Towards the classification of static vacuum spacetimes with negative cosmological constant

Piotr T. Chruściel *
Département de Mathématiques
Faculté des Sciences
Parc de Grandmont
F37200 Tours, France

Walter Simon†
Institut für theoretische Physik
Universität Wien,
Boltzmanngasse 5,
A-1090 Wien, Austria

December 20, 2000

Abstract

We present a systematic study of static solutions of the vacuum Einstein equations with negative cosmological constant which asymptotically approach the generalized Kottler ("Schwarzschild—anti-de Sitter") solution, within (mainly) a conformal framework. We show connectedness of conformal infinity for appropriately regular such space-times. We give an explicit expression for the Hamiltonian mass of the (not necessarily static) metrics within the class considered; in the static case we show that they have a finite and well defined Hawking mass. We prove inequalities relating the mass and the horizon area of the (static) metrics considered to those of appropriate reference generalized Kottler metrics. Those inequalities yield an inequality which is opposite to the conjectured generalized Penrose inequality. They can thus be used to prove a uniqueness theorem for the generalized Kottler black holes if the generalized Penrose inequality can be established.

*Supported in part by KBN grant # 2 P03B 073 15. *E–mail*: Chrusciel@Univ-Tours.fr
†Supported by Jubiläumsfonds der Österreichischen Nationalbank, project # 6265, and by a grant from Région Centre, France. *E–mail*: simon@ap.univie.ac.at

Contents

1 Introduction

Consider the families of metrics

$$ds^2 = -(k - \frac{2m}{r} - \frac{\Lambda}{3}r^2)dt^2 + (k - \frac{2m}{r} - \frac{\Lambda}{3}r^2)^{-1}dr^2 + r^2 d\Omega_k^2 \ , \quad k = 0, \pm 1 \ , \tag{1.1}$$

$$ds^2 = -(\lambda - \Lambda r^2)dt^2 + (\lambda - \Lambda r^2)^{-1}dr^2 + |\Lambda|^{-1}d\Omega_k^2 \ , \quad k = \pm 1 \ , \quad k\Lambda > 0 \ , \lambda \in \mathbb{R} \tag{1.2}$$

where $d\Omega_k^2$ denotes a metric of constant Gauss curvature k on a two dimensional manifold ${}^2M$. (Throughout this work we assume that ${}^2M$ is compact.) These are well known static solutions of the vacuum Einstein equation with a cosmological constant Λ; some subclasses of (1.1) and (1.2) have been discovered by de Sitter [64] ((1.1) with $m = 0$ and $k = 1$), by Kottler [56] (Equation (1.1) with an arbitrary m and $k = 1$), and by Nariai [60] (Equation (1.2) with $k = 1$). As discussed in detail in Section 5.4, the parameter $m \in \mathbb{R}$ is related to the Hawking mass of the foliation $t =$ const, $r =$ const. We will refer to those solutions as the generalized Kottler and the generalized Nariai solutions. The constant Λ is an arbitrary real number, but in this paper we will mostly be interested in $\Lambda < 0$, and this assumption will be made unless explicitly stated otherwise. There has been recently renewed interest in the black hole aspects of the generalized Kottler solutions [19, 33, 59, 66]. The object of this paper is to initiate a systematic study of static solutions of the vacuum Einstein equations with a negative cosmological constant.

The first question that arises here is that of asymptotic conditions one wants to impose. In the present paper we consider metrics which tend to the generalized Kottler solutions, leaving the asymptotically Nariai case to future work. We present the following three approaches to asymptotic structure, and study their mutual relationships: three dimensional conformal compactifications, four dimensional conformal completions, and a coordinate approach. We show that under rather natural hypotheses the conformal boundary at infinity is connected.

The next question we address is that of the definition of mass for such solutions, *without assuming staticity* of the metrics. We review again the possible approaches that occur here: a naive coordinate approach, a Hamiltonian approach, a "Komar–type" approach, and the Hawking approach. We show that the Hawking mass converges to a finite value for the metrics considered here, and we also give conditions on the conformal completions under which the "coordinate mass", or the Hamiltonian mass, are finite. Each of those masses come with different normalization factor, whenever all are defined, except for the Komar and Hamiltonian masses which coincide. We suggest that the correct normalization is the Hamiltonian one.

Returning to the static case, we recall that appropriately behaved vacuum black holes with $\Lambda = 0$ are completely described by the parameter m appearing above [20, 26, 48], and it is natural to enquire whether this remains true for other values of Λ. In fact, for $\Lambda < 0$, Boucher, Gibbons, and Horowitz [15] have given arguments suggesting uniqueness of the anti-de Sitter solution within an appropriate class. As a step towards a proof of a uniqueness theorem in the general case we derive, under appropriate hypotheses, 1) lower bounds on (loosely speaking) the area of cross-sections of the horizon, and 2) upper bounds on the mass of static vacuum black holes with negative cosmological constant. When these inequalities are combined the result goes precisely the opposite way as a (conjectured) generalization of the Geroch–Huisken–Ilmanen–Penrose inequality [16, 17, 37, 46, 47, 62] appropriate to space-times with non-vanishing cosmological constant. In fact, such a generalization was obtained by Gibbons [38] along the lines of Geroch [37], and of Jang and Wald [49], *i.e.* under the very stringent assumption of the global existence and smoothness of the inverse mean curvature flow, see Section 6. We note that it is far from clear that the arguments of Huisken and Ilmanen [46, 47], or those of Bray [16, 17], which establish the original Penrose conjecture can be adapted to the situation at hand. If this were the case, a combination of this with the results of the present work would give a fairly general uniqueness result. In any case this part of our work demonstrates the usefulness of a generalized Penrose inequality, if it can be established at all.

To formulate our results more precisely, consider a static space-time $(M, {}^4g)$ which might — but does not have to — contain a black hole region. In the asymptotically flat case there exists a well established theory (see [22], or [26, Sections 2 and 6] and references therein) which, under appropriate hypotheses, allows one to reduce the study of such space-times to the problem of finding all suitable triples (Σ, g, V), where (Σ, g) is a three dimensional Riemannian manifold and V is a *non-negative* function on Σ. Further V is required to vanish precisely on the boundary of Σ, when non-empty:

$$V \geq 0 , \qquad V(p) = 0 \Longleftrightarrow p \in \partial\Sigma . \tag{1.3}$$

Finally g and V satisfy the following set of equations on Σ:

$$\Delta V = -\Lambda V , \tag{1.4}$$
$$R_{ij} = V^{-1} D_i D_j V + \Lambda g_{ij} \tag{1.5}$$

($\Lambda = 0$ in the asymptotically flat case). Here R_{ij} is the Ricci tensor of the ("three dimensional") metric g. We shall not attempt to formulate the conditions on $(M, {}^4g)$ which will allow one to perform such a reduction (some of the aspects of the relationship between (Σ, g, V) and the associated space-time are discussed in Section 3.2, see in particular Equation (3.37)), but we shall directly address the question of properties of solutions of (1.4)–(1.5). Our first main result concerns the topology of $\partial\Sigma$ (*cf.* Theorem 4.1, Section 4; compare [32, 69]):

Theorem 1.1 *Let $\Lambda < 0$, consider a set (Σ, g, V) which is C^3 conformally compactifiable in the sense of Definition 3.1 below, suppose that (1.3)–(1.5) hold. Then the conformal boundary at infinity $\partial_\infty\Sigma$ of Σ is connected.*

Our second main result concerns the Hawking mass of the level sets of V, *cf.* Theorem 5.2, Section 5.4:

Theorem 1.2 *Under the conditions of Theorem 1.1, the Hawking mass m of the level sets of V is well defined and finite.*

It is natural to enquire whether there exist static vacuum space-times with complete spacelike hypersurfaces and no black hole regions; it is expected that no such solutions exist when $\Lambda < 0$ and $\partial_\infty\Sigma \neq S^2$. We hope that points *2.* and *3.* of the following theorem can be used as a tool to prove their non-existence:

Theorem 1.3 *Under the conditions of Theorem 1.1, suppose further that*

$$\partial\Sigma = \emptyset ,$$

and that the scalar curvature R' of the metric $g' = V^{-2}g$ is constant on $\partial_\infty\Sigma$. Then:

1. *If $\partial_\infty\Sigma$ is a sphere, then the Hawking mass m of the level sets of V is* non-positive, *vanishing if and only if there exists a diffeomorphism $\psi : \Sigma \to \Sigma_0$ and a positive constant λ such that $g = \psi^* g_0$ and $V = \lambda V_0 \circ \psi$, with (Σ_0, g_0, V_0) corresponding to the anti-de Sitter space-time.*

2. *If $\partial_\infty\Sigma$ is a torus, then the Hawking mass m is* strictly negative.

3. If the genus g_∞ of $\partial_\infty\Sigma$ is higher than or equal to 2 *we have*

$$m < -\frac{1}{3\sqrt{-\Lambda}}\,, \tag{1.6}$$

with $m = m(V)$ normalized as in Equation (6.7).

A mass inequality similar to that in point *1.* above has been established in [15], and in fact we follow their technique of proof. However, our hypotheses are rather different. Further, the mass here is *a priori* different from the one considered in [15]; in particular it isn't clear at all whether the mass defined as in [15] is also defined for the metrics we consider, *cf.* Sections 3.3 and 5.1 below.

We note that metrics satisfying the hypotheses of point 2 above, with arbitrarily large (strictly) negative mass, have been constructed in [45].

As a straightforward corollary of Theorem 1.3 one has:

Corollary 1.4 *Suppose that the generalized positive energy inequality*

$$m \geq m_{crit}(g_\infty)$$

holds in the class of three dimensional manifolds (Σ, g) which satisfy the requirements of point 1. *of Definition 3.1 with a connnected conformal infinity $\partial_\infty\Sigma$ of genus g_∞, and, moreover, the scalar curvature R of which satisfies $R \geq 2\Lambda$. Then:*

1. *If $m_{crit}(g_\infty = 0) = 0$, then the only solution of Equations (1.4)–(1.5) satisfying the hypotheses of point* 1. *of Theorem 1.3 is the one obtained from anti-de Sitter space-time.*

2. *If $m_{crit}(g_\infty > 1) = -1/(3\sqrt{-\Lambda})$, then there exist no solutions of Equations (1.4)–(1.5) satisfying the hypotheses of point* 3. *of Theorem 1.3.*

When $\partial_\infty\Sigma = S^2$ one expects that the inequality $m \geq 0$, *with m being the mass defined by spinorial identities* can be established using Witten type techniques (*cf.* [6, 39]), regardless of whether or not $\partial\Sigma = \emptyset$. (On the other hand it follows from [11] that when $\partial_\infty\Sigma \neq S^2$ there exist no asymptotically covariantly constant spinors which can be used in the Witten argument.) This might require imposing some further restrictions on *e.g.* the asymptotic behavior of the metric. To be able to conclude in this case that there are no static solutions without horizons, or that the only solution with a connected non-degenerate horizon is the anti-de Sitter one, requires working out those restrictions, and showing that the Hawking mass of the level sets of V coincides with the mass occuring in the positive energy theorem.

When horizons occur, our comparison results for mass and area read as follows:

Theorem 1.5 *Under the conditions of Theorem 1.1, suppose further that the genus g_∞ of $\partial_\infty\Sigma$ satisfies*

$$g_\infty \geq 2\,,$$

and that the scalar curvature R' of the metric $g' = V^{-2}g$ is constant on $\partial_\infty\Sigma$. Let $\partial_1\Sigma$ be any connected component of $\partial\Sigma$ for which the surface gravity κ defined by Equation (7.1) is largest, and assume that

$$0 < \kappa \le \sqrt{-\frac{\Lambda}{3}} \ . \tag{1.7}$$

Let m_0, respectively A_0, be the Hawking mass, respectively the area of $\partial\Sigma_0$, for that generalized Kottler solution (Σ_0, g_0, V_0), with the same genus g_∞, the surface gravity κ_0 of which equals κ. Then

$$m \le m_0 \ , \qquad A_0(g_{\partial_1\Sigma} - 1) \le A(g_\infty - 1) \ , \tag{1.8}$$

where A is the area of $\partial_1\Sigma$ and $m = m(V)$ is the Hawking mass of the level sets of V. Further $m = m_0$ if and only if there exists a diffeomorphism $\psi : \Sigma \to \Sigma_0$ and a positive constant λ such that $g = \psi^* g_0$ and $V = \lambda V_0 \circ \psi$.

The asymptotic conditions assumed in Theorems 1.3 and 1.5 are somewhat related to those of [9, 15, 43, 44]. The precise relationships are discussed in Sections 3.2 and 3.3. Let us simply mention here that the condition that R' is constant on $\partial_\infty\Sigma$ is the (local) higher genus analogue of the (global) condition in [9, 43] that the group of conformal isometries of $\mathscr{I}$ coincides with that of the standard conformal completion of the anti-de Sitter space-time; the reader is referred to Proposition 3.6 in Section 3.2 for a precise statement.

We note that the hypothesis (1.7) is equivalent to the assumption that the generalized Kottler solution with the same value of κ has non-positive mass; *cf.* Section 2 for a discussion. We emphasize, however, that we do not make any *a priori* assumptions concerning the sign of the mass of (Σ, g, V). Our methods do not lead to any conclusions for those values of κ which correspond to generalized Kottler solutions with positive mass.

With $m = m(V)$ normalized as in Equation (6.7), the inequality $m \le m_0$ takes the following explicit form

$$m \le \frac{(\Lambda + 2\kappa^2)\sqrt{\kappa^2 - \Lambda} + 2\kappa^3}{3\Lambda^2} \ , \tag{1.9}$$

while $A(g_\infty - 1) \ge A_0(g_{\partial_1\Sigma} - 1)$ can be explicitly written as

$$A(g_\infty - 1) \ge 4\pi(g_{\partial_1\Sigma} - 1)\left[\frac{\kappa + \sqrt{\kappa^2 - \Lambda}}{\Lambda}\right]^2 \ . \tag{1.10}$$

(The right-hand sides of Equations (1.9) and (1.10) are obtained by straightforward algebraic manipulations from (2.1) and (2.11).)

It should be pointed out that in [70] a lower bound for the area has also been established. However, while the bound there is sharp only for the generalized Kottler solutions with $m = 0$, our bound is sharp for all Kottler solutions. On the other hand in [70] it is not assumed that the space-time is static.

If the generalized Penrose inequality (which we discuss in some detail in Section 6) holds,

$$2M_{Haw}(u) \geq \sum_{i=1}^{k} \left((1 - g_{\partial_i \Sigma}) \left(\frac{A_{\partial_i \Sigma}}{4\pi} \right)^{1/2} - \frac{\Lambda}{3} \left(\frac{A_{\partial_i \Sigma}}{4\pi} \right)^{3/2} \right) \qquad (1.11)$$

(with the $\partial_i\Sigma$'s, $i = 1, \ldots, k$, being the connected components of $\partial\Sigma$, the $A_{\partial_i\Sigma}$'s — their areas, and the $g_{\partial_i\Sigma}$'s — the genera thereof) we obtain uniqueness of solutions:

Corollary 1.6 *Suppose that the generalized Penrose inequality (1.11) holds in the class of three dimensional manifolds (Σ, g) with scalar curvature R satisfying $R \geq 2\Lambda$, which satisfy the requirements of point* 1. *of Definition 3.1 with a connnected conformal infinity $\partial_\infty\Sigma$ of genus $g_\infty > 1$, and which have a compact connected boundary. Then the only static solutions of Equations (1.4)–(1.5) satisfying the hypotheses of Theorem 1.5 are the corresponding generalized Kottler solutions.*

This paper is organized as follows: in Section 2 we discuss those aspects of the generalized Kottler solutions which are relevant to our work. The main object of Section 3 is to set forth the boundary conditions which are appropriate for the problem at hand. In Section 3.1 this is analyzed from a three dimensional point of view. We introduce the class of objects considered in Definition 3.1, and analyze the consequences of this Definition in the remainder of that section. In Section 3.2 four-dimensional conformal completions are considered; in particular we show how the setup of Section 3.1 relates to a four dimensional one, *cf.* Proposition 3.4 and Theorem 3.5. We also show there how the requirement of local conformal flatness of the geometry of $\mathscr{I}$ relates to the restrictions on the geometry of $\partial_\infty\Sigma$ considered in Section 3.1. In Section 3.3 a four dimensional coordinate approach is described; in particular, when (M, g) admits suitable conformal completions, we show there how to construct useful coordinate systems in a neighborhood of $\mathscr{I}$ — *cf.* Proposition 3.7. In Section 4 connectedness of the conformal boundary $\partial_\infty\Sigma$ is proved under suitable conditions. Section 5 is devoted to the question how to define the total mass for the class of space-times at hand. This is discussed from a coordinate point of view in Section 5.1, from a Hamiltonian point of view in Section 5.2, and using the Hawking approach in Section 5.4; in Section 5.3 we present a generalization of the Komar integral appropriate to our setting. The main results of the analysis in Section 5 are the boundary conditions (5.19) together with Equation (5.22), which gives an ADM-type expression for the Hamiltonian mass for space-times with generalized Kottler asymptotics; we emphasize that this formula holds without any hypotheses of staticity or stationarity of the space-time metric. Theorem 1.2 is proved in Section 5.4. In Section 6 we recall an argument due to Gibbons [38] for the validity of the generalized Penrose inequality. (However, our conclusions are different from those of [38].) In Section 7 we prove Theorems 1.3 and 1.5, as well as Corollary 1.6.

Acknowledgements W.S. is grateful to Tom Ilmanen for helpful discussions on the Penrose inequality. We thank Gary Horowitz for pointing out Reference [45].

2 The generalized Kottler solutions

We recall some properties of the solutions (1.1). Those solutions will be used as reference solutions in our arguments, so it is convenient to use a subscript 0 when referring to them. As already mentioned, we assume $\Lambda < 0$ unless indicated otherwise. For $m_0 \in \mathbb{R}$, let r_0 be the largest positive root of the equation [1]

$$V_0^2 \equiv k - \frac{2m_0}{r} - \frac{\Lambda}{3}r^2 = 0 \ . \tag{2.1}$$

We set

$$\Sigma_0 = \{(r,v) | r > r_0, v \in {}^2M\} \ , \tag{2.2}$$
$$g_0 = (k - \frac{2m_0}{r} - \frac{\Lambda}{3}r^2)^{-1}dr^2 + r^2 d\Omega_k^2 \ , \tag{2.3}$$

where, as before, $d\Omega_k^2$ denotes a metric of constant Gauss curvature k on a smooth two dimensional compact manifold ${}^2M$. We denote the corresponding surface gravity by κ_0. (Recall that the surface gravity of a connected component of a horizon $\mathcal{N}[X]$ is usually defined by the equation

$$(X^\alpha X_\alpha)_{,\mu}\Big|_{\mathcal{N}[X]} = -2\kappa X_\mu \ , \tag{2.4}$$

where X is the Killing vector field which is tangent to the generators of $\mathcal{N}[X]$. This requires normalizing X; here we impose the normalization[2] that $X = \partial/\partial t$ in the coordinate system of (1.1).) We set

$$W_0(r) \equiv g_0^{ij} D_i V_0 D_j V_0 = (\frac{m_0}{r^2} - \frac{\Lambda r}{3})^2 \ . \tag{2.5}$$

When $m_0 = 0$ we note the relationship

$$W_0 = -\frac{\Lambda}{3}(V_0^2 - k) \ , \tag{2.6}$$

which will be useful later on, and which holds regardless of the topology of ${}^2M$.

2.1 $k = -1$

Suppose, now, that $k = -1$, and that m_0 is in the range

$$m_0 \in [m_{\rm crit}, 0] \ , \tag{2.7}$$

[1] See [66] for an exhaustive analysis, and explicit formulae for the roots of Equation (2.1).

[2] When ${}^2M = T^2$ a unique normalization of X needs a further normalization of $d\Omega_k^2$, *cf.* Sections 5.1 and 5.2 for a detailed discussion of this point.

where

$$m_{\rm crit} \equiv -\frac{1}{3\sqrt{-\Lambda}} \ . \tag{2.8}$$

Here $m_{\rm crit}$ is defined as the smallest value of m_0 for which the metrics (1.1) can be extended across a Killing horizon [19, 66]. Let us show that Equation (2.7) is equivalent to

$$r_0 \in [\frac{1}{\sqrt{-\Lambda}}, \sqrt{-\frac{3}{\Lambda}}] \ . \tag{2.9}$$

In order to simplify notation it is useful to introduce

$$\frac{1}{\ell^2} \equiv -\frac{\Lambda}{3} \ . \tag{2.10}$$

Now, the equation $V_0(\ell/\sqrt{3}) = 0$ implies $m = m_{\rm crit}$. Next, an elementary analysis of the function $r^3/\ell^2 - r - 2m_0$ (recall that $k = -1$ in this section) shows that 1) V has no positive roots for $m < m_{\rm crit}$; 2) for $m = m_{\rm crit}$ the only positive root is $\ell/\sqrt{3}$; 3) if r_0 is the largest positive root of the equation $V_0(r_0) = 0$, then for each $m_0 > m_{\rm crit}$ the radius $r_0(m_0)$ exists and is a differentiable function of m_0. Differentiating the equation $r_0 V_0(r_0) = 0$ with respect to m_0 gives $(\frac{3r_0^2}{\ell^2} + k)\frac{\partial r_0}{\partial m_0} = (\frac{3r_0^2}{\ell^2} - 1)\frac{\partial r_0}{\partial m_0} = 2$. It follows that for $r \geq \ell/\sqrt{3}$ the function $r_0(m_0)$ is a monotonically increasing function on its domain of definition $[m_{\rm crit}, \infty)$, which establishes our claim.

