A poor man's positive energy theorem: II. Null geodesics

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Abstract

We show that positivity of energy for stationary, or *strongly uniformly Schwarzschildian*, asymptotically flat, non-singular domains of outer communications can be proved using Galloway's null rigidity theorem.

1 Introduction

In a recent note [4] we have shown that positivity of energy for uniformly Schwarzschildian, asymptotically flat, non-singular domains of outer communications can be proved using timelike lines together with the Lorentzian splitting theorem. The object of this paper is to show that a variation of the argument of [4], using null lines, can successfully be completed in a similar, "strongly uniformly Schwarzschildian", setting. The extension of the result to more general metrics, as suggested in [13], still awaits a more complete justification.

For $m \in \mathbb{R}$, let g_m denote the $n+1$ dimensional, $n \geq 3$, Schwarzschild metric with mass parameter m; in isotropic coordinates [11],

$$g_m = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \left(\sum_{1=1}^{n} dx_i^2\right) - \left(\frac{1 - m/2|x|^{n-2}}{1 + m/2|x|^{n-2}}\right)^2 dt^2 \ . \quad (1.1)$$

We shall say that a metric g on $\mathbb{R} \times (\mathbb{R}^n \setminus B(0, R))$, $R^{n-2} > m/2$, is *strongly uniformly Schwarzschildian* if, in the coordinates of (1.1),

$$g - g_m = O(|m|r^{-(n-1)}) \,, \quad \partial_\mu (g - g_m) = O(|m|r^{-n}) \,, \qquad (1.2)$$
$$\partial_\mu \partial_\nu (g - g_m) = O(|m|r^{-n-1}) \,. \qquad (1.3)$$

(Here O is meant at fixed g and m, uniformly in t and in angular variables, with r going to infinity.) What is meant in the case $m = 0$ is that $g = g_0$, i.e., g is flat[1], for $r > R$. We note that this condition is more restrictive than the "uniformly Schwarzschildian" one in [4].

We will use the symbol r to denote some fixed smooth function on $\mathscr{M}$ which coincides with $|x|$ in $\mathscr{M}_{\text{ext}}$.

The *domain of outer communications* $\langle\!\langle \mathscr{M}_{\text{ext}} \rangle\!\rangle$ associated to $\mathscr{M}_{\text{ext}}$ is defined as the intersection of the causal past $J^-(\mathscr{M}_{\text{ext}})$ of the asymptotic region

$$\mathscr{M}_{\text{ext}} = \mathbb{R} \times (\mathbb{R}^n \setminus B(0, R)) \qquad (1.4)$$

with its causal future $J^+(\mathscr{M}_{\text{ext}})$.

We need a version of weak asymptotic simplicity [6] for uniformly Schwarzschildian spacetimes. Following [4], we shall say that such a space-time $(\mathscr{M}, g)$ is *weakly asymptotically regular* if every null line starting in the domain of outer communications $\langle\!\langle \mathscr{M}_{\text{ext}} \rangle\!\rangle$ either crosses an event horizon (if any), or reaches arbitrarily large values of r in the asymptotically flat regions. Recall that a null line in $(\mathscr{M}, g)$ is an inextendible null geodesic that is globally achronal.

The first main result of this paper is a simple proof of the following:

THEOREM 1.1 *Let* $(\mathscr{M}^{n+1} = \mathscr{M}, g)$ *be a* $(n+1)$-*dimensional weakly asymptotically regular space-time containing a strongly uniformly Schwarzschildian region* $\mathscr{M}_{\text{ext}}$ *with*

$$m \le 0 \,.$$

If the domain of outer communications associated to $\mathscr{M}_{\text{ext}}$ *has a Cauchy surface* $\mathscr{S}$, *the closure of which is the union of one asymptotic end and of a compact interior region (with a differentiable boundary lying at the intersection of the future and past event horizons, if any), then* $\mathscr{M}$ *contains a null line.*

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[1]The asymptotic conditions for the case $m = 0$ of our theorem are way too strong for a rigidity statement of real interest, even for stationary metrics, so our results only exclude $m < 0$ in practice. Nevertheless, in this context one should keep in mind an argument of Lohkamp [9] for initial data sets with trace of $K = 0$, which shows that the proof of positivity of mass is obtained in its full generality once it has been established for data which are flat outside of a compact set.

Theorem 1.1 implies non-existence of appropriately regular, uniformly Schwarzschildian space-times with negative mass in various situations of interest. For example, consider a uniformly Schwarzschildian four-dimensional, asymptotically simple, vacuum space-time $(\mathscr{M}, g)$ with $m \leq 0$. It then follows from Theorem 1.1 and from Galloway's null rigidity theorem [5] that $(\mathscr{M}, g)$ is the Minkowski space-time. This result is somewhat stronger than the one in [4], because no hypotheses on timelike geodesics are made (we will show below that, similarly to [4], in our context the hypothesis of asymptotic simplicity needed in [5] can be replaced by the weak version thereof). On the other hand, there is a restriction on space-time dimension, not made in [4].

The original idea in [12] was that space-times containing null lines cannot satisfy the so-called "genericity condition". However, it is not clear how generic is "generic" when, e.g., vacuum equations are imposed, and therefore, it is not clear how much information is carried by theorems basing upon "genericity". For this reason it appears useful to develop an argument which does not invoke this condition. As a first step towards this, assuming the *null energy condition*,

$$R_{\mu\nu} X^{\mu} X^{\nu} \geq 0 \text{ for all null vectors } X^{\mu}, \tag{1.5}$$

we prove the following:

THEOREM 1.2 *Under the hypotheses of Theorem 1.1, suppose moreover that the null energy condition (1.5) holds. Then*

$$\mathscr{M} = \ll \mathscr{M}_{\text{ext}} \gg ,$$

and every maximally extended null geodesic is a line. Further, for each $\vec{k} \in S^{n-2}$ there exists a one-parameter family $\mathscr{N}_{\vec{k}} \equiv \{\mathscr{N}_{\vec{k},T}\}_{T \in \mathbb{R}}$, with $\mathscr{N}_{\vec{k}'} \neq \mathscr{N}_{\vec{k}}$ if $\vec{k} \neq \vec{k}'$, of smooth, closed, null, totally geodesic, achronal hypersurfaces covering $\mathscr{M}$.

With a little more effort one can show that each family $\mathscr{N}_{\vec{k}}$ forms a foliation; we give no details as we will not pursue this line of approach. Now, each foliation of $\mathscr{M}$ by null hypersurfaces as described in Theorem 1.2 defines a shear-free, rotation-free, and expansion-free congruence of null geodesics covering $\mathscr{M}$. Space-times admitting one such congruence are expected to be very special[2], while we actually have a family of congruences parameter-

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[2]See, e.g., [8, Section 7.6 and Chapter 31]. The results there are based upon a supplementary assumption on the vanishing of some components of the Ricci tensor. Note, however, that it follows from the Raychaudhuri equation that this condition will be satisfied in our case if the dominant energy condition is assumed.

ized by S^{n-1}. We therefore expect that the only space-time satisfying the hypotheses of Theorem 1.2 is the Minkowski one, but we are not aware any results to this effect without further hypotheses (compare [10]). Assuming n equal to four, as well as existence of a smooth conformal completion, we will be able to reach this conclusion by an argument that does not proceed via the algebraically special structure of the Riemann tensor; this is the main result of this paper:

THEOREM 1.3 *Let $(\mathscr{M}^{3+1} = \mathscr{M}, g)$ be a four dimensional space-time satisfying the null energy condition (1.5), and suppose that $\mathscr{M}$ contains a strongly uniformly Schwarzschildian region $\mathscr{M}_{\mathrm{ext}} = \mathbb{R} \times \left(\mathbb{R}^3 \setminus B(0, R)\right)$ which admits a smooth conformal completion at null infinity. Assume that $(\mathscr{M}, g)$ is weakly asymptotically regular, and that the domain of outer communications of $\mathscr{M}$ has a Cauchy surface $\mathscr{S}$, the closure of which is the union of one asymptotic end and of a compact interior region (with a differentiable boundary lying at the intersection of the future and past event horizons, if any). Then*

$$m > 0 \, ,$$

unless $(\mathscr{M}, g)$ is the Minkowski space-time.

