced data $(\tilde{\gamma}, \tilde{K})$, and let*

$$p_a \in \tilde{M}, \ a = 1, 2, \ \text{be two points at which the inequality (1.7) is strict.}$$

Then for all $\epsilon > 0$ there exists a smooth initial data set $(M, \gamma(\epsilon), K(\epsilon))$ on M satisfying the dominant energy condition such that $(\gamma(\epsilon), K(\epsilon))$ coincides with $(\tilde{\gamma}, \tilde{K})$ away from $B(p_1, \epsilon) \cup B(p_2, \epsilon)$.

The reader will have observed that Theorem 1.1 concerns initial data sets only, while in Theorem 1.4 the starting point is a space-time. This is related to the fact that we have not made any assumptions on the matter fields except energy dominance. If $(\tilde{M}, \tilde{g}, \tilde{K})$ has constant mean curvature, then the proof of Theorem 1.4 is such that we could restate the result purely in terms of initial data, with no reference to the space-time $(\mathscr{M}, g)$.

Theorems 1.1 and 1.4 are established in Section 4. The proofs are a mixture of gluing techniques developed in [17–19] and those of [12–14]. In fact, the proof proceeds via a generalisation of the analysis in [18, 19] to compact manifolds with boundary; this is carried through in Section 2 in vacuum with cosmological constant $\Lambda = 0$, and in Section 3 with matter and $\Lambda \in \mathbb{R}$. These results may be of independent interest.

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2 The (global) gluing construction for the vacuum constraints with $\Lambda = 0$ for manifolds with boundaries

In this section we formulate some generalisations of the results in [18] and [19] to vacuum initial data sets on manifolds with boundary, with vanishing cosmological constant; the vacuum case with $\Lambda \neq 0$ will be covered by the analysis in Section 3. Although the paper [18] only treats the case $n = 3$, since the generalization to higher dimensions is not difficult (the necessary modifications are discussed in [17]), we work here in general dimension $n \geq 3$. We begin with an initial data set $(\tilde{M}, \tilde{\gamma}, \tilde{K})$ where $\tilde{M}$ has non-empty smooth boundary $\partial \tilde{M}$ and we assume first that $\tilde{K}$ has constant trace $\tilde{\tau} = \mathrm{tr}_{\tilde{\gamma}} \tilde{K}$ (i.e., these are constant mean curvature, or CMC, initial data sets). Decomposing $\tilde{K}$ into its trace and trace-free components, we write $\tilde{K} = \tilde{\mu} + \frac{\tilde{\tau}}{n} \tilde{\gamma}$. Since $\mathrm{tr}_{\tilde{\gamma}} \tilde{K}$ is constant, the vacuum momentum constraint equation implies that $\tilde{\mu}$ is divergence free as well (i.e., $\tilde{\mu}$ is a transverse–traceless tensor). We "mark" $\tilde{M}$ with two points p_a, $a = 1, 2$, about which we will perform the gluing. The global gluing construction can be carried out in this setting, with Dirichlet boundary conditions, generalizing the result of [18].

THEOREM 2.1 *Let $(\tilde{M}, \tilde{\gamma}, \tilde{K}; p_a)$ be a smooth, marked, constant mean curvature solution of the Einstein vacuum constraint equations with cosmological constant $\Lambda = 0$ on $\tilde{M}$, an n-manifold with boundary. Then there is a geometrically natural choice of a parameter T and, for T sufficiently large, a one-parameter family of solutions (M_T, Γ_T, K_T) of the Einstein constraint equations with the following properties. The n-manifold M_T is constructed from $\tilde{M}$ by adding a neck connecting the two points p_1 and p_2. For large values of T, the Cauchy data (Γ_T, K_T) is a small perturbation of the initial Cauchy data $(\tilde{\gamma}, \tilde{K})$ away from small balls about the points p_a. In fact, for any $\epsilon > 0$ and $k \in \mathbb{N}$ we have $(\Gamma_T, K_T) \to (\tilde{\gamma}, \tilde{K})$ as $T \to \infty$ in $C^k \left(\overline{M \setminus (B(p_1, \epsilon) \cup B(p_2, \epsilon))} \right)$.* [2]

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[2]One should note the absence of any nondegeneracy condition in Theorem 2.1. As is evident in the proof, this is accounted for by the imposition of Dirichlet boundary conditions.