We note that the surface gravity κ_0 is given by the formula

$$\kappa_0 = \sqrt{W_0(r_0)} = \frac{m_0}{r_0^2} + \frac{r_0}{\ell^2} \ , \tag{2.11}$$

which gives

$$\frac{\partial \kappa_0}{\partial m_0} = \frac{1}{r_0^2} + \left(\frac{1}{\ell^2} - \frac{2m_0}{r_0^3}\right)\frac{\partial r_0}{\partial m_0} \ .$$

Equation (2.11) shows that κ_0 vanishes when $m_0 = m_{\rm crit}$. [3] *Under the hypothesis that* $m_0 \leq 0$, it follows from what has been said above a) that $\frac{\partial \kappa_0}{\partial m_0}$ is positive; b) that we have

$$\kappa_0 \in [0, \sqrt{-\frac{\Lambda}{3}}] \ , \tag{2.12}$$

when (2.7) holds, and c) that, under the current hypotheses on k and Λ, (2.7) is equivalent to (2.12) for the metrics (1.1). While this can probably be established

[3] The methods of [68] show that in this case the space-times with metrics (1.1) can be extended to black hole space-times with a degenerate event horizon, thus a claim to the contrary in [66] is wrong. It has been claimed without proof in [19] that $\mathscr{I}^+$, as constructed by the methods of [68], can be extended to a larger one, say $\widehat{\mathscr{I}^+}$, which is connected. Recall that that claim would imply that $\partial I^-(\widehat{\mathscr{I}^+}) = \emptyset$ (see Figure 2 in [19]), thus the space-time would not contain an event horizon with respect to $\widehat{\mathscr{I}^+}$. Regardless of whether such an extended $\widehat{\mathscr{I}^+}$ exists or not, we wish to point out the following: a) there will still be degenerate *event* horizons as defined with respect to any connected component of $\mathscr{I}^+$; b) regardless of how null infinity is added there will exist *degenerate Killing horizons* in those space-times; c) there will exist an *observer* horizon associated to the world-line of any observer which moves along the orbits of the Killing vector field in the asymptotic region. It thus appears reasonable to give those space-times a black hole interpretation in any case.

directly, we note that it follows from Theorem 1.5 that (2.12) is equivalent to (2.7) without having to assume that $m_0 \leq 0$.

In what follows we shall need the fact that in the above ranges of parameters the relationship $V_0(r)$ can be inverted to define a smooth function $r(V_0) : [0, \infty) \to \mathbb{R}$. Indeed, the equation $\frac{dV_0}{dr}(r_{\rm crit}) = 0$ yields $r_{\rm crit}^3 = 3m_0/\Lambda$; when $k = -1$, $\Lambda < 0$, and when (2.7) holds one finds $V_0(r_{\rm crit}) \leq 0$, with the inequality being strict unless $m = m_{\rm crit}$. Therefore, $V_0(r)$ is a smooth strictly monotonic function in $[r_0, \infty)$, which implies in turn that $r(V_0)$ is a smooth strictly monotonic function on $(0, \infty)$; further $r(V_0)$ is smooth up to 0 except when $m = m_{\rm crit}$.

3 Asymptotics

3.1 Three dimensional formalism

As a motivation for the definition below, consider one of the metrics (1.1) and introduce a new coordinate $x \in (0, x_0]$ by

$$\frac{r^2}{\ell^2} = \frac{1 - kx^2}{x^2} \; . \tag{3.1}$$

with x_0 defined by substituting r_0 in the left-hand-side of (3.1). It then follows that

$$g = \ell^2 x^{-2} \left[\frac{dx^2}{(1 - kx^2)(1 - \frac{2mx^3}{\ell\sqrt{1-kx^2}})} + (1 - kx^2) d\Omega_k^2 \right] \; .$$

Thus the metric

$$g' \equiv (\ell^{-2} x^2) g$$

is a smooth up to boundary metric on the compact manifold with boundary $\overline{\Sigma}_0 \equiv [0, x_0] \times {}^2M$. Furthermore, xV_0 can be extended by continuity to a smooth up to boundary function on $\overline{\Sigma}_0$, with $xV_0 = 1$. This justifies the following definition:

Definition 3.1 *Let Σ be a smooth manifold*[4]*, with perhaps a compact boundary which we denote by $\partial\Sigma$ when non empty.*[5] *Suppose that g is a smooth metric on Σ, and that V is a smooth nonnegative function on Σ, with $V(p) = 0$ if and only if $p \in \partial\Sigma$.*

1. *(Σ, g) will be said to be C^i, $i \in \mathbb{N} \cup \{\infty\}$, conformally compactifiable or, shortly, compactifiable, if there exists a C^{i+1} diffeomorphism χ from $\Sigma \setminus \partial\Sigma$ to the interior of a compact Riemannian manifold with boundary $(\overline{\Sigma} \approx \Sigma \cup \partial_\infty\Sigma, \overline{g})$, with $\partial_\infty\Sigma \cap \Sigma = \emptyset$, and a C^i function $\omega : \overline{\Sigma} \to \mathbb{R}^+$ such that*

$$g = \chi^*(\omega^{-2}\overline{g}) \; . \tag{3.2}$$

We further assume that $\{\omega = 0\} = \partial_\infty\Sigma$, with $d\omega$ nowhere vanishing on $\partial_\infty\Sigma$, and that $\overline{g}$ is of C^i differentiability class on $\overline{\Sigma}$.

[4] All manifolds are assumed to be Hausdorff, paracompact, and orientable throughout.

[5] We use the convention that a manifold with boundary Σ contains its boundary as a point set.

2. *A triple (Σ, g, V) will be said to be C^i, $i \in \mathbb{N} \cup \{\infty\}$, compactifiable if (Σ, g) is C^i compactifiable, and if $V\omega$ extends by continuity to a C^i function on $\overline{\Sigma}$,*

3. *with*

$$\lim_{\omega \to 0} V\omega > 0 \ . \tag{3.3}$$

We emphasize that Σ itself is allowed to have a boundary on which V vanishes,

$$\partial\Sigma = \{p \in \Sigma | V(p) = 0\} \ .$$

If that is the case we will have

$$\partial\overline{\Sigma} = \partial\Sigma \cup \partial_\infty \Sigma \ .$$

To avoid ambiguities, we stress that one point compactifications of the kind encountered in the asymptotically flat case (*cf.*, *e.g.*, [13]) are not allowed in our context.

The conditions above are not independent when the "static field equations" (Equations (1.4)–(1.5)) hold:

Proposition 3.2 *Consider a triple (Σ, g, V) satisfying Equations (1.3)–(1.5).*

1. *The condition that $|d\omega|_{\overline{g}}$ has no zeros on $\partial_\infty\Sigma$ follows from the remaining hypotheses of point* 1. *of Definition 3.1, when those hold with $i \geq 2$.*

2. *Suppose that (Σ, g) is C^i compactifiable with $i \geq 2$. Then $\lim_{\omega\to 0} V\omega$ exists. Further, one can choose a (uniquely defined) conformal factor so that ω is the $\overline{g}$-distance from $\partial_\infty\Sigma$. With this choice of conformal factor, when (3.3) holds a necessary condition that (Σ, g, V) is C^i compactifiable is that*

$$(4\overline{R}_{ij} - \overline{R}\overline{g}_{ij})\overline{n}^i\overline{n}^j\Big|_{\partial_\infty\Sigma} = 0 \ , \tag{3.4}$$

where $\overline{n}$ is the field of unit normals to $\partial_\infty\Sigma$.

3. *(Σ, g, V) is C^∞ compactifiable if and only if (Σ, g) is C^∞ compactifiable and Equations (3.3) and (3.4) hold.*

Remarks: 1. When (Σ, g) is C^∞ compactifiable but Equation (3.4) does not hold, the proof below shows that $V\omega$ is of the form $\alpha_0 + \alpha_1\omega^2 \log\omega$, for some smooth up-to-boundary functions α_0 and α_1. This isn't perhaps so surprising because the nature of the equations satisfied by g and V suggests that both $\overline{g}$ and $V\omega$ should be polyhomogeneous, rather than smooth. ("Polyhomogeneous" means that $\overline{g}$ and $V\omega$ are expected to admit asymptotic expansions in terms of powers of ω and $\log\omega$ near $\partial_\infty\Sigma$ under some fairly weak conditions on their behavior at $\partial_\infty\Sigma$; *cf.*, *e.g.* [4] for precise definitions and related results.) From this point of view the hypothesis that (Σ, g) is C^∞ compactifiable is somewhat unnatural and should be replaced by that of polyhomogeneity of $\overline{g}$ at $\partial_\infty\Sigma$.

2. One can prove appropriate versions of point *3.* above for (Σ, g)'s which are C^i compactifiable for finite i. This seems to lead to lower differentiability

of $1/V$ near $\partial_\infty\Sigma$ as compared to $\overline{g}$, and for this reason we shall not discuss it here.

3. We leave it as an open problem whether or not there exist solutions of (1.3)–(1.5) such that (Σ, g) is smoothly compactifiable, such that V can be extended by continuity to a smooth function on $\overline{\Sigma}$, while (3.3) does not hold.

4. We note that (3.4) is a conformally invariant condition because ω and $\overline{g}$ are uniquely determined by g. However, it is not conformally covariant, in the sense that if $\overline{g}$ is conformally rescaled, then (3.4) will not be of the same form in the new rescaled metric. It would be of interest to find a form of (3.4) which does not have this drawback.

5. The result above has counterparts for one-point compactifications in the asymptotically flat case, (*cf.*, *e.g.*, the theorem in the Appendix of [13]).

PROOF: Let

$$\alpha \equiv V\omega \ .$$

After suitable identifications we can without loss of generality assume that the map χ in (3.2) is the identity. Equations (1.4)–(1.5) together with the definition of $\overline{g} = \omega^2 g$ lead to the following

$$\overline{\Delta}\alpha - 3\frac{\overline{D}^i\omega\overline{D}_i\alpha}{\omega} + \left(\frac{\overline{\Delta}\omega}{\omega} + \frac{\overline{R}}{2}\right)\alpha = 0 \ , \tag{3.5}$$

$$\overline{D}_i\overline{D}_j\alpha - \frac{\overline{D}^k\omega\overline{D}_k\alpha}{\omega}\overline{g}_{ij} = \left(\overline{R}_{ij} + 2\frac{\overline{D}_i\overline{D}_j\omega}{\omega} - \left(\frac{\overline{\Delta}\omega}{\omega} + \frac{\overline{R}}{2}\right)\overline{g}_{ij}\right)\alpha \ . \tag{3.6}$$

We have also used $R = 2\Lambda$ which, together with the transformation law of the curvature scalar under conformal transformations, implies

$$\omega^2\overline{R} = 6|d\omega|^2_{\overline{g}} + 2\Lambda - 4\omega\overline{\Delta}\omega \ . \tag{3.7}$$

In all the equations here barred quantities refer to the metric $\overline{g}$. Point *1* of the proposition follows immediately from Equation (3.7).

To avoid factors of $-\Lambda/3$ in the remainder of the proof we rescale the metric g so that $\Lambda = -3$. Next, to avoid annoying technicalities we shall present the proof only for smoothly compactifiable (Σ, g) — $i = \infty$; the finite i cases can be handled using the results in [4, Appendix A] and [28, Appendix A]. Suppose, thus, that $i = \infty$. As shown in [5, Lemma 2.1] we can choose ω and $\overline{g}$ so that ω coincides with the $\overline{g}$-distance from $\partial_\infty\Sigma$ in a neighborhood of $\partial_\infty\Sigma$; we shall use the symbol x to denote this function. In this case we have

$$\overline{\Delta}\omega = \overline{p} \ , \tag{3.8}$$

where $\overline{p}$ is the mean curvature of the level sets of $\omega = x$. Further $|d\omega|_{\overline{g}} = 1$ so that (3.8) together with (3.7) give

$$\overline{R} = -4\frac{\overline{p}}{x} \ , \tag{3.9}$$

in particular

$$\overline{p}\Big|_{x=0} = 0 \ . \tag{3.10}$$

We can introduce Gauss coordinates (x^1, x^A) near $\partial_\infty\Sigma$ in which $x^1 = x \in [0, x_0)$, while the $(x^A) = v$'s form local coordinates on $\partial_\infty\Sigma$, with the metric taking the form

$$\overline{g} = dx^2 + \overline{h} \ , \qquad \overline{h}(\partial_x, \cdot) = 0 \ . \tag{3.11}$$

To prove point *2*, from Equation (3.6) we obtain

$$\begin{aligned} &\omega\overline{D}^i\omega\overline{D}^j\omega\overline{D}_i(\omega^{-1}\overline{D}_j\alpha) \ = \\ &= \ \overline{D}^i\omega\overline{D}^j\omega\left(\overline{R}_{ij} + 2\frac{\overline{D}_i\overline{D}_j\omega}{\omega} - \left(\frac{\overline{\Delta}\omega}{\omega} + \frac{\overline{R}}{2}\right)\overline{g}_{ij}\right)\alpha \ . \end{aligned} \tag{3.12}$$

Equations (3.8)–(3.12) lead to

$$x\partial_x(x^{-1}\partial_x\alpha) = (\overline{R}_{xx} - \frac{\overline{R}}{4})\alpha \ . \tag{3.13}$$

At each $v \in \partial_\infty\Sigma$ this is an ODE of Fuchsian type for $\alpha(x, v)$. Standard results about such equations show that for each v the functions $x \to \alpha(x, v)$ and $x \to \partial_x\alpha(x, v)$ are bounded and continuous on $[0, x_0)$. Integrating (3.13) one finds

$$\partial_x\alpha = x\beta(v) + (\overline{R}_{xx} - \frac{\overline{R}}{4})\alpha(0, v)x\ln x + O(x^2\ln x) \ , \tag{3.14}$$

where $\beta(v)$ is a (v-dependent) integration constant. By hypothesis there exist no points at $\partial_\infty\Sigma$ such that $\alpha(0, v) = 0$, Equations (3.13) and (3.14) show that $\partial_x^2\alpha$ blows up at $x = 0$ unless (3.4) holds, and point *2.* follows.

We shall only sketch the proof of point *3.*: Standard results about Fuchsian equations show that solutions of Equation (3.13) will be smooth in x whenever $(\overline{R}_{xx} - \frac{\overline{R}}{4})(x = 0, v)$ vanishes throughout $\partial_\infty\Sigma$. A simple bootstrap argument applied to Equation (3.6) with $(ij) = (1A)$ shows that α is also smooth in v. Commuting Equation (3.6) with $(x\partial_x)^i\partial_v^\beta$, where β is an arbitrary multi-index, and iteratively repeating the reasoning outlined above establishes smoothness of α jointly in v and x. □

A consequence of condition 3 of Definition 3.1 is that the function

$$V' \equiv 1/V \ ,$$

when extended to $\overline{\Sigma}$ by setting $V' = 0$ on $\partial_\infty\Sigma$, can be used as a compactifying conformal factor, at least away from $\partial\Sigma$: If we set

$$g' = V^{-2}g \ ,$$

then g' is a Riemannian metric smooth up to boundary on $\overline{\Sigma} \setminus \partial\Sigma$. In terms of this metric Equations (1.4)–(1.5) can be rewritten as

$$\Delta'V' \ = \ 3V'W + \Lambda V \ , \tag{3.15}$$

$$R'_{ij} \ = \ -2VD'_iD'_jV' \ . \tag{3.16}$$

Here R'_{ij} is the Ricci tensor of the metric g', D' is the Levi–Civita covariant derivative associated with g', while Δ' is the Laplace operator associated with g'. Taking the trace of (3.16) and using (3.15) we obtain

$$R' = -6W - 2\Lambda V^2 \ , \tag{3.17}$$

where

$$W \equiv D_i V D^i V \ . \tag{3.18}$$

Defining

$$W' \equiv g'^{ij} D'_i V' D'_j V' = (V')^2 W \ , \tag{3.19}$$

Equation (3.17) can be rewritten as

$$6W' = -2\Lambda - R'(V')^2 \ . \tag{3.20}$$

If (Σ, g, V) is C^2 compactifiable then R' is bounded in a neighborhood of $\partial_\infty\Sigma$, and since V blows up at $\partial_\infty\Sigma$ it follows from Equation (3.17) that so does W, in particular W is strictly positive in a neighborhood of $\partial_\infty\Sigma$. Further Equation (3.20) implies that the level sets of V are smooth manifolds in a neighborhood of $\partial_\infty\Sigma$, diffeomorphic to $\partial_\infty\Sigma$ there.

Equations (1.4)–(1.5) are invariant under a rescaling $V \to \lambda V$, $\lambda \in \mathbb{R}^*$. This is related to the possibility of choosing freely the normalization of the Killing vector field in the associated space-time. Similarly the conditions of Definition 3.1 are invariant under such rescalings with $\lambda > 0$. For various purposes — *e.g.*,, for the definition (7.1) of surface gravity — it is convenient to have a unique normalization of V. We note that if (Σ, g, V) corresponds to a generalized Kottler solution (Σ_0, g_0, V_0), then (1.1) and (2.5) together with (3.18) give $6W'_0 = -2\Lambda(1 - k(V'_0)^2) + O((V'_0)^3)$ so that from (3.17) one obtains

$$R'_0|_{\partial_\infty\Sigma} = -2\Lambda k \ . \tag{3.21}$$

We have the following:

Proposition 3.3 *Consider a C^i-compactifiable triple (Σ, g, V), $i \geq 3$, satisfying equations (1.4)–(1.5).*

1. *We have*

$$ {}^2\mathcal{R}'\Big|_{x=0} = \frac{1}{3} R'\Big|_{x=0} \ , \tag{3.22}$$

where ${}^2\mathcal{R}'$ is the scalar curvature of the metric induced by $g' \equiv V^{-2}g$ on the level sets of V, and R' is the Ricci scalar of g'.

2. *If R' is constant on $\partial_\infty\Sigma$, replacing V by a positive multiple thereof if necessary we can achieve*

$$R'|_{\partial_\infty\Sigma} = -2\Lambda k \ , \tag{3.23}$$

where $k = 0$, 1 or -1 according to the sign of the Gauss curvature of the metric induced by g' on $\partial_\infty\Sigma$.

REMARK: When $k=0$ Equation (3.23) holds with an arbitrary normalization of V.

PROOF: Consider a level set $\{V = \text{const}\}$ of V which is a smooth hypersurface in $\overline{\Sigma}$, with unit normal n_i, induced metric h_{ij}, scalar curvature ${}^2\mathcal{R}$, second fundamental form p_{ij} defined with respect to an inner pointing normal, mean curvature $p = h^{ij}p_{ij} = h_i{}^k h_j{}^m D_{(k}n_{m)}$; we denote by q_{ij} the trace-free part of p_{ij}: $q_{ij} = p_{ij} - 1/2 h_{ij} p$. Let R_{ijk}, respectively R'_{ijk}, be the Cotton tensor of the metric g_{ij}, respectively g'_{ij}; by definition,

$$R_{ijk} = 2\left(R_{i[j} - \frac{1}{4} R g_{i[j}\right)_{;k]} \;, \tag{3.24}$$

where square brackets denote antisymmetrization with an appropriate combinatorial factor (1/2 in the equation above), and a semi-column denotes covariant differentiation. We note the useful identity due to Lindblom [57]

$$\begin{aligned} R'_{ijk}R'^{ijk} &= V^6 R_{ijk}R^{ijk} \\ &= 8(VW)^2 q_{ij}q^{ij} + V^2 h^{ij} D_i W D_j W \;. \end{aligned} \tag{3.25}$$

When (Σ, g, V) is C^3 compactifiable the function $R'_{ijk}R'^{ijk}$ is uniformly bounded on a neighborhood of $\overline{\Sigma}$, which gives

$$(VW)^2 q_{ij}q^{ij} \le C \tag{3.26}$$

in that same neighborhood, for some constant C. Equations (3.26) and (3.19) give

$$|q|_g = O((V')^3) \;, \tag{3.27}$$

Let q'_{ij} be the trace-free part of the second fundamental form p'_{ij} of the level sets of V' with respect to the metric g'_{ij}, defined with respect to an inner pointing normal; we have $q'_{ij} = q_{ij}/V$, so that

$$|q'|_{g'} = O((V')^2) \;. \tag{3.28}$$

Throughout we use $|\cdot|_k$ to denote the norm of a tensor field with respect to a metric k.

