Gluing Initial Data Sets for General Relativity

Piotr T. Chruściel*
Départmént de Mathématiques, Faculté des Sciences,
Université de Tours, Parc de Grandmont, F37200, Tours, France

James Isenberg†
Department of Mathematics, University of Oregon, Eugene, Oregon 97403-5203, USA

Daniel Pollack‡
Department of Mathematics, University of Washington,
Box 354350, Seattle, Washington 98195-4350, USA
(Dated: 18 August, 2004)

We establish an optimal gluing construction for general relativistic initial data sets. The construction is optimal in two distinct ways. First, it applies to *generic* initial data sets and the required (generically satisfied) hypotheses are geometrically and physically natural. Secondly, the construction is completely *local* in the sense that the initial data is left unaltered on the complement of arbitrarily small neighborhoods of the points about which the gluing takes place. Using this construction we establish the existence of cosmological, maximal globally hyperbolic, vacuum space-times with no constant mean curvature spacelike Cauchy surfaces.

PACS numbers: 04.20.Ex

INTRODUCTION

As with Maxwell's theory, a set of initial data for Einstein's theory of gravity must satisfy a set of constraint equations. Unlike the Maxwell constraints $\nabla \cdot E = \rho$ and $\nabla \cdot B = 0$, the Einstein constraint equations are nonlinear and fairly complicated. Hence, although much has been learned about solutions of the Einstein constraint equations during the past thirty years using the conformal method and related techniques, many questions concerning them have remained formidable. A number of these questions can now be answered using a powerful tool from geometric analysis: Gluing. Analytic gluing techniques have played a prominent role in many areas of differential geometry over the past twenty years. Some particularly notable applications include: the study of smooth topology of four manifolds (via Donaldson and later Seiberg-Witten theory), pseudo-holomorphic curves in symplectic geometry, the existence of half-conformally flat structures on four manifolds, manifolds with exceptional holonomy, singular Yamabe metrics and the study of minimal and constant mean curvature surfaces in Euclidean three-space, and many others.

It is only recently that these techniques have been applied to general relativity, and the impact of this work has been significant. In the first two applications gluing techniques were used in very different ways toward quite different aims. In [1], Corvino applied a new gluing method to asymptotically flat, time symmetric initial data. The core feature of [1] is a local deformation result for the scalar curvature operator. By exploiting the underdetermined nature of this operator (or more precisely, the fact that its adjoint is overdetermined) Corvino is able to show that one can solve for prescribed, small *compactly supported* deformations of the scalar curvature. This deformation result is used in [1] to prove the existence of asymptotically flat, scalar flat metrics on $\mathbb{R}^n$ ($n \geq 3$) which are Schwarzschild outside of a compact set. The evolution of this initial data produces nontrivial space-times which are identically Schwarzschild near spatial infinity. In the more general setting of constant mean curvature initial data sets a gluing construction was developed [2], in the context of the well known conformal method of Lichnerowicz, Choquet-Bruhat and York which reduces the constraints equations to a determined elliptic system. The construction of [2], and subsequently [3], allowed one to demonstrate how space-times can be joined by means of a geometric connected sum, or how a wormhole can be added between two points in a given space-time (on the level of the initial data). This was flexible enough to address a number of issues concerning the relation of the spatial topology to the geometry of solutions of the constraints and the constructibility of multi black hole solutions (see also [4]).

We have subsequently developed this technique so that it can be applied to a much wider class of solutions; indeed, we have now obtained the sharpest possible gluing theorem for the Einstein constraint equations. Using it, we can do all of the following: 1) Show that for a generic solution of the constraint equations and any pair of points in this solution, one can add a wormhole connecting these points to the solution with *no* change in the data away from a neighborhood of each of the points. 2) Show that for almost any pair of initial data sets (including, say, a pair of black hole data sets, or a cosmological data set paired with a set of black hole data) one can construct a new set which joins them. 3) Prove that there exist spatially compact maximal globally hyperbolic space-times which satisfy the vacuum Einstein equations and contain no closed constant mean curvature hypersurface. It is likely that these new gluing techniques will continue to be very useful for the practical construction of physically interesting initial data sets.

