Recent results in mathematical relativity

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January 3, 2005

Abstract

We review selected recent results concerning the global structure of solutions of the vacuum Einstein equations. The topics covered include quasi-local mass, strong cosmic censorship, non-linear stability, new constructions of solutions of the constraint equations, improved understanding of asymptotic properties of the solutions, existence of solutions with low regularity, and construction of initial data with trapped surfaces or apparent horizons.

This is an expanded version of a plenary lecture, sponsored by *Classical and Quantum Gravity*, held at the GR17 conference in Dublin in July 2004.

1 Introduction

Reviewing recent progress in mathematical relativity is a difficult task, in view of the large number of excellent papers appearing in the field. Choices have to be made because of obvious time limits set for a lecture. In order to narrow down the number of topics covered I will concentrate on those results that concern the *global* properties of solutions of the *vacuum* Einstein equations, and which have appeared in preprint or final form within the last three years.

2 Quasi-local mass

The question of localisation of mass in general relativity has a long history, with no unanimously accepted candidate emerging so far, see [119, 122] and references therein. There are at least two strategies which one might adopt here: trying to isolate a mathematically interesting object, or trying to find a physically relevant one. In the best of the worlds the same quantity would result, but no such thing has been found yet. From a physical point of view the strongest case can be made, I believe, for definitions obtained by Hamiltonian methods. Recall that the geometric symplectic framework of Kijowski and Tulczyjew [80] has been applied to general relativity by Kijowski and collaborators [37, 49, 77–79]), and it allows a systematic treatment of boundary terms, together with associated Hamiltonians, at least at a

*Partially supported by a Polish Research Committee grant 2 P03B 073 24, and by the Institute of Physics, London; email `Piotr.Chrusciel@lmpt.univ-tours.fr`, URL `www.phys.univ-tours.fr/~piotr`

formal level[1]. One of the Hamiltonians that emerges in this way is the following [79]: Consider a three dimensional initial data set (M, g, K) in a four-dimensional space-time $(\mathscr{M}, {}^4g)$. Let Σ be a two dimensional surface within M and suppose that the mean extrinsic curvature vector κ of Σ is *spacelike*. Let

$$\lambda := \sqrt{{}^4g(\kappa, \kappa)}$$

be the ${}^4g$-length of κ. Assuming that the dominant energy condition holds in $(\mathscr{M}, {}^4g)$, it follows from the embedding equations that the Gauss curvature of the metric induced by g on Σ is positive. One can then invoke the Weyl embedding theorem [101, 104] to isometrically embed $(\Sigma, {}^4g|_\Sigma)$ into $\mathbb{R}^3$. We shall denote by λ_0 the associated λ as calculated using the flat metric in $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$. Let m_K be the Kijowski mass of Σ,

$$m_\mathrm{K} = \frac{1}{8\pi} \int_\Sigma (\lambda_0 - \lambda) d^2\mu \ . \tag{1}$$

A surprising theorem of Liu and Yau [92][2] asserts that

$$m_\mathrm{K} \geq 0 \ , \tag{2}$$

with equality if and only if (M, g, K) is a subset of Euclidean $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$. The key to the proof is a similar result by Shi and Tam [117], which is the Riemannian analogue of this statement: Shi and Tam prove that for manifolds of positive scalar curvature, the mean curvature H of a convex surface bounding a compact set satisfies

$$m_\mathrm{BY} = \frac{1}{8\pi} \int_\Sigma (H_0 - H) d^2\mu \geq 0 \ . \tag{3}$$

Here H_0 is the mean curvature of an isometric embedding of ∂M into $\mathbb{R}^3$ (thus H_0 coincides with λ_0). Liu and Yau show that the inequality (2) can be reduced to the Shi-Tam inequality using Jang's equation, in a way somewhat similar to the transition from the "Riemannian" to the "full" Schoen-Yau positive mass theorems [115, 116]. The "quasi-local mass" m_BY appearing in (3) has been introduced, and studied, by Brown and York [26, 27].

In [102] O'Murchadha, Szabados and Tod show that $m_\mathrm{K} \neq 0$ for some surfaces in Minkowski space-time, which indicates that the normalisation in (1) is not optimal; this leaves room for future improvements.