Let us work out some implications of (3.28); Equations (3.15)–(3.17) lead to

$$(\Delta' + \frac{R'}{2})V' = 0 \;. \tag{3.29}$$

Equations (3.19) and (3.20) show that dV' is nowhere vanishing on a suitable neighborhood of $\partial_\infty\Sigma$. We can thus introduce coordinates there so that

$$V' = x \;.$$

If the remaining coordinates are Lie dragged along the integral curves of $D'x$ the metric takes the form

$$g' = (W')^{-1}\,dx^2 + h' \;, \qquad h'(\partial_x, \cdot) = 0 \;. \tag{3.30}$$

Equations (3.29)–(3.30) give then

$$\begin{aligned} p' &= -\frac{1}{2\sqrt{W'}}\left(\frac{\partial W'}{\partial x}+R'x\right) \\ &= -\frac{x}{12\sqrt{W'}}\left(4R'-x\frac{\partial R'}{\partial x}\right) , \end{aligned} \tag{3.31}$$

and in the second step we have used (3.20). Here $p'=\sqrt{W'}\partial_x(\sqrt{\det h'})/\sqrt{\det h'}$ is the mean curvature of the level sets of x measured with respect to the inner pointing normal $n'=\sqrt{W'}\partial_x$. Equation (3.16) implies

$$\begin{aligned} R'_{ij}n'^in'^j &= -2Vn'^in'^jD'_iD'_jV' \\ &= -2\frac{D'^iV'D'^jV'}{V'W'}D'_iD'_jV' \\ &= -\frac{D'^iV'D'_iW'}{V'W'} = \frac{-\partial_xW'}{x} \end{aligned}$$

in the coordinate system of Equation (3.30). From (3.20) we get

$$R'_{ij}n'^in'^j=\frac{R'}{3}+O(x) . \tag{3.32}$$

From the Codazzi–Mainardi equation,

$$(-2R'_{ij}+R'g'_{ij})n'^in'^j = {}^2\mathcal{R}'+q'_{ij}q'^{ij}-\frac{1}{2}p'^2 , \tag{3.33}$$

where ${}^2\mathcal{R}'$ is the scalar curvature of the metric induced by g' on $\partial_\infty\Sigma$, one obtains

$$(-2R'_{ij}+R'g'_{ij})n'^in'^j = {}^2\mathcal{R}'+O(x) , \tag{3.34}$$

where we have used (3.28) and (3.31). This, together with Equation (3.32), establishes Equation (3.22). In particular $R'|_{\partial_\infty\Sigma}$ is constant if and only if ${}^2\mathcal{R}'$ is, and R' at $x=0$ has the same sign as the Gauss curvature of the relevant connected component of $\partial_\infty\Sigma$. Under a rescaling $V\to\lambda V$, $\lambda>0$, we have $W\to\lambda^2V$; Equation (3.17) shows that $R'\to\lambda^2R'$, and choosing λ appropriately establishes the result. □

We do not know whether or not there exist smoothly compactifiable solutions of Equations (1.4)–(1.5) for which R' is not locally constant at $\partial_\infty\Sigma$, it would be of interest to settle this question. Let us point out that the remaining Codazzi–Mainardi equations do not lead to such a restriction. For example, consider the following equation:

$$\begin{aligned} R'_{1a} &= -\mathcal{D}'_ap'+\mathcal{D}'_bp'_a{}^b \\ &= -\frac{1}{2}\mathcal{D}'_ap'+\mathcal{D}'_bq'_a{}^b . \end{aligned} \tag{3.35}$$

Here we are using the adapted coordinate system of Equation (3.30) with $x^1=x$ and with the indices $a,b=2,3$ corresponding to the remaining coordinates;

further $\mathcal{D}'$ denotes the Levi–Civita derivative associated with the metric h'. Since $D'_a D'_x x = \mathcal{D}'_a \sqrt{W'}$, Equation (3.16) yields

$$\mathcal{D}'_a(\sqrt{W'} - \frac{1}{4}xp') = -\frac{1}{2}x\mathcal{D}'_b q'_a{}^b = O(x^3) \; ; \tag{3.36}$$

in the last equality Equation (3.28) has been used. Unfortunately the terms containing R' exactly cancel out in Equations (3.31) and (3.20) leading to

$$\sqrt{W'} + \frac{1}{4}xp' = \sqrt{-\frac{\Lambda}{3}} + O(x^3) \; ,$$

which does not provide any new information.

3.2 Four dimensional conformal approach

Consider a space-time $(M, {}^4g)$ of the form $M = \mathbb{R} \times \Sigma$ with the metric ${}^4g$

$${}^4g = -V^2dt^2 + g \; , \quad g(\partial_t, \cdot) = 0 \; , \quad \partial_t V = \partial_t g = 0 \; . \tag{3.37}$$

By definition of a space-time ${}^4g$ has Lorentzian signature, which implies that g has signature $+3$; it then naturally defines a Riemannian metric on Σ which will still be denoted by g. Equations (1.4)–(1.5) are precisely the vacuum Einstein equations with cosmological constant Λ for the metric ${}^4g$. It has been suggested that an appropriate [9, 43] framework for asymptotically anti-de Sitter space-times is that of conformal completions introduced by Penrose [61]. The work of Friedrich [31] has confirmed that it is quite reasonable to do that, by showing that a large class of space-times (not necessarily stationary) with the required properties exist; some further related results can be found in [50, 58]. In this approach one requires that there exists a space-time with boundary $(\overline{M}, {}^4\overline{g})$ and a positive function $\Omega : \overline{M} \to \mathbb{R}^+$, with Ω vanishing precisely at $\mathscr{I} \subset \partial\overline{M}$, and with $d\Omega$ without zeros on $\mathscr{I}$, together with a diffeomorphism $\Xi : M \to \overline{M} \setminus \mathscr{I}$ such that

$${}^4g = \Xi^*(\Omega^{-2}\, {}^4\overline{g}) \; . \tag{3.38}$$

The vector field $X = \partial_t$ is a Killing vector field for the metric (3.37) on M, and it is well known (*cf., e.g.,* [36, Appendix B]) that X extends as smoothly as the metric allows to $\mathscr{I}$; we shall use the same symbol to denote that extension. We have the following trivial observation:

Proposition 3.4 *Assume that (Σ, g, V) is smoothly compactifiable, then $M = \mathbb{R} \times \Sigma$ with the metric (3.37) has a smooth conformal completion with $\mathscr{I}$ diffeomorphic to $\mathbb{R} \times \partial_\infty\Sigma$. Further $(M, {}^4g)$ satisfies the vacuum equations with a cosmological constant Λ if and only if Equations (1.4)–(1.5) hold.*

The implication the other way round requires some more work:

Theorem 3.5 *Consider a space-time $(M, {}^4g)$ of the form $M = \mathbb{R} \times \Sigma$, with a metric ${}^4g$ of the form (3.37), and suppose that there exists a smooth conformal completion $(\overline{M}, {}^4\overline{g})$ with nonempty $\mathscr{I}$. Then:*

1. *X is timelike on $\mathscr{I}$; in particular it has no zeros there;*

2. *The hypersurfaces t =const extend smoothly to $\mathscr{I}$;*

3. *(Σ, g, V) is smoothly compactifiable;*

4. *There exists a (perhaps different) conformal completion of $(M, {}^4g)$, still denoted by $(\overline{M}, {}^4\overline{g})$, such that $\overline{M} = \mathbb{R} \times \overline{\Sigma}$, where $(\overline{\Sigma}, \overline{g})$ is a conformal completion of (Σ, g), with $X = \partial_t$ and with*

$$ {}^4\overline{g} = -\alpha^2 dt^2 + \overline{g}\ , \quad \overline{g}(\partial_t, \cdot) = 0\ , \quad X(\alpha) = \mathcal{L}_X \overline{g} = 0\ . \tag{3.39} $$

REMARK: The new completion described in point *4.* above will coincide with the original one if and only if the orbits of X are complete in the original completion.

PROOF: As the isometry group maps M to M, it follows that X has to be tangent to $\mathscr{I}$. On M we have ${}^4\overline{g}(X,X) > 0$ hence ${}^4\overline{g}(X,X) \geq 0$ on $\mathscr{I}$, and to establish point *1.* we have to exclude the possibility that ${}^4\overline{g}(X,X)$ vanishes somewhere on $\mathscr{I}$.

Suppose, first, that $X(p) = 0$ for a point $p \in \mathscr{I}$. Clearly X is a conformal Killing vector of ${}^4\overline{g}$. We can choose a neighborhood $\mathscr{U}$ of $\mathscr{I}$ so that X is strictly timelike on $\mathscr{U} \setminus \mathscr{I}$. There exists $\epsilon > 0$ and a neighborhood $\mathcal{O} \subset \mathscr{U}$ of p such that the flow $\phi_t(q)$ of X is defined for all $q \in \mathcal{O}$ and $t \in [-\epsilon, \epsilon]$. The ϕ_t's are local conformal isometries, and therefore map timelike vectors to timelike vectors. Since X vanishes at p the ϕ_t's leave p invariant. It follows that the ϕ_t's map causal curves through p into causal curves through p; therefore they map $\partial J^+(p)$ into itself. This implies that X is tangent to $\partial J^+(p)$. However this last set is a null hypersurface, so that every vector tangent to it is spacelike or null, which contradicts timelikeness of X on $\partial J^+(p) \cap \mathscr{U} \neq \emptyset$. It follows that X has no zeros on $\mathscr{I}$.

Suppose, next, that $X(p)$ is lightlike at p. There exists a neighborhood of p and a strictly positive smooth function ψ such that X is a Killing vector field for the metric ${}^4\overline{g}\psi^2$. Now the staticity condition

$$ X_{[\alpha} \nabla_\beta X_{\gamma]} = 0 \tag{3.40} $$

is conformally invariant, and therefore also holds in the ${}^4\overline{g}$ metric. We can thus use the Carter–Vishweshvara Lemma [21, 67] to conclude that the set $\mathscr{N} = \{q \in \overline{M} | X(q) \neq 0\} \cap \partial\{{}^4\overline{g}(X,X) < 0\} \neq \emptyset$ is a null hypersurface. By hypothesis there exists a neighborhood $\mathscr{U}$ of $\mathscr{I}$ in $\overline{M}$ such that $\mathscr{N} \cap M \cap \mathscr{U} = \emptyset$, hence $\mathscr{N} \subset \mathscr{I}$. This contradicts the fact [61] that the conformal boundary of a vacuum space-time with a strictly negative cosmological constant Λ is timelike. It follows that X cannot be lightlike on $\mathscr{I}$ either, and point *1.* is established.

To establish point *2.*, we note that Equation (3.40) together with point *1.* show that the one-form

$$ \lambda \equiv \frac{1}{{}^4\overline{g}_{\alpha\beta} X^\alpha X^\beta} {}^4\overline{g}_{\mu\nu} X^\mu dx^\nu $$

is a smooth closed one-form on a neighborhood $\mathscr{O}$ of $\mathscr{I}$, hence on any simply connected open subset of $\mathscr{O}$ there exists a smooth function $\overline{t}$ such that $\lambda = d\overline{t}$.

Now (3.37) shows that the restriction of λ to M is dt, which establishes our claim. From now on we shall drop the bar on $\bar{t}$, and write t for the corresponding time function on $\overline{M}$.

Let

$$\overline{\Sigma} = \overline{M} \cap \{t=0\}\ , \quad \chi = \Xi\Big|_{t=0}\ , \quad \omega = \Omega\Big|_{t=0}\ ,$$

where Ξ and Ω are as in (3.38); from Equation (3.38) one obtains

$$g = \chi^*(\omega^{-2}\overline{g})\ ,$$

which shows that $(\overline{\Sigma}, \overline{g})$ is a conformal completion of (Σ, g). We further have $V^2\omega^2 = {}^4g(X,X)\Big|_{t=0}\omega^2 = {}^4\overline{g}(X,X)\Big|_{t=0}$, which has already been shown to be smoothly extendible to $\mathscr{I}^+$ and strictly positive there, which establishes point *3.*

There exists a neighborhood $\mathscr{V}$ of $\overline{\Sigma}$ in $\overline{M}$ on which a new conformal factor Ω can be defined by requiring $\Omega\Big|_{t=0} = \omega$, $X(\Omega) = 0$. Redefining ${}^4\overline{g}$ appropriately and making suitable identifications so that Ξ is the identity, Equation (3.38) can then be rewritten on $\mathscr{V}$ as

$$^4\overline{g} = -(V\Omega)^2 dt^2 + \Omega^2 g\ . \tag{3.41}$$

All the functions appearing in Equation (3.41) are time-independent. The new manifold $\overline{M}$ defined as $\overline{\Sigma}\times\mathbb{R}$ with the metric (3.41) satisfies all the requirements of point *4.*, and the proof is complete. □

In addition to the conditions described above, in [9, 43] it was proposed to further restrict the geometries under consideration by requiring the group of conformal isometries of $\mathscr{I}$ to be the same as that of the anti-de Sitter space-time, namely the universal covering group of $O(2,3)$; *cf.* also [58] for further discussion. While there are various ways of adapting this proposal to our setup, we simply note that the requirement on the group of conformal isometries to be $O(2,3)$ or a covering therof implies that the metric induced on $\mathscr{I}$ is locally conformally flat. Let us then see what are the consequences of the requirement of local conformal flatness of ${}^{\mathscr{I}}g$ in our context; this last property is equivalent to the vanishing of the Cotton tensor of the metric ${}^{\mathscr{I}}g$ induced by ${}^4\overline{g}$ on $\mathscr{I}$. As has been discussed in detail in Section 3.1, we can choose the conformal factor Ω to coincide with V^{-1}, in which case Equation (3.41) reads

$$\begin{aligned} {}^4g' &\equiv {}^4g/V^2 \\ &= -dt^2 + V^{-2}g \\ &= -dt^2 + g'\ , \end{aligned} \tag{3.42}$$

with $g' \equiv V^{-2}g$ already introduced in Section 3.1. It follows that

$$^{\mathscr{I}}g \equiv {}^4g'\Big|_{\mathscr{I}} = -dt^2 + h'\ , \tag{3.43}$$

where h' is the metric induced on $\partial_\infty\Sigma \equiv \mathscr{I}\cap\overline{\Sigma}$ by g'. Let ${}^{\mathscr{I}}R_{ij}$ denote the Ricci tensor of ${}^{\mathscr{I}}g$; from (3.43) we obtain

$$^{\mathscr{I}}R_{it} = 0\ , \quad {}^{\mathscr{I}}R_{AB} = {}^2\mathcal{R}_{AB}\ , \tag{3.44}$$

where ${}^2\mathcal{R}_{AB}$ is the Ricci tensor of h'. In particular the xxA component of the Cotton tensor ${}^{\mathscr{I}}R_{ijk}$ of ${}^{\mathscr{I}}g$ satisfies

$$ {}^{\mathscr{I}}R_{xxA} = -\frac{{}^2\mathcal{R}_{,A}}{4} \; . $$

Point *1.* of Proposition 3.3, see Equation (3.22), shows that the requirement of conformal flatness of ${}^{\mathscr{I}}g$ implies that R' is constant on $\partial_\infty\Sigma$. Conversely, it is easily seen from (3.44) that a locally constant R' — or equivalently ${}^2\mathcal{R}$ — on $\partial_\infty\Sigma$ implies the local conformal flatness of ${}^{\mathscr{I}}g$. We have therefore proved:

Proposition 3.6 *Let (Σ, g, V) be C^i conformally compactifiable, $i \geq 3$, and satisfy (1.3)–(1.5). The conformal boundary $\mathbb{R}\times\partial_\infty\Sigma$ of the space-time $(M = \mathbb{R}\times\Sigma, {}^4g)$, ${}^4g$ given by (3.37), is locally conformally flat if and only if the scalar curvature R' of the metric $V^{-2}g$ is locally constant on $\partial_\infty\Sigma$. This is equivalent to requiring that the metric induced by $V^{-2}g$ on $\partial_\infty\Sigma$ has locally constant Gauss curvature.*

3.3 A coordinate approach

An alternative approach to the conformal one discussed above is by introducing preferred coordinate systems. As discussed in [44, Appendix D], coordinate approaches are often equivalent to conformal approaches when sufficiently strong hypotheses are made. We stress that this equivalence is a delicate issue when finite degrees of differentiability are assumed, as arguments leading from one approach to the other often involve constructions in which some differentiability is lost.

In any case, the coordinate approach has been used by Boucher, Gibbons and Horowitz [15] in their argument for uniqueness of the anti-de Sitter metric within a certain class of static space-times. More precisely, in [15] one considers metrics which are asymptotic to generalized Kottler metrics with $k = 1$ in the following strong sense: if g_0 denotes one of the metrics (1.1) with $k = 1$, then one assumes that there exists a coordinate system (t, r, x^A) such that

$$ \begin{aligned} g \;=\;& g_0 + O(r^{-2})dt^2 + O(r^{-6})dr^2 \\ & +O(r)\,(\text{remaining differentials not involving } dr) \\ & +O(r^{-1})\,(\text{remaining differentials involving } dr)\;. \end{aligned} \tag{3.45} $$

We note that in the uniqueness assertions of [15] one makes appeal to the positive energy theorem to conclude. Now we are not aware of a version of such a theorem which would hold without some further hypotheses on the behavior of the metric. For example, in such a theorem one is likely to require that the derivatives of the metric also fall off at some sufficiently high rates. In any case the argument presented in [15] seems to implicitly assume that the asymptotic behavior of g^{tt} described above is preserved under differentiation, so that the corrections terms in (3.45) give a vanishing contribution when calculating $|dV|^2_g - |dV_0|^2_{g_0}$ and passing to the limit $r \to \infty$, with g_0 — the anti-de Sitter metric. While it might well be possible that Equations (1.4)–(1.5) force the metrics satisfying (3.45) to have sufficiently good asymptotic properties to be able to justify this, or to apply a positive energy theorem[6], this remains to be established.[7]

It is far from being clear whether or not a general metric of the form (3.45) has any well behaved conformal completions. For example, the coordinate transformation (3.1) together with a multiplication by the square of the conformal factor $\omega = x$ brings the metric (3.45) to one which can be continuously extended to the boundary, but if only (3.45) is assumed then the resulting metric will not be differentiable up to boundary on the compactified manifold in general. There could, however, exist coordinate systems which lead to better conformal behavior when Equations (1.4)–(1.5) are imposed.

In any case, it is natural to ask whether or not a metric satisfying the requirements of Section 3.1 will have a coordinate representation similar to (3.45). A partial answer to this question is given by the following[8]:

Proposition 3.7 *Let (Σ, g, V) be a C^i compactifiable solution of Equations (1.4)–(1.5), $i \geq 3$. Define a C^{i-2} function $\tilde{k} = \tilde{k}(x^A)$ on $\partial_\infty\Sigma$ by the formula*

$$R'|_{\partial_\infty\Sigma} = -2\Lambda\tilde{k} \ . \tag{3.46}$$

1. *Rescaling V by a positive constant if necessary, there exists a coordinate system (r, x^A) near $\partial_\infty\Sigma$ in which we have*

$$V^2 = \tfrac{r^2}{\ell^2} + \tilde{k} \ , \tag{3.47}$$

$$g = \frac{dr^2}{\left(\frac{r^2}{\ell^2} + \tilde{k} - \frac{2\mu}{r}\right)} + O(r^{-3})dr\, dx^A + \left(r^2\check{h}_{AB} + O(r^{-1})\right) dx^A dx^B \tag{3.48}$$

(recall that $\ell^2 = -3\Lambda^{-1}$), for some r-independent smooth two-dimensional metric $\check{h}_{AB}$ with Gauss curvature equal to $\tilde{k}$ and for some function $\mu = \mu(r, x^A)$. Further

$$\check{h}^{AB}g_{AB} = 2\left(r^2 - \frac{\mu_\infty}{r} + O(r^{-2})\right) \ , \tag{3.49}$$

where $\check{h}^{AB}$ denotes the matrix inverse to $\check{h}_{AB}$ while

$$\mu_\infty \equiv \lim_{r\to\infty}\mu = \frac{\ell^3}{12}\frac{\partial R'}{\partial x}\bigg|_{x=0} \ . \tag{3.50}$$

---

[6]Recall that in the asymptotically flat case one can derive an asymptotic expansion for stationary metrics from rather weak hypotheses on the leading order behavior of the metric [25, 52, 63]. See especially [2, 3], where the Lichnerowicz theorem is proved without any hypotheses on the asymptotic behavior of the metric, under the condition of geodesic completeness of space-time.

[7]The key point of the argument in [15] is to prove that the coordinate mass is negative. When $\partial_\infty\Sigma = S^2$, and the asymptotic conditions are such that the positive energy theorem applies, one can conclude that the initial data set under consideration must be coming from one in anti-de Sitter space-times *provided* one shows that the coordinate mass coincides with the mass which occurs in the positive energy theorem. To our knowledge such an equality has not been proved so far for metrics with the asymptotics (3.45), or else.

[8]See [44, Appendix] for a related discussion. While the conclusions in [44] appear to be weaker than ours, it should be stressed that in [44] staticity of the space-times under consideration is not assumed.

2. *If one moreover assumes that R' is locally constant on $\partial_\infty\Sigma$, then Equation (3.48) can be improved to*

$$g = \frac{dr^2}{\left(\frac{r^2}{\ell^2} + k - \frac{2\mu}{r}\right)} + \left(r^2\check{h}_{AB} + O(r^{-1})\right) dx^A dx^B \;, \tag{3.51}$$

with $\check{h}_{AB}$ having constant Gauss curvature $k = 0, \pm 1$ according to the genus of the connected component of $\partial_\infty\Sigma$ under consideration.

REMARKS: 1. The function $(x, x^A) \to \mu(r = 1/x, x^A)$ is of differentiability class C^{i-3} on $\overline{\Sigma}$, with the function $(x, x^A) \to (\mu/r)(r = 1/x, x^A)$ being of differentiability class C^{i-2} on $\overline{\Sigma}$.

2. In Equations (3.48) and (3.51) the error terms $O(r^{-j})$ satisfy

$$\partial_r^s \partial_{A_1} \ldots \partial_{A_t} O(r^{-j}) = O(r^{-j-s})$$

for $0 \le s + t \le i - 3$.

3. We emphasize that the function $\tilde{k}$ defined in Equation (3.46) could *a priori* be x^A-dependent. In such a case neither the definition of coordinate mass of Section 5.1 nor the definition of Hamiltonian mass of Section 5.2 apply.

4. It seems that to be able to obtain (3.45), in addition to the hypothesis that R' is locally constant on $\partial_\infty\Sigma$ one would at least need the quantity appearing at the right hand side of Equation (3.50) to be locally constant on $\partial_\infty\Sigma$ as well. We do not know whether this is true in general; we have not investigated this question as this is irrelevant for our purposes.