MAIN RESULTS

We recall that a set of initial data (M^n, γ, K, Ψ) for Einstein's theory consists of an n-manifold M^n, a Riemannian metric γ on M^n, a symmetric tensor field K on M^n, and possibly a set of non-gravitational fields Ψ (e.g., E and B for Einstein-Maxwell). The constraint equations require that (M^n, γ, K, Ψ) satisfy

$$16\pi\rho = R(\gamma) - (2\Lambda + |K|^2_\gamma - (\mathrm{tr}_\gamma K)^2) \ , \qquad (1)$$

$$16\pi J^j = 2D_i(K^{ij} - \mathrm{tr}_\gamma K \gamma^{ij}) \ . \qquad (2)$$

where D is the covariant derivative corresponding to γ, R is its scalar curvature, $\rho = \rho(\gamma, \Psi)$ is the energy density function of the non-gravitational fields, and $J = J(\gamma, \Psi)$ is the corresponding momentum flow vector field [**?**]. Now let (M^n, γ, K, Ψ) be a solution of (1)-(2), and let p_1 and p_2 be a pair of points contained in M^n. The basic idea of gluing is simple: let $\tilde{M}$ be the manifold obtained by removing from M geodesic balls of radius ϵ around p_1 and p_2, and gluing in a neck $\mathbb{S}^{n-1} \times I$, One then tries to construct initial data $(\tilde{\gamma}(\epsilon), \tilde{K}(\epsilon), \tilde{\Psi}(\epsilon))$ on $\tilde{M}$ which coincide with the original data away from a small neighborhood of the neck. If the points lie in distinct connected components of M, then the manifold $\tilde{M}$ is the connected sum of those components. If the points lie in the same component, then $\tilde{M}$ consists of M together with a "handle" or wormhole connecting neighborhoods of the two points.

We cannot expect to be able to glue every pair of solutions of the constraints. For example, if we could *locally* glue a set of data for Minkowski space to a solution of the constraints on a manifold which does not admit a flat metric, then the resulting data would have zero ADM mass, and yet would not be data for Minkowski space, thereby violating the positive energy theorem [5]. So there are necessarily conditions a data set must satisfy if it is to admit a gluing construction as above.

As noted above, in earlier work, it was required that a solution have constant mean curvature (CMC) [2], or at least have a CMC region surrounding each of the chosen gluing points [3]. Further, global "nondegeneracy" conditions needed to be satisfied as well. Consequently, gluing could not be applied generically. Our results here are much less restrictive. The condition which must be met is local, in the sense that it only involves the data in regions close to the gluing points. Further, the condition is satisfied at all points in generic solutions.

To define the gluing condition in the vacuum case, we fix a solution (M^n, γ, K) of the vacuum constraint equations (1)-(2) (with $\rho = 0$, and $J = 0$), and consider the L^2 adjoint $\mathcal{P}^*_{(\gamma,K)}$ of the linearization of the constraint equations at this solution. Viewed as an operator acting on a scalar function N and a vector field Y, $\mathcal{P}^*_{(\gamma,K)}$ takes the explicit form

$$\mathcal{P}^*_{(\gamma,K)}(N,Y) = \begin{pmatrix} 2(\nabla_{(i}Y_{j)} - \nabla^l Y_l g_{ij} - K_{ij}N + \mathrm{tr}K\, N g_{ij}) \\ \\ \nabla^l Y_l K_{ij} - 2K^l{}_{(i}\nabla_{j)}Y_l + K^q{}_l \nabla_q Y^l g_{ij} - \Delta N g_{ij} + \nabla_i\nabla_j N \\ +(\nabla^p K_{lp} g_{ij} - \nabla_l K_{ij})Y^l - N\mathrm{Ric}\,(g)_{ij} + 2NK^l{}_i K_{jl} - 2N(\mathrm{tr}\,K)K_{ij} \end{pmatrix} . \qquad (3)$$

Now let Ω be an open subset of M^n. By definition, the set of "KIDs" on Ω, denoted $\mathscr{K}(\Omega)$, is the set of all solutions of the equation

$$\mathcal{P}^*_{(\gamma,K)|\Omega}(N,Y) = 0. \qquad (4)$$

Such a solution (N, Y), if nontrivial, generates a space-time Killing vector field [**?**] in the domain of dependence of $(\Omega, \gamma|_\Omega, K|_\Omega)$ [6]. In terms of KIDs, our only "nondegeneracy condition" is simply that there exist neighborhoods $\Omega_1 \ni p_1$ and $\Omega_2 \ni p_2$ of the gluing points such that the KIDs (with no boundary conditions imposed) are all trivial on Ω_1 and Ω_2.