The celebrated papers by Bray [24] and Huisken and Ilmanen [70] proving the Riemannian Penrose inequality have appeared recently (see also [25] and references therein). Those papers settle the problem for initial data sets with *positive Ricci scalar* (this is a restrictive condition which is satisfied, e.g., for maximal initial data sets), proving that the ADM mass is not less than the square root of the area of the outermost minimal surface divided by 4π. Suggestions how to approach the *full* Penrose inequality, without the $R \geq 0$ restriction, have recently been made by Frauendiener and by Malec, Mars and Simon in [61, 94]. In [95] Malec and Ó Murchadha show that a direct approach based on Jang's equation cannot succeed. In [124] Weinstein and Yamada point out that for *connected* charged black holes (with charge Q and with $R = \sqrt{A/4\pi}$ – the area radius of the outermost minimal surface), a Penrose-type inequality involving global charge,

$$m \geq \frac{1}{2}\left(R + \frac{Q^2}{R}\right) \ , \tag{4}$$

[1] Kijowski's analysis leads to symplectic structures on spaces of fields with prescribed boundary data. To obtain a bona fide Hamiltonian system one should prove that the resulting initial-boundary value problems are well posed, which has not been done so far. It would be of interest to analyse how the Friedrich-Nagy [65] initial-boundary value problems fits into this framework.

[2] A gap in [92] has been filled in [93].

follows from the Huisken-Ilmanen proof. They also prove that (4) fails for some initial data sets with two black holes; the proof is a variation upon the Isenberg-Mazzeo-Pollack gluings [71], discussed in Section 5 below.

We finish this section by noting some recent improvements in our understanding of the mass of asymptotically hyperbolic manifolds [44, 50, 52, 123]. Recall, now, that the asymptotically flat positive mass theorem can be used to prove uniqueness of static asymptotically flat black holes [28]; similarly the positive mass theorem for asymptotically hyperbolic manifolds can be used to prove uniqueness results in this context [5, 23] (compare [2, 3, 66, 67] for completely different approaches). A particularly elegant proof of rigidity of hyperbolic space, reducing the problem to the standard asymptotically Euclidean positive energy theorem, has been recently given by Qing [105]. The approach has been exploited in [22] by Bonini, Miao and Qing to considerably extend the rigidity result. Yet another recent approach to rigidity of asymptotically hyperbolic manifolds, based on volume comparison, has been presented by Shi and Tian in [118].

3 Strong cosmic censorship

The *strong cosmic censorship (SCC) problem* concerns predictability: it is a fundamental requirement that solutions of good physical theories should be *uniquely determined by initial data.* This is not the case for Einstein equations — in general relativity there exist examples where uniqueness fails [45,98]. In this context, a key result is a theorem of Choquet-Bruhat and Geroch [32], which states that *to any initial data one can associate, uniquely up to a diffeomorphism, a maximal globally hyperbolic development of those data.* The resulting space-time is unique in the class of globally hyperbolic space-times, but in some situations it can be extended in more than one way to strictly larger vacuum solutions. In such cases the extension always takes places across a null hypersurface called *Cauchy horizon.*

A mathematical formulation of strong cosmic censorship, essentially due to Moncrief and Eardley [100] (*cf.* also [39, 40, 107]), is the following:

> *Consider the collection of initial data for, say, vacuum or electro–vacuum space–times, with the initial data surface $\mathscr{S}$ being compact, or with the initial data $(\mathscr{S}, {}^3g, K)$ — asymptotically flat. For generic such data the maximal globally hyperbolic development thereof is inextendible.*

The reader is referred to [4] for an excellent recent review of SCC.

Because of the difficulty of the strong cosmic censorship problem, a full understanding of the issues which arise in this context seems to be completely out of reach at this stage. There is therefore some interest in trying to understand that question under various restrictive hypotheses, *e.g.*, symmetry. Such a program has been initiated by Moncrief in [99, 100], and some further results in the spatially compact case have been obtained in [39, 46, 53, 73]. The simplest case, of spatially homogeneous space-times, has turned out to be surprisingly difficult, because of the intricacies of the dynamics of some of the Bianchi models. Spectacular progress in the understanding of this last class of solutions has been made a few years ago by Ringström. His results imply, amongst others, that curvature blows up in the contracting direction in generic Bianchi models. This forbids Cauchy horizons (this was already known, by completely different abstract arguments, from the work in [53]), and also provides further information about the geometry near "the end of space-time".

