UNIVERSITE DE YAOUNDE I
UNIVERSITY OF YAOUNDE I

FACULTE DES SCIENCES
FACULTY OF SCIENCE

DEPARTEMENT DE MATHEMATIQUES
DEPARTMENT OF MATHEMATICS

POLYHOMOGENEOUS SOLUTIONS OF WAVE EQUATIONS NEAR NULL INFINITY

THESIS

Presented and defended for the award of the

DOCTORAT/ PhD IN MATHEMATICS

Option: Analysis

By:

TAGNE WAFO Roger

Registration number: 91T525

DEA in Mathematics

Under the supervision of:

Piotr T. CHRUSIEL

Professor, Vienna University

Marcel C. DOSSA

Professor, University of Yaoundé I

Year 2011

DEDICACES

Je dédie cette thèse à mes parents **M. et Mme WAFO**. Que ce travail soit pour vous le témoignage de mon infinie reconnaissance pour tout ce que vous nous avez donné.

REMERCIEMENTS

En tout premier lieu, j'adresse mes très sincères remerciements aux Professeurs **Piotr T. Chruściel** et **Marcel Dossa** qui ont accepté de codiriger ma thèse, le premier ayant proposé le sujet. Je souhaiterais qu'ils trouvent en ces quelques mots l'expression de ma profonde gratitute car tout au long de la préparation de ces travaux ils auront fait preuve d'un très grand professionalisme, de beaucoup de disponibilité et de rigueur et aussi et surtout de beaucoup de patience et d'efficacité.

Je remercie les Professeurs **Jean Philippe Nicolas** de l'Université de Brest en France, **Bitjong Ndombol** Doyen de la Faculté des Sciences de l'Université de Yaoundé I, **François Wamon** Chef de Département de Mathématiques de l'Université de Yaoundé I, **Norbert Noutchegueme** de l'Université de Yaoundé I et **Gideon Akumah Ngwa** de l'Université de Buéa qui ont consacré une part très importante de leur précieux temps à la lecture intégrale de cette thèse. Parmi eux, les professeurs **Jean Philippe Nicolas** et **François Wamon** ont accepté d'en faire des rapports, j'en suis très honoré.

Mes très sincères remerciements vont également à tous les enseignants de la Faculté des Sciences de l'Université de Yaoundé I qui ont contribué à ma formation.

Je remercie très sincèrement tous mes collègues du Département de Mathématiques et Informatique de la Faculté des Sciences de l'Université de Douala dont les nombreux conseils et discussions auront contribué de façon significative à la réalisation du présent travail.

Toutes les personnes qui me sont proches ont contribué chacun en sa manière propre à la réalisation de ce travail, qu'elles trouvent ici l'expression de ma très grande reconnaissance. Je pense très particulièrement à mon frère ainé **Jean Samuel Talom** pour ses nombreux encouragements et surtout pour avoir mis à ma disposition au sein de son entreprise un bureau et aussi du matériel de bureau. Je pense aussi à mon très grand ami **Williams T. Poka** pour ses nombreux conseils.

Je remercie mon épouse pour sa patience et surtout pour toutes les privations consenties pendant les nombreuses années de préparation de cette thèse.

Enfin, je remercie **Dieu** le père tout puissant qui nous donne le souffle de vie et sans qui aucune chose n'est possible.

Contents

List of Figures

Résumé

Dans la première partie de la thèse, on démontre l'existence de solutions dans un espace de Sobolev à poids de problème de Cauchy hyperboloidal pour une classe de systèmes d'équations aux dérivées partielles symétriques hyperboliques non linéaires, compatibles avec les équations d'Einstein-Maxwell en dimension d'espace-temps supérieure ou égale à 7. De même, on démontre pour de tels systèmes l'existence des solutions polyhomogènes au voisinage de l'infini isotrope en dimensions d'espace-temps $n + 1 \geq 9$. Il en découle pour ces dimensions, que les solutions globales des équations couplées Einstein-Maxwell du vide obtenues par évolution des données initiales petites, stationnaires en dehors d'un compact sont polyhomogènes au voisinage de l'infini isotrope. Dans la seconde partie de la thèse, sous des hypothèses de nullité sur le terme source, on démontre un résultat d'existence et d'unicité pour une classe d'équations d'ondes semi-linéaires dont les données initiales sont prescrites sur le cône lumière futur de sommet l'origine des coordonnées dans l'espace-temps de Minkowski. Les hypothèses imposées sur la partie non linéaire du système considéré garantissent que l' épaisseur du voisinage du cône future tout entier sur lequel nous obtenons notre solution ne s'annule pas lorsqu'on atteint l'infini isotrope. Le résultat obtenu est appliqué aux applications d'ondes sur l'espace-temps de Minkowski $\mathbb{R}^{n+1}$avec $n \geq 3$.

Mots clés:

Equations d'ondes, Equations d'Einstein-Maxwell, Jauge harmonique, Jauge de Lorenz, Problèmes de Cauchy hyperbolïdaux, Problèmes de Cauchy Caractéristiques, Espaces de Sobolev à poids, Solutions polyhomogènes.

Abstract

In the first part of the thesis, we prove propagation of weighted Sobolev regularity for solutions of the hyperboloidal Cauchy problem for a class of quasi-linear symmetric hyperbolic systems, under structure conditions compatible with the Einstein-Maxwell equations in space-time dimensions $n + 1 \geq 7$. Similarly we prove propagation of polyhomogeneity in dimensions $n + 1 \geq 9$. As a byproduct we obtain, in those last dimensions, polyhomogeneity at null infinity of small data solutions of vacuum Einstein, or Einstein-Maxwell equations evolving out of initial data which are stationary outside of a ball. In the second part of the thesis, we prove existence and uniqueness of solution of a class of semi-linear wave equations with initial data prescribed on the light-cone with vertex the origin of the Minkowski space-time. The nonlinear term is assumed to satisfy a nullity condition which guarantee that the neighborhood of the initial cone on which we obtain our solution does not shrink to zero as one approaches infinity. This result is applied to wave maps on Minkowski space-times $\mathbb{R}^{n+1}$ with $n \geq 3$.

Keywords

Wave equations, Einstein-Maxwell equations, Harmonic gauge, Lorenz gauge, Hyperboloidal Cauchy Problem, Characteristic Cauchy problem, Weighted Sobolev spaces, polyhomogeneous solutions.

General Introduction

In this thesis, we study the asymptotic behavior of the solutions of Cauchy problems for systems of second order hyperbolic equations. In the first part, we are interested with a class of quasi-linear wave equations compatible with the coupled vacuum Einstein-Maxwell equations in harmonic and Lorenz gauges. The Cauchy data which are considered for such systems are prescribed on a hyperboloid $\mathscr{S}_0$ and are polyhomogeneous (i.e. around infinity, they are expandable in terms of $r^{-j}\log^i r$ rather than in terms of r^{-j}). We intend to prove an existence and a uniqueness theorem on a future neighborhood of the initial data hypersurface by guaranteeing that the thickness of this neighborhood does not shrink to zero as one approaches infinity and that the asymptotic properties of the initial data are preserve by evolution near infinity. The motivation of studying such problem arises from the fact that in [1], L. Andersson and P. T. Chruściel have constructed a large class of solutions of the constraint of the Einstein equations which are polyhomogeneous. This leads naturally to the question, whether polyhomogeneity of initial data is preserved under evolution dictated by wave equations. The results of [19, 20] constitute a first step towards an affirmative answer to this question. In these references, the authors consider a hyperboloidal Cauchy problem for semi-linear scalar wave equation and wave map equation on Minkowski space-time and, using the techniques of conformal transformation they prove that there exists a neighborhood (with a uniform thickness) of the whole initial hyperboloid on which existence in weighted Sobolev spaces (the weight being choose in order to control the singular behavior of the data near conformal boundary) and polyhomogeneity of solutions with appropriate polyhomogeneous initial data is obtained. We adapt this method to the quasi-linear case. First, by following step by step the proof of Theorem 3.7 of [19] (semi-linear case), we generalize this theorem to quasi-linear case (see Theorem 1.1.1 page 8): If the coefficients and the initial data are polyhomogeneous, if the source terms depend upon the unknown function as well as its first order derivatives and satisfy the $NL-$

condition (which is compatible with the Einstein-Maxwell equations), then the solution is polyhomogeneous provided that it belongs to some spaces of differentiable functions with singular behavior on the boundary, these singularities being controlled with appropriate weight. Next, to conclude that polyhomogeneous Cauchy data lead to polyhomogeneous solutions of the coupled Einstein-Maxwell equations, we need to show that one can construct solutions of these equations which satisfy in a neighborhood of null infinity the necessary estimates for the polyhomogeneity theorem. It turns out that the estimates on the global solutions obtained in [37, 38] in the case of space dimension $n = 3$ and in [40] in high space dimensions case are not sufficient to apply Theorem 1.1.1. To obtain those properties, we consider a more general quasi-linear wave equation (see Equation (4.1.13) page 85) and after a gauge transformation which transforms the hyperboloid $\mathscr{S}_s$ into relatively compact sets, by the means of energy momentum tensor contracted with a suitable vector field, we establish some energy inequalities on the slices $\{\tau = const\}$. Using afterwards these energy estimates, we solve locally (in time) the transformed equation on a future neighborhood of the initial data hypersurface $\{\tau = \tau_0\}$, obtaining at the same time the needed estimates for our polyhomogeneity theorem. In that way, we have proved that, hyperboloidal Cauchy data in weighted Sobolev space lead to solutions of the vacuum Einstein-Maxwell equations near null infinity in space-time dimension $n + 1 \geq 7$ odd or even and that polyhomogeneous hyperboloidal Cauchy data for the same partial differential equations lead to polyhomogeneous solutions near $\mathscr{I}^+$ in space-time dimensions $n + 1 \geq 9$. In those last dimensions, as a consequence of our approach, we obtain that the global solution of the Einstein-Maxwell equations obtained by J. Loizelet in [39, 40] by evolving small initial data which are stationary out of a compact set are polyhomogeneous near null infinity.

Because of its applications to physical phenomenon, notably to the theory of general relativity (see [27, 29, 47] and the references therein for the importance of characteristic Cauchy problem in GR), it would be very interesting to state and prove the characteristic analog of the results mentioned above. In other words, one can enquire whether polyhomogeneous Cauchy data prescribed on one or several intersecting characteristic hypersurfaces lead to polyhomogeneous solutions of the Einstein equations. The second part of the thesis can be seen as a first step towards the resolution of this problem. Indeed, in the second part of the thesis, we consider a class of semi-linear wave equations for which the data are given on the light cone with vertex the origin of the canonical coordinates in the Minkowski space-time. By assuming that the nonlinear term satisfies an appropriate structure condition and that the prescribed data on the cone satisfy the hypothesis of those of [27] near the tip of the cone and that near $\{r = \infty\}$ they are in some weighted Sobolev spaces, we state and prove for the considered problem, an existence and uniqueness of solution theorem on a neighborhood of the whole future light cone. The approach which is used here is a good combination of the techniques of conformal compactification of the first part of the thesis, the techniques of local solutions developed by M. Dossa in [28] and those of iterative scheme of [42]. This is achieved in two steps. First, from the results of [28] we obtain a local solution of the problem at hand near the tip of the cone and following [6], we use this local solution to reduce the transformed characteristic Cauchy problem to characteristic initial problem with data prescribed on two intersecting characteristic hypersurfaces. Next, as in [5] we use the method of iterative scheme introduced by A. J. Majda in [42]. The solution of our problem is obtained as a limit of solutions of linear Goursat problems on a neighborhood of the entire cone which intersect the future null infinity $\mathscr{I}^+$.

Part I

Solutions of quasi-linear wave equations polyhomogeneous at null infinity in high dimensions

Introduction of the first part

A problem of current interest is the asymptotic behavior of solutions of hyperbolic equations in the radiation zone. For large (however, not for all) sets of initial data, this question can be reduced to one where the initial data are given on a Cauchy surface that resembles a hyperboloid in Minkowski space-time. In recent works [19, 20], polyhomogeneity of solutions of such Cauchy problems, with polyhomogeneous initial data, has been proved for a large class of semi-linear symmetric hyperbolic systems. The object of this work is to extend those results to quasi-linear equations satisfying certain structure conditions which are compatible with the vacuum Einstein equations, or with the Einstein-Maxwell equations, in space-time dimensions $n+1 \geq 9$.

A special case of our results is Theorem 5.0.12 below, where polyhomogeneity at null infinity of small data global solutions of the Einstein-Maxwell equations, evolving out of initial data which are stationary outside of a compact set, is established; this is perhaps the most significant result in this work. For clarity we repeat the relevant part of that theorem here:

Theorem 0.0.1 *In dimensions $n + 1 \geq 9$ the global solutions of Einstein-Maxwell equations constructed in [39, 40] out from small initial data stationary outside of a compact set are polyhomogeneous at null infinity.*

The polyhomogeneous expansions above are in terms of powers of $\log r$ and negative integer powers of r in odd space dimension, while one has powers of $\log r$ and negative half-integer powers of r in even space dimension.

Theorem 0.0.1 should be compared with [9], where even space-time dimension $n+1 \geq 6$ is assumed, where initial data *Schwarzschildian* outside of a compact set are considered, and where solutions which are smooth at null infinity are obtained. The methods of that last reference completely fail in odd space-time dimensions. Furthermore, in odd space dimensions, generic initial data which are only *stationary*, as opposed to *Schwarzschildian*, are likely to be polyhomogeneous, *but not smooth*, at null infinity, and generic such initial data are expected to be too singular to be covered by the approach in [9]. We also note the analysis in [4], which implies smoothness at null infinity of *exactly* stationary vacuum or electro-vacuum space-times, in even space-dimension, in space-time harmonic gauge. But the dimensions covered in [4] are precisely those not covered by the evolution theorems in [2, 9].

Chapter 1

Polyhomogeneous solutions

The purpose of this chapter is to state and prove the main theorem of the first part of this thesis. In order to have a complete presentation (we need a self-contained document) of this theorem on the polyhomogeneity of solution of a class of quasilinear hyperbolic systems of first order, we choose to give a detailed presentation of the spaces of smooth and polyhomogeneous functions with their properties in Appendix A page 190. We also refer the reader to this Appendix for notations and definitions involved in this chapter.

1.1 The main theorem

Let $\psi = (\psi_1, \psi_2)$ and set

$$f := (\psi, \varphi) \ , \quad \bar{f} := (\psi_1, x\psi_2, x\varphi) \ . \tag{1.1.1}$$

We shall say that a function G *satisfies the* NL-*condition* if there exist $N, p_i, q_i, m_i \in \mathbb{N}^*$ and functions $H_i(z,w)$ $\mathscr{A}^{\delta}_{\{0\le x\le y\}}$*-polyhomogeneous in* z *with a uniform zero of order* m_i *in the variable*

$$\begin{aligned} w &:= (\bar{f}, x^2\partial_x f, x^2\partial_y f, x\partial_A f) \\ &\equiv (\psi_1, x\psi_2, x\varphi, x^2\partial_x f, x^2\partial_y f, x\partial_A f) \end{aligned}$$

such that

$$G = \sum_{i=1}^{N} x^{-p_i\delta} H_i(z, x^{q_i\delta} w) \ , \tag{1.1.2}$$

with, for $i = 1, \ldots, N$,

$$m_i > \frac{p_i - \frac{1}{\delta}}{q_i} \ . \tag{1.1.3}$$

Our first main result is the following:

Theorem 1.1.1 *Let $\mathscr{U}$ be defined in (A.2.1), suppose that $p \in \mathbb{Z}$, $q, 1/\delta \in \mathbb{N}^*$, $k \in \mathbb{N} \cup \{\infty\}$, and let*

$$\psi = (\psi_1, \psi_2)$$

and φ, with

$$\psi_1 \in \mathscr{C}^{<-1}_{\{0\le x\le y\},\infty} \cap \mathscr{C}^{<0}_{\{0\le x\le y\},0} , \qquad \psi_2, \varphi \in \mathscr{C}^{<-1}_{\{0\le x\le y\},\infty} , \tag{1.1.4}$$

be a solution on $\mathscr{U}$ of the following system of equations:

$$\begin{cases} \partial_y\varphi + B_{\varphi\varphi}\varphi + B_{\varphi\psi}\psi = L_{\varphi\varphi}\varphi + L_{\varphi\psi}\psi + a + G_\varphi \\ \partial_x\psi + B_{\psi\varphi}\varphi + B_{\psi\psi}\psi = L_{\psi\varphi}\varphi + L_{\psi\psi}\psi + b + G_\psi \end{cases} , \tag{1.1.5}$$

with the operators

$$L_{ij} = L^A_{ij}\partial_A + xL^y_{ij}\partial_y + xL^x_{ij}\partial_x \tag{1.1.6}$$

satisfying

$$L^\mu_{\varphi\varphi} \in x^\delta \mathscr{A}^\delta_{\{0\le x\le y\}} , \quad L^\mu_{\psi\varphi} , L^\mu_{\varphi\psi} , L^\mu_{\psi\psi} \in \mathscr{A}^\delta_{\{0\le x\le y\}} \tag{1.1.7}$$

(no symmetry hypotheses are made on the matrices L^μ_{ij}), while

$$B_{\varphi\varphi} \in C_\infty(\overline{\mathscr{U}}) + x^\delta \mathscr{A}^\delta_{\{0\le x\le y\}} , \qquad B_{\varphi\psi}, B_{\psi\psi}, B_{\psi\varphi} \in \mathscr{A}^\delta_{\{0\le x\le y\}} , \tag{1.1.8}$$

$$a, b \in x^{-1+\delta}\mathscr{A}^\delta_{\{0\le x\le y\}} , \tag{1.1.9}$$

$$\varphi|_{x=y} = \mathring{\varphi} \in x^{-1+\delta}\mathscr{A}^\delta_{\{x=0\}} , \ \psi|_{x=y} = \mathring{\psi} \in x^{-1+\delta}\mathscr{A}^\delta_{\{x=0\}} . \tag{1.1.10}$$

If the non-linear terms G_φ, G_ψ satisfy the NL*-condition, then*

$$(\psi, \varphi) \in \mathscr{A}^\delta_{\{0\le x\le y\}} \times x^{\delta-1}\mathscr{A}^\delta_{\{0\le x\le y\}} ;$$

more precisely

$$\psi \in x^\delta \mathscr{A}^\delta_{\{0\le x\le y\}} + \mathscr{A}^\delta_{\{y=0\}} , \tag{1.1.11a}$$

$$\varphi \in x^{\delta-1}\mathscr{A}^\delta_{\{x=0\}} + x^{\delta-1}y\mathscr{A}^\delta_{\{0\le x\le y\}} . \tag{1.1.11b}$$

In particular for any $\tau > 0$ we have

$$(\psi, \varphi)|_{\{y\ge\tau\}} \in \mathscr{A}^\delta_{\{x=0\}} \times x^{\delta-1}\mathscr{A}^\delta_{\{x=0\}} ,$$

which shows that the solution is polyhomogeneous with respect to $\{x = 0\}$ on $\{y \ge \tau\}$.

Proof: *This theorem is a generalization of the semi-linear case, Theorem 3.7 of [19], and we will follow step by step the proof given there.*

By hypothesis we have

$$B_{\varphi\varphi} = \mathring{B}_{\varphi\varphi} + B^{\delta}_{\varphi\varphi} \text{ with } \mathring{B}_{\varphi\varphi} \in C_\infty(\overline{\mathscr{U}}) \text{ and } B^{\delta}_{\varphi\varphi} \in \mathscr{A}^{\delta}_{\{0\le x\le y\}} \ .$$

We rewrite the system (1.1.5) as:

$$\begin{cases} \partial_y \varphi + \mathring{B}_{\varphi\varphi}\varphi = c_\varphi \\ \partial_x \psi = c_\psi \end{cases} \tag{1.1.12}$$

with

$$\begin{cases} c_\varphi = L_{\varphi\varphi}\varphi + L_{\varphi\psi}\psi + a + G_\varphi - B^{\delta}_{\varphi\varphi}\varphi - B_{\varphi\psi}\psi \\ c_\psi = L_{\psi\varphi 1}\varphi + L_{\psi\psi}\psi + b + G_\psi - B_{\psi\varphi}\varphi - B_{\psi\psi}\psi \end{cases} \ . \tag{1.1.13}$$

The first step in the proof is to prove the following:

Lemma 1.1.2 *Under the hypotheses of Theorem 1.1.1, we have:*

$$\psi \in \mathscr{C}^{<0}_{\{0\le x\le y\},\infty} + x^{\delta}\mathscr{A}^{\delta}_{\{0\le x\le y\}} + \mathscr{A}^{\delta}_{\{y=0\}} \tag{1.1.14a}$$

$$\varphi \in \mathscr{C}^{<-1+\delta}_{\{0\le x\le y\},\infty} + x^{\delta-1}y\mathscr{A}^{\delta}_{\{0\le x\le y\}} + x^{\delta-1}\mathscr{A}^{\delta}_{\{x=0\}} \ . \tag{1.1.14b}$$

Proof: *Integration of the second equation of* (1.1.12) *yields:*

$$\begin{aligned} \psi(x,v^A,y) &= \psi(y,v^A,y) + \int_y^x c_\psi(s,v^A,y)ds \\ &= \mathring{\psi}(y,v^A) + \int_y^x b(s,v^A,y)ds \\ &\quad + \int_y^x \left\{L_{\psi\varphi}\varphi + L_{\psi\psi}\psi - B_{\psi\varphi}\varphi - B_{\psi\psi}\psi + G_\psi\right\}(s,v^A,y)ds \ . \end{aligned}$$

By hypotheses, we have: $\mathring{\psi} \in \mathscr{A}^{\delta}_{\{y=0\}}$ *and* $b \in x^{\delta-1}\mathscr{A}^{\delta}_{\{0\le x\le y\}}$, *we deduce from Proposition A.6.2 that*

$$I_1(b) = -\int_y^x b(s,v^A,y)ds \in x^{\delta}\mathscr{A}^{\delta}_{\{0\le x\le y\}} + y^{\delta}\mathscr{A}^{\delta}_{\{y=0\}}.$$

If we set $\psi_{0,phg} = \mathring{\psi} + I_1(b)$ *then, we have:*

$$\psi = \psi_{0,phg} + \psi_{\epsilon,1} + \psi_{\epsilon,2} \ ,$$

with

$$\psi_{0,phg} \in x^\delta \mathscr{A}^\delta_{\{0\le x\le y\}} + y^\delta \mathscr{A}^\delta_{\{y=0\}} + \mathscr{A}^\delta_{\{y=0\}} \tag{1.1.15a}$$

$$\psi_{\epsilon,1} := \int_y^x \left\{L_{\psi\varphi}\varphi + L_{\psi\psi}\psi - B_{\psi\varphi}\varphi - B_{\psi\psi}\psi\right\}(s,v^A,y)ds \tag{1.1.15b}$$

$$\psi_{\epsilon,2} := \int_y^x G_\psi(s,v^A,y)ds\,. \tag{1.1.15c}$$

Since the space $\mathscr{C}^{<\theta}_{\{0\le x\le y\},\infty}$ *is invariant under the operators* ∂_A, $x\partial_x$ *and* $x\partial_y$, *and since we have the embedding*

$$\mathscr{A}^\delta_{\{0\le x\le y\}} \subset \mathscr{C}^{<0}_{\{0\le x\le y\},\infty}\,,$$

one obtains that

$$L_{\psi\varphi}\varphi,\ L_{\psi\psi}\psi,\ B_{\psi\varphi}\varphi,\ B_{\psi\psi}\psi \in \mathscr{C}^{<-1}_{\{0\le x\le y\},\infty}, \tag{1.1.16}$$

and from Lemma A.6.3, we deduce that:

$$\psi_{1,\epsilon} \in \mathscr{C}^{<0}_{\{0\le x\le y\},\infty}\,. \tag{1.1.17}$$

On the other hand, recall

$$G_\psi(z) = \sum_{i=0}^{N} x^{-p_i\delta} H_{i\psi}(z, x^{q_i\delta}\omega).$$

Let $\epsilon>0$, *if we apply (A.3.5) page 196 to* $H_{i\psi}$ *with* $k=i=0$, *one obtains:*

$$\begin{aligned}\|x^{-p_i\delta}H_{i\psi}(z,x^{q_i\delta}\omega)\|_{L^\infty(\mathscr{U})} &= \|x^{-p_i\delta}H_{i\psi}(z,x^{q_i\delta-\frac{\epsilon}{m_i}}x^{\frac{\epsilon}{m_i}}w)\|_{L^\infty(\mathscr{U})}\\ &\le Cx^{-p_i\delta}\|x^{q_i\delta-\frac{\epsilon}{m_i}}x^{\frac{\epsilon}{m_i}}w\|^{m_i}_{L^\infty(\mathscr{U})}\\ &\le Cx^{(-p_i+q_im_i)\delta-\epsilon}\,.\end{aligned}$$

We recall that from (1.1.4) we have $w\in\mathscr{C}^{<0}_{\{0\le x\le y\},0}(\mathscr{U})$ *which implies that* $x^\epsilon w$ *is in* $L^\infty(\mathscr{U})$; *thus,*

$$x^{-p_i\delta}H_{i\psi}(z,x^{q_i\delta}\omega) = O(x^{(-p_i+q_im_i)\delta-\epsilon}) \tag{1.1.18}$$

and

$$\begin{aligned}\psi_{\epsilon,2} &= \int_y^x \sum_{i=0}^{N} \underbrace{x^{-p_i\delta}H_{i\psi}(s,y,v^A,s^{q_i\delta}\omega)}_{O(s^{-p_i\delta+q_im_i\delta-\epsilon})}ds\\ &= \sum_{i=0}^{N}\left\{O(x^{(-p_i+q_im_i)\delta-\epsilon+1}) + O(y^{(-p_i+q_im_i)\delta-\epsilon+1})\right\}.\end{aligned}$$

Since $(-p_i + q_i m_i)\delta + 1 \; > \; 0$, *we have*

$$\psi_{\epsilon,2} \; = \; O(x^{-\epsilon}) + O(y^{-\epsilon}) \; = \; O(x^{-\epsilon})$$

and we obtain that,

$$\psi \; - \; \psi_{0,phg} \; = \; O(x^{-\epsilon}) \; , \tag{1.1.19}$$

i.e.

$$\begin{aligned} \psi \;\; &\in \;\; \mathscr{C}^{<0}_{\{0\le x\le y\},0} + x^\delta \mathscr{A}^\delta_{\{0\le x\le y\}} + y^\delta \mathscr{A}^\delta_{\{y=0\}} + \mathscr{A}^\delta_{\{y=0\}} \\ &\subset \;\; \mathscr{C}^{<0}_{\{0\le x\le y\},0} + x^\delta \mathscr{A}^\delta_{\{0\le x\le y\}} + \mathscr{A}^\delta_{\{y=0\}}. \end{aligned}$$

We would like to have the same estimations on the first order derivatives of $\psi \; - \; \psi_{0,phg}$*. From (1.1.16) and (1.1.18) we have*

$$x\partial_x(\psi - \psi_{0,phg}) \; = \; x\big[L_{\psi\varphi}\varphi + L_{\psi\psi}\psi - B_{\psi\varphi}\varphi - B_{\psi\psi}\psi + G_\psi\big] \; \in \mathscr{C}^{<0}_{\{0\le x\le y\},0} \; ,$$

i.e.

$$x\partial_x(\psi \; - \; \psi_{0,phg}) \; = \; O(x^{-\epsilon}), \; \forall \epsilon > 0.$$

We have,

$$y\partial_y(\partial_x\psi) = y\partial_y[L_{\psi\varphi}\varphi + L_{\psi\psi}\psi - B_{\psi\varphi}\varphi - B_{\psi\psi}\psi \; + \; b \; + \; G_\psi] \; .$$

Again (1.1.16), gives

$$y\partial_y[L_{\psi\varphi}\varphi + L_{\psi\psi}\psi - B_{\psi\varphi}\varphi - B_{\psi\psi}\psi] \in \mathscr{C}^{<-1}_{\{0\le x\le y\},\infty} \; ;$$

and from (1.1.8) we have

$$y\partial_y(b) \in x^{-1+\delta}\mathscr{A}^\delta_{\{0\le x\le y\}} \subset \mathscr{C}^{<-1+\delta}_{\{0\le x\le y\},\infty} \; .$$

On the other hand,

$$y\partial_y(G_\psi) = \sum_{i=0}^{N} x^{-p_i\delta} y\partial_y(H_{i\psi}(z, x^{q_i\delta}w)) \; .$$

Recall that

$$w = (\bar{f}, x^2\partial_x f, x^2\partial_y f, x\partial_A f) = \big(w_j\big)_{j=1}^{12}$$

thus

$$\begin{aligned} y\partial_y(G_\psi) \;\; &= \;\; \sum_{i=0}^{N} x^{-p_i\delta}(y\partial_y H_{i\psi})(z, x^{q_i\delta}w) \\ &\quad + \sum_{i=0}^{N}\sum_{j=1}^{12} x^{-(p_i-q_i)\delta}\frac{\partial H_{i\psi}}{\partial w_j}(z, x^{q_i\delta}w) y\partial_y(w_j) \qquad (1.1.20) \\ &= \;\; A + B \; . \end{aligned}$$

A can be estimated as follows: $\forall \epsilon > 0$,

$$\begin{aligned} |x^{-p_i\delta}(y\partial_y H_{i\psi})(z, x^{q_i\delta}w)| &\leq \bar{C}x^{-p_i\delta}\|H_{i\psi}(., x^{q_i\delta-\frac{\epsilon}{m_i}}x^{\frac{\epsilon}{m_i}}w)\|_{\mathscr{C}^0_{\{0\leq x\leq y\},1-0}} \\ &\leq \bar{C}x^{-p_i\delta}\|x^{q_i\delta-\frac{\epsilon}{m_i}}x^{\frac{\epsilon}{m_i}}w\|^{m_i}_{L^\infty} \\ &\leq \bar{C}x^{-p_i\delta+q_im_i\delta-\epsilon}\ . \end{aligned}$$

Which implies that,

$$A = \sum_{i=1}^{N} O(x^{-p_i\delta+q_im_i\delta-\epsilon}) = O(x^{-1-\epsilon}) \ \ since \ -p_i\delta + q_im_i\delta > -1.$$

If we use again the hypothesis (A.3.5) with $k = i = 1$ *, we find that*

$$\frac{\partial H_{i\psi}}{\partial w_j}(z, x^{q_i\delta}w) \ = \ O(x^{q_im_i\delta-q_i\delta-\epsilon})\ .$$

For

$$j \ \neq \ 1, \quad y\partial_y w_j \in \mathscr{C}^{<0}_{\{0\leq x\leq y\},\infty}$$

thus,

$$\sum_{i=0}^{N}\sum_{j=2}^{12} x^{-(p_i-q_i)\delta}\frac{\partial H_{i\psi}}{\partial w_j}(z, x^{q_i\delta}w)y\partial_y(w_j) = \sum_{i=0}^{N} O(x^{(-p_i+q_im_i)\delta-\epsilon}) \quad (1.1.21)$$

We can then write

$$B = O(x^{-1-\epsilon}) + \mathcal{K}(z)(y\partial_y)\psi_1 \quad (1.1.22)$$

where $\mathcal{K}$ *is a sum of terms each of which being of order* $O(x^{(-p_i+q_i'm_i')\delta-\epsilon})$ *with* p_i*,* q_i *and* m_i *satisfying (1.1.3). We obtain finally the equation*

$$\partial_x(y\partial_y\psi) + \mathcal{K}(x^\mu)(y\partial_y\psi_1) = O(x^{-1-\epsilon}), \quad (1.1.23)$$

and Lemma A.2.4 apply to this equation gives

$$y\partial_y\psi = O(x^{-\epsilon}). \quad (1.1.24)$$

Since $\partial_A\psi$ *can be estimated in the same way, we have shown that:*

$$\begin{aligned} \psi \ &\in \ x^\delta\mathscr{A}^\delta_{\{0\leq x\leq y\}} + y^\delta\mathscr{A}^\delta_{\{y=0\}} + \mathscr{A}^\delta_{\{y=0\}} + \mathscr{C}^{<0}_{\{0\leq x\leq y\},1}(\mathscr{U}) \\ &\subset \ x^\delta\mathscr{A}^\delta_{\{0\leq x\leq y\}} + \mathscr{A}^\delta_{\{y=0\}} + \mathscr{C}^{<0}_{\{0\leq x\leq y\},1}(\mathscr{U})\ . \end{aligned}$$

We pass now to the analysis of φ .
φ *can be calculated as:*

$$\varphi(x, v^A, y) = R(x, v^A; y, x)\varphi(x, v^A, x) + \int_x^y R(x, v^A; s, x)c_\varphi(x, v^A, s)ds \tag{1.1.25}$$

where $R(x, v^A; y, y_1)$ *is the family of resolvents (smooth up to boundary in all variables) of the family of ODE's* $\partial_y\varphi(x, v^A; y) = \mathring{B}_{\varphi,\varphi}(x, v^A; y)\varphi(x, v^A; y)$, *with parameters* (x, v^A) *and with initial value at* y_1. *By hypothesis we have* $\varphi(x, v^A, x) = \mathring{\varphi}(x, v^A) \in x^{-1+\delta}\mathscr{A}^\delta_{\{x=0\}}$ *which implies that the first term* $R\mathring{\varphi}$ *is in* $x^{-1+\delta}\mathscr{A}^\delta_{\{x=0\}}$.
On the other hand we recall that: $c_\varphi = L_{\varphi\varphi}\varphi + L_{\varphi\psi}\psi + a + G_\varphi - B^\delta_{\varphi\varphi}\varphi - B_{\varphi\psi}\psi$.
By hypothesis, $a \in x^{-1+\delta}\mathscr{A}^\delta_{\{0\le x\le y\}}$, *and from (1.1.7) we have,* $L^\mu_{\varphi\varphi} \in x^\delta\mathscr{A}^\delta_{\{0\le x\le y\}} \subset \mathscr{C}^{<\delta}_{\{0\le x\le y\},\infty}$. *Thus,*

$$(L^A_{\varphi\varphi}\partial_A + xL^x_{\varphi\varphi}\partial_x + xL^y_{\varphi\varphi}\partial_y)\varphi \in \mathscr{C}^{<-1+\delta}_{\{0\le x\le y\},\infty} .$$

To estimate the terms of $L_{\varphi\psi}\psi$, *we proceed in a similar way, using the supplementary hypothesis that* $\psi \in \mathscr{C}^{<0}_{\{0\le x\le y\},1}$ *and obtain that*

$$L_{\varphi\psi}\psi \in \mathscr{C}^{<0}_{\{0\le x\le y\},0}(\mathscr{U}).$$

On the other hand, $B^\delta_{\varphi\varphi}\varphi \in \mathscr{C}^{<-1+\delta}_{\{0\le x\le y\},\infty}(\mathscr{U})$, $B_{\varphi\psi}\psi \in \mathscr{C}^{<0}_{\{0\le x\le y\},1}(\mathscr{U})$.
Now,

$$G_\varphi(z) = \sum_{i=0}^N x^{-p_i\delta}H_{i\varphi}(z, x^{q_i\delta}z)$$

and as in (1.1.18) we have :

$$G_\varphi = \sum_{i=0}^N O(x^{(-p_i+q_im_i)\delta-\epsilon}) = O(x^{-1-\epsilon+\delta}), \forall\epsilon > 0. \tag{1.1.26}$$

The last equality follows from the following

Remark 1.1.3 $m_i > \frac{p_i-\frac{1}{\delta}}{q_i} \implies q_im_i - p_i > -\frac{1}{\delta}$ and since $q_im_i - p_i$ and $-\frac{1}{\delta}$ are integers, we have, $q_im_i - p_i \ge -\frac{1}{\delta} + 1 \implies (q_im_i - p_i)\delta \ge -1 + \delta$.