PROOF: Consider, near $\partial_\infty\Sigma$, the coordinate system of Equation (3.30); from Equations (3.31) and (3.20) we obtain

$$\partial_x \left(\ln \sqrt{\det h'_{AB}}\right) = -2\tilde{k}x - \frac{3\mu_\infty}{\ell}x^2 + O(x^3) \;, \tag{3.52}$$

ℓ as in (2.10), $\tilde{k}$ as in (3.46), μ_∞ as in (3.50). This, together with Equation (3.28), leads to

$$\frac{\partial h'_{AB}}{\partial x} = -2x\tilde{k}h'_{AB} + O(x^2) \quad \Longrightarrow \quad h'_{AB} = (1 - \tilde{k}x^2)\ell^2\check{h}_{AB} + O(x^3) \;,$$

where

$$\check{h}_{AB} \equiv \frac{1}{\ell^2} h'_{AB}\Big|_{x=0} \;.$$

Proposition 3.3 shows that $\tilde{k}$ is proportional to the Gauss curvature of $\check{h}_{AB}$. It follows now from (3.20) that

$$g = x^{-2}g' = \frac{\ell^2}{x^2\left(1 - \frac{R'\ell^2x^2}{6}\right)}dx^2 + \left\{\frac{(1 - \tilde{k}x^2)}{x^2}h'_{AB}\Big|_{x=0} + O(x^3)\right\} dx^A dx^B \;.$$

The above suggests to introduce a coordinate r via the formula[9]

$$\frac{r^2}{\ell^2} = \frac{1 - \tilde{k}x^2}{x^2} \ . \tag{3.53}$$

Suppose, first, that $\tilde{k}$ is locally constant on $\partial_\infty\Sigma$, then $\tilde{k}$ equals $k = 0, \pm 1$ according to the genus of the connected component of $\partial_\infty\Sigma$ under consideration, and one finds

$$\begin{aligned} g &= \frac{dr^2}{\left(\frac{r^2}{\ell^2} + k\right)\left\{1 + \frac{\ell^2}{r^2}\left(k - \frac{R'\ell^2x^2}{6}\right)\right\}} + \left(\frac{r^2}{\ell^2} h'_{AB}\Big|_{x=0} + O(r^{-1})\right) dx^A dx^B \\ &= \frac{dr^2}{\left(\frac{r^2}{\ell^2} + k - \frac{2\mu}{r}\right)} + \left(\frac{r^2}{\ell^2} h'_{AB}\Big|_{x=0} + O(r^{-1})\right) dx^A dx^B \ , \end{aligned}$$

where the "mass aspect" function $\mu = \mu(r, x^A)$ is defined as

$$\begin{aligned} \mu &\equiv -\frac{r}{2}\left(1 + k\frac{\ell^2}{r^2}\right)\left(k - \frac{R'\ell^2x^2}{6}\right) \\ &= -\frac{r}{2}\left(k - \frac{R'\ell^2}{6} + \frac{k^2\ell^2}{r^2}\right) \\ &= \frac{r\ell^2}{2}\left(\frac{1}{6}(R' - R'|_{x=0}) - \frac{k^2}{r^2}\right) \ . \end{aligned} \tag{3.54}$$

This establishes Equations (3.47) and (3.51). When $\tilde{k}$ is not locally constant an identical calculation using the coordinate r defined in Equation (3.53) establishes Equation (3.48) — the only difference is the occurrence of non-vanishing error terms in the $drdx^A$ part of the metric, introduced by the angle dependence of $\tilde{k}$. It follows from Equation (3.54) — or from the $\tilde{k}$ version thereof when $\tilde{k}$ is not locally constant — that

$$\mu = \frac{\ell^3}{12}\frac{\partial R'}{\partial x}\Big|_{x=0} + O(r^{-1}) \ ,$$

which establishes Equation (3.50). Equation (3.49) is obtained by integration of Equation (3.52).

4 Connectedness of $\partial_\infty\Sigma$

The class of manifolds considered so far could in principle contain Σ's for which neither $\partial_\infty\Sigma$ nor $\partial\Sigma$ are connected. Under the hypothesis of staticity the question of connectedness of $\partial\Sigma$ is open; we simply note here the existence of dynamical (non-stationary) solutions of Einstein–Maxwell equations with a non-connected black hole region with positive cosmological constant Λ [18, 51]. As far as $\partial_\infty\Sigma$ is concerned, we have the following:

[9] We note that $\tilde{k}$ is of differentiability class lower by two orders as compared to the metric itself, which leads to a loss of three derivatives when passing to a new coordinate system in which r is defined by Equation (3.53). One can actually introduce a coordinate system closely related to (3.53) with a loss of only one degree of differentiability of the metric by using the techniques of [4, Appendix A], but we shall not discuss this here.

Theorem 4.1 *Let (Σ, g, V) be a C^i compactifiable solution of Equations (1.4)–(1.5), $i \geq 3$. Then $\partial_\infty \Sigma$ is connected.*

PROOF: Consider the manifold $M = \mathbb{R} \times \Sigma$ with the metric (3.37); its conformal completion $\overline{M} = \mathbb{R} \times \overline{\Sigma}$ with the metric ${}^4g/V^2$ is a stably causal manifold with boundary. We wish to show that it is also globally hyperbolic in the sense of [33], namely that 1) it is strongly causal and 2) for each $p, q \in M$ the set $J^+(p) \cap J^-(q)$ is compact. The existence of the global time function t clearly implies strong causality, so it remains to verify the compactness condition. Now a path $\Gamma(s) = (t(s), \gamma(s)) \in \mathbb{R} \times \Sigma$ is an achronal null geodesic from $p = (t(0), \gamma(0))$ to $q = (t(1), \gamma(1))$ if and only if $\gamma(s)$ is a minimizing geodesic between $\gamma(0)$ and $\gamma(1)$ for the "optical metric" $V^{-2}g$. Compactness of $J^+(p) \cap J^-(q)$ is then equivalent to compactness of the $V^{-2}g$-distance balls; this latter property will hold when $(\Sigma \cup \partial_\infty \Sigma, V^{-2}g)$ is a geodesically complete manifold (with boundary) by (an appropriate version of) the Hopf–Rinow theorem.

Let us thus show that $(\Sigma, V^{-2}g)$ is geodesically complete. Suppose, first, that $\partial\Sigma = \emptyset$; the hypothesis that Σ has compact interior together with the fact that V tends to infinity in the asymptotic regions implies that $V \geq V_0 > 0$ for some constant V_0. This shows that $(\Sigma, V^{-2}g)$ is a compact manifold with boundary $\partial_\infty \Sigma$, and the result follows. (When the metric induced by $V^{-2}g$ on $\partial_\infty \Sigma$ has positive scalar curvature connectedness of $\partial_\infty \Sigma$ can also be inferred from [69].)

Consider, next, the case $\partial\Sigma \neq \emptyset$. It is well known that $|dV|_g$ is a non-zero constant on every connected component of $\partial\Sigma$ (*cf.* the discussion around Equation (7.2)); therefore we can introduce coordinates near $\partial\Sigma$ so that $V = x$, with the metric taking the form

$$V^{-2}g = x^{-2}\left((dx)^2 + h_{AB}(x, x^A)dx^A dx^B\right) , \tag{4.1}$$

where the x^A's are local coordinates on $\partial\Sigma$. It is elementary to show now from (4.1) that $(\Sigma \cup \partial_\infty \Sigma, V^{-2}g)$ is a complete manifold with boundary, as claimed.

When (Σ, g) is smoothly compactifiable we can now use [33, Theorem 2.1] to infer connectedness of $\partial_\infty \Sigma$, compare [32, Corollary, Section III]. For compactifications with finite differentiability we argue as follows: For small s let λ be the mean curvature of the sets $\equiv \{x = s\}$, where x is the coordinate of Equation (3.11). In the coordinate system used there the unit normal to those sets pointing away from $\partial_\infty \Sigma$ equals $x\partial_x$; if (Σ, g, V) is C^3 compactifiable the tensor field $\overline{h}$ appearing in Equation (3.11) will be C^1 so that[10]

$$\begin{aligned}
\lambda &= \frac{1}{\sqrt{\det g}}\partial_i\left(\sqrt{\det g}\, n^i\right) \\
&= \frac{x^3}{\sqrt{\det \overline{h}}}\partial_x\left(x^{-2}\sqrt{\det \overline{h}}\right) \\
&= -2 + O(x) .
\end{aligned}$$

[10]The differentiability threshold $k = 3$ can be lowered using the "almost Gaussian coordinate systems" of [4, Appendix A], we shall however not be concerned with this here.

It follows that for s small enough the sets $\{x = s, t = \tau\}$ are trapped, with respect to the inward pointing normal, in the space-time $\mathbb{R}\times\Sigma$ with the metric (3.37). Suppose that $\partial_\infty\Sigma$ were not connected, then those (compact) sets would be outer trapped with respect to every other connected component of $\partial_\infty\Sigma$. This is, however, not possible by the usual global arguments, *cf.*, *e.g.*, [34, 35] or [28, Section 4] for details. $\square$

5 The mass

5.1 A coordinate mass M_c

There exist several proposals how to assign a mass M to a space-time which is asymptotic to an anti-de Sitter space-time [1, 8, 9, 38, 44]; it seems that the simplest way to do that (as well as to extend the definition to the generalized Kottler context considered here) proceeds as follows: consider a metric defined on a coordinate patch covering the set

$$\Sigma_{\text{ext}} \equiv \{t = t_0, r \geq R, (x^A) \in {}^2M\} \tag{5.1}$$

(which we will call an "end"), and suppose that in this coordinate system the functions $g_{\mu\nu}$ are of the form (1.1) plus lower order terms[11]

$$\begin{gathered} g_{tt} = -(k - \tfrac{2m}{r} - \tfrac{\Lambda}{3}r^2) + \tfrac{o(1)}{r}\,, \qquad g_{rr} = 1/(k - \tfrac{2m}{r} - \tfrac{\Lambda}{3}r^2 + \tfrac{o(1)}{r})\,, \\ g_{t\mu} = o(1)\,, \quad \mu \neq t\,, \qquad g_{r\mu} = o(1)\,, \quad \mu \neq r, t\,, \\ g_{AB} - r^2 h_{AB} = o(r^2)\,, \end{gathered} \tag{5.2}$$

for some constant m, and for some constant curvature (t and r independent) metric $h_{AB}dx^A dx^B$ on ${}^2M$. Then we define the coordinate mass M_c of the end Σ_{ext} to be the parameter m appearing in (1.1). This procedure gives a unique prescription how to assign a mass to a metric *and a coordinate system* on Σ_{ext}.

There is an obvious coordinate-dependence in this definition when $k = 0$: Indeed, in that case a coordinate transformation $r \to \lambda r$, $t \to t/\lambda$, $d\Omega_k^2 \to \lambda^{-2}d\Omega_k^2$, where λ is a positive constant, does not change the asymptotic form of the metric, while m gets replaced by $\lambda^{-3}m$. When ${}^2M$ is compact this freedom can be removed *e.g.* by requiring that the area of ${}^2M$ with respect to the metric $d\Omega_k^2$ be equal to 4π, or to 1, or to some other chosen constant. For $k = \pm 1$ this ambiguity does not arise because in this case such rescalings will change the asymptotic form of the metric, and are therefore not allowed.

It is far from being clear that the above definition will assign the same parameter M_c to every coordinate system satisfying our requirements: if that is the case, to prove such a statement one might perhaps need to further require that the coordinate derivatives of the coordinate components of g in the above described coordinate system have some appropriate decay properties; further one might perhaps have to replace the $o(1)$'s by $o(r^{-\sigma})$'s or $O(r^{-\sigma})$'s, for some

[11] Because the natural length of the vectors ∂_A is $O(r)$ it would actually be natural to require $g_{r\mu} = o(r)$, $\mu \neq r, t$ instead of $g_{r\mu} = o(1)$, $\mu \neq r, t$.

appropriate σ's, perhaps as in (3.45); this is however irrelevant for our discussion at this stage.

A possible justification of this definition proceeds as follows: when ${}^2M = S^2$ and $\Lambda = 0$ it is widely accepted that the mass of $\Sigma_{\rm ext}$ equals m, because m corresponds to the active gravitational mass of the gravitational field in a quasi-Newtonian limit. (It is also known in this case that the definition is coordinate-independent [10, 24].) For $\Lambda \neq 0$ and/or ${}^2M \neq S^2$ one calls m the mass by extrapolation.

Consider, then, the metric (3.37), with V and g as in (3.47)–(3.48); suppose further that the limit

$$\mu_\infty \equiv \lim_{r\to\infty} \mu$$

exists and is a constant. To achieve the form of the metric ${}^4g$ just described one needs to replace the coordinate r of (3.47)–(3.48) with a new coordinate ρ defined as

$$r^2 + k = \rho^2 + k + \frac{\mu_\infty}{\rho} \ .$$

This leads to

$$\begin{aligned} {}^4g \ &= \ -\left(\frac{\rho^2}{\ell^2} + k + \frac{\mu_\infty}{\rho}\right) dt^2 + \frac{d\rho^2}{\left(\frac{\rho^2}{\ell^2} + k + \frac{\mu_\infty}{\rho} + O(\frac{1}{\rho^2})\right)} \\ &\qquad + O(\rho^{-3}) d\rho\, dx^A + \left(\rho^2 \check{h}_{AB} + O(\rho^{-1})\right) dx^A dx^B \ , \end{aligned} \tag{5.3}$$

and therefore

$$M_c \equiv -\frac{\mu_\infty}{2} = -\frac{\ell^3}{24}\frac{\partial R'}{\partial x}\Big|_{x=0} \ , \tag{5.4}$$

where the second equality above follows from (3.50). We note that the approach described does not give a definition of mass when $\lim_{r\to\infty}\mu$ does not exist, or is not a constant function on $\partial_\infty\Sigma$.

The above described dogmatic approach suffers from various shortcomings. First, when ${}^2M \neq S^2$, the arguments given are compatible with M_c being any function $M_c(m, \Lambda)$ with the property that $M_c(m, 0) = m$. Next, the transition from $\Lambda \neq 0$ to $\Lambda = 0$ has dramatic consequences as far as global properties of the corresponding space-times are concerned, and one can argue that there is no reason why the function $M_c(m, \Lambda)$ should be continuous at zero. For example, according to [44, Equation (III.8c)], the mass of the metric (1.1) with ${}^2M = S^2$ should be $16\pi m\ell$, with ℓ as in (2.10), which blows up when Λ tends to zero with m being held fixed. Finally, when ${}^2M \neq S^2$ the Newtonian limit argument does not apply because the metrics (1.1) with $\Lambda = 0$ and ${}^2M \neq S^2$ do not seem to have a Newtonian equivalent. In particular there is no reason why M_c should not depend upon the genus g_∞ of ${}^2M$ as well.

All the above arguments make it clear that a more fundamental approach to the definition of mass would be useful. It is common in field theory to define energy by Hamiltonian methods, and this is the approach we shall pursue in the next section.

5.2 The Hamiltonian mass M_{Ham}.

The Hamiltonian approach allows one to define the energy from first principles. For a solution of the field equations, we can simply take as the energy the numerical value of the Hamiltonian. It must be recognized, however, that the Hamiltonians might depend on the choice of the phase space, if several such choices are possible, and they are defined only up to an additive constant on each connected component of the phase space. They also depend on the choice of the Hamiltonian structure, if more than one such structure exists.

Let us start by briefly recalling the results of the analysis of [23], based on the Hamiltonian approach of Kijowski and Tulczyjew [54, 55], see also [53]. One assumes that a manifold M on which an (unphysical) background metric b is given, and one considers metrics ${}^4g$ which asymptote to b in the relevant asymptotic regions of M. We stress that the background metric is only a tool to prescribe the asymptotic boundary conditions, and does not have any physical significance. Let X be any vector field on M and let Σ be any hypersurface in M. By a well known procedure the motion of Σ along the flow of X can be used to construct a Hamiltonian dynamical system in an appropriate phase space of fields over Σ; the reader is referred to [29, 53–55] for a geometric approach to this question. In [23] it is also assumed that X is a Killing vector field of the background metric; this is certainly not necessary (*cf., e.g.*, [29] for general formulae), but is sufficient for our purposes, as we are going to take X to be equal to $\partial/\partial t$ in the coordinate system of Equation (3.37). In the context of metrics which asymptote to the generalized Kottler metrics at large r, a rigorous functional description of the spaces involved has not been carried out so far, and lies outside the scope of this paper. Let us simply note that one expects, based on the results in [29, 31, 50], to obtain a well defined Hamiltonian system in this context. Therefore the formal calculations of [23] lead one to expect that on an appropriate space of fields, such that the associated physical space-time metrics ${}^4g$ asymptote to the background metric b at a suitable rate, the Hamiltonian $H(X,\Sigma)$ will coincide with (or, at worse, will be closely related to) the one given by the formula derived in [23]:

$$H(X,\Sigma) \quad = \quad \frac{1}{2}\int_{\partial\Sigma} \mathbb{U}^{\alpha\beta}\, dS_{\alpha\beta}\ , \tag{5.5}$$

where the integral over $\partial\Sigma$ should be understood by a limiting process, as the limit as R tends to infinity of integrals of coordinate spheres $t = 0$, $r = R$ on $\Sigma_{\rm ext}$. Here $dS_{\alpha\beta}$ is defined as $\frac{\partial}{\partial x^\alpha}\lrcorner\frac{\partial}{\partial x^\beta}\lrcorner \mathrm{d}x^0\wedge\cdots\wedge \mathrm{d}x^n$, with $\lrcorner$ denoting contraction, and $\mathbb{U}^{\alpha\beta}$ is given by

$$\mathbb{U}^{\nu\lambda} \quad = \quad \mathbb{U}^{\nu\lambda}{}_{\beta}X^\beta + \frac{1}{8\pi}\left(\sqrt{|\det g_{\rho\sigma}|}\ g^{\alpha[\nu} - \sqrt{|\det b_{\rho\sigma}|}\ b^{\alpha[\nu}\right)\delta^{\lambda]}_{\beta}X^\beta{}_{;\alpha} \tag{5.6}$$

$$\mathbb{U}^{\nu\lambda}{}_{\beta} \quad = \quad \frac{2|\det b_{\mu\nu}|}{16\pi\sqrt{|\det g_{\rho\sigma}|}}g_{\beta\gamma}(e^2 g^{\gamma[\nu}g^{\lambda]\kappa})_{;\kappa}\ . \tag{5.7}$$

Here, and *throughout this section*, g stands for the space-time metric ${}^4g$ unless explicitly indicated otherwise. Further, a semicolon denotes covariant differentiation *with respect to the background metric* b, while

$$e \equiv \frac{\sqrt{|\det g_{\rho\sigma}|}}{\sqrt{|\det b_{\mu\nu}|}} . \tag{5.8}$$

Some comments concerning Equation (5.6) are in order: in [23] the starting point of the analysis is the Hilbert Lagrangian for vacuum Einstein gravity,

$$\mathcal{L} = \sqrt{-\det g_{\mu\nu}}\ \frac{g^{\alpha\beta}R_{\alpha\beta}}{16\pi} .$$

As the normalization factors play an important role in giving a correct definition of mass, we recall that the factor $1/16\pi$ is determined by the requirement that the theory has the correct Newtonian limit (units $G = c = 1$ are used throughout). With our signature $(- + ++)$ the Einstein equations with a cosmological constant read

$$R_{\mu\nu} - \frac{g^{\alpha\beta}R_{\alpha\beta}}{2} g_{\mu\nu} = -\Lambda g_{\mu\nu} ,$$

so that one needs to repeat the analysis in [23] with $\mathcal{L}$ replaced by

$$\frac{\sqrt{-\det g_{\mu\nu}}}{16\pi}\left(g^{\alpha\beta}R_{\alpha\beta} - 2\Lambda\right) .$$

The general expression for the Hamiltonian (5.5) in terms of X^μ, $g_{\mu\nu}$ and $b_{\mu\nu}$ ends up to coincide with that obtained with $\Lambda = 0$, which can be seen either by direct calculations, or by the Legendre transformation arguments of [23, end of Section 3] together with the results in [53]. Note that Equation (5.6) does not exactly coincide with that derived in [23], as the formulae there do not contain the term $-\sqrt{|\det b_{\rho\sigma}|}\ b^{\alpha[\nu}\delta^{\lambda]}_{\beta} X^\beta{}_{;\alpha}$. However, this term does not depend on the metric g, and such terms can be freely added to the Hamiltonian because they do not affect the variational formula that defines a Hamiltonian. From an energy point of view such an addition corresponds to a choice of the zero point of the energy. We note that in our context $H(X,\Sigma)$ would not converge if the term $-\sqrt{|\det b_{\rho\sigma}|}\ b^{\alpha[\nu}\delta^{\lambda]}_{\beta} X^\beta{}_{;\alpha}$ were not present in (5.6).

In order to apply this formalism in our context, we let b be any t-independent metric on $M = \mathbb{R}\times\Sigma$ such that (with $0 \neq \Lambda = -3/\ell^2$)

$$b = -(k + \frac{r^2}{\ell^2})dt^2 + (k + \frac{r^2}{\ell^2})^{-1}dr^2 + r^2\check{h} \tag{5.9}$$

on $\mathbb{R}\times\Sigma_{\mathrm{ext}} \approx \mathbb{R}\times[R,\infty)\times{}^2M$, for some $R \geq 0$, where $\check{h} = \check{h}_{AB}dx^A dx^B$ denotes a metric of constant Gauss curvature $k = 0, \pm 1$ on the two dimensional connected compact manifold ${}^2M$.