It is of interest to compare the results and proofs here to those in [4]. Recall that we are trying to implement the original idea of Penrose [12], that negative mass should imply existence of a *null line* in space-time, which then should be incompatible with energy conditions. In [4] we have shown, under the hypotheses there, that negative mass implies existence of a *timelike line*, and that this is compatible with the timelike convergence condition,

$$R_{\mu\nu} X^\mu X^\nu \geq 0 \text{ for all timelike vectors } X^\mu, \tag{1.6}$$

only if the space-time is the Minkowski one. Both in [4] and here we make the unsatisfactory hypothesis that the metric is uniformly Schwarzschildian. This is, however, a quite reasonable hypothesis for *stationary* space-times with $m \neq 0$ (see, e.g., [14]). In the stationary case the only global hypothesis needed for both constructions is that of existence of an asymptotically flat Cauchy surface for the domain of outer communications, with conditionally compact interior. For stationary metrics the proof of existence of lines is completely elementary for timelike lines in [4], and marginally more complicated for null lines here.

It is not clear at all whether there exist *non-stationary* solutions of physically reasonable field equations that are uniformly Schwarzschildian, and

therefore assuming that last condition without assuming stationarity is presumably an academic exercise. We have nevertheless carried this out, in order to test the limits of the techniques employed, and in the hope that the arguments will eventually generalise to a proof under less stringent asymptotic conditions. As already pointed out, in the non-stationary case the construction of a null line is only slightly more involved than that of a timelike one. On the other hand, the conclusion that the space-time must be flat requires considerably more work. However, the proof of existence of a timelike line needs a supplementary global hypothesis concerning timelike geodesics, which is avoided in the current setting. This last fact provides yet another motivation for the work here.

The next step in the positivity argument is to show that space-times containing lines and satisfying energy conditions are very special. In considerations involving timelike lines the natural energy condition is the unphysical inequality (1.6). From this point of view null lines are much more satisfactory, as their global causality properties are tied to the physically motivated *null energy condition* (1.5). This is the final motivation for our work.

This paper is organised as follows: In Section 2 we give the proofs of our results. Appendix A contains some technical results on geodesics in uniformly Schwarzschildian metrics, as needed in the proofs of Theorems 1.2 and 1.3.

2 Proofs

We start with a proof of Theorem 1.1. We will be sketchy in places, assuming that the reader is familiar with the argument in [4].

Let x^μ be the coordinates of (1.1) ($x^0 = t$), and let the indices $a, b, ..$ run from 1 to $n - 1$, where n is the space-dimension. Using (A.6)-(A.7) in the appendix below, one finds that the function

$$\rho := \sqrt{\sum_{a=1}^{n-1} (x^a)^2}$$

has convexity properties somewhat similar to those of r, as exploited in [4]:

$$\text{Hess } \rho \quad = \quad -\frac{m\rho}{r^n} \left\{ (n-2)dt^2 + \sum_{i=1}^{n} (dx^i)^2 \right\} + \frac{2m}{r^{n-1}} \; d\rho dr$$

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$$+\frac{1}{\rho}\left\{\sum_{a=1}^{n-1}(dx^a)^2 - d\rho^2\right\} + \sum_{\mu,\nu=0}^{n}\sum_{a=1}^{n-1}\frac{x^a}{\rho}O(r^{-n})dx^\mu dx^\nu \ . \quad (2.1)$$

On level sets of ρ all terms involving $d\rho$ drop out, then both expressions in braces are manifestly positive for negative m. In order to show that they dominate all the remaining terms when $\rho \geq \mathring{R}$ for some $\mathring{R}$, one needs to check that

$$\frac{|m|\rho}{r^n} \geq c(n)\frac{x^a}{\rho}O\left(\frac{1}{r^n}\right) \iff |m|\rho^2 \geq c(n)x^a O(1) \ , \quad (2.2)$$

where $c(n)$ is a (large) dimension-dependent constant. Equation (2.2) will clearly hold for ρ large enough if $m \neq 0$. It follows that $\mathrm{Hess}\,\rho$, when restricted to $\{\rho = R\}$, with $R \geq \mathring{R}$, is positive definite when $m < 0$. This shows, as in [4], for negative or vanishing m, that any geodesic segment Γ with initial point p and final point q such that $\rho(p), \rho(q) < \mathring{R}$ will satisfy

$$\rho \circ \Gamma < \mathring{R} \ . \quad (2.3)$$

We start with a lemma[3]:

LEMMA 2.1 *For every $p \in J^-(\mathscr{M}_{\mathrm{ext}})$ there exists an achronal, future inextendible, null geodesic ray*

$$\Gamma_p^+ : [0, \infty) \to J^-(\mathscr{M}_{\mathrm{ext}})$$

such that

$$\Gamma_p^+(0) = p \ , \qquad \rho \circ (\Gamma_p^+) \leq M \ ,$$

for some constant $M = M(p)$.

PROOF: Since $p \in J^-(\mathscr{M}_{\mathrm{ext}})$ for any $N \in \mathbb{N}$ there exists a causal curve from p to a point

$$p_N^+ = (t_N^+, 0, \cdots, 0, N) \in \mathscr{M}_{\mathrm{ext}} \ , \quad (2.4)$$

for some t_N^+. Minimising the time of arrival t_N^+ over all such causal curves one obtains, by global hyperbolicity, a null achronal geodesic segment Γ_N^+ from p to perhaps a different point p_N^+, still of the form (2.4). By the convexity properties of the function ρ, discussed above, we have the bound $\rho \circ \Gamma_N^+ \leq M$. Global hyperbolicity implies that there exists a subsequence of the sequence Γ_N accumulating at the desired inextendible null geodesic ray Γ_p^+. Achronality of Γ_p^+ follows from the fact that an accumulation curve of achronal curves is achronal. $\square$

It immediately follows that:

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[3]I am grateful to G.Galloway for providing this simple proof.

COROLLARY 2.2 *For every $p \in \lll \mathscr{M}_{\text{ext}} \ggg$ there exist achronal, future inextendible, null geodesic rays*

$$\Gamma^+_p : [0, \infty) \to \lll \mathscr{M}_{\text{ext}} \ggg \quad and \quad \Gamma^-_p : (-\infty, 0] \to \lll \mathscr{M}_{\text{ext}} \ggg$$

such that

$$\Gamma^{\pm}_p(0) = p , \qquad \rho \circ (\Gamma^{\pm}_p) \leq M ,$$

for some constant $M = M(p)$.

Choose $\mathring{R}$ large enough so that the hypersurfaces $\{x^n = \pm\mathring{R}\}$ are closed boundaryless in $\mathscr{M}_{\text{ext}}$. Increasing $\mathring{R}$ we can also assume that ∂_t is timelike for $r \geq \mathring{R}$, that the slopes of the light cones in the region $r \geq \mathring{R}$ are between $1/2$ and 2, and that $\operatorname{Hess} \rho$ is positive definite on the level sets $\{\rho = R\}$ for all $R \geq \mathring{R}$.

Choose $\tau \in \mathbb{R}$, let $p = (\tau, 0, \ldots, 0, -\mathring{R})$, set

$$\Gamma^+ := \Gamma^+_p , \quad \Gamma^- := \Gamma^-_p ,$$

and define

$$\mathscr{N}^+ := \dot{J}^-(\Gamma^+) , \quad \mathscr{N}^- := \dot{J}^+(\Gamma^-) .$$

Since the $\Gamma^{\pm}$'s are achronal we have $\Gamma^{\pm} \subset \mathscr{N}^{\pm}$, thus the $\mathscr{N}^{\pm}$'s are nonempty, closed, achronal, locally Lipschitz submanifolds of $\mathscr{M}$. Denote by

$$\mathscr{P}^{\pm} := \mathscr{N}^{\pm} \cap \{x^n = -\mathring{R}\} .$$

The manifolds $\{x^0 = \pm\mathring{R}\}$ with the induced metric are globally hyperbolic submanifolds of $(\mathscr{M}, g)$, with the $\mathscr{P}^{\pm}$ non-empty (as $\Gamma^{\pm} \cap \{r = \pm\mathring{R}\} = p \in \mathscr{P}^{\pm}$), which shows that the $\mathscr{P}^{\pm}$'s are closed, achronal, locally Lipschitz submanifolds of $\{x^n = -\mathring{R}\}$, and can thus be written as graphs

$$\mathscr{P}^{\pm} = \{t = s^{\pm}(x^a)\} ,$$

for some locally Lipschitz functions $s^{\pm}$.

