A poor man's positive energy theorem

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Abstract

We show that positivity of energy for stationary, asymptotically flat, non-singular domains of outer communications is a simple corollary of the Lorentzian splitting theorem.

A milestone in mathematical general relativity is the positive energy theorem [8, 10]. It is somewhat surprising that the standard proofs of this result do not have anything to do with Lorentzian geometry. In this note we prove energy positivity using purely Lorentzian techniques, albeit for a rather restricted class of geometries; it seems that in practice our proof only applies to stationary (with or without black holes) space-times. This is a much weaker statement than the theorems in [8, 10] and their various extensions, but the proof here seems of interest because the techniques involved are completely different and of a quite elementary nature. Using arguments rather similar in spirit to those of the classical singularity theorems [5], our proof is a very simple reduction of the problem to the Lorentzian splitting theorem [4]. (In lieu of the Lorentzian splitting theorem, one can impose

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the *generic condition* [5, p. 101], thereby making the proof completely elementary.) The approach taken here bares some relation to the Penrose-Sorkin-Woolgar [7] argument for positivity of mass, and indeed arose out of an interest in understanding their work.

For $m \in \mathbb{R}$, let g_m denote the $n+1$ dimensional, $n \geq 3$, Schwarzschild metric with mass parameter m; in isotropic coordinates [6],

$$g_m = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \left(\sum_{1=1}^{n} dx_i^2\right) - \left(\frac{1 - m/2|x|^{n-2}}{1 + m/2|x|^{n-2}}\right)^2 dt^2 \ . \qquad (1)$$

We shall say that a metric g on $\mathbb{R} \times (\mathbb{R}^n \setminus B(0,R))$, $R^{n-2} > m/2$, is *uniformly Schwarzchildian* if, in the coordinates of (1),

$$g - g_m = o(|m| r^{-(n-2)}) \ , \quad \partial_\mu \left(g - g_m\right) = o(|m| r^{-(n-1)}) \ . \qquad (2)$$

(Here o is meant at fixed g and m, uniformly in t and in angular variables, with r going to infinity.) It is a flagrant abuse of terminology to allow $m = 0$ in this definition, and we will happily abuse; what is meant in this case is that $g = g_0$, i.e., g is flat [1], for $r > R$.

Some comments about this notion are in order. First, metrics as above have constant Trautman-Bondi mass and therefore do not contain gravitational radiation; one expects such metrics to be stationary if physically reasonable field equations are imposed. Next, every metric in space-time dimension four which is stationary, asymptotically flat and vacuum or electrovacuum in the asymptotically flat region is uniformly Schwarzschildian there when $m \neq 0$ (cf., e.g., [9]).

The hypotheses of our theorem below are compatible with stationary black hole space-times with non-degenerate Killing horizons.

We say that the matter fields satisfy the *timelike convergence condition* if the Ricci tensor $R_{\mu\nu}$, as expressed in terms of the matter energy-momentum tensor $T_{\mu\nu}$, satisfies the condition

$$R_{\mu\nu} X^\mu X^\nu \geq 0 \text{ for all timelike vectors } X^\mu. \qquad (3)$$

We define the *domain of outer communications* of $\mathscr{M}$ as the intersection of the past $J^-(\mathscr{M}_{\rm ext})$ of the asymptotic region $\mathscr{M}_{\rm ext} = \mathbb{R} \times (\mathbb{R}^n \setminus B(0,R))$ with its future $J^+(\mathscr{M}_{\rm ext})$.

[1]The asymptotic conditions for the case $m = 0$ of our theorem are way too strong for a rigidity statement of real interest, even within a stationary context. So it is fair to say that our result only excludes $m < 0$ for stationary space-times.

We need a version of weak asymptotic simplicity [5] for uniformly Schwarzschildian spacetimes. We shall say that such a spacetime $(\mathscr{M}, g)$ is *weakly asymptotically regular* if every null line starting in the domain of outer communications (DOC) either crosses an event horizon (if any), or reaches arbitrarily large values of r in the asymptotically flat region. By definition, a null line in $(\mathscr{M}, g)$ is an inextendible null geodesic that is globally achronal; a timelike line is an inextendible timelike geodesic, each segment of which is maximal. Finally, we shall say that the DOC is *timelike geodesically regular* if every timelike line in $\mathscr{M}$ which is entirely contained in the DOC, and along which r is bounded, is complete.