We can now formally state our main result, for vacuum initial data:

THEOREM 1 *Let (M, γ, K) be a smooth vacuum initial data set, and consider two open sets $\Omega_a \subset M$ such that*

the set of KIDs, $\mathscr{K}(\Omega_a)$, is trivial. (5)

Then for all $p_a \in \Omega_a$, $\epsilon > 0$ and $k \in \mathbb{N}$ there exists a smooth vacuum initial data set $(\tilde{\gamma}(\epsilon), \tilde{K}(\epsilon))$ on $\tilde{M}$ such that $(\tilde{\gamma}(\epsilon), \tilde{K}(\epsilon))$ is ϵ-close to (γ, K) in a $C^k \times C^k$ topology away from $B(p_1, \epsilon) \cup B(p_2, \epsilon)$. Moreover $(\tilde{\gamma}(\epsilon), \tilde{K}(\epsilon))$ coincides with (γ, K) away from $\Omega_1 \cup \Omega_2$.

While the tie between nontrivial KIDs and the presence of local Killing fields suggests that the absence of nontrivial KIDs is generic, such a result requires proof. Theorems to this effect are proven in [7].

Besides the significant relaxing of the gluing conditions which this new result provides, the fact that the glued data is identical to the original data away from the points p_1 and p_2 in Theorem 1 provides an important improvement over the earlier gluing results [2] and [3]. These earlier results only guaranteed that the glued data set is arbitrarily close (relative to an appropriate function space) to what it was originally.

What happens for solutions of the constraints with non-gravitational "matter" fields present? If the fields are entirely described by the choice ρ and J, then the conditions needed for gluing are relatively mild: It is sufficient that $\rho(x) > |J(x)|$ for all x in a pair of neighborhoods of the points p_1 and p_2 at which gluing is to be done. Since the energy condition $\rho(x) \geq |J(x)|$ is generally imposed for physical reasons, requiring that the inequality be strict is a mild additional restriction. For non-gravitational fields with the introduction of additional constraint equations (e.g., the Einstein-Maxwell theory which adds $\nabla \cdot E = 0$ and $\nabla \cdot B = 0$), it is likely that the required condition for a local gluing construction is a natural generalization of the "no KIDs" condition.

THE CONSTRUCTION

The detailed proofs of Theorem 1 and of the analogous Einstein-matter theorem (with no extra constraints) are described in [8]. Here, to illustrate some of the ideas involved, we provide a brief sketch of the vacuum case. We choose balls $B(p_1, r_1) \subset \Omega_1$ and $B(p_2, r_2) \subset \Omega_2$ within which to do the gluing. In [7] it is shown that, under the nondegeneracy assumption (5), we can ϵ-perturb the data on Ω_1 and Ω_2, without changing them away from those regions, so that the constraint equations still hold, and so that there are no space-time isometries in any open set contained within $B(p_a, r)$, for sufficiently small r. The next step is to use a theorem of Bartnik [9] to deform the balls $B(p_a, r)$, in the space-time evolution of this new data, so that the trace of K is constant on $B(p_a, r)$, reducing r if necessary. The non-existence of space-time isometries is preserved under this deformation. This deformation is done so that we are in the setting in which a generalization of the gluing theorem of [2] to compact manifolds with boundary (and to include matter fields) may be applied. This constitutes the third step in the construction and is essentially done by repeating the arguments of [2, 10] in this new setting. We thus obtain a one parameter family of initial data which satisfies the constraint equations, and which contains a neck connecting the spheres $S(p_a, r)$. This family of data has the property that the initial data approach the original ones in a neighborhood of the $S(p_a, r)$'s when the parameter ϵ tends to zero. However, the transverse derivatives of those data do not match those of the original ones at the boundary spheres. This problem is cured, for ϵ small enough, by a theorem in [11], which holds precisely under the "no local space-time isometries" condition (5). This provides the desired gluing, localized within the sets Ω_a.

SPACE-TIMES WITH NO CMC SLICES

One of the original motivations for attempting to apply gluing constructions to initial data has been to show that there are spatially compact, maximally extended, globally hyperbolic solutions of the vacuum Einstein equations with no constant mean curvature Cauchy surfaces. This result is interesting, since the traditional view of both mathematical and numerical relativists has been that the most useful and reliable choice of time for a globally hyperbolic space-time is one based on a foliation by CMC slices. Such a foliation, if it exists in a given solution, has the virtue that it is unique [12]; CMC slices also appear to avoid singularities in numerical simulations.