In the coordinates of (1.1) the null geodesics $\Gamma^{\pm}$ can be written as

$$\Gamma^{\pm}(s) = (t^{\pm}(s), \gamma^{\pm}(s)) .$$

For $k \in \mathbb{N}$ let $p_k = \Gamma^+(k)$, set

$$q_k = \dot{J}^-(p_k) \cap \left(\mathbb{R} \times \{\gamma^-(-k)\}\right) .$$

By global hyperbolicity there exists a null achronal geodesic Γ_k from q_k to p_k. By Corollary 2.2 there exists a constant M such that $\rho(q_k), \rho(p_k) \leq M$,

and by the argument presented in [4] we then have $\rho \circ \Gamma_k \leq M$ for all k large enough. Let

$$r_k = \Gamma_k \cap \{x^n = -\mathring{R}\} \ .$$

We claim that r_k satisfies

$$r_k \in \underbrace{\{(x^0, x^a) : \ s^-(x^a) \leq x^0 \leq s^+(x^a) \ , \ \rho(x^a) \leq M\}}_{=:K} \subset \{x^n = -\mathring{R}\} \ . \quad (2.5)$$

The upper bound on x^0 follows immediately from $r_k \in J^-(\Gamma^+)$. To obtain the lower one we show that $r_k \in J^+(\Gamma^-)$: Indeed, since achronal null geodesics maximise the time of arrival (see, e.g., [1, Proposition A.2]) we have

$$t(q_k) = \sup_{p \in J^-(p_k) \cap \{\vec{x} = \gamma^-(-k)\}} t(p) \geq t^-(-k) = t(\Gamma^-(-k)) \ .$$

This shows that a future directed causal curve from $\Gamma^-(-k)$ to r_k is obtained by first following the coordinate line

$$[t^-(-k), t(r_k)] \ni t \to (t, \gamma^-(-k)) \ ,$$

and then following Γ_k until it meets $\{x^n = -\mathring{R}\}$.

We have thus shown that all the achronal geodesic segments Γ_k pass through the compact set K defined in (2.5), and the existence of a null line Γ at which the Γ_k's accumulate follows from standard results in Lorentzian geometry. $\qquad \square$

PROOF OF THEOREM 1.2: Let Γ be the line constructed in the proof of Theorem 1.1. Writing $\Gamma \cap \mathscr{M}_{\text{ext}}$ in coordinates on $\mathscr{M}_{\text{ext}}$ as $x^\mu(s)$, by Corollary A.4 there exists k, T and β such that (A.3)-(A.4) hold. Since ρ is bounded along Γ we must have $k = (1, 0, \ldots, 0, 1)$.

The construction of Γ applies for every choice of asymptotic direction $k = (1, \vec{k})$, we shall denote by $\mathring{\Gamma}_{\vec{k}, T}$ the resulting line. By Corollary A.4 we have

$$\mathring{\Gamma}_{\vec{k}, T} = \Gamma_{(1, \vec{k}), T, \beta(\vec{k}, T)} \quad (2.6)$$

for some $\beta(\vec{k}, T) \in \mathbb{R}^{n-1}$. Note that Γ depends upon a parameter $\tau \in \mathbb{R}$, so this equation defines a function $T(\tau)$. The construction of Γ, together with Corollary A.4, similarly shows that

$$\mathring{\Gamma}_{\vec{k}, T} = \Gamma_{(-1, -\vec{k}), T', \beta'(\vec{k}, T)} \ , \quad (2.7)$$

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for some $T' \in \mathbb{R}$ and $\beta' \in \mathbb{R}^{n-1}$. Here $\mathbb{R}^{n-1}$ is understood as the plane orthogonal to $\vec{k}$ with respect to the Euclidean metric on $\mathbb{R}^n$.

Let $\tilde{\mathscr{N}}_{\vec{k},T}$ denote the collection of points in $\dot{J}^-(\mathring{\Gamma}_{\vec{k},T}) \cap \dot{J}^+(\mathring{\Gamma}_{\vec{k},T})$ which lie on a generator of $\dot{J}^-(\mathring{\Gamma}_{\vec{k},T})$ which is past complete, future complete, and entirely contained in $\ll\!\mathscr{M}_{\mathrm{ext}}\!\gg$. Then $\tilde{\mathscr{N}}_{\vec{k},T}$ is non-empty, as it contains $\mathring{\Gamma}_{\vec{k},T}$. We define

$$\mathscr{N}_{\vec{k},T} = \{\text{the connected component of } \tilde{\mathscr{N}}_{\vec{k},T} \text{ containing } \mathring{\Gamma}_{\vec{k},T}\}.$$

Assuming the null energy condition, we claim that $\mathscr{N}_{\vec{k},T}$ is a smooth null hypersurface. This results from the discussion in [5], and can be seen as follows: Let $p \in \mathring{\Gamma}_{\vec{k},T}$, as $\mathscr{M}_{\mathrm{ext}}$ is open there exists a neighborhood $\mathscr{O}$ of p entirely contained in $\mathscr{M}_{\mathrm{ext}}$. Consider the generators of $\dot{J}^-(\mathring{\Gamma}_{\vec{k},T})$ passing through $\mathscr{O}$. As p is an interior point of such a generator, all the null tangents in $T\mathscr{O}$ can be made as close to the null tangent at p as desired if $\mathscr{O}$ is made small enough (see, e.g., [3]). It follows from continuous dependence of geodesics upon initial data that all those generators have asymptotic behavior as described in Proposition A.1 when $\mathscr{O}$ is chosen small enough (with perhaps an asymptotic direction vector $\vec{k}' \neq \vec{k}$), in particular they will be future complete. By Lemma 4.2 in [5] the divergence of $\dot{J}^-(\mathring{\Gamma}_{\vec{k},T})$ (as measured towards the future, defined in the sense of support hypersurfaces) is non-negative at those points of $\dot{J}^-(\mathring{\Gamma}_{\vec{k},T})$ which lie on generators meeting $\mathscr{O}$. Similarly, reducing $\mathscr{O}$ if necessary, the divergence of generators of $\dot{J}^+(\mathring{\Gamma}_{\vec{k},T})$ (as measured again towards the future) is non-positive at those points of $\dot{J}^-(\mathring{\Gamma}_{\vec{k},T})$ which lie on generators meeting $\mathscr{O}$. By Theorem 3.4 of [5] $\dot{J}^-(\mathring{\Gamma}_{\vec{k},T}) \cap \mathscr{O}$ is a smooth null hypersurface which coincides with $\dot{J}^+(\mathring{\Gamma}_{\vec{k},T}) \cap \mathscr{O}$. It is then easily seen that the collection of points of $\dot{J}^-(\mathring{\Gamma}_{\vec{k},T})$ which lie on generators meeting $\mathscr{O}$ is a smooth null hypersurface contained in $\dot{J}^+(\mathring{\Gamma}_{\vec{k},T})$, with all the generators there being both future and past complete, contained in $\mathscr{M}_{\mathrm{ext}}$, extending arbitrary far in the asymptotic region both to the future and to the past.