Let us return to the discussion in Section 5.1 concerning the freedom of rescaling the coordinate r by a positive constant λ. First, if k in Equation (5.9) is any constant different from zero, then there exists a (unique) rescaling of r and t which brings k to one, or to minus one. Next, if $k = 0$ we can —

without changing the asymptotic form of the metric — rescale the coordinate r by a positive constant λ, the coordinate t by $1/\lambda$, and the metric $\check{h}$ by λ^{-2}, so that there is still some freedom left in the coordinate system above; a unique normalization can then be achieved by asking *e.g.* that the area

$$A_\infty \equiv \int_{{}^2M} d^2\mu_{\check{h}} \tag{5.10}$$

equals 4π — this will be the most convenient normalization for our purposes. Here $d^2\mu_{\check{h}}$ is the Riemannian measure associated with the metric $\check{h}$. We wish to point out that that *regardless* of the value of k, the Hamiltonian $H(X,\Sigma)$ is *independent* of this scaling: this follows immediately from the fact that $\mathbb{U}^{\alpha\beta}$ behaves as a density under linear coordinate transformations. An alternative way of seeing this is that in the new coordinate system X equals $\lambda\partial/\partial t$, which accounts for a factor $1/\lambda$ in the transformation law of the coordinate mass, as discussed at the beginning of Section 5.1. The remaining factor $1/\lambda^2$ needed there is accounted for by a change of the area of $\partial_\infty\Sigma$ under the rescaling of the metric $\check{h}$ which accompanies that of r.

In order to evaluate H it is useful to introduce the following b-orthonormal frame:

$$e_{\hat{0}} = \frac{1}{\sqrt{k+\frac{r^2}{\ell^2}}}\partial_t \ , \quad e_{\hat{1}} = \sqrt{k+\frac{r^2}{\ell^2}}\partial_r \ , \quad e_{\hat{A}} = \frac{1}{r}\check{e}_{\hat{A}} \ , \tag{5.11}$$

where $\check{e}_{\hat{A}}$ is an ON frame for the metric $\check{h}$. The connection coefficients, defined by the formula $\nabla_{e_{\hat{a}}} e_{\hat{b}} = \omega^{\hat{c}}{}_{\hat{b}\hat{a}} e_{\hat{c}}$, read

$$\begin{aligned}
\omega_{\hat{0}\hat{1}\hat{0}} &= -\frac{r}{\ell^2\sqrt{k+\frac{r^2}{\ell^2}}} = -\tfrac{1}{\ell} + O(r^{-2}) \ , \\
\omega_{\hat{1}\hat{2}\hat{2}} = \omega_{\hat{1}\hat{3}\hat{3}} &= -\frac{\sqrt{k+\frac{r^2}{\ell^2}}}{r} = -\tfrac{1}{\ell} + O(r^{-2}) \ , \\
\omega_{\hat{2}\hat{3}\hat{3}} &= \begin{cases} -\frac{\coth\theta}{r} \ , & k=-1 \ , \\ 0 \ , & k=0 \ , \\ -\frac{\cot\theta}{r} \ , & k=1 \ . \end{cases}
\end{aligned} \tag{5.12}$$

Those connection coefficients which are not obtained from the above ones by permutations of indices are zero; we have used a coordinate system θ,φ on ${}^2M$ in which $\check{h}$ takes, locally, the form $d\theta^2+\sinh^2\theta\,d\varphi^2$ for $k=-1$, $d\theta^2+d\varphi^2$ for $k=0$, and $d\theta^2+\sin^2\theta\,d\varphi^2$ for $k=1$. We also have

$$X^{\hat{0}} = \sqrt{k+\tfrac{r^2}{\ell^2}} = \tfrac{r}{\ell} + O(r^{-1}) \ , \tag{5.13}$$

$$e_{\hat{1}}(X^{\hat{0}}) = X^{\hat{0}}{}_{;\hat{1}} = -X_{\hat{0};\hat{1}} = X_{\hat{1};\hat{0}} = \tfrac{r}{\ell^2} \ , \tag{5.14}$$

with the third equality in (5.14) following from the Killing equations $X_{\mu;\nu} + X_{\nu;\mu} = 0$; all the remaining $X^{\hat{\mu}}$'s and $X_{\hat{\mu};\hat{\nu}}$'s are zero. Let the tensor field $e^{\mu\nu}$ be defined by the formula

$$e^{\mu\nu} \equiv g^{\mu\nu} - b^{\mu\nu} \ . \tag{5.15}$$

We shall use hatted indices to denote the components of a tensor field in the frame $e_{\hat{a}}$ defined in (5.11), *e.g.* $e^{\hat{a}\hat{c}}$ denotes the coefficients of $e^{\mu\nu}$ with respect

to that frame:
$$e^{\mu\nu}\partial_\mu \otimes \partial_\nu = e^{\hat a\hat c} e_{\hat a} \otimes e_{\hat c} \ .$$
Suppose that the metric ${}^4g$ is such that the $e^{\hat a\hat c}$'s tend to zero as r tends to infinity. By a Gram–Schmidt procedure we can find a frame $f_{\tilde a}$, $\tilde a = 0, \ldots, 3$, orthonormal *with respect to the metric* g, such that f_0 is proportional to e_0, and such that the $e_{\hat a}$ components of $f_0 - e_0$, ..., $f_3 - e_3$ tend to zero as r tends to infinity:
$$f_{\tilde a} = f_{\tilde a}{}^{\hat a} e_{\hat a} \to_{r\to\infty} \delta^{\hat a}_{\tilde a} e_{\hat a} \ . \tag{5.16}$$
From (5.5) and (5.16) we expect that[12]
$$H(X,\Sigma) \quad = \quad \lim_{R\to\infty} \int_{\Sigma\cap\{r=R\}} r^2 \mathbb{U}^{\hat 1\hat 0} d^2\mu_r \ , \tag{5.17}$$
where $d^2\mu_r$ is the Riemannian measure induced on $\Sigma \cap \{r = R\}$ by ${}^4g$. We wish to analyze when the above limit exists; we have
$$r^2\mathbb{U}^{\hat 1\hat 0}{}_\beta X^\beta = r^2 \mathbb{U}^{\hat 1\hat 0}{}_{\hat 0} X^{\hat 0} \approx \frac{r^3}{\ell}\mathbb{U}^{\hat 1\hat 0}{}_{\hat 0} \ ,$$
hence we need to keep track of all the terms in $\mathbb{U}^{\hat 1\hat 0}{}_{\hat 0}$ which decay as r^{-3} or slower. Similarly one sees from Equations (5.13)–(5.14) that only those terms in
$$\Delta^{\hat\alpha\hat\nu} \equiv \sqrt{|\det g_{\hat\rho\hat\sigma}|}\; g^{\hat\alpha\hat\nu} - \sqrt{|\det b_{\hat\rho\hat\sigma}|}\; b^{\hat\alpha\hat\nu}$$
which are $O(r^{-3})$, or which are decaying slower, will give a non-vanishing contribution to the term involving the derivatives of X in the integral (5.17). This suggests to consider metrics ${}^4g$ such that
$$e^{\hat\mu\hat\nu} = o(r^{-3/2}) \ , \quad e_{\hat\rho}(e^{\hat\mu\hat\nu}) = o(r^{-3/2}) \ . \tag{5.18}$$
The boundary conditions (5.18) ensure that one needs to keep track only of those terms in $\mathbb{U}^{\hat 1\hat 0}$ which are linear in $e^{\hat\mu\hat\nu}$ and $e_{\hat\rho}(e^{\hat\mu\hat\nu})$, when $\mathbb{U}^{\hat 1\hat 0}$ is Taylor expanded around b. For a generalized Kottler metric (1.1) we have
$$e^{\hat 0\hat 0} \approx e^{\hat 1\hat 1} \approx -\frac{2m\ell^2}{r^3} \ , \qquad e_{\hat 1}(e^{\hat 0\hat 0}) \approx e_{\hat 1}(e^{\hat 1\hat 1}) \approx \frac{6m\ell}{r^3} \ , \tag{5.19}$$
with the remaining $e^{\hat\mu\hat\nu}$'s and $e_{\hat\sigma}(e^{\hat\mu\hat\nu})$'s vanishing, so that Equations (5.18) are satisfied. Under (5.18) one obtains
$$\begin{aligned} g_{\hat a\hat c} &= \eta_{\hat a\hat c} - \eta_{\hat a\hat r}\eta_{\hat c\hat s} e^{\hat r\hat s} + o(r^{-3}) \ , \qquad (5.20)\\ \sqrt{|\det g_{\mu\nu}|} &= \sqrt{|\det b_{\mu\nu}|}\left(1 + \frac{1}{2}(e^{\hat 0\hat 0} - e^{\hat 1\hat 1} - e^{\hat A\hat A}) + o(r^{-3})\right) \ , \\ \mathbb{U}^{\hat 1\hat 0}{}_{\hat 0} &= -\frac{1}{16\pi}\left(2e_{;\hat 1} + e^{\hat 1\hat\imath}{}_{;\hat\imath} - e^{\hat 0\hat 0}{}_{;\hat 1}\right) + o(r^{-3}) \end{aligned}$$

[12]Equation (5.17) will indeed turn out to be correct under the conditions (5.18) imposed below.

$$
\begin{aligned}
&= \frac{1}{16\pi}\left(e_{\hat 1}(e^{\hat A\hat A}) + \frac{1}{\ell}(e^{\hat A\hat A} - 2e^{\hat 1\hat 1}) - \frac{1}{r}\check{\mathcal{D}}_{\hat A}e^{\hat 1\hat A}\right) + o(r^{-3}) \;, \\
\frac{1}{8\pi}\Delta^{\alpha[\hat 1}X^{\hat 0]}{}_{;\alpha} &= \frac{1}{16\pi}\left(\Delta^{\hat 1\hat 1} - \Delta^{\hat 0\hat 0}\right)X^{\hat 0}{}_{;\hat 1} \\
&= \frac{r}{16\pi\ell^2}\left(\Delta^{\hat 1\hat 1} - \Delta^{\hat 0\hat 0}\right) + o(r^{-3}) \\
&= -\frac{r}{16\pi\ell^2}e^{\hat A\hat A} + o(r^{-3}) \;. \qquad (5.21)
\end{aligned}
$$

The indices $\hat\imath$ run from 1 to 3 while the indices $\hat A$ run from 2 to 3; $\check{\mathcal{D}}_{\hat A}$ denotes the covariant derivative on ${}^2M$, and $\check{\mathcal{D}}_{\hat A}e^{\hat 1\hat A}$ is understood to be the covariant derivative associated with the metric $\check h$ of a vector field on ${}^2M$, with repeated $\hat A$ indices being summed over. In Equation (5.20) $\eta_{\hat\mu\hat\nu}$ =diag$(-1,+1,+1,+1)$, while the $g_{\hat\mu\hat\nu}$'s are the components of the tensor $g_{\hat\mu\hat\nu}$ in a co-frame dual to (5.11). Inserting all this into (5.17) one is finally led to the simple expression

$$
\begin{aligned}
M_{Ham} &\equiv H(\frac{\partial}{\partial t},\{t=0\}) \\
&= \lim_{R\to\infty}\frac{r^3}{16\pi\ell^2}\int_{\Sigma\cap\{r=R\}}\left(r\frac{\partial e^{\hat A\hat A}}{\partial r} - 2e^{\hat 1\hat 1}\right)d^2\mu_{\check h} \;. \qquad (5.22)
\end{aligned}
$$

In particular if ${}^4g$ is the generalized Kottler metric (1.1) one obtains (*cf.* Equation (5.19))

$$M_{Ham} = \frac{A_\infty m}{4\pi} \;, \qquad (5.23)$$

A_∞ defined in (5.10). If ${}^2M = T^2$ with area normalized to 4π we obtain $M_{Ham} = m$. For $k = \pm 1$ it follows from the Gauss–Bonnet theorem that $A_\infty = 4\pi|1-g_\infty|$, where g_∞ is the genus of ${}^2M$, hence

$$M_{Ham} = |1-g_\infty|m \;. \qquad (5.24)$$

This gives again $M_{Ham} = m$ for ${}^2M = S^2$, but this will not be true anymore for ${}^2M$'s of higher genus. We believe that the Hamiltonian approach is the one which provides *the* correct definition of mass in field theories, and therefore Equations (5.23)–(5.24) are the ones which provide the correct normalization of mass.

Let us finally consider static metrics ${}^4g$ of the form (3.37), and suppose that the hypotheses of point *2.* of Proposition 3.7 hold. We can then use the coordinates of that proposition to calculate M_{Ham}, and obtain

$$M_{Ham} = -\frac{1}{8\pi}\int_{\partial_\infty\Sigma}\mu_\infty\, d^2\mu_{\check h} \;. \qquad (5.25)$$

If we further assume that μ_∞ is constant on $\partial_\infty\Sigma$, Equation (5.25) gives

$$M_{Ham} = -\frac{\mu_\infty}{2} = M_c$$

for ${}^2M = S^2$ and for an appropriately normalized T^2, while

$$M_{Ham} = -|1-g_\infty|\frac{\mu_\infty}{2} = |1-g_\infty|M_c$$

for higher genus $\partial_\infty\Sigma$'s. Here M_c is the coordinate mass as defined in Section 5.1.

5.3 A generalized Komar mass

Recall that the Komar mass is a number which can be assigned to every stationary, asymptotically flat metric the energy-momentum tensor of which decays sufficiently rapidly:

$$M_K = \lim_{R\to\infty} \frac{1}{8\pi} \int_{S_{R,T}} \sqrt{|\det g_{\alpha\beta}|}\ \nabla^\mu X^\nu \ dS_{\mu\nu}\ , \tag{5.26}$$

where $X^\mu\partial_\mu$ is the Killing vector field which asymptotes to $\partial/\partial t$ in the asymptotically flat region, and the $S_{R,T} \equiv \{t = T, r = R\}$'s are coordinate spheres in that region. The normalization factor $1/(8\pi)$ has been chosen so that M_K reproduces the familiar mass parameter m when Schwarzschild metrics are considered. For metrics considered here with $\Lambda \neq 0$ the integral (5.26) diverges when $X^\mu\partial_\mu = \partial/\partial t$ and when the $S_{R,T}$'s are taken to be coordinate spheres in the region $\Sigma_{\rm ext}$ where the metric exhibits the generalized Kottler asymptotics. An obvious way to generalize M_K to the situation considered in this paper is to remove the divergent part of the integral using a background metric b:

$$M_K = \lim_{R\to\infty} \frac{1}{8\pi} \int_{S_{R,T}} \left(\sqrt{|\det g_{\alpha\beta}|}\ \nabla^\mu X^\nu - \sqrt{|\det b_{\alpha\beta}|}\ \overline{\nabla}^\mu X^\nu\right) dS_{\mu\nu}\ . \tag{5.27}$$

Here $\overline{\nabla}$ denotes a covariant derivative with respect to the background metric. More precisely, let $\Sigma_{\rm ext}$, b, $\check h$, *etc.*, be as in Equation (5.9), and consider time-independent metrics g which in the coordinate system of Equation (5.9) are of the form (3.37) with

$$\begin{aligned}
V^2 &= \tfrac{r^2}{\ell^2} + \tilde k - \tfrac{2\beta}{r} + o(\tfrac{1}{r})\ ,\\
\partial_r(V^2 - \tfrac{r^2}{\ell^2} - \tilde k + \tfrac{2\beta}{r}) &= o(\tfrac{1}{r^2})\ ,\\
g^{rr} &= \tfrac{r^2}{\ell^2} + \tilde k - \tfrac{2\gamma}{r} + o(\tfrac{1}{r})\ ,\\
\sqrt{|\det g_{\alpha\beta}|} &= \left(r^2 + \tfrac{2\delta\ell^2}{r} + o(\tfrac{1}{r})\right)\sqrt{|\det \check h_{AB}|}\ ,
\end{aligned} \tag{5.28}$$

for some r-independent differentiable functions $\tilde k = \tilde k(x^A)$, $\beta = \beta(x^A)$, $\gamma = \gamma(x^A)$ and $\delta = \delta(x^A)$ defined on a coordinate neigbhourhood of $\partial_\infty\Sigma$. (The conditions (5.28) roughly reflect the behavior of the metric in the coordinate system of Proposition 3.7). Under (5.28) the limit as R tends to infinity in the definition (5.27) of M_K exists, and one finds

$$\begin{aligned}
M_K &= \lim_{R\to\infty} \frac{1}{4\pi} \int_{S_{R,T}} \left(\sqrt{|\det g_{\alpha\beta}|}\ g^{r\mu}g^{\nu t}\partial_{[\mu}g_{\nu]t} - \sqrt{|\det b_{\alpha\beta}|}\ b^{r\mu}b^{\nu t}\partial_{[\mu}b_{\nu]t}\right) dx^2dx^3\\
&= \lim_{R\to\infty} \frac{1}{8\pi} \int_{S_{R,T}} \left(\sqrt{|\det g_{\alpha\beta}|}\ g^{rr}g^{tt}\partial_r g_{tt} - \sqrt{|\det b_{\alpha\beta}|}\ b^{rr}b^{tt}\partial_r b_{tt}\right) dx^2dx^3\\
&= \frac{1}{4\pi}\int_{\partial_\infty\Sigma}(3\beta - 2\gamma + 2\delta)\ d^2\mu_{\check h}\ .
\end{aligned} \tag{5.29}$$

It turns out that the value of M_K so obtained depends on the background metric chosen. (Our definition of background, Equation (5.9), is tied to the choice of a particular coordinate system, so another way of stating this is that the number M_K as defined so far is assigned to a metric *and* to a coordinate system, in a manner somewhat similar to the coordinate mass of Section 5.1). Indeed, given any differentiable function $\alpha(x^A)$ there exists a neighborhood of $\partial_\infty\Sigma$ on which a new coordinate $\hat{r}$ can be introduced by the formula

$$\frac{\hat{r}^2}{\ell^2} - 2\frac{\alpha}{\hat{r}} = \frac{r^2}{\ell^2} . \tag{5.30}$$

We can then chose the new background to be $b = -(k+\frac{\hat{r}^2}{\ell^2})dt^2 + (k+\frac{\hat{r}^2}{\ell^2})^{-1}d\hat{r}^2 + \hat{r}^2\check{h}$, and obtain a new M_K which will in general *not* coincide with the old one. (It is noteworthy that the coordinate transformation (5.30) with the associated change of background do *not* change the value of the Hamiltonian mass M_{Ham}.) For example, if α is constant, using hats to denote the corresponding functions appearing in the metric expressed in the new coordinate system we obtain

$$\begin{array}{ccc} \hat{\beta} = \beta + \alpha \,, & \hat{\gamma} = \gamma + 3\alpha \,, & \hat{\delta} = \delta - 2\alpha \\ \Longrightarrow & \hat{M}_K = M_K - \frac{7\alpha A_\infty}{4\pi} \,, & \end{array}$$

where A_∞ is the area of $\partial_\infty\Sigma$ with respect to the metric $\check{h}$. It turns out that one can remove this coordinate dependence in an appropriate class of metrics, tailoring the prescription in such a way that Equation (5.29) reproduces, up to a genus dependent factor, the coordinate mass M_c. In order to do that we shall suppose that the metric ${}^4g$ satisfies the hypotheses of point *2.* of Proposition 3.7 (in particular $\tilde{k} = k = 0, \pm 1$ according to the genus of the connected component of $\partial_\infty\Sigma$ under consideration), and we let the background be associated with a coordinate system (ρ, x^A) with ρ given by (3.47). It follows from Equations (5.3) and (3.49) that in this coordinate system it holds

$$\sqrt{|\det g_{\alpha\beta}|} = r^2 + o(\frac{1}{r}) \,, \tag{5.31}$$

where we have used the generic symbol r to denote the coordinate ρ. We then *impose (5.31) as a restriction on the coordinate system in which the generalized Komar mass M_K has to be calculated.* When this condition is imposed we obtain from (5.3) and (5.25)

$$M_K = -\frac{1}{8\pi}\int_{\partial_\infty\Sigma} \mu_\infty \, d^2\mu_{\check{h}} = M_{Ham} \,.$$

We have thus proved

Proposition 5.1 *Consider a metric ${}^4g$ satisfying the hypotheses of point* 2. *of Proposition 3.7, then the generalized Komar mass (5.27) associated to a background metric (5.9) such that (5.31) holds equals the Hamiltonian mass (5.22).*

Proposition 5.1 is the $\Lambda < 0$ analogue of the theorem of Beig [12], that for static $\Lambda = 0$ vacuum metrics which are asymptotically flat in spacelike directions the ADM mass and the Komar masses coincide. Our treatment here is inspired by, and somewhat related to, the analysis of [58].

5.4 The Hawking mass $M_{Haw}(\psi)$

Let ψ be a function defined on the asymptotic region $\Sigma_{\rm ext}$, with $\Sigma_{\rm ext}$ defined as in (5.1), such that the level sets of ψ are smooth compact surfaces diffeomorphic to each other (at least for ψ large enough), with $\psi \to_{r\to\infty} \infty$. Following Hawking [42], Gibbons [38, Equation (17)] assigns a mass $M_{Haw}(\psi)$ to such a foliation via the formula

$$M_{Haw}(\psi) \equiv \lim_{\epsilon\to 0}\frac{\sqrt{A_{1/\epsilon}}}{32\pi^{3/2}}\int_{\{\psi=1/\epsilon\}}({}^2\mathcal{R}-\frac{1}{2}p^2-\frac{2}{3}\Lambda)dA\ , \qquad (5.32)$$

where A_α is the area of the connected component under consideration of the level set $\{\psi=\alpha\}$

By considering simple examples in Minkowski space-times it can be seen that this definition is ψ dependent. However, when ${}^2M=S^2$, $\Lambda=0$, and the coordinate system on $\Sigma_{\rm ext}$ is such that the ADM mass m_{ADM} (which equals m_H as defined in Section 5.2) of $\Sigma_{\rm ext}$ is well defined (see [10, 24]), then $M_{Haw}(\psi)$ will be independent of ψ, in the class of ψ's singled out by the condition that the level sets of ψ approach round spheres at a suitable rate. No results of this kind are known when $\Lambda\neq 0$.