PROOF: These solutions are constructed via the conformal method following the technique developed in [18]. The adaptation of the proof of Theorem 1 of [18] to allow for initial data on manifolds with boundary requires only minor variations which we indicate here. The construction begins with a conformal deformation of the initial data within small balls about the points p_a, $a = 1, 2$. The metric is conformally deformed to make deleted neighborhoods of these points asymptotically cylindrical. One then truncates these neighborhoods at a distance T (in the asymptotically cylindrical metric, for T large) and identifies the remaining ends to form the new manifold M_T with metric γ_T. The first variation in the proof occurs when deforming the approximate transverse–traceless μ_T formed by gluing the conformally transformed $\tilde{\mu}$ across the neck via cut-off functions. This requires solving (with appropriate estimates) the elliptic system

$$LX = W$$

where $W = \operatorname{div}_{\gamma_T} \mu_T$ is supported near the center of the asymptotic cylinder, $L = -\operatorname{div}_{\gamma_T} \circ \mathcal{D}$ and $\mathcal{D}X = \frac{1}{2}\mathcal{L}_X \gamma_T - \frac{1}{n}(\operatorname{div}_{\gamma_T} X)\gamma_T$ is the conformal Killing operator applied to the (unknown) vector field X. In [18] the required uniform invertibility of L is established under a nondegeneracy condition which amounts to the absence of conformal Killing vectors fields (which are in the kernel of L) vanishing at p_a. When $\tilde{M}$ has a non-empty boundary we are actually interested in solutions to the boundary value problem

$$\begin{cases} LX &=& W & \text{in} & M_T \\ X &=& 0 & \text{on} & \partial M_T. \end{cases} \tag{2.1}$$

The core to solvability of this problem is provided by Theorem 2 of [18]. The proof in the present setting is identical to the one there with the exception that the step where the nondegeneracy condition (Definition 1 of [18]) is evoked is now replaced by the nonexistence of conformal Killing fields which vanish on the boundary, $\partial \tilde{M}$ (see, *e.g.*, Proposition 6.2.2 of [1]). The required estimates on the solution follow from Corollary 1 of [18] coupled with the boundary Schauder estimates. Setting $\sigma_T = \mathcal{D}X$ and $\tilde{\mu}_T = \mu_T - \sigma_T$, we see that $\tilde{\mu}_T$ is our desired transverse-traceless tensor.

The other modification occurs in solving the nonlinear boundary value problem

$$\begin{cases} \mathcal{N}_T(\psi_T + \eta_T) &=& 0 & \text{in} & M_T \\ \eta_T &=& 0 & \text{on} & \partial M_T. \end{cases} \tag{2.2}$$

where η_T is presumed to be a small perturbation of an explicit approximate solution ψ_T, and $\mathcal{N}_T$ is the Lichnerowicz operator.

$$
\begin{aligned}
\mathcal{N}_T(\psi) \quad = \quad \Delta_T \psi - \frac{n-2}{4(n-1)} R_T \psi \quad &+ \frac{n-2}{4(n-1)} |\tilde{\mu}_T|^2 \psi^{\frac{-3n+2}{n-2}} \\
&- \frac{n-2}{4n} \tau^2 \psi^{\frac{n+2}{n-2}}.
\end{aligned}
\tag{2.3}
$$

Equation (2.2) is solved by means of a contraction mapping argument. The key ingredient is a good understanding of the linearised operator $\mathcal{L}_T$ on M_T. $\mathcal{L}_T$ is the operator

$$
\begin{aligned}
\mathcal{L}_T \quad = \quad \Delta_{\gamma_T} - \frac{n-2}{4(n-1)} \Big(R(\gamma_T) \quad &+ \frac{3n-2}{n-2} |\tilde{\mu}_T|^2 \, \psi_T^{-\frac{4(n-1)}{n-2}} \\
&+ \frac{(n-1)(n+2)}{n(n-2)} \, \tau^2 \, \psi_T^{\frac{4}{n-2}} \Big).
\end{aligned}
\tag{2.4}
$$

The basic point is to show that, corresponding to the solutions to the boundary value problem

$$
\begin{cases}
\mathcal{L}_T \eta &= \quad f \quad \text{in} \quad M_T \\
\eta &= \quad 0 \quad \text{on} \quad \partial M_T,
\end{cases}
$$

we have an isomorphism between certain weighted Hölder spaces on M_T where the weight factor controls decay/growth across the neck, and moreover for a certain range of weights, there is a T_0 such that this map has a uniformly bounded inverse for all $T \geq T_0$. The proof of this follows §5 of [18] and relies on the fact that the boundary value problem

$$
\begin{cases}
\Delta_\gamma \eta - \left(|\mu|_\gamma^2 + \frac{1}{n} \tau^2 \right) \eta &= \quad 0 \quad \text{in} \quad \tilde{M} \\
\eta &= \quad 0 \quad \text{on} \quad \partial \tilde{M}
\end{cases}
\tag{2.5}
$$

has no non-trivial solutions. The linear operator appearing in (2.5) is precisely the linearised Lichnerowicz operator about the original solution $(\tilde{M}, \tilde{\gamma}, \tilde{K})$.