The next simplest case is that of Gowdy metrics on $\mathbb{T}^3 := S^1 \times S^1 \times S^1$: in coordinates $t \in (-\infty, 0)$ and $(\theta, x^1, x^2) \in \mathbb{T}^3$ the metrics are of the form

$$\begin{aligned} g &= e^{-\gamma/2} \mid t \mid^{-1/2} (-dt^2 + d\theta^2) + \mid t \mid e^P (dx^1)^2 + 2 \mid t \mid e^P Q \, dx^1 dx^2 \\ &\quad + \mid t \mid \left(e^P Q^2 + e^{-P} \right) (dx^2)^2 \ , \end{aligned}$$

with ∂_{x^1} and ∂_{x^2} being Killing vectors. The essential part of the evolution equations consists of two coupled non-linear equations:

$$\begin{aligned} \partial_t^2 P - \partial_\theta^2 P &= -\frac{\partial_t P}{t} + e^{2P} \left((\partial_t Q)^2 - (\partial_\theta Q)^2 \right) \ , \\ \partial_t^2 Q - \partial_\theta^2 Q &= -\frac{\partial_t Q}{t} - 2 \left(\partial_t P \partial_t Q - \partial_\theta P \partial_\theta Q \right) \ . \end{aligned}$$

The question of SCC in this class of metrics has been settled by Ringström [113], who proved that the set of smooth initial data for Gowdy models on $\mathbb{T}^3$ that do *not* lead to the formation of Cauchy horizons contains a set which is open and dense within the set of all smooth initial data. Some key steps in Ringström's analysis are provided by the analysis of Fuchsian PDEs of Kichenassamy and Rendall [76, 108], and the analysis of the action of Geroch transformations by Rendall and Weaver [111] (compare [29]). See also [51] for the related problem of an exhaustive description of Cauchy horizons in those models.

Now, the existence of two Killing vectors is also compatible with S^3, $L(p,q)$ ("lens" spaces), and $S^1 \times S^2$ topologies. Thus, to achieve a complete understanding of the set of spatially compact initial data with precisely two Killing vectors one needs to extend Ringström's analysis to those cases. There is an additional difficulty that arises because of the occurrence of axes of symmetry, where the $(1+1)$–reduced equations have the usual singularity associated with polar coordinates. Nevertheless, in view of the analysis by Christodoulou and Tahvildar-Zadeh [35,36] (see also [38]), the global geometry of *generic* maximal globally hyperbolic solutions with those topologies is reasonably well understood. This leads one to expect that one should be able to achieve a proof of SCC in those models using simple abstract arguments, but this remains to be seen.

Recall, finally, that general models with two Killing vectors X_1 and X_2 on $\mathbb{T}^3$ have non-vanishing *twist constants*:

$$c_a = \epsilon_{\alpha\beta\gamma\delta} X_1^\alpha X_2^\beta \nabla^\gamma X_a^\delta \ , \qquad a = 1, 2 \ .$$

The Gowdy metrics are actually "zero measure" in the set of all $U(1) \times U(1)$ symmetric metrics on $\mathbb{T}^3$ because $c_a \equiv 0$ for the Gowdy models. The equations for the resulting metrics are considerably more complicated when the c_a's do not vanish, and only scant rigorous information is available on the global properties of the associated solutions [18,75,106]. It seems urgent to study the dynamics of those models, as they are expected to display [19] highly oscillatory behavior as the singularity is approached. Thus, they should provide the simplest non-isotropic model in which to study this behavior.

There has also been progress in the understanding of models with *exactly one* Killing vector. Here one only has *stability* results in the expanding direction, away from the singularity: In [31] Choquet-Bruhat considers $U(1)$ symmetric initial data for the vacuum Einstein equations on a manifold of the form $\Sigma \times S^1$, where Σ is a compact surface of genus $g > 1$. One assumes that the initial data have constant mean curvature and are sufficiently close to (g_0, K_0), where g_0 is a product metric

$$g_0 = h + dx^2 \ ,$$

with h — a metric of constant Gauss curvature on Σ, and with K_0 — pure trace. The sign of the trace of K_0 determines an expanding time direction and a contracting one. Under those conditions, Choquet-Bruhat proves that the solution exists for an infinite proper time in the expanding direction. The analysis builds upon previous work by Choquet-Bruhat and Moncrief [33], where a supplementary polarisation condition has been imposed. Almost nothing is known in the contracting direction in those models [74], where mixmaster behavior is expected [17, 20].