The definition (5.32) applied to the function $\psi = r$ and the metric (1.1) with $k\neq 0$ gives

$$M_{Haw} = m|1-g_\infty|^{3/2}\ .$$

We have also used the Gauss–Bonnet theorem to calculate $\sqrt{A_{1/\epsilon}}$. Thus the definition (5.32) differs from the coordinate one by the somewhat unnatural factor $|1-g_\infty|^{3/2}$. It is not clear why such a factor should be included in the definition of mass.

Consider, next, the metrics (3.37) with V and g given by (3.47)–(3.48). Let $\psi = V$; from the Codazzi–Mainardi Equation (3.33), the Equation (1.5), and the definition (3.18) of W we obtain, for V large enough so that $|dV|>0$,

$$\begin{aligned}
{}^2\mathcal{R}-\frac{1}{2}p^2-\frac{2}{3}\Lambda &= (-2R_{ij}+Rg_{ij})n^in^j-|q_{ij}|^2_g-\frac{2}{3}\Lambda \\
&= -2\frac{D^iVD^jV}{VW}D_iD_jV-|q_{ij}|^2_g-\frac{2}{3}\Lambda \\
&= -\frac{D^iVD_iW}{VW}-|q_{ij}|^2_g-\frac{2}{3}\Lambda\ .
\end{aligned}$$

In the coordinate system of Equation (3.30), where $V=1/x$, one is led to

$$\begin{aligned}
{}^2\mathcal{R}-\frac{1}{2}p^2-\frac{2}{3}\Lambda &= x^3\frac{\partial W}{\partial x}-\frac{2}{3}\Lambda+O(x^6) \\
&= -\frac{x^3}{6}\frac{\partial R'}{\partial x}+O(x^6)\ ,
\end{aligned}$$

and we have used (3.27) and (3.17). From $A_{1/\epsilon}\approx x^{-2}A'_{\partial_\infty\Sigma}$ we finally obtain

$$M_{Haw}(V) = -\frac{\sqrt{A'_{\partial_\infty\Sigma}}}{32\pi^{3/2}}\int_{\partial_\infty\Sigma}\frac{1}{6}\frac{\partial R'}{\partial x}d^2\mu_{h'}$$

$$= -\frac{\sqrt{A'_{\partial_\infty\Sigma}}}{32\pi^{3/2}}\int_{\partial_\infty\Sigma}\ell\frac{n'(R')}{6}d^2\mu_{h'}\ , \qquad (5.33)$$

where $d^2\mu_{h'}$ is the Riemannian area element induced by g' on $\partial_\infty\Sigma$, and n' denotes the inward-pointing g'-unit normal to $\partial_\infty\Sigma$. We have thus proved the following result:

Theorem 5.2 *Let a triple (Σ, g, V) satisfying Equations (1.3)–(1.5) be C^i compactifiable, $i \geq 3$. Then the Hawking mass $M_{Haw}(V)$ of the V-foliation is finite and well defined; it is given by the formula (5.33), with R' — the curvature scalar of the metric $g' = V^{-2}g$.*

We can relate $M_{Haw}(V)$ to the coordinate mass M_c if we assume in addition that the latter is well defined; recall that this required R' and $\partial_x R'$ to be constant on $\partial_\infty\Sigma$. In this case Equation (5.4) gives

$$M_{Haw}(V) = \left(\frac{A'_{\partial_\infty\Sigma}}{4\pi\ell^2}\right)^{3/2} M_c\ . \qquad (5.34)$$

From Equation (3.22) we have ${}^2\mathcal{R}'|_{x=0} = 2k/\ell^2$, and the Gauss–Bonnet theorem implies

$$\int_{\partial_\infty\Sigma} {}^2\mathcal{R}' d^2\mu_{h'} = \frac{2k}{\ell^2}A'_{\partial_\infty\Sigma} = 8\pi(1-g_\infty)\ ,$$

so that when $g_\infty \neq 1$ we obtain

$$M_{Haw}(V) = |1-g_\infty|^{3/2}M_c\ . \qquad (5.35)$$

We emphasize that $M_{Haw}(V)$ is finite and well defined even when the conditions of Section (5.1), which we have set forth to define M_c, are not met.

Similarly, the Hamiltonian mass M_{Ham}, associated to the background singled out by the coordinate system of Proposition 3.7, can be defined when R' is constant on $\partial_\infty\Sigma$. (This holds regardless of whether or not $\partial_x R'$ is constant on $\partial_\infty\Sigma$.) Proceeding as above, making use of Equations (3.46)–(3.51), one is led to

$$\begin{aligned} g_\infty \neq 1 \quad &\Longrightarrow \quad M_{Haw}(V) = |1-g_\infty|^{1/2}M_{Ham}\ , \\ g_\infty = 1\ ,\ A'_\infty = 4\pi\ell^2 \quad &\Longrightarrow \quad M_{Haw}(V) = M_{Ham}\ . \end{aligned} \qquad (5.36)$$

6 The generalized Penrose inequality

We recall here an argument of Geroch [37], as extended by Jang and Wald [49] and Gibbons [38], for the validity of the Penrose inequality[13]:

Proposition 6.1 *Assume we are given a three dimensional manifold (Σ, g) with connected boundary $\partial\Sigma$ such that:*

[13] The argument we review has been used by Gibbons in [38] to obtain a somewhat different inequality, in which the genus factors are not present. The inequality in [38] is violated for generalized Kottler metrics with $g_\infty \geq 3$.

1. *$R \geq 2\Theta$ for some strictly negative constant Θ.*

2. *There exists a smooth, global solution of the inverse mean curvature flow without critical points,* i.e., *there exists a surjective function $u : \Sigma \to [0, \infty)$ such that du has no zeros and*

$$\begin{cases} u|_{\partial\Sigma} = 0 \ , \\ \nabla_i \left(\frac{\nabla^i u}{|du|} \right) = |du| \ . \end{cases} \tag{6.1}$$

3. *The level sets of u*

$$N_s = \{u(x) = s\}$$

   are compact.

4. *The boundary $\partial\Sigma = u^{-1}(0)$ of Σ is minimal.*

5. *The Hawking mass of the level sets of u as defined in (5.32) exists.*

Then

$$2M_{Haw}(u) \geq (1 - g_{\partial\Sigma}) \left(\frac{A_{\partial\Sigma}}{4\pi} \right)^{1/2} - \frac{\Theta}{3} \left(\frac{A_{\partial\Sigma}}{4\pi} \right)^{3/2} \ . \tag{6.2}$$

Here $A_{\partial\Sigma}$ is the area of $\partial\Sigma$ and $g_{\partial\Sigma}$ is the genus thereof.

REMARKS: 1. The Proposition above can be applied to solutions of (1.4) and (1.5) with $\Theta = \Lambda$: in this case we have $R = 2\Lambda$; further Equation (1.5) multiplied by V and contracted with two vectors tangent to $\partial\Sigma$ shows that the boundary $\{V = 0\}$ is totally geodesic and hence minimal.

2. Equation (6.2) is sharp — the inequality there becomes an equality for the generalized Kottler metrics.

PROOF: Let A_s denote the area of N_s, and define

$$\sigma(s) = \sqrt{A_s} \int_{N_s} ({}^2\mathcal{R}_s - \frac{1}{2} p_s^2 - \frac{2}{3}\Theta) d^2\mu_s \ , \tag{6.3}$$

where ${}^2\mathcal{R}_s$ is the scalar curvature of the metric induced on N_s, $d^2\mu_s$ is the Riemannian volume element associated to that same metric, and p_s is the mean curvature of N_s. The hypothesis that du is nowhere vanishing implies that all the objects involved are smooth in s. At $s = 0$ we have

$$\begin{aligned} \sigma(0) &= \sqrt{A_{\partial\Sigma}} \int_{\partial\Sigma} ({}^2\mathcal{R}_0 - \frac{2}{3}\Theta) d^2\mu_0 \\ &= \sqrt{A_{\partial\Sigma}} \left(8\pi(1 - g_{\partial\Sigma}) - \frac{2}{3}\Theta A_{\partial\Sigma} \right) \ . \end{aligned} \tag{6.4}$$

On the other hand,

$$\lim_{s\to\infty} \sigma(s) = 32\pi^{3/2} M_{Haw}(u) \ .$$

The generalization in [38] of the classical calculation of [37] gives

$$\frac{\partial\sigma}{\partial s} \geq 0 \ . \tag{6.5}$$

This implies $\lim_{s\to\infty}\sigma(s) \geq \sigma(0)$, which gives (6.2). $\square$

To be able to carry out the above argument one had to assume that du had no zeros, which implies in particular that $\partial_\infty\Sigma$ is connected with $g_{\partial\Sigma} = g_\infty$. It is not known whether or not the other hypotheses of Proposition 6.1, or the conditions of Definition 3.1 together with Equations (1.3)–(1.5), force $\partial\Sigma$ to be connected. If they do not, one is tempted to conjecture that the right inequality should be

$$2M_{Haw}(u) \geq \sum_{i=1}^{k}\left((1-g_{\partial_i\Sigma})\left(\frac{A_{\partial_i\Sigma}}{4\pi}\right)^{1/2} - \frac{\Theta}{3}\left(\frac{A_{\partial_i\Sigma}}{4\pi}\right)^{3/2}\right) . \tag{6.6}$$

Here the $\partial_i\Sigma$'s, $i = 1,\ldots,k$, are the connected components of $\partial\Sigma$, $A_{\partial_i\Sigma}$ is the area of $\partial_i\Sigma$, and $g_{\partial_i\Sigma}$ is the genus thereof. This would be the inequality one would obtain from the Geroch–Gibbons argument if it could be carried through for u's which are allowed to have critical points, on manifolds with $\partial_\infty\Sigma$ connected but $\partial\Sigma$ — not connected.

We note that when $\Lambda = 0$ there is a version of the proof of Proposition 6.1 due to Huisken and Ilmanen in which du is allowed to have zeros (with $\partial\Sigma$ — connected)[14]. Note that at points where du vanishes Equation (6.1) does not make sense classically, and has to be understood in a proper way. Further the monotonicity calculation of [37] breaks down at critical level sets of u, as those do not have to be smooth submanifolds. Nevertheless (when $\Lambda = 0$) existence of appropriate functions u (perhaps with critical points) together with the monotonicity of σ can be established [46, 47] when $\partial\Sigma$ is an outermost (necessarily connected) minimal sphere. It is conceivable that the argument of Huisken and Ilmanen can be modified to include the case $\Lambda \neq 0$. One of the difficulties here is to handle the possibly changing genus of the level sets of u.

Let us discuss some of the consequences of the (hypothetical) Equation (6.6). To proceed further it is convenient to introduce a mass parameter m defined as follows:

$$m = \begin{cases} M_{Haw}\ , & \partial_\infty\Sigma = S^2\ , \\ M_{Haw}\ , & \partial_\infty\Sigma = T^2, \text{ with the normalization } A'_\infty = 4\pi\ell^2\ , \\ \dfrac{M_{Haw}}{|g_{\partial_\infty\Sigma}-1|^{3/2}}\ , & g_{\partial_\infty\Sigma} > 1\ . \end{cases} \tag{6.7}$$

Strictly speaking, we should write $m(u)$ if $M_{Haw}(u)$ is used above, $m(V)$ if $M_{Haw}(V)$ is used, *etc.*; we shall do this when confusions are likely to occur. For generalized Kottler metrics the mass $m = m(u)$ so defined coincides with the mass parameter appearing in (1.1) when u is the radial solution $u = u(r)$ of the problem (6.1); $m(V)$ coincides with the coordinate mass M_c for the metrics considered here when M_c is defined, *cf.* Equation (5.34).

Note, first, that if all connected components of the horizon have spherical or toroidal topology, then the lower bound (6.6) is strictly positive. For example,

[14]Bray's proof [17] of the inequality (6.6) with $\Theta = 0$ but $\partial\Sigma$ — not necessarily connected, uses a completely different technique; in particular it makes appeal to the positive energy theorem which does not hold in the class of manifolds considered here.

if $\partial\Sigma = T^2$, and $\partial_\infty\Sigma = T^2$ as well we obtain

$$2m \geq \frac{1}{\ell^2}\left(\frac{A_{\partial\Sigma}}{4\pi}\right)^{3/2} .$$

On the other hand if $\partial\Sigma = T^2$ but $g_{\partial_\infty\Sigma} > 1$ from Equation (6.6) one obtains

$$2m \geq \frac{1}{\ell^2|g_\infty - 1|}\left(\frac{A_{\partial\Sigma}}{4\pi}\right)^{3/2} .$$

Let us return to the case[15] where Equations (1.3)–(1.5) hold; we can then use the Galloway–Schleich–Witt–Woolgar inequality [33]

$$\sum_{i=1}^{k} g_{\partial_i\Sigma} \leq g_\infty . \tag{6.8}$$

It implies that if $\partial_\infty\Sigma$ has spherical topology, then all connected components of the horizon must be spheres. Similarly, if $\partial_\infty\Sigma$ is a torus, then all components of the horizon are spheres, except perhaps for at most one which could be a torus. It follows that to have a component of the horizon which has genus higher than one we need $g_\infty > 1$ as well.

When some — or all — connected components of the horizon have genus higher than one the right hand side of Equation (6.6) might become negative. Minimizing the generalized Penrose inequality (6.6) with respect to the areas of the horizons gives the following interesting inequality

$$M_{Haw}(u) \geq -\frac{1}{3\sqrt{-\Lambda}}\sum_i |g_{\partial_i\Sigma} - 1|^{3/2} , \tag{6.9}$$

where the sum is over those connected components $\partial_i\Sigma$ of $\partial\Sigma$ for which $g_{\partial_i\Sigma} \geq 1$. Equation (6.9), together with the elementary inequality $\sum_{i=1}^N |\lambda_i|^{3/2} \leq \left(\sum_{i=1}^N |\lambda_i|\right)^{3/2}$, lead to

$$m \geq -\frac{1}{3\sqrt{-\Lambda}} . \tag{6.10}$$

The Geroch–Gibbons argument establishing the inequality (6.4) when a suitable u exists can also be *formally* carried through when $\partial\Sigma = \emptyset$. In this case one still considers solutions u of the differential equation that appears in Equation (6.1), however the Dirichlet condition on u at $\partial\Sigma$ is replaced by a condition on the behavior of u near some chosen point $p_0 \in \Sigma$. If the level set of u around p_0 approach distance spheres centered at p_0 at a suitable rate, then $\sigma(s)$ tends to zero when the N_s's shrink to p_0, which together with the monotonicity of σ leads to the positive energy inequality:

$$M_{Haw}(u) \geq 0 . \tag{6.11}$$

[15]The discussion that follows actually applies to all (Σ, g)'s that can be isometrically embedded into a globally hyperbolic space-time M in which the null convergence condition holds; further the image of Σ should be a partial Cauchy surface in M. Finally the intersection of Σ with $\mathscr{I}$ should be compact. The global hyperbolicity here, and the notion of Cauchy surfaces, is understood in the sense of manifolds with boundary, see [33] for details.

It should be emphasized that the Horowitz-Myers solutions [45] with negative mass show that this argument breaks down when $g_\infty = 1$.

When $\partial_\infty\Sigma = S^2$ one expects that (6.11), with $M_{Haw}(u)$ replaced by the spinorially defined mass (which might perhaps coincide with $M_{Haw}(u)$, but this remains to be established), can be proved by Witten type techniques, compare [6, 39]. On the other hand it follows from [11] that when $\partial_\infty\Sigma \neq S^2$ there exist no asymptotically covariantly constant spinors which can be used in the Witten argument. The Geroch–Gibbons argument has a lot of "ifs" attached in this context, in particular if $\partial_\infty\Sigma \neq S^2$ then some level sets of u are necessarily critical and it is not clear what happens with σ when a jump of topology from a sphere to a higher genus surface occurs. We note that the area of the horizons does not occur in (6.10) which, when $g_{\partial_\infty\Sigma} > 1$, suggests that the correct inequality is actually (6.10) rather than (6.11).

7 Mass and area inequalities

7.1 Preliminaries

We first give here a sketch of the content of this section. We define W_0 via a suitably chosen generalized Kottler solution, (called the "reference solution", RS) and $\widetilde{W} = \Psi^{-4}W$ and $\widetilde{W}_0 = \Psi^{-4}W_0$ for a certain function $\Psi(V)$. We then establish three lemmas. The first one (Lemma 7.1) expresses the surface integral at infinity of the normal derivative $n^iD_i(\widetilde{W} - \widetilde{W}_0)$ in terms of the mass difference between the given solution and a RS, while Lemma 7.2 expresses this same normal derivative taken on the horizon in terms of the difference of the areas of the given and the RS, with appropriate genus factors. We next recall from [14], an elliptic equation of the form $(\triangle - a)(\widetilde{W} - \widetilde{W}_0) \geq 0$, for some function a. This equation is first employed in Lemma 7.3 where we show that the generalized Kottler solutions can be characterized either by the condition $\widetilde{W} = \widetilde{W}_0$ or by conformal flatness of (Σ, g) (this Lemma is actually formulated more genrerally such as to include the Nariai case as well). The crucial step in the proofs then consists of applying the maximum principle to the elliptic equation for $\widetilde{W} - \widetilde{W}_0$. This is possible if the function a is non-negative, which is the case in the present situation ($\Lambda < 0$) iff the mass of the reference Kottler solution is non-positive. By the asymptotic conditions (and by a suitable choice of the RS) we can achieve that $\widetilde{W} - \widetilde{W}_0$ takes its maximum value (namely zero) both at the horizon (if there is one) and at infinity. The maximum principle then yields that the derivatives $n^iD_i(\widetilde{W} - \widetilde{W}_0)$ with respect to the outward normals at the horizon and at infinity are positive, and zero precisely if $\widetilde{W} = \widetilde{W}_0$. Theorems 1.3 and 1.5 then readily follow from the lemmata. As a final step we combine the mass and area inequalities to derive the inverse Penrose inequality.

Turning, then, to the details, let (Σ, g, V) satisfy (1.3)–(1.5) together with the topological, the differential, and the asymptotic requirements spelled out in the statements of Theorems 1.3 or 1.5. (As mentioned above, Lemma 7.3 holds under more general conditions). We first introduce the surface gravity κ of $\partial\Sigma$ to be the corresponding restriction of the function $\sqrt{W}$ defined by (3.18) to $\partial\Sigma$:

$$\kappa \equiv |dV|_g \Big|_{\partial\Sigma} . \tag{7.1}$$

where we have normalized V so that Equation (3.23) holds, *cf.* Proposition 3.3. By the strong maximum principle [40, Lemma 3.4] W is nowhere vanishing on $\partial\Sigma$. Moreover, it is well known (and easily seen using Equation (1.5)) that κ is locally constant on $\partial\Sigma$:

$$\begin{aligned} 0 = n^j D_i D_j V \Big|_{V=0} &= \frac{D^j V}{\sqrt{W}} D_i D_j V \Big|_{V=0} \\ &= \frac{1}{2\sqrt{W}} D_i W \Big|_{V=0} . \end{aligned} \tag{7.2}$$

Here n^i is the unit normal to $\partial\Sigma$, where V vanishes.

It is convenient to introduce the notion of a *reference solution* (RS) as follows: This is a generalized Kottler solution with the same genus g_∞ as (Σ, g, V). Moreover, if $\partial\Sigma \neq \emptyset$, the surface gravity κ of the RS is chosen to be equal to the maximum of the surface gravities of (Σ, g, V). On the other hand, if $\partial\Sigma = \emptyset$, the mass of the RS will be defined suitably below, in the proof of theorem 1.3. We only consider RS with mass m_0 in the range (2.7) (if $\partial\Sigma \neq \emptyset$, this property follows from the restriction (1.7) on κ). It should be stressed that we are *not* comparing manifolds and/or metrics, but we are only using the resulting scalar functions V and W.

Let $r(\cdot)$ be the function $V_0 \to r(V_0)$ constructed at the end of Section 2. Composing r with V we obtain functions $r(V(\cdot))$ and $W_0(r(V(\cdot))$ defined on Σ. By an abuse of notation we shall still denote those functions by r and W_0.

In the same manner, we can define a RS from other solutions with the property that W is a function of V only. (In Lemma 7.3 below we will also include the Nariai case).

Following [14] we define $\psi(V)$ to be that unique solution of the equation

$$\psi^{-1} \frac{d\psi}{dV} = -V W_0^{-1} \frac{m_0}{r^3} \tag{7.3}$$

which goes[16] to 1 as V goes to ∞. (In particular $\psi \equiv 1$ when $m_0 = 0$.) Here $r = r(V)$ is again the function defined at the end of Section 2. Standard results on ODE's show that solutions of (7.3) have no zeros unless identically vanishing, and that

$$\Psi \equiv \psi \circ V$$

can be extended by continuity to a smooth function on $\overline{\Sigma}$, still denoted by Ψ, which satisfies

$$\Psi > 0 , \quad \Psi|_{\partial_\infty \Sigma} = 1 .$$

[16] Using the asymptotic behavior of $V(r)$ and $r(V)$ it is not too difficult to show that solutions of (7.3) are uniformly bounded on $[0, \infty)$, and approach a non-zero constant at infinity unless identically vanishing. Since solutions of (7.3) are defined up to a multiplicative constant, we can choose this constant so that our normalization holds.