Letting $\tilde{\psi}_T = \psi_T + \eta_T$ be the solution to (2.2) one finds that the desired solution to the constraint equations is then given by

$$
\Gamma_T = \tilde{\psi}_T^{\frac{4}{n-2}} \gamma_T \qquad \text{and} \qquad K_T = \tilde{\psi}_T^{-2} \tilde{\mu}_T + \frac{1}{n} \tau \tilde{\psi}_T^{\frac{4}{n-2}} \gamma_T.
$$

The fact that these solutions converge uniformly to the original initial data sets in $C^{k,\alpha}(\overline{\tilde{M}})$ away from small balls about the points p_1, p_2 follows from the calculations of §8 of [18] together with the boundary Schauder estimates. $\square$

The gluing construction of [19], which only requires the initial data to have constant mean curvature in small balls about the points at which the gluing is to be done, also easily generalizes to manifolds with boundary. To show this, we need to introduce the notion of *nondegeneracy* for solutions of the constraint equations which are not necessarily CMC on manifolds with boundary. We do this in the context of the conformal method for non CMC data, which works as follows: Given a fixed background metric γ, a trace-free symmetric tensor μ, and a function τ, if we can solve the coupled equations

$$\Delta_\gamma \phi - \frac{n-2}{4(n-1)} R_\gamma \phi + \frac{n-2}{4(n-1)} |\mu + \mathcal{D}W|^2 \phi^{\frac{-3n+2}{n-2}} - \frac{n-2}{4n} \tau^2 \phi^{\frac{n+2}{n-2}} = 0$$

$$LW - (\operatorname{div} \mu - \frac{n-1}{n} \phi^{\frac{2n}{n-2}} \nabla \tau) = 0$$

for a positive function ϕ and a vector field W, then the initial data

$$\tilde{\gamma} = \phi^{\frac{4}{n-2}} \gamma, \qquad \widetilde{K} = \phi^{-2}(\mu + \mathcal{D}W) + \frac{\tau}{n} \phi^{\frac{4}{n-2}} \gamma,$$

satisfies the ($\Lambda = 0$) vacuum Einstein constraints (1.1)-(1.2). The first of these is again referred to as the Lichnerowicz equation. We write this coupled system as $\mathcal{N}(\phi, W; \tau) = 0$. The mean curvature τ is emphasized here, while the dependence of $\mathcal{N}$ on γ and μ is suppressed. We are interested here in the boundary value problem

$$\begin{cases} \mathcal{N}(\phi, W; \tau) &= 0 \quad \text{in} \quad \tilde{M} \\ \phi &= 1 \quad \text{on} \quad \partial\tilde{M} \\ W &= 0 \quad \text{on} \quad \partial\tilde{M}. \end{cases} \tag{2.6}$$

The linearization $\mathcal{L}$ of $\mathcal{N}$ in the directions (ϕ, W) (but not τ) is of central concern. We consider this linearization relative to a specified choice of Banach spaces $\mathbf{X}$ and $\mathbf{Y}$, each consisting of scalar functions and vectors fields vanishing on the boundary. If our manifold is not compact then one would also build into $\mathbf{X}$ and $\mathbf{Y}$ appropriate asymptotic conditions.

Definition 1 A solution to the constraint equation boundary value problem (2.6), is nondegenerate with respect to the Banach spaces $\mathbf{X}$ and $\mathbf{Y}$ provided $\mathcal{L} : \mathbf{X} \to \mathbf{Y}$ is an isomorphism.

The main result of the first gluing paper [18] shows that any two nondegenerate solutions of the vacuum constraint equations with the same constant mean curvature τ can be glued. For compact CMC solutions on manifolds without boundary, nondegeneracy is equivalent to $K \not\equiv 0$ together with

the absence of conformal Killing fields, while asymptotically Euclidean or asymptotically hyperbolic CMC solutions are always nondegenerate (cf. §7 of [18]). In [19], using a definition of nondegeneracy similar to that stated above, we show how to glue non-CMC initial data sets, provided the data is CMC (same constant) near the gluing points. The argument from [19] readily applies to similar sets of non-CMC data on manifolds with boundary which are nondegenerate in the sense of Definition 1, yielding the following;