The proof of the above result bears some similarity to the *future stability* theorem of Andersson and Moncrief [8] for a class of hyperbolic models *without any symmetries*. Those authors consider initial data near a negatively curved compact space form satisfying a specific rigidity condition, with the extrinsic curvature being close to a multiple of the metric, obtaining future geodesic completeness in the expanding direction. The control of the solution is obtained by studying the Bel-Robinson tensor, and its higher-derivatives analogues. One of the ingredients of the proof is the use of an elliptic-hyperbolic system of equations to obtain local existence in time [7].

This last reference is interesting in its own right because of the following: the standard local existence theory for the general relativistic Cauchy problem proceeds through two steps. First one solves the equations in some chosen gauge, e.g. in harmonic coordinates. As a second step one patches the resulting solutions together to construct the space-time manifold. When working with harmonic coordinates, this second step leads to an increase by one of the differentiability threshold for existence and uniqueness of the solution. The elliptic-hyperbolic system of [7] is expected to solve that problem, the elliptic character of some of the equations providing improved regularity of the solution. There is, however, some more work needed to settle this issue, related to the fact that Andersson and Moncrief use a constant-mean curvature (CMC) slicing. The current state-of-the art uniqueness theory [12] for CMC slices requires, roughly speaking, $C^{1,1}$ metrics, and this provides a threshold for the analysis involved. It is conceivable that some weak form of the maximum principle such as, e.g., that in [6], could be used to lower the $C^{1,1}$ threshold, but this remains to be seen.

4 Stability of Minkowski space-time

One of the flagship results in mathematical general relativity is nonlinear stability of Minkowski space-time, first proved by Christodoulou and Klainerman [34] on some 500 pages of their celebrated book. An alternative approach to the problem, due to Klainerman and Nicolò [81] has appeared a year ago in print. That proof is based on an analysis of outgoing and ingoing null cones, and takes around 380 pages. A completely new argument of Lindblad and Rodnianski [91] is available now. Their proof takes some 65 Latex pages, though it should be said that the conditions on the initial data there are much stronger than in previous work: those authors require that the initial data coincide with Schwarzschildian ones outside of a compact set. We note that under those asymptotic conditions, together with a smallness condition, the global existence follows already from the stability results of Friedrich [63]. However, it has been announced by the authors (private communication) that the method is flexible enough to allow the inclusion of a scalar field, and to handle the following, rather weak, asymptotic behavior of the initial data:

$$g = (1 + 2M/r)\delta + O(r^{-1-\alpha}) \ , \qquad K = O(r^{-2-\alpha}) \ . \tag{5}$$

It should be emphasised that the decay conditions (5) are much weaker than those in [34, 81].

5 Initial data engineering

General relativistic initial data sets have to satisfy the constraint equations,

$$\begin{pmatrix} J \\ \\ \rho \end{pmatrix} (K,g) := \begin{pmatrix} 2(-\nabla^j K_{ij} + \nabla_i \,\mathrm{tr} K) \\ \\ R(g) - K^{ij}K_{ij} + (\mathrm{tr}K)^2 \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix} . \tag{6}$$

This makes it difficult to construct space-times with controlled global properties, such as geodesically complete space-times with conformally smooth asymptotics (Penrose's "asymptotic simplicity"), or solutions containing many black holes, or wormholes, or trapped surfaces, or apparent horizons, and so on. A new technique for deforming initial data sets has been invented by Corvino and Schoen [54, 55]. One of the applications of that technique is the unexpected statement [55] that

> *asymptotically flat initial data with well defined mass, momentum, centre of mass, and angular momentum can be deformed in the asymptotic region to a Kerr metric, with arbitrarily small change in global Poincaré charges*