We also define

$$\begin{aligned}\widetilde{g}_{ij} &= V^{-2}\Psi^4 g_{ij}\ ,\\ \widetilde{W} &= \Psi^{-4}W\ ,\\ \widetilde{W}_0 &= \Psi^{-4}W_0\ .\end{aligned}\tag{7.4}$$

We proceed with a computation which is required in Lemma 7.1 as well as in Lemma 7.2. Consider a level set $\{V = \text{const}\}$ of V which is a smooth hypersurface in $\overline{\Sigma}$, with unit normal n_i, induced metric h_{ij}, scalar curvature ${}^2\mathcal{R}$, second fundamental form p_{ij} defined with respect to an inner pointing normal, mean curvature $p = h^{ij}p_{ij}$; we denote by q_{ij} the trace-free part of p_{ij}: $q_{ij} = p_{ij} - \frac{1}{2}h_{ij}p$. Using Equation (2.5), the Equation (1.4) with $g = g_0$ and $V = V_0$, together with the relation

$$\frac{dV_0}{dr} = \frac{\sqrt{W_0}}{V_0}\tag{7.5}$$

we obtain

$$V^{-1}\frac{dW_0}{dV} = -\frac{2}{3}\Lambda - \frac{4m_0}{r^3}\ .\tag{7.6}$$

To obtain (7.7) we use, in this order, the definitions (7.4), the Equations (1.4)–(1.5), Equations (7.6) and (7.3), and the Codazzi-Mainardi equation:

$$\begin{aligned}&V^{-1}\widetilde{W}^{-1}D^iVD_i(\widetilde{W}-\widetilde{W}_0)\\ &\quad= V^{-1}W^{-1}D^iV(D_iW) - V^{-1}\frac{dW_0}{dV} - 4V^{-1}\Psi^{-1}\frac{d\Psi}{dV}(W-W_0)\\ &\quad= (2R_{ij}-Rg_{ij})n^in^j + \frac{2}{3}\Lambda + \frac{4m_0}{r^3} - \frac{4m_0}{r^3}(1-W_0^{-1}W)\\ &\quad= -{}^2\mathcal{R} - q_{ij}q^{ij} + \frac{1}{2}p^2 + \frac{2}{3}\Lambda + \frac{4m_0}{r^3} - \frac{4m_0}{r^3}(1-W_0^{-1}W)\ .\end{aligned}\tag{7.7}$$

Lemma 7.1 *Under the conditions of Theorem 1.1, suppose further that the scalar curvature R' of the metric $g' = V^{-2}g$ is constant on $\partial_\infty\Sigma$. Let V be normalized so that (3.23) holds, with $A'_\infty = 4\pi\ell^2$ when $\partial_\infty\Sigma = T^2$. If m is the Hawking mass parameter defined as in (6.7), then*

$$\int_{\partial_\infty\Sigma} D'_i(\widetilde{W}-\widetilde{W}_0)dS'^i = -\left(\frac{2\Lambda}{3}\right)^2 A'_{\partial_\infty\Sigma}(m-m_0)\ ,\tag{7.8}$$

where dS'^i denotes the outer-oriented area element of the metric $g' = V^{-2}g$, and $A'_{\partial_\infty\Sigma}$ is the area of $\partial_\infty\Sigma$ with respect to that metric.

Proof: Using

$$D'^i(\widetilde{W}-\widetilde{W}_0)n'_i = \frac{1}{\sqrt{W'}}D_i(\widetilde{W}-\widetilde{W}_0)D^iV\tag{7.9}$$

and (7.7), the left hand side of (7.8) reads

$$\int_{\partial_\infty\Sigma}\frac{V\widetilde{W}}{\sqrt{W'}}\left[-{}^2\mathcal{R} - q_{ij}q^{ij} + \frac{1}{2}p^2 + \frac{2}{3}\Lambda + \frac{4m_0}{r^3} - \frac{4m_0}{r^3}(1-W_0^{-1}W)\right]d^2\mu_{g'}\tag{7.10}$$

where $d^2\mu_{g'}$ is the two-dimensional surface measure associated with the metric g'. Chasing through the definitions one finds that

$$\frac{V\widetilde{W}}{\sqrt{W'}} \approx \sqrt{-\frac{\Lambda}{3}}V^3 \tag{7.11}$$

near $\partial_\infty\Sigma$. From the definition of V_0 we further have

$$r \approx \sqrt{-\frac{3}{\Lambda}}V \ ,$$

again near $\partial_\infty\Sigma$, so that $\lim_{V\to\infty} V\widetilde{W}/(\sqrt{W'}r^3) = (-\Lambda/3)^2$. It follows that the second to last term in (7.10) gives a contribution

$$\left(\frac{2\Lambda}{3}\right)^2 A'_{\partial_\infty\Sigma}m_0 \tag{7.12}$$

where $A'_{\partial_\infty\Sigma}$ denotes the g'-area of the connected component of $\partial_\infty\Sigma$ under consideration. Equation (3.17) and its equivalent with W replaced by W_0 show that

$$(1 - W_0^{-1}W) \to_{V\to\infty} 0$$

so that the last term drops out from (7.10). Furthermore, by Equation (3.27) we have

$$\frac{V\widetilde{W}}{\sqrt{W'}}q_{ij}q^{ij} = O(V^{-3}) \to_{V\to\infty} 0 \ ,$$

and it remains to analyze the contribution of $-V\widetilde{W}\left({}^2\mathcal{R} - \frac{1}{2}p^2 - \frac{2}{3}\Lambda\right)/\sqrt{W'}$ to the integral (7.8). To do this, note that

$$\begin{aligned} A_{1/\epsilon} \equiv A(\{V = 1/\epsilon\}) &= \int_{V'=\epsilon} d^2\mu_g \\ &= \int_{V'=\epsilon} V^2 d^2\mu_{g'} \ \approx \ \epsilon^{-2}A'_{\partial_\infty\Sigma} \ , \end{aligned}$$

where $d^2\mu_g$ is the induced measure on $\partial_\infty\Sigma$ associated with the metric g. It follows that

$$\begin{aligned} -\int_{V'=\epsilon} & \frac{V\widetilde{W}}{\sqrt{W'}}\left({}^2\mathcal{R} - \frac{1}{2}p^2 - \frac{2}{3}\Lambda\right) d^2\mu_{g'} \\ \approx \quad & -\sqrt{-\frac{\Lambda}{3}}\frac{1}{\epsilon}\int_{V'=\epsilon}\left({}^2\mathcal{R} - \frac{1}{2}p^2 - \frac{2}{3}\Lambda\right) d^2\mu_g \\ \approx \quad & -\sqrt{-\frac{\Lambda}{3}}\sqrt{\frac{A_{1/\epsilon}}{A'_{\partial_\infty\Sigma}}}\int_{V'=\epsilon}\left({}^2\mathcal{R} - \frac{1}{2}p^2 - \frac{2}{3}\Lambda\right) d^2\mu_g \\ \to_{\epsilon\to 0} \quad & -\left(\frac{2\Lambda}{3}\right)^2 A'_{\partial_\infty\Sigma}m \ , \end{aligned} \tag{7.13}$$

where

$$m \equiv \lim_{\epsilon\to 0}\frac{1}{4}\left(-\frac{\Lambda A'_{\partial_\infty\Sigma}}{3}\right)^{-3/2}\sqrt{A_{1/\epsilon}}\int_{\{V=1/\epsilon\}}({}^2\mathcal{R} - \frac{1}{2}p^2 - \frac{2}{3}\Lambda)dA \ . \tag{7.14}$$

To finish the proof we need to show that m in (7.14) is indeed the Hawking mass as defined in Equation (6.7). In the torus case this follows immediately from the normalization condition $A'_\infty = 4\pi\ell^2$; for the remaining topologies this can be seen as follows: if V is normalized so that (3.23) holds, then (3.22) implies

$$ {}^2\mathcal{R}'\Big|_{x=0} = -\frac{2}{3}\Lambda k \; . $$

When $g_\infty \neq 1$ the Gauss–Bonnet theorem gives

$$ \begin{aligned} 8\pi|1-g_\infty| &= \left|\int {}^2\mathcal{R}' d^2\mu_{g'}\right| \\ &= -\frac{2}{3}\Lambda A'_{\partial_\infty\Sigma} \; , \end{aligned} $$

which shows that the mass defined by Equation (7.14) coincides with that of (6.7). □

As to the subsequent Lemma, recall from Theorem 1.5 that $\partial_1\Sigma$ refers to the the component of $\partial\Sigma$ of the given solution with the largest surface gravity.

Lemma 7.2 *Under the conditions of Theorem 1.1, we have*

$$ \int_{\partial_1\Sigma} \widetilde{W}^{-1/2}\widetilde{D}_i(\widetilde{W}-\widetilde{W}_0)d\widetilde{S}^i = 8\pi\left[(g_{\partial_1\Sigma}-1) - \frac{A_{\partial_1\Sigma}}{A_0}(g_\infty - 1)\right] \tag{7.15} $$

Proof: We integrate (7.7) over $\partial_1\Sigma$. We note that Equation (1.5) multiplied by V and contracted with two vectors tangent to $\partial\Sigma$ shows that $\partial\Sigma$ is totally geodesic; equivalently, $q_{ij}=0$. We introduce

$$ {}^2\mathcal{R}_0 = \frac{2}{3}\Lambda + \frac{4m_0}{r_0^3} \; , $$

the scalar curvature of the metric $d\Omega_k^2$. Using (7.7) and the Gauss–Bonnet theorem, the left hand side of (7.15) can be written as

$$ \int_{\partial_1\Sigma}(-{}^2\mathcal{R} + \frac{2}{3}\Lambda + \frac{4m_0}{r_0^3})dA = \int_{\partial_1\Sigma}(-{}^2\mathcal{R} + {}^2\mathcal{R}_0)dA = 8\pi\,(g_{\partial_1\Sigma}-1) + {}^2\mathcal{R}_0 A_{\partial_1\Sigma} \tag{7.16} $$

Equation (7.15) is then obtained by eliminating ${}^2\mathcal{R}_0$ from (7.16), using the Gauss–Bonnet theorem for the generalized Kottler metrics:

$$ 8\pi(1-g_\infty) = {}^2\mathcal{R}_0 A_0 \; . $$

□

The following elliptic equation for $\widetilde{W}-\widetilde{W}_0$ will be the crucial ingredient in the proof of the theorems. It is also useful for Lemma 7.3.

$$ \begin{aligned} &(\widetilde{\Delta} - a)(\widetilde{W}-\widetilde{W}_0) = \\ &\quad = \frac{1}{4}\widetilde{W}^{-1}\widetilde{R}_{ijk}\widetilde{R}^{ijk} + \frac{3}{4}\widetilde{W}^{-1}\widetilde{D}_i(\widetilde{W}-\widetilde{W}_0)\widetilde{D}^i(\widetilde{W}-\widetilde{W}_0) \; , \end{aligned} \tag{7.17} $$

with

$$a = \frac{5}{3r^3} m_0 \Lambda V^4 W_0^{-2}\, \widetilde{W}\ , \tag{7.18}$$

$\widetilde{\Delta}$ being the Laplace operator of the metric $\widetilde{g}_{ij}$, and $\widetilde{R}_{ijk}$ — the Cotton tensor of $\widetilde{g}_{ij}$. This equation is obtained by specializing[17] Equation (5.4) of [14] (which has been used in that paper in the context of a uniqueness proof for static perfect fluid solutions) to the present case with $8\pi\rho = -8\pi p = \Lambda$.

It is important to stress that Equation (7.17), as it stands, makes only sense on the set $\{dV \neq 0\}$, because of the factors $\widetilde{W}^{-1}$ appearing there. However, it follows from Equation (1.4) that the set $\{dV = 0\}$ has no interior: indeed, if dV vanishes on a connected open set then V is constant there, which is compatible with Equation (1.5) only if V vanishes there. This contradicts our hypothesis that V vanishes only[18] on $\partial\Sigma$. Hence Equation (7.17) holds on an open dense set of Σ. Since the left hand side of Equation (7.17) is a smooth function on $\Sigma \setminus \partial\Sigma$, the right hand side thereof is smoothly extendible by continuity to a smooth function on $\Sigma \setminus \partial\Sigma$, and Equation (7.17) holds everywhere on this set with the right hand side being understood in the sense explained here.

Lemma 7.3 *Let $\Lambda \in \mathbb{R}$, and let (Σ, g, V) be a solution of (1.3)–(1.5) such that*

a. *either $W \equiv W_0$ for W_0 defined from the generalized Kottler or from the Nariai solution (1.2), or*

b. *(Σ, g) is locally conformally flat.*

Suppose further that Σ is a union of compact boundary-less level sets of V. Then:

1. *Every connected component $\mathscr{V}$ of the set $\{p \in \Sigma \mid dV(p) \neq 0\}$ "corresponds to" one of the generalized Kottler solutions (1.1), or to one of the generalized Nariai solutions (1.2), or is flat. More precisely, there exists an interval $J \subset \mathbb{R}$, a two-dimensional compact Riemannian manifold $({}^2M, d\Omega^2_k)$, with $d\Omega^2_k$ an (r-independent) metric of constant Gauss curvature $k = 0, \pm 1$, and a diffeomorphism $\psi : \mathscr{V} \to J \times {}^2M$ such that, transporting g and V to $J \times {}^2M$ using ψ, we have:*

   (i) Either there exists a constant $\lambda > 0$ such that $V = \lambda V_0$ and

   $$g = V_0^{-2} dr^2 + r^2 d\Omega^2_k\ ,\ r \in J\ , \tag{7.19}$$
   $$V_0^2 = k - \tfrac{2m}{r} - \tfrac{\Lambda}{3} r^2\ , \tag{7.20}$$

   (ii) or, when $k\Lambda > 0$, there exists a constant $\lambda \in \mathbb{R}$ ($\lambda > 0$ if $\Lambda > 0$) such that

   $$g = V^{-2} dz^2 + |\Lambda|^{-1} d\Omega^2_k\ ,\ z \in J\ , \tag{7.21}$$
   $$V^2 = \lambda - \Lambda z^2\ , \tag{7.22}$$

[17] The assumption of spherical symmetry of the level sets of the reference solution made in [14] is not needed to obtain Equation (7.18).

[18] Let us mention that if V is zero on an open set, then the Aronszajn unique continuation theorem [7] shows in any case that V must be identically zero on Σ.

(iii) or, when $k = \Lambda = 0$, there exists a constant $\lambda > 0$ such that $V = \lambda z$ and

$$g = dz^2 + d\Omega_k^2 \ , \ z \in J \ . \qquad (7.23)$$

(In each case the interval J is constrained by the condition that V and V^2 be non-negative).

2. *Under condition a. above, if Σ is connected and if W_0 (considered as a function of V) has no zeros in the interval where V takes its values,*

$$\forall\, p \in \Sigma \qquad W_0(V(p)) \neq 0 \ , \qquad (7.24)$$

then $\mathscr{V} = \Sigma$, thus Equations (7.21) or (7.19) hold globally on Σ.

REMARKS: 1. Here we do not make any hypotheses on the sign of Λ.

2. The result is local, in particular it is sufficient to be able to invert $r_0(V_0)$ locally on the range of the values of V under consideration to obtain $W_0(V)$.

3. The set (Σ, g, V) corresponding to the metric (7.23) arises from a boost Killing vector in suitably identified Minkowski space-time.

4. We note that the set $\mathscr{V}$ could be empty, as is the case for $\mathbb{R} \times T^3$ with the obvious flat metric. Our analysis does not say anything about the metric on regions where dV vanishes.

5. We note that the generalized Kottler and the generalized Nariai metrics also arise naturally in the generalized Birkhoff theorem, see [30, 41], and also [65] for a very clear treatment in the $\Lambda > 0$ case.

6. The Lemma can easily be reformulated by taking any conformally flat solution of (1.4) and (1.5) as a reference solution. The condition of conformal flatness is required to ensure that (7.17) holds and excludes, in particular, the Horowitz-Myers solutions with toroidal $\mathscr{I}^+$ [45] as RS.

PROOF: The proof is an adaptation of an argument of [27] to the current setting. Suppose that $W = W_0$ for some W_0; Equation (7.17) shows then that $\widetilde{R}_{ijk}\widetilde{R}^{ijk}$ vanishes, so that (Σ, g) is locally conformally flat. It then follows that condition $b.$ holds in both cases.

We start by removing from Σ some undesirable points: set

$$\begin{aligned} \Sigma_{\rm sing} &\equiv \{p \in \Sigma |\ \text{the connected component of the set } \{q | V(q) = V(p)\} \\ &\qquad \text{containing } p \text{ contains a point } r \text{ such that } dV(r) = 0.\ \} \ , \\ \Sigma' &\equiv \Sigma \setminus \Sigma_{\rm sing} \ . \end{aligned}$$

$\Sigma_{\rm sing}$ is a closed subset of Σ, so that Σ' is still a manifold. It follows from Sard's theorem that $\Sigma' \neq \emptyset$. We note that Σ' still satisfies all the hypotheses of the Lemma, except perhaps for being connected. By construction all the level sets of V are non-critical in Σ'. (Recall that a level set $\{V = c\}$ of V is non-critical if dV is nowhere vanishing on $\{V = c\}$.)

Let $\mathscr{U}$ to be any connected component of Σ'. Compactness of the level sets of V implies[19] that $\mathscr{U}$ is diffeomorphic to $I \times {}^2M$, for some two-dimensional

[19]The possibility that $\mathscr{U}$ is diffeomorphic to $S^1 \times {}^2M$ (or some twisted version thereof) is excluded by the fact that dV does not vanish on $\mathscr{U}$.

compact connected manifold ${}^2M$ and some interval $I \subset \mathbb{R}$, with V equal to c on $\{c\} \times {}^2M$, $c \in I$, and that on $\mathscr{U}$ the function V can be used as a coordinate. Further we can introduce on ${}^2M$ a finite number of coordinate patches with coordinates x^A, $A = 1, 2$, so that on $\mathscr{U}$ the metric takes the form

$$g = W^{-1}dV^2 + h_{AB}dx^A dx^B \ . \tag{7.25}$$

Let, as before, $q_{AB}dx^A dx^B$ be the trace free part of the extrinsic curvature tensor of the level sets of V — in the coordinate system of (7.25)

$$q_{AB} = \sqrt{W}\Big(\frac{\partial h_{AB}}{\partial V} - \frac{1}{2}h^{CD}\frac{\partial h_{CD}}{\partial V}h_{AB}\Big) \ . \tag{7.26}$$

Equations (7.26) and (3.25) imply that q_{AB} vanishes hence $\frac{\partial h_{AB}}{\partial V}$ is pure trace, that $W = W(V)$, and that $\det \gamma_{AB}$ is a product of a function of V with a function of the remaining coordinates. We thus have

$$h = W(V)^{-1}dV^2 + r(V)^2 d\Omega^2 \ . \tag{7.27}$$

for some positive function $r(V)$, where $d\Omega^2$ is a V-independent metric on ${}^2M$. Next, from (1.5) and from the Codazzi-Mainardi equations (3.35), respectively (3.33), applied to ${}^2M \subset \mathscr{U}$, we find that the mean curvature p of all level surfaces, respectively their Ricci scalars, are constant. Hence $({}^2M, d\Omega^2)$ is a space of constant curvature, and scaling r appropriately we can without loss of generality assume that the Gauss curvature k of the metric $d\Omega^2$ equals $0, \pm 1$, as appropriate to the genus of ${}^2M$. We define

$$L = \frac{dW}{dV} + 2\Lambda V \ . \tag{7.28}$$

Evaluating (1.4) for the metric (7.27), we find

$$\frac{dr}{dV} = -\frac{rL}{4W} \ . \tag{7.29}$$

Equations (1.4)–(1.5) for the metric (7.27) are equivalent to (7.28)–(7.29) together with

$$2W\left(\Lambda - \tfrac{k}{r^2}\right) = L\left(V^{-1}W - \tfrac{L}{8}\right) \ , \tag{7.30}$$
$$W\tfrac{dL}{dV} = \tfrac{3}{4}L^2 + (V^{-1}W - \Lambda V)L \ . \tag{7.31}$$

These equations arise *e.g.* by adapting Equations (3.16) and (3.17) of [14] to the present case (namely by setting $8\pi\rho = -8\pi p = \Lambda$, $L_1 = L$ and $C^2 = k$, and allowing the constant k to take negative values). Suppose, first, that there exists V_* such that $L(V_*) = 0$. Equation (7.31) shows then that $L \equiv 0$, and from (7.30) one obtains

$$\Lambda = \frac{k}{r^2} \ . \tag{7.32}$$

If $k = 0$ then Λ vanishes as well; further r is constant by Equation (7.29) and can therefore be absorbed into $d\Omega^2$. Integrating Equation (7.28) one finds that there exists a strictly positive constant λ such that $W = \lambda^2$, defining a coordinate z by the equation $z = V/\lambda$ proves point *1iii.* on $\mathscr{U}$. Next, if $k \neq 0$ Equation (7.32) gives $k\Lambda > 0$ as desired, together with $r^2 = -1/|\Lambda|$. Integrating Equation (7.28) one obtains

$$W = \Lambda(\lambda - V^2) \ ,$$

for some constant $\lambda \in \mathbb{R}$. Introducing the coordinate z via the equation $V^2 = \lambda - \Lambda z^2$ establishes point *1ii.* on $\mathscr{U}$.