THEOREM 2.2 *Let $(\tilde{M}, \tilde{\gamma}, \tilde{K}; p_a)$ be a smooth, marked solution of the Einstein vacuum constraint equations with cosmological constant $\Lambda = 0$ on $\tilde{M}$, an n-manifold with boundary. We assume that the solution is nondegenerate and that the mean curvature, $\tau = \mathrm{tr}_\gamma K$ is constant in the union of small balls (of any radius) about the points p_a, $a = 1, 2$. Then there is a geometrically natural choice of a parameter T and, for T sufficiently large, a one-parameter family of solutions (M_T, Γ_T, K_T) of the Einstein constraint equations with the following properties. The n-manifold M_T is constructed from $\tilde{M}$ by adding a neck connecting the two points p_1 and p_2. For large values of T, the Cauchy data (Γ_T, K_T) is a small perturbation of the initial Cauchy data $(\tilde{\gamma}, \tilde{K})$ away from small balls about the points p_a. In fact, for any $\epsilon > 0$ and $k \in \mathbb{N}$ we have $(\Gamma_T, K_T) \rightarrow (\tilde{\gamma}, \tilde{K})$ as $T \rightarrow \infty$ in $C^k \left(\overline{M \setminus (B(p_1, \epsilon) \cup B(p_2, \epsilon))} \right).$*

3 The (global) gluing construction for the Einstein-matter constraints for manifolds with boundaries

We need to show that the gluing construction which we have just described for the Einstein vacuum constraints on a manifold with boundary can be extended to the case of the Einstein-matter constraints (1.4), including a cosmological constant Λ. In [17], Isenberg, Maxwell and Pollack describe in detail how to carry out gluing constructions analogous to that of [18] for solutions of the constraints for Einstein's theory coupled to a wide variety of source fields (Maxwell, Yang-Mills, fluids, *etc.*) on complete manifolds. Here, we briefly describe how this works, and we adapt these results to the case of a manifold with boundary. For present purposes, we ignore any extra constraints which might have to be satisfied by the matter fields, and we describe those fields exclusively in terms of their stress-energy contributions[3]

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[3]This can be interpreted as a perfect fluid model. However, we are not making any hypotheses upon the dynamics of the matter fields.

ρ and J^i. These are required to satisfy the dominant energy condition (1.3). We also allow for the inclusion of a non-zero cosmological constant Λ.

So we start with a set of initial data $(\tilde{M}, \tilde{\gamma}, \tilde{K}, \tilde{\rho}, \tilde{J}, \tilde{\Lambda})$ which satisfies the constraint equations (1.4)-(1.5) on $\tilde{M}$, an n-dimensional manifold with smooth non-empty boundary. We presume that this set of data has constant mean curvature $\tilde{\tau}$, from which it follows that if we do a trace decomposition $\tilde{K} = \tilde{\nu} + \frac{1}{3}\tilde{\tau}\tilde{\gamma}$, with $\tilde{\tau} = \operatorname{tr}_{\tilde{\gamma}}\tilde{K}$, the trace-free field $\tilde{\nu}$ satisfies the condition

$$\tilde{\nabla}_j \tilde{\nu}^{ji} = 8\pi \tilde{J}^i.$$

Along with the initial data, we specify the pair of points $p_1, p_2 \in M$ at which we carry out the gluing.

We recall that the first step of the gluing construction for the vacuum constraints in [18] involves a conformal blowup of the gravitational fields at each of the points p_1 and p_2, followed by gluing these fields together using cutoff functions along the join of the two asymptotic cylinders created by this blowup. For the non vacuum case, we need to conformally transform and glue the matter quantities ρ and J as well. The conformal transformations which keep the conformal constraints semi-decoupled for CMC data, which preserve the dominant energy condition, and which lead to the simplest form for the Lichnerowicz equation, are $\tilde{\rho} \to \phi^8 \tilde{\rho}$ and $\tilde{J}^i \to \phi^{10} \tilde{J}^i$ (coupled with $\tilde{\gamma}_{ij} \to \phi^{-4}\tilde{\gamma}_{ij}$, $\tilde{\nu}_{ij} \to \phi^2 \tilde{\nu}_{ij}$, and $\tilde{\tau} \to \tilde{\tau}$, together with $\tilde{\Lambda} \to \tilde{\Lambda}$.) As for the gluings, we also apply a simple cutoff function procedure to $\phi^8 \tilde{\rho}$ and $\phi^{10} \tilde{J}^i$, thereby producing the smooth fields $\tilde{\rho}_T$ and $\tilde{J}_T$ on M_T, along with $\tilde{\gamma}_T, \tilde{\nu}_T$, and the constants $\tau = \tilde{\tau}$ and $\Lambda = \tilde{\Lambda}$.