Thus, $\forall\epsilon > 0$, *we can write ,* $c_\varphi = c_{\varphi,phg} + c_{\varphi,\epsilon}$ *with*

$$c_{\varphi,phg} \in x^{\delta-1}\mathscr{A}^\delta_{\{0\le x\le y\}} \tag{1.1.27}$$

and

$$c_{\varphi,\epsilon} = O(x^{-1+\delta-\epsilon}) \ . \tag{1.1.28}$$

If we come back to the expression of φ *we have:*

$$\varphi(x, v^A, y) \ = \ \underbrace{R(x, v^A; y, x)\varphi(x, v^A, x) + \int_x^y R(x, v^A; s, x) c_{\varphi,phg}(x, v^A, s)\, ds}_{:=\varphi_{0,phg}}$$
$$+ \int_x^y \underbrace{R(x, v^A; s, x) c_{\varphi,\epsilon}(x, v^A, s)}_{=O(x^{-1+\delta-\epsilon})}\, ds.$$

Now, we have

$$R\mathring{\varphi} \ \in \ x^{\delta-1}\mathscr{A}^{\delta}_{\{x=0\}}$$

$Rc_{\varphi,phg} \ \in \ x^{\delta-1}\mathscr{A}^{\delta}_{\{0\leq x\leq y\}}$ *thus, from Proposition A.6.2, we have*

$$I_2(Rc_{\varphi,phg}) \in x^{\delta-1} y \mathscr{A}^{\delta}_{\{0\leq x\leq y\}} + \ x^{\delta}\mathscr{A}^{\delta}_{\{x=0\}}.$$

Therefore,

$$\varphi - \varphi_{0,phg} = O(x^{-1+\delta-\epsilon})$$

with

$$\varphi_{0,phg} \ \in \ x^{\delta-1}\mathscr{A}^{\delta}_{\{x=0\}} \ + \ x^{\delta-1} y \mathscr{A}^{\delta}_{\{0\leq x\leq y\}}.$$

At this stage we have proved that

$$\psi \ \in \ x^{\delta}\mathscr{A}^{\delta}_{\{0\leq x\leq y\}} \ + \ y^{\delta}\mathscr{A}^{\delta}_{\{y=0\}} \ + \ \mathscr{A}^{\delta}_{\{y=0\}} \ + \ \mathscr{C}^{<0}_{\{0\leq x\leq y\},1}(\mathscr{U}) \tag{1.1.29a}$$
$$\varphi \ \in \ x^{\delta-1}\mathscr{A}^{\delta}_{\{x=0\}} \ + \ x^{\delta-1} y \mathscr{A}^{\delta}_{\{0\leq x\leq y\}} \ + \ \mathscr{C}^{<-1+\delta}_{\{0\leq x\leq y\},0} \ . \tag{1.1.29b}$$

As in the proof of theorem 3.1 of [19], we have the following

Lemma 1.1.4 *Under the hypothesis of the theorem 1.1.1, the fields*

$$\tilde{\varphi} \ := \begin{pmatrix} \varphi \\ \partial_A \varphi \\ x\partial_x \varphi \\ y\partial_y \varphi \end{pmatrix} \ ; \quad \tilde{\psi} := \begin{pmatrix} \psi \\ \partial_A \psi \\ x\partial_x \psi \\ y\partial_y \psi \end{pmatrix} \tag{1.1.30}$$

satisfy a system of equations of the form (1.1.5), with coefficients $\mathcal{L}_{ij}$, $\mathcal{B}_{ij}$ *and sources* $\tilde{a}$, $\tilde{b}$, $\tilde{G}_{\varphi}$, $\tilde{G}_{\psi}$ *satisfying the hypothesis of the main theorem, with* $\tilde{\psi}_1 \in (\mathscr{C}^{<-1}_{\{0\leq x\leq y\},\infty} + \mathscr{C}^{<0}_{\{0\leq x\leq y\},0})(\mathscr{U})$, $\tilde{\psi}_2$, $\tilde{\varphi} \in \mathscr{C}^{<-1}_{\{0\leq x\leq y\},\infty}(\mathscr{U})$ *and* $\tilde{\varphi}|_{\{y=x\}} \in x^{-1+\delta}\mathscr{A}^{\delta}_{\{x=0\}}$, $\tilde{\psi}|_{\{y=x\}} \in \mathscr{A}^{\delta}_{\{x=0\}}$.

Proof: The original system can be written as:

$$\begin{cases} \partial_y\varphi \;=\; \mathring{c}_\varphi \;:=\; c_\varphi \;-\; \mathring{B}_{\varphi\varphi}\varphi \\ \partial_x\psi = c_\psi \end{cases} . \tag{1.1.31}$$

Thus, differentiating $\tilde{\varphi}$ and $\tilde{\psi}$ leads to:

$$\partial_y\tilde{\varphi} = \begin{pmatrix} \mathring{c}_\varphi \\ \partial_A\mathring{c}_\varphi \\ x\partial_x\mathring{c}_\varphi \\ \mathring{c}_\varphi \;+\; y\partial_y\mathring{c}_\varphi \end{pmatrix} \quad \text{and} \quad \partial_x\tilde{\psi} = \begin{pmatrix} c_\psi \\ \partial_A c_\psi \\ c_\psi \;+\; x\partial_x c_\psi \\ y\partial_y c_\psi \end{pmatrix} . \tag{1.1.32}$$

We want to obtain a system of the form

$$\begin{cases} \partial_y\tilde{\varphi} + \mathcal{B}_{\varphi\varphi}\tilde{\varphi} + \mathcal{B}_{\varphi\psi}\tilde{\psi} = \mathcal{L}_{\varphi\varphi}\tilde{\varphi} + \mathcal{L}_{\varphi\psi}\tilde{\psi} + \tilde{a} + \tilde{G}_{\tilde{\varphi}} \\ \partial_x\tilde{\psi} + \mathcal{B}_{\psi\varphi}\tilde{\varphi} + \mathcal{B}_{\psi\psi}\tilde{\psi} = \mathcal{L}_{\psi\varphi}\tilde{\varphi} + \mathcal{L}_{\psi\psi}\tilde{\psi} + \tilde{b} + \tilde{G}_{\tilde{\psi}} \end{cases}$$

with $\mathcal{L}^\mu_{ij}$ and $\mathcal{B}_{ij}$ having the same structure as in (1.1.6)-(1.1.7), and $\tilde{G}_{\tilde{\varphi}}$, $\tilde{G}_{\tilde{\psi}}$ satisfying the $\mathrm{NL}-$*condition . Following the proof of Lemma 3.5 of [19], one easily obtains the coefficients $\mathcal{L}^\mu_{ij}$, $\mathcal{B}_{ij}$, $\tilde{a}$ and $\tilde{b}$ of the linear terms, and verify that they are in the right spaces. It remains to show that the non-linear terms satisfy the* $\mathrm{NL}-$*condition. We have*

$$\tilde{G}_\varphi \;=\; \begin{pmatrix} G_\varphi \\ \partial_B G_\varphi \\ x\partial_x G_\varphi \\ G_\varphi \;+\; y\partial_y G_\varphi \end{pmatrix} \quad \text{and} \quad \tilde{G}_\psi \;=\; \begin{pmatrix} G_\psi \\ \partial_B G_\psi \\ G_\psi \;+\; x\partial_x G_\psi \\ y\partial_y G_\psi \end{pmatrix} . \tag{1.1.33}$$

We write,

$\tilde{f} = (\tilde{\psi} = (\tilde{\psi}_1, \tilde{\psi}_2), \tilde{\varphi})$, *and we set* $\bar{\tilde{f}} := (\tilde{\psi}_1, x\tilde{\psi}_2, x\tilde{\varphi})$ *and* $\widetilde{w} := \left(\bar{\tilde{f}}, x^2\partial_x\tilde{f}, x^2\partial_y\tilde{f}, x\partial_A\tilde{f}\right)$

and we notice that, all the terms in w are in $\widetilde{w}$. Thus, a function of w can be considered as a function of $\widetilde{w}$. On the other hand,

$$\begin{aligned} \partial_B w &= \left(\partial_B(\bar{f}), x^2\partial_x(\partial_B f), x^2\partial_y(\partial_B f), x\partial_A(\partial_B f)\right) \\ &= \mathcal{A}_1{}^t\widetilde{w} \end{aligned}$$

where the coefficients of the rectangular matrix $\mathcal{A}_1$ are all equal to 0 *or* 1*, and ${}^t\widetilde{w}$ is the transpose of w . Similarly,*

$$\begin{aligned} y\partial_y w &= \left(y\partial_y(\bar{f}), x^2\partial_x(y\partial_y f), x^2\partial_y(y\partial_y f) - x^2\partial_y f, x\partial_A(y\partial_y f)\right) \\ &= \mathcal{A}_2{}^t\widetilde{w} \end{aligned}$$

where again the coefficients of the rectangular matrix $\mathcal{A}_2$ *are all equal to* -1,0 *or* 1. *Continuing this way,*

$$\begin{aligned} x\partial_x w &= \big(x\partial_x(\bar{f}), x(x\partial_x f) + x^2\partial_x(x\partial_x f), 2x^2\partial_y f + x^2\partial_y(x\partial_x f), x\partial_A f + x\partial_A(x\partial_x f)\big) \\ &= \mathcal{A}_3{}^t\widetilde{w} \end{aligned}$$

where the coefficients of the rectangular matrix $\mathcal{A}_3$ *are all equal to* 0, 1 *or* 2. *Now, we have*

$$\begin{aligned} \partial_B\{G_\varphi(z)\} &= \sum_{i=0}^{N} x^{-p_i\delta}(\partial_B H_{i\varphi})(z, x^{q_i\delta}w) \\ &\quad + \sum_{i=0}^{N}\sum_{j=1}^{12} x^{-(p_i-q_i)\delta}\frac{\partial H_{i\varphi}}{\partial w_j}(z, x^{q_i\delta}w)\partial_B w \\ &= \sum_{i=0}^{N} x^{-p_i\delta}(\partial_B H_{i\varphi})(z, x^{q_i\delta}w) \\ &\quad + \sum_{i=0}^{N}\sum_{j=1}^{12} x^{-p_i\delta}\frac{\partial H_{i\varphi}}{\partial w_j}(z, x^{q_i\delta}w)x^{q_i\delta}\Big(\mathcal{A}_1{}^t\widetilde{w}t\Big)_j \\ &= A + B\,. \end{aligned} \tag{1.1.34}$$

From the definition of $\mathscr{A}^{\delta}_{\{0\le x\le y\}}$*-polyhomogeneous in* z *with a uniform zero of order* ℓ, *we conclude that* A *has the desired form.*

From (A.3.5) of Definition A.3.8, $\frac{\partial H_{i\varphi}}{\partial w_j}$ *is* $\mathscr{A}^{\delta}_{\{0\le x\le y\}}$*-polyhomogeneous in* z *with a uniform zero of order* $m_i - 1$ *in* w *and then* $\frac{\partial H_{i\varphi}}{\partial w_j}(z, x^{q_i\delta}w)\Big(\mathcal{A}_1{}^t\widetilde{w}\Big)_j$ *is* $\mathscr{A}^{\delta}_{\{0\le x\le y\}}$*-polyhomogeneous in* z *with a uniform zero of order* m_i *in* $\widetilde{w}$, *this allows us to conclude that* B *has the desired structure, thus* $\partial_B G_\varphi$ *satisfies the* NL − *condition i.e.* $\partial_B G_\varphi = \sum_{i=0}^{N} x^{-p_i\delta}\mathcal{H}_{i\varphi}(x^{q_i\delta}\widetilde{w})$ *where the* $\mathcal{H}_{i\varphi}$*'s are* $\mathscr{A}^{\delta}_{\{0\le x\le y\}}$*-polyhomogeneous in* z *with a uniform zero of order* m_i *in* $\widetilde{w}$.

The same analysis holds for

$$\begin{aligned} y\partial_y(G_\psi) &= \sum_{i=0}^{N} x^{-p_i\delta}(y\partial_y H_{i\psi})(z, x^{q_i\delta}w) \\ &\quad + \sum_{i=0}^{N}\sum_{j=1}^{12} x^{-(p_i-q_i)\delta}\frac{\partial H_{i\psi}}{\partial w_j}(z, x^{q_i\delta}w)y\partial_y(w_j). \end{aligned}$$

As far as the term $x\partial_x G_\varphi$ is concerned we notice that it has supplementary terms:

$$
\begin{aligned}
x\partial_x(G_\varphi) &= \sum_{i=0}^{N} -p\delta x^{-p_i\delta} H_{i\psi}(z, x^{q_i\delta}w) + \sum_{i=0}^{N} x^{-p_i\delta}(x\partial_x H_{i\varphi})(z, x^{q_i\delta}w) \\
&\quad + \sum_{i=0}^{N}\sum_{j=1}^{12} x^{-(p_i-q_i)\delta}\frac{\partial H_{i\varphi}}{\partial w_j}(z, x^{q_i\delta}w)(x\partial_x w_j) \\
&\quad + \sum_{i=0}^{N}\sum_{j=1}^{12} q_i\delta x^{-(p_i-q_i)\delta}\frac{\partial H_{i\varphi}}{\partial w_j}(z, x^{q_i\delta}w)w_j.
\end{aligned}
$$

The above analysis holds for each term of this expression, and we conclude that $\tilde{G}_\varphi$ satisfies the NL – *condition. Similarly, $\tilde{G}_\psi$ satisfies the* NL – *condition.*

The last step in the proof of this lemma is to show that the restriction to the hypersurface $\{y = x\}$ of $\tilde{\varphi}$, and $\tilde{\psi}$ are in the right spaces. We proceed exactly as in [19]. The difference here is that we have supplementary terms coming from the nonlinearity of the problem at hand. We have to make sure that these terms will have the right structure.

For the components φ, ψ, $\partial_A\varphi$, and $\partial_A\psi$ this is again hypothesis (1.1.10). Therefore it remains to show that $x\partial_x\varphi,\ y\partial_y\varphi,\ x\partial_x\psi,\ y\partial_y\psi \in x^{-1+\delta}\mathscr{A}^{\delta}_{\{x=0\}}$. From the second equation of (1.1.31), we have

$$\psi(x, v, y) = \mathring{\psi}(y, v) + \int_x^y c_\psi(s, v, y)ds$$

which implies that

$$\partial_y\psi(, v, y) = \partial_y\mathring{\psi}(y, v) - c_\psi(x, v, y) + \int_y^x \partial_y c_\psi(s, v, y)ds\ . \tag{1.1.35}$$

Now, we take the limit $x \to y$ in (1.1.35) to obtain

$$y\partial_y\psi|_{\mathscr{S}} = y(\partial_y\mathring{\psi}) - yc_\psi|_{\mathscr{S}}\ . \tag{1.1.36a}$$

Similarly from

$$\varphi(x, v^A, y) = \underbrace{\varphi(x, v^A, x)}_{\mathring{\varphi}(x,v^A)} + \int_x^y \mathring{c}_\varphi(x, v^A, s)\,ds\ .$$

we find (again for $x = y$)

$$x\partial_x\varphi|_{\mathscr{S}} = x(\partial_x\mathring{\varphi}) - x\mathring{c}_\varphi|_{\mathscr{S}} \; . \qquad (1.1.36\text{b})$$

Equations (1.1.5) further give

$$y(\partial_x\psi)|_{\mathscr{S}} = yc_\psi|_{\mathscr{S}} \; , \qquad (1.1.36\text{c})$$

$$y(\partial_y\varphi)|_{\mathscr{S}} = y\mathring{c}_\varphi|_{\mathscr{S}} \; . \qquad (1.1.36\text{d})$$

The terms $y(\partial_x\mathring{\psi})(y, v^A)$ and $y(\partial_x\mathring{\varphi})(y, v^A)$ in (1.1.36a)-(1.1.36b) are in $y^{-1+\delta}\mathscr{A}^\delta_{\{y=0\}}$. Now,

$$y\mathring{c}_\varphi|_{\mathscr{S}} = y(L_{\varphi\varphi}\varphi + L_{\varphi\psi}\psi + a + G_\varphi - B_{\varphi\varphi}\varphi - B_{\varphi\psi}\psi)|_{\mathscr{S}} \; .$$

The restrictions to $\mathscr{S}$ of the terms a, $B_{\varphi\varphi}\varphi$, $B_{\varphi\psi}\psi$ and the derivatives of φ and ψ with respect to v^A, give a contribution which is in $y^{-1+\delta}\mathscr{A}^\delta_{\{y=0\}}$. As far as the restriction to $\mathscr{S}$ of G_φ is concerned, we use Lemma A.3.9 and obtain again a contribution which is in $y^{-1+\delta}\mathscr{A}^\delta_{\{y=0\}}$. The remaining terms are of the form $y(\partial_y\psi)|_{\mathscr{S}}$, $y(\partial_y\varphi)|_{\mathscr{S}}$, $y(\partial_x\psi)|_{\mathscr{S}}$, $y(\partial_x\varphi)|_{\mathscr{S}}$ multiplied by coefficients from $\mathscr{A}^\delta_{\{y=0\}}$. The same analysis applies to $yc_\psi|_{\mathscr{S}}$, so that we can write the system of equations (1.1.36) as

$$(\mathrm{Id} - yK)\begin{pmatrix} y(\partial_y\psi)|_{\mathscr{S}} \\ y(\partial_y\varphi)|_{\mathscr{S}} \\ y(\partial_x\psi)|_{\mathscr{S}} \\ y(\partial_x\varphi)|_{\mathscr{S}} \end{pmatrix} \in y^{-1+\delta}\mathscr{A}^\delta_{\{y=0\}} \; .$$

Here K is a matrix with components in $\mathscr{A}^\delta_{\{y=0\}}$. There exists $\epsilon > 0$ so that for $0 \le y < \epsilon$ the matrix $\mathrm{Id} - yK$ has an inverse in $\mathrm{Id} + y\mathscr{A}^\delta_{\{y=0\}}$, and polyhomogeneity (with appropriate power structure) of the initial data for $(\tilde{\varphi}, \tilde{\psi})$ follows. □

Thus, applying Lemma 1.1.4 , we have again (1.1.29) with $\tilde{\varphi}$ and $\tilde{\psi}$ instead of φ and ψ i.e.

$$\tilde{\psi} \in x^\delta\mathscr{A}^\delta_{\{0\le x\le y\}} + y^\delta\mathscr{A}^\delta_{\{y=0\}} + \mathscr{A}^\delta_{\{y=0\}} + \mathscr{C}^{<0}_{\{0\le x\le y\},1} \qquad (1.1.37\text{a})$$

$$\tilde{\varphi} \in x^{\delta-1}\mathscr{A}^\delta_{\{x=0\}} + x^{\delta-1}y\mathscr{A}^\delta_{\{0\le x\le y\}} + \mathscr{C}^{<-1+\delta}_{\{0\le x\le y\},0} \qquad (1.1.37\text{b})$$

and from Proposition A.3.7 one obtains that

$$\psi \in x^\delta\mathscr{A}^\delta_{\{0\le x\le y\}} + y^\delta\mathscr{A}^\delta_{\{y=0\}} + \mathscr{A}^\delta_{\{y=0\}} + \mathscr{C}^{<0}_{\{0\le x\le y\},2} \qquad (1.1.38\text{a})$$

$$\varphi \in x^{\delta-1}\mathscr{A}^{\delta}_{\{x=0\}} + x^{\delta-1}y\mathscr{A}^{\delta}_{\{0\le x\le y\}} + \mathscr{C}^{<-1+\delta}_{\{0\le x\le y\},1}\,. \quad (1.1.38b)$$

Continuing this way, we obtain

$$\psi \in x^{\delta}\mathscr{A}^{\delta}_{\{0\le x\le y\}} + y^{\delta}\mathscr{A}^{\delta}_{\{y=0\}} + \mathscr{A}^{\delta}_{\{y=0\}} + \mathscr{C}^{<0}_{\{0\le x\le y\},\infty} \quad (1.1.39a)$$

$$\varphi \in x^{\delta-1}\mathscr{A}^{\delta}_{\{x=0\}} + x^{\delta-1}y\mathscr{A}^{\delta}_{\{0\le x\le y\}} + \mathscr{C}^{<-1+\delta}_{\{0\le x\le y\},\infty} \quad (1.1.39b)$$

and the proof is complete. □

To next step of the proof of the main theorem, is to show that for all integers k, we can decompose φ and ψ in the following way:

$$\varphi = \underbrace{\varphi_{1,k}}_{\in x^{\delta-1}\mathscr{A}^{\delta}_{\{x=0\}}+x^{\delta-1}y\mathscr{A}^{\delta}_{\{0\le x\le y\}}} + \underbrace{\varphi_{2,k}}_{\in \mathscr{A}^{\delta}_{\{x=0\}},\dot{\bigoplus}_i x^{i\delta}\mathring{\mathscr{F}}^{<k\delta-1+\delta-i\delta}_{\{0\le x\le y\},\infty}} + \underbrace{\varphi_{3,k}}_{\in \mathscr{T}^{<k\delta-1+\delta,(0;0)}_{\{0\le x\le y\},\infty}}\,, \quad (1.1.40)$$

$$\psi = \underbrace{\psi_{1,k}}_{\in \mathscr{A}^{\delta}_{\{y=0\}}+x^{\delta}\mathscr{A}^{\delta}_{\{0\le x\le y\}}} + \underbrace{\psi_{2,k}}_{\in \mathscr{A}^{\delta}_{\{x=0\}},\dot{\bigoplus}_i x^{i\delta}\mathring{\mathscr{F}}^{<k\delta-i\delta}_{\{0\le x\le y\},\infty}} + \underbrace{\psi_{3,k}}_{\in \mathscr{T}^{<k\delta,(0;0)}_{\{0\le x\le y\},\infty}}\,. \quad (1.1.41)$$

The embeddings

$$\mathscr{C}^{<0}_{\{0\le x\le y\},\infty} \subset \mathscr{T}^{<0,(0;0)}_{\{0\le x\le y\},\infty},\ \mathscr{C}^{<-1+\delta}_{\{0\le x\le y\},\infty} \subset \mathscr{T}^{<-1+\delta,(0;0)}_{\{0\le x\le y\},\infty}$$

and Equations (1.1.38) justify the case $k=0$ of the induction, where $\varphi_{2,0}=0$ and $\psi_{2,0}=0$.
We suppose now that (1.1.40) *and* (1.1.41) *hold for an integer k and we want to show that these decompositions also hold for k replaced by $k+1$ there . From* (1.1.40) *we have*

$$x\varphi \in x^{\delta}y\mathscr{A}^{\delta}_{\{0\le x\le y\}} + x^{\delta}\mathscr{A}^{\delta}_{\{x=0\}} + \mathscr{A}^{\delta}_{\{x=0\},\dot{\bigoplus}_i x^{i\delta+1}\mathring{\mathscr{F}}^{<k\delta-1+\delta-i\delta}_{\{0\le x\le y\},\infty}} + \mathscr{T}^{<k\delta+\delta,(0;0)}_{\{0\le x\le y\},\infty}\,.$$

Note that if we set $i\delta+1 =: j\delta$ then $-i\delta = +1-j\delta$ and the above equation can be written as

$$x\varphi \in x^{\delta}y\mathcal{A}^{\delta}_{\{0\le x\le y\}} + x^{\delta}\mathscr{A}^{\delta}_{\{x=0\}} + \mathscr{A}^{\delta}_{\{x=0\},\dot{\bigoplus}_i x^{i\delta}\mathring{\mathscr{F}}^{<(k+1)\delta-i\delta}_{\{0\le x\le y\},\infty}} + \mathscr{T}^{<(k+1)\delta,(0;0)}_{\{0\le x\le y\},\infty}\,. \quad (1.1.42)$$

It then follows from (a) of Lemma (A.4.3) that

$$\bar{f} = (\psi_1, x\psi_2, x\varphi) \in \mathscr{A}^{\delta}_{\{0\le x\le y\}} + \mathscr{A}^{\delta}_{\{x=0\},\dot{\bigoplus}_i x^{i\delta}\mathring{\mathscr{F}}^{<k\delta-i\delta}_{\{0\le x\le y\},\infty}} + \mathscr{T}^{<k\delta,(0;0)}_{\{0\le x\le y\},\infty}\,. \quad (1.1.43)$$

On the other hand, from (1.1.40) and (1.1.41)

$$f=(\psi_1,\psi_2,\varphi)\ \in x^{\delta-1}\mathscr{A}^{\delta}_{\{0\le x\le y\}}+\mathscr{A}^{\delta}_{\{x=0\},\dot{\bigoplus}_i x^{i\delta}\mathring{\mathscr{F}}^{<(k+1)\delta-1-i\delta}_{\{0\le x\le y\},\infty}}+\mathscr{T}^{<(k+1)\delta-1,(0;0)}_{\{0\le x\le y\},\infty}\ , \tag{1.1.44}$$

and since all the spaces in the above equation are invariant under $x\partial_x$, $x\partial_y$ *and* ∂_A, *we have:*

$$(x^2\partial_x f, x^2\partial_y f, x\partial_A f)\in x^{\delta}\mathscr{A}^{\delta}_{\{0\le x\le y\}}+\mathscr{A}^{\delta}_{\{x=0\},\dot{\bigoplus}_i x^{i\delta}\mathring{\mathscr{F}}^{<(k+1)\delta-i\delta}_{\{0\le x\le y\},\infty}}+\mathscr{T}^{<(k+1)\delta,(0;0)}_{\{0\le x\le y\},\infty}\ . \tag{1.1.45}$$

Thus, from (1.1.43) and (1.1.45), we have,

$$w\in\mathscr{A}^{\delta}_{\{0\le x\le y\}}+\mathscr{A}^{\delta}_{\{x=0\},\dot{\bigoplus}_i x^{i\delta}\mathring{\mathscr{F}}^{<k\delta-i\delta}_{\{0\le x\le y\},\infty}}+\mathscr{T}^{<k\delta,(0;0)}_{\{0\le x\le y\},\infty}\ . \tag{1.1.46}$$

It then follows from Lemma A.4.4 that

$$\begin{aligned}x^{-p_i\delta}H_i\psi(.,x^{q_i\delta}w)\in x^{(-p_i+q_im_i)\delta}\mathscr{A}^{\delta}_{\{0\le x\le y\}} &+ \mathscr{A}^{\delta}_{\{x=0\},\dot{\bigoplus}_j x^{(-p_i+q_im_i+j)\delta}\mathring{\mathscr{F}}^{<k\delta-j\delta}_{\{0\le x\le y\},\infty}}\\ &+ \mathscr{T}^{<(-p_i+q_im_i+k)\delta,(0;0)}_{\{0\le x\le y\},\infty}\ .\end{aligned} \tag{1.1.47}$$

Applying Propositions A.6.2, A.6.5 and A.6.7 we obtain the following:

$$\begin{aligned}I_1\big(x^{-p_i\delta}H_i\psi(.,x^{q_i\delta}w)\big)\ \in\ & y^{(m_iq_i-p_i)\delta+1}\mathscr{A}^{\delta}_{\{y=0\}}+x^{(m_iq_i-p_i)\delta+1}\mathscr{A}^{\delta}_{\{0\le x\le y\}}\\ &+\mathscr{A}^{\delta}_{\{x=0\},\dot{\bigoplus}_j x^{(j+m_iq_i-p_i)\delta+1}\mathring{\mathscr{F}}^{<k\delta-j\delta}_{\{0\le x\le y\},\infty}}\\ &+\mathscr{T}^{<(k+m_iq_i-p_i)\delta+1,(0;0)}_{\{0\le x\le y\},\infty}+y^{\epsilon}\mathring{\mathscr{F}}^{<(k+m_iq_i-p_i)\delta+1}_{\{0\le x\le y\},\infty}\\ &.\end{aligned} \tag{1.1.48}$$

Using again the inequality $(-p_i+q_im_i)\delta+1\ge\delta$, *one obtains:*

$$\begin{aligned}I_1\big(x^{-p_i\delta}H_i\psi(.,x^{q_i\delta}w)\big)\ \in\ & y^{\delta}\mathscr{A}^{\delta}_{\{y=0\}}+x^{\delta}\mathscr{A}^{\delta}_{\{0\le x\le y\}}+\mathscr{A}^{\delta}_{\{x=0\},\dot{\bigoplus}_i x^{(i+1)\delta}\mathring{\mathscr{F}}^{<k\delta-\epsilon-i\delta}_{\{0\le x\le y\},\infty}}\\ &+\mathscr{T}^{<(k+1)\delta,(0;0)}_{\{0\le x\le y\},\infty}+y^{\epsilon/2}\mathring{\mathscr{F}}^{<(k+1)\delta}_{\{0\le x\le y\},\infty}\ .\end{aligned}$$

The embedding $y^{\epsilon/2}\mathring{\mathscr{F}}^{(k+1)\delta-\epsilon}=y^{\epsilon/2}\mathring{\mathscr{F}}^{(k+1)\delta-\epsilon/2-\epsilon/2}\subset\mathring{\mathscr{F}}^{(k+1)\delta-\epsilon}$ *(see part (b) of Proposition A.4.3) gives:*

$$\begin{aligned}I_1\big(x^{-p_i\delta}H_i\psi(.,x^{q_i\delta}w)\big)\ \in\ & y^{\delta}\mathscr{A}^{\delta}_{\{y=0\}}+x^{\delta}\mathscr{A}^{\delta}_{\{0\le x\le y\}}+\mathscr{A}^{\delta}_{\{x=0\},\dot{\bigoplus}_i x^{i\delta}\mathring{\mathscr{F}}^{<(k+1)\delta-i\delta}_{\{0\le x\le y\},\infty}}\\ &+\mathscr{T}^{<(k+1)\delta,(0;0)}_{\{0\le x\le y\},\infty}\ .\end{aligned} \tag{1.1.49}$$

Recall that

$$\psi(x,v^A,y)=\psi_{0,phg}(x,v^A,y)+\int_y^x\{L_{21}\varphi+L_{22}\psi-B_{\psi\varphi}\varphi-B_{\psi\psi}\psi+G_\psi\}(s,v^A,y)ds\ ,$$

thus

$$\psi=\psi_{0,phg}+I_1(L_{\psi\varphi}\varphi+L_{\psi\psi}\psi-B_{\psi\varphi}\varphi_-B_{\psi\psi}\psi+G_\psi)\ .$$

Equation (1.1.49) shows that $I_1(G_\psi)$ *gives a contribution in* $\psi_{1,k+1}$, $\psi_{2,k+1}$ *and* $\psi_{3,k+1}$. *Therefore it remains to show that the same works for* $\psi_{0,phg}+I_1(L_{\psi\varphi}\varphi+L_{\psi\psi}\psi-B_{\psi\varphi}\varphi_-B_{\psi\psi}\psi)$. *We proceed exactly as in [19].*

Integrating the equation for ψ *and using Propositions A.6.2, A.6.5 and A.6.7 one finds*

$$\begin{aligned}\psi(x,v^A,y) &= \underbrace{\psi_{0,\mathrm{phg}}(x,v^A,y)+I_1\Big(L_{\psi\varphi}\varphi_{1,k}+L_{\psi\psi}\psi_{1,k}-B_{\psi\varphi}\varphi_{1,k}-B_{\psi\psi}\psi_{1,k}\Big)}_{=:\psi_{1,k+1}\in x^{\beta+1}\mathscr{A}^\delta_{\{0\le x\le y\}}+y^\beta\mathscr{A}^\delta_{\{0\le x\le y\}}}\\ &\quad+\underbrace{I_1\Big(L_{\psi\varphi}\varphi_{2,k}+L_{\psi\psi}\psi_{2,k}-B_{\psi\varphi}\varphi_{2,k}-B_{\psi\psi}\psi_{2,k}\Big)}_{\in\mathscr{A}^\delta_{\{x=0\},\dot\oplus_i x^{i\delta}\mathring{\mathscr{F}}^{k\delta+1-\epsilon-i\delta}_{\{0\le x\le y\},\infty}}\subset\mathscr{A}^\delta_{\{x=0\},\dot\oplus_i x^{i\delta}\mathring{\mathscr{F}}^{(k+1)\delta-\epsilon-i\delta}_{\{0\le x\le y\},\infty}}}\\ &\quad+\underbrace{I_1\Big(L_{\psi\varphi}\varphi_{3,k}+L_{\psi\psi}\psi_{3,k}-B_{\psi\varphi}\varphi_{3,k}-B_{\psi\psi}\psi_{3,k}\Big)}_{\in\mathring{\mathscr{F}}^{k\delta+1-\epsilon}_{\{0\le x\le y\},\infty}+\mathscr{T}^{k\delta+1-\epsilon,(0;0)}_{\{0\le x\le y\},\infty}\subset\mathring{\mathscr{F}}^{(k+1)\delta-\epsilon}_{\{0\le x\le y\},\infty}+\mathscr{T}^{(k+1)\delta-\epsilon,(0;0)}_{\{0\le x\le y\},\infty}}\quad , \qquad (1.1.50)\end{aligned}$$

showing that the result is true for ψ *with* k *replaced by* $k+1$. *Thus (1.1.41) holds for* k *replaced by* $k+1$,

$$i.e.\qquad \psi=\underbrace{\psi_{1,k+1}}_{\in\mathscr{A}^\delta_{\{y=0\}}+x^\delta\mathscr{A}^\delta_{\{0\le x\le y\}}}+\underbrace{\psi_{2,k+1}}_{\in\mathscr{A}^\delta_{\{x=0\},\dot\bigoplus_i x^{i\delta}\mathring{\mathscr{F}}^{<(k+1)\delta-i\delta}_{\{0\le x\le y\},\infty}}}+\underbrace{\psi_{3,k+1}}_{\in\mathscr{T}^{<(k+1)\delta,(0;0)}_{\{0\le x\le y\},\infty}}\ . \qquad (1.1.51)$$

Equations (1.1.42) and (1.1.51) show that, (1.1.43) holds with k replaced by k+1,

$$i.e.\qquad \bar f=(\psi_1,x\psi_2,x\varphi)\ \in\mathscr{A}^\delta_{\{0\le x\le y\}}+\mathscr{A}^\delta_{\{x=0\},\dot\bigoplus_i x^{i\delta}\mathring{\mathscr{F}}^{<(k+1)\delta-i\delta}_{\{0\le x\le y\},\infty}}+\mathscr{T}^{<(k+1)\delta,(0;0)}_{\{0\le x\le y\},\infty}. \qquad (1.1.52)$$

(1.1.45) and (1.1.52) show that (1.1.46) holds for k *replaced by* $k+1$

$$i.e.\qquad w\in\mathscr{A}^\delta_{\{0\le x\le y\}}+\mathscr{A}^\delta_{\{x=0\},\dot\bigoplus_i x^{i\delta}\mathring{\mathscr{F}}^{<(k+)\delta-i\delta}_{\{0\le x\le y\},\infty}}+\mathscr{T}^{<(k+1)\delta,(0;0)}_{\{0\le x\le y\},\infty}. \qquad (1.1.53)$$

We want now to show that we can obtain (1.1.40) with k replaced by k + 1 there. Recall (1.1.25):

$$\varphi(x, v^A, y) = R(x, v^A; y, x)\varphi(x, v^A, x) + \int_x^y R(x, v^A; s, x)c_\varphi(x, v^A, s)ds,$$

i.e.

$$\varphi = R\mathring{\varphi} + I_2[R(L_{\varphi\varphi}\varphi + L_{\varphi\psi}\psi + a + G_\varphi - B^\delta_{\varphi\varphi}\varphi - B_{\varphi\psi}\psi)] \, . \qquad (1.1.54)$$

R stands for the family of resolvent (smooth up to boundary in all variables) of the family of ODE's $\partial_y\varphi(x, v^A; y) = \mathring{B}_{\varphi\varphi}(x, v^A; y)\varphi(x, v^A; y)$*, with parameters* (x, v^A) *and with initial value at* y_1*.*

•*From Proposition A.6.2,* $I_2(Ra) \in x^\delta \mathscr{A}^\delta_{\{x=0\}} + x^{-1+\delta} y \mathscr{A}^\delta_{\{0\le x\le y\}}$ *and since* $R\mathring{\varphi}$ *is in* $x^{-1+\delta}\mathscr{A}^\delta_{\{x=0\}}$*, the term* $R\mathring{\varphi} + I_2(Ra)$ *will give a contribution in* $\varphi_{1,k+1}$*.*

• *As in the previous case, the analysis of the term* $I_2(L_{\varphi\varphi}\varphi + L_{\varphi\psi}\psi - B_{\varphi\psi}\varphi - B_{\varphi\psi}\psi)$ *will be made as in the linear case in [19]:*

Inserting (1.1.53) into (1.1.54) one similarly finds, using Propositions A.6.2 and A.6.8, that (1.1.40) with k replaced by k + 1 holds for φ *:*

$$\begin{aligned}\varphi(x, v^A, y) \;=\; & \underbrace{\varphi_{0,\mathrm{phg}}(x, v^A, y) + I_2\Big[R\cdot\big(L_{\varphi\varphi}\varphi_{1,k} + L_{\varphi\psi}\psi_{1,k} - B^\delta_{\varphi\varphi}\varphi_{1,k} - B_{\varphi\psi}\psi_{1,k}\big)\Big]}_{=:\varphi_{1,k+1}\in x^\beta\mathscr{A}^\delta_{\{0\le x\le y\}}+y^\beta\mathscr{A}^\delta_{\{0\le x\le y\}}} \\ & + \underbrace{I_2\Big[R\cdot\big(L_{\varphi\varphi}\varphi_{2,k} + L_{\varphi\psi}\psi_{2,k} - B^\delta_{\varphi\varphi}\varphi_{2,k} - B_{\varphi\psi}\psi_{2,k}\big)\Big]}_{\in\mathscr{A}^\delta_{\{x=0\},\oplus_i x^{i\delta}\mathring{\mathscr{F}}^{(k+1)\delta-i\delta}_{\{0\le x\le y\},\infty}}} \\ & + \underbrace{I_2\Big[R\cdot\big(\underbrace{L_{\varphi\varphi}\varphi_{3,k} - B^\delta_{\varphi\varphi}\varphi_{3,k} + L_{\varphi\psi}\psi_{3,k} - B_{\varphi\psi}\psi_{3,k}}_{\in\mathscr{T}^{(k+1)\delta-\epsilon,(0;0)}_{\{0\le x\le y\},\infty}}\big)\Big]}_{\in\mathscr{T}^{(k+1)\delta-\epsilon,(1;1)}_{\{0\le x\le y\},\infty}\subset\mathscr{T}^{(k+1)\delta-\epsilon,(0;0)}_{\{0\le x\le y\},\infty}} \, . \qquad (1.1.55)\end{aligned}$$

• *If we use again Lemma A.4.4 with 1.1.53 then, we obtain that:*

$$\begin{aligned}x^{-p_i\delta}H_i\varphi(.,x^{q_i\delta}w) \in x^{(-p_i+q_im_i)\delta}\mathscr{A}^\delta_{\{0\le x\le y\}} \;&+\; \mathscr{A}^\delta_{\{x=0\},\bigoplus_j x^{(-p_i+q_im_i+j)\delta}\mathring{\mathscr{F}}^{<(k+1)\delta-j\delta}_{\{0\le x\le y\},\infty}} \\ &+\; \mathscr{T}^{<(-p_i+q_im_i+1+k)\delta,(0;0)}_{\{0\le x\le y\},\infty} \, . \qquad (1.1.56)\end{aligned}$$

Applying now Propositions A.6.2 and A.6.8 we obtain:

$$I_2\big(x^{-p_i\delta}H_i\varphi(.,x^{q_i\delta}w)\big) \in x^{(m_iq_i-p_i)\delta+1}\mathscr{A}^{\delta}_{\{x=0\}}+x^{(m_iq_i-p_i)\delta}y\mathscr{A}^{\delta}_{\{0\le x\le y\}} +\mathscr{A}^{\delta}_{\{x=0\},\dot{\bigoplus}_j x^{(j+m_iq_i-p_i)\delta}\mathring{\mathscr{F}}^{<(k+1)\delta-j\delta}_{\{0\le x\le y\},\infty}} +\mathscr{T}^{<(k+1+m_iq_i-p_i)\delta,(1;1)}_{\{0\le x\le y\},\infty}\,. \tag{1.1.57}$$

Using again the inequality $(-p_i+q_im_i)\delta+1\ge\delta$*, one obtains:*

$$I_2\big(x^{-p_i\delta}H_i\varphi(.,x^{q_i\delta}w)\big) \in x^{\delta}\mathscr{A}^{\delta}_{\{x=0\}}+x^{-1+\delta}y\mathscr{A}^{\delta}_{\{0\le x\le y\}} +\mathscr{A}^{\delta}_{\{x=0\},\dot{\bigoplus}_i x^{i\delta-1+\delta}\mathring{\mathscr{F}}^{<(k+1)\delta-i\delta}_{\{0\le x\le y\},\infty}}+\mathscr{T}^{<(k+1)\delta-1+\delta,(0;0)}_{\{0\le x\le y\},\infty}\,.$$

This shows that $I_2(G_\varphi)$ *will give a contribution in* $\varphi_{1,k+1}$, $\varphi_{2,k+1}$ *and* $\varphi_{3,k+1}$ *, thus* (1.1.40) *holds for* k *replaced by* $k+1$ *and the induction step is completed. We then conclude that* (1.1.40) *and* (1.1.41) *hold for all integer* k*. To end the proof of the main theorem, for any* $m\in\mathbb{N}$*, we can choose* k *large enough so that the last terms in* (1.1.40) *and* (1.1.41) *are in* $C_m(\overline{\mathscr{U}})$*, and that all the coefficients of an expansion of* $f_{2,k}$ *in terms of powers of* x *and* $\ln x$ *are also in* $C_m(\overline{\mathscr{U}})$*. The result follows now by an application of Proposition A.3.6. Thus we have proved*

$$\varphi\in x^{\delta-1}\mathscr{A}^{\delta}_{\{x=0\}}+x^{\delta-1}y\mathscr{A}^{\delta}_{\{0\le x\le y\}}+\mathscr{A}^{\delta}_{\{x=0\}}\subset x^{\delta-1}\mathscr{A}^{\delta}_{\{x=0\}}+x^{\delta-1}y\mathscr{A}^{\delta}_{\{0\le x\le y\}}$$

and

$$\psi\in\mathscr{A}^{\delta}_{\{y=0\}}+x^{\delta}\mathscr{A}^{\delta}_{\{0\le x\le y\}}+\mathscr{A}^{\delta}_{\{x=0\}}\,.$$

□

1.2 Propagation of the polyhomogeneity for the Einstein-Maxwell equations

Let us show that Theorem 1.1.1 applies to the source-free Einstein-Maxwell equations; we will make extensive appeal to [9]. More generally, consider a system of second order wave equations of the form

$$\eta^{\alpha\beta}\frac{\partial^2 f}{\partial x^\alpha\partial x^\beta}=-H^{\alpha\beta}(x^\mu,f,\partial f,\partial\partial f)\frac{\partial^2 f}{\partial x^\alpha\partial x^\beta}+F(f,\partial f,x^\mu)\,, \tag{1.2.1}$$

for a map f *with values in* $\mathbb{R}^N$ *for some* N*, where* η *is the* $(n+1)$*–dimensional Minkowski metric. (The map* f *in this section should* not *be confused with* *the map f appearing in (1.1.1), compare (1.2.37) below.) The Einstein-Maxwell equations in the harmonic-Lorenz gauge can be written in this form, with $f := (g_{\mu\nu} - \eta_{\mu\nu}, A_\mu)$, then*

$$H^{\mu\nu} := g^{\mu\nu} - \eta^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}(g_{\alpha\beta} - \eta_{\alpha\beta}) + \textit{ quadratic terms}$$

depends only upon $g_{\mu\nu} - \eta_{\mu\nu}$, while F is a quadratic form in ∂f with coefficients depending upon $g_{\mu\nu} - \eta_{\mu\nu}$; see [37, 38, 40]. Thus, in the Einstein-Maxwell case the source function F has a uniform zero of order two, while the functions $H^{\mu\nu}$ all have a uniform zero of order one.