(compare [42]). While this theorem is very interesting on its own, what is even more important is the new technique introduced, which has already led to a few noteworthy applications. One of them is *existence of asymptotically simple space-times*: recall that a space-time is *asymptotically simple* [103] if it has smooth conformal structure at null infinity $\mathscr{I}$, and if all maximally extended null geodesics have initial and final end points on $\mathscr{I}$. Examples of such space-times include Minkowski space-time, and static asymptotically flat stars, and the static solutions of the Einstein – Yang-Mills equations of Bartnik and McKinnon [15]. More generally, spaces-times which are stationary and vacuum outside of a world tube have a smooth conformal structure at null infinity [57,59], but the Kruskal-Szekeres-Schwarzschild space-time is not asymptotically simple because of null geodesics terminating in the singularities under the event horizons. In any case, asymptotically simple space-times possess a smooth global $\mathscr{I}$, and various previous attempts to construct asymptotically simple *vacuum* solutions have been unsuccessful. For instance, the C-metrics have both singularities inside space-time and at $\mathscr{I}$ [9, 21]; the Robinson-Trautman metrics possess a smooth $\mathscr{I}^+$ which is complete to the future, but which is expected to be complete to the past only for the Schwarzschild metric; the space-times constructed by Christodoulou-Klainerman are not known to possess smooth asymptotic structure. In fact, the only previous dynamical example satisfying reasonable field equations has been given by Cutler and Wald, in electro-vacuum [56].

The paper [41] gives the first proof of *existence of non-trivial asymptotically simple vacuum space-times.* The construction uses the Corvino-Schoen gluing techniques [42] and Friedrich's stability theorem [63], and the simplest version of the result is the following: One considers a non-trivial initial data set $(\mathbb{R}^3, \mathring{g}, \mathring{K} = 0)$, with ADM mass $\mathring{m}$, satisfying a *parity condition*

$$\mathring{g}_{ij}(x) = \mathring{g}_{ij}(-x) . \tag{7}$$

One also assumes that one is given on $\mathbb{R}^3 \setminus B(R)$ a parity-symmetric *reference family* $(\mathbb{R}^3, g^m, K^m = 0)$: this is, by definition, a family of metrics labeled by their ADM masses m, such that the m's cover continuously a neighborhood of $\mathring{m}$. In [41] it is proved that

> *the initial data set* $(\mathbb{R}^3, \mathring{g}, 0)$ *can be deformed to* some *member of the reference family on* $B(2R) \setminus B(R)$ *if the metrics are close enough near* $S(R)$.

If $\mathring{m}$ is sufficiently small one can invoke Friedrich's stability theorem to obtain existence of a global smooth $\mathscr{I}$, as well as asymptotic simplicity.

We note that large families of parity-symmetric, scalar flat metrics $\mathring{g}$, with mass $\mathring{m}$ as small as desired, can be constructed using the conformal method [42]. Similarly, the collection of reference families that can be used here is rather large: one can use Schwarzschild metrics, or Weyl metrics, or other static metrics [86,112]. In fact, any metric on $\mathbb{R}^3$ with zero scalar curvature and with ADM mass m equal to $\mathring{m}$ is part of a reference family obtained by scaling the metric and the coordinates,

$$x^i \to \lambda x^i \ , \quad g \to \lambda^{-2} g \qquad \Longrightarrow \qquad m \to \lambda^{-1}\mathring{m} \ .$$

The stationary character of the reference metrics is needed to be able to assert smoothness of $\mathscr{I}$, but is not needed in the gluing construction.

The time-symmetry condition $K = 0$ has only been made for simplicity of presentation of the result, and the construction also provides families of asymptotically simple space-times which are not time-symmetric.

The parity condition (7) does look rather unnatural at first sight. It can be replaced by the requirement that $\mathring{g}$ is sufficiently close to a metric which satisfies (7). It is related to the fact that in the gluing process one needs to adjust the centres of mass of the metrics which are being glued. Parity guarantees that the centres of mass of all metrics involved are automatically zero, and therefore there is nothing to adjust. There is a simple Newtonian analogue which illustrates the problem: Let ρ be the Newtonian energy density, m the Newtonian mass (integral over the "space manifold" $M = \mathbb{R}^3$ of ρ), and let $\vec{c}$ be the Newtonian centre of mass of the gravitating system,

$$\vec{c} = \int_M \vec{x}\rho \ .$$

If $\vec{x}$ is changed to $\vec{x} + \vec{a}$, then the new centre of mass $\vec{c}\,'$ equals

$$\vec{c}\,' = \int_M (\vec{x} + \vec{a})\rho = \vec{c} + m\vec{a} \ .$$

In order to obtain $\vec{c}\,' = 0$, one needs to translate the system by $-\frac{\vec{c}}{m}$, which can be very large even if $\vec{c}$ is small when m is small. Because the gluing techniques are based on the implicit function theorem, they break down when the translations cease to be very small.