In [13], Bartnik shows that there exist maximally extended, globally hyperbolic solutions of the Einstein equations *with dust* which admit no CMC slices. Later, Eardley and Witt proposed a scheme for showing that similar vacuum solutions exist [14], but their proof was incomplete. Using our gluing result in Theorem 1, we can now complete the argument.

We consider a set of vacuum initial data $(T^3\#T^3, \gamma, K)$, where $T^3\#T^3$ is the connected sum of a pair of three tori. We assume that, relative to some chosen two sphere S on the connecting cylinder of $T^3\#T^3$, there is a "reflection map" $\mu : T^3\#T^3 \to T^3\#T^3$ with the following properties: (i) μ is a diffeomorphism; (ii) $\mu(S) = S$; (iii) $\mu^*(\gamma) = \gamma$; and (iv) $\mu^*(K) = -K$. Note that as a consequence of these properties, $K|_S = 0$.

It follows from [15] that there is a unique, maximally extended, globally hyperbolic, development $(T^3\#T^3 \times \mathbb{R}, g)$ of the data $(T^3\#T^3, \gamma, K)$, with g satisfying the Einstein vacuum field equations on $T^3\#T^3 \times \mathbb{R}$. Further, as a consequence of this uniqueness, the map μ described above extends to a diffeomorphism from $(T^3\#T^3 \times \mathbb{R}, g)$ to itself with the property that if Σ^3 is a Cauchy surface in $(T^3\#T^3 \times \mathbb{R}, g)$ with induced data $(\gamma_{\Sigma^3}, K_{\Sigma^3})$, then $\mu(\Sigma^3)$ is a Cauchy surface as well, with its data $(\gamma_{\mu(\Sigma^3)}, K_{\mu(\Sigma^3)})$ satisfying

$$\mu^*(\gamma_{\mu(\Sigma^3)}) = \gamma \qquad \text{and} \qquad \mu^*(K_{\mu(\Sigma^3)}) = -K. \qquad (6)$$

Say there exists a CMC Cauchy surface Σ_τ in the space-time $(T^3\#T^3 \times \mathbb{R}, g)$ with mean curvature $tr(K_{\Sigma_\tau}) = \tau$ constant. Applying the map μ, we obtain another Cauchy surface, which must also have constant mean curvature, but with $tr(K_{\mu(\Sigma_\tau)}) = -\tau$. It now follows from barrier arguments that there must be a *maximal* Cauchy surface in the space-time (with $\tau = 0$). However, if there is such a Cauchy surface, then it follows from the constraint equations that the scalar curvature on this maximal surface must be non-negative. This is known to be incompatible with the topology $T^3 \# T^3$ [16]. Thus we have a contradiction, from which it follows that the space-time development of initial data with the reflection properties described above cannot contain a CMC Cauchy surface.

We now use Theorem 1 to show that we can produce such data. To start, we use the conformal method to find a CMC solution $(T^3, \hat{\gamma}, \hat{K})$ of the constraints on the torus which has no conformal Killing fields, has nonvanishing mean curvature, and has the traceless part of $\hat{K}$ nonvanishing. (It follows from [7, 17, 18] that such data sets exist.) We easily verify that consequently this data has no global KIDs, $\mathscr{K}(T^3) = 0$.

Let $(\mathscr{M}, g)$ be the maximal, globally hyperbolic, development of this data. We deform the initial data hypersurface T^3 in $\mathscr{M}$ to create a small neighborhood of a point p in which the trace of the new induced $\hat{K}$ vanishes, while maintaining the condition $\mathscr{K}(T^3) = \{0\}$.

Now let $\tilde{M}$ consist of two copies of T^3, with initial data $(\hat{\gamma}, \hat{K})$ on the first copy, T^3_1, and with data $(\hat{\gamma}, -\hat{K})$ on the second copy, T^3_2. We let $\Omega_a = T^3_a$ for $a = 1, 2$ and we let p_a denote the points in M_a corresponding to p. Noting that the mean curvature vanishes in symmetric neighborhoods of p_1 and p_2, we may now apply an argument similar to that used in proving Theorem 1 to this initial data set on $\tilde{M}$ relative to the points p_a. For this procedure to produce the desired initial data set on $T^3 \# T^3$, it is crucial to verify that all the steps are done with the correct symmetry around the middle of the connecting neck. In particular, we must check that the glued data obtained from this procedure gives a solution of the constraints which has the symmetry indicated by the presence of the reflection map μ. This is ensured by using approximate solutions with the same reflection symmetry in the construction used in the first step of the proof of Theorem 1. That the end result has the same symmetry follows from the uniqueness (within the given conformal class) of the solutions obtained there.