The next step is finding the traceless tensor $\tilde{\sigma}_T$ which satisfies the momentum constraint

$$\nabla_j^{(\gamma_T)} \tilde{\sigma}_T^{ji} = 8\pi \tilde{J}_T^i. \tag{3.1}$$

Here $\nabla^{(\gamma_T)}$ is the Levi-Civita covariant derivative of the metric γ_T. We obtain $\tilde{\sigma}_T$ by solving the boundary value problem

$$\begin{cases} LX &= V &\text{in} &M_T \\ X &= 0 &\text{on} &\partial M_T. \end{cases} \tag{3.2}$$

with L as in Section 2, and with

$$V = J_T - div_{\gamma_T}\tilde{\nu}_T,$$

and then setting $\tilde{\sigma}_T = \tilde{\nu}_T + \mathcal{D}X$ (recall that $\mathcal{D}$ has been defined in the paragraph preceding (2.1)). Noting that V is supported near the points p_1 and p_2, we readily verify that the arguments for solvability of the boundary

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value problem (2.1) in Section 2 apply here as well. We also obtain the required pointwise estimates for $\tilde{\sigma}_T - \tilde{\nu}_T$.

The remaining step in the gluing construction of [18] involves solving the Lichnerowicz equation and then obtaining the requisite estimates for the solution ψ_T (ie, showing that away from the neck, $\psi_T \to 1$ in a suitable sense). For the Einstein-matter constraints (1.4)-(1.5), with the decompositions described here, the Lichnerowicz operator takes the form (compare (2.3))

$$
\mathcal{N}_T(\psi) = \Delta_T \psi - \frac{n-2}{4(n-1)} R_T \psi + \frac{n-2}{4(n-1)} |\tilde{\sigma}_T|^2 \psi^{\frac{-3n+2}{n-2}}
$$
$$
+ \frac{4\pi(n-2)}{n-1} \rho_T \psi^{\frac{18-7n}{n-2}} - (n-2)\Big(\frac{1}{4n}\tau^2 - \frac{1}{2(n-1)}\Lambda\Big)\psi^{\frac{n+2}{n-2}} . \quad (3.3)
$$

The matter-related term in (3.3), $2\pi\rho_T\psi^{-3}$, causes very few changes in the analysis. We note, for example, that in the expression for the linearised Lichnerowicz operator

$$
\mathcal{L}_T = \Delta_{\gamma_T} - \Big(\frac{n-2}{4(n-1)} R(\gamma_T) + \frac{-3n+2}{4(n-1)} |\tilde{\sigma}_T|^2 \, \psi_T^{\frac{4(-n+1)}{n-2}}
$$
$$
+ \frac{4\pi(18-7n)}{n-1} \pi\rho_T\psi_T^{\frac{20-8n}{n-2}} + (n+2)\Big(\frac{1}{4n}\tau^2 - \frac{1}{2(n-1)}\Lambda\Big)\psi_T^{\frac{4}{n-2}}\Big),
$$

the ρ term has very much the same effect as does the σ term, so its presence does not alter the proof of the existence of a solution or the subsequent error analysis.

The constant Λ, on the other hand, can cause trouble. However the argument presented in Section 2 shows that $\mathcal{L}_T$ has a uniformly bounded inverse for T sufficiently large provided that

$$
(\mathrm{tr}_g K)^2 \geq \frac{2n}{(n-1)}\Lambda , \quad (3.4)
$$

If this condition holds, then the rest of the analysis goes through. We thus have, finally

THEOREM 3.1 *Let $(M, \gamma, K, \rho, J, \Lambda; p_1, p_2)$ be a smooth, marked, constant mean curvature solution of the Einstein matter constraint equations on M, an n-manifold with boundary. We assume that (3.4) holds, and that the dominant energy condition $\rho \geq |J|$ holds. Then there is a geometrically natural choice of a parameter T and, for T sufficiently large, a one-parameter family of solutions $(M_T, \Gamma_T, K_T, \rho_T, J_T, \Lambda)$ of the Einstein constraint equations with the following properties. The n-manifold M_T is constructed from*

M by adding a neck connecting the two points p_1 and p_2. For large values of T, the Cauchy data $(\Gamma_T, K_T, \rho_T, J_T, \Lambda)$ is a small perturbation of the initial Cauchy data $(\gamma, K, \rho, J, \Lambda)$ away from small balls about the points p_1, p_2. In fact, for any $\epsilon > 0$ and $k \in \mathbb{N}$ we have $(\Gamma_T, K_T, \rho_T, J_T, \Lambda) \to (\gamma, K, \rho, J, \Lambda)$ as $T \to \infty$ in $C^k\left(\overline{M} \setminus (B(p_1, \epsilon) \cup B(p_2, \epsilon))\right)$.