Our main result is the following:

THEOREM 1 *Let $(\mathscr{M}^{n+1} = \mathscr{M}, g)$ be an $(n+1)$-dimensional space-time with matter fields satisfying the timelike convergence condition, and suppose that $\mathscr{M}$ contains a uniformly Schwarzschildian region*

$$\mathscr{M}_{\mathrm{ext}} = \mathbb{R} \times (\mathbb{R}^n \setminus B(0, R)) \ . \tag{4}$$

Assume that $(\mathscr{M}, g)$ is weakly asymptotically regular and that the domain of outer communications is timelike geodesically regular. If the domain of outer communications of $\mathscr{M}$ has a Cauchy surface $\mathscr{S}$, the closure of which is the union of one asymptotic end and of a compact interior region (with a boundary lying at the intersection of the future and past event horizons, if any), then $m > 0$, unless $(\mathscr{M}, g)$ isometrically splits as $\mathbb{R} \times \mathscr{S}$ with metric $g = -d\tau^2 + \gamma$, $\mathscr{L}_{\partial_\tau}\gamma = 0$, and $(\mathscr{S}, \gamma)$ geodesically complete. Furthermore, the last case does not occur if event horizons are present.

Before passing to the proof, we note the following Corollary:

COROLLARY 2 *In addition to the hypotheses of Theorem 1, assume that*

$$T_{\mu\nu} \in L^1 \left(\mathbb{R}^n \setminus B(0, R)\right) \ , \qquad \partial_\nu \partial_\mu g = O(r^{-\alpha}) \ , \quad \alpha > 1 + \frac{n}{2} \ . \tag{5}$$

Then $m > 0$ unless $\mathscr{M}$ is the Minkowski space-time.

PROOF OF THEOREM 1: The idea is to show that for $m \leq 0$ the domain of outer communications contains a timelike line, and the result then follows from the version of the Lorentzian splitting theorem obtained in [4]. A straightforward computation shows that the Hessian $\mathrm{Hess}\, r = \nabla dr$ of r is given by

$$\mathrm{Hess}\, r = -\frac{m}{r^{n-1}} \left((n-2)dt^2 - dr^2 + r^2\, h\right) + r\, h + o(r^{-(n-1)}) \ , \tag{6}$$

where h is the canonical metric on S^n, and the size of the error terms refers to the components of the metric in the coordinates of (1). Note that when $m < 0$, $\mathrm{Hess}\, r$, when restricted to the hypersurfaces of constant r, is strictly positive definite for $r \geq R_1$, for some sufficiently large R_1. Increasing R_1 if necessary, we can obtain that ∂_t is timelike for $r \geq R_1$. If $m = 0$ we set $R_1 = R$. Let $p_{\pm k}$ denote the points $t = \pm k$, $\vec{x} = (0, 0, R_1)$; by global hyperbolicity there exists a maximal future directed timelike geodesic segment σ_k from p_{-k} to p_{+k}. We note, first, that the σ_k's are obviously contained in the domain of outer communications and therefore cannot cross the event horizons, if any. If $m = 0$ then σ_k clearly cannot enter $\{r > R_1\}$, since timelike geodesics in that region are straight lines which never leave that region once they have entered it. It turns out that the same occurs for $m < 0$: suppose that σ_k enters $\{r > R_1\}$, then the function $r \circ \sigma_k$ has a maximum. However, if s is an affine parameter along σ_k we have

$$\frac{d^2(r \circ \sigma_k)}{ds^2} = \mathrm{Hess}\, r(\dot{\sigma}_k, \dot{\sigma}_k) > 0$$