We have thus shown that $\mathscr{N}_{\vec{k},T}$ has all the properties listed in the statement of the theorem except perhaps for being closed. In order to establish that last property, consider a sequence of points $p_n \in \mathscr{N}_{\vec{k},T}$ converging to $p \in \mathscr{M}$, and let Γ_n denote the generator of $\dot{J}^-(\mathring{\Gamma}_{\vec{k},T})$ passing through p_n. Set

$$q_n = \Gamma_n \cap \mathscr{S} \,,$$

where $\mathscr{S}$ is the Cauchy surface for $\ll\!\mathscr{M}_{\mathrm{ext}}\!\gg$, then a subsequence, still denoted by q_n, converges to a point $q \in \overline{\mathscr{S}}$. Let Γ denote an accumulation

curve of the Γ_n's passing through q, then Γ is an achronal, maximally extended in $\overline{\langle \mathscr{M}_{\text{ext}} \rangle}$, geodesic passing through q. Suppose that $q \in \partial \mathscr{S}$, then the portion of Γ_n which lies to the future of $\mathscr{S}$ accumulates to a generator of $\partial \mathscr{D}^+(\mathscr{S})$, while the portion of Γ_n which lies to the past of $\mathscr{S}$ accumulates to a generator of $\partial \mathscr{D}^-(\mathscr{S})$, resulting in an accumulation curve Γ which is not differentiable at q. This, however, contradicts the fact that Γ is a geodesic. Hence $q \in \mathscr{S}$, and Γ is both future and past complete. It follows that $q \in \mathscr{N}_{\vec{k},T}$, with p lying on Γ. This shows that $p \in \mathscr{N}_{\vec{k},T}$, and closedness of $\mathscr{N}_{\vec{k},T}$ follows.

Note that so far T was an arbitrarily chosen fixed number. If the domain of outer communications is stationary one can move $\mathscr{N}_{\vec{k},T}$ by isometries to obtain a one-parameter family $\mathscr{N}_{\vec{k},t} := \phi_{t-T}(\mathscr{N}_{\vec{k},T})$, $t \in \mathbb{R}$, of such hypersurfaces. A similar construction can be done in the stationary-rotating case.

In general, we start by noting that:

LEMMA 2.3 *Let $s_0 \in \mathbb{R}$, $\vec{k} \in S^{n-1}$, $a = 1,2$. For $T_a \in \mathbb{R}$ and $\beta_a \in \mathbb{R}^{n-1}$ set $\Gamma_a = \Gamma_{(1,\vec{k}),T_a,\beta_a}|_{[s_0,\infty)}$. Then*

$$T_1 < T_2 \quad \implies \quad \Gamma_1 \subset I^-(\Gamma_2; \mathscr{M}_{\text{ext}}) \ .$$

PROOF: We write x_a^μ for Γ_a in the asymptotically Minkowskian coordinates on $\mathscr{M}_{\text{ext}}$, and use an affine parameter as in (A.4). By Proposition A.3 for any $\lambda > 0$ we have

$$x_2^\mu(s + \lambda) - x_1^\mu(s) \to_{s\to\infty} (T_2 - T_1 + \lambda, \beta_2 - \beta_1, \lambda) \ . \tag{2.8}$$

Consider the coordinate-line segment

$$[0,1] \ni t \to \gamma_s^\mu(t) = t x_2^\mu(s + \lambda) + (1 - t) x_1^\mu(s) \ .$$

Calculating with respect to the Minkowski metric η, the quantity $\eta_{\mu\nu} \dot{\gamma}_s^\mu \dot{\gamma}_s^\nu$ (a dot denotes a t-derivative) approaches, as s tends to infinity, the Minkowskian length of the vector appearing at the right-hand-side of (2.8), which is

$$-(T_2 - T_1 + \lambda)^2 + |\beta_2 - \beta_1|_\delta^2 + \lambda^2 = -2(T_2 - T_1)\lambda + |\beta_2 - \beta_1|_\delta^2 \ .$$

When $T_2 > T_1$ this is negative for all λ sufficiently large. It follows that γ_s is a timelike curve with respect to the Minkowski metric for all λ and s large. Since the metric g uniformly tends to the Minkowski one along γ_s as s tends to infinity, γ_s will be a timelike curve with respect to g for all λ and s large enough. $\qquad \square$

In what follows $\mathring{R}$ is chosen at least as large as in the proof of Theorem 1.1, and in (A.22):

COROLLARY 2.4 *Let* T', β' *be such that the geodesic* Γ_p^+, $p = (\tau, 0, \ldots, 0, -\mathring{R})$, *of Lemma 2.1 is a subset of* $\Gamma_{(1,\vec{k}),T',\beta'}$ *(when both are understood as point sets in* $\mathscr{M}$*), and let* $T = T(\tau)$ *be as defined by (2.6). Then*

$$T' = T .$$

PROOF: Clearly $T \leq T'$ by construction. Suppose $T < T'$, then $\mathring{\Gamma}_{\vec{k},T}$ would lie to the timelike past of Γ_p^+ by Lemma 2.3. But, by construction, $\mathring{\Gamma}_{\vec{k},T} \subset \dot{J}^-(\Gamma_p^+)$, whence the result. □

COROLLARY 2.5 *The map which to the parameter* τ *assigns* T, *as defined by (2.6), is a diffeomorphism from* $\mathbb{R}$ *to* $\mathbb{R}$.

PROOF: The map $\tau \to T'(\tau)$ is the inverse of the map Φ defined after Equation (A.21), Appendix A, which is shown to be a diffeomorphism of $\mathbb{R}$ there, and the result follows from Corollary 2.4. □

We continue with

PROPOSITION 2.6 $\mathscr{N}_{\vec{k},T} = \cup_{\beta \in \mathbb{R}^n} \Gamma_{(1,\vec{k}),T,\beta}$.

PROOF: Let $p \in \mathscr{N}_{\vec{k},T}$, then there exists a causal line Γ_p passing through p contained in $\langle \mathscr{M}_{\mathrm{ext}} \rangle$. By weak asymptotic regularity Γ_p satisfies (A.18), and thus $\Gamma_p = \Gamma_{(1,\vec{k}'),T',\beta'}$ for some $\Gamma_{(1,\vec{k}'),T',\beta'}$ by Corollary A.4.

We want, first, to show that

$$\vec{k}' = \vec{k} . \tag{2.9}$$

Since $\mathscr{N}_{\vec{k},T}$ is a smooth hypersurface its field of null tangents is also smooth. This, together with connectedness of $\mathscr{N}_{\vec{k},T}$, allows one to reduce the proof of (2.9) to a situation where $\vec{k}'$ is close to $\vec{k}$. We shall therefore assume that $\vec{k} = (0, \ldots, 0, 1)$ and that $k'^n > \sqrt{15}/4$. It follows from (A.4) that choosing $\mathring{R}$ and s_0 large enough one can also assume that $\Gamma_{(1,\vec{k}'),T',\beta'}(s) \in \{x^n \geq \mathring{R}\}$ for $s \geq s_0$.

Let s be an affine parameter as in (A.3)–(A.5), and for $i \in \mathbb{N}$ let $p_i = p_i(s) \in \mathscr{M}_{\mathrm{ext}}$ denote a point with coordinates $(x^0(s) - 1/i, \vec{x}(s))$. Since $p_i \in$

$I^-(\Gamma_{(1,\vec{k}'),T',\beta'})$ we have $p_i \in I^-(\mathring{\Gamma}_{\vec{k},T})$, thus there exists a future directed causal curve γ_i from p_i to a point $q_i \in \mathring{\Gamma}_{\vec{k},T}$.