1.2.1 Gauge transformation and its properties

As in [9] (in that reference one works within $I^+_{\eta,x}(0)$, while in [20] the complement of $I^+_{\eta,x}(0)$ is considered. However, the methods of [20] apply to both situations), and similarly to [20], we use a mapping $\phi : x \mapsto y$ from the future timelike cone with vertex 0, $I^+_{\eta,x}(0)$, of a Minkowski space-time, which we denote $(\mathbb{R}^{n+1}_x, \eta_x)$, into the past timelike cone with vertex 0 of another Minkowski space-time, $(\mathbb{R}^{n+1}_y, \eta_y)$, defined by

$$\phi : I^+_{\eta,x}(0) \to \mathbb{R}^{n+1}_y \quad \textit{by} \;\; x^\alpha \mapsto y^\alpha := \frac{x^\alpha}{\eta_{\lambda\mu}x^\lambda x^\mu} \;. \tag{1.2.2}$$

We have the following

Proposition 1.2.1 *The map ϕ is a bijection from $I^+_{\eta,x}(0)$ onto $I^-_{y,\eta}(0)$, with inverse*

$$\phi^{-1} : y^\alpha \mapsto x^\alpha \quad \textit{by} \quad x^\alpha := \frac{y^\alpha}{\eta_{\lambda\mu}y^\lambda y^\mu} \;. \tag{1.2.3}$$

Proof: *Let $(x^\alpha) \in I^+_{\eta,x}(0)$ and set $(y^\alpha) = \phi(x^\alpha)$. We have* $\begin{cases} \eta_{\alpha\beta}x^\alpha x^\beta < 0 \\ x^0 > 0 \\ y^\mu = \frac{x^\mu}{\eta_{\alpha\beta}x^\alpha x^\beta} \end{cases}$. *This implies that $(\eta_{\alpha\beta}x^\alpha x^\beta)(\eta_{\mu\nu}y^\mu y^\nu) = 1$, and then $\eta_{\mu\nu}y^\mu y^\nu < 0$. Therefore $y^\alpha = \frac{x^\alpha}{\eta_{\lambda\mu}x^\lambda x^\mu}$ implies* $\begin{cases} \eta_{\mu\nu}y^\mu y^\nu < 0 \\ x^\alpha = y^\alpha(\eta_{\lambda\mu}x^\lambda x^\mu) = \frac{y^\alpha}{\eta_{\lambda\mu}y\lambda y^\mu} \end{cases}$. *On the other hand, $x^0 = \frac{y^0}{\eta_{\lambda\mu}y\lambda y^\mu} > 0$and $\eta_{\mu\nu}y^\mu y^\nu < 0$imply that $y^0 < 0$. Thus $(y^\alpha) \in I^-_{y,\eta}(0)$ and then ϕ is a bijection from $I^+_{\eta,x}(0)$ onto $I^-_{y,\eta}(0)$ with inverse given by (1.2.3).* □

Remark 1.2.2 We emphasize on the fact that the transformation ϕ maps the hyperboloid

$$\mathcal{H}_t = \left\{ (x^\alpha) \in \mathbb{R}^{n+1}_{x,\eta} / x^0 - t = \sqrt{t^2 + r^2} \right\}$$

onto

$$\phi(\mathcal{H}_t) = \left\{ (y^\alpha) \in \mathbb{R}^{n+1}_{y,\eta} / y^0 = -\frac{1}{2t} \right\} \cap I^-_{y,\eta}(0) \ .$$

Here t is a non-negative parameter.

We also have the following

Proposition 1.2.3 *The map ϕ is a conformal mapping between Minkowski metrics:*

$$\eta_{\alpha\beta} dx^\alpha dx^\beta = \Omega^{-2} \eta_{\alpha\beta} dy^\alpha dy^\beta \ , \tag{1.2.4}$$

where Ω is a function defined on all $\mathbb{R}^{n+1}_y$, given by

$$\Omega := -\eta_{\alpha\beta} y^\alpha y^\beta \ . \tag{1.2.5}$$

Proof: *We have*

$$\begin{aligned} dx^\alpha = d\Big(\frac{y^\alpha}{\eta_{\lambda\mu} y^\lambda y^\mu} \Big) &= \frac{\partial}{\partial y^\gamma} \Big(\frac{y^\alpha}{\eta_{\lambda\mu} y^\lambda y^\mu} \Big) dy^\gamma \\ &= \frac{-\delta^\alpha_\gamma \Omega - 2 y^\alpha \eta_{\lambda\mu} \delta^\lambda_\gamma y^\mu}{\Omega^2} dy^\gamma . \end{aligned}$$

Thus,

$$\begin{aligned} \eta_{\alpha\beta} dx^\alpha dx^\beta &= \frac{\eta_{\alpha\beta} \big(-\delta^\alpha_\gamma \Omega - 2 y^\alpha \eta_{\lambda\mu} \delta^\lambda_\gamma y^\mu \big) \big(-\delta^\beta_\tau \Omega - 2 y^\beta \eta_{\sigma\theta} \delta^\sigma_\tau y^\theta \big)}{\Omega^4} dy^\gamma dy^\tau \\ &= \Omega^{-2} \eta_{\alpha\beta} dy^\alpha dy^\beta . \end{aligned}$$

□

We work within $I^-_{y,\eta}(0)$ and to the future of a hypersurface

$$\mathscr{S}_{\tau_0} := \{ y^0 = \tau_0 \} \ , \quad \tau_0 < 0 \ ,$$

where we set

$$\rho \equiv |\vec{y}| := \sqrt{\sum_{i=1}^n (y^i)^2} \ , \quad x := -|\vec{y}| - y^0 \geq 0 \ , \quad y := y^0 - |\vec{y}| + 1 \geq 0 \ ,$$

so that we have the following

Proposition 1.2.4 *We have the following identities*

$$\Omega = x(1-y)\ , \quad \partial_x = -\frac{1}{2}\Big(\sum_{i=1}^{n}\frac{y^i}{|\vec{y}|}\frac{\partial}{\partial y^i}+\frac{\partial}{\partial y^0}\Big)\ , \quad \partial_y = -\frac{1}{2}\Big(\sum_{i=1}^{n}\frac{y^i}{|\vec{y}|}\frac{\partial}{\partial y^i}-\frac{\partial}{\partial y^0}\Big)\ ,$$

and

$$y^\alpha\frac{\partial}{\partial y^\alpha} = (y-1)\partial_y + x\partial_x\ . \tag{1.2.6}$$

Proof:

$$\Omega = -\eta_{\alpha\beta}y^\alpha y^\beta = (y^0)^2 - \sum_{i=1}^{n}(y^i)^2 = (y^0)^2 - \rho^2 = x(1-y)\ .$$

We introduce here the spherical coordinate on the sphere S^{n-1}: $v^1,\ v^2,\ldots,\ v^{n-1}$ and we set:

$$\left\{\begin{array}{c} y^0 = \frac{1}{2}(y-x-1) \\ y^i = \frac{1}{2}(1-y-x)\omega^i(v^A),\ \forall i\in\{1,\ldots,n\}\end{array}\right. \tag{1.2.7}$$

Note that,

$$\sum_{i=i}^{n}(\omega^i)^2 = 1 \quad \textit{and} \quad \sum_{i=i}^{n}\omega^i d\omega^i = 0 \textit{ on } S^{n-1}. \tag{1.2.8}$$

We have:

$$\frac{\partial}{\partial y} = \frac{\partial y^\alpha}{\partial y}\frac{\partial}{\partial y^\alpha} = \sum_{i=1}^{n}\frac{\partial y^i}{\partial y}\frac{\partial}{\partial y^i}+\frac{\partial y^0}{\partial y}\frac{\partial}{\partial y^0}, \qquad \frac{\partial}{\partial x} = \frac{\partial y^\alpha}{\partial x}\frac{\partial}{\partial y^\alpha} = \sum_{i=1}^{n}\frac{\partial y^i}{\partial x}\frac{\partial}{\partial y^i}+\frac{\partial y^0}{\partial x}\frac{\partial}{\partial y^0}$$

and since

$$\frac{\partial y^0}{\partial y} = \frac{1}{2}, \quad \frac{\partial y^i}{\partial y} = -\frac{1}{2}\omega^i = -\frac{1}{2}\frac{y^i}{|\overrightarrow{y}|}, \qquad \frac{\partial y^0}{\partial x} = -\frac{1}{2}, \quad \frac{\partial y^i}{\partial x} = -\frac{1}{2}\omega^i = -\frac{1}{2}\frac{y^i}{|\overrightarrow{y}|}\ ,$$

the second and third identities follow.
Finally, from $y-1 = y^0 - |\vec{y}|$ and $x = -y^0 - |\vec{y}|$ we have:

$$\left\{\begin{array}{rl} (y-1)\frac{\partial}{\partial y} &= \frac{1}{2}\Big(\sum\limits_{i=1}^{n}\frac{(-y^0+|\vec{y}|)y^i}{|\vec{y}|}\frac{\partial}{\partial y^i} - (-y^0+|\vec{y}|)\frac{\partial}{\partial y^0}\Big) \\ x\frac{\partial}{\partial x} &= \frac{1}{2}\Big(\sum\limits_{i=1}^{n}\frac{(y^0+|\vec{y}|)y^i}{|\vec{y}|}\frac{\partial}{\partial y^i} + (y^0+|\vec{y}|)\frac{\partial}{\partial y^0}\Big)\end{array}\right.$$

and obtain that, $\quad x\partial_x + (y-1)\partial_y = y^\alpha\frac{\partial}{\partial y^\alpha}$ □

Furthermore we have

Proposition 1.2.5 *The flat d'Alembertian* $\Box_{\eta,y}$ *associated with the coordinates* y^μ *equals*

$$\Box_{\eta,y} = 4\partial_x\partial_y - \frac{2(n-1)}{1-x-y}\Big(\partial_x + \partial_y\Big) + \frac{4\Delta_h}{(1-x-y)^2}\ ,$$

where Δ_h *is the canonical Laplacian on* S^{n-1}.

Proof:

$$\Box_{\eta,y} = \frac{1}{\sqrt{|\eta|}}\partial_\mu\big(\sqrt{|\eta|}\eta^{\mu\nu}\partial_\nu\big) = -\frac{\partial^2}{(\partial y^0)^2} + \sum_{i=1}^{n}\frac{\partial^2}{(\partial y^i)^2}. \tag{1.2.9}$$

From (1.2.7), we have,

$$dy^0 = \frac{1}{2}(dy - dx) \quad \text{and} \quad dy^i = -\frac{1}{2}\omega^i(dy+dx) + \frac{1}{2}(1-x-y)d\omega^i$$

thus,

$$(dy^0)^2 = \frac{1}{4}\big((dy)^2 + (dx)^2 - dx\otimes dy - dy\otimes dx\big)$$

and

$$\begin{aligned}(dy^i)^2 \quad &= \quad \frac{1}{4}(\omega^i)^2\big((dy)^2 + (dx)^2 + dx\otimes dy + dy\otimes dx\big)\\ &\quad +\frac{1}{4}(1-x-y)^2 d(\omega^i)^2 - \frac{1}{4}\omega^i(1-x-y)(dx+dy)\otimes(d\omega^i)\\ &\quad +\frac{1}{4}\omega^i(1-x-y)(d\omega^i)\otimes(dx+dy)\end{aligned}$$

and then,

$$-(dy^0)^2 + \sum_{i=1}^{n}(dy^i)^2 = \frac{1}{2}(dx\otimes dy + dy\otimes dx) + \frac{1}{4}\sum_{i=1}^{n}(1-x-y)^2 d(\omega^i)^2 \tag{1.2.10}$$

From this equation we deduce that, if we write $\widetilde{\eta} = \widetilde{\eta}_{\alpha\beta}d\xi^\alpha\xi^\beta$ *with* $(\xi^\alpha) = (x, y, v^A)$ *then*

$$\Big(\widetilde{\eta}_{\alpha\beta}\Big) = \begin{pmatrix} 0 & \frac{1}{2} & 0 & & \dots & & 0\\ \frac{1}{2} & 0 & 0 & & \dots & & 0\\ 0 & 0 & & & & & \\ \vdots & \vdots & & \frac{1}{4}(1-x-y)^2 h_{AB} & & & \\ 0 & 0 & & & & & \end{pmatrix} \tag{1.2.11}$$

where the h_{AB}'s are the components of the canonical metric on S^{n-1}. From the above equality we deduce that:

$$det\Big(\widetilde{\eta}_{\alpha\beta}\Big) = -\frac{1}{4}\Big(\frac{(1-x-y)^2}{4}\Big)^{n-1} det(h_{AB}) \; i.e. \; \sqrt{|det\widetilde{\eta}_{\alpha\beta}|} = \frac{1}{2}\Big(\frac{1-x-y}{2}\Big)^{n-1}\sqrt{|det(h_{AB})|}$$

and

$$\Big(\widetilde{\eta}^{\alpha\beta}\Big) = \begin{pmatrix} 0 & 2 & 0 & \dots & 0 \\ 2 & 0 & 0 & \dots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & \Big(\frac{4}{(1-x-y)^2}\Big)h^{AB} & \\ 0 & 0 & & & \end{pmatrix}. \tag{1.2.12}$$

Therefore,

$$\begin{aligned} \Box_{\eta,y} &= \frac{1}{\sqrt{|\widetilde{\eta}_{\alpha\beta}|}}\partial_\mu\big(\sqrt{|\widetilde{\eta}_{\alpha\beta}|}\;\widetilde{\eta}^{\mu\nu}\partial_\nu\big) \\ &= 2\Big(\frac{2}{1-x-y}\Big)^{n-1}\frac{1}{\sqrt{|h_{AB}|}}\partial_\mu\Big(\frac{1}{2}\big(\frac{1-x-y}{2}\big)^{n-1}\sqrt{|h_{AB}|}\;\widetilde{\eta}^{\mu\nu}\partial_\nu\Big) \\ &= \frac{1}{(1-x-y)^{n-1}\sqrt{|h_{AB}|}}\partial_\mu\Big((1-x-y)^{n-1}\sqrt{|h_{AB}|}\;\widetilde{\eta}^{\mu\nu}\partial_\nu\Big) \\ &= 4\partial_x\partial_y - \frac{2(n-1)}{1-x-y}\Big(\partial_x+\partial_y\Big) + \frac{4\Delta_h}{(1-x-y)^2}\;, \end{aligned} \tag{1.2.13}$$

where Δ_h is the canonical Laplacian on $\mathbb{S}^{n-1}$. It should be kept in mind that we are interested in x small and y bounded away from one. □

We point out the following

Proposition 1.2.6 *For any function f sufficiently differentiable on a neighborhood of $I^+_{\eta,x}(0)$, we have the following identities*

$$\frac{\partial f}{\partial x^\mu}\circ\phi^{-1} = \Big(-\Omega\frac{\partial}{\partial y^\mu} - 2\eta_{\mu\alpha}y^\alpha((y-1)\frac{\partial}{\partial y} + x\frac{\partial}{\partial x})\Big)f\circ\phi^{-1}\;, \tag{1.2.14}$$

and

$$\begin{aligned} \frac{\partial^2 f}{\partial x^\lambda\partial x^\mu}\circ\phi^{-1} = &\Big\{x^2(1-y)^2\frac{\partial^2}{\partial y^\lambda y^\mu} + 4\eta_{\mu\sigma}\eta_{\lambda\theta}y^\sigma y^\theta((y-1)\partial_y + x\partial_x)^2 \\ &+4x(1-y)\eta_{\theta(\lambda}y^\theta\frac{\partial}{\partial y^{\mu)}}\big((y-1)\partial_y+x\partial_x\big) \\ &+2\big(\Omega\eta_{\mu\lambda}+2y_\lambda y_\mu\big)\big((y-1)\partial_y+x\partial_x\big)\Big\}f\circ\phi^{-1}. \end{aligned} \tag{1.2.15}$$

Proof: *We have*

$$\frac{\partial f}{\partial x^\mu}\circ\phi^{-1} = \frac{\partial(f\circ\phi^{-1})}{\partial y^\alpha}A^\alpha_\mu \quad \textit{where} \quad A^\alpha_\mu := \frac{\partial y^\alpha}{\partial x^\mu}\circ\phi^{-1}$$

$$\frac{\partial y^\alpha}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}\Big(\frac{x^\alpha}{\eta_{\lambda\sigma}x^\lambda x^\sigma}\Big) = \frac{\delta^\alpha_\mu}{\eta_{\lambda\sigma}x^\lambda x^\sigma} - 2\frac{\eta_{\mu\sigma}x^\alpha x^\sigma}{(\eta_{\lambda\sigma}x^\lambda x^\sigma)^2}\ .$$

Thus,

$$A^\alpha_\mu = -\Omega\delta^\alpha_\mu - 2\eta_{\mu\sigma}y^\alpha y^\sigma \tag{1.2.16}$$

and we obtain (1.2.14) from (1.2.6). We emphasize the occurrence of a factor of x in front of each derivative except ∂_y. On the other hand,

$$\begin{aligned}\frac{\partial^2 f}{\partial x^\lambda\partial x^\mu} &= \frac{\partial}{\partial x^\lambda}\left(\frac{\partial f}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^\mu}\right) = \frac{\partial}{\partial x^\lambda}\left(\frac{\partial f}{\partial y^\alpha}\right)\frac{\partial y^\alpha}{\partial x^\mu} + \frac{\partial f}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^\mu\partial x^\lambda}\\ &= \frac{\partial^2 f}{\partial y^\alpha\partial y^\beta}\frac{\partial y^\beta}{\partial x^\lambda}\frac{\partial y^\alpha}{\partial x^\mu} + \frac{\partial f}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^\mu\partial x^\lambda}\end{aligned}$$

and then,

$$\frac{\partial^2 f}{\partial x^\lambda\partial x^\mu}\circ\phi^{-1} = \frac{\partial^2(f\circ\phi^{-1})}{\partial y^\alpha y^\beta}\frac{\partial y^\beta}{\partial x^\lambda}\circ\phi^{-1}\frac{\partial y^\alpha}{\partial x^\mu}\circ\phi^{-1} + \frac{\partial(f\circ\phi^{-1})}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^\mu\partial x^\lambda}\circ\phi^{-1}$$

i.e.

$$\frac{\partial^2 f}{\partial x^\lambda\partial x^\mu}\circ\phi^{-1} = \frac{\partial^2(f\circ\phi^{-1})}{\partial y^\alpha\partial y^\beta}A^\alpha_\mu A^\beta_\lambda + \frac{\partial(f\circ\phi^{-1})}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^\mu\partial x^\lambda}\circ\phi^{-1}\ .$$

Next

$$\begin{aligned}\frac{\partial^2 y^\alpha}{\partial x^\mu\partial x^\lambda} &= \frac{\partial}{\partial x^\lambda}\left(\frac{\partial y^\alpha}{\partial x^\mu}\right)\\ &= \frac{\partial}{\partial x^\lambda}\left(\frac{\delta^\alpha_\mu}{\eta_{\theta\sigma}x^\theta x^\sigma} - 2\frac{\eta_{\mu\tau}x^\alpha x^\tau}{(\eta_{\theta\sigma}x^\theta x^\sigma)^2}\right)\\ &= \frac{-2\delta^\alpha_\mu\delta^\theta_\lambda\eta_{\theta\sigma}x^\sigma}{K^2} - 2\frac{(\eta_{\mu\tau}\delta^\alpha_\lambda x^\tau + \eta_{\mu\tau}\delta^\tau_\lambda x^\alpha)K^2 - 4(\eta_{\theta\sigma}\delta^\theta_\lambda x^\sigma)(\eta_{\mu\tau}x^\alpha x^\tau)K}{K^4}\end{aligned}$$

where $K = \eta_{\theta\sigma}x^\theta x^\sigma = -\frac{1}{\Omega}$, *we thus deduce that,*

$$\begin{aligned}\frac{\partial^2 y^\alpha}{\partial x^\mu\partial x^\lambda}\circ\phi^{-1} &= 2\Omega\delta^\alpha_\mu\eta_{\lambda\sigma}y^\sigma + 2\Omega\delta^\alpha_\lambda\eta_{\mu\tau}y^\tau + 2\Omega\eta_{\mu\lambda}y^\alpha + 8\eta_{\lambda\sigma}\eta_{\mu\tau}y^\sigma y^\alpha y^\tau\\ &= 4\Omega\delta^\alpha_{(\mu}\eta_{\lambda)\sigma}y^\sigma + 2\Omega\eta_{\mu\lambda}y^\alpha + 8\eta_{\lambda\sigma}\eta_{\mu\tau}y^\sigma y^\alpha y^\tau \qquad (1.2.17)\end{aligned}$$

and then,

$$\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\lambda}\circ\phi^{-1}\frac{\partial}{\partial y^\alpha} = 4\Omega\delta^\alpha_{(\mu}\eta_{\lambda)\sigma}y^\sigma\frac{\partial}{\partial y^\alpha} + 2\Omega\eta_{\mu\lambda}y^\alpha\frac{\partial}{\partial y^\alpha} + 8\eta_{\lambda\sigma}\eta_{\mu\tau}y^\sigma y^\alpha y^\tau\frac{\partial}{\partial y^\alpha}$$

and since $y^\alpha\frac{\partial}{\partial y^\alpha} = (y-1)\partial_y + x\partial_x$ *one obtains:*

$$\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\lambda}\circ\phi^{-1}\frac{\partial}{\partial y^\alpha} = 4\Omega\delta^\alpha_{(\mu}\eta_{\lambda)\sigma}y^\sigma\frac{\partial}{\partial y^\alpha} + 2\Omega\eta_{\mu\lambda}y^\alpha\frac{\partial}{\partial y^\alpha} + 8y_\lambda y_\mu\left((y-1)\partial_y + x\partial_x\right) \tag{1.2.18}$$

On the other hand

$$\begin{aligned} A^\alpha_\mu A^\beta_\lambda &= (-\Omega\delta^\alpha_\mu - 2\eta_{\mu\sigma}y^\alpha y^\sigma)(-\Omega\delta^\beta_\lambda - 2\eta_{\lambda\theta}y^\beta y^\theta) \\ &= \Omega^2\delta^\alpha_\mu\delta^\beta_\lambda + 4\eta_{\mu\sigma}\eta_{\lambda\theta}y^\sigma y^\theta y^\alpha y^\beta + 2\Omega(\delta^\alpha_\mu\eta_{\lambda\theta}y^\beta y^\theta + \delta^\beta_\lambda\eta_{\mu\sigma}y^\alpha y^\sigma) \end{aligned}$$

This allows us to write,

$$\begin{aligned} A^\alpha_\mu A^\beta_\lambda\frac{\partial^2}{\partial y^\alpha y^\beta} &= \Omega^2\delta^\alpha_\mu\delta^\beta_\lambda\frac{\partial^2}{\partial y^\alpha y^\beta} + 4\eta_{\mu\sigma}\eta_{\lambda\theta}y^\sigma y^\theta y^\alpha y^\beta\frac{\partial^2}{\partial y^\alpha y^\beta} \\ &\quad + 2\Omega(\delta^\alpha_\mu\eta_{\lambda\theta}y^\beta y^\theta + \delta^\beta_\lambda\eta_{\mu\sigma}y^\alpha y^\sigma)\frac{\partial^2}{\partial y^\alpha y^\beta} \\ &= A + B + C + D \end{aligned}$$

$$A = \Omega^2\frac{\partial^2}{\partial y^\lambda y^\mu}$$

$$\begin{aligned} B = 4\eta_{\mu\sigma}\eta_{\lambda\theta}y^\sigma y^\theta y^\alpha y^\beta\frac{\partial^2}{\partial y^\alpha y^\beta} &= 4\eta_{\mu\sigma}\eta_{\lambda\theta}y^\sigma y^\theta y^\alpha\frac{\partial}{\partial y^\alpha}\big(y^\beta\frac{\partial}{\partial y^\beta}\big) - 4\eta_{\mu\sigma}\eta_{\lambda\theta}\delta^\beta_\alpha y^\sigma y^\theta y^\alpha\frac{\partial}{\partial y^\beta} \\ &= 4\eta_{\mu\sigma}\eta_{\lambda\theta}y^\sigma y^\theta y^\alpha\frac{\partial}{\partial y^\alpha}\big(y^\beta\frac{\partial}{\partial y^\beta}\big) - 4\eta_{\mu\sigma}\eta_{\lambda\theta}y^\sigma y^\theta y^\beta\frac{\partial}{\partial y^\beta} \\ &= 4\eta_{\mu\sigma}\eta_{\lambda\theta}y^\sigma y^\theta((y-1)\partial_y + x\partial_x)^2 - 4\eta_{\mu\sigma}\eta_{\lambda\theta}y^\sigma y^\theta((y-1)\partial_y + x\partial_x) \end{aligned}$$

$$\begin{aligned} C &= 2\Omega\delta^\alpha_\mu\eta_{\lambda\theta}y^\beta y^\theta\frac{\partial^2}{\partial y^\alpha y^\beta} \\ &= 2\Omega\delta^\alpha_\mu\eta_{\lambda\theta}y^\theta\frac{\partial}{\partial y^\alpha}\big(y^\beta\frac{\partial}{\partial y^\beta}\big) - 2\Omega\delta^\alpha_\mu\delta^\beta_\alpha\eta_{\lambda\theta}y^\theta\frac{\partial}{\partial y^\beta} \\ &= 2\Omega\eta_{\lambda\theta}y^\theta\frac{\partial}{\partial y^\mu}\big((y-1)\partial_y + x\partial_x\big) - 2\Omega\eta_{\lambda\theta}y^\theta\frac{\partial}{\partial y^\mu}. \end{aligned} \tag{1.2.19}$$

Similarly

$$D = 2\Omega\delta^{\beta}_{\lambda}\eta_{\mu\sigma}y^{\alpha}y^{\sigma}\frac{\partial^2}{\partial y^{\alpha}y^{\beta}} = 2\Omega\eta_{\mu\theta}y^{\theta}\frac{\partial}{\partial y^{\lambda}}\big((y-1)\partial_y + x\partial_x\big) - 2\Omega\eta_{\mu\theta}y^{\theta}\frac{\partial}{\partial y^{\lambda}}. \tag{1.2.20}$$

Thus,

$$\begin{aligned} A+B+C+D &= \Omega^2\frac{\partial^2}{\partial y^{\lambda}y^{\mu}} + 4\eta_{\mu\sigma}\eta_{\lambda\theta}y^{\sigma}y^{\theta}((y-1)\partial_y + x\partial_x)^2 - 4\eta_{\mu\sigma}\eta_{\lambda\theta}y^{\sigma}y^{\theta}((y-1)\partial_y + x\partial_x) \\ &\quad -2\Omega\eta_{\lambda\theta}y^{\theta}\frac{\partial}{\partial y^{\mu}}\big((y-1)\partial_y + x\partial_x\big) + 2\Omega\eta_{\lambda\theta}y^{\theta}\frac{\partial}{\partial y^{\mu}} \\ &\quad -2\Omega\eta_{\mu\theta}y^{\theta}\frac{\partial}{\partial y^{\lambda}}\big((y-1)\partial_y + x\partial_x\big) + 2\Omega\eta_{\mu\theta}y^{\theta}\frac{\partial}{\partial y^{\lambda}} \\ &= \Omega^2\frac{\partial^2}{\partial y^{\lambda}y^{\mu}} + 4\eta_{\mu\sigma}\eta_{\lambda\theta}y^{\sigma}y^{\theta}((y-1)\partial_y + x\partial_x)^2 - 4\eta_{\mu\sigma}\eta_{\lambda\theta}y^{\sigma}y^{\theta}((y-1)\partial_y + x\partial_x) \\ &\quad +4\Omega\eta_{\theta(\lambda}y^{\theta}\frac{\partial}{\partial y^{\mu)}}\big((y-1)\partial_y + x\partial_x\big) - 4\Omega\eta_{\theta(\lambda}y^{\theta}\frac{\partial}{\partial y^{\mu)}} \end{aligned}$$

and finally,

$$\begin{aligned} \frac{\partial^2 f}{\partial x^{\lambda}\partial x^{\mu}}\circ\phi^{-1} &= \{\Omega^2\frac{\partial^2}{\partial y^{\lambda}y^{\mu}} + 4\eta_{\mu\sigma}\eta_{\lambda\theta}y^{\sigma}y^{\theta}((y-1)\partial_y + x\partial_x)^2 - 4\eta_{\mu\sigma}\eta_{\lambda\theta}y^{\sigma}y^{\theta}((y-1)\partial_y + x\partial_x) \\ &\quad +4\Omega\eta_{\theta(\lambda}y^{\theta}\frac{\partial}{\partial y^{\mu)}}\big((y-1)\partial_y + x\partial_x\big) - 4\Omega\eta_{\theta(\lambda}y^{\theta}\frac{\partial}{\partial y^{\mu)}} \\ &\quad +4\Omega\delta^{\alpha}_{(\mu}\eta_{\lambda)\sigma}y^{\sigma}\frac{\partial}{\partial y^{\alpha}} + 2\Omega\eta_{\mu\lambda}y^{\alpha}\frac{\partial}{\partial y^{\alpha}} + 8y_{\lambda}y_{\mu}((y-1)\partial_y + x\partial_x)\}f\circ\phi^{-1} \end{aligned} \tag{1.2.21}$$

i.e.

$$\begin{aligned} \frac{\partial^2 f}{\partial x^{\lambda}\partial x^{\mu}}\circ\phi^{-1} &= \Big\{\Omega^2\frac{\partial^2}{\partial y^{\lambda}y^{\mu}} + 4\eta_{\mu\sigma}\eta_{\lambda\theta}y^{\sigma}y^{\theta}((y-1)\partial_y + x\partial_x)^2 \\ &\quad +4\Omega\eta_{\theta(\lambda}y^{\theta}\frac{\partial}{\partial y^{\mu)}}\big((y-1)\partial_y + x\partial_x\big) \\ &\quad +2\big(\Omega\eta_{\mu\lambda} + 2y_{\lambda}y_{\mu}\big)\big((y-1)\partial_y + x\partial_x\big)\Big\}f\circ\phi^{-1}. \end{aligned}$$

□

1.2.2 Application of the main Theorem

The general relation between the wave operator on scalar functions in two conformal metrics transforms the left-hand-side of (1.2.1) into the following *partial differential operator*

$$\eta^{\alpha\beta}\frac{\partial^2(\Omega^{-\frac{n-1}{2}}f\circ\phi^{-1})}{\partial y^\alpha\partial y^\beta}\equiv\Omega^{-\frac{n+3}{2}}(\eta^{\alpha\beta}\frac{\partial^2 f}{\partial x^\alpha\partial x^\beta})\circ\phi^{-1}\ . \tag{1.2.22}$$

We introduce the following new set of scalar functions on $\mathbb{R}^{n+1}_y$

$$\hat{f}:=\Omega^{-\frac{n-1}{2}}f\circ\phi^{-1}\quad i.e.\quad f\circ\phi^{-1}=\Omega^{\frac{n-1}{2}}\hat{f}\ , \tag{1.2.23}$$

so that the system (1.2.1) reads

$$\eta^{\alpha\beta}\frac{\partial^2\hat{f}}{\partial y^\alpha\partial y^\beta}=-\Omega^{-\frac{n+3}{2}}\Big\{H^{\alpha\beta}(x,f,\partial f,\partial\partial f)\frac{\partial^2 f}{\partial x^\alpha\partial x^\beta}-F(f,\partial f)\Big\}\circ\phi^{-1}\ , \tag{1.2.24}$$

and we need to analyze the structure of the right-hand side. We have the following:

Proposition 1.2.7 *The set of functions* $\hat{f}$ *satisfies the following identity:*

$$\begin{aligned}\frac{\partial^2 f}{\partial x^\lambda\partial x^\mu}\circ\phi^{-1}\ &=\ \big(x(1-y)\big)^{\frac{n-1}{2}}\Bigg\{x^2(1-y)^2\frac{\partial^2}{\partial y^\lambda\partial y^\mu}+4x(1-y)\eta_{\alpha(\lambda}y^\alpha\frac{\partial}{\partial y^{\mu)}}(x\partial_x+(y-1)\partial_y)\\&+4y_\lambda y_\mu((y-1)\partial_y+x\partial_x)^2+2(n-1)x(1-y)y^\alpha\eta_{\alpha(\lambda}\frac{\partial}{\partial y^{\mu)}}\\&+\Big[4ny_\lambda y_\mu+2x(1-y)\eta_{\lambda\mu}\Big]((y-1)\partial_y+x\partial_x)\\&+(n-1)\big[(n+1)y_\lambda y_\mu+x(1-y)\eta_{\lambda\mu}\big]\Bigg\}\hat{f}\ .\end{aligned} \tag{1.2.25}$$