CONCLUSIONS

We see from this result that gluing is a powerful tool for the mathematical analysis of solutions of Einstein's equations. Other results relying on gluing, such as the proof that for any closed manifold Σ there exists an asymptotically Euclidean [3] as well as an asymptotically hyperbolic solution [2] of the constraints on Σ with a point removed, support this contention. The notion of gluing which we have explored here, which is topologically the connected sum or 'handle addition' (for wormholes) operation, is the simplest sort of surgery one can perform on manifolds. In space-time dimensions $n + 1$, for $n > 3$, it is likely that similar results can be established for other types of surgeries. This extension would be of interest, for example, in the construction of black strings [19] and will be discussed elsewhere. Not yet fully explored is the extent of the utility of this procedure for constructing physically interesting solutions. Theorem 1 together with results from [11] shows that we can use gluing to produce a wide variety of multi black hole solutions with prescribed asymptotics. We can also use it to glue a prescribed black hole to a cosmological solution. Will these glued solutions be useful for modeling astrophysical or cosmological phenomena? We believe so, and we are working to demonstrate this utility.

---

* Partially supported by a Polish Research Committee grant 2 P03B 073 24; Electronic address: `Piotr.Chrusciel@lmpt.univ-tours.fr`; URL: `www.phys.univ-tours.fr/~piotr`

† Partially supported by the NSF under Grant PHY-0099373; Electronic address: `jim@newton.uoregon.edu`; URL: `www.physics.uoregon.edu/~jim`

‡ Partially supported by the NSF under Grant DMS-0305048 and the UW Royalty Research Fund; Electronic address: `pollack@math.washington.edu`; URL: `www.math.washington.edu/~pollack`

[1] J. Corvino, Commun. Math. Phys. **214**, 137 (2000).
[2] J. Isenberg, R. Mazzeo, and D. Pollack, Comm. Math. Phys. **231**, 529 (2002), gr-qc/0109045.
[3] J. Isenberg, R. Mazzeo, and D. Pollack, Ann Henri Poincaré **4**, 369 (2003), gr-qc/0206034.
[4] P. Chruściel and R. Mazzeo, Class. Quantum Grav. **20**, 729 (2003), gr-qc/0210103.
[5] R. Schoen and S.-T. Yau, Comm. Math. Phys. **65**, 45 (1979).
[6] V. Moncrief, Jour. Math. Phys. **16**, 493 (1975).
[7] R. Beig, P. Chruściel, and R. Schoen (2004), gr-qc/0403042.
[8] P. Chruściel, J. Isenberg, and D. Pollack (2004), gr-qc/0403066.
[9] R. Bartnik, Acta Math. **161**, 145 (1988).
[10] J. Isenberg, D. Maxwell, and D. Pollack (2004), in preparation.
[11] P. Chruściel and E. Delay, Mém. Soc. Math. de France. **94**, 1 (2003), gr-qc/0301073v2.
[12] D. Brill and F. Flaherty, Ann. Inst. H. Poincaré Sect. A (N.S.) **28**, 335 (1978).
[13] R. Bartnik, Comm. Math. Phys. **117**, 615 (1988).
[14] D. Eardley and D. Witt, unpublished, 1992.
[15] Y. Choquet-Bruhat and R. Geroch, Commun. Math. Phys. **14**, 329 (1969).
[16] R. Schoen and S.-T. Yau, Man. Math. **28**, 159 (1979).
[17] M. Berger and D. Ebin, J. Diff. Geo. **3**, 379 (1969).
[18] J. Isenberg, Class. Quantum Grav. **12**, 2249 (1995).
[19] G. Horowitz (2001), hep-th/0205069.
[] For Maxwell theory, $\rho = \frac{1}{2}(E^2 + B^2)$, and $J = E \times B$
[] Hence the name "KIDs", which abbreviates "Killing initial data set"