Consider, now, the optical function S^+ defined in Appendix A.2, see (A.22). As ∇S^+ is lightlike, the function S^+ is strictly increasing on every timelike curve contained within the domain of definition of S^+. Suppose, first, that γ_i is entirely contained in $\{x_n \geq \mathring{R}\}$. Then $S^+(p_i) < S^+(q_i)$. By construction of S^+ we have $S^+(q_i) = T$. On the other hand, the asymptotic behavior (A.23) of S^+ and of $\Gamma_{(1,\vec{k}'),T',\beta'}$ gives, to leading order,

$$S^+(p_i) = x^0(s) - x^n(s) + o(s) = s - k'^n s + o(s) \to_{s\to\infty} \infty$$

except if (2.9) is satisfied. Thus, if (2.9) does not hold, then γ_i crosses the boundary of the set $\{x^n \geq \mathring{R}\}$; let r_i denote the first crossing point, and let t_i denote the last one. Note that $\dot{J}^-(\mathring{\Gamma}_{\vec{k},T}; \{x^n \geq \mathring{R}\}) \cap \{x^n = \mathring{R}\}$ can be represented as a graph:

$$\dot{J}^-(\mathring{\Gamma}_{\vec{k},T}; \{x^n \geq \mathring{R}\}) \cap \{x^n = \mathring{R}\} = \{x^0 = s^-(x^a) , \ x^a \in \mathbb{R}^{n-1}\} .$$

Since $S^+ \leq T$ on $\dot{J}^-(\mathring{\Gamma}_{\vec{k},T}; \{x^n \geq \mathring{R}\}) \cap \{x^n = \mathring{R}\}$ we have, by (A.23),

$$s^- = O(\ln(2 + \rho)) . \tag{2.10}$$

Somewhat similarly, $\dot{J}^+(p_1; \{r \geq \mathring{R}\}) \cap \{x^n = \mathring{R}\}$ is a graph of a function s^+. With our choice of $\mathring{R}$ the slopes of the light cones in $\mathscr{M}_{\text{ext}}$ are bounded from below by one half, which implies

$$
\begin{aligned}
s^+(x^a) &\geq \underbrace{x^0(s) - 1 + \frac{x^n(s) - \mathring{R}}{2}}_{I} + \underbrace{\frac{1}{2}\sqrt{\sum_a (x^a - x^a(s))^2}}_{II} \\
&\geq \frac{5}{4}s + \frac{1}{2}\sqrt{\sum_a (x^a - x^a(s))^2} - C ,
\end{aligned}
\tag{2.11}
$$

for some s–independent constant C. Here the contribution denoted as I is obtained by following a coordinate line, of slope one half lying in a $x^0 - x^n$ plane, from p_1 to a point $r \in \{x^n = \mathring{R}\}$, while the contribution denoted as II is obtained from a cone of slope one half issued from r.

Now

$$
\begin{aligned}
\rho &= \sqrt{\sum_a (x^a)^2} \leq \sqrt{\sum_a (x^a - x^a(s))^2} + \sqrt{\sum_a (x^a(s))^2} \\
&\leq \sqrt{\sum_a (x^a - x^a(s))^2} + \frac{1}{2}s + C' ,
\end{aligned}
$$

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so that

$$\rho - \frac{1}{2}s - C' \leq \sqrt{\sum_a (x^a - x^a(s))^2} \;,$$

and (2.11) gives

$$s^+(x^a) \geq s + \frac{1}{2}\rho - \frac{1}{2}C' - C \;. \tag{2.12}$$

It follows that for any future directed causal curve starting from one of the points p_i and entirely contained in $\{r \geq \mathring{R}\}$, at each crossing point (2.12) holds.

Suppose that γ_i never exits $\{r \geq \mathring{R}\}$. Then at the last point $t_i = (x_i^0, x_i^a)$ at which γ_i crosses $\{x^n = \mathring{R}\}$ we must have

$$s^+(x_i^a) \leq s^-(x_i^a) \;,$$

which is not possible by (2.10) and (2.12) if s is chosen sufficiently large. Thus, either (2.9) holds, or γ_i enters and exits $\{r \geq \mathring{R}\}$ (perhaps more than once). Let $\hat{r}_i$ denote the first exit point. Let $\hat{s}^-$ be the graphing function of $\dot{J}^-(\mathring{\Gamma}_{\vec{k},T}; \mathscr{M}) \cap \{r = \mathring{R}\}$ (exceptionally, for emphasis, we write $\dot{J}^-(\mathring{\Gamma}_{\vec{k},T}; \mathscr{M})$ for $\dot{J}^-(\mathring{\Gamma}_{\vec{k},T})$):

$$\dot{J}^-(\mathring{\Gamma}_{\vec{k},T}; \mathscr{M}) \cap \{r = \mathring{R}\} = \{x^0 = \hat{s}^-(v) \,, v \in S^{n-1}(\mathring{R}) \subset \mathbb{R}^n\} \;.$$

Since $\hat{r}_i = (\hat{x}_i^0, \hat{v}_i) \in J^-(\mathring{\Gamma}_{\vec{k},T}; \mathscr{M})$ we have $\hat{x}_i^0 \leq \hat{s}^-(v_i) \leq \sup \hat{s}^- < \infty$, the last inequality following from compactness and from the fact that

$$J^-(\mathring{\Gamma}_{\vec{k},T}; \mathscr{M}) \cap \{r = \mathring{R}\} \supset \mathring{\Gamma}_{\vec{k},T} \cap \{r = \mathring{R}\} \quad \Longrightarrow \quad J^-(\mathring{\Gamma}_{\vec{k},T}; \mathscr{M}) \cap \{r = \mathring{R}\} \neq \emptyset \;,$$

so that $\hat{s}^- \not\equiv \infty$. However, x^0 is increasing along γ_i, so that $x_i^0 \geq x^0(s) - 1 \to_{s \to \infty} \infty$. It follows that an appropriate choice of s guarantees that all the γ_i's are entirely contained within $\{r \geq \mathring{R}\}$. Consequently, (2.9) must be true.

From (2.9) and the definition of the p_i's we have $S^+(p_i) \to_{i \to \infty} T'$, which implies $T' \leq T$. Lemma 2.3 shows that no point of the form $\Gamma_{(1,\vec{k}),T',\beta'}(s)$, with $T' < T$ and s large, can be in $\dot{J}^-(\mathring{\Gamma}_{\vec{k},T})$, hence

$$T' = T \;.$$

We have thus shown

$$\mathscr{N}_{\vec{k},T} \subset \cup_{\beta \in \mathbb{R}^n} \Gamma_{(1,\vec{k}),T,\beta} \;. \tag{2.13}$$

Equality ensues now from the fact that the intersections of both sets with the region $\{x^n > \mathring{R}\}$ are smooth closed hypersurfaces. $\quad\square$

COROLLARY 2.7 *Each maximally extended geodesic* Γ *in* $\ll\!\mathscr{M}_{\mathrm{ext}}\!\gg$ *such that* $r(\Gamma(s)) \to_{s\to\infty} \infty$ *is a line.*

PROOF: By Corollary A.4 every such future directed Γ coincides with some $\Gamma_{(1,\vec{k}),T,\beta}$. But Corollary 2.5 and Proposition 2.6 show that each $\Gamma_{(1,\vec{k}),T,\beta}$ is a line. The result for past-directed geodesics follows by changing time orientation. $\qquad\square$

Returning to the proof of Theorem 1.2, we claim that each point p in $\ll\!\mathscr{M}_{\mathrm{ext}}\!\gg$ lies on some $\Gamma_{(1,\vec{k}),T,\beta}$. Indeed, Lemma 2.1 with the direction $(0,\ldots,0,1)$ rotated into $\vec{k}$ provides a geodesic ray through p with asymptotic direction $\vec{k}$, and the result follows from Proposition A.3.