Proof: *We have to write the four terms of identity (1.2.15) with* $f\circ\phi^{-1}$ *there replaced by* $\hat{f}$ *. Since*

$$\frac{\partial\Omega}{\partial y^\alpha}=-2y_\alpha\qquad and\qquad y^\alpha\frac{\partial\Omega}{\partial y^\alpha}=2\Omega \tag{1.2.26}$$

we have

$$\frac{\partial(f\circ\phi^{-1})}{\partial y^\alpha}\equiv\frac{\partial(\Omega^{\frac{n-1}{2}}\hat{f})}{\partial y^\alpha}=\Omega^{\frac{n-1}{2}}\frac{\partial\hat{f}}{\partial y^\alpha}+(1-n)\Omega^{\frac{n-3}{2}}y_\alpha\hat{f} \tag{1.2.27}$$

and differentiating a second time with respect to y^β *we obtain*

$$\begin{aligned}\frac{\partial^2(f\circ\phi^{-1})}{\partial y^\alpha\partial y^\beta}\ &\equiv\ \Omega^{\frac{n-1}{2}}\frac{\partial^2\hat{f}}{\partial y^\alpha\partial y^\beta}-(n-1)\Omega^{\frac{n-3}{2}}\left(y_\beta\frac{\partial\hat{f}}{\partial y^\alpha}+y_\alpha\frac{\partial\hat{f}}{\partial y^\beta}\right)\\&+\frac{(1-n)}{2}\Omega^{\frac{n-5}{2}}D_{\alpha\beta}\hat{f}\ ,\end{aligned}$$

with

$$D_{\alpha\beta} := 2(3-n)y_\alpha y_\beta + 2\Omega\eta_{\alpha\beta}\ . \tag{1.2.28}$$

Thus the first term of (1.2.15) is

$$\begin{aligned}\Omega^2\frac{\partial^2(f\circ\phi^{-1})}{\partial y^\lambda\partial y^\mu} &\equiv \Omega^{\frac{n+3}{2}}\frac{\partial^2\hat f}{\partial y^\lambda\partial y^\mu} - (n-1)\Omega^{\frac{n+1}{2}}\left(y_\mu\frac{\partial\hat f}{\partial y^\lambda} + y_\lambda\frac{\partial\hat f}{\partial y^\mu}\right)\\ &\quad +\frac{(1-n)}{2}\Omega^{\frac{n-1}{2}}D_{\lambda\mu}\hat f\ . \end{aligned}\tag{1.2.29}$$

Next,

$$\begin{aligned}(x\partial_x+(y-1)\partial_y)(f\circ\phi^{-1}) &= (x\partial_x+(y-1)\partial_y)(\Omega^{\frac{n-1}{2}}\hat f)\\ &= (n-1)\Omega^{\frac{n-1}{2}}\hat f + \Omega^{\frac{n-1}{2}}(x\partial_x+(y-1)\partial_y)\hat f \end{aligned}\tag{1.2.30}$$

and thus

$$\begin{aligned}(x\partial_x+(y-1)\partial_y)^2(f\circ\phi^{-1}) &= (x\partial_x+(y-1)\partial_y)\{(n-1)\Omega^{\frac{n-1}{2}}\hat f + \Omega^{\frac{n-1}{2}}(x\partial_x+(y-1)\partial_y)\hat f\}\\ &= \Omega^{\frac{n-1}{2}}\Big\{(n-1)^2+2(n-1)(x\partial_x+(y-1)\partial_y)+(x\partial_x+y\partial_y)^2\Big\}\hat f\end{aligned}$$

and the second term of (1.2.15) is

$$4y_\lambda y_\mu(x\partial_x+(y-1)\partial_y)^2(f\circ\phi^{-1}) = 4y_\lambda y_\mu\Omega^{\frac{n-1}{2}}\Big\{(n-1)^2+2(n-1)(x\partial_x+(y-1)\partial_y)+(x\partial_x+y\partial_y)^2\Big\}\hat f\ . \tag{1.2.31}$$

As far as the third therm of (1.2.15) is concerned, we have

$$\begin{aligned}y_\lambda\frac{\partial}{\partial y^\mu}(x\partial_x+(y-1)\partial_y)f\circ\phi^{-1} &= \Omega^{\frac{n-3}{2}}\Big\{-(n-1)^2y_\lambda y_\mu+(n-1)\Omega y_\lambda\frac{\partial}{\partial y^\mu}\\ &\quad +(1-n)y_\lambda y_\mu(x\partial_x+(y-1)\partial_y)\\ &\quad +\Omega y_\lambda\frac{\partial}{\partial y^\mu}(x\partial_x+(y-1)\partial_y)\Big\}\hat f\ .\end{aligned}$$

From this, we deduce that the third term of (1.2.15) reads

$$\begin{aligned}4\Omega\eta_{\alpha(\lambda}y^\alpha\frac{\partial}{\partial y^{\mu)}}(x\partial_x+(y-1)\partial_y)f\circ\phi^{-1} &= \Omega^{\frac{n-1}{2}}\Big\{-4(n-1)^2y_\lambda y_\mu\\ &\quad +4(n-1)\Omega\eta_{\alpha(\lambda}y^\alpha\frac{\partial}{\partial y^{\mu)}}\\ &\quad -4(n-1)y_\lambda y_\mu(x\partial_x+(y-1)\partial_y)\\ &\quad +4\Omega\eta_{\alpha(\lambda}y^\alpha\frac{\partial}{\partial y^{\mu)}}(x\partial_x+(y-1)\partial_y)\Big\}\hat f.\end{aligned}\tag{1.2.32}$$

Using (1.2.30), the fourth term of (1.2.15) is:

$$2\big(\Omega\eta_{\mu\lambda}+2y_\lambda y_\mu\big)\big((y-1)\partial_y+x\partial_x\big)\Big\}f\circ\phi^{-1} = 2\big(\Omega\eta_{\mu\lambda}+2y_\lambda y_\mu\big)\Big\{(n-1)\Omega^{\frac{n-1}{2}}\hat f + \Omega^{\frac{n-1}{2}}(x\partial_x+(y-1)\partial_y)\hat f\Big\}. \quad (1.2.33)$$

Summing side by side equations (1.2.29), (1.2.31), (1.2.32) and (1.2.33), we obtain (1.2.25) and the proof is complete. □

The second term on the right-hand side of (1.2.24) is

$$II=\Omega^{-\frac{n+3}{2}}F\Big(f,\frac{\partial f}{\partial x^\mu}\Big)\circ\phi^{-1}=\Omega^{-\frac{n+3}{2}}F\Big(f\circ\phi^{-1},\frac{\partial f}{\partial x^\mu}\circ\phi^{-1}\Big)\ .$$

Since

$$\frac{\partial f}{\partial x^\mu}\circ\phi^{-1}=A^\alpha_\mu\frac{\partial(f\circ\phi^{-1})}{\partial y^\alpha}\quad \textit{with}\quad A^\alpha_\mu:=\frac{\partial y^\alpha}{\partial x^\mu}\circ\phi^{-1}\equiv-\Omega\delta^\alpha_\mu-2y^\alpha\eta_{\mu\beta}y^\beta\ , \quad (1.2.34)$$

which is bounded on any bounded set of $\mathbb{R}^{n+1}_y$*, it follows from (1.2.27) that*

$$\frac{\partial f}{\partial x^\mu}\circ\phi^{-1}=A^\alpha_\mu\frac{\partial(f\circ\phi^{-1})}{\partial y^\alpha}=-\Omega^{\frac{n-1}{2}}\Big(\Omega\frac{\partial}{\partial y^\mu}+2y_\mu((y-1)\partial_y+x\partial_x)+(n-1)y_\mu\Big)\hat f\ , \quad (1.2.35)$$

and we obtain that

$$II = \Omega^{-\frac{n+3}{2}}F\Big(\Omega^{\frac{n-1}{2}}\hat f,-\Omega^{\frac{n-1}{2}}\Big(\Omega\frac{\partial}{\partial y^\mu}+2y_\mu((y-1)\partial_y+x\partial_x)+(n-1)y_\mu\Big)\hat f\ .$$

Now, we see that the right-hand side of the last equation can be rewritten as

$$\big(x(1-y)\big)^{-\frac{n+3}{2}}\times$$
$$F\Big(\big(x(1-y)\big)^{\frac{n-1}{2}}\hat f,\big(x(1-y)\big)^{\frac{n-1}{2}}\Big(x(y-1)\frac{\partial\hat f}{\partial y^\mu}-2\eta_{\mu\alpha}y^\alpha((y-1)\partial_y\hat f+x\partial_x\hat f)-(n-1)y_\mu\hat f\Big)\Big)\ . \quad (1.2.36)$$

Since

$$\frac{\partial}{\partial y^0}=\partial_y-\partial_x\quad \textit{and}\quad \frac{\partial}{\partial y^i}=-\frac{2y^i}{1-x-y}(\partial_y+\partial_x)+\sum_{A=1}^{n-1}\frac{\partial v^A}{\partial y^i}\partial_A\ .$$

To make contact with (1.1.2) we set

$$\psi_1=\hat f\ ,\quad \psi_2=\partial_y\hat f\ ,\quad \varphi=(\partial_x\hat f,\partial_A\hat f)\ . \quad (1.2.37)$$

Here $\partial_A f = \partial_{v^A} f$*, where the* v^A*'s are local coordinates on the sphere. To bring (1.2.36) to the desired form (1.1.2), the choice*

$$p_2\delta = \frac{n+3}{2}\ , \quad q_2\delta = \frac{n-3}{2}\ ,$$

provides the supplementary power of x *needed in the arguments of* F *to satisfy the structure conditions of Theorem 1.1.1, provided that we choose* $1/(2\delta) \in \mathbb{N}^*$ *in even space-dimensions; any* $1/\delta \in \mathbb{N}^*$ *is admissible in odd ones. If we assume that* F *has a uniform zero of order* m_2*, condition (1.1.3) will now be satisfied for*

$$m_2 > \frac{n+1}{n-3} = 1 + \frac{4}{n-3} \qquad \Longleftrightarrow \qquad n \geq 4 \ \ \text{and } m_2 \geq \begin{cases} 6, & n = 4; \\ 4, & n = 5; \\ 3, & n = 6, 7; \\ 2, & n \geq 8. \end{cases} \tag{1.2.38}$$

(In the Einstein-Maxwell case we have $m_2 = 2$*, which enforces* $n \geq 8$*.)*

Let us turn our attention to the first term at the right-hand side of (1.2.24). Recall that this term is

$$I = -\Omega^{-\frac{n+3}{2}} H^{\mu\lambda}(f \circ \phi^{-1}, \frac{\partial f}{\partial x^\nu} \circ \phi^{-1}) \frac{\partial^2 f}{\partial x^\mu \partial x^\lambda} \circ \phi^{-1}\ ,$$

and from (1.2.15) and (1.2.27), it can be written as

$$I = A \times B \tag{1.2.39}$$

with,

$$\begin{aligned} A \ = \ & -(x(1-y))^{-\frac{n+3}{2}} H^{\mu\lambda}\Big((x(1-y))^{\frac{n-1}{2}} \hat{f}, \quad (x(1-y))^{\frac{n-1}{2}} \Big(x(1-y) \frac{\partial \hat{f}}{\partial y^\nu} \\ & -2\eta_{\nu\alpha} y^\alpha ((y-1)\partial_y \hat{f} + x\ \partial_x \hat{f}) - (n-1) y_\nu \hat{f} \Big) \Big) \end{aligned} \tag{1.2.40}$$

and

$$
\begin{aligned}
B &= \frac{\partial^2 f}{\partial x^\lambda \partial x^\mu} \circ \phi^{-1} \\
&= \big(x(1-y)\big)^{\frac{n-1}{2}} \Bigg\{ x^2(1-y)^2 \frac{\partial^2}{\partial y^\lambda \partial y^\mu} + 4x(1-y)\eta_{\alpha(\lambda} y^\alpha \frac{\partial}{\partial y^{\mu)}} \left(x\partial_x + (y-1)\partial_y\right) \\
&\quad + 4y_\lambda y_\mu((y-1)\partial_y + x\partial_x)^2 + 2(n-1)x(1-y)y^\alpha \eta_{\alpha(\lambda} \frac{\partial}{\partial y^{\mu)}} \\
&\quad + \Big[4n y_\lambda y_\mu + 2x(1-y)\eta_{\lambda\mu}\Big]((y-1)\partial_y + x\partial_x) \\
&\quad + (n-1)\big[(n+1)y_\lambda y_\mu + x(1-y)\eta_{\lambda\mu}\big] \Bigg\} \hat{f} \ .
\end{aligned}
$$

In what follows we will consider the following restricted class of non-linearities: we assume that, after replacing f *by* $\Omega^{\frac{n-1}{2}}\hat{f}$ *and changing variables* $x^\mu \to y^\mu$ *as above, the terms* $H^{\alpha\beta}$ *takes the form*

$$
H^{\alpha\beta} = G^{\alpha\beta}(\Omega^{\frac{n-1}{2}}\hat{f}, \Omega^{\frac{n-1}{2}+1}\partial_{y^\mu}\hat{f}, \Omega^{\frac{n-1}{2}+2}\partial_{y^\nu}\partial_{y^\rho}\hat{f}) \ , \tag{1.2.41}
$$

with a uniform zero of order m_0*. Such a structure will clearly be obtained from a function in (1.2.1) which depends only upon* f*, in particular this will be the case for the Einstein or the Einstein-Maxwell equations, with* $m_0 = 1$*.*

Using (1.2.39) we can write

$$
\Omega^{-\frac{n+3}{2}} H^{\mu\nu}\partial_{x^\mu}\partial_{x^\nu} f = x^{-\frac{n+3}{2}+\frac{n-1}{2}} F_1(H, \hat{f}, \partial_{y^\alpha}\partial_{y^\beta}\hat{f}, \partial_{y^\gamma}\hat{f})
$$

where F_1 *is linear in the second, third, and fourth argument. Assuming (1.2.41), this can be rewritten as*

$$
\begin{aligned}
\Omega^{-\frac{n+3}{2}} H^{\mu\nu}\partial_{x^\mu}\partial_{x^\nu} f &= x^{-\frac{n+7}{2}} F_1(H, x^{\frac{n-1}{2}+2}\hat{f}, x^{\frac{n-1}{2}+2}\partial_{y^\alpha}\partial_{y^\beta}\hat{f}, x^{\frac{n-1}{2}+2}\partial_{y^\gamma}\hat{f}) \\
&= x^{-\frac{n+7}{2}} F_2(x^{\frac{n-1}{2}}\hat{f}, x^{\frac{n-1}{2}+2}\partial_{y^\alpha}\partial_{y^\beta}\hat{f}, x^{\frac{n-1}{2}+1}\partial_{y^\gamma}\hat{f}) \ ,
\end{aligned}
$$

where F_2 *has a uniform zero of order* $m_1 = m_0 + 1$*. With the restrictions on* δ *as before, we will obtain the right structure by setting*

$$
p_1\delta = \frac{n+7}{2} \ , \quad q_1\delta = \frac{n-1}{2} \ ,
$$

and the NL*-condition will hold provided that* $m_1 := m_0 + 1$ *satisfies*

$$
m_1 > \frac{n+5}{n-1} = 1 + \frac{6}{n-1} \ , \quad \Longleftrightarrow \quad m_0 \geq \begin{cases} 7, & n = 2; \\ 4, & n = 3; \\ 3, & n = 4; \\ 2, & n = 5, 6, 7; \\ 1, & n \geq 8. \end{cases} \tag{1.2.42}
$$

In particular the structure conditions will be satisfied by the Einstein-Maxwell equations in space-dimensions larger than or equal to eight.

The hypothesis (1.2.41) will not be satisfied in general if $H^{\mu\nu}$ in (1.2.1) is a non-linear function of f and $\partial_{x^\mu} f$, for then H will belong instead to the following class of functions (compare (1.2.36))

$$H = G(\Omega^{\frac{n-1}{2}} \hat{f}, \Omega^{\frac{n-1}{2}} \partial_{y^\mu} \hat{f}, \Omega^{\frac{n-1}{2}+1} \partial_{y^\nu} \partial_{y^\rho} \hat{f}) \, , \tag{1.2.43}$$

An analysis similar to the one above shows that, for $H^{\mu\nu}$'s which are a finite sum of terms of the form (1.2.43), we will obtain the right structure by setting

$$p_1 \delta = \frac{n+5}{2} \, , \quad q_1 \delta = \frac{n-3}{2} \, ,$$

and the $\mathrm{NL}-$ *condition will hold provided that $m_1 = m_0 + 1$ satisfies*

$$m_1 > \frac{n+3}{n-3} = 1 + \frac{6}{n-3} \qquad \Longleftrightarrow \qquad n \geq 4 \quad \textit{and } m_0 \geq \begin{cases} 7, & n = 4; \\ 4, & n = 5,; \\ 3, & n = 6; \\ 2, & n = 7, 8, 9; \\ 1, & n \geq 10. \end{cases} \tag{1.2.44}$$

The reader should have no troubles similarly working out the conditions on the nonlinearity for general H's which depend on f, $\partial_{x^\mu} f$ and $\partial_{x^\mu}\partial_{x^\nu} f$: In the general case where the nonlinearity H depends on f, $\partial_{x^\mu} f$ and $\partial_{x^\mu}\partial_{x^\nu} f$ (not necessary linearly), we choose as before,

$$p\delta = \frac{n+3}{2}, \quad q\delta = \frac{n-5}{2}$$

and if H has a uniform zero of order m_g, then the NL-condition will hold provided that m_g satisfies

$$m_g > \frac{n+1}{n-5} \qquad \Longleftrightarrow \qquad n \geq 6 \quad \textit{and } m_g \geq \begin{cases} 8, & n = 6; \\ 5, & n = 7; \\ 4, & n = 8; \\ 3, & n = 9, 10, 11; \\ 2, & n \geq 12. \end{cases} \tag{1.2.45}$$

Summarizing, we have proved:

Theorem 1.2.8 *Let f be a solution of equation (1.2.1), define ψ_1, ψ_2, and φ by (1.2.37), where $\hat{f}$ is given by (1.2.23). Suppose that (1.2.38) holds, and* *assume that* either *(1.2.41) with (1.2.42) hold,* or *(1.2.43) with (1.2.44) hold. If (1.1.4) and (1.1.10) hold, then the conclusions of Theorem 1.1.1 apply. In particular Theorem 1.1.1 applies to the Einstein-Maxwell equations in space-time dimensions* $n + 1 \geq 9$.

Chapter 2

Towards solutions with a polyhomogeneous Scri

In order to establish existence of solutions of the vacuum Einstein equations, in sufficiently high dimensions, with a polyhomogeneous Scri, it remains to construct appropriate initial data, and show that the corresponding solutions are in the right function spaces.

Recall, now, that large classes of polyhomogeneous hyperboloidal initial data have been constructed in [1] (the emphasis in that reference is on $n = 3$ at several places, but the general results there show that the conformal method, starting from smooth or polyhomogeneous seed fields, provides polyhomogeneous solutions of the general relativistic vacuum constraint equations in any dimension $n \geq 3$). There is little doubt that large collections of initial data so constructed provide polyhomogeneous data for the harmonically reduced equations of the last section, but we have not checked this in detail. Instead, we will follow the standard-by-now strategy of using initial data which are stationary outside of a compact set. So, in Section 2.2, we provide large classes of Corvino-Schoen type initial data with polyhomogeneous asymptotics on hyperboloids. One of the reasons for proceeding this way is that small such initial data lead to global, geodesically complete solutions [40, 41].

One then needs to verify that the associated solutions satisfy the space-time weighted regularity conditions needed in Theorem 1.1.1. One could hope that the Lindblad-Rodnianski type estimates of Loizelet [40,41] would provide that information. It turns out that the available estimates, for space-times obtained by evolving small initial data of Section 2.2, are not sufficient for our polyhomogeneity result; this is analyzed in Section 2.3. This means that *the desired estimates have to be derived from scratch, which will be done in the remainder of this first part of the thesis.*

2.1 Stationary vacuum metrics in higher dimensions

The only way, so far, of obtaining space-times with controlled asymptotic behavior near i^0 *is to use initial data sets which are stationary at large distances. We will outline the construction of such data in Section 2.2, but before doing this it is convenient to start with a short discussion of stationary metrics in higher dimensions; our presentation follows [4].*

Consider a vacuum Lorentzian metric ${}^{n+1}g$ *in any space-time-dimension* $n+1 \geq 4$*, with Killing vector* $X = \partial/\partial t$*. In the region where* X *is timelike there exist adapted coordinates in which* ${}^{n+1}g$ *takes the form*

$$ {}^{n+1}g = -V^2(dt + \underbrace{\theta_i dx^i}_{=\theta})^2 + \underbrace{g_{ij}dx^i dx^j}_{=g} , \tag{2.1.1}$$

$$ \partial_t V = \partial_t \theta = \partial_t g = 0 . \tag{2.1.2}$$

The vacuum Einstein equations (with vanishing cosmological constant) read (see, e.g., [22])

$$ \begin{cases} V\nabla^*\nabla V = \frac{1}{4}|\lambda|^2_g , \\ \mathrm{Ric}(g) - V^{-1}\mathrm{Hess}\,_g V = \frac{1}{2V^2}\lambda \circ \lambda , \\ \mathrm{div}(V\lambda) = 0 , \end{cases} \tag{2.1.3}$$

where

$$ \lambda_{ij} = -V^2(\partial_i\theta_j - \partial_j\theta_i) , \quad (\lambda\circ\lambda)_{ij} = \lambda_i{}^k\lambda_{kj} . $$

We assume that there exists $\alpha > 0$ *such that*

$$ g_{ij} - \delta_{ij} = O(r^{-\alpha}) , \ \partial_k g_{ij} = O(r^{-\alpha-1}) , \tag{2.1.4}$$

similarly for $V-1$ *and* θ_i*. A redefinition* $t \to t + \psi$*, introduces a gauge transformation*

$$ \theta \to \theta + d\psi , $$

and one can exploit this freedom to impose restrictions on θ*. For our purposes it is convenient to impose the* harmonic gauge, $\Box t = 0$*, which reads*

$$ \partial_i(\sqrt{\det g}Vg^{ij}\theta_j) = 0 . \tag{2.1.5}$$

Equation (2.1.5) can always be achieved by replacing θ by $\theta + d\psi$, and solving the resulting linear equation for ψ, cf., e.g., *[3, 10] for the relevant isomorphism theorems.) One can then introduce new coordinates [3] which are harmonic for g.*

In space-harmonic coordinates, and in the gauge (2.1.5), the system (2.1.3) is elliptic, and standard considerations show that the functions g_{ij}, V and θ_i have a polyhomogeneous expansion in terms of $\log r$ *and inverse powers of r. Furthermore, ${}^{n+1}g_{\mu\nu}$ is Schwarzschild in the leading order, and there exist constants α_{ij} such that*

$$\theta_i = \frac{\alpha_{ij}x^j}{r^n} + O(r^{-n}) \ .$$

It is of interest to enquire whether or not the logarithmic powers are essential in the polyhomogeneous expansion. It has long been know in space-dimension three that, for metrics which are stationary and vacuum in the asymptotic region, coordinate systems exist *where no* $\log r$ *terms arise whenever the ADM mass is non-zero [48]. The same property is true for* static *solutions with non-zero ADM mass in space-dimension four [4]. Now, in the evolution theorems used below we need all coordinates to satisfy the wave equation,*

$$\Box x^\mu = 0 \ , \tag{2.1.6}$$

and the transition from the coordinates used in [4] to the coordinates satisfying (2.1.6) might introduce log terms: This is exactly what happens for the Schwarzschild metric in $n = 4$, which does have *a logarithmic term in its asymptotic expansion in a natural choice of wave coordinates [9], but this is the only dimension where this happens for Schwarzschild.*

In general, (2.1.6) is achieved by changing space-coordinates $x^i \to x^i + \psi^i(x^j)$ (recall that t is already harmonic), thus solving a linear equation for ψ^i; by standard results (see, e.g., [13]) the ψ^i's will have a full asymptotic expansion in terms of powers of $\ln r$ *and inverse powers of r, and so will the space-time metric in the new coordinate system, when transformed from the space-harmonic ones. In view of the calculations in [9], this implies the existence of polyhomogeneous asymptotics of the initial data on hyperboloids at $\mathscr{I}$, as needed in Theorem 1.1.1.*

Rather surprisingly, in even space-dimensions larger than or equal to six the space-coordinates used in [4] satisfy (2.1.6), and so does the time coordinate. It follows that the analysis of stationary solutions in [4] directly provides wave coordinates in which no log terms occur in those dimensions.

2.2 Corvino-Schoen data in higher dimensions

So far we have considered metrics which are exactly stationary. Now, there exists a construction due to Corvino and Schoen [23, 24] (see also [16, 17], and also the more recent Reference [15], where the construction is carried out under considerably weaker asymptotic conditions) which allows one to glue exactly stationary ends to asymptotically Euclidean initial data sets. Some details of this construction have been presented in those references in dimension three only, but the construction generalises to any dimension, as follows: Recall that the construction requires a family of stationary reference metrics which cover the whole range of asymptotic charges. In dimension $3 + 1$ *this is provided by the family of metrics obtained by boosting and translating the Kerr metrics. In higher dimensions one such family can be obtained by boosting and translating the Myers-Perry metrics [43]. Note that the question, whether or not the reference solutions have naked singularities is irrelevant for the problem at hand because here one only needs the solutions at large distances. (Similarly to the Kerr family, all the metrics in the family so obtained have a timelike ADM momentum, and therefore can only be glued to asymptotically flat initial data which also have this property; this is no restriction for well behaved initial data sets which are spin, or for space-dimensions up to seven, and is expected not to be a restriction for well behaved initial data sets in general, but this has not been proved at the moment of writing of this work.)*

So let R_x*,* ϵ_k *be positive constants and consider the collection, say* $\mathbf{C}_{R_x,\epsilon_k}$ *of general relativistic electro-vacuum initial data sets* $(\mathbb{R}^n, g, K)$ *which are stationary outside a coordinate ball* $B(R_x)$ *and with weighted Sobolev norm controlling* k*-derivatives of the metric smaller than* ϵ_k*. Here* k *should be sufficiently large as in [9, 41], and the norm should be the one described in those references. From what has been said this collection is non-empty, and contains an open set (in the topology associated to the norm) around Minkowski space-time.*

Now, for the Schwarzschild metric in dimension $n + 1$ *with* $n \geq 4$*, and in harmonic coordinates, the boundary of the domain of influence of a ball is sandwiched between two hypersurfaces* $t - r = \text{const}$ *[9, Section 5.3]. This remains true for stationary electro-vacuum metrics because the leading order behavior of the metric coincides with the Schwarzschild one (compare [14, Appendix A]). This implies that the maximal globally hyperbolic development of all initial data in* $\mathbf{C}_{R_x,\epsilon_k}$ *contains hyperboloidal hypersurfaces, the asymptotic region of which is contained in that part of the space-time where the metric is stationary. So our considerations of the previous section apply* *to this region, leading to polyhomogeneous initial data on such hypersurfaces. Since the leading order deviation of the metric from the flat one is Schwarzschildian, the tensor field* $\hat{h} := \Omega^{-\frac{n-1}{2}}(g-\eta)$*, that plays a key role in our analysis, is* $O(x^{(n-2)-(n-1)/2}) = O(x^{(n-3)/2})$*, and in fact*

$$\hat{h} \in x^{(n-3)/2}(\mathscr{A}^{\delta}_{\{x=0\}} \cap L^\infty) \ , \quad \partial_{y^0}\hat{h} \in x^{(n-5)/2}(\mathscr{A}^{\delta}_{\{x=0\}} \cap L^\infty) \ , \qquad (2.2.1)$$

with $\delta = 1$ *on any hyperboloid whose asymptotic part is contained in the stationary region.*

2.3 Lindblad-Rodnianski-Loizelet metrics near $\mathscr{I}$

In this section we analyze how the asymptotic behavior of the small-data space-times constructed in [40] (compare [37, 38]) relates to the differentiability conditions needed in Theorem 1.1.1. We find that sharper decay rates along outgoing null geodesics would be needed for a direct proof of polyhomogeneity using our approach. The estimates established here are then combined with the results of our analysis in subsequent sections to provide a rather more involved proof of polyhomogeneity.

We start by recalling some notation of [37, 38, 40]. Let $\mathcal{Z}$ *denote the following set of vectors on Minkowski space-time:*

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha},\ \alpha = 0,1,\ldots,n;$$

$$Z_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha,\ \alpha,\beta = 0,1,\ldots,n;$$

$$Z_0 = \sum_{\alpha=0}^{n} x^\alpha \partial_\alpha = t\partial_t + \sum_{i=1}^{n} x_i \partial_i = t\partial_t + r\partial_r \ .$$

Here, as usual, $x_0 = -x^0 = -t$, $x_i = x^i$ *for* $i = 1\ldots,n$*. Let the spherical coordinates* (r,θ^A) *be defined as*

$$\begin{cases} t = x^0 \ , \\ r = \big(\sum_{i=1}^{n}(x^i)^2\big)^{1/2} \ , \\ x^i = r\omega^i(\theta^A),\ i = 1,\ldots,n, \end{cases} \qquad (2.3.1)$$

where θ^A *denotes any local coordinates on the sphere* S^{n-1}*. The vector fields*

$$L = \partial_t + \partial_r = \partial_t + \omega^i\partial_i \ , \qquad \underline{L} = \partial_t - \partial_r = \partial_t - \omega^i\partial_i \ .$$

are tangent, respectively transverse, to the light cones $t - r = \text{const}$. *We note*

$$Z_0 = t\partial_t + r\partial_r \ .$$

Furthermore, the Z_{ij}*'s,* $i, j = 1 \ldots, n$ *are tangent to the spheres* $S^{n-1} \subset \mathbb{R}^n$*, and can be purely expressed in terms of the* θ^A*'s.*

Let $T \geq 0$*, set* $T^\mu = (T, 0, \ldots, 0)$*, in this section it is more convenient to consider instead the following variation of (1.2.2):*

$$y^\mu = \frac{x^\mu + T^\mu}{(x^\alpha + T^\alpha)(x_\alpha + T_\alpha)} \qquad \Longleftrightarrow \qquad x^\mu + T^\mu = \frac{y^\mu}{y^\alpha y_\alpha} \ . \tag{2.3.2}$$

This provides a conformal transformation from the future causal cone centred at T^μ *in the Minkowski space-time with coordinates* x^μ *to the past causal cone of the origin in the Minkowski space-times with coordinates* y^μ*, and with conformal factor* $\Omega = y^\alpha y_\alpha = \frac{1}{-(t+T)^2 + r^2}$.

To make contact with Section 1 we set

$$x = -y^0 - \rho, \quad y = y^0 - \rho + 1 \quad \textit{where} \quad \rho = \big(\sum_{i=1}^{n} (y^i)^2\big)^{1/2} \ ,$$

so that

$$\begin{cases} y^0 = \frac{1}{2}(y - x - 1) \\ \rho = \frac{1}{2}(-y - x + 1) \\ y^i = \frac{1}{2}(-y - x + 1)\omega^i(v^A), \ i = 1, \ldots, n \end{cases} \ . \tag{2.3.3}$$

Here ω^i *is a unit vector, and the* v^A*'s denote local coordinates on* S^{n-1} *in the* y*–coordinates. One can take* $\omega^i(\theta^A) = \omega^i(v^A)$*,* $i = 1, \ldots, n$*; we will make this choice, and simply write* ω^i *in both* x^μ *and* y^μ *coordinates.*

Letting $\mathscr{H}_s$ *be the following family of hyperboloids,*

$$\mathscr{H}_s = \left\{ x^0 - s = \sqrt{s^2 + r^2} \right\} \ , s > 0 \ ,$$

we will have

$$\phi(\mathscr{H}_s) = \{y^0 = -\frac{1}{2s}\}$$

in particular $\phi(\mathscr{H}_1) = \{y^0 = -\frac{1}{2}\}$.

The methods of Section 1 involve the vector fields

$$x\partial_x, \ y\partial_y, \ \partial_A = \frac{\partial}{\partial v^A}, \ A = 1, \ldots n - 1.$$

By straightforward calculations one finds, keeping in mind that $\rho = \frac{r}{(t+T)^2-r^2}$ *for* $t+T \geq r$,

$$x = \frac{1}{t+T+r}, \quad 1-y = \frac{1}{t+T-r} \quad \Longleftrightarrow \quad r = \frac{1}{2x} - \frac{1}{2(1-y)}, \quad t = \frac{1}{2x} + \frac{1}{2(1-y)} - T$$

$$\begin{cases} x\partial_x = -\frac{1}{2}(t+T+r)(\partial_t + \partial_r) \ , \\ (1-y)\partial_y = \frac{1}{2}(t+T-r)(\partial_t - \partial_r) \ , \\ \partial_A = \textit{linear combinations of } Z_{ij}, \ i,j = 1,\dots,n \ \ . \end{cases} \tag{2.3.4}$$

The coefficients in the equation for ∂_A *above depend only upon the angular variables, and a finite number of coordinate patches* v^A *can be chosen so that in each of those patches the coefficients are uniformly bounded together with derivatives of any order.*

This leads us to

Proposition 2.3.1 *Let* $T, T_0 > 0$, $t \geq 0$ *and suppose that*

$$1 - T \leq t - r \leq T_0 \quad \Longleftrightarrow \quad 0 \leq y \leq 1 - \frac{1}{T+T_0} \ . \tag{2.3.5}$$

For all $k \in \mathbb{N}$, $\forall\ (i,j,\gamma) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}^{n-1}$ *satisfying* $i+j+|\gamma| \leq k$, *and for any function* $f \in C^k$ *we have*

$$[x\partial_x]^i \partial_y^j \partial_v^\gamma f = \sum_{|I|\leq k,\ Z\in\mathcal{Z}} H_I^{ij\gamma}(\theta,y) Z^I f \tag{2.3.6}$$

with $|H_I^{ij\gamma}(\theta,y)| \leq C(i,j,I,T,T_0)$.

Proof: *Using (2.3.4) one can rewrite* $x\partial_x$ *and* ∂_y *as*

$$x\partial_x = -\frac{1}{2}(Z_0 - \omega^i Z_{0i} + T(\partial_t + \omega^i\partial_i)) \ , \tag{2.3.7}$$

$$\partial_y = \underbrace{\frac{1}{2(1-y)}}_{=:\varphi_1(y)} \underbrace{(Z_0 + \omega^i Z_{0i} + T(\partial_t - \omega^i\partial_i))}_{=:\tilde{Z}} \ . \tag{2.3.8}$$

It is thus clear that $x\partial_x$, *and any of its powers, have the right structure. Next, the factor* $\varphi_1(y)$ *appearing in (2.3.8) is bounded on any compact subinterval of* $[0,1)$ *(note that* $y=1$ *corresponds to the tip of the past causal cone centred at the origin of the* y^μ*-coordinates). One easily finds by induction that*

$$\partial_y^j = \sum_{i=1}^{j} \varphi_i(y) \tilde{Z}^i \ ,$$

where the functions φ_i are bounded on compact subsets of $[0,1)$, whence the result. $\square$

We wish to obtain the asymptotic behavior of the fields occurring in Theorem 1.1.1 for the global solutions

$$f := (h_{\mu\nu}, A_\mu)$$

of the Einstein-Maxwell equations constructed in [40]. In order to apply Theorem 1.1.1 we need

$$\psi_1 = \hat{f} \in \mathscr{C}^{<0}_{\{0\le x\le y\},0};\ \big(\psi_2 = (\partial_y \hat{f}, \partial_A \hat{f}),\ \varphi = \partial_x \hat{f}\big) \in \mathscr{C}^{<-1}_{\{0\le x\le y\},\infty}\ ,$$

where

$$\hat{f} = \Omega^{-\frac{n-1}{2}} f \circ \phi^{-1}.$$

Now,

$$\Omega = -x(1-y)$$

which implies that for any $\alpha \in \mathbb{R}$ we have

$$(x\partial_x)^i(\Omega^\alpha f) = \Omega^\alpha \sum_{j=0}^{i} C(\alpha, i, j)(x\partial_x)^j f\ . \tag{2.3.9}$$

Similarly,

$$(y\partial_y)^i(\Omega^\alpha f) = \Omega^\alpha \sum_{j=0}^{i} C'(\alpha, i, j, x, y)(y\partial_y)^j f\ , \qquad \partial_y^i(\Omega^\alpha f) = \Omega^\alpha \sum_{j=0}^{i} C''(\alpha, i, j, x, y)\partial_y^j f\ , \tag{2.3.10}$$

where the functions C' and C'' are bounded for x in, say, $[0, x_0]$, and for y bounded away from 1.