We note, without further discussion, that one can also readily produce a theorem analogous to Theorem 2.2, but with matter included in the constraint equations.

4 Proof of Theorems 1.1 and 1.4

In the vacuum case let $(\mathcal{M}, g)$ be the maximal globally hyperbolic vacuum development of the initial data $(\tilde{M}, \tilde{\gamma}, \tilde{K})$; in the non-vacuum case let $(\mathcal{M}, g)$ be the development of the data, the existence of which has been assumed. In the vacuum case $\tilde{M}$ is achronal in $\mathcal{M}$ by construction; in the non-vacuum case this can be achieved, without loss of generality, by passing to a subset of $\mathcal{M}$.

There exists $r_0 > 0$ such that for all $0 < r \leq r_0$, the open geodesic balls $B(p_a, r)$ in $(\tilde{M}, \tilde{\gamma})$ have smooth boundaries and relatively compact domains of dependence in $(\mathcal{M}, g)$. In the setting of Theorem 1.4 we set $\Omega_a = B(p_a, r_0)$. By reducing r_0 if necessary we can assume that $\rho > |J|$ on the domains of dependence $\mathscr{D}(\Omega_a)$. Without loss of generality we can further assume that $r_0 \leq \epsilon/2$, where ϵ is the radius chosen in the statement of the theorems.

By a result in [7], we can make an ϵ-small deformation of the initial data, supported in $\Omega_1 \cup \Omega_2$, such that the deformed initial data set satisfies the dominant energy condition, remains vacuum if it was to begin with, still satisfies $\mathscr{K}(B(p_a, r_0)) = \{0\}$, and now moreover there exists an R such that for every r_- and r_+ satisfying $0 < r_- < r_+ < R < r_0$, we have

$$\mathscr{K}(\Gamma(p_a, r_-, r_+)) = \{0\}, \tag{4.1}$$

where $\Gamma(p_a, r_-, r_+) := B(p_a, r_+) \setminus \overline{B(p_a, r_-)}$. (In fact, the deformation can be arranged so that $\mathscr{K}(\mathscr{U}) = \{0\}$ for any open set $\mathscr{U} \subset B(p_a, r_0)$.) In vacuum, replacing Ω_a with $B(p_a, r_0)$ if necessary, we may work in $B(p_a, r_0)$ with r_0 being taken as small as desired. We assume in what follows that this is the case.

For any set Ω with a distance function d, we define

$$\Omega(s) := \{p \in \Omega : d(p, \partial\Omega) < s\}\;;$$

the sets Ω considered here will always be equipped with a Riemannian metric, and then d will be taken to be the distance function associated with this metric. In particular we thus have $\Omega_a(s) = \Gamma(p_a, r_0 - s, r_0)$.

Let us denote by (γ_a, K_a) the initial data induced on Ω_a. We next wish to reduce the problem to that in which $(\Omega_a, \gamma_a, K_a)$ have constant (sufficiently large) mean curvature. We choose a constant τ so that

$$\tau^2 - \frac{2n}{(n-1)}\Lambda \geq 0 \; . \tag{4.2}$$

As the domains of dependence $\mathscr{D}(\Omega_a)$ are compact, we can use the work of Bartnik [2, Theorem 4.1] to conclude that there exist smooth spacelike hypersurfaces $\widehat{\Omega}_a \subset \mathscr{M}$, with boundaries $\partial\Omega_a$, on which the induced data (γ_a, K_a) satisfy

$$\mathrm{tr}_{\gamma_a} K_a = \tau \; . \tag{4.3}$$

In the Einstein matter case, with $\rho > |J|$, we appeal to the results in [7] to obtain a small perturbation of the data induced on $\hat{\Omega}_a$, preserving (4.3), such that there are no KIDs on any open subset of the regions $\hat{\Omega}_a$. By continuity the dominant energy condition $\rho > |J|$ will still hold provided the perturbation is small enough.