So far we have been gluing metrics which were close to each other, with the final metric not being drastically different from the original one. It turns out that gluings can lead to objects with global properties rather distinct from the ones we started with. The idea here is to make a small gluing to add a "neck" to a first initial data set (M_1, g_1, K_1), and then another small gluing to add a second initial data (M_2, g_2, K_2) set on the other end of the neck. In this way one obtains a *new initial data set on the connected sum manifold* $M_1 \# M_2$. A similar construction adds wormholes within a single initial data set. This has been successfully carried out by Isenberg, Mazzeo and Pollack [71,72] a few years ago, using the conformal method. Their approach works for CMC initial data (and, with an additional hypothesis, for data which is only CMC near the neck region) in either the compact, asymptotically flat or asymptotically hyperbolic setting. The conformal factor can be made as close to one as desired away from the necks by making the necks small enough, but those techniques lead to deformations of the original data sets which are never localised. The question then arises, whether this can be done by deformations which are supported in a small neighborhood of the neck.

It is easily seen that there is an obstruction do to that, related to the positive energy theorem: if one could glue, via local deformations, a non-trivial initial data set to $(\mathbb{R}^n, \delta, 0)$, where δ denotes the Euclidean metric, one would end up with a non-trivial initial data set with zero ADM mass, which is impossible.

Nevertheless, in [48] it is shown that

> *generic* initial data sets can be glued together, with *no deformations away from arbitrarily small sets* where the gluing takes place.

Here a data set is *generic* if there are no *local Killing vectors* in the associated space-time. (A local Killing vector field is a Killing vector field defined only on a subset of the space-time, not necessarily globally. The intuitively obvious fact, that non-existence of local Killing vectors is a generic condition, is rigorously justified in [16] in several important cases.) The proofs use a version with boundary of the results in [71, 72], the results by Bartnik [12] on existence of maximal surfaces, together with Corvino-Schoen type techniques [42, 55].

One interesting application of this gluing technique is the construction of spatially compact, *vacuum*, maximal globally hyperbolic space-times without any CMC surfaces [47, 48]. The argument follows an idea of Bartnik [13], who obtains such space-times which contain dust.

The Corvino-Schoen gluing technique is based on an analysis of the operator which arises by linearising the map $(g, K) \mapsto (J, \rho)$ of (6), and it is well known that the properties of this map determine the manifold structure of the set of solutions of the constraint equations [60]. The analysis in that last reference allows one to introduce a Fréchet manifold structure on solutions of the constraint equations away from initial data sets with symmetries, but fails to provide a Banach manifold structure. In a recent beautiful paper [14] Bartnik manages to overcome the technical difficulties that arise, obtaining a Hilbert manifold. Various different Banach manifold structures have also been recently constructed in [43], again as an application of the Corvino-Schoen techniques; further applications of the technique can be found in [42, 124].

6 Asymptotic structure

The following questions have been keeping mathematical relativists busy for years:

> *What are the asymptotic conditions on the initial data at $\{t = 0\}$ that guarantee that the resulting space-time has*
>
> 1. *"a piece of smooth $\mathscr{I}^+$"?*
> 2. *a global smooth $\mathscr{I}^+$?*

As already pointed out, Corvino-Schoen gluings lead to initial data with a maximal globally hyperbolic development containing a smooth piece of $\mathscr{I}$ when the resulting initial data are stationary at large distances (and even global smooth $\mathscr{I}$'s for small initial data). They provide thus the first example of non-trivial initial data which are relevant in this context. The Christodoulou-Klainerman theorem [34] gives a $\mathscr{I}$ with very low differentiability, not enough for peeling theorems. It is conceivable that one shouldn't care about that, the properties of $\mathscr{I}$ in Christodoulou-Klainerman space-times being enough for e.g. for the Trautman-Bondi mass-loss formula. However, for the sake of completeness of our understanding of the asymptotics of the gravitational field, one would like to have an answer to questions *1* and *2*.

Some progress on this has been recently made by Klainerman and Nicolò [82], who show that *peeling holds for initial data with asymptotics*

$$g = (1 + \frac{M}{2r})^4 \delta + O(r^{-3-\alpha}) , \qquad K = O(r^{-4-\alpha}) , \quad \alpha > 0 . \tag{8}$$

Large classes of initial data as above, not necessarily stationary outside of a compact set, can be constructed by a variant of the Corvino-Schoen gluing [41]. We emphasise that (8) provides *peeling*, but does not guarantee smoothness of $\mathscr{I}$. In fact, the *first known explicit obstructions to smoothness of $\mathscr{I}$ in terms of data at $t = 0$* have been found by Valiente-Kroon [87–90]. This leads to the striking statement, that

> *Bowen-York* and *Brill-Lindquist* initial data do not lead to a smooth $\mathscr{I}$ (if any $\mathscr{I}$ at all).