The solutions constructed in [40] satisfy the following: there exists $0 < \delta < 1/4$ such that for $t \ge 0$ and $|t-r| \le C_1$, and for all I there exists a constant C, depending upon I and C_1, such that

$$|Z^I f(t, x^i)| \le C(1+t+r)^{\frac{1-n}{2}+\delta}\ , \tag{2.3.11}$$

$$|\bar{\partial} Z^I f(t, x^i)| \le C(1+t+r)^{\frac{-1-n}{2}+\delta}\ , \tag{2.3.12}$$

where

$$\bar{\partial} \in \Big\{\partial_t + \partial_r,\ r^{-1}\partial_A\Big\} = \Big\{-2x^2\partial_x, \frac{2x(1-y)}{1-x-y}\partial_A\Big\}\ . \tag{2.3.13}$$

Now,

$$\frac{1+t+r}{t+T+r} = 1 + \frac{1+T}{t+T+r} \in \left[1, \frac{1+T}{T}\right] \quad \textit{for} \quad T>0,\ t\geq 0\ ,$$

so (2.3.11)-(2.3.12) imply

$$|Z^I f(t,x^i)| \quad \leq \quad Cx^{\frac{n-1}{2}-\delta}\ , \quad |\bar{\partial} Z^I f(t,x^i)| \leq Cx^{\frac{n-1}{2}+1-\delta}\ . \tag{2.3.14}$$

From (2.3.9)-(2.3.10) and Proposition 2.3.1 we obtain

$$\begin{aligned}
&[x\partial_x]^i \partial_y^j \partial_v^\gamma \hat{f}(x,y,v) \\
&\quad = [x\partial_x]^i \partial_y^j \partial_v^\gamma \Omega^{\frac{1-n}{2}} f(t,x^i) \\
&\quad = \Omega^{\frac{1-n}{2}} \sum_{0\leq m\leq i} \sum_{0\leq \ell\leq j} c(i,j,m,\ell,n,x,y) [x\partial_x]^m \partial_y^\ell \partial_v^\gamma f(t,x^i) \\
&\quad = \Omega^{\frac{1-n}{2}} \sum_{0\leq m\leq i} \sum_{0\leq \ell\leq j} c(i,j,m,\ell,n,x,y) \sum_{|I|\leq k,\ Z\in\mathcal{Z}} H_I^{m\ell\gamma}(\theta,y) Z^I f(t,x^i)\ .
\end{aligned}$$

Using the first inequality in (2.3.14) we conclude that for any $0<\epsilon\leq 1$ *and for* $0\leq y\leq 1-\epsilon$ *we have*

$$\left|[x\partial_x]^i [y\partial_y]^j \partial_v^\gamma \hat{f}(x,y,v)\right| \leq \left|[x\partial_x]^i \partial_y^j \partial_v^\gamma \hat{f}(x,y,v)\right| \leq Cx^{-\delta}\ ,$$

while it should be clear from (2.3.13) that the second inequality in (2.3.14) does not provide any new information in the coordinate ranges assumed above. In any case the property

$$\Big(\psi_1 = \hat{f}, \psi_2 = (\partial_y \hat{f}, \partial_A \hat{f})\Big) \in \mathscr{C}^{-\delta}_{\{0\leq x\leq y\},0}\ , \quad \varphi = \partial_x \hat{f} \in \mathscr{C}^{-1-\delta}_{\{0\leq x\leq y\},\infty} \tag{2.3.15}$$

immediately follows. Unfortunately, to apply Theorem 1.1.1 one would need δ *to be an arbitrary positive number, while in (2.3.15)* δ *is a small number determined by the initial data. So, as already pointed out, we need to derive the necessary estimates by different methods. This is the purpose of the chapters that follow.*

Chapter 3

Weighted energy estimates near a null boundary

Let $(\mathscr{M}, \mathfrak{g})$ be an $(n+1)$-dimensional space-time. We consider systems of quasi-linear wave equations, with diagonal principal part of the form

$$\Box_{\mathfrak{g}} u = F(.\,, u, \partial u)\ , \tag{3.0.1}$$

on a neighborhood of a null hypersurface of $\mathscr{M}$. We suppose that the background metric $\mathfrak{g}$ is a smooth function of the coordinates, of the unknown vector valued function u, as well as its first order derivatives.

All calculations below will be done for a real valued function u, the result for a vector valued function is obtained by summing over the components.

3.1 The hypotheses, and the geometry of the problem

3.1.1 The hypotheses

We will consider the Cauchy problem associated to equation (3.0.1), the initial data will be given on a hypersurface $\mathscr{S}_0$. We will evolve these initial data to obtain a solution of our problem in a past *one-sided neighborhood of a null hypersurface*

$$\mathscr{N} = \{x = 0\}$$

forming the boundary, or a subset thereof, of the domain of dependence of $\mathscr{S}_0$. Here, and throughout, x stands for a positive function such that dx has *no zeros on $\{x = 0\}$. We will be working in a neighborhood of $\{x = 0\}$, chosen so that x is a coordinate there, of the form*

$$\mathscr{V} \equiv [\tau_0,\ \tau_1\,[\ \times\ ]\,0, x_0\,[\ \times \mathscr{O}\ ,$$

where $[\tau_0,\ \tau_1\,[$ corresponds to the time interval, $]\,0, x_0\,[$ the range of the variable x, and $\mathscr{O}$ is an $(n-1)$-dimensional compact submanifold of $\mathscr{M}$ without boundary. The coordinates will be denoted by (τ, x, v), with $v = (v^A)_{A=1}^{n-1}$ the coordinates on $\mathscr{O}$. We assume that ∂_τ is timelike, and we choose the time-orientation on $\mathscr{M}$ such that the vector ∂_τ is everywhere future directed.

One can think of the set $\mathscr{U}$ of (A.2.1) as a subset of the coordinate patch above, compare Figure 4.2, page 108.

On the components of the metric $\mathfrak{g}$ with respect to the coordinates (τ, x, v), we assume the following:

1. *We suppose that*

$$\exists \epsilon_0 > 0\ , \quad \textit{such that} \quad -\mathfrak{g}^{\tau\tau} \geq \epsilon_0 \tag{3.1.1}$$

everywhere on $\mathscr{V}$.

2. *The components $\mathfrak{g}^{\tau\tau}$ and $\mathfrak{g}^{\tau x}$ can be written as*

$$\mathfrak{g}^{\tau\tau} = -1 + x\mathfrak{h}^0(\tau, x, v^A) \quad \textit{and} \quad \mathfrak{g}^{\tau\tau} + \mathfrak{g}^{\tau x} = x\mathfrak{h}^1(\tau, x, v^A) \tag{3.1.2}$$

where the functions $\mathfrak{h}^0$ and $\mathfrak{h}^1$ are bounded on bounded sets.

3. *On the components $\mathfrak{g}^{xA}$ and $\mathfrak{g}^{xx}$ we assume that*

$$\mathfrak{g}^{xA} = O(x) \quad \textit{and} \quad \mathfrak{g}^{\tau\tau} + 2\mathfrak{g}^{\tau x} + \mathfrak{g}^{xx} = 1 + O(x) \tag{3.1.3}$$

and we set $\mathfrak{g}^{xA} = x\mathfrak{h}^A$ and $\mathfrak{g}^{\tau\tau} + 2\mathfrak{g}^{\tau x} + \mathfrak{g}^{xx} = 1 + x\mathfrak{h}$, where $\mathfrak{h}$ and $\mathfrak{h}^A$ are bounded functions on bounded sets. We further suppose that

$$\mathfrak{g}^{\tau\tau} + 2\mathfrak{g}^{\tau x} + \mathfrak{g}^{xx} > 0\ .$$

4. *The vector field*

$$Y^\nu \partial_\nu := \partial_\tau - \partial_x \tag{3.1.4}$$

is assumed to be everywhere timelike on $\mathscr{V}$ and future directed. This vector will be used to contract the energy momentum tensor.

The set of functions $(\mathfrak{h}\ , \mathfrak{h}^\mu)$ will be denoted by $\mathfrak{h}^\sharp$ and $\mathfrak{g}^\sharp$ will denote the inverse matrix of the matrix $(\mathfrak{g}_{\mu\nu})$.

Remark 3.1.1 It follows from the above that the vector ∇x (where ∇ is the covariant derivative compatible with the metric $\mathfrak{g}$) can be decomposed as

$$\nabla x = \omega^{(1)} + \beta(x)\omega^{(2)} \tag{3.1.5}$$

where $\omega^{(1)}$ is causal future directed, and that there exists a constant C_0 such that

$$|\beta(x)| \le C_0 x \ , \ \ |\omega^{(2)}| \le C_0|\mathfrak{h}^\sharp| \ . \tag{3.1.6}$$

Example 3.1.2 As an example, consider a conformally rescaled asymptotically flat solution of asymptotically vacuum Einstein equations in *Bondi coordinates* near Scri [49], with the metric taking the form

$$\tilde{\mathfrak{g}}_B = e^{2\beta} dx \mathring{\otimes} dy + \chi dy \mathring{\otimes} dy + 2\gamma \mathring{\otimes} dy + \mu \ , \tag{3.1.7}$$

for some functions β and χ, and a one-form field γ. (Here y corresponds to the Bondi retarded time u, and $x = 1/2r$ is half the inverse of the luminosity distance r. E.g., for the Minkowski metric in any dimensions, $\beta = \chi = 0 = \gamma$.) In $3+1$ dimensions, for smoothly compactifiable metrics, the Einstein equations imply, for matter fields decaying sufficiently fast, that $\beta = O(x^2)$ as well as

$$\chi = O(x^2) \ , \quad \gamma_A = O(x^2) \ , \tag{3.1.8}$$

with derivatives behaving in the obvious way. Equation (3.1.8) remains valid for asymptotically vacuum metrics which, after conformal rescaling, are polyhomogeneous and C^1 (see [21, Section 6] or [18, Appendix C.1.2]), while for general $\mathscr{A}^{\delta}_{\{x=0\}} \cap L^\infty$–polyhomogeneous asymptotically vacuum metrics one has [21, Equations (2.15)-(2.19) with $H = X^a = 0$] the asymptotic behaviors $\beta = O(x^2 \ln^N x)$ and

$$\chi = O(x^2) \ , \quad \gamma_A = O(x^2 \ln^N x) \ , \tag{3.1.9}$$

for some N. Here "asymptotically vacuum" requires, for polyhomogeneous metrics, that the components of the energy-momentum tensor in asymptotically Minkowskian coordinates satisfy (see [21, end of Section 2])

$$T_{\mu\nu} = o(r^{-2}) \ . \tag{3.1.10}$$

We have

$$\det \mathfrak{g} = -\frac{1}{4} \det \mu \ ,$$

which, for a Lorentzian metric, shows that μ must be a non-degenerate $(n-1)\times(n-1)$ tensor field. It is simple to check that the inverse metric $\mathfrak{g}^\sharp = \mathfrak{g}^{\alpha\gamma}\partial_\alpha\mathring{\otimes}\partial_\gamma$ is given by the formula

$$\begin{aligned}\mathfrak{g}^\sharp &= 4(-\chi+|\gamma|^2_\mu)\partial_x\mathring{\otimes}\partial_x + 4\partial_x\mathring{\otimes}\partial_y - 4\gamma^\sharp\mathring{\otimes}\partial_x + \mu^\sharp \\ &= 4\partial_x\mathring{\otimes}\Big(\partial_y + (-\chi+|\gamma|^2_\mu)\partial_x - \gamma^\sharp\Big) + \mu^\sharp\ , \end{aligned} \qquad (3.1.11)$$

with $\mu^\sharp = \mu^{AB}\partial_A\mathring{\otimes}\partial_B$, where μ^{AB} is the matrix inverse to μ_{AB}, $\gamma^\sharp = \mu^{AB}\gamma_A\partial_B$, $|\gamma|^2_\mu = \mu^\sharp(\gamma,\gamma) = \mu^{AB}\gamma_A\gamma_B$, and $\mathring{\otimes}$ denotes the symmetric tensor product. We note

$$\mathfrak{g}(\nabla y,\nabla y) = \mathfrak{g}^{yy} = 0\ ,$$

which makes clear the null character of the level sets of y, and implies, by a well-known argument, that the integral curves of

$$\nabla y = \mathfrak{g}^{\alpha\gamma}\partial_\alpha y\partial_\gamma = \mathfrak{g}^{y\gamma}\partial_\gamma = 2\partial_x$$

are null geodesics.

Consider a new coordinate system (x, v^A, τ), where

$$(x,y)\longrightarrow (x,\tau = \frac{y-x}{2})\ , \qquad (3.1.12)$$

so that

$$\partial_x \longrightarrow \partial_x - \frac{1}{2}\partial_\tau\ ,\quad \partial_y = \frac{1}{2}\partial_\tau\ . \qquad (3.1.13)$$

Thus

$$\mathfrak{g}^\sharp = 4(-\chi+|\gamma|^2_\mu)(\partial_x-\frac{1}{2}\partial_\tau)\mathring{\otimes}(\partial_x-\frac{1}{2}\partial_\tau) + 4(\partial_x-\frac{1}{2}\partial_\tau)\mathring{\otimes}(\frac{1}{2}\partial_\tau) - 4\gamma^\sharp\mathring{\otimes}(\partial_x-\frac{1}{2}\partial_\tau) + \mu^\sharp\ ,$$

giving

$$\mathfrak{g}^{xx} = 4(-\chi+|\gamma|^2_\mu)\ ,\quad \mathfrak{g}^{x\tau} = 1-2(-\chi+|\gamma|^2_\mu)\ ,\quad \mathfrak{g}^{xA} = -2\mu^{AB}\gamma_B \qquad (3.1.14)$$

$$\mathfrak{g}^{\tau A} = \mu^{AB}\gamma_B\ ,\quad \mathfrak{g}^{\tau\tau} = -1 + (-\chi+|\gamma|^2_\mu)\ ,\quad \mathfrak{g}^{AB} = \mu^{AB}\ . \qquad (3.1.15)$$

This, together with (3.1.9), shows that (3.1.2)-(3.1.3) hold for such metrics.

3.1.2 The slices

In this section we describe the sets within which we obtain our estimates, see Figure 3.1. Let $t\in[\tau_0, 0[$ *run over the range of the time coordinate* τ *of the previous section.*

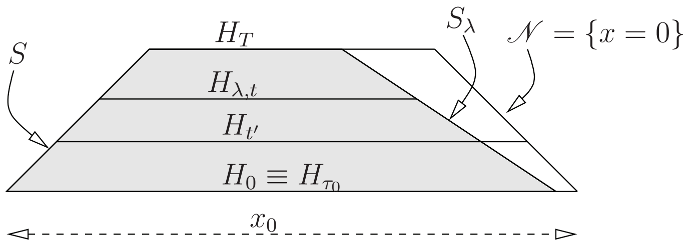

Figure 3.1: The sets $\mathscr{U}_{\lambda,T}$ (shaded) and $\mathscr{U}_T$ (the outermost trapezium). In this picture (but *not* in our hypotheses) the light-cones have forty-five degrees slopes, as in Minkowski space-time.

- *Let* $\lambda \in [0, 1]$ *parameterize a family of spacelike hypersurfaces* S_λ*, which approach* $\{x = 0\}$ *when* λ *approaches zero, of the form*

$$S_\lambda = \{(\tau, x, v^A) : \ x = \sigma_\lambda(\tau)\} ,$$

where σ_λ *is a* C^1*function such that:*

  - $\sigma_0(\tau) \equiv 0$ *i.e.* $S_0 = \{x = 0\}$
  - S_λ *is everywhere spacelike.*

One can legitimately raise concerns about existence of the family S_λ *with global behaviour as above when the space-time under consideration is being constructed as a solution of a Cauchy problem. While the aim of this work is to prove that the resulting space-time will have properties as in Figure 3.1, this is not known a priori. Now, one way to proceed is to construct the solution as the limit of solutions of linear equations on a sequence of metrics, each of those metrics satisfying controlled weighted energy estimates as proved below. In particular each space-time in this sequence is globally hyperbolic, with the set* $\{x = 0\}$ *being part of the boundary of the domain of dependence of the initial surface. For each metric in the sequence a relevant family* S_λ *can be constructed using e.g. Cauchy time functions; no details will be given as no significant difficulties are involved. This can then be used to justify our estimates for each metric in the sequence, and for the solution.*

- *By S we denote a smooth spacelike hypersurface transverse to* $\{\tau = \tau_0\}$ *defined by*
$$S = \{(\tau, x, v^A) : \ x = \sigma(\tau)\} \ , \tag{3.1.16}$$
where σ is a smooth function of τ such that
$$0 < \sigma(\tau_1) \le \sigma(\tau) \le \sigma(\tau_0) = x_0 \ .$$

- $\mathbf{H}_{\lambda,t} = \{(\tau, x, v^A); \ \tau = t, \ \sigma_\lambda(\tau) \le x \le \sigma(\tau)\}, \ \mathscr{U}_{\lambda,\tau_1} = \bigcup_{\tau_0 \le t \le \tau_1} \mathbf{H}_{\lambda,t}.$

- $\mathbf{H}_t = \{(\tau, x, v^A); \ \tau = t, \ 0 \le x \le \sigma(\tau)\}, \quad \mathscr{U}_{\tau_1} = \bigcup_{\tau_0 \le t \le \tau_1} \mathbf{H}_t.$

Note that the boundary $\partial\mathscr{U}_{\lambda,t}$ of the region $\mathscr{U}_{\lambda,t}$ is made of four pieces, S_λ, S, $\mathbf{H}_{\lambda,\tau_0}$ and $\mathbf{H}_{\lambda,t}$. We recall that, for $\theta \in \mathbb{R}$, $j \in \mathbb{N}$ the spaces $\mathscr{C}^\theta_j(\mathbf{H}_{\lambda,\tau})$, $\mathscr{B}^\theta_j(\mathbf{H}_{\lambda,\tau})$, $\mathscr{H}^\theta_j(\mathbf{H}_{\lambda,\tau})$ and $\mathscr{G}^\theta_j(\mathbf{H}_{\lambda,\tau})$ are defined in the Appendix B Section B.1 page 206 .

3.1.3 The causality properties of the boundary

We want to show that under the assumptions we made on certain components of the metric, all the hypersurfaces defined above have the nature which will be needed when applying the Stokes' theorem or when we will like to use the positivity of the stress energy momentum tensor.

The vector $\nabla\tau = \nabla^\mu(\tau)\partial_\mu = \mathfrak{g}^{\mu\nu}\delta^\tau_\nu\partial_\mu = \mathfrak{g}^{\tau\tau}\partial_\tau + \mathfrak{g}^{x\tau}\partial_x + \mathfrak{g}^{A\tau}\partial_A$ *is normal to the hypersurfaces $\mathbf{H}_t$ and $\mathbf{H}_{\lambda,t}$, and the square of its norm is* $\mathfrak{g}(\nabla\tau, \nabla\tau) = \mathfrak{g}^{\tau\tau} < 0$. *Therefore $\nabla\tau$ is time-like and thus these hypersurfaces are spacelike. Their* past directed *unit normal is*
$$\eta = \eta^\mu\partial_\mu = \frac{1}{\sqrt{|\mathfrak{g}^{\tau\tau}|}}(\mathfrak{g}^{\tau\tau}\partial_\tau + \mathfrak{g}^{x\tau}\partial_x + \mathfrak{g}^{A\tau}\partial_A) \ . \tag{3.1.17}$$
We also note de following
$$\eta_\mu = \mathfrak{g}_{\mu\nu}\eta^\nu = \frac{1}{\sqrt{|\mathfrak{g}^{\tau\tau}|}}\mathfrak{g}_{\mu\nu}\mathfrak{g}^{\nu\tau} = \frac{1}{\sqrt{|\mathfrak{g}^{\tau\tau}|}}\delta^\tau_\mu$$
that is
$$\eta_\mu dx^\mu = \frac{1}{\sqrt{|\mathfrak{g}^{\tau\tau}|}}d\tau \ . \tag{3.1.18}$$
As far as the hypersurfaces S_λ are concerned, the functions σ_λ are assumed to be such that the normal $N = \nabla\{-x + \sigma_\lambda(\tau)\}$ is timelike and the outward unit normal to this hypersurface is such that the integral of the contracted energy momentum tensor is negative (see (3.2.18)). The same remark holds for the hypersurface S.

3.2 Estimates on the space derivatives of the solution

We want to derive weighted energy inequalities for solutions of (3.0.1). These inequalities will be used to prove existence of a solution satisfying the hypothesis of the theorem of polyhomogeneous solution of quasi-linear wave equation near scri.

3.2.1 The stress energy momentum tensor and its properties

The stress-energy tensor of the system (3.0.1) is given by

$$T_{\mu\nu} := \nabla_\mu u \nabla_\nu u - \frac{1}{2}\mathfrak{g}_{\mu\nu}\nabla^\alpha u \nabla_\alpha u \ .$$

The explicit form of ${T_0}^0$*, (the component of the tensor* T *which in general determines the energy density of the system) in local coordinates system is given by:*

$$\begin{aligned} {T_0}^0 &= \nabla^0 u \nabla_0 u - \frac{1}{2}\nabla^\alpha u \nabla_\alpha u \\ &= \mathfrak{g}^{0\beta}\nabla_\beta u \nabla_0 u - \frac{1}{2}\mathfrak{g}^{\alpha\beta}\nabla_\alpha u \nabla_\beta u \\ &= \{\mathfrak{g}^{00}\nabla_0 u \nabla_0 u + \mathfrak{g}^{0i}\nabla_i u \nabla_0 u\} - \frac{1}{2}\{\mathfrak{g}^{00}\nabla_0 u \nabla_0 u + 2\mathfrak{g}^{0i}\nabla_0 u \nabla_i u + \mathfrak{g}^{ij}\nabla_i u \nabla_j u\} \\ &= \frac{1}{2}\{\mathfrak{g}^{00}(\nabla_0 u)^2 - \mathfrak{g}^{ij}\nabla_i u \nabla_j u\} \ = \ -\frac{1}{2}\{-\mathfrak{g}^{00}(\nabla_0 u)^2 + |Du|^2\} \end{aligned} \tag{3.2.1}$$

with $|Du|^2 := \mathfrak{g}^{ij}\nabla_i u \nabla_j u$*.*

The tensor T *is symmetric and its divergence is given by*

$$\begin{aligned} \nabla_\mu {T_\nu}^\mu &= \Box_{\mathfrak{g}} u \nabla_\nu u \\ &= F\nabla_\nu u \qquad \textit{when } u \textit{ solves (3.0.1)} \ . \end{aligned} \tag{3.2.2}$$

Further, one of the useful properties of the tensor T *is its positivity: For any vectors fields* v^α *and* w^α *both causal future-pointing we have:*

$${T_\nu}^\mu v^\nu w_\mu \geq 0 \ . \tag{3.2.3}$$

Remark 3.2.1 In the particular frame (τ, x, v^A) we will be interested with, let us calculate the quantity $T^Y := T(\partial_\tau - \partial_x, d\tau) = {T_\tau}^\tau - {T_x}^\tau$ which we will use as energy density. From (3.2.1) we have:

$${T_\tau}^\tau = \frac{1}{2}\left\{\mathfrak{g}^{\tau\tau}(\partial_\tau u)^2 - \mathfrak{g}^{xx}(\partial_x u)^2 - 2\mathfrak{g}^{xA}\partial_x u \partial_A u - \mathfrak{g}^{AB}\partial_A u \partial_B u\right\} \ .$$

This expression shows that in the case we are concerned with, $T_\tau{}^\tau$ cannot be used to control the energy of the system near $\{x = 0\}$ since the metric component $\mathfrak{g}^{xx}$ can degenerate there. On the other hand we have

$$T_x{}^\tau = \mathfrak{g}^{\tau\tau}\partial_\tau u \partial_x u + \mathfrak{g}^{\tau x}\left(\partial_x u\right)^2 + \mathfrak{g}^{\tau A}\partial_x u \partial_A u \ ,$$

therefore we deduce the following expression of T^Y :

$$\begin{aligned} T^Y = \frac{1}{2}\Big\{\mathfrak{g}^{\tau\tau}\left(\partial_\tau u\right)^2 \quad - \quad & 2\mathfrak{g}^{\tau\tau}\partial_\tau u \partial_x u - \left(\mathfrak{g}^{xx} + 2\mathfrak{g}^{\tau x}\right)\left(\partial_x u\right)^2 \\ & -2\left(\mathfrak{g}^{xA} + \mathfrak{g}^{\tau A}\right)\partial_x u \partial_A u - \mathfrak{g}^{AB}\partial_A u \partial_B u\Big\} \ . \end{aligned} \tag{3.2.4}$$

Now, if we set

$$\left\{ \begin{array}{l} \lambda = \mathfrak{g}^{\tau\tau} + \mathfrak{g}^{xx} + 2\mathfrak{g}^{x\tau} = 1 + \mathrm{O}(x) > 0 \quad \text{(by hypothesis)} \\ \xi^A = \mathfrak{g}^{xA} + \mathfrak{g}^{A\tau} \\ \kappa^{AB} = \frac{\xi^A \xi^B}{\lambda} \end{array} \right. ,$$

then we obtain the following decomposition of T^Y

$$T^Y = -\frac{1}{2}\left\{ -\mathfrak{g}^{\tau\tau}\left(\partial_\tau u - \partial_x u\right)^2 + \lambda\left(\partial_x u + \frac{\left(\mathfrak{g}^{xA} + \mathfrak{g}^{A\tau}\right)}{\lambda}\partial_A u\right)^2 + \left(\mathfrak{g}^{AB} - \kappa^{AB}\right)\partial_A u \partial_B u \right\} . \tag{3.2.5}$$

The above decomposition shows that the quantity T^Y controls uniformly the energy of the system if and only if there exists $\epsilon_0 > 0$ (which can be made to coincide with the one occurring in (3.1.1)) such that

$$\lambda > \epsilon_0, \text{ and } \left(\mathfrak{g}^{AB} - \kappa^{AB}\right)\zeta_A \zeta_B \geq \epsilon_0 \textstyle\sum_A (\zeta_A)^2 \ ; \tag{3.2.6}$$

the existence of such a constant follows already from our previous hypotheses. It turns out that if we have a priori bounds on the L^∞ norms of $\mathfrak{g}^\sharp$ from above and below, this expression can be used to control all the components of the stress energy tensor. In fact we have

$$|T_\nu{}^\mu| = |\mathfrak{g}^{\mu\sigma}\partial_\sigma u \partial_\nu u - \frac{1}{2}\delta_\nu{}^\mu \mathfrak{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u| \leq C|\mathfrak{g}^\sharp||\partial u|^2 \leq C|\mathfrak{g}^\sharp||T_\tau{}^\tau - T_x{}^\tau| \ ; \tag{3.2.7}$$

here the constant C depends upon ϵ_0, and is allowed to change after each inequality symbol in general.

Remark 3.2.2 For further purposes we note that, using the vector field $\partial_\tau - \partial_x$, the principal part of the d'Alembertian has the following form:

$$\begin{aligned} \mathfrak{g}^{\alpha\beta}\partial_{\alpha\beta} \quad = \quad & \mathfrak{g}^{\tau\tau}(\partial_\tau - \partial_x)^2 + 2\left(\mathfrak{g}^{\tau\tau} + \mathfrak{g}^{\tau x}\right)\left(\partial_\tau - \partial_x\right)\partial_x + 2\mathfrak{g}^{\tau A}\left(\partial_\tau - \partial_x\right)\partial_A \\ & + \left(\mathfrak{g}^{\tau\tau} + 2\mathfrak{g}^{\tau x} + \mathfrak{g}^{xx}\right)\partial_x^2 + 2\left(\mathfrak{g}^{xA} + \mathfrak{g}^{\tau A}\right)\partial_x \partial_A + \mathfrak{g}^{AB}\partial_A \partial_B \ . \end{aligned} \tag{3.2.8}$$

3.2.2 Estimates on the first derivatives of the solution

We want to derive some energy inequalities for the solution u of the system (3.0.1). For this purpose, we consider the weighted energy at an instant t of the evolution of the system defined using the vector field $\partial_\tau - \partial_x$*; recall* $T^Y = T_\tau{}^\tau - T_x{}^\tau$*:*

$$E[u(t)] = -\int_{\mathbf{H}_t} x^{-2\alpha} T^Y \, \frac{dx}{x} d^{n-1}\nu_{t,x} \tag{3.2.9}$$

where $d^{n-1}\nu_{t,x}$ *is the measure defined on* $\{t\} \times \{x\} \times \mathscr{O}$ *by the metric* $\mathfrak{g}$ *(as will be made precise shortly), and* $\alpha \leq 0$ *a real parameter the range of which will be given later. We set*

$$E_\lambda[u(t)] = -\int_{\mathbf{H}_{\lambda,t}} x^{-2\alpha} T^Y \, \frac{dx}{x} d^{n-1}\nu_{t,x} \; . \tag{3.2.10}$$

Our strategy will be to obtain a bound of $E[u(t)]$ *from an uniform bound (with respect to* λ*) of* $E_\lambda[u(t)]$*. We will apply the divergence theorem to the energy-momentum tensor; this holds e.g. for* $C^{1,1}_{\rm loc}$ *functions u (first derivatives locally Lipschitz continuous). We want to establish the following (recall that* ϵ_0 *is the constant arising in (3.1.1) and in (3.2.6), while* C_0 *is defined in (3.1.6)):*

Proposition 3.2.3 *Let* $\alpha \leq -\frac{1}{2}$*. Under hypotheses (3.1.1)-(3.1.3) and (3.2.6), there exists a constant* C_1 *depending upon* ϵ_0, C_0, α *such that for all*

$$\tau \in [\tau_0, \tau_1] \quad \textit{and} \quad u \in C^{1,1}_{\rm loc}$$

satisfying (3.0.1), we have

$$\begin{aligned} E_\lambda[u(\tau)] \;\; \leq \;\; C_1 \Bigg\{ & E_\lambda[u(\tau_0)] + \int_{\tau_0}^{\tau} \Big\{ \|F(s)\|^2_{\mathscr{H}^\alpha_0(\mathbf{H}_{\lambda,s})} + \Big(1 + \|\mathfrak{h}^\sharp\|_{L^\infty} + \|\mathfrak{g}^\sharp\|_{L^\infty}\Big) \\ & \times \Big(1 + \|\mathfrak{g}\|^2_{L^\infty(\mathbf{H}_{\lambda,\mathbf{s}})} + \|\mathfrak{g}^\sharp\|^2_{L^\infty(\mathbf{H}_{\lambda,\mathbf{s}})} + \|\left(\partial_\tau - \partial_x\right)\mathfrak{g}^\sharp\|^2_{L^\infty(\mathbf{H}_{\lambda,\mathbf{s}})}\Big) \, E_\lambda[u(s)]\Big\} ds \Bigg\} \end{aligned} \tag{3.2.11}$$

Proof: *Stokes' theorem for the vector field* $\Lambda^\mu = x^{-2\alpha-1} T_\nu{}^\mu Y^\nu$ *on* $\mathscr{U}_{\lambda,\tau}$ *(compare Fig. 3.1) gives*

$$\int_{\partial\mathscr{U}_{\lambda,\tau}} x^{-2\alpha-1} T_\nu{}^\mu Y^\nu \eta_\mu dS = \int_{\mathscr{U}_{\lambda,\tau}} \nabla_\mu \{x^{-2\alpha-1} T_\nu{}^\mu Y^\nu\} dV \tag{3.2.12}$$

for an arbitrary differentiable vector field Y. Here

$$dV = \sqrt{|\det \mathfrak{g}|} d\tau \wedge dx \wedge d^{n-1}v \ , \tag{3.2.13}$$

where $\det \mathfrak{g}$ *is the determinant of the metric* $\mathfrak{g}$. *Further, on non-characteristic parts of the boundary,* η_μ *is the unit outwards pointing conormal, and*

$$dS = \sqrt{|\det \gamma|} d^n y \ , \tag{3.2.14}$$

with y^i, $i = 1, \ldots, n$, *a system of coordinates on the corresponding boundary, and* γ *the metric induced on it by the metric* $\mathfrak{g}$*; i.e.* $\gamma = j^* \mathfrak{g}$, j *being the canonical injection of the boundary into the manifold. (On characteristic parts of the boundary, a convenient choice of* η_μ *and* dS *will be made as need arises). In the case under consideration,* $\partial \mathscr{U}_{\lambda,\tau}$ *is made of four pieces* $\mathbf{H}_{\lambda,\tau_0}$, $\mathbf{H}_{\lambda,\tau}$, *together with*

$$S_{\lambda,\tau} := S_\lambda \cap \{0 \le t \le \tau\} \ \text{and} \ S^\tau := S \cap \{0 \le t \le \tau\} \ .$$

Therefore the identity (3.2.12) reads:

$$\begin{aligned}\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1} T_\nu{}^\mu Y^\nu \eta_\mu dS \ &+ \ \int_{\mathbf{H}_{\lambda,\tau_0}} x^{-2\alpha-1} T_\nu{}^\mu Y^\nu \eta_\mu dS + \int_{S_{\lambda,\tau}} x^{-2\alpha-1} T_\nu{}^\mu Y^\nu \eta_\mu dS \\ &+ \ \int_{S^\tau} x^{-2\alpha-1} T_\nu{}^\mu Y^\nu \eta_\mu dS = \int_{\mathscr{U}_{\lambda,\tau}} \nabla_\mu \{ x^{-2\alpha-1} T_\nu{}^\mu Y^\nu \} dV \ .\end{aligned} \tag{3.2.15}$$

The left-hand-side of equation (3.2.15) is made of four terms which will be labeled in their order of appearance L_1, L_2, L_3 *and* L_4. *As mentioned before, we choose the vector field* $Y = Y^\mu \partial_\mu$ *to be equal to* $\partial_\tau - \partial_x$. *Once this choice is made, let us look at each of the terms* $L_i, i = 1, 2, 3, 4$. *Recall that (see equation (3.1.18)) on* $\mathbf{H}_{\lambda,\tau}$ *we have:*

$$\eta_\mu dx^\mu = \frac{1}{\sqrt{|\mathfrak{g}^{\tau\tau}|}} d\tau \quad \text{which implies that} \quad T_\nu{}^\mu Y^\nu \eta_\mu = \frac{1}{\sqrt{|\mathfrak{g}^{\tau\tau}|}} \{ T_\tau{}^\tau - T_x{}^\tau \}$$

and $dS = \sqrt{|\det \gamma|} dx \wedge d^{n-1}v$ *is the surface element denoted in equations (3.2.9) and (3.2.4) by* $dx \, d^{n-1}v_{t,x}$. *Since* $\eta_0 \sqrt{\det \mathfrak{g}} = \sqrt{\det \gamma}$ *on* $\mathbf{H}_{\lambda,\tau}$, *we obtain that (remember that* $\eta^\mu \partial_\mu$ *is past directed)*

$$L_1 = -E_\lambda[u(\tau)] \ . \tag{3.2.16}$$

From this, the sign coming from the Stokes' identity shows that

$$L_2 = E_\lambda[u(\tau_0)] \ . \tag{3.2.17}$$

On the hypersurfaces S_λ and S, since the unit outward normal is also past directed and the vector field $Y^\nu\partial_\nu = \partial_\tau - \partial_x$ future directed, we deduce from the positivity of the stress energy tensor that:

$$L_3 \leq 0 \quad and \quad L_4 \leq 0 \, . \tag{3.2.18}$$

We can now rewrite (3.2.15) as:

$$-E_\lambda[u(\tau)] + E_\lambda[u(\tau_0)] + L_3 + L_4 = \int_{\mathscr{U}_{\lambda,\tau}} \nabla_\mu\{x^{-2\alpha-1}T_\nu{}^\mu Y^\nu\}dV \, . \tag{3.2.19}$$

Now, let us consider the right-hand side of the above equation. We have:

$$\begin{aligned}
&\nabla_\mu\{x^{-2\alpha-1}T_\nu{}^\mu Y^\nu\} \\
&= \; x^{-2\alpha-1}\Big\{(\nabla_\mu T_\nu{}^\mu)Y^\nu + T_\nu{}^\mu(\nabla_\mu Y^\nu) - (2\alpha+1)x^{-1}T_\nu{}^\mu Y^\nu\nabla_\mu(x)\Big\} \\
&= \; x^{-2\alpha-1}\Big\{(\nabla_\mu T_\nu{}^\mu)Y^\nu + T_\nu{}^\mu\left\{\Gamma^\nu_{\mu\tau} - \Gamma^\nu_{\mu x}\right\} \\
&\quad -(2\alpha+1)x^{-1}\nabla_\mu x\,\left\{T_\tau{}^\mu - T_x{}^\mu\right\}\Big\} \\
&=: \; R_1 + R_2 + R_3,
\end{aligned} \tag{3.2.20}$$

where

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}\mathfrak{g}^{\sigma\rho}(\partial_\mu\mathfrak{g}_{\sigma\nu} + \partial_\nu\mathfrak{g}_{\mu\sigma} - \partial_\sigma\mathfrak{g}_{\mu\nu}),$$

are the Christoffel's symbols of the metric $\mathfrak{g}$. From (3.2.2), we have:

$$\begin{aligned}
x^{2\alpha+1}|R_1| = |F||\nabla_\nu uY^\nu| = |F||\left(\partial_\tau u - \partial_x u\right)| \; &\leq \; \frac{1}{2}\Big\{F^2 + (\partial_\tau u - \partial_x u)^2\Big\}) \\
&\leq \; c(\epsilon_0)\left(F^2 + \left|T_\tau{}^\tau - T_x{}^\tau\right|\right) \, .
\end{aligned} \tag{3.2.21}$$