In the vacuum case, we claim that the domains $\widehat{\Omega}_a$ have no local KIDs on every collar neighborhood of their boundary. Indeed, suppose that this is not the case. Then there exists a collar neighborhood, say $\widehat{\Omega}_1(s) \subset \widehat{\Omega}_1$, with a non trivial set of KIDs there. Therefore there exists a non-trivial Killing vector field X on the domain of dependence $\mathscr{D}(\widehat{\Omega}_1(s))$. But the intersection

$$\mathscr{D}(\widehat{\Omega}_1(s)) \cap \Omega_1$$

contains some collar neighborhood $\Omega_1(s_1)$, and therefore X induces a KID there, contradicting (4.1).

For all $s_0 > 0$ the argument just given also guarantees the existence of an s_1 satisfying $0 < s_1 < s_0$ such that

$$\mathscr{K}\left(\widehat{\Omega}_a(s_0) \setminus \overline{\widehat{\Omega}_a(s_1)}\right) = \{0\} \; . \tag{4.4}$$

The process described so far reduces the problem to one with CMC initial data satisfying (4.2)-(4.4), on a compact manifold with boundary. (As pointed out in the introduction, the hypothesis of existence of the associated space-time, made in Theorem 1.4, is not needed for such data.) Choose, now, a pair of points $\hat{p}_a \in \widehat{\Omega}_a \setminus \partial\widehat{\Omega}_a$. Applying Theorem 2.1 in vacuum or Theorem 3.1 with matter to $(\widehat{\Omega}_a, \gamma_a, K_a)$ for any sufficiently small ϵ, we

obtain a glued initial data set $(\widehat{M}, \gamma(\epsilon), K(\epsilon))$, where $\widehat{M}$ is the manifold, with boundary $\partial \widehat{M} = \partial \widehat{\Omega}_1 \cup \partial \widehat{\Omega}_2$, which is the connected sum of $\widehat{\Omega}_1$ and $\widehat{\Omega}_2$ across a small neck around the points $\hat{p}_a$. Let $s_0 > 0$ be any number such that $\hat{p}_a \notin \widehat{\Omega}_a(s_0)$. On $\widehat{\Omega}_a(s_0)$ the deformed data $(\gamma(\epsilon), K(\epsilon))$ arising from Theorem 2.1 approach (γ_a, K_a) in any C^k norm as ϵ goes to zero. As a consequence of (4.4), the construction presented in Section 8.6 of [12] can be carried through at fixed ρ and J^i and it gives, for all ϵ small enough, a smooth deformation of $(\gamma(\epsilon), K(\epsilon))$ on $\widehat{M}$, which coincides with (γ_a, K_a) on $\widehat{\Omega}_a(s_1)$, and coincides with $(\gamma(\epsilon), K(\epsilon))$ away from $\widehat{\Omega}_a(s_0)$. The deformation preserves the strict dominant energy condition (reducing ϵ if necessary), or is vacuum if the original data were.

Consider, finally, the manifold M which is obtained by gluing together $\tilde{M} \setminus (\Omega_1 \cup \Omega_2)$ and $\widehat{M}$, across $\partial \widehat{M}$. M carries an obvious initial data set (γ, K), which is smooth except perhaps at the gluing boundary $\partial \widehat{M}$, at which both γ and some components of K are at least continuous. But in a neighborhood of $\partial \widehat{M}$, bounded away from the neck region, the data (γ, K) coincide with those arising from a continuous, piecewise smooth hypersurface in $\mathcal{M}$, which consists of a gluing of $\tilde{M}$ on one side of $\partial \widehat{M}$, with $\widehat{\Omega}_a$ on the other. If we smooth out that hypersurface in $\mathcal{M}$ around $\partial \widehat{\Omega}_a$, then the new data near $\partial \Omega_a$ arising from the smoothed-out hypersurface, provides a smoothing of the initial data constructed so far. $\qquad\square$

5 Applications

5.1 Vacuum space-times without CMC surfaces

In [3] Bartnik has constructed an inextendible spatially compact space-time, satisfying the dominant energy condition, which has no closed CMC hypersurfaces (see also [11, 19, 22]). Here, using a construction analogous to that proposed by Eardley and Witt [15], we prove a similar result (Corollary 1.3) for *vacuum* spacetimes. The key step is proving the existence of vacuum initial data on a connected copy of T^3 with itself, with the property that the metric is symmetric under a reflection across the middle of the connecting neck, while K changes sign under this reflection. The non-existence of closed CMC surfaces in the maximal globally hyperbolic development of those initial data follows then from the arguments presented in [3].