We have thus proved that each of the distinct families of null hypersurfaces

$$\mathscr{N}_{\vec{k}} := \{\mathscr{N}_{\vec{k},T}\}_{T\in\mathbb{R}}$$

covers $\ll\!\mathscr{M}_{\mathrm{ext}}\!\gg$. Suppose that $\ll\!\mathscr{M}_{\mathrm{ext}}\!\gg \neq \mathscr{M}$, then there exists a point

$$p \in I^-(\partial\mathscr{S}) \cap I^-(\mathscr{M}_{\mathrm{ext}}) \setminus \ll\!\mathscr{M}_{\mathrm{ext}}\!\gg \subset J^-(\mathscr{M}_{\mathrm{ext}})$$

and, by Lemma 2.1, a future directed null half-line from p extending to infinity in $\mathscr{M}_{\mathrm{ext}}$. This is, however, impossible by Corollary 2.7, and the proof is complete. $\qquad\square$

PROOF OF THEOREM 1.3: By Theorem 1.2 every maximally extended geodesic is a line. In view of the Raychaudhuri equation this is only possible if for all null vectors k we have

$$\mathrm{Ric}(k,k) = 0 . \tag{2.14}$$

Elementary algebra implies then that the Ricci tensor is proportional to the metric. For future reference we note a somewhat stronger result, which does not assume that (2.14) holds for all k:

LEMMA 2.8 *Suppose that there exists an open set* Ω *of null vectors* k *at* $p \in \mathscr{M}$ *for which we have*

$$R_{\mu\nu}k^\mu k^\nu = 0 .$$

Then $R_{\mu\nu}(p)$ *is proportional to* $g_{\mu\nu}$.

PROOF: Let e^μ be an ON-frame at p, (changing e^0 to $-e^0$ if necessary) in this frame we write $k = (|\vec{k}|_\delta, \vec{k})$, and there exists an open cone $\mathscr{U} \subset \mathbb{R}^{n-1}$ of $\vec{k}$'s such that

$$\alpha_{ij}k^i k^j + 2\beta_i k^i |\vec{k}|_\delta = 0 , \tag{2.15}$$

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where $\beta_i = R_{0i}(p)$, $\alpha_{ij} = R_{ij}(p) + R_{00}(p)\delta_{ij}$. Let $\vec{k}_0 \in \mathscr{U}$, after rescaling we can assume that $|\vec{k}_0|_\delta = 1$. Let $\vec{m}$ be any vector such that $\vec{m} \cdot \vec{k}_0 = 0$, $|\vec{m}|_\delta = 1$. Setting $\vec{k} = \vec{k}_0 + x\vec{m}$ in (2.15) one obtains an equation of the form

$$f(x) = a + bx + cx^2 + (d + ex)\sqrt{1 + x^2} = 0 \ ,$$

for all x in a neighborhood of zero. As f is analytic in a neighborhood of zero, all the coefficients of its power series there vanish, leading to $a = b = c = d = e = 0$, which is equivalent to

$$\alpha_{ij}k_0^i k_0^j = \alpha_{ij}k_0^i m^j = \alpha_{ij}m^i m^j = \beta_i k_0^i = \beta_i m^i = 0 \ .$$

Since m is arbitrary we obtain $\beta_i = 0$, and $\alpha_{ij} = 0$ follows by polarisation.$\square$

Returning to the proof of Theorem 1.3, Lemma 2.8 shows that the Ricci tensor is proportional to the metric. By a Bianchi identity the proportionality factor must be a constant, and asymptotic flatness shows that the constant is zero. Therefore $(\mathscr{M}, g)$ is null geodesically complete and vacuum, and we conclude by [5]. $\square$

A   Null geodesics extending to infinity in uniformly Schwarzschildian space-times

In this appendix we study null geodesics in $\mathscr{M}_{\text{ext}}$, as needed in the argument above. We start by showing existence of null geodesics $\Gamma(s)$ with freely prescribable asymptotic direction, which can without loss of generality be assumed to be $e_0 + e_n$, as well as freely prescribable values

$$\beta^a := \lim_{s\to\infty} \Gamma^a(s) \ , \quad a = 1, \ldots, n-1 \ .$$

We will assume that there exist constants $C_0 > 0$ and $\epsilon > 0$ such that, in the coordinates of (1.1) on $\mathscr{M}_{\text{ext}}$,

$$|g - g_m| \le C_0 r^{-1-\epsilon} \ , \quad |\partial_\mu (g - g_m)| \le C_0 r^{-2-\epsilon} \ , \tag{A.1}$$

$$|\partial_\mu \partial_\nu (g - g_m)| \le C_0 r^{-3-\epsilon} \ . \tag{A.2}$$

When $m = 0$ and g is flat in the asymptotic region all the results established in this appendix hold trivially, and therefore from now on we assume that $m \ne 0$. On the other hand, for the purposes of the analysis of the behavior of the null geodesics in this appendix, the separation of g into a Schwarzschild part and a remainder is only relevant in dimension $n = 3$, so to avoid unnecessary discussions we assume here that

$$0 < \epsilon \le \max(1, n-3) \ .$$

A.1 Asymptotic behavior of null geodesics

Let

$$\alpha = (\alpha^\mu) = (T, \beta = (\beta^a), R) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} , \qquad (A.3)$$

$$b = |\beta|_\delta ,$$

we want to construct geodesics

$$[0, \infty) \ni s \to x^\mu(s) \in \mathscr{M}_{\text{ext}}$$

of the form

$$x^\mu(s) = \alpha^\mu + s k^\mu + \underbrace{\psi_0^\mu(s) + \delta\psi^\mu(s)}_{\psi^\mu(s)} , \qquad k = (k^\mu) = (1, \vec{k}) , \; |\vec{k}|_\delta = 1 . \quad (A.4)$$

The function ψ_0^μ is chosen so as to eliminate one of the leading order terms in the geodesic equation for $n = 3$ and $\mu = 0$. So we set $\psi_0^\mu \equiv 0$ unless $n = 3$ and $\mu = 0$, in which case we set

$$\psi_0^0(s) = 2m \ln \left(\frac{R + s + \sqrt{(R+s)^2 + b^2}}{2} \right) ,$$

so that

$$\dot\psi_0^0(s) = \frac{2m}{\sqrt{(R+s)^2 + b^2}} .$$

PROPOSITION A.1 *Let $n \geq 3$ and let g satisfy (A.1)-(A.2) on $\mathscr{M}_{\text{ext}}$ with some $0 < \epsilon \leq \max(1, n-3)$. There exist constants $\mathring{R}$ and $\mathring{C}$ such that for every $\vec{k} \in S^{n-1} \subset \mathbb{R}^n$, $(T, \beta) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $R \geq \mathring{R}$ there exists an affinely parameterized null geodesic of the form (A.3)-(A.4) satisfying*

$$|\delta\psi| + \sqrt{(R+s)^2 + |\beta|_\delta^2} \, |\delta\dot\psi| \leq \frac{\mathring{C}}{((R+s)^2 + |\beta|_\delta^2)^{\epsilon/2}} . \qquad (A.5)$$

REMARK A.2 If the metric has a smooth, or polyhomogeneous, conformal completion, one immediately obtains a polyhomogeneous expansion for $\delta\psi$ in terms of s.

PROOF: $x^\mu(s)$ will be an affinely parameterized geodesic if and only if

$$\frac{d^2\delta\psi^\mu}{ds^2} = F^\mu(\delta\psi, \delta\dot\psi, s) ,$$

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where a dot denotes an s–derivative, and

$$F^\mu(\chi, \lambda, s) = -\left(\Gamma^\mu_{\nu\rho} \circ (\alpha + sk + \psi_0 + \chi)\right)(k^\nu + \dot\psi_0^\nu + \lambda^\nu)(k^\rho + \dot\psi_0^\rho + \lambda^\rho) - \frac{d^2\psi_0^\mu}{ds^2} \ .$$

From (1.1) and (A.1)-(A.2) we have $\Gamma^\mu_{\nu\rho} = O(r^{-2-\epsilon})$ except for

$$\Gamma^k_{00} = \Gamma^0_{k0} = \Gamma^0_{0k} = \frac{(n-2)m}{r^{n-1}} \frac{x^k}{r} + O(r^{-2-\epsilon}) \ , \qquad \text{(A.6)}$$

$$\Gamma^k_{ij} = \frac{m}{r^{n-1}} \left(\delta_{ij}\frac{x^k}{r} - \delta_{jk}\frac{x^i}{r} - \delta_{ik}\frac{x^j}{r}\right) + O(r^{-2-\epsilon}) \ . \qquad \text{(A.7)}$$

We will write $\delta\Gamma^\mu_{\nu\rho}$ for all the remainder terms not explicitly listed above.