As far as the second term is concerned, we have:

$$T_\nu{}^\mu\Gamma^\nu_{\mu\theta} \;=\; \frac{1}{2}T^{\mu\sigma}\partial_\theta\mathfrak{g}_{\mu\sigma} \;=\; -\frac{1}{2}T_{\mu\sigma}\partial_\theta\mathfrak{g}^{\mu\sigma}.$$

Thus, replacing successively in the above expression θ with τ and x and subtracting the two expressions we find that

$$x^{2\alpha+1}R_2 = -\frac{1}{2}T_\mu{}^\nu\mathfrak{g}_{\nu\sigma}\left(\partial_\tau - \partial_x\right)\mathfrak{g}^{\mu\sigma} \, .$$

From (3.2.7) we obtain:

$$x^{2\alpha+1}|R_2| = |T_{\nu\sigma}\left(\partial_\tau - \partial_x\right)\mathfrak{g}^{\mu\sigma}| \leq (n+1)C|\mathfrak{g}^\sharp|\left(|\mathfrak{g}|^2 + |(\partial_\tau - \partial_x)\mathfrak{g}^\sharp|^2\right)\left|T_\tau{}^\tau - T_x{}^\tau\right| \, . \tag{3.2.22}$$

For the third term we have, keeping in mind (3.1.5):

$$\begin{aligned}
x^{2\alpha+1}R_3 &= -(2\alpha+1)x^{-1}T_\nu{}^\mu\nabla_\mu x\,Y^\nu\\
&= -(2\alpha+1)x^{-1}\mathfrak{g}^{\mu\sigma}T_{\nu\sigma}\nabla_\mu x\,Y^\nu\\
&= -(2\alpha+1)x^{-1}T_{\mu\nu}\nabla^\mu x\,Y^\nu\\
&= \underbrace{-(2\alpha+1)x^{-1}T_{\mu\nu}Y^\nu\omega^{(1)\mu}}_{\geq 0}-(2\alpha+1)\frac{\beta(x)}{x}T_{\mu\nu}Y^\nu\omega^{(2)\mu} \quad \textit{for} \quad \alpha\leq\ -1/2\\
&\geq -(2\alpha+1)\frac{\beta(x)}{x}T_{\mu\nu}Y^\nu\omega^{(2)\mu}=-(2\alpha+1)\frac{\beta(x)}{x}\left(T_{\mu\tau}-T_{\mu x}\right)\omega^{(2)\mu}\\
&\geq -C(\ \alpha,C_0,n)|\mathfrak{h}^\sharp|\left(1+|\mathfrak{g}|^2+|\mathfrak{g}^\sharp|^2\right)|T_\tau{}^\tau-T_x{}^\tau|\ . \qquad (3.2.23)
\end{aligned}$$

Let us justify the last inequality. In other words let us show that the expression $T_{\mu\tau}-T_{\mu x}$ is controlled by $|T_\tau{}^\tau-T_x{}^\tau|$. We have:

$$\begin{aligned}
|T_{\mu\tau}-T_{\mu x}| &= |\partial_\mu u\left(\partial_\tau-\partial_x\right)u-\frac{1}{2}\left(\mathfrak{g}_{\mu\tau}-\mathfrak{g}_{\mu x}\right)\mathfrak{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u|\\
&\leq (\partial_\mu u)^2+[(\partial_\tau-\partial_x)\,u]^2+\left(|\mathfrak{g}|^2+|\mathfrak{g}^\sharp|^2\right)\left(\delta^{\alpha\beta}\partial_\alpha u\partial_\beta u\right)\\
&\leq C(\epsilon_0)\left(1+|\mathfrak{g}|^2+|\mathfrak{g}^\sharp|^2\right)|T_\tau{}^\tau-T_x{}^\tau|. \qquad \textit{See (3.2.5)}
\end{aligned}$$

Inequalities (3.2.21), (3.2.22) and (3.2.23) show that the right-hand side of (3.2.20) can be estimated as:

$$R_1+R_2+R_3\geq -C_1x^{-(2\alpha+1)}\left\{\left(1+|\mathfrak{h}^\sharp|+|\mathfrak{g}^\sharp|\right)\left(1+|\mathfrak{g}|^2+|\mathfrak{g}^\sharp|^2+|(\partial_\tau-\partial_x)\mathfrak{g}^\sharp|^2\right)|T^Y|+F^2\right\}\ , \qquad (3.2.24)$$

where $C_1=C(\alpha,\epsilon_0,C_0,n)$. Now from (3.2.19) we have

$$-E_\lambda[u(t)]+E_\lambda[u(\tau_0)]+L_3+L_4=R_1+R_2+R_3,$$

thus, using (3.2.18), we obtain the following:

$$\begin{aligned}
E_\lambda[u(t)]\leq E_\lambda[u(\tau_0)] &+ C_1\int_{\tau_0}^t\int_{\mathbf{H}_{\lambda,s}}x^{-2\alpha}\left\{\left(1+|\mathfrak{h}^\sharp|+|\mathfrak{g}^\sharp|\right)\left(1+|\mathfrak{g}|^2+|\mathfrak{g}^\sharp|^2\right.\right.\\
&+ \left.\left.|(\partial_\tau-\partial_x)\mathfrak{g}^\sharp|^2\right)|T^Y|+F^2(s)\right\}ds\frac{dx}{x}d^{n-1}\nu\ .
\end{aligned}$$

Therefore, there exists a constant $C_1 > 0$ depending upon n, ϵ_0 , α and C_0 such that

$$E_\lambda[u(\tau)] \leq C_1\Bigg\{ E_\lambda[u(\tau_0)] + \int_{\tau_0}^{\tau} \Big\{ \|F(s)\|^2_{\mathscr{H}_0^\alpha(\mathbf{H}_{\lambda,s})} + \Big(1 + \|\mathfrak{h}^\sharp\|_{L^\infty} + \|\mathfrak{g}^\sharp\|_{L^\infty}\Big) \times \Big(1 + \|\mathfrak{g}\|^2_{L^\infty(\mathbf{H}_{\lambda,\mathbf{s}})} + \|\mathfrak{g}^\sharp\|^2_{L^\infty(\mathbf{H}_{\lambda,\mathbf{s}})} + \|\,(\partial_\tau - \partial_x)\,\mathfrak{g}^\sharp\|^2_{L^\infty(\mathbf{H}_{\lambda,\tau})}\Big)\, E_\lambda[u(s)]\Big\} ds \Bigg\} \quad (3.2.25)$$

and the proof is completed. □

3.2.3 Estimates on the higher space derivatives of the solution

To proceed further, we would like to have an estimate similar to (3.2.11) on space derivatives of the unknown function in equation (3.0.1). For this purpose, for $k \in \mathbb{N}$, $\beta = (\beta_1, \beta_2, \ldots, \beta_r) \in \mathbb{N}^r$, with $|\beta| \leq k$; we set:

$$\overset{(\beta)}{T}{}_\nu{}^\mu = x^{-2\alpha-1+2\beta_1}\left\{\nabla^\mu \mathscr{D}^\beta u \nabla_\nu \mathscr{D}^\beta u - \frac{1}{2}\delta_\nu{}^\mu \nabla^\alpha \mathscr{D}^\beta u \nabla_\alpha \mathscr{D}^\beta u\right\},$$

where $\alpha \leq -1/2$ is the real parameter of the previous section, $\mathscr{D}^\beta = X_1^{\beta_1} X_2^{\beta_2} \ldots X_r^{\beta_r}$, with the X_i's being the vector fields defined in [20] page 51: for $i = 2, \ldots, r$, $X_i = \sum\limits_{A=2}^{r} X_i^A(v)\partial_A$, where the X_i^A's are smooth functions bounded on bounded set with all their derivatives, and $X_1 = \partial_x$. Since the operator ∇ is linear, as in (3.2.2), we have

$$\nabla_\mu \overset{(\beta)}{T}{}_\nu{}^\mu = x^{2\alpha-1+2\beta_1}\Box_{\mathfrak{g}}(\mathscr{D}^\beta u)\nabla_\nu(\mathscr{D}^\beta u) + (-2\alpha - 1 + 2\beta_1)\frac{\nabla_\mu(x)}{x}\overset{(\beta)}{T}{}_\nu{}^\mu .$$

Now

$$\Box_{\mathfrak{g}}(\mathscr{D}^\beta u) = \mathscr{D}^\beta(\Box_{\mathfrak{g}} u) + [\Box_{\mathfrak{g}}, \mathscr{D}^\beta]u = \mathscr{D}^\beta F + [\Box_{\mathfrak{g}}, \mathscr{D}^\beta]u \,, \quad (3.2.26)$$

for any solution of the equation (3.0.1). Thus

$$\nabla_\mu \overset{(\beta)}{T}{}_\nu{}^\mu = x^{-2\alpha-1+2\beta_1}\left\{\mathscr{D}^\beta F + [\Box_{\mathfrak{g}}, \mathscr{D}^\beta]u\right\}\nabla_\nu(\mathscr{D}^\beta u) + (-2\alpha-1+2\beta_1)\frac{\nabla_\mu(x)}{x}\overset{(\beta)}{T}{}_\nu{}^\mu . \quad (3.2.27)$$

Similarly to the previous section, we set:

$$\overset{(\beta)}{T}{}^{Y} = \overset{(\beta)}{T}{}_{\tau}{}^{\tau} - \overset{(\beta)}{T}{}_{x}{}^{\tau} \ ,$$

$$E_k^{\alpha}[u(\tau)] = \sum_{|\beta|=0}^{k} \int_{\mathbf{H}_t} -\overset{(\beta)}{T}{}^{Y} \, dx d^{n-1}\nu_{t,x} \quad and \quad E_{k,\lambda}^{\alpha}[u(t)] = \sum_{|\beta|=0}^{k} \int_{\mathbf{H}_{\lambda,\tau}} -\overset{(\beta)}{T}{}^{Y} \, dx d^{n-1}\nu_{t,x} \ . \tag{3.2.28}$$

Remark 3.2.4 From (3.2.5) we deduce the following decomposition for $\overset{(\beta)}{T}{}^{Y}$:

$$\begin{aligned}\overset{(\beta)}{T}{}^{Y} \quad = \quad & -\frac{1}{2}\Bigg\{ - \mathfrak{g}^{\tau\tau}\left(x^{-\alpha-\frac{1}{2}+\beta_1}\mathscr{D}^{\beta}(\partial_\tau - \partial_x)u\right)^2 \\ & + \lambda\left(x^{-\alpha-\frac{1}{2}+\beta_1}\mathscr{D}^{\beta}(\partial_x u) + \frac{\left(\mathfrak{g}^{xA} + \mathfrak{g}^{A\tau}\right)}{\lambda}\partial_A\left(x^{-\alpha-\frac{1}{2}+\beta_1}\mathscr{D}^{\beta}u\right)\right)^2 \\ & + \left(\mathfrak{g}^{AB} - \kappa^{AB}\right)\partial_A\left(x^{-\alpha-\frac{1}{2}+\beta_1}\mathscr{D}^{\beta}u\right)\partial_B\left(x^{-\alpha-\frac{1}{2}+\beta_1}\mathscr{D}^{\beta}u\right)\Bigg\} \ . \end{aligned} \tag{3.2.29}$$

Since the coefficients of the terms arising in commutating ∂_A and $\mathscr{D}^{\beta}$ are uniformly bounded, from the above we find that the energy of order k controls the $\mathscr{H}_k^{\alpha}$-norms of the first order derivatives of the unknown function u. That is:

$$\|(\partial_\tau - \partial_x)u\|^2_{\mathscr{H}_k^{\alpha}(\mathbf{H}_{\lambda,\tau})} + \|\partial_x u\|^2_{\mathscr{H}_k^{\alpha}(\mathbf{H}_{\lambda,\tau})} + \sum_A \|\partial_A u\|^2_{\mathscr{H}_k^{\alpha}(\mathbf{H}_{\lambda,\tau})} \le E_{k,\lambda}^{\alpha}[u(\tau)] \ . \tag{3.2.30}$$

Let us set

$$\Upsilon^{\nu} := -\mathfrak{g}^{\alpha\mu}\Gamma^{\nu}_{\alpha\mu} = \frac{1}{\sqrt{|\det \mathfrak{g}|}}\partial_\mu\left(\sqrt{|\det \mathfrak{g}|}\mathfrak{g}^{\mu\nu}\right) \ . \tag{3.2.31}$$

Let us define

$$\begin{aligned} M(\tau) \quad := \quad & \|F\|^2_{\mathscr{B}_0^{\alpha}(\mathbf{H}_\tau)} + \|(\mathfrak{g}, (\partial_\tau - \partial_x)\mathfrak{g}^{\sharp})\|^2_{L^{\infty}(\mathbf{H}_\tau)} \\ & + \|(\mathfrak{g}^{\sharp}, \mathfrak{h}^{\sharp}, \Upsilon)\|^2_{\mathscr{C}^0_{\{x=0\},1}(\mathbf{H}_\tau)} \ . \end{aligned} \tag{3.2.32}$$

We claim that:

Proposition 3.2.5 *Let $\lambda > 0$, $k \in \mathbb{N}$ and suppose that $\alpha \le -\frac{1}{2}$. Under hypotheses (3.1.1)-(3.1.3) and (3.2.6), there exists a function $C_2(\epsilon_0, C_0, \alpha, k, n, M)$. monotonously increasing in M, which we write as $C_2(M)$, such that for all*

$$\tau \in [\tau_0, \tau_1]$$

and for all u satisfying (3.0.1) we have

$$\begin{aligned} & E^{\alpha}_{k,\lambda}[u(\tau)] \\ \le \quad & E^{\alpha}_{k,\lambda}[u(\tau_0)] + \int_{\tau_0}^{\tau} C_2(M(s)) \Bigg\{ E^{\alpha}_{k,\lambda}[u(s)] + \|F(s)\|^2_{\mathscr{H}^{\alpha}_k(\mathbf{H}_{\lambda,\tau})} \\ & + \|\big((\partial_\tau - \partial_x)u, \partial_x u, \partial_A u\big)\|^2_{\mathscr{B}^{\alpha}_1(\mathbf{H}_{\lambda,\tau})} \times \| \Big(\mathfrak{g}^\sharp, \mathfrak{h}^\sharp, \Upsilon\Big) \|^2_{\mathscr{G}^0_k(\mathbf{H}_{\lambda,\tau})} \Bigg\} ds \ . \end{aligned} \tag{3.2.33}$$

Remark 3.2.6 The reader should note that C_2 does not depend upon λ.

Proof: *If the right-hand side of (3.2.33) is infinite there is nothing to prove. Otherwise, the calculations that follow should be done assuming smoothness of u, and the inequality for general u's can be obtained by a density argument.*

The equivalent of (3.2.15) for space-derivatives of the solution of (3.0.1) reads:

$$\begin{aligned} \sum_{|\beta|=0}^{k} \int_{\mathbf{H}_{\lambda,\tau}} \overset{(\beta)}{T_\nu}{}^{\mu} Y^\nu \eta_\mu dS \quad + \quad & \sum_{|\beta|=0}^{k} \int_{\mathbf{H}_{\lambda,T_0}} \overset{(\beta)}{T_\nu}{}^{\mu} Y^\nu \eta_\mu dS + \sum_{|\beta|=0}^{k} \int_{S_{\lambda,\tau}} \overset{(\beta)}{T_\nu}{}^{\mu} Y^\nu \eta_\mu dS \\ + \quad & \sum_{|\beta|=0}^{k} \int_{S_{\theta,\tau}} \overset{(\beta)}{T_\nu}{}^{\mu} Y^\nu \eta_\mu dS = \sum_{|\beta|=0}^{k} \int_{\Omega_{\lambda,\tau}} \nabla_\mu \left(\overset{(\beta)}{T_\nu}{}^{\mu} Y^\nu \right) dV \end{aligned} \tag{3.2.34}$$

which gives the following equation:

$$\begin{aligned} -E^{\alpha}_{k,\lambda}[u(\tau)] + E^{\alpha}_{k,\lambda}[u(\tau_0)] \quad + \quad & \underbrace{\sum_{|\beta|=0}^{k} \int_{S_{\lambda,\tau}} \overset{(\beta)}{T_\nu}{}^{\mu} Y^\nu \eta_\mu dS + \sum_{|\beta|=0}^{k} \int_{S_{\theta,\tau}} \overset{(\beta)}{T_\nu}{}^{\mu} Y^\nu \eta_\mu dS}_{:= \hat{L}_3 + \hat{L_4} \le 0} \\ = & \sum_{|\beta|=0}^{k} \int_{\Omega_{\lambda,\tau}} \nabla_\mu \{ \overset{(\beta)}{T_\nu}{}^{\mu} Y^\nu \} dx\, d\nu. \end{aligned} \tag{3.2.35}$$

Again as in the previous section we take $Y^\nu\partial_\nu = \partial_\tau - \partial_x$, then the divergence in the right-hand side of (3.2.35) reads:

$$
\begin{aligned}
\nabla_\mu\{\overset{(\beta)}{T}{}_\nu{}^\mu Y^\nu\} &= \nabla_\mu \overset{(\beta)}{T}{}_\nu{}^\mu Y^\nu + \overset{(\beta)}{T}{}_\nu{}^\mu \nabla_\mu Y^\nu \\
&= x^{-2\alpha-1-2\beta_1}\left\{\mathscr{D}^\beta F + [\Box_{\mathfrak{g}}, \mathscr{D}^\beta]u\right\}(\partial_\tau - \partial_x)(\mathscr{D}^\beta u) \\
&+ \overset{(\beta)}{T}{}_\nu{}^\mu\left(\Gamma^\nu_{\mu\tau} - \Gamma^\nu_{\mu x}\right) + (-2\alpha - 1 + 2\beta_1)\frac{\nabla_\mu(x)}{x}\left(\overset{(\beta)}{T}{}_\tau{}^\mu - \overset{(\beta)}{T}{}_x{}^\mu\right) \\
&=: \widehat{R}_1 + \widehat{R}_2 + \widehat{R}_3.
\end{aligned}
\tag{3.2.36}
$$

If we repeat the calculations in the previous section that led to (3.2.22) and (3.2.23), we obtain that there exists a constant $C = C(n, k, C_0, \alpha, \epsilon_0) > 0$ such that:

$$
|\widehat{R}_2| \le C|\mathfrak{g}^\sharp|\left(|\mathfrak{g}|^2 + |(\partial_\tau - \partial_x)\mathfrak{g}^\sharp|^2\right)|\overset{(\beta)}{T}{}^Y| \tag{3.2.37}
$$

and, keeping in mind that the term with the worst power of x can be discarded because of a favorable sign,

$$
\hat{R}_3 \ge -C|\mathfrak{h}^\sharp|\left(1 + |\mathfrak{g}|^2 + |\mathfrak{g}^\sharp|^2\right)|\overset{(\beta)}{T}{}^Y|\,. \tag{3.2.38}
$$

As far as the term $\widehat{R}_1$ is concerned, from the inequality $ab \le \frac{1}{2}(a^2 + b^2)$, we have:

$$
\begin{aligned}
x^{2\alpha+1-2\beta_1}|\widehat{R}_1| &= |\{\mathscr{D}^\beta F + [\Box_{\mathfrak{g}}, \mathscr{D}^\beta]u\}(\partial_\tau - \partial_x)(\mathscr{D}^\beta u)| \\
&\le \frac{1}{2}(\mathscr{D}^\beta F)^2 + \frac{1}{2}\left(\left[\Box_{\mathfrak{g}}, \mathscr{D}^\beta\right]u\right)^2 + \left[(\partial_\tau - \partial_x)\left(\mathscr{D}^\beta u\right)\right]^2 \\
&\le (\mathscr{D}^\beta F)^2 + C(\epsilon_0)|\overset{(\beta)}{T}{}^Y| + \left(\left[\Box_{\mathfrak{g}}, \mathscr{D}^\beta\right]u\right)^2 .
\end{aligned}
\tag{3.2.39}
$$

From inequalities (3.2.37), (3.2.38) , (3.2.39) and the fact that $\hat{L}_3$, $\hat{L}_4 \le 0$ we obtain that:

$$
\begin{aligned}
E^\alpha_{k,\lambda}[u(\tau)] - E^\alpha_{k,\lambda}[u(T_0)] \;\le\; & C\int_{T_0}^\tau \Big[\left(1 + \|\mathfrak{h}^\sharp\|_{L^\infty} + \|\mathfrak{g}^\sharp\|_{L^\infty}\right)\left(1 + \|\mathfrak{g}\|^2_{L^\infty} + \|\mathfrak{g}^\sharp\|^2_{L^\infty}\right. \\
& \left. + \|(\partial_\tau - \partial_x)\,\mathfrak{g}^\sharp\|^2_{L^\infty}\right)E^\alpha_{k,\lambda}[u(s)] + \|F(s)\|^2_{\mathscr{H}^\alpha_k(\mathbf{H}_{\lambda,\mathbf{s}})}\Big]ds \\
& + \sum_{|\beta|=0}^k \int_{T_0}^\tau \int_{\mathbf{H}_{\lambda,s}} x^{-2\alpha-1+2\beta_1}([\Box_{\mathfrak{g}}, \mathscr{D}^\beta]u)^2(s)\,dx\,d\nu_{t,x}ds
\end{aligned}
\tag{3.2.40}
$$

with $C = C(n, \alpha, k, C_0, \epsilon_0)$. Now, let us estimate the last term of the right-hand side of the above inequality. From the definition (3.2.31) of Υ^μ we have

$$\Box_{\mathfrak{g}} = \mathfrak{g}^{\mu\nu}\partial^2_{\mu\nu} + \Upsilon^\nu\partial_\nu, \tag{3.2.41}$$

and then

$$\begin{aligned}[\Box_{\mathfrak{g}}, \mathscr{D}^\beta]u &= \mathfrak{g}^{\alpha\mu}[\partial_\alpha\partial_\mu, \mathscr{D}^\beta]u - \Upsilon^\nu[\mathscr{D}^\beta, \partial_\nu]u - \Big\{\mathscr{D}^\beta\left(\Upsilon^\nu\partial_\nu u\right) - \Upsilon^\nu\mathscr{D}^\beta\left(\partial_\nu u\right)\Big\} \\ &\quad - \Big\{\mathscr{D}^\beta\left(\mathfrak{g}^{\alpha\mu}\partial_\alpha\partial_\mu u\right) - \mathfrak{g}^{\alpha\mu}\mathscr{D}^\beta\left(\partial_\alpha\partial_\mu u\right)\Big\} \\ &=: A_1 + A_2 + A_3 + A_4\,. \end{aligned} \tag{3.2.42}$$

To estimate the first and second terms, we use the explicit form of the differential operator $\mathscr{D}$: $\mathscr{D}^\beta = \partial_x^{\beta_1}X_2^{\beta_2}\ldots X_r^{\beta_r} = \partial_x^{\beta_1}X_v^{\beta_v}$. Since ∂_τ and ∂_x commute with $\mathscr{D}^\beta$, we have (see (3.2.8))

$$A_1 = \mathfrak{g}^{\mu\alpha}[\partial_\mu\partial_\alpha, \mathscr{D}^\beta]u = 2\mathfrak{g}^{\tau A}[(\partial_\tau - \partial_x)\partial_A, \mathscr{D}^\beta]u + 2(\mathfrak{g}^{xA} + \mathfrak{g}^{\tau A})[\partial_x\partial_A, \mathscr{D}^\beta]u + \mathfrak{g}^{AB}[\partial_A\partial_B, \mathscr{D}^\beta]u,$$

and since

$$\mathfrak{g}^{\tau A}[(\partial_\tau - \partial_x)\partial_A, \mathscr{D}^\beta]u = \mathfrak{g}^{\tau A}\partial_x^{\beta_1}\partial_A X_v^{\beta_v}[(\partial_\tau - \partial_x)u] - \mathfrak{g}^{\tau A}\partial_x^{\beta_1}X_v^{\beta_v}\partial_A[(\partial_\tau - \partial_x)u]$$

we obtain that (see (3.2.30):

$$\begin{aligned}\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}\Big(\mathfrak{g}^{\tau A}[(\partial_\tau - \partial_x)\partial_A, \mathscr{D}^\beta]u\Big)^2 dx\, d\nu &\leq c\|\mathfrak{g}^\sharp\|^2_{L^\infty(\mathbf{H}_{\lambda,\tau})}\|(\partial_\tau - \partial_x)u\|^2_{\mathscr{H}_k^\alpha} \\ &\leq c\|\mathfrak{g}^\sharp\|^2_{L^\infty(\mathbf{H}_{\lambda,\tau})}E^\alpha_{k,\lambda}[u(\tau)]\,.\end{aligned}$$

Similarly, we have

$$\left(\mathfrak{g}^{xA} + \mathfrak{g}^{\tau A}\right)[\partial_x\partial_A, \mathscr{D}^\beta]u = \left(\mathfrak{g}^{xA} + \mathfrak{g}^{\tau A}\right)\Big(\partial_A\mathscr{D}^\beta(\partial_x u) - \mathscr{D}^\beta\partial_A(\partial_x u)\Big)\,,$$

which leads to:

$$\begin{aligned}\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}\Big\{\left(\mathfrak{g}^{xA} + \mathfrak{g}^{\tau A}\right)[\partial_x\partial_A, \mathscr{D}^\beta]u\Big\}^2(s)dx\, d\nu &\leq C\|\mathfrak{g}^\sharp\|^2_{L^\infty(\mathbf{H}_{\lambda,\tau})}\|\partial_x u\|^2_{\mathscr{H}_k^\alpha} \\ &\leq c\|\mathfrak{g}^\sharp\|^2_{L^\infty(\mathbf{H}_{\lambda,\tau})}E^\alpha_{k,\lambda}[u(\tau)]\,.\end{aligned}$$

Similar calculations give:

$$\begin{aligned}\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}(\mathfrak{g}^{AB}[\partial_A\partial_B, \mathscr{D}^\beta]u)^2(s)dx\, d\nu &\leq c\|\mathfrak{g}^\sharp\|^2_{L^\infty(\mathbf{H}_{\lambda,\tau})}\sum_A\|\partial_A u\|^2_{\mathscr{H}_k^\alpha} \\ &\leq c\|\mathfrak{g}^\sharp\|^2_{L^\infty(\mathbf{H}_{\lambda,\tau})}E^\alpha_{k,\lambda}[u(\tau)]\,.\end{aligned}$$

We obtain thus the following estimate for the first term of the identity (3.2.42):

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} A_1^2 dx\, d\nu \leq C \|\mathfrak{g}^\sharp\|^2_{L^\infty(\mathbf{H}_{\lambda,\tau})} E^{\alpha}_{k,\lambda}[u(\tau)] \ . \tag{3.2.43}$$

Again since ∂_τ and ∂_x commute with $\mathscr{D}^\beta$, if we develop the second term of (3.2.42), we find that:

$$A_2 = \Upsilon^\nu [\mathscr{D}^\beta, \partial_\nu] u = \Upsilon^A [\mathscr{D}^\beta, \partial_A] u$$

and we then have the estimates:

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} A_2^2 dx\, d\nu \leq \|\Upsilon^A\|^2_{L^\infty} \|\partial_A u\|^2_{\mathscr{H}^\alpha_{k-1}} \leq \|\Upsilon^A\|^2_{L^\infty} E^{\alpha}_{k,\lambda}[u(\tau)] \ . \tag{3.2.44}$$

As far as the third term is concerned, we write

$$\begin{aligned} A_3 = \mathscr{D}^\beta \left(\Upsilon^\nu \partial_\nu u\right) - \Upsilon^\nu \mathscr{D}^\beta \left(\partial_\nu u\right) &= \mathscr{D}^\beta \left(\Upsilon^\tau (\partial_\tau - \partial_x) u\right) - \Upsilon^\tau \mathscr{D}^\beta \left((\partial_\tau - \partial_x) u\right) \\ &\quad + \mathscr{D}^\beta \left((\Upsilon^x + \Upsilon^\tau) \partial_x u\right) - (\Upsilon^x + \Upsilon^\tau) \mathscr{D}^\beta \left(\partial_x u\right) \\ &\quad + \mathscr{D}^\beta \left(\Upsilon^A \partial_A u\right) - \Upsilon^A \mathscr{D}^\beta \left(\partial_A u\right) \\ &=: I + II + III \ . \end{aligned}$$

Now we will use the weighted Moser-type inequality B.2.10 of Proposition B.2.3 to estimate the components of A_3. Its first component gives the following

$$\begin{aligned} &\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} \left\{I\right\}^2 dx\, d\nu \\ &\qquad = \|x^{\beta_1} \mathscr{D}^\beta \left(\Upsilon^\tau (\partial_\tau - \partial_x) u\right) - x^{\beta_1} \Upsilon^\tau \mathscr{D}^\beta \left((\partial_\tau - \partial_x) u\right) \|^2_{\mathscr{H}^{\alpha+0}_0(\mathbf{H}_{\lambda,\tau})} \\ &\quad \leq \ C_s \left(\|(\partial_\tau - \partial_x) u\|^2_{\mathscr{B}^\alpha_0} \|\Upsilon^\tau\|^2_{\mathscr{G}^0_k} + \|(\partial_\tau - \partial_x) u\|^2_{\mathscr{H}^\alpha_{k-1}} \|\Upsilon^\tau\|^2_{\mathscr{C}^0_{\{x=0\},1}} \right) \\ &\quad \leq \ C \left(\|(\partial_\tau - \partial_x) u\|^2_{\mathscr{B}^\alpha_0} \|\Upsilon^\tau\|^2_{\mathscr{G}^0_k} + \|\Upsilon^\tau\|^2_{\mathscr{C}^0_{\{x=0\},1}} E^\alpha_k[u(\tau)] \right) \end{aligned} \tag{3.2.45}$$

For the second term:

$$\begin{aligned} &\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} \left\{II\right\}^2 dx\, d\nu \\ &\qquad = \|x^{\beta_1} \mathscr{D}^\beta \left(\Upsilon^x + \Upsilon^\tau\right) \partial_x u - x^{\beta_1} (\Upsilon^x + \Upsilon^\tau) \mathscr{D}^\beta (\partial_x u)\|^2_{\mathscr{H}^{\alpha+0}_0(\mathbf{H}_{\lambda,\tau})} \\ &\quad \leq \ C_s \left(\|\partial_x u\|^2_{\mathscr{B}^\alpha_0} \|\Upsilon^x + \Upsilon^\tau\|^2_{\mathscr{G}^0_k} + \|\partial_x u\|^2_{\mathscr{H}^\alpha_{k-1}} \|\Upsilon^x + \Upsilon^\tau\|^2_{\mathscr{C}^0_{\{x=0\},1}} \right) \\ &\quad \leq \ C \left(\|\partial_x u\|^2_{\mathscr{B}^\alpha_0} \|\Upsilon^x + \Upsilon^\tau\|^2_{\mathscr{G}^0_k} + \|\Upsilon^x + \Upsilon^\tau\|^2_{\mathscr{C}^0_{\{x=0\},1}} E^\alpha_k[u(\tau)] \right) \ . \end{aligned}$$

The same holds for the third term of A_3:

$$\begin{aligned}
&\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} \{III\}^2 \, dx \, d\nu \\
&\quad = \|x^{\beta_1} \mathscr{D}^\beta \left(\Upsilon^A \partial_A u\right) - x^{\beta_1} \Upsilon^A \mathscr{D}^\beta \left(\partial_A u\right) \|^2_{\mathscr{H}_0^{\alpha+0}(\mathbf{H}_{\lambda,\tau})} \\
&\quad \le \; C_s \left(\|\partial_A u\|^2_{\mathscr{B}_0^\alpha} \|\Upsilon^A\|^2_{\mathscr{G}_k^0} + \|\partial_A u\|^2_{\mathscr{H}_{k-1}^\alpha} \|\Upsilon^A\|^2_{\mathscr{C}^0_{\{x=0\},1}} \right) \\
&\quad \le \; C \left(\|\partial_A u\|^2_{\mathscr{B}_0^\alpha} \|\Upsilon^A\|^2_{\mathscr{G}_k^0} + \|\Upsilon^A\|^2_{\mathscr{C}^0_{\{x=0\},1}} E_k^\alpha[u(\tau)] \right) .
\end{aligned}$$

We then obtain the following estimate for the third term of equation (3.2.42)

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} (A_3)^2 dx \, d\nu \tag{3.2.46}$$

$$\begin{aligned}
&\le \|(\partial_\tau - \partial_x) u\|_{\mathscr{B}_0^\alpha} \|\Upsilon^\tau\|^2_{\mathscr{G}_k^0} + \|\partial_x u\|^2_{\mathscr{B}_0^\alpha} \|\Upsilon^\tau + \Upsilon^x\|^2_{\mathscr{G}_k^0} + \|\partial_A u\|^2_{\mathscr{B}_0^\alpha} \|\Upsilon^A\|^2_{\mathscr{G}_k^0} \\
&+ \left\{ \|\Upsilon^\tau\|^2_{\mathscr{C}^0_{\{x=0\},1}} + \|\Upsilon^x + \Upsilon^\tau\|^2_{\mathscr{C}^0_{\{x=0\},1}} + \|\Upsilon^A\|^2_{\mathscr{C}^0_{\{x=0\},1}} \right\} E^\alpha_{k,\lambda}[u(\tau)] \, .
\end{aligned} \tag{3.2.47}$$

In order to estimate the fourth term A_4 of (3.2.42), we need to look separately at each of its components as we have to make sure that every ∂_x^2 comes with a factor of x. We write

$$A_4 = A^{00} + 2A^{\tau x} + 2A^{\tau A} + A^{xx} + 2A^{xA} + A^{AB}, \tag{3.2.48}$$

where the labeling A^{ab} corresponds to the terms obtained when in A_4 we replace $\mathfrak{g}^{\alpha\beta}\partial^2_{\alpha\beta}$ with its expression as in (3.2.8). Now we use again the weighted Moser-type inequality of Proposition B.2.3 Equation B.2.10 to estimate these terms. We have:

$$\begin{aligned}
&\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} \left\{A^{AB}\right\}^2 dx \, d\nu \\
&\quad = \|x^{\beta_1} \mathscr{D}^\beta \left(\mathfrak{g}^{AB} \partial_A \partial_B u\right) - x^{\beta_1} \mathfrak{g}^{AB} \mathscr{D}^\beta \left(\partial_A \partial_B u\right) \|_{\mathscr{H}_0^{\alpha+0}(\mathbf{H}_{\lambda,\tau})} \\
&\le \; C_s \sum_A \left(\|\partial_A u\|^2_{\mathscr{B}_1^\alpha} \|\mathfrak{g}^\sharp\|^2_{\mathscr{G}_k^0} + \|\partial_A u\|^2_{\mathscr{H}_k^\alpha} \|\mathfrak{g}^\sharp\|^2_{\mathscr{C}^0_{\{x=0\},1}} \right) \\
&\le \; C \left(\sum_A \|\partial_A u\|^2_{\mathscr{B}_1^\alpha} \|\mathfrak{g}^\sharp\|^2_{\mathscr{G}_k^0} + \|\mathfrak{g}^\sharp\|^2_{\mathscr{C}^0_{\{x=0\},1}} E^\alpha_{k,\lambda}[u(\tau)] \right)
\end{aligned} \tag{3.2.49}$$

and

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}\left\{A^{\tau A}\right\}^2 dx\,d\nu \tag{3.2.50}$$

$$\begin{aligned}
&= \|x^{\beta_1}\mathscr{D}^\beta\left(\mathfrak{g}^{\tau A}\partial_A(\partial_\tau-\partial_x)u\right) - x^{\beta_1}\mathfrak{g}^{\tau A}\mathscr{D}^\beta\left(\partial_A(\partial_\tau-\partial_x)u\right)\|_{\mathscr{H}^\alpha_0}\\
\leq\ & C_s\left(\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{B}^\alpha_1}\|\mathfrak{g}^\sharp\|^2_{\mathscr{G}^0_k} + \|(\partial_\tau-\partial_x)u\|^2_{\mathscr{H}^\alpha_k}\|\mathfrak{g}^\sharp\|^2_{\mathscr{C}^0_{\{x=0\},1}}\right)\\
\leq\ & C_s\left(\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{B}^\alpha_1}\|\mathfrak{g}^\sharp\|^2_{\mathscr{G}^0_k} + \|\mathfrak{g}^\sharp\|^2_{\mathscr{C}^0_{\{x=0\},1}}E^\alpha_{k,\lambda}[u(\tau)]\right)\ . \qquad (3.2.51)
\end{aligned}$$

Continuing in this way we have:

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}\left\{A^{xA}\right\}^2 dx\,d\nu \tag{3.2.52}$$

$$\begin{aligned}
&= \|x^{\beta_1}\mathscr{D}^\beta\left\{\left(\mathfrak{g}^{xA}+\mathfrak{g}^{\tau A}\right)\partial_A\partial_x u\right\} - x^{\beta_1}\left(\mathfrak{g}^{xA}+\mathfrak{g}^{\tau A}\right)\mathscr{D}^\beta\partial_A\partial_x u\|_{\mathscr{H}^\alpha_0}\\
\leq\ & C\left(\|\partial_A\partial_x u\|^2_{\mathscr{B}^\alpha_0}\|\left(\mathfrak{g}^{xA}+\mathfrak{g}^{\tau A}\right)\|^2_{\mathscr{G}^0_k} + \|\partial_A\partial_x u\|^2_{\mathscr{H}^\alpha_{k-1}}\|\left(\mathfrak{g}^{xA}+\mathfrak{g}^{\tau A}\right)\|^2_{\mathscr{C}^0_{\{x=0\},1}}\right)\\
\leq\ & C\sum_A\left(\|\partial_x u\|^2_{\mathscr{B}^\alpha_1}\|\left(\mathfrak{g}^{xA}+\mathfrak{g}^{\tau A}\right)\|^2_{\mathscr{G}^0_k} + \|\partial_x u\|^2_{\mathscr{H}^\alpha_k}\|\left(\mathfrak{g}^{xA}+\mathfrak{g}^{\tau A}\right)\|^2_{\mathscr{C}^0_{\{x=0\},1}}\right)\\
\leq\ & C\sum_A\left(\|\partial_x u\|^2_{\mathscr{B}^\alpha_1}\|\left(\mathfrak{g}^{xA}+\mathfrak{g}^{\tau A}\right)\|^2_{\mathscr{G}^0_k} + \|\left(\mathfrak{g}^{xA}+\mathfrak{g}^{\tau A}\right)\|^2_{\mathscr{C}^0_{\{x=0\},1}}E^\alpha_{k,\lambda}[u(\tau)]\right)\ .
\end{aligned} \tag{3.2.53}$$

We recall that $\mathfrak{g}^{\tau\tau}+\mathfrak{g}^{x\tau} = x\mathfrak{h}^1(\tau,x,v^A)$, *we then obtain the following expression for* $A^{\tau x}$.

$$\begin{aligned}
A^{\tau x} &= \mathscr{D}^\beta\left[\mathfrak{h}^1 x\partial_x(\partial_\tau-\partial_x)u\right] - x\mathfrak{h}^1\mathscr{D}^\beta\left[\partial_x(\partial_\tau-\partial_x)u\right]\\
&= \mathscr{D}^\beta\left[\mathfrak{h}^1 x\partial_x(\partial_\tau-\partial_x)u\right] - \mathfrak{h}^1\mathscr{D}^\beta\left[x\partial_x(\partial_\tau-\partial_x)u\right]\\
&\qquad + \underbrace{\mathfrak{h}^1\mathscr{D}^\beta\left[x\partial_x(\partial_\tau-\partial_x)u\right] - x\mathfrak{h}^1\mathscr{D}^\beta\left[\partial_x(\partial_\tau-\partial_x)u\right]}_{=\beta_1\mathfrak{h}^1\mathscr{D}^\beta(\partial_\tau-\partial_x)u}\ .
\end{aligned}$$

Since

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}\left\{\mathfrak{h}^1\mathscr{D}^\beta(\partial_\tau-\partial_x)u\right\}^2 dx\,d\nu \leq \|\mathfrak{h}^1\|^2_{L^\infty}\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{H}^\alpha_k}\ ,$$

we have

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}\left\{A^{\tau x}\right\}^2 dx\, d\nu$$
$$\leq C_s\left(\|(\partial_\tau-\partial_x)u)\|^2_{\mathscr{B}^\alpha_1}\|\mathfrak{h}^1\|^2_{\mathscr{G}^0_k}+\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{H}^\alpha_k}\|\mathfrak{h}^1\|^2_{\mathscr{C}^0_{\{x=0\},1}}\right)$$
$$\leq C\left(\|(\partial_\tau-\partial_x)u)\|^2_{\mathscr{B}^\alpha_1}\|\mathfrak{h}^1\|^2_{\mathscr{G}^0_k}+\|\mathfrak{h}^1\|^2_{\mathscr{C}^0_{\{x=0\},1}}E^\alpha_{k,\lambda}[u(\tau)]\right). \quad (3.2.54)$$

On the other hand, since $\mathfrak{g}^{\tau\tau}+2\mathfrak{g}^{\tau x}+\mathfrak{g}^{xx}=1+x\mathfrak{h}$, *we have*

$$\begin{aligned}\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}\left\{A^{xx}\right\}^2 dx\, d\nu \;&\leq\; \|x^{\beta_1}\mathscr{D}^\beta\left(\mathfrak{h}x\partial_x[\partial_x u]\right)-x^{\beta_1}\mathfrak{h}\mathscr{D}^\beta\left(x\partial_x[\partial_x u]\right)\|^2_{\mathscr{H}^{\alpha+0}_0(\mathbf{H}_{\lambda,\tau})}\\ &\quad+\|\mathfrak{h}\underbrace{\left\{x^{\beta_1}\mathscr{D}^\beta\left(x\partial_x[\partial_x u]\right)-x^{\beta_1}x\mathscr{D}^\beta\left(\partial_x^2 u\right)\right\}}_{=\beta_1 x^{\beta_1}\mathscr{D}^\beta(\partial_x u)}\|^2_{\mathscr{H}^{\alpha+0}_0(\mathbf{H}_{\lambda,\tau})}\\ &\leq C_s\Big(\|x\partial_x[\partial_x u]\|^2_{\mathscr{B}^\alpha_0}\|\mathfrak{h}\|^2_{\mathscr{G}^0_k}+\|x\partial_x[\partial_x u]\|^2_{\mathscr{H}^\alpha_{k-1}}\|\mathfrak{h}\|^2_{\mathscr{C}^0_{\{x=0\},1}}\\ &\qquad+\|\mathfrak{h}\|^2_{L^\infty}\|\partial_x u\|^2_{\mathscr{H}^\alpha_k}\Big)\\ &\leq C_s\left(\|\partial_x u\|^2_{\mathscr{B}^\alpha_1}\|\mathfrak{h}\|^2_{\mathscr{G}^0_k}+\|\mathfrak{h}\|^2_{\mathscr{C}^0_{\{x=0\},1}}E^\alpha_{k,\lambda}[u(\tau)]\right). \quad (3.2.55)\end{aligned}$$

We note that $\|x^j\partial_x^j\Phi\|_{\mathscr{H}^\alpha_k}\leq\|\Phi\|_{\mathscr{H}^\alpha_{k+j}}$ *which can be shown by induction. In order to estimate the term* A^{00}, *we proceed as follows:*

$$\begin{aligned}A^{00}=\left[\mathscr{D}^\beta,\mathfrak{g}^{\tau\tau}\left(\partial_\tau-\partial_x\right)^2\right]u \;&=\; \mathscr{D}^\beta\left([-1+x\mathfrak{h}^0]\left(\partial_\tau-\partial_x\right)^2 u\right)-[-1+x\mathfrak{h}^0]\mathscr{D}^\beta\left(\partial_\tau-\partial_x\right)^2 u\\ &=\; \mathscr{D}^\beta\left([x\mathfrak{h}^0]\left(\partial_\tau-\partial_x\right)^2 u\right)-[x\mathfrak{h}^0]\mathscr{D}^\beta\left(\partial_\tau-\partial_x\right)^2 u\,. \quad (3.2.56)\end{aligned}$$

Now using equation (3.0.1), (3.2.41) and (3.2.8), we obtain the following expression of $\left(\partial_\tau-\partial_x\right)^2 u$:

$$\begin{aligned}\left(\partial_\tau-\partial_x\right)^2 u \;=\;& -2\left(\hat{\mathfrak{g}}^{\tau\tau}+\hat{\mathfrak{g}}^{\tau x}\right)\left(\partial_\tau-\partial_x\right)\partial_x-\left(\hat{\mathfrak{g}}^{\tau\tau}+2\hat{\mathfrak{g}}^{\tau x}+\hat{\mathfrak{g}}^{xx}\right)\partial_x^2-2\hat{\mathfrak{g}}^{\tau A}\left(\partial_\tau-\partial_x\right)\partial_A\\ & -2\left(\hat{\mathfrak{g}}^{xA}+\hat{\mathfrak{g}}^{\tau A}\right)\partial_x\partial_A-\mathfrak{g}^{AB}\partial_A\partial_B\;-\hat{\Upsilon}^\sigma\partial_\sigma u+\hat{F}\,. \quad (3.2.57)\end{aligned}$$

Here the hat means multiplication with $1/\mathfrak{g}^{\tau\tau}$ *(recall* $|\mathfrak{g}^{\tau\tau}|>\epsilon_0>0$ *). We will need the following:*

Lemma 3.2.7 *Let*

$$\tilde{\partial}=(x\partial_x,\partial_A),\quad k\in\mathbb{N}^*,\quad \theta\in\mathbb{R},\quad \hat{\psi}=\frac{\psi}{\mathfrak{g}^{\tau\tau}}\,,\quad \left|\frac{1}{\mathfrak{g}^{\tau\tau}}\right|\leq\frac{1}{\epsilon_0}\,. \quad (3.2.58)$$

We have the following estimates:

$$\|\hat\psi\|_{\mathscr{C}^\theta_{\{x=0\},0}} \le \frac{1}{\epsilon_0}\|\psi\|_{\mathscr{C}^\theta_{\{x=0\},0}}\ , \tag{3.2.59}$$

$$\|\hat\psi\|_{\mathscr{C}^\theta_{\{x=0\},1}} \le \frac{1}{\epsilon_0}\|\psi\|_{\mathscr{C}^\theta_{\{x=0\},1}} + \frac{1}{\epsilon_0^2}\|\tilde\partial(x\mathfrak{h}^0)\|_{L^\infty}\|\psi\|_{\mathscr{C}^\theta_{\{x=0\},0}}\ , \tag{3.2.60}$$

and

$$\|\hat\psi\|_{\mathscr{H}^\theta_k} \le \frac{1}{\epsilon_0}\|\psi\|_{\mathscr{H}^\theta_k} + \|\psi\|_{\mathscr{B}^\theta_0} C(\|\mathfrak{h}^0\|_{L^\infty})\left(1+\|\mathfrak{h}^0\|_{\mathscr{H}^{-1}_k}\right)\ , \tag{3.2.61}$$

with identical estimates with $\mathscr{C}^\theta_{\{x=0\},0}$ *replaced by* $\mathscr{B}^\theta_0$ *and* $\mathscr{H}^\theta_k$ *replaced by* $\mathscr{G}^\theta_k$.

Proof: *The first inequality is obvious. Next:*

$$\begin{aligned}
\|\hat\psi\|_{\mathscr{C}^\theta_{\{x=0\},1}} &\le \|x^{-\theta}\frac{1}{\mathfrak{g}^{\tau\tau}}\psi\|_{L^\infty} + \|x^{-\theta}\tilde\partial\left\{\frac{1}{\mathfrak{g}^{\tau\tau}}\psi\right\}\|_{L^\infty}\\
&\le \frac{1}{\epsilon_0}\|\psi\|_{\mathscr{C}^\theta_{\{x=0\},0}} + \|x^{-\theta}\left\{\psi\tilde\partial(\frac{1}{\mathfrak{g}^{\tau\tau}}) + \frac{1}{\mathfrak{g}^{\tau\tau}}\tilde\partial\psi\right\}\|_{L^\infty}\\
&\le \frac{1}{\epsilon_0}\|\psi\|_{\mathscr{C}^\theta_{\{x=0\},0}} + \frac{1}{\epsilon_0^2}\|\psi\|_{\mathscr{C}^\theta_{\{x=0\},0}}\|\tilde\partial(x\mathfrak{h}^0)\|_{L^\infty} + \frac{1}{\epsilon_0}\|\tilde\partial\psi\|_{\mathscr{C}^\theta_{\{x=0\},0}}\\
&\le \frac{1}{\epsilon_0}\|\psi\|_{\mathscr{C}^\theta_{\{x=0\},1}} + \frac{1}{\epsilon_0^2}\|\tilde\partial(x\mathfrak{h}^0)\|_{L^\infty}\|\psi\|_{\mathscr{C}^\theta_{\{x=0\},0}}\ .
\end{aligned}$$

On the other hand, from Inequality B.2.4 of Proposition B.2.2, we have:

$$\begin{aligned}
\|\hat\psi\|_{\mathscr{H}^\theta_k} &= \|\frac{1}{\mathfrak{g}^{\tau\tau}}\psi\|_{\mathscr{H}^\theta_k} \le \|\psi\|_{\mathscr{B}^\theta_0}\|\frac{1}{\mathfrak{g}^{\tau\tau}}\|_{\mathscr{G}^0_k} + \|\psi\|_{\mathscr{H}^\theta_k}\|\frac{1}{\mathfrak{g}^{\tau\tau}}\|_{\mathscr{C}^0_{\{x=0\},0}}\\
&\le \frac{1}{\epsilon_0}\|\psi\|_{\mathscr{H}^\theta_k} + \|\psi\|_{\mathscr{B}^\theta_0}\|\frac{1}{\mathfrak{g}^{\tau\tau}}\|_{\mathscr{G}^0_k}\ . \qquad (3.2.62)
\end{aligned}$$

Now, from hypothesis we have,

$$\begin{aligned}
\frac{1}{\mathfrak{g}^{\tau\tau}(\tau,x,v^A)} &= \frac{1}{-1+x\mathfrak{h}^0(\tau,x,v^A)} = -1 + \frac{x\mathfrak{h}^0(\tau,x,v^A)}{-1+x\mathfrak{h}^0(\tau,x,v^A)}\\
&= -1 + G(\tau,x,v^A,x\mathfrak{h}^0)\ ,
\end{aligned}$$

where G is any function which takes the correct values in the range of interest, e.g.,

$$G(\tau,x,v^A,p) = \frac{p\chi(p)}{-1+p} \quad \text{with} \quad \chi\in C^\infty(\mathbb{R}) \quad \text{such that} \quad \chi(p) = \begin{cases} 1 & \text{if} \quad p\le 1-\frac{3\epsilon_0}{4} \\ 0 & \text{if} \quad p\ge 1-\frac{\epsilon_0}{4} \end{cases}\ .$$

Recall that hypothesis (3.1.1) reads $x\mathfrak{h}^0 \le 1-\epsilon_0$. We have (note that the space of functions $\mathscr{G}^\theta_k$ contains constant functions)

$$\|\frac{1}{\mathfrak{g}^{\tau\tau}}\|_{\mathscr{G}^0_k} \le \|1\|_{\mathscr{G}^0_k} + \|G(\,.\,,x\mathfrak{h}^0)\|_{\mathscr{G}^0_k} \le C\left(1+\|G(.,\;x\mathfrak{h}^0)\|_{\mathscr{G}^0_k}\right)\;. \tag{3.2.63}$$

The function G satisfies the following:

$$\|G(\,.\,,p)\|_{\mathscr{C}^0_{\{x=0\},k}} = \|G(\,.\,,p)\|_{\mathscr{C}^0_{\{x=0\},0}} \le C(\epsilon_0)$$

and for $i=0,1$;

$$\left\|\frac{\partial^i G(\,.\,,p)}{\partial p^i}\right\|_{\mathscr{C}^0_{\{x=0\},k-i}} \le C(\epsilon_0)|p|^{1-i}\;.$$

These two inequalities show that G has a uniform zero of order 1 at $p=0$. Therefore, we can apply Inequality B.2.8 of Proposition B.2.2 and obtain that

$$\|G(\,.\,,x\mathfrak{h}^0)\|_{\mathscr{G}^0_k} \le C(\|\mathfrak{h}^0\|_{L^\infty})\|h^0\|_{\mathscr{H}^{-1}_k}\;.$$

This implies (see (3.2.63))

$$\|\frac{1}{\mathfrak{g}^{\tau\tau}}\|_{\mathscr{G}^0_k} \le C(\|\mathfrak{h}^0\|_{L^\infty})\left(1+\|\mathfrak{h}^0\|_{\mathscr{H}^{-1}_k}\right), \tag{3.2.64}$$

and (3.2.62) leads to (3.2.61).

If we insert (3.2.57) into equation (3.2.56), we obtain seven commutators which we label A^{00}_a , $a=1,\dots,7$. These terms can be estimated in the same way as we did before, using B.2.9, B.2.10 and Lemma 3.2.7. They will be analyzed in the order $7-3-5-1-2-4-6$. Let us estimate the term A^{00}_7 containing the source term F. We have

$$\begin{aligned}\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}\left\{A^{00}_7\right\}^2 dx\,d\nu &= \|x^{\beta_1}\mathscr{D}^\beta\left([x\mathfrak{h}^0]\hat F\right) - x^{\beta_1}[x\mathfrak{h}^0]\mathscr{D}^\beta\hat F\|^2_{\mathscr{H}^\alpha_0}\\ &\le C\left(\|\hat F\|^2_{\mathscr{B}^\alpha_0}\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_k} + \|\hat F\|^2_{\mathscr{H}^\alpha_{k-1}}\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\right)\\ &\le C(\epsilon_0)\|F\|^2_{\mathscr{B}^\alpha_0}\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_k} + C(\epsilon_0)\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\\ &\quad\times\left\{\|F\|^2_{\mathscr{H}^\alpha_{k-1}} + \|F\|^2_{\mathscr{B}^\alpha_0}C(\|\mathfrak{h}^0\|_{L^\infty})\left(1+\|x\mathfrak{h}^0\|_{\mathscr{H}^0_{k-1}}\right)\right\}\;.\end{aligned} \tag{3.2.65}$$

The third term can be estimated as follows:

$$\begin{aligned}
&\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}\left\{A_3^{00}\right\}^2 dx\,d\nu \\
&= 2\|\mathscr{D}^\beta\left(x\mathfrak{h}^0\hat{\mathfrak{g}}^{\tau A}\partial_A(\partial_\tau-\partial_x)u\right) - x\mathfrak{h}^0\mathscr{D}^\beta\left(\hat{\mathfrak{g}}^{\tau A}\partial_A(\partial_\tau-\partial_x)u\right)\|^2_{\mathscr{H}_0^\alpha} \\
&\le C\left(\|\hat{\mathfrak{g}}^{\tau A}\partial_A(\partial_\tau-\partial_x)u\|^2_{\mathscr{B}_0^\alpha}\|x\mathfrak{h}^0\|^2_{\mathscr{G}_k^0} + \|\hat{\mathfrak{g}}^{\tau A}\partial_A(\partial_\tau-\partial_x)u\|^2_{\mathscr{H}_{k-1}^\alpha}\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\right) \\
&\le C\|\hat{\mathfrak{g}}^\sharp\|^2_{L^\infty}\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{B}_1^\alpha}\|x\mathfrak{h}^0\|^2_{\mathscr{G}_k^0} \\
&\quad + C\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\left\{\|(\partial_\tau-\tau)u\|^2_{\mathscr{B}_1^\alpha}\|\hat{\mathfrak{g}}^\sharp\|^2_{\mathscr{G}^0_{k-1}} + \|(\partial_\tau-\partial_x)u\|^2_{\mathscr{H}_k^\alpha}\|\hat{\mathfrak{g}}^\sharp\|^2_{\mathscr{C}^0_{\{x=0\},0}}\right\} \\
&\le C(\epsilon_0)\|\mathfrak{g}^\sharp\|^2_{L^\infty}\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{B}_1^\alpha}\|x\mathfrak{h}^0\|^2_{\mathscr{G}_k^0} + C(\epsilon_0)\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\|\mathfrak{g}^\sharp\|^2_{L^\infty}\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{H}_k^\alpha} \\
&\quad + \|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\|(\partial_\tau-\tau)u\|^2_{\mathscr{B}_1^\alpha}\left\{\|\mathfrak{g}^\sharp\|^2_{\mathscr{G}^0_{k-1}} + \|\mathfrak{g}^\sharp\|^2_{L\infty}C(\|\mathfrak{h}^0\|_{L^\infty})\left(1+\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_{k-1}}\right)\right\}\ .
\end{aligned} \tag{3.2.66}$$

A similar analysis gives (A_3^{00} and A_5^{00} have the same structure):

$$\begin{aligned}
&\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}\left\{A_5^{00}\right\}^2 dx\,d\nu \\
&= \|\mathscr{D}^\beta\left([x\mathfrak{h}^0][\hat{\mathfrak{g}}^{AB}\partial_A\partial_B u]\right) - [x\mathfrak{h}^0]\mathscr{D}^\beta\left([\hat{\mathfrak{g}}^{AB}\partial_A\partial_B u]\right)\|^2_{\mathscr{H}_0^\alpha} \\
&\le C(\epsilon_0)\|\mathfrak{g}^\sharp\|^2_{L^\infty}\|\partial_A u\|^2_{\mathscr{B}_1^\alpha}\|x\mathfrak{h}^0\|^2_{\mathscr{G}_k^0} + C(\epsilon_0)\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\|\mathfrak{g}^\sharp\|^2_{L^\infty}\|\partial_A u\|^2_{\mathscr{H}_k^\alpha} \\
&\quad + \|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\|\partial_A u\|^2_{\mathscr{B}_1^\alpha}\left\{\|\mathfrak{g}^\sharp\|^2_{\mathscr{G}^0_{k-1}} + \|\mathfrak{g}^\sharp\|^2_{L\infty}C(\|\mathfrak{h}^0\|_{L^\infty})\left(1+\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_{k-1}}\right)\right\}\ .
\end{aligned} \tag{3.2.67}$$

As far as the first term A_1^{00} is concerned, we have

$$-\frac{1}{2}A_1^{00} = \mathscr{D}^\beta\left(x\mathfrak{h}^0\hat{\mathfrak{h}}^1(x\partial_x)(\partial_\tau-\partial_x)u\right) - x\mathfrak{h}^0\mathscr{D}^\beta\left(\hat{\mathfrak{h}}^1(x\partial_x)(\partial_\tau-\partial_x)u\right)\ .$$

Using again the weighted Moser-type Inequality B.2.10, we can evaluate the square of its norm as follows:

$$\begin{aligned}
&\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}(A_1^{00})^2 dx\,d\nu \\
&= 2\|x^{\beta_1}\mathscr{D}^\beta\left([x\mathfrak{h}^0][\hat{\mathfrak{h}}^1(x\partial_x)(\partial_\tau-\partial_x)u]\right) - x^{\beta_1}[x\mathfrak{h}^0]\mathscr{D}^\beta\left(\hat{\mathfrak{h}}^1(x\partial_x)(\partial_\tau-\partial_x)u\right)\|^2_{\mathscr{H}_0^\alpha} \\
&\le C\left(\|\hat{\mathfrak{h}}^1(x\partial_x)(\partial_\tau-\partial_x)u\|^2_{\mathscr{B}_0^\alpha}\|x\mathfrak{h}^0\|^2_{\mathscr{G}_k^0} + \|\hat{\mathfrak{h}}^1(x\partial_x)(\partial_\tau-\partial_x)u\|^2_{\mathscr{H}_{k-1}^\alpha}\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\right) \\
&\le C(\epsilon_0)\left(\|\mathfrak{h}^1\|_{L^\infty}\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{B}_1^\alpha}\|x\mathfrak{h}^0\|^2_{\mathscr{G}_k^0} + \|\hat{\mathfrak{h}}^1(x\partial_x)(\partial_\tau-\partial_x)u\|^2_{\mathscr{H}_{k-1}^\alpha}\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\right)\ .
\end{aligned}$$

Using now Inequality B.2.9 of Proposition B.2.3 gives (the last inequality is obtained by using (3.2.61):

$$\begin{aligned}
&\|\hat{\mathfrak{h}}^1(x\partial_x)(\partial_\tau-\partial_x)u\|^2_{\mathscr{H}^\alpha_{k-1}}\\
&\quad\le\; C\left(\|x\partial_x(\partial_\tau-\partial_x)u\|^2_{\mathscr{B}^\alpha_0}\|\hat{\mathfrak{h}}^1\|^2_{\mathscr{G}^0_{k-1}}+\|x\partial_x(\partial_\tau-\partial_x)u\|^2_{\mathscr{H}^\alpha_{k-1}}\|\hat{\mathfrak{h}}^1\|^2_{\mathscr{C}^0_0}\right)\\
&\quad\le\; C\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{B}^\alpha_1}\|\hat{\mathfrak{h}}^1\|^2_{\mathscr{G}^0_{k-1}}+C(\epsilon_0)\|\mathfrak{h}^1\|^2_{L^\infty}\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{H}^\alpha_k}\\
&\quad\le\; C(\epsilon_0)\|\mathfrak{h}^1\|^2_{L^\infty}\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{H}^\alpha_k}+C(\epsilon_0)\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{B}^\alpha_1}\\
&\qquad\times\left\{\|\mathfrak{h}^1\|^2_{\mathscr{G}^0_{k-1}}+C(\|\mathfrak{h}^0\|_{L^\infty})\|\mathfrak{h}^1\|^2_{L^\infty}(1+\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_{k-1}})\right\},
\end{aligned}$$

which gives

$$\begin{aligned}
&\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}(A^{00}_1)^2dx\,d\nu\\
&\le\; C(\epsilon_0)\|\mathfrak{h}^1\|^2_{L^\infty}\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{B}^\alpha_1}\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_k}+C(\epsilon_0)\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\|\mathfrak{h}^1\|^2_{L^\infty}\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{H}^\alpha_k}\\
&\quad+C(\epsilon_0)\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{B}^\alpha_1}\left\{\|\mathfrak{h}^1\|^2_{\mathscr{G}^0_{k-1}}+C(\|\mathfrak{h}^0\|_{L^\infty})\|\mathfrak{h}^1\|^2_{L^\infty}(1+\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_{k-1}})\right\}\\
&\le\; C(\epsilon_0)\|x\mathfrak{h}^0\|_{\mathscr{C}^0_{\{x=0\},1}}\|\mathfrak{h}^1\|^2_{L^\infty}\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{H}^\alpha_k}\\
&\quad+C(\epsilon_0)\left(1+\|x\mathfrak{h}^0\|_{\mathscr{C}^0_{\{x=0\},1}}\right)\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{B}^\alpha_1}\left\{\|\mathfrak{h}^1\|^2_{\mathscr{G}^0_{k-1}}+C(\|\mathfrak{h}^0\|_{L^\infty})\|\mathfrak{h}^1\|^2_{L^\infty}(1+\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_k})\right\}.
\end{aligned}\tag{3.2.68}$$

Since the terms A^{00}_1 and A^{00}_4 have the same structure, to estimate the second one, we just have to replace in the estimate on A^{00}_1, $\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{H}^\alpha_k}$ by $\|\partial_x u\|^2_{\mathscr{H}^\alpha_k}$ and $\|x\mathfrak{h}^1\|^2_{\mathscr{G}^0_{k-1}}$ by $\|\hat{\mathfrak{g}}^{\tau A}+\hat{\mathfrak{g}}^{xA}\|^2_{\mathscr{G}^0_{k-1}}$. Thus we have

$$\begin{aligned}
&\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}\left\{A^{00}_4\right\}^2dx\,d\nu\\
&\quad=\|\mathscr{D}^\beta\left([x\mathfrak{h}^0][(\hat{\mathfrak{g}}^{\tau A}+\hat{\mathfrak{g}}^{xA})\partial_A(\partial_x u)]\right)-[x\mathfrak{h}^0]\mathscr{D}^\beta\left((\hat{\mathfrak{g}}^{\tau A}+\hat{\mathfrak{g}}^{xA})\partial_A(\partial_x u)\right)\|^2_{\mathscr{H}^\alpha_0}\\
&\quad\le C(\epsilon_0)\|x\mathfrak{h}^0\|_{\mathscr{C}^0_{\{x=0\},1}}\|(\hat{\mathfrak{g}}^{\tau A}+\hat{\mathfrak{g}}^{xA})\|^2_{L^\infty}\|\partial_x u\|^2_{\mathscr{H}^\alpha_k}\\
&\qquad+C(\epsilon_0)\left(1+\|x\mathfrak{h}^0\|_{\mathscr{C}^0_{\{x=0\},1}}\right)\|\partial_x u\|^2_{\mathscr{B}^\alpha_1}\\
&\qquad\times\left\{\|(\hat{\mathfrak{g}}^{\tau A}+\hat{\mathfrak{g}}^{xA})\|^2_{\mathscr{G}^0_{k-1}}+C(\|\mathfrak{h}^0\|_{L^\infty})\|(\hat{\mathfrak{g}}^{\tau A}+\hat{\mathfrak{g}}^{xA})\|^2_{L^\infty}(1+\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_k})\right\}\ .
\end{aligned}\tag{3.2.69}$$

We continue with the most dangerous term A^{00}_2 . We have (recall that $\hat{1}=1/\mathfrak{g}^{\tau\tau}$)

$$-A^{00}_2=\mathscr{D}^\beta\left([x\mathfrak{h}^0](\hat{1}+x\hat{\mathfrak{h}})\partial^2_x u\right)-[x\mathfrak{h}^0]\mathscr{D}^\beta\left([\hat{1}+x\hat{\mathfrak{h}}]\partial^2_x u\right)\ ;$$

$$
\begin{aligned}
&\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} \left\{A_2^{00}\right\}^2 dx\, d\nu \\
&\quad = \quad \|x^{\beta_1}\mathscr{D}^\beta \left([\mathfrak{h}^0][\hat{1}+x\hat{\mathfrak{h}}](x\partial_x)\partial_x u\right) - x^{\beta_1}[x\mathfrak{h}^0]\mathscr{D}^\beta \left([\hat{1}+x\hat{\mathfrak{h}}]\partial_x^2 u\right) \|^2_{\mathscr{H}_0^\alpha} \\
&\quad \le \quad \|x^{\beta_1}\mathscr{D}^\beta \left(x\mathfrak{h}^0(\hat{1}.\partial_x^2 u)\right) - x^{\beta_1}[x\mathfrak{h}^0]\mathscr{D}^\beta \left(\hat{1}.\partial_x^2 u\right) \|^2_{\mathscr{H}_0^\alpha} \\
&\qquad\quad +\|x^{\beta_1}\mathscr{D}^\beta \left(x\mathfrak{h}^0\hat{\mathfrak{h}}x\partial_x(\partial_x u)\right) - x^{\beta_1}[x\mathfrak{h}^0]\mathscr{D}^\beta \left(\hat{\mathfrak{h}}x\partial_x(\partial_x u)\right) \|^2_{\mathscr{H}_0^\alpha} \\
&\quad =: \quad (a)+(b)\ .
\end{aligned}
$$

Now, estimating these two expressions as we did with A_1^{00}, we obtain the following

$$
\begin{aligned}
(b) \quad &\le \quad C\left(\|\hat{\mathfrak{h}}(x\partial_x)\partial_x u\|^2_{\mathscr{B}_0^\alpha}\|x\mathfrak{h}^0\|^2_{\mathscr{G}_k^0} + \|\hat{\mathfrak{h}}(x\partial_x)\partial_x u\|^2_{\mathscr{H}_{k-1}^\alpha}\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\right) \\
&\le \quad C(\epsilon_0)\|\mathfrak{h}\|^2_{L^\infty}\|\partial_x u\|^2_{\mathscr{B}_1^\alpha}\|x\mathfrak{h}^0\|^2_{\mathscr{G}_k^0} + C\|\hat{\mathfrak{h}}(x\partial_x)\partial_x u\|^2_{\mathscr{H}_{k-1}^\alpha}\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\ .
\end{aligned}
$$

Inequations B.2.9 and 3.2.61 give,

$$
\begin{aligned}
\|\hat{\mathfrak{h}}(x\partial_x)\partial_x u\|^2_{\mathscr{H}_{k-1}^\alpha} \quad &\le \quad C\left(\|(x\partial_x)\partial_x u\|^2_{\mathscr{B}_0^\alpha}\|\hat{\mathfrak{h}}\|^2_{\mathscr{G}_{k-1}^0} + \|(x\partial_x)\partial_x u\|^2_{\mathscr{H}_{k-1}^\alpha}\|\hat{\mathfrak{h}}\|^2_{\mathscr{C}^0_{\{x=0\},0}}\right) \\
&\le \quad C(\epsilon_0)\|\mathfrak{h}\|^2_{L^\infty}\|\partial_x u\|^2_{\mathscr{H}_k^\alpha} \\
&\qquad +C(\epsilon_0)\|\partial_x u\|^2_{\mathscr{B}_1^\alpha}\left\{\|\mathfrak{h}\|^2_{\mathscr{G}_{k-1}^0} + \|\mathfrak{h}\|^2_{L^\infty}C(\|\mathfrak{h}^0\|_{L^\infty})(1+\|x\mathfrak{h}^0\|^2_{\mathscr{G}_{k-1}^0})\right\}\ ,
\end{aligned}
\tag{3.2.70}
$$

which gives the following estimate for (b):

$$
\begin{aligned}
(b) \quad &\le \quad C(\epsilon_0)\|\mathfrak{h}\|^2_{L^\infty}\left(\|\partial_x u\|^2_{\mathscr{B}_1^\alpha}\|x\mathfrak{h}^0\|^2_{\mathscr{G}_k^0} + \|\partial_x u\|^2_{\mathscr{H}_k^\alpha}\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\right) \\
&\quad +C(\epsilon_0)\|\partial_x u\|^2_{\mathscr{B}_1^\alpha}\left\{\|\mathfrak{h}\|^2_{\mathscr{G}_{k-1}^0} + \|\mathfrak{h}\|^2_{L^\infty}C(\|\mathfrak{h}^0\|_{L^\infty})(1+\|x\mathfrak{h}^0\|^2_{\mathscr{G}_{k-1}^0})\right\}\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\ .
\end{aligned}
\tag{3.2.71}
$$

In order to estimate the term (a) we write here $\beta=(\beta_1,\beta')$ and $\mathscr{D}^\beta = \partial_x^{\beta_1}\partial_v^{\beta'}$, with $\partial_v^{\beta'} = X_2^{\beta_2}\dots X_r^{\beta_r}$:

$$
\begin{aligned}
\mathscr{D}^\beta\left(\mathfrak{h}^0 x(\hat{1}.\partial_x^2 u)\right) - [x\mathfrak{h}^0]\mathscr{D}^\beta\left(\hat{1}.\partial_x^2 u\right) \quad &= \quad \mathscr{D}^\beta\left(\mathfrak{h}^0 x(\hat{1}.\partial_x^2 u)\right) - \mathfrak{h}^0\mathscr{D}^\beta\left(\hat{1}.x\partial_x^2 u\right) \\
&\qquad +\mathfrak{h}^0\mathscr{D}^\beta\left(\hat{1}.x\partial_x^2 u\right) - [x\mathfrak{h}^0]\mathscr{D}^\beta\left(\hat{1}.\partial_x^2 u\right) \\
&=: \quad (1)+(2)\ .
\end{aligned}
\tag{3.2.72}
$$

We have

$$
x^{\beta_1}(2) = \beta_1\mathfrak{h}^0 x^{\beta_1}\partial_v^{\beta'}\partial_x^{\beta_1-1}(\hat{1}.\partial_x^2 u)
$$

and we have

$$x^{-2\alpha-1+2\beta_1}(2)^2 = \beta_1^2(\mathfrak{h}^0)^2 x^{-2(\alpha-1)-1+2(\beta_1-1)}\left(\partial_v^{\beta'}\partial_x^{\beta_1-1}(\hat{1}.\partial_x^2 u)\right)^2 .$$

This identity leads to

$$\begin{aligned}\|x^{\beta_1}(2)\|^2_{\mathscr{H}_0^\alpha} &\leq C\|\mathfrak{h}^0\|^2_{L^\infty}\|\hat{1}.\partial_x^2 u\|^2_{\mathscr{H}_{k-1}^{\alpha-1}}\\ &\leq C\|\mathfrak{h}^0\|^2_{L^\infty}\left(\|\partial_x^2 u\|^2_{\mathscr{B}_0^{\alpha-1}}\|\hat{1}\|^2_{\mathscr{G}_{k-1}^0}+\|\partial_x^2 u\|^2_{\mathscr{H}_{k-1}^{\alpha-1}}\|\hat{1}\|^2_{\mathscr{C}_0^0}\right)\\ &\leq C\|\mathfrak{h}^0\|^2_{L^\infty}\left(\|\partial_x u\|^2_{\mathscr{B}_1^\alpha}\|\hat{1}\|^2_{\mathscr{G}_{k-1}^0}+\frac{1}{\epsilon_0}\|\partial_x u\|^2_{\mathscr{H}_k^\alpha}\right) .\end{aligned}$$

Using again (3.2.61) we have:

$$\|\hat{1}\|^2_{\mathscr{G}_{k-1}^0}\leq\frac{1}{\epsilon_0}\|1\|^2_{\mathscr{G}_{k-1}^0}+C(\|\mathfrak{h}^0\|_{L^\infty})\left(1+\|x^2\mathfrak{h}^0\|^2_{\mathscr{G}_{k-1}^0}\right),$$

that is

$$\|\hat{1}\|^2_{\mathscr{G}_{k-1}^0}\leq C(\|\mathfrak{h}^0\|_{L^\infty})\left(1+\|x^2\mathfrak{h}^0\|^2_{\mathscr{G}_{k-1}^0}\right) . \qquad (3.2.73)$$

Thus,

$$\|x^{\beta_1}(2)\|^2_{\mathscr{H}_0^\alpha} \leq C\left(\|\mathfrak{h}^0\|_{L^\infty}\right)\left\{\|\partial_x u\|^2_{\mathscr{B}_1^\alpha}\left(1+\|x\mathfrak{h}^0\|^2_{\mathscr{G}_{k-1}^0}\right)+\frac{1}{\epsilon_0}E^\alpha_{k,\lambda}[u(\tau)]\right\} . \qquad (3.2.74)$$