In the case of L without zeros we obtain, from (7.28), (7.29) and (7.31), that

$$\frac{d}{dV}\left(\frac{V\sqrt{W}}{rL}\right) = 0 \ ,$$

which implies that there exists a non-vanishing constant α such that

$$L = \alpha V \frac{\sqrt{W}}{r} \ . \tag{7.33}$$

Using (7.29) one is led to

$$\frac{dV}{dr} = -\frac{4\sqrt{W}}{\alpha V} \ . \tag{7.34}$$

Next we define

$$m(V) = -\frac{\alpha}{4} r^2 \sqrt{W} + \frac{\Lambda r^3}{3} \ ; \tag{7.35}$$

from (7.28), (7.33) and (7.34) we obtain $dm/dV = 0$, *i.e.* m is a constant. Equation (7.30) gives

$$V^2 = \frac{16}{\alpha^2}\left(k - \frac{2m}{r} - \frac{\Lambda}{3}r^2\right) \ .$$

Equation (7.29) shows that we can use r as a coordinate, and Equation (7.34) implies that the metric is of the desired form (7.19). This establishes point *1i.* on $\mathscr{U}$.

Let $\mathscr{V}$ be the connected component of $\{dV \neq 0\} \subset \Sigma$ that contains $\mathscr{U}$. To establish point *1.* of the Lemma we need to show that $\mathscr{V} = \mathscr{U}$. We claim that $\mathscr{U}$ is open in $\mathscr{V}$ — and hence in Σ — which can be seen as follows: Let $p \in \mathscr{U}$, we thus have $dV(q) \neq 0$ for all q such that $V(p) = V(q)$. By Equation (7.27) $|dV|_g = \sqrt{W}$ is constant on the intersection with $\mathscr{U}$ of the level set $V^{-1}(V(p))$ of V through p, so that

$$\inf_{V^{-1}(V(p))\cap\mathscr{U}} |dV|_g > 0 \ ,$$

which easily implies that all nearby level sets in $\mathscr{U} \subset \Sigma'$ are non-critical.

Let us show now that $\mathscr{U}$ is closed in $\mathscr{V}$. To see that, consider a sequence $p_i \in \mathscr{U}$ such that $p_i \to p \in \mathscr{V}$. By definition of $\mathscr{V}$ the function $|dV|_g$ has no zeros on $\mathscr{V}$, hence $dV(p) \neq 0$. Now it follows from (3.25) that $|dV|_g$ is locally constant on smooth subsets of level sets of V, which easily implies a) that the connected component of $V^{-1}(V(p))$ containing p is smooth and b) that $|dV|_g$

is nowhere vanishing there. Compactness of the level sets of V implies that all the connected components of level sets intersecting a neighborhood of p are non-critical, and hence are in Σ'. It then follows that $p \in \mathscr{U}$.

We have thus shown that $\mathscr{U}$ is both open and closed in $\mathscr{V}$; connectedness of $\mathscr{V}$ shows that $\mathscr{U} = \mathscr{V}$, and point *1.* is established.

To prove part *2.*, we note that the equality $W(p) = W_0(V(p))$ together with Equation (7.24) shows that V has no critical points on Σ; as Σ is connected the set $\mathscr{V}$ of point *1.* coincides with Σ, and point *2.* follows from point *1.* $\square$

7.2 Proofs

PROOF OF THEOREM 1.3: Suppose that $\partial\Sigma = \emptyset$. We first consider as RS a generalized Kottler solution with $m_0 = 0$ (see Equation (2.6)). This leads to

$$\Psi \equiv 1 , \qquad \widetilde{W}_0(V_0) = -\frac{\Lambda}{3}(V_0^2 - k) . \tag{7.36}$$

We further normalize V as in Proposition 3.3, so that by (3.17), (3.21) and (3.23) we have

$$\widetilde{W} - \widetilde{W}_0 \to_{r\to\infty} 0 .$$

(Actually when $\partial_\infty\Sigma = T^2$, the normalization of V does not play any role, as we make claims only about the sign of m in this case.) Equation (7.17) together with the maximum principle shows that

$$\widetilde{W} - \widetilde{W}_0 \le 0 \ \text{ on } \ \Sigma , \tag{7.37}$$

$$n'^i D'_i(\widetilde{W} - \widetilde{W}_0)|_{\partial_\infty\Sigma} \ge 0 , \tag{7.38}$$

where n' is the *outer* pointing g'-unit normal to $\partial_\infty\Sigma$. Further, equality is attained in (7.37) or in (7.38) if and only if $W \equiv W_0$ [40, Theorems 3.5 and 3.6]. Thus Lemma 7.1 together with Equation (7.38) shows that

$$m \le 0 .$$

Assume now that $m = 0$ in the case $\partial_\infty\Sigma = S^2$; as an indirect argument, we also assume that $m = 0$ in the T^2 case, or that $m \ge m_{crit}$ in the remaining cases. In the sphere or torus case from the strong maximum principle we obtain

$$W \equiv W_0 . \tag{7.39}$$

In the higher genus cases we consider (7.17) again but take here as RS a generalized Kottler solution with the same mass as the given one, $m_0 = m$. Equations (7.37)–(7.38) hold again; then Lemma 7.1 shows that equality must hold in (7.38). Applying the maximum principle again yields Equation (7.39). We note that both point *a.* as well as the structural hypotheses of Lemma 7.3 hold under the hypotheses of Theorem 1.3. Equation (7.39) and the discussion of Section 2 show that point *2.* of that Lemma applies, so that the given solution must be a member of the generalized Kottler family with m in the range (2.7) (the generalized Nariai metrics are excluded as they do not satisfy the asymptotic hypotheses of Theorem 1.3). In the case $\partial_\infty\Sigma = S^2$ point *1.* readily follows. In the remaining cases none of these solutions has the topology required in Theorem 1.3, which gives a contradiction and establishes Theorem 1.3. □

PROOF OF THEOREM 1.5: By choice of the RS we have $(\widetilde{W} - \widetilde{W}_0)|_{\partial\Sigma} = 0$. We normalize V again so that $\lim_{\to\infty}(\widetilde{W} - \widetilde{W}_0) = 0$ holds, *cf.* Proposition 3.3 and equation (3.17). Negativity of m_0 implies that a in (7.17) is nonnegative. The maximum principle applied to Equation (7.17) gives $\widetilde{W} - \widetilde{W}_0 \le 0$ on Σ, with equality being achieved somewhere if and only if $W \equiv W_0$. Moreover, as in the proof of part 2, the boundary version of the strong maximum principle [40, Theorem 3.6] implies that $n^{i\prime} D_i'(\widetilde{W} - \widetilde{W}_0) > 0$ on $\partial_\infty\Sigma$ unless $W = W_0$. Lemma 7.1 allows us to conclude that either $m < m_0$ or $W \equiv W_0$. In that last case point *2.* of Lemma 7.3 implies that (Σ, g, V) corresponds to a generalized Kottler solution. In any case there holds $m \le m_0$.

To prove the area inequality in (1.8) requires some care as the metric $\widetilde{g}$ defined in Equation (7.4) is singular at Σ, so that standard maximum principle arguments such as [40, Theorem 3.6] do not apply. We proceed as follows. By choice of W_0 we have $\widetilde{W} = \widetilde{W}_0$ on $\partial_1\Sigma$. Further, Equation (7.2) shows that $n^i D_i(\widetilde{W} - \widetilde{W}_0)$ vanishes there. De l'Hospital's rule, the non-vanishing of dV at $\partial\Sigma$, and the requirement $\widetilde{W} - \widetilde{W}_0 \le 0$ lead to

$$n^j n^i D_i D_j(\widetilde{W} - \widetilde{W}_0)\Big|_{\partial\Sigma} = \lim_{V\to 0} \frac{D^i V D_i(\widetilde{W} - \widetilde{W}_0)}{V} \le 0 \, .$$

It follows that the left-hand-side of Equation (7.15) is non-positive, which establishes the second part of (1.8). □

PROOF OF COROLLARY 1.6: Assume that $\partial\Sigma$ is connected and that (6.2) holds; we want to show that (1.8) implies an inequality inverse to (6.2). In order to do this, note first that by (1.8) the mass m is non-positive, and Equation (6.2) implies that $g_{\partial\Sigma} > 1$. It is useful to introduce a genus-rescaled area radius $r_{\partial\Sigma}$ by the formula

$$r_{\partial\Sigma} = \sqrt{\frac{A_{\partial\Sigma}}{4\pi(g_{\partial\Sigma} - 1)}} \, .$$

In terms of this object, the inequality (6.2) reads

$$2m|g_\infty - 1|^{3/2} + \left(r_{\partial\Sigma} + \frac{\Lambda}{3} r_{\partial\Sigma}^3\right)|g_{\partial\Sigma} - 1|^{3/2} \ge 0 \, , \tag{7.40}$$

It follows that $r_{\partial\Sigma} + \frac{\Lambda}{3} r_{\partial\Sigma}^3 \ge 0$, and the Galloway–Schleich–Witt–Woolgar inequality $g_{\partial\Sigma} \le g_\infty$ implies

$$2m + r_{\partial\Sigma} + \frac{\Lambda}{3} r_{\partial\Sigma}^3 \ge 0 \, , \tag{7.41}$$

Let us denote by r_0 the $r_{\partial\Sigma}$ corresponding to the relevant generalized Kottler solution:

$$r_0 = \sqrt{\frac{A_0}{4\pi(g_{\partial_\infty\Sigma} - 1)}} \, .$$

The inequality (7.41) is actually an equality for the generalized Kottler solutions, therefore it holds that

$$2m_0 + r_0 + \frac{\Lambda}{3} r_0^3 = 0 \ .$$

We have $r_0 \geq 1/\sqrt{-\Lambda}$ from (2.9), and $m \leq m_0$, $r_{\partial\Sigma} \geq r_0$ from (1.8), so that

$$\begin{aligned} 2m + r_{\partial\Sigma} + \frac{\Lambda}{3} r_{\partial\Sigma}^3 &= 2m + r_{\partial\Sigma} + \frac{\Lambda}{3} r_{\partial\Sigma}^3 - 2m_0 - r_0 - \frac{\Lambda}{3} r_0^3 = \\ &= 2(m - m_0) + (r_{\partial\Sigma} - r_0)[1 + \frac{\Lambda}{3}(r_{\partial\Sigma}^2 + r_{\partial\Sigma} r_0 + r_0^2)] \leq \\ &\leq (r_{\partial\Sigma} - r_0)(1 + \Lambda r_0^2) \leq 0 \ . \end{aligned} \tag{7.42}$$

It follows from Equations (7.41)–(7.42) that $r_{\partial\Sigma} = r_0$, $m = m_0$, and the rigidity part of Theorem 1.5 establishes Corollary 1.6. □

References

[1] L.F. Abbott and S. Deser, *Stability of gravity with a cosmological constant*, Nucl. Phys. **B195** (1982), 76–96.

[2] M.T. Anderson, *On stationary vacuum solutions to the Einstein equations*, URL `http://www.math.sunysb.edu/~anderson`, gr-qc/0001091, to appear in Annales Henri Poincaré, 2000.

[3] ______, *On the structure of solutions to the static vacuum Einstein equations*, URL `http://www.math.sunysb.edu/~anderson`, gr-qc/0001018, to appear in Annales Henri Poincaré, 2000.

[4] L. Andersson and P.T. Chruściel, *On asymptotic behaviour of solutions of the constraint equations in general relativity with "hyperboloidal boundary conditions"*, Dissert. Math. **355** (1996), 1–100.

[5] L. Andersson, P.T. Chruściel, and H. Friedrich, *On the regularity of solutions to the Yamabe equation and the existence of smooth hyperboloidal initial data for Einsteins field equations*, Comm. Math. Phys. **149** (1992), 587–612.

[6] L. Andersson and M. Dahl, *Scalar curvature rigidity for asymptotically locally hyperbolic manifolds*, Annals of Global Anal. and Geom. **16** (1998), 1–27, dg-ga/9707017.

[7] N. Aronszajn, *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order*, Jour. de Math. **XXXVI** (1957), 235–249.

[8] A. Ashtekar and S. Das, *Asymptotically anti-de Sitter space-times: Conserved quantities*, Class. Quantum Grav. **17** (2000), L17–L30, hep-th/9911230.

[9] A. Ashtekar and A. Magnon, *Asymptotically anti–de Sitter space-times*, Classical Quantum Gravity **1** (1984), L39–L44.

[10] R. Bartnik, *The mass of an asymptotically flat manifold*, Comm. Pure Appl. Math. **39** (1986), 661–693.

[11] H. Baum, *Complete Riemannian manifolds with imaginary Killing spinors*, Ann. Global Anal. Geom. **7** (1989), 205–226.

[12] R. Beig, *Arnowitt–Deser–Misner energy and g_{00}*, Phys. Lett. **69A** (1978), 153–155.

[13] R. Beig and W. Simon, *Proof of a multipole conjecture due to Geroch*, Commun. Math. Phys. **78** (1980), 75–82.

[14] ______, *On the uniqueness of static perfect–fluid solutions in general relativity*, Commun. Math. Phys. **144** (1992), 373–390.

[15] W. Boucher, G.W. Gibbons, and G.T. Horowitz, *Uniqueness theorem for anti–de Sitter spacetime*, Phys. Rev. D **30** (1984), 2447–2451.

[16] H. Bray, *The Penrose inequality in general relativity and volume comparison theorem involving scalar curvature*, Ph.D. thesis, Stanford University, 1997.

[17] ______, *Proof of the Riemannian Penrose conjecture using the positive mass theorem*, math.DG/9911173, 1999.

[18] D. Brill, G.T. Horowitz, D. Kastor, and J. Traschen, *Testing cosmic censorship with black hole collisions*, Phys. Rev. **D49** (1994), 840–852, gr-qc/9307014.

[19] D. Brill, J. Louko, and P. Peldan, *Thermodynamics of (3+1)-dimensional black holes with toroidal or higher genus horizons*, Phys. Rev. **D56** (1997), 3600–3610, gr-qc/9705012.

[20] G. Bunting and A.K.M. Masood–ul–Alam, *Nonexistence of multiple black holes in asymptotically euclidean static vacuum space-time*, Gen. Rel. Grav. **19** (1987), 147–154.

[21] B. Carter, *Killing horizons and orthogonally transitive groups in space–time*, Jour. Math. Phys. **10** (1969), 70–81.

[22] ______, *The general theory of the mechanical, electromagnetic and thermodynamic properties of black holes*, General relativity (S.W. Hawking and W. Israel, eds.), Cambridge University Press, Cambridge, 1979, pp. 294–369.

[23] P.T. Chruściel, *On the relation between the Einstein and the Komar expressions for the energy of the gravitational field*, Ann. Inst. H. Poincaré **42** (1985), 267–282.

[24] ———, *Boundary conditions at spatial infinity from a hamiltonian point of view*, Topological Properties and Global Structure of Space–Time (P. Bergmann and V. de Sabbata, eds.), Plenum Press, New York, 1986, URL `http://www.phys.univ-tours.fr/~piotr/scans`, pp. 49–59.

[25] ———, *"No Hair" Theorems – folklore, conjectures, results*, Differential Geometry and Mathematical Physics (J. Beem and K.L. Duggal, eds.), Cont. Math., vol. 170, AMS, Providence, 1994, pp. 23–49, gr–qc/9402032.

[26] ———, *The classification of static vacuum space–times containing an asymptotically flat spacelike hypersurface with compact interior*, Class. Quantum Grav. **16** (1999), 661–687, gr-qc/9809088.

[27] ———, *Towards the classification of static electro–vacuum space–times containing an asymptotically flat spacelike hypersurface with compact interior*, Class. Quantum Grav. **16** (1999), 689–704, gr-qc/9810022.

[28] P.T. Chruściel, E. Delay, G. Galloway, and R. Howard, *Regularity of horizons and the area theorem*, Annales Henri Poincaré **in press** (2000), gr-qc/0001003.

[29] P.T. Chruściel, J. Jezierski, and J. Kijowski, *A Hamiltonian framework for field theories in the radiating regime*, in preparation.

[30] J.M. Foyster and C.B.G. McIntosh, *A class of solutions of Einstein's equations which admit a 3-parameter group of isometries*, Commun. Math. Phys. **27** (1972), 241–246.

[31] H. Friedrich, *Einstein equations and conformal structure: Existence of anti-de-Sitter-type space-times*, Jour. Geom. and Phys. **17** (1995), 125–184.

[32] G.J. Galloway, K. Schleich, D.M. Witt, and E. Woolgar, *The AdS/CFT correspondence conjecture and topological censorship*, (1999), hep-th/9912119.

[33] ———, *Topological censorship and higher genus black holes*, Phys. Rev. **D60** (1999), 104039, gr-qc/9902061.

[34] D. Gannon, *Singularities in nonsimply connected space-times*, Jour. Math. Phys. **16** (1975), 2364–2367.

[35] ———, *On the topology of spacelike hypersurfaces, singularities, and black holes*, Gen. Rel. Grav. **7** (1976), 219–232.

[36] R. Geroch, *Limits of spacetimes*, Commun. Math. Phys. **13** (1969), 180–193.

[37] ———, *Energy extraction*, Ann. New York Acad. Sci. **224** (1973), 108–117.

[38] G. Gibbons, *Gravitational entropy and the inverse mean curvature flow*, Class. Quantum Grav. **16** (1999), 1677–1687.

[39] G.W. Gibbons, S.W. Hawking, G.T. Horowitz, and M.J. Perry, *Positive mass theorem for black holes*, Commun. Math. Phys. **88** (1983), 295–308.

[40] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer, Berlin, 1983.

[41] H. Gönner and J. Stachel, *Einstein tensor and* 3*-parameter groups of isometries with* 2*-dimensional orbits*, Jour. Math. Phys. **11** (1970), 3358–3370.

[42] S.W. Hawking, *Gravitational radiation in an expanding universe*, Jour. Math. Phys. **9** (1968), 598–604.

[43] ______, *The boundary conditions for gauged supergravity*, Phys. Lett. B **126** (1983), 175–177.

[44] M. Henneaux and C. Teitelboim, *Asymptotically anti–de Sitter spaces*, Commun. Math. Phys. **98** (1985), 391–424.

[45] G.T. Horowitz and R.C. Myers, *The AdS/CFT correspondence and a new positive energy conjecture for general relativity*, Phys. Rev. **D59** (1999), 026005 (12 pp.).

[46] G. Huisken and T. Ilmanen, *The Riemannian Penrose inequality*, Int. Math. Res. Not. **20** (1997), 1045–1058.

[47] ______, *The inverse mean curvature flow and the Riemannian Penrose inequality*, Jour. Diff. Geom. (1999), in press.

[48] W. Israel, *Event horizons in static vacuum space-times*, Phys. Rev. **164** (1967), 1776–1779.

[49] P.S. Jang and R.M. Wald, *The positive energy conjecture and the cosmic censor hypothesis*, Jour. Math. Phys **18** (1977), 41–44.

[50] J. Kánnár, *Hyperboloidal initial data for the vacuum Einstein equations with cosmological constant*, Class. Quantum Grav. **13** (1996), 3075–3084.

[51] D. Kastor and J. Traschen, *Cosmological multi–black hole solutions*, Phys. Rev. **D47** (1993), no. 10, 5370–5375, hep-th/9212035.

[52] D. Kennefick and N. Ó Murchadha, *Weakly decaying asymptotically flat static and stationary solutions to the Einstein equations*, Class. Quantum Grav. **12** (1995), 149–158.

[53] J. Kijowski, *On a new variational principle in general relativity and the energy of the gravitational field*, Gen. Rel. Grav. **9** (1978), 857–877.

[54] ______, *A simple derivation of canonical structure and quasi-local Hamiltonians in general relativity*, Gen. Rel. Grav. **29** (1997), 307–343.

[55] J. Kijowski and W.M. Tulczyjew, *A symplectic framework for field theories*, Lecture Notes in Physics, vol. 107, Springer, New York, Heidelberg, Berlin, 1979.

[56] F. Kottler, *Über die physikalischen Grundlagen der Einsteinschen Gravitationstheorie*, Annalen der Physik **56** (1918), 401–462.

[57] L. Lindblom, *Static uniform-density stars must be spherical in general relativity*, Jour. Math. Phys. **29** (1988), 436–439.

[58] A. Magnon, *On Komar integrals in asymptotically anti–de Sitter space-times*, Jour. Math. Phys. **26** (1985), 3112–3117.

[59] R. B. Mann, *Pair production of topological anti–de Sitter black holes*, Class. Quantum Grav. **14** (1997), L109–L114, gr-qc/9607071.

[60] H. Nariai, *On a new cosmological solution of Einstein's field equations of gravitation*, Sci. Rep. Tohoku Univ. Ser. I. **35** (1951), 62–67.

[61] R. Penrose, *Zero rest-mass fields including gravitation*, Proc. Roy. Soc. London **A284** (1965), 159–203.

[62] ———, *Gravitational collapse — the role of general relativity*, Riv. del Nuovo Cim. **1 (numero speziale)** (1969), 252–276.

[63] W. Simon and R. Beig, *The multipole structure of stationary space–times*, Jour. Math. Phys. **24** (1983), 1163–1171.

[64] W. De Sitter, *On the curvature of space*, Proc. Kon. Ned. Akad. Wet. **20** (1917), 229–243.

[65] C. Stanciulescu, *Spherically symmetric solutions of the vacuum Einstein field equations with positive cosmological constant*, 1998, Diploma Thesis, University of Vienna.

[66] L. Vanzo, *Black holes with unusual topology*, Phys. Rev. **D56** (1997), 6475–6483, gr-qc/9705004.

[67] C.V. Vishveshwara, *Generalization of the "Schwarzschild surface" to arbitrary static and stationary metrics*, Jour. Math. Phys. **9** (1968), 1319–1322.

[68] M. Walker, *Bloc diagrams and the extension of timelike two–surfaces*, Jour. Math. Phys. **11** (1970), 2280–2286.

[69] E. Witten and S.T. Yau, *Connectedness of the boundary in the AdS/CFT correspondence*, (1999), hep-th/9910245.

[70] E. Woolgar, *Bounded area theorems for higher genus black holes*, Class. Quantum Grav. **16** (1999), 3005–3012, gr-qc/9906096.