Let $\hat{\gamma}$ be any metric on $M = T^3$ which has no conformal Killing vectors (such metrics exist, *e.g.* by [7]), let $\hat{\mu} \not\equiv 0$ be any transverse traceless tensor on M (such tensors exist, *e.g.* by [8]), and let $\hat{K} = \hat{\mu} + \tau\hat{\gamma}$, for some constant

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$\tau \neq 0$. It follows, *e.g.* from [16], that the conformal method applies, leading to a vacuum initial data set $(\tilde{\gamma}, \tilde{K})$, with $\tilde{\gamma}$ being a conformal deformation of $\hat{\gamma}$. Now, it is easily checked that for CMC data $(\tilde{\gamma}, \tilde{K})$ on a closed manifold, a KID (N, Y) must have $N = 0$ and must have Y a Killing vector field of $\tilde{\gamma}$. Consequently, Y is a conformal Killing vector for $\hat{\gamma}$, so that $Y = 0$ by our hypothesis on $\hat{\gamma}$. It follows that $(\tilde{\gamma}, \tilde{K})$ does not have any nontrivial global KIDs; *i.e.*, $\mathscr{K}(M) = \{0\}$.

Let $(\mathscr{M}, g)$ be the maximal globally hyperbolic development of the data. As in Section 4, we can deform the initial data hypersurface in $\mathscr{M}$ to create a small neighborhood of a point p in which the trace of the new induced $\tilde{K}$ vanishes, while maintaining the condition $\mathscr{K}(M) = \{0\}$. We use the same symbols $(\tilde{\gamma}, \tilde{K})$ to denote the new data.

Now let $\tilde{M}$ consist of two copies of M, with initial data $(\tilde{\gamma}, \tilde{K})$ on the first copy, say M_1, and with data $(\tilde{\gamma}, -\tilde{K})$ on the second copy, say M_2. We let $\Omega_a = M_a$, and we let p_a denote the points in M_a corresponding to p. Noting that the mean curvature vanishes in symmetric neighborhoods of p_1 and p_2, we now apply the construction for Theorem 1.1 presented in Section 4. To produce the desired initial data set on $T^3 \# T^3$, it is crucial to verify that all the steps are done with the correct symmetry around the middle of the connecting neck. In particular, we must check that the glued solution obtained from Theorem 2.1, when applied to such initial data, leads to a solution of the constraints which has the desired symmetry: this is achieved by using approximate solutions with the correct symmetry in the construction used to prove Theorem 2.1. The end result follows from the uniqueness (within the given conformal class) of the solutions obtained there. We thus have verified Corollary 1.3.

5.2 Bray's quasi-local inner mass

In [9] Bray defines a notion of "inner mass" for a surface Σ which is outer-minimising with respect to area in an asymptotically Euclidean initial data set (M, γ, K) satisfying the dominant energy condition (1.3) (see also [10]). Given a surface $\Sigma \subset M$ which is outer-minimising with respect to a fixed asymptotically flat end of (M, γ), define the region I 'inside' Σ to be the union of the components of $M \setminus \Sigma$ containing all the ends of M except for the chosen one. The inner mass $m_{inner}(\Sigma)$ is then defined to be the supremum of $\sqrt{A/16\pi}$ taken over all fill-ins of Σ (or replacements of I with the data on $M \setminus I$ unchanged) with initial data sets satisfying (1.3), where A is the minimum area of the fill-in needed to enclose all the ends other than the chosen 'exterior' end. Note the similarity of this definition to Bartnik's

notion of quasi-local mass [4, 5].

It is by no means clear that extremal data which realise $m_{inner}(\Sigma)$ exist. However Bartnik has observed that the construction described in this paper results in the following:

THEOREM 5.1 *Suppose that* (M, γ, K) *is an asymptotically flat initial data set which realises the inner mass* $m_{inner}(\Sigma)$ *for an outer minimising surface* $\Sigma \subset M$. *Thus there is a surface* $S \subset I$, *the interior region of* (M, γ, K) *relative to* Σ, *such that* $A = Area(S)$ *satisfies* $m_{inner}(\Sigma) = \sqrt{A/16\pi}$. *If there is an open set* $\Omega \subset I$ *satisfying* $\partial\Omega = S \cup \Sigma$, *then there is at least one non-trivial KID on* Ω *ie.* $\mathscr{K}(\Omega) \neq \{0\}$. *In particular, in the time-symmetric case,* $K \equiv 0$, *the resulting vacuum space-time is static in the domain of dependence of* Ω.

The proof of Theorem 5.1 is an immediate consequence of the fact that were there to be no KIDS on Ω, $\mathscr{K}(\Omega) = \{0\}$, we could apply Theorem 1.1 and locally glue in an additional black hole whose apparent horizon would contribute an additional area to A. This would contradict the assumption that the original data was extremal for the inner mass.

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