at the maximum if $m < 0$, since $dr(\dot{\sigma}_k) = 0$ there, which is impossible. It follows that all the σ_k's (for k sufficiently large) intersect the Cauchy surface $\mathscr{S}$ in the compact set $\overline{\mathscr{S}} \setminus \{r > R_1\}$. A standard argument shows then that the σ_k's accumulate to a timelike or null line σ through a point $p \in \overline{\mathscr{S}}$. Let $\{p_k\} = \sigma_k \cap \mathscr{S}$; suppose that $p \in \partial\mathscr{S}$, then the portions of σ_k to the past of p_k accumulate at a generator of the past event horizon $\dot{J}^+(\mathscr{M}_{\mathrm{ext}})$, and the portions of σ_k to the future of p_k accumulate at a generator of the future event horizon $\dot{J}^-(\mathscr{M}_{\mathrm{ext}})$. This would result in σ being non-differentiable at p, contradicting the fact that σ is a geodesic. Thus the p_k's stay away from $\partial\mathscr{S}$, and $p \in \mathscr{S}$. By our "weak asymptotic regularity" hypothesis σ cannot be null (as it does not cross the event horizons, nor does it extend arbitrarily far into the asymptotic region). It follows that σ is a timelike line in $\mathscr{M}$ entirely contained in the globally hyperbolic domain of outer communications $\mathscr{D}$, with $r \circ \sigma$ bounded, and hence is complete by the assumed timelike geodesic regularity of $\mathscr{D}$. Thus, one may apply [4] to conclude that $(\mathscr{D}, g|_{\mathscr{D}})$ is a metric product,

$$g = -d\tau^2 + \gamma\,, \tag{7}$$

for some τ–independent complete Riemannian metric γ. The completeness of this metric product implies $\mathscr{D} = \mathscr{M}$ (and in particular excludes the existence of event horizons). $\square$

Proof of Corollary 2: The lapse function N associated with a Killing vector field on a totally geodesic hypersurface $\mathscr{S}$ with induced metric γ and unit normal n satisfies the elliptic equation

$$\Delta_\gamma N - \mathrm{Ric}(n,n)N = 0 \ .$$

The vector field ∂_τ is a static Killing vector in $\mathscr{M}_{\rm ext}$, and the usual analysis of groups of isometries of asymptotically flat space-times shows that the metric γ in (7) is asymptotically flat. Again in (7) we have $N = 1$ hence $\mathrm{Ric}(n,n) = 0$, and the Komar mass of $\mathscr{S}$ vanishes. By a theorem of Beig [2] (originally proved in dimension four, but the result generalises to any dimensions under (5)) this implies the vanishing of the ADM mass. Let e_a, $a = 0,\ldots,n$, be an orthonormal frame with $e_0 = \partial_\tau$. The metric product structure implies that $R_{0i} = 0$. Thus, by the energy condition, for any fixed i we have

$$0 \leq \mathrm{Ric}(e_0 + e_i, e_0 + e_i) = R_{00} + R_{ii} = R_{ii} \ .$$

But again by the product structure, the components R_{ii} of the space-time Ricci tensor equal those of the Ricci tensor $\mathrm{Ric}_{\mathscr{S}}$ of γ. It follows that $\mathrm{Ric}_{\mathscr{S}} \geq 0$. A generalisation by Bartnik [1] of an argument of Witten [10] shows that $(\mathscr{S},\gamma)$ is isometric to Euclidean space; we reproduce the proof to make clear its elementary character: Let y^i be global harmonic functions forming an asymptotically rectangular coordinate system near infinity. Let $K^i = \nabla y^i$; then by Bochner's formula,

$$\Delta |K^i|^2 = 2|\nabla K^i|^2 + 2\mathrm{Ric}_{\mathscr{S}}(K^i,K^i) \ .$$

Integrating the sum over $i = 1,\ldots,n$ of this gives the ADM mass as boundary term at infinity, and the ∇y^i are all parallel. Since $\mathscr{S}$ is simply connected at infinity, it must be Euclidean space. $\square$

We close this note by showing that the conditions on geodesics in Theorem 1 are always satisfied in stationary domains of outer communications.

Proposition 3 *Let the domain of outer communications $\mathscr{D}$ of $(\mathscr{M},g)$ be globally hyperbolic, with a Cauchy surface $\mathscr{S}$ such that $\overline{\mathscr{S}}$ is the union of a finite number of asymptotically flat regions and of a compact set (with a boundary lying at the intersection of the future and past event horizons, if any). Suppose that there exists on $\mathscr{M}$ a Killing vector field X with complete orbits which is timelike, or stationary-rotating*[2] *in the asymptotically flat regions. Then the weak asymptotic regularity and the timelike regularity conditions hold.*

[2] See [3] for the definition.