In dimension three one has

$$\begin{aligned}
F^0 &= -\frac{2m}{r^2}\left(\left(\sum_i \frac{x^i}{r}(1 + \dot\psi_0^0 + \lambda^0)(k^i + \lambda^i)\right) - \frac{r^2(R+s)}{((R+s)^2 + b^2)^{3/2}}\right) \\
&\quad -\delta\Gamma^0_{\mu\nu}(k^\mu + \dot\psi_0^\mu + \lambda^\mu)(k^\nu + \dot\psi_0^\nu + \lambda^\nu) \ , \qquad\qquad \text{(A.8)} \\
F^i &= -\frac{m}{r^2}\left(\frac{x^i}{r}\left(|\vec{k} + \lambda|_\delta^2 + (1 + \dot\psi_0^0 + \lambda^0)^2\right) - 2(k^i + \lambda^i)\left(\sum_j \frac{x^j}{r}(k^j + \lambda^j)\right)\right) \\
&\quad -\delta\Gamma^i_{\mu\nu}(k^\mu + \dot\psi_0^\mu + \lambda^\mu)(k^\nu + \dot\psi_0^\nu + \lambda^\nu) \ . \qquad\qquad \text{(A.9)}
\end{aligned}$$

Here, and in all the calculations that follow, x^μ has to be replaced by $\alpha^\mu + sk^\mu + \psi_0^\mu + \chi^\mu$.

On the other hand, in higher dimension it will suffice to estimate directly from

$$F^\alpha = -\Gamma^\alpha_{\mu\nu}(k^\mu + \lambda^\mu)(k^\nu + \lambda^\nu) \ .$$

We choose

$$0 < \sigma < \epsilon \ ,$$

and we let $X \subset C([0,\infty)) \times C([0,\infty))$ denote the set of pairs of maps into $\mathbb{R}^{n+1}$ such that the following norm is finite:

$$\|(\chi, \lambda)\|_{R,b} = \|(R + s + b)^\sigma \chi\|_{L^\infty([0,\infty))} + \frac{2\|(R + s + b)^{1+\sigma}\lambda\|_{L^\infty([0,\infty))}}{1+\sigma} \ . \qquad \text{(A.10)}$$

Let $B(1) \subset X$ be the unit closed ball in X and define the map

$$B(1) \ni (\chi, \lambda) \to \mathscr{T}(\chi, \lambda) = \left(-\int_s^\infty \lambda(u)\,du \ , \ -\int_s^\infty F^\mu(\chi, \lambda, u)\,du\right) \ . \qquad \text{(A.11)}$$

Clearly a fixed point (χ_0, λ_0) of $\mathscr{T}$ provides an affinely parameterized geodesic of the form (A.4), with $|\delta\psi(s)| = |\chi_0| \le (R + s + b)^{-\sigma}$, and with

$$\dot{x}^\mu \to_{s\to\infty} k^\mu . \tag{A.12}$$

In order to check that $\mathscr{T}$ does indeed have a fixed point, note first the elementary inequality: for $R + s \ge 0$, $b \ge 0$,

$$\sqrt{(R+s)^2 + b^2} \le R + s + b \le \sqrt{2}\sqrt{(R+s)^2 + b^2} . \tag{A.13}$$

It follows that for $(\chi, \lambda) \in B(1)$ and for R sufficiently large we have

$$\frac{1}{2}(R + s + b) \le r \le 2(R + s + b) , \tag{A.14}$$

$$|\rho - b| \le \frac{C}{(R + s + b)^\sigma} , \tag{A.15}$$

$$|x^n - R - s| = |\chi^n(s)| \le \frac{1}{(R + s + b)^\sigma} . \tag{A.16}$$

This implies

$$
\begin{aligned}
|r - \sqrt{(R+s)^2 + b^2}| &= \left| \frac{r^2 - (R+s)^2 - b^2}{r + \sqrt{(R+s)^2 + b^2}} \right| \\
&= \left| \frac{(x^n - (R+s))(x^n + (R+s)) + (\rho - b)(\rho + b)}{r + \sqrt{(R+s)^2 + b^2}} \right| \\
&\le \frac{1 + C}{(R + s + b)^\sigma} .
\end{aligned} \tag{A.17}
$$

Equation (A.14) immediately gives an estimate

$$|\delta\Gamma^\sigma_{\mu\nu}(k^\mu + \dot\psi^\mu_0 + \lambda^\mu)(k^\nu + \dot\psi^\nu_0 + \lambda^\nu)| \le \frac{C_2}{(R + s + b)^{2+\epsilon}} .$$

Writing

$$F^\mu(\chi, \lambda) = \chi \int_0^1 \frac{\partial F^\mu}{\partial \chi}(s\chi, \lambda)ds + \lambda \int_0^1 \frac{\partial F^\mu}{\partial \lambda}(0, s\lambda)ds + F^\mu(0, 0) ,$$

with a little more work a similar estimate is obtained for the whole of F^μ using (A.15)-(A.17). This leads to

$$\left| \int_s^\infty F^\mu(\chi, \lambda, u)du \right| \le C_3(R + s + b)^{-1-\epsilon} \le C_3 R^{\sigma-\epsilon}(R + s + b)^{-(1+\sigma)} ,$$

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for some $C_3 = C_3(C_1, |m|, \epsilon)$, with C_3 also depending on the constant dominating the error terms in (A.1). Increasing R_1 if necessary one finds that $\mathscr{T}$ maps $B(1)$ into $B(1)$ for all $R \geq R_1$.

One similarly checks that the map $\mathscr{T}$ defined in (A.11) is a contraction for R sufficiently large, uniformly in b, and the existence of a geodesic of the form (A.4), with $\delta\psi(s) \rightarrow_{s\to\infty} 0$, ensues. Once we know that the solution just described exists, it is straightforward to show, using the geodesic equation and the estimates already available, that $\delta\psi$ satisfies (A.5). $\qquad\square$

We will need to know that all geodesics extending to infinity are as above:

PROPOSITION A.3 *Let g satisfy (A.1)-(A.2) on $\mathscr{M}_{\mathrm{ext}}$. Then every null geodesic $\Gamma(s)$ such that*

$$r(\Gamma(s)) \rightarrow_{s\to\infty} \infty \qquad\qquad (A.18)$$

satisfies (A.3)-(A.5).

PROOF: The existence of k^μ is provided by Proposition B.1 in [2]. From that last proposition one also has a uniform bound on dx^μ/ds, as well as the estimate (A.14), and the remaining claims easily follow by inspection of the geodesic equation. $\qquad\square$

COROLLARY A.4 *The map which to every maximally extended, null geodesic satisfying (A.18) assigns k, β and T is bijective.*

PROOF: It remains to prove injectivity. But Proposition A.3 implies that geodesics satisfying (A.18) are solutions of the fixed point problem considered in the proof of Proposition A.1, and are therefore unique. $\qquad\square$

Let $\Gamma_{k,T,\beta}(s)$ denote the unique affinely parameterized maximally extended geodesic provided by Corollary A.4. The differentiability properties of $\Gamma_{k,T,\beta}$ with respect to k, T and β are best studied by considering Jacobi fields along Γ. The method of proof of Proposition A.1 applies to give:

PROPOSITION A.5 *Let $\Gamma : [0,\infty)$ be a null affinely parameterized geodesic satisfying (A.18). Then for every $A, B \in \mathbb{R}^{n+1}$ there exists a solution Z of the Jacobi equation,*

$$\frac{D^2 Z}{ds^2}(s) = \mathrm{Riem}(\dot\gamma, Z)\dot\gamma , \qquad\qquad (A.19)$$

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where Riem *denotes the Riemann tensor, such that*

$$Z^\mu = \begin{cases} (R+s)A^\mu + O(\ln(R+s+\sqrt{(R+s)^2+b^2})), & A^\mu \neq 0,\ n=3; \\ (R+s)A^\mu + O((R+s+b)^{-\epsilon}), & A^\mu \neq 0,\ n>3; \\ B^\mu + O((R+s+b)^{-(1+\epsilon)}), & A^\mu = 0, \end{cases} \tag{A.20}$$

with the error terms being uniform in b.