In Friedrich's Cargèse lecture notes [62, 64] and in [88] the following conjecture is made: There exists $N(k) \in \mathbb{N}$, with $N(k) \to \infty$ if $k \to \infty$, such that

> *time-symmetric* initial data lead to a $\mathscr{I}$ of differentiability C^k if and only if the data are *static* up to terms $O(r^{-N(k)})$.

One expects an analogous statement without the time-symmetry condition, but this requires more care. As discussed in [89], smoothness of $\mathscr{I}^+$ imposes conditions on the initial data that are not identical to those for smoothness of $\mathscr{I}^-$. The requirement that *both* $\mathscr{I}^+$ and $\mathscr{I}^-$ are C^k is expected then to be compatible only with those initial data which are *stationary* up to terms $O(r^{-N(k)})$, for some $N(k) \to_{k\to\infty} \infty$.

So far we have been assuming that the cosmological constant vanishes. Some new results by Rendall on existence and properties of conformal completions for vacuum space-times with $\Lambda \geq 0$ can be found in [109, 110]. Recall that Friedrich's analysis [63] applies only to dimension four, while Rendall's approach works in higher dimensions as well. A new method of approaching the question of existence of conformal completions in *all even space-time dimensions* has been presented by Anderson in [1].

7 Low regularity solutions

Standard theory of hyperbolic equations [69, 121] shows that local existence of solutions of the general relativistic Cauchy problem holds for initial data $(g, K) \in H_s \times H_{s-1}$ with $s > 5/2$. There are several reasons why it is of interest to try to lower this threshold. An improved existence theorem:

- renders certain singularities innocuous
- can potentially be used to simplify and/or extend global existence results
- gives insight into the mathematical structure of the equations.

A first step in this direction has been taken by Klainerman and Rodnianski [83, 84], who show that the existence time of the solution can be bounded from below by a function that depends only upon

$$\|(g, K)\|_{H_{2+\epsilon}\times H_{1+\epsilon}}$$

where ϵ is as small as desired. (See [10, 120] and references therein for related results by Bahouri and Chemin, and by Tataru, and others, for large classes of nonlinear wave equations.)

The existence of initial data in the $H_2 \times H_1$ class is proved by Choquet-Bruhat in [30] and by Maxwell in [97]. (In fact, Maxwell's paper goes further and produces initial data in $H_s \times H_{s-1}$ for $s > 3/2$, and shows that one can approximate (in that topology) this data by smooth data.) The already mentioned construction of manifold structure by Bartnik [11] allows $H_2 \times H_1$ differentiability as well. Klainerman conjectured that

L^2 regularity of the Riemann tensor should suffice for existence.

Some partial results towards the proof of this conjecture can be found in the paper [85] by Klainerman and Rodnianski.

8 Vacuum initial data set with apparent horizons and trapped surfaces

A classical statement in Hawking and Ellis [68] asserts that *the boundary of a trapped region is an apparent horizon.* The argument given there does not quite prove that, as it seems to make the implicit hypothesis that the boundary of the trapped region is a C^2 surface, and it is a still an open question whether or not this is correct. Now, the main point is the existence of an apparent horizon within $\mathscr{S}$, and this has been shown to be correct very recently by Schoen [114]:

> *Suppose that an asymptotically flat initial data set $(\mathscr{S}, g, K)$ contains a compact trapped surface $\mathscr{T}$, then there exists in $\mathscr{S}$ a smooth, outermost, compact apparent horizon enclosing $\mathscr{T}$.*

The proof proceeds by constructing singular solutions of the Jang equation, with delicate control of their blow-up set. This generalises a previous result by Huisken and Ilmanen [70], who prove the corresponding statement in the time-symmetric case.

We finish this section, and this review, by noting that Dain [58] and Maxwell [96] have provided direct constructions of initial data with smooth, trapped or marginally trapped, boundaries.

Acknowledgements Useful comments by J. Isenberg and D. Pollack on a previous version of this manuscript are gratefully acknowledged.

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