REMARK 4 We note that there might exist maximally extended null geodesics in $(\mathscr{D}, g)$ which are trapped in space within a compact set (as happens for the Schwarzschild metric), but those geodesics will not be achronal.

PROOF: By [3, Propositions 4.1 and 4.2] we have $\mathscr{D} = \mathbb{R} \times \mathscr{S}$, with the flow of X consisting of translations along the $\mathbb{R}$ axis:

$$g = \alpha d\tau^2 + 2\beta d\tau + \gamma , \quad X = \partial_\tau , \tag{8}$$

where γ is a Riemannian metric on $\mathscr{S}$ and β is a one-form on $\mathscr{S}$. (We emphasise that we do not assume X to be timelike, so that $\alpha = g(X, X)$ can change sign.) Let ϕ_t denote the flow of X and let $\sigma(s) = (\tau(s), p(s)) \in \mathbb{R}\times\mathscr{S}$ be an affinely parameterised causal line in $\mathscr{D}$, then for each $t \in \mathbb{R}$ the curve $\phi_t(\sigma(s)) = (\tau(s) + t, p(s))$ is also an affinely parameterised causal line in $\mathscr{D}$. Suppose that there exists a sequence s_i such that $p(s_i) \to \partial\mathscr{S}$, setting $t_i = -\tau(s_i)$ we have $\tau(\phi_{t_i}(\sigma(s_i)) = 0$, then the points $\{p_{k_i}\} = \phi_{t_i}(\sigma) \cap \mathscr{S}$ accumulate at $\partial\mathscr{S}$, which is not possible as in the proof of Theorem 1. Therefore there exists an open neighborhood $\mathscr{K}$ of $\partial\mathscr{S}$ such that $\sigma \cap (\mathbb{R} \times \mathscr{K}) = \emptyset$. This implies in turn that σ meets all the level sets of τ. Standard considerations using the fact that $\mathscr{D}$ is a stationary, or stationary-rotating domain of outer communications (cf., e.g., [3]) show that for every $p, q \in \mathscr{S}$ there exists $T > 0$ and a timelike curve from $(0, p)$ to (T, q). The constant T can be chosen independently of p and q within the compact set $\overline{\mathscr{S}} \setminus (\mathscr{K} \cup \{r > R_1\})$, with $R_1 = \sup_\sigma r$. It follows that an inextendible null geodesic which is bounded in space within a compact set cannot be achronal, so that σ has to reach arbitrarily large values of r, and weak asymptotic regularity follows. Similarly, if σ is a timelike line bounded in space within a compact set, then there exists $s_1 > 0$ such that for any point $(\tau(s), p(s))$ with $s = s_1 + u$, $u > 0$ one can find a timelike curve from $(0, p(0))$ to $(\tau(s), p(s))$ by going to the asymptotic region, staying there for a time u, and coming back. The resulting curve will have Lorentzian length larger than $u/2$ if one went sufficiently far into the asymptotic region, and since σ is length-maximising it must be complete. □

The key point in the proof of Proposition 3 is non-existence of observer horizons contained in the DOC. Somewhat more generally, we have the following result, which does not assume existence of a Killing vector:

PROPOSITION 5 *Suppose that causal lines σ, with $r \circ \sigma$ bounded, and which are contained entirely in $\mathscr{D}$, do not have* observer horizons *extending to the asymptotic region $\mathscr{M}_{\rm ext}$ (see (4)):*

$$\dot{J}^\pm(\sigma; \mathscr{D}) \cap \mathscr{M}_{\rm ext} = \emptyset . \tag{9}$$

Then the weak asymptotic regularity and the timelike regularity conditions hold.

Proof: It follows from (9) that for any $u > 0$ and for any s_1 there exists s_2 and a timelike curve Γ_{u,s_1} from $\sigma(s_1)$ to $\sigma(s_2)$ which is obtained by following a timelike curve from $\sigma(s_1)$ to the asymptotic region, then staying there at fixed space coordinate for a coordinate time u, and returning back to σ along a timelike curve. One concludes as in the proof of Proposition 3. □

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