Similarly to Proposition A.3, every Jacobi field along Γ as in Proposition A.5 is a linear combination of solutions of the form (A.20); this is proved by usual techniques.

It is standard to show now

PROPOSITION A.6 *The map*

$$S^{n-1} \times \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} \supset \mathscr{U} \ni (\vec{k}, T, \beta, s) \to \Gamma_{(1,\vec{k}),T,\beta}(s)$$

is differentiable on the open set $\mathscr{U}$ on which it is defined.

Fix β, R, with $R \geq R_1$, and let

$$\Phi : \mathbb{R} \to \mathbb{R} \tag{A.21}$$

be the map which to $T \in \mathbb{R}$ assigns $x^0(0)$, the coordinate time at which the null geodesic, as constructed in the proof of Proposition A.1, intersects $\{x^n = R\}$. Proposition A.5 (with $A = 0$) and standard ODE considerations show that Φ is differentiable, with strictly positive derivative (perhaps increasing R_1 if necessary) and hence strictly increasing, in particular τ is continuous. As the constant $\mathring{C}$ in (A.5) is uniform in T, one clearly has

$$\tau(T) \to_{T\to\pm\infty} \pm\infty .$$

It follows that Φ is a smooth bijection from $\mathbb{R}$ to itself.

A.2 The optical functions $S^\pm$

There exists $\mathring{R}_1$ such that, using the implicit function theorem, one can define a function

$$S^+ : \{x^n \geq \mathring{R}_1\} \to \mathbb{R} \tag{A.22}$$

by assigning T to each point lying on $\Gamma_{k,T,\beta}$. Then S^+ is differentiable, with null gradient, and satisfies

$$S^+ = x^0 - x^n + \mathring{R} - \begin{cases} 2m\ln\left(\frac{x^n+\sqrt{(x^n)^2+\rho^2}}{2}\right) + o(1), & n=3; \\ o(1), & n \geq 4, \end{cases} \tag{A.23}$$

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with the error term tending to zero as $r(x)$ goes to infinity, uniformly with respect to $(x^1, \ldots, x^{n-1}) \in \mathbb{R}^{n-1}$. We give some details of the construction, because the function S^+ plays an important role in Section 2; furthermore, the proof of existence of S^+ (and its time-reverse counterpart S^-), with the required properties, is the missing step in the argument in [13]. Let

$$(y^\mu) = (T, \beta, s) \,,$$

then (A.4) with $k^\mu = (1, 0, \ldots, 0, 1)$ and $R = \mathring{R}$ (as given by Proposition A.1) defines a map $x^\mu(y^\alpha)$. We have

1. $\partial x^\mu / \partial y^0$ at the point $x^\mu(y^\alpha)$ is given by the value of the Jacobi field Z^μ which asymptotes to δ_0^μ as $s = y^n$ tends to infinity; from the proof of Proposition A.5 we thus have

$$\frac{\partial x^\mu}{\partial y^0} = \delta_0^\mu + O\left(\left(\mathring{R} + y^n + \sqrt{\sum_a (y^a)^2}\right)^{-1}\right) \,.$$

2. Similarly $\partial x^\mu / \partial y^a$ at $x^\mu(y^\alpha)$ is given by the value of the Jacobi field Z^μ which asymptotes to δ_a^μ as $s = y^n$ tends to infinity; from the proof of Proposition A.5 one finds

$$\frac{\partial x^\mu}{\partial y^a} = \delta_a^\mu + O\left(\left(\mathring{R} + y^n + \sqrt{\sum_a (y^a)^2}\right)^{-1}\right) \,.$$

3. Directly from its definition, $\partial x^\mu / \partial y^n = \partial x^\mu / \partial s$ equals $k^\mu + d\psi^\mu / ds$, so that

$$\frac{\partial x^\mu}{\partial y^n} = (1, 0, \ldots, 0, 1) + O\left(\left(\mathring{R} + y^n + \sqrt{\sum_a (y^a)^2}\right)^{-1}\right) \,.$$

Increasing $\mathring{R}$ if necessary, it follows that $y^\mu \to x^\mu$ is a smooth local diffeomorphism for $r(y) \geq \mathring{R}$.

To prove bijectivity with the image we need the following, certainly well known, result:

PROPOSITION A.7 *Let $\mathscr{O} \subset \mathbb{R}^n$ be an open convex set and let $\Phi : \mathscr{O} \to \mathbb{R}^n$ be continuously differentiable. Then there exists $\epsilon = \epsilon(n) > 0$ such that if*

$$|\frac{\partial \Phi^\mu}{\partial y^\nu} - \delta_\nu^\mu| < \epsilon \,, \tag{A.24}$$

then Φ is injective. If $\mathscr{O} = \mathbb{R}^n$ then $\Phi(\mathscr{O}) = \mathbb{R}^n$.

PROOF: The calculation follows those in [7, Theorem 3.1]. We have

$$\left| \Phi(x) - \Phi(y) - (x - y) \right| = \left| \int_0^1 \Big(\Phi'(tx + (1-t)y) - \mathrm{Id} \Big)(x-y)dt \right|$$
$$\leq C(n)\epsilon|x-y| \ .$$

This implies

$$|x - y| \leq |\Phi(x) - \Phi(y)| + |\Phi(x) - \Phi(y) - (x-y)|$$
$$\leq |\Phi(x) - \Phi(y)| + C(n)\epsilon|x-y| \ .$$

If $C(n)\epsilon < 1$ we obtain

$$|x - y| \leq \frac{1}{1 - C(n)\epsilon}|\Phi(x) - \Phi(y)| \ , \tag{A.25}$$

and injectivity follows. By the implicit function theorem Φ is open. It further follows from (A.25) that if the sequence $(\Phi(x_i))_{i \in \mathbb{N}}$ is Cauchy, then so is the sequence $(x_i)_{i \in \mathbb{N}}$. This shows that if $\mathcal{O} = \mathbb{R}^n$ then $\Phi(\mathcal{O})$ is closed, proving surjectivity. $\square$

For R large enough consider the map Φ_R which to $(T, \beta) \in \mathbb{R}^{n-1}$ assigns $x^\mu(s)$, where $s = s(T, \beta)$ is chosen so that $x^n(s) = R$. By arguments similar to the ones already given one finds that Φ_R satisfies the hypotheses of Proposition A.7 for $R \geq \mathring{R}_1$, increasing $\mathring{R}_1$ if necessary. Hence the Φ_R's are bijective for $R \geq \mathring{R}_1$, which implies that every point in the region $\{x^n \geq \mathring{R}_1\}$ lies on *precisely one* null geodesic with asymptotic velocity $(1, 0, \ldots, 0, 1)$. It follows that the map $y^\mu \to x^\mu(y^\alpha)$ is injective onto $\{x^n \geq \mathring{R}_1\}$, and therefore is a global diffeomorphism from some set $\mathcal{U} \subset [0, \infty) \times \mathbb{R}^n$ to $\{x^n \geq \mathring{R}_1\}$. Inverting this map one obtains smooth functions $y^\mu(x^\alpha)$ on $\{x^n \geq \mathring{R}_1\}$. The function S^+ is then set to be equal to $y^0(x^\mu)$. Equation (A.23) is obtained by eliminating s and β from the definition

$$S^+(x^\mu) = T(x^\mu) = x^0(s) - s - \psi^0(s)$$

using equations (A.4).

As already pointed out, the function S^- is obtained by a time-reverse of the construction just carried out.

Because the estimates above are uniform in $r(y)$ for $r(y)$ large, it should be clear that the domain of definition of S^+ can be extended to a region of the form $\{r \geq \mathring{R}_1 , x^n \geq 0\}$, increasing $\mathring{R}_1$ if necessary. Similarly S^- can be

defined on a region of the form $\{r(x) \geq \mathring{R}_1 \,, x^n \leq 0\}$. Those extensions are relevant to the original Penrose-Sorkin-Woolgar argument.

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