As far as the first term of (3.2.72) is concerned, we have:

$$\begin{aligned}\|x^{\beta_1}(1)\|^2_{\mathscr{H}_0^\alpha} &= \|x^{\beta_1}\mathscr{D}^\beta\left(\mathfrak{h}^0(\hat{1}.(x\partial_x)\partial_x u)\right)-x^{\beta_1}\mathfrak{h}^0\mathscr{D}^\beta\left(\hat{1}.(x\partial_x)\partial_x u\right)\|^2_{\mathscr{H}_0^\alpha}\\ &\leq C\left\{\|\hat{1}.(x\partial_x)\partial_x u\|^2_{\mathscr{B}_0^\alpha}\|\mathfrak{h}^0\|^2_{\mathscr{G}_k^0}+\|\hat{1}.(x\partial_x)\partial_x u\|^2_{\mathscr{H}_{k-1}^\alpha}\|\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\right\}\\ &\leq C(\epsilon_0)\Bigg\{\|\partial_x u\|^2_{\mathscr{B}_1^\alpha}\|\mathfrak{h}^0\|^2_{\mathscr{G}_k^0}\\ &\quad+\|\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\left\{\|(x\partial_x)\partial_x u\|^2_{\mathscr{B}_0^\alpha}\|\hat{1}\|^2_{\mathscr{G}_{k-1}^0}+\|(x\partial_x)\partial_x u\|^2_{\mathscr{H}_{k-1}^\alpha}\|\hat{1}\|^2_{\mathscr{C}^0_{\{x=0\},0}}\right\}\Bigg\}\\ &\leq C(\epsilon_0)\left\{\|\partial_x u\|^2_{\mathscr{B}_1^\alpha}\|\mathfrak{h}^0\|^2_{\mathscr{G}_k^0}+\|\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\left\{\|\partial_x u\|^2_{\mathscr{B}_1^\alpha}\|\hat{1}\|^2_{\mathscr{G}_{k-1}^0}+\frac{1}{\epsilon_0^2}\|\partial_x u\|^2_{\mathscr{H}_k^\alpha}\right\}\right\}\\ &\leq C(\epsilon_0)\left\{\|\partial_x u\|^2_{\mathscr{B}_1^\alpha}\|\mathfrak{h}^0\|^2_{\mathscr{G}_k^0}+\|\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}E^\alpha_{k,\lambda}[u(\tau)]\right\}\\ &\quad+C(\epsilon_0)\|\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\|\partial_x u\|^2_{\mathscr{B}_1^\alpha}\left\{C(\|\mathfrak{h}^0\|_{L^\infty})\left(1+\|x\mathfrak{h}^0\|^2_{\mathscr{G}_{k-1}^0}\right)\right\} .\end{aligned} \qquad (3.2.75)$$

Equations 3.2.74 and 3.2.75 show that

$$\begin{aligned}(a) \quad \leq \quad & C(\epsilon_0)\left\{\|\partial_x u\|^2_{\mathscr{B}^\alpha_1}\|\mathfrak{h}^0\|^2_{\mathscr{G}^0_k} + \|\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}} E^\alpha_{k,\lambda}[u(\tau)]\right\} \\ & + C(\epsilon_0)\|\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\|\partial_x u\|^2_{\mathscr{B}^\alpha_1}\left\{C(\|\mathfrak{h}^0\|_{L^\infty})\left(1+\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_{k-1}}\right)\right\} .\end{aligned} \tag{3.2.76}$$

Inequalities 3.2.71 and 3.2.76 show that

$$\begin{aligned}&\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}\left\{A^{00}_2\right\}^2 dx\, d\nu \\ &\quad \leq C(\epsilon_0)\|\mathfrak{h}\|^2_{L^\infty}\left(\|\partial_x u\|^2_{\mathscr{B}^\alpha_1}\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_k} + \|\partial_x u\|^2_{\mathscr{H}^\alpha_k}\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\right) \\ &\quad + C(\epsilon_0)\|\partial_x u\|^2_{\mathscr{B}^\alpha_1}\left\{\|\mathfrak{h}\|^2_{\mathscr{G}^0_{k-1}} + \|\mathfrak{h}\|^2_{L^\infty}C(\|\mathfrak{h}^0\|_{L^\infty})(1+\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_{k-1}})\right\}\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}} .\end{aligned} \tag{3.2.77}$$

Now let us consider the sixth term A^{00}_6 of A^{00}. We have

$$\widehat{\Upsilon}^\mu\partial_\mu = \widehat{\Upsilon}^\tau\left(\partial_\tau - \partial_x\right) + \left(\widehat{\Upsilon}^x + \widehat{\Upsilon}^\tau\right)\partial_x + \widehat{\Upsilon}^A\partial_A,$$

and we decompose A^{00}_6 as

$$A^{00}_6 = a + b + c . \tag{3.2.78}$$

We have

$$a := \mathscr{D}^\beta\left([x\mathfrak{h}^0]\widehat{\Upsilon}^\tau(\partial_\tau-\partial_x)u\right) - [x\mathfrak{h}^0]\mathscr{D}^\beta\left(\widehat{\Upsilon}^\tau(\partial_\tau-\partial_x)u\right),$$

and

$$\begin{aligned}&\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}a^2 dx\, d\nu \\ &\quad = \quad \|x^{\beta_1}\mathscr{D}^\beta\left([x\mathfrak{h}^0]\widehat{\Upsilon}^\tau(\partial_\tau-\partial_x)u\right) - x^{\beta_1}[x\mathfrak{h}^0]\mathscr{D}^\beta\left(\widehat{\Upsilon}^\tau(\partial_\tau-\partial_x)u\right)\|^2_{\mathscr{H}^\alpha_0} \\ &\quad \leq \quad C\left(\|\widehat{\Upsilon}^\tau(\partial_\tau-\partial_x)u\|^2_{\mathscr{B}^\alpha_0}\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_k} + \|\widehat{\Upsilon}^\tau(\partial_\tau-\partial_x)u\|^2_{\mathscr{H}^\alpha_{k-1}}\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\right) \\ &\quad \leq \quad C(\epsilon_0)\|\Upsilon^\tau\|^2_{L^\infty}\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{B}^\alpha_0}\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_k} + \|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}} \\ &\qquad\qquad \times\left\{(\partial_\tau-\partial_x)u\|^2_{\mathscr{B}^\alpha_0}\|\widehat{\Upsilon}^\tau\|^2_{\mathscr{G}^0_{k-1}} + \|(\partial_\tau-\partial_x)u\|^2_{\mathscr{H}^\alpha_{k-1}}\|\widehat{\Upsilon}^\tau\|^2_{\mathscr{C}^0_{\{x=0\},0}}\right\} .\end{aligned}$$

Now, from (3.2.59) and (3.2.61) we have

$$\|\widehat{\Upsilon}^\tau\|^2_{\mathscr{C}^0_{\{x=0\},0}} \le C(\epsilon_0)\|\Upsilon^\tau\|_{L^\infty},$$

and

$$\|\widehat{\Upsilon}^\tau\|^2_{\mathscr{G}^0_{k-1}} \le C(\epsilon_0)\left\{\|\Upsilon^\tau\|_{\mathscr{G}^0_{k-1}} + \|\Upsilon^\tau\|^2_{L^\infty} C(\|\mathfrak{h}^0\|_{L^\infty})\left(1+\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_{k-1}}\right)\right\} .$$

Thus

$$\begin{aligned}\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}a^2 dx\, d\nu \quad \le \quad & C(\epsilon_0)\|\Upsilon^\tau\|^2_{L^\infty}\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{B}^\alpha_0}\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_k} \\ & +C(\epsilon_0)\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\|\Upsilon^\tau\|^2_{L^\infty}\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{H}^\alpha_{k-1}} \\ & +C(\|\mathfrak{h}^0\|_{L^\infty})\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{B}^\alpha_0}\|\Upsilon^\tau\|^2_{\mathscr{G}^0_{k-1}} \\ & +C(\epsilon_0)\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{B}^\alpha_0}\|\Upsilon^\tau\|^2_{L^\infty}\left(1+\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_{k-1}}\right) .\end{aligned} \tag{3.2.79}$$

On the other hand,

$$b := \mathscr{D}^\beta\left([x\mathfrak{h}^0](\widehat{\Upsilon}^\tau+\widehat{\Upsilon}^x)\partial_x u\right) - [x\mathfrak{h}^0]\mathscr{D}^\beta\left((\widehat{\Upsilon}^\tau+\widehat{\Upsilon}^x)\partial_x u\right)$$

and we have

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}b^2 dx\, d\nu \tag{3.2.80}$$

$$\begin{aligned} = \quad & \|x^{\beta_1}\mathscr{D}^\beta\left([x\mathfrak{h}^0](\widehat{\Upsilon}^\tau+\widehat{\Upsilon}^x)\partial_x u\right) - x^{\beta_1}x\mathfrak{h}^0\mathscr{D}^\beta\left((\widehat{\Upsilon}^\tau+\widehat{\Upsilon}^x)\partial_x u\right)\|^2_{\mathscr{H}^\alpha_0} \\ \le \quad & C\|(\widehat{\Upsilon}^\tau+\widehat{\Upsilon}^x)\partial_x u\|^2_{\mathscr{B}^\alpha_0}\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_k} + C\|(\widehat{\Upsilon}^\tau+\widehat{\Upsilon}^x)\partial_x u\|^2_{\mathscr{H}^\alpha_{k-1}}\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}} \\ \le \quad & C(\epsilon_0)\|\Upsilon^\tau+\Upsilon^x\|^2_{L^\infty}\|\partial_x u\|^2_{\mathscr{B}^\alpha_0}\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_k} + C\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}} \\ & \quad \times\left\{\|\partial_x u\|^2_{\mathscr{B}^\alpha_0}\|\widehat{\Upsilon}^\tau+\widehat{\Upsilon}^x\|^2_{\mathscr{G}^0_{k-1}} + \|\partial_x u\|^2_{\mathscr{H}^\alpha_{k-1}}\|\widehat{\Upsilon}^\tau+\widehat{\Upsilon}^x\|^2_{\mathscr{C}^0_{\{x=0\},0}}\right\} \\ \le \quad & C(\epsilon_0)\|\Upsilon^\tau+\Upsilon^x\|^2_{L^\infty}\|\partial_x u\|^2_{\mathscr{B}^\alpha_0}\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_k} \\ & +C(\epsilon_0)\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\|\Upsilon^\tau+\Upsilon^x\|^2_{L^\infty}E^\alpha_{k,\lambda}[u(\tau)] \\ & +C(\|\mathfrak{h}^0\|_{L^\infty})\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\|\partial_x u\|^2_{\mathscr{B}^\alpha_0} \\ & \qquad \times\left\{\|\Upsilon^\tau+\Upsilon^x\|^2_{\mathscr{G}^0_{k-1}} + \|\Upsilon^\tau+\Upsilon^x\|^2_{L^\infty}\left(1+\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_{k-1}}\right)\right\} .\end{aligned} \tag{3.2.81}$$

The same holds for the term

$$c := \mathscr{D}^\beta \left([x\mathfrak{h}^0]\widehat{\Upsilon}^A\partial_A u\right) - [x\mathfrak{h}^0]\mathscr{D}^\beta\left(\widehat{\Upsilon}^A\partial_A u\right)$$

and we have

$$\begin{aligned}\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}c^2 dx\, d\nu \quad \le \quad & C(\epsilon_0)\|\Upsilon^A\|^2_{L^\infty}\|\partial_A u\|^2_{\mathscr{B}^\alpha_0}\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_k} \\ & + C(\epsilon_0)\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\|\Upsilon^A\|^2_{L^\infty}\|\partial_A u\|^2_{\mathscr{H}^\alpha_{k-1}} \\ & + C(\|\mathfrak{h}^0\|_{L^\infty})\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\|\partial_A u\|^2_{\mathscr{B}^\alpha_0}\|\Upsilon^A\|^2_{\mathscr{G}^0_{k-1}} \\ & + C(\epsilon_0)\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\|\partial_A u\|^2_{\mathscr{B}^\alpha_0}\|\Upsilon^A\|^2_{L^\infty}\left(1+\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_{k-1}}\right) .\end{aligned} \tag{3.2.82}$$

This provides the right estimate for A^{00}_6*, and hence for of* A^{00}.

An identical estimate is obtained on the fourth term A_4 *of the commutator (3.2.42). This finishes the estimation of the commutator* $[\Box_\mathfrak{g}, \mathscr{D}^\beta]u$ *appearing in (3.2.40), and the proof is complete.* $\Box$

Conclusion

The proof of the Proposition 3.2.5 used essentially Stokes's theorem, the weighted Moser-type Inequalities A.34 and A.35 of Proposition A.3 of [20], and the weighted substitution inequality type (A.31) of the same reference (see also Appendix B). One of the points there is that all the constants appearing in these inequalities are independent of x_2 *(recall that the sets* M_{x_2,x_1} *there corresponds to the sets* $\mathbf{H}_{\lambda,\tau}$ *here) which is the distance between the boundary of* M_{x_2,x_1} *and the null hypersurface* $\mathscr{N} = \{x = 0\}$*. So, in our case, all the constants involved in the proof of the previous proposition are independent of* λ*. This allows us to take the limit as* λ *goes to* 0 *in (3.2.33) and obtain an identical inequality with* $E^\alpha_{k,\lambda}[u(\tau)]$ *there replaced with* $E^\alpha_k[u(\tau)]$*. Therefore we have proved the following:*

Proposition 3.2.8 *Proposition 3.2.5 remains true with* $\lambda = 0$.

Inequality (3.2.33) with $\lambda = 0$ *is the key in deriving an existence theorem for the Einstein-Maxwell equations with data on a hyperboloid, singular near* $\{x = 0\}$*. In this case, we will show that all the* $\mathscr{H}_k$ *and* $\mathscr{G}_k$ *norms appearing in this inequality are controlled by the energy.*

It turns out that the proof, in Chapter 5, of global polyhomogeneity of the geodesically complete metrics constructed by Loizelet requires a slightly different inequality. For this we need to split the metric into two parts as

$$\mathfrak{g}^{\alpha\beta} = \mathring{\mathfrak{g}}^{\alpha\beta} + \delta\mathfrak{g}^{\alpha\beta} \; . \tag{3.2.83}$$

The rationale behind such a splitting is, that the Lorentzian metric $\mathring{\mathfrak{g}}$ *will be fixed (in fact, it will be the flat Minkowski metric in our applications), while the correction* $\delta\mathfrak{g}$ *will eventually depend on the fields. This leads to the obvious corresponding decomposition of* Υ,

$$\Upsilon^\alpha = \mathring{\Upsilon}^\alpha + \delta\Upsilon^\alpha \; . \tag{3.2.84}$$

We assume that there exist constants σ, M *and* N *such that for* $\tau \in [\tau_0, \tau_1]$ *we have*

$$\begin{aligned} M &\geq \|(\mathring{\mathfrak{h}}^\sharp, \mathring{\mathfrak{g}}^\sharp, \mathring{\Upsilon})\|_{\mathscr{G}^0_k(\mathbf{H}_\tau)} + \|(\delta\mathfrak{h}^\sharp, \delta\mathfrak{g}^\sharp, \delta\Upsilon)\|_{\mathscr{C}^0_{\{x=0\},1}(\mathbf{H}_\tau)} \\ &\quad + \|(\partial_x - \partial_\tau)\mathfrak{g}^\sharp\|_{L^\infty(\mathbf{H}_\tau)} \; , \qquad (3.2.85) \\ N &\geq \|(\partial_\tau u, \partial_x u, \partial_A u)\|_{\mathscr{B}^\sigma_1(\mathbf{H}_\tau)} + \|(\mathfrak{g}^\sharp, \Upsilon)\|_{L^\infty(\mathbf{H}_\tau)} \\ &\quad + \|(\mathring{\mathfrak{g}}^{\tau A}, \mathring{\mathfrak{g}}^{xA})\|_{\mathscr{G}^{\alpha-\sigma}_{k-1}(\mathbf{H}_\tau)} + \|(\delta\mathfrak{g}^\sharp, \delta\Upsilon)\|_{\mathscr{C}^0_{\{x=0\},1}(\mathbf{H}_\tau)} \; . \qquad (3.2.86) \end{aligned}$$

We then have:

Proposition 3.2.9 *Let* $k > n/2+1$, $\sigma \in \mathbb{R}$, $\alpha \leq -1/2$. *There exist functions* $C_3(\epsilon_0, C_0, \alpha, k, n, M)$ *and* $C_4(\epsilon_0, C_0, \alpha, \sigma, k, n, N)$, *monotonously increasing in* M *and* N, *which we write as* $C_3(M)$ *and* $C_4(N)$, *such that for all*

$$\tau \in [\tau_0, \tau_1]$$

and for all u *satisfying (3.0.1) we have*

$$\begin{aligned} E^\alpha_k[u(\tau)] &\leq E^\alpha_k[u(\tau_0)] + \int_{\tau_0}^{\tau} \Big\{ C_3(M) \Big(E^\alpha_k[u(s)] + \|F(s)\|^2_{\mathscr{H}^\alpha_k(\mathbf{H}_\tau)} \Big) \\ &\quad + C_4(N) \Big(1 + \|(\delta\mathfrak{g}^\sharp, \delta\mathfrak{h}^\sharp, \delta\Upsilon)\|^2_{\mathscr{G}^{\alpha-\sigma}_k(\mathbf{H}_\tau)} \Big) \Big\} ds \; . \qquad (3.2.87) \end{aligned}$$

Proof: *The result is obtained by calculations very similar to those of Proposition 3.2.8. We follow that proof until (3.2.41), which is rewritten as*

$$\Box_{\mathfrak{g}} = \mathring{\mathfrak{g}}^{\mu\nu}\partial^2_{\mu\nu} + \delta\mathfrak{g}^{\mu\nu}\partial^2_{\mu\nu} + \mathring{\Upsilon}^\nu\partial_\nu + \delta\Upsilon^\nu\partial_\nu \; . \tag{3.2.88}$$

This leads to the following rewriting of (3.2.42):

$$
\begin{aligned}
[\Box_{\mathfrak{g}}, \mathscr{D}^\beta]u \;=\; & \underbrace{\mathring{\mathfrak{g}}^{\alpha\mu}[\partial_\alpha\partial_\mu, \mathscr{D}^\beta]u}_{=:\mathring{A}_1} + \underbrace{\delta\mathfrak{g}^{\alpha\mu}[\partial_\alpha\partial_\mu, \mathscr{D}^\beta]u}_{=:\delta A_1} \\
& \underbrace{-\mathring{\Upsilon}^\nu[\mathscr{D}^\beta, \partial_\nu]u}_{=:\mathring{A}_2} \underbrace{-\left\{\mathscr{D}^\beta\left(\mathring{\Upsilon}^\nu\partial_\nu u\right) - \mathring{\Upsilon}^\nu\mathscr{D}^\beta\left(\partial_\nu u\right)\right\}}_{=:\mathring{A}_3} \\
& \underbrace{-\delta\Upsilon^\nu[\mathscr{D}^\beta, \partial_\nu]u}_{=:\delta A_2} \underbrace{-\left\{\mathscr{D}^\beta\left(\delta\Upsilon^\nu\partial_\nu u\right) - \delta\Upsilon^\nu\mathscr{D}^\beta\left(\partial_\nu u\right)\right\}}_{=:\delta A_3} \\
& \underbrace{-\left\{\mathscr{D}^\beta\left(\mathring{\mathfrak{g}}^{\alpha\mu}\partial_\alpha\partial_\mu u\right) - \mathring{\mathfrak{g}}^{\alpha\mu}\mathscr{D}^\beta\left(\partial_\alpha\partial_\mu u\right)\right\}}_{=:\mathring{A}_4} \\
& \underbrace{-\left\{\mathscr{D}^\beta\left(\delta\mathfrak{g}^{\alpha\mu}\partial_\alpha\partial_\mu u\right) - \delta\mathfrak{g}^{\alpha\mu}\mathscr{D}^\beta\left(\partial_\alpha\partial_\mu u\right)\right\}}_{=:\delta A_4} . \qquad (3.2.89)
\end{aligned}
$$

The terms $A_i := \mathring{A}_i + \delta A_i$, i=1,2 are estimated as in (3.2.43)-(3.2.44). For $\mathring{A}_3$, instead of (3.2.45) the estimates proceed as before, except that at the end one invokes the weighted Sobolev embedding of Proposition B.2.1; e.g.,

$$
\begin{aligned}
& \int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}\left\{\mathring{I}\right\}^2 dx\, d\nu \\
&= \|x^{\beta_1}\mathscr{D}^\beta\left(\mathring{\Upsilon}^\tau(\partial_\tau-\partial_x)u\right) - x^{\beta_1}\mathring{\Upsilon}^\tau\mathscr{D}^\beta\left((\partial_\tau-\partial_x)u\right)\|^2_{\mathscr{H}^\alpha_0(\mathbf{H}_{\lambda,\tau})} \\
&\le C_s\left(\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{B}^\alpha_0}\|\mathring{\Upsilon}^\tau\|^2_{\mathscr{G}^0_k} + \|(\partial_\tau-\partial_x)u\|^2_{\mathscr{H}^\alpha_{k-1}}\|\mathring{\Upsilon}^\tau\|^2_{\mathscr{C}^0_{\{x=0\},1}}\right) \\
&\le C\left(\|\mathring{\Upsilon}^\tau\|^2_{\mathscr{G}^0_k} + \|\mathring{\Upsilon}^\tau\|^2_{\mathscr{C}^0_{\{x=0\},1}}\right)E^\alpha_k[u(\tau)] . \qquad (3.2.90)
\end{aligned}
$$

For δA_3, we use Proposition B.2.3. Instead of (3.2.45) we then have

$$
\begin{aligned}
& \int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}\left\{\delta I\right\}^2 dx\, d\nu \\
&= \|x^{\beta_1}\mathscr{D}^\beta\left(\delta\Upsilon^\tau(\partial_\tau-\partial_x)u\right) - x^{\beta_1}\delta\Upsilon^\tau\mathscr{D}^\beta\left((\partial_\tau-\partial_x)u\right)\|^2_{\mathscr{H}^\alpha_0(\mathbf{H}_{\lambda,\tau})} \\
&\le C_s\left(\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{B}^\sigma_0}\|\delta\Upsilon^\tau\|^2_{\mathscr{G}^{\alpha-\sigma}_k} + \|(\partial_\tau-\partial_x)u\|^2_{\mathscr{H}^\alpha_{k-1}}\|\delta\Upsilon^\tau\|^2_{\mathscr{C}^0_{\{x=0\},1}}\right) \\
&\le C\left(\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{B}^\sigma_0}\|\delta\Upsilon^\tau\|^2_{\mathscr{G}^{\alpha-\sigma}_k} + \|\delta\Upsilon^\tau\|^2_{\mathscr{C}^0_{\{x=0\},1}}E^\alpha_k[u(\tau)]\right) . \qquad (3.2.91)
\end{aligned}
$$

An identical treatment applies to the remaining three displayed equations following (3.2.45).

The term A_4 is split into $A^{\mu\nu}$'s as in (3.2.48), and then for $\mu\nu \neq 00$ we split $A^{\mu\nu} = \mathring{A}^{\mu\nu} + \delta A^{\mu\nu}$ in the obvious way. All the terms $\mathring{A}^{\mu\nu}$ with $\mu\nu \neq 00$ are then treated as in the proof of Proposition 3.2.8, and at the end we invoke the inequality, for $k \geq n/2+1$,

$$\|f\|^2_{\mathscr{B}^\alpha_1} \leq C\|f\|^2_{\mathscr{H}^\alpha_k} \,.$$

The terms involving $\delta A^{\mu\nu}$ with $\mu\nu \neq 00$ are treated as in (3.2.91); for example, (3.2.49) becomes

$$\begin{aligned}
&\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}\left\{\delta A^{AB}\right\}^2 dx\, d\nu \\
&\quad = \|x^{\beta_1}\mathscr{D}^\beta\left(\delta\mathfrak{g}^{AB}\partial_A\partial_B u\right) - x^{\beta_1}\delta\mathfrak{g}^{AB}\mathscr{D}^\beta\left(\partial_A\partial_B u\right)\|_{\mathscr{H}^\alpha_0(\mathbf{H}_{\lambda,\tau})} \\
&\quad \leq C_s\sum_A\left(\|\partial_A u\|^2_{\mathscr{B}^{\alpha-\sigma}_1}\|\delta\mathfrak{g}^\sharp\|^2_{\mathscr{G}^\sigma_k} + \|\partial_A u\|^2_{\mathscr{H}^\alpha_k}\|\delta\mathfrak{g}^\sharp\|^2_{\mathscr{C}^0_{\{x=0\},1}}\right) \\
&\quad \leq C\left(\sum_A\|\partial_A u\|^2_{\mathscr{B}^{\alpha-\sigma}_1}\|\delta\mathfrak{g}^\sharp\|^2_{\mathscr{G}^\sigma_k} + \|\delta\mathfrak{g}^\sharp\|^2_{\mathscr{C}^0_{\{x=0\},1}}E^\alpha_{k,\lambda}[u(\tau)]\right)\,. \qquad (3.2.92)
\end{aligned}$$

In (3.2.66) it is convenient to use the splitting $\mathfrak{h} = \mathring{\mathfrak{h}} + \delta\mathfrak{h}$. The terms involving $\mathring{\mathfrak{h}}$ are estimated, using the Sobolev embedding, by $E^\alpha_k[u(\tau)]$, while for those involving $\delta\mathfrak{h}$ we write

$$\begin{aligned}
&\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}\left\{\delta A^{00}_3\right\}^2 dx\, d\nu \\
&\quad = \quad 2\|\mathscr{D}^\beta\left(x\delta\mathfrak{h}^0\hat{\mathfrak{g}}^{\tau A}\partial_A(\partial_\tau-\partial_x)u\right) - x\delta\mathfrak{h}^0\mathscr{D}^\beta\left(\hat{\mathfrak{g}}^{\tau A}\partial_A(\partial_\tau-\partial_x)u\right)\|^2_{\mathscr{H}^\alpha_0} \\
&\quad \leq \quad C\|\hat{\mathfrak{g}}^{\tau A}\partial_A(\partial_\tau-\partial_x)u\|^2_{\mathscr{B}^\sigma_0}\|x\delta\mathfrak{h}^0\|^2_{\mathscr{G}^{\alpha-\sigma}_k} \\
&\qquad\quad +C\|\hat{\mathfrak{g}}^{\tau A}\partial_A(\partial_\tau-\partial_x)u\|^2_{\mathscr{H}^\alpha_{k-1}}\|x\delta\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\,. \qquad (3.2.93)
\end{aligned}$$

The first line above is estimated as

$$C\|\hat{\mathfrak{g}}^\sharp\|^2_{L^\infty}\|(\partial_\tau-\partial_x)u\|^2_{\mathscr{B}^\sigma_1}\|\delta\mathfrak{g}^\sharp\|^2_{\mathscr{G}^{\alpha-\sigma}_k}\,,$$

as desired. The second is estimated as

$$\begin{aligned}
&C\|\delta\mathfrak{g}^\sharp\|^2_{\mathscr{C}^0_{\{x=0\},1}}\left\{\|(\partial_\tau-\partial_\tau)u\|^2_{\mathscr{B}^\sigma_1}\|\hat{\mathfrak{g}}^{\tau A}\|^2_{\mathscr{G}^{\alpha-\sigma}_{k-1}} + \|(\partial_\tau-\partial_x)u\|^2_{\mathscr{H}^\alpha_k}\|\hat{\mathfrak{g}}^\sharp\|^2_{\mathscr{C}^0_{\{x=0\},0}}\right\} \\
&\quad \leq \quad C\|\delta\mathfrak{g}^\sharp\|^2_{\mathscr{C}^0_{\{x=0\},1}}\left\{\|(\partial_\tau-\partial_\tau)u\|^2_{\mathscr{B}^\sigma_1}\|\hat{\mathfrak{g}}^{\tau A}\|^2_{\mathscr{G}^{\alpha-\sigma}_{k-1}} + \|\mathfrak{g}^\sharp\|^2_{\mathscr{C}^0_{\{x=0\},0}}E^\alpha_{k,\lambda}[u(\tau)]\right\}\,. \qquad (3.2.94)
\end{aligned}$$

To estimate the term A_5^{00} (compare (3.2.68)) we need to split both $\mathfrak{h}^\sharp$ and $\mathfrak{g}^\sharp$ into two. The terms there involving $\mathring{\mathfrak{h}}^\sharp$ and $\mathring{\mathfrak{g}}^\sharp$ can be estimated by $E_k^\alpha[u(\tau)]$. The terms involving $\delta\mathfrak{g}^\sharp$ are estimated as in the analysis of δA_3^{00}. The mixed term involving $\mathring{\mathfrak{g}}^\sharp$ and $\delta\mathfrak{h}$ is handled in the obvious way

$$\begin{aligned}
&\|x^{\beta_1}\mathscr{D}^\beta\left([x\delta\mathfrak{h}^0][\hat{\mathring{\mathfrak{g}}}^{AB}\partial_A\partial_B u]\right) - x^{\beta_1}[x\delta\mathfrak{h}^0]\mathscr{D}^\beta\left([\hat{\mathring{\mathfrak{g}}}^{AB}\partial_A\partial_B u]\right)\|^2_{\mathscr{H}_0^\alpha} \\
&\quad\le\ C\big(\|\hat{\mathring{\mathfrak{g}}}^{AB}\partial_A\partial_B u\|_{\mathscr{B}_0^\sigma}\|\|x\delta\mathfrak{h}^0\|_{\mathscr{G}_k^{\alpha-\sigma}} + \|x\delta\mathfrak{h}^0\|_{\mathscr{C}^0_{\{x=0\},1}}\|\hat{\mathring{\mathfrak{g}}}^{AB}\partial_A\partial_B u\|_{\mathscr{H}_{k-1}^\alpha}\big) \\
&\quad\le\ C\big(\|\hat{\mathring{\mathfrak{g}}}^\sharp\|_{L^\infty}\|\partial_A u\|_{\mathscr{B}_1^\sigma}\|x\delta\mathfrak{h}^0\|_{\mathscr{G}_k^{\alpha-\sigma}} + \|\hat{\mathring{\mathfrak{g}}}^\sharp\|_{\mathscr{G}^0_{k-1}}\|\partial_A u\|_{\mathscr{H}_k^\alpha}\|x\delta\mathfrak{h}^0\|_{\mathscr{C}^0_{\{x=0\},1}}\big) \\
&\quad\le\ C\big(\|\mathring{\mathfrak{g}}^\sharp\|_{L^\infty}\|\partial_A u\|_{\mathscr{B}_1^\sigma}\|\delta\mathfrak{g}^\sharp\|_{\mathscr{G}_k^{\alpha-\sigma}} + \|\mathring{\mathfrak{g}}^\sharp\|_{\mathscr{G}^0_{k-1}}\|\delta\mathfrak{g}^\sharp\|_{\mathscr{C}_1^0}E^\alpha_{k,\lambda}[u(\tau)]\big)\ . \qquad (3.2.95)
\end{aligned}$$

A similar analysis of the remaining terms proves the proposition. □

Chapter 4

Application to the Einstein-Maxwell Equations in wave coordinates and Lorenz gauge

4.1 Change of coordinates

4.1.1 On the gauge condition

Throughout this section, the (unphysical) conformally rescaled metric is denoted by $\mathfrak{g}$, and the (physical) metric is denoted by g; thus $\mathfrak{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$.

Remember that in the original system of coordinates (x^μ) we have

$$\Box_g x^\mu = 0 \qquad \textit{with} \qquad g = \eta + h \ ,$$

which leads to

$$\partial_\mu\big(g^{\mu\nu}\sqrt{|\det g|}\big) = 0 \ . \tag{4.1.1}$$

We want to rewrite the above equation in the new system of coordinate (y^α) (see (4.1.4)). We have

$$\sqrt{|\det g|} = 1 + \frac{1}{2}\eta^{\alpha\beta}h_{\alpha\beta} + Q(h) \ ,$$

where Q has a uniform zero of order two in h. We set

$$g^{\mu\nu} = \eta^{\mu\nu} + H^{\mu\nu} \ . \tag{4.1.2}$$

In what follows, we use a generic symbol Q for functions which have a uniform zero of order two. We have

$$
\begin{aligned}
\partial_\mu\big(g^{\mu\nu}\sqrt{|\det g|}\big) &= \partial_\mu\big[g^{\mu\nu}\{1+\frac{1}{2}\eta^{\alpha\beta}h_{\alpha\beta}+Q(h)\}\big] \\
&= \partial_\mu\big[\{\eta^{\mu\nu}+H^{\mu\nu}\}\{1+\frac{1}{2}\eta^{\alpha\beta}h_{\alpha\beta}+Q(h)\}\big] \\
&= \partial_\mu H^{\mu\nu}\{1+\frac{1}{2}\eta^{\alpha\beta}h_{\alpha\beta}+Q(h)\}+\{\eta^{\mu\nu}+H^{\mu\nu}\}\{\frac{1}{2}\eta^{\alpha\beta}\partial_\mu h_{\alpha\beta}+\partial_\mu Q(h)\}\ .
\end{aligned}
$$

Using this identity, equation (4.1.1) takes the form:

$$
\begin{aligned}
&\partial_\mu H^{\mu\nu}+\frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\partial_\mu h_{\alpha\beta} \\
&\quad= -\partial_\mu H^{\mu\nu}\{\frac{1}{2}\eta^{\alpha\beta}h_{\alpha\beta}+Q(h)\}-H^{\mu\nu}\{\frac{1}{2}\eta^{\alpha\beta}\partial_\mu h_{\alpha\beta}+\partial_\mu Q(h)\}-\eta^{\mu\nu}\partial_\mu Q(h)\ . \qquad (4.1.3)
\end{aligned}
$$

Let us rewrite this equation in the system of coordinates (τ, x, v^A) where

$$
y^\mu=\frac{x^\mu}{\eta_{\alpha\beta}x^\alpha x^\beta},\quad \tau=y^0\le 0,\quad x=-y^0-\rho\ge 0 \quad \textit{and} \quad y^i=\rho\omega^i(v^A)\ . \tag{4.1.4}
$$

Recall that

$$
\Omega=-y_\alpha y^\alpha=\tau^2-\rho^2=x(-\tau+\rho)\ge 0\ , \tag{4.1.5}
$$

*and $\hat f=\Omega^{-\frac{n-1}{2}}f$ (*not *to be confused with division by $\mathfrak{g}^{\tau\tau}$, as used in the previous chapter), so that*

$$
\frac{\partial f}{\partial x^\mu}=\Omega^{\frac{n-1}{2}}\Big\{-(n-1)y_\mu-\Omega\frac{\partial}{\partial y^\mu}-2y_\mu y^\alpha\frac{\partial}{\partial y^\alpha}\Big\}\hat f\ , \tag{4.1.6}
$$

thus the left-hand-side of (4.1.3) can be rewritten as

$$
-(n-1)\Omega^{\frac{n-1}{2}}y_\mu\Big(\widehat H^{\mu\nu}+\frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\hat h_{\alpha\beta}\Big)-\Omega^{\frac{n-1}{2}}\Big\{\Omega\frac{\partial}{\partial y^\mu}+2y_\mu y^\alpha\frac{\partial}{\partial y^\alpha}\Big\}\Big(\widehat H^{\mu\nu}+\frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\hat h_{\alpha\beta}\Big).
$$

We want to analyze the structure of the right-hand side of (4.1.3). This expression is made of three terms which will be labeled R_1, R_2, and R_3. We have (see (4.1.6) and recall that $y^\alpha\frac{\partial\Omega}{\partial y^\alpha}=2\Omega$):

$$
\begin{aligned}
R_1 &= \Omega^{\frac{n-1}{2}}\left\{\frac{1}{2}\Omega^{\frac{n-1}{2}}tr_\eta(\hat h)+Q(\Omega^{\frac{n-1}{2}}\hat h)\right\}\left\{(n-1)y_\mu+\Omega\frac{\partial}{\partial y^\mu}+2y_\mu y^\alpha\frac{\partial}{\partial y^\alpha}\right\}\widehat H^{\mu\nu} \\
&=: Q(\Omega^{\frac{n-1}{2}}\hat h,\Omega^{\frac{n-1}{2}}y_\mu\widehat H^{\mu\nu})+Q(\Omega^{\frac{n-1}{2}}\hat h,\Omega^{\frac{n+1}{2}}\partial_\mu\widehat H^{\mu\nu}) \\
&\quad +Q(\Omega^{\frac{n-1}{2}}\hat h,\Omega^{\frac{n-1}{2}}y_\mu y^\alpha\frac{\partial}{\partial y^\alpha}\widehat H^{\mu\nu})\ . \qquad (4.1.7)
\end{aligned}
$$

Now, since $\frac{\partial Q}{\partial h}$ has a uniform zero of order one, we have

$$\begin{aligned}\frac{\partial}{\partial x^\mu}Q(h) &= \frac{\partial Q}{\partial h}\frac{\partial h}{\partial x^\mu} = -\frac{\partial Q}{\partial h}(\Omega^{\frac{n-1}{2}}\hat h)\Omega^{\frac{n-1}{2}}\left\{(n-1)y_\mu + \Omega\frac{\partial}{\partial y^\mu} + 2y_\mu y^\alpha\frac{\partial}{\partial y^\alpha}\right\}\hat h \\ &=: Q(\Omega^{\frac{n-1}{2}}\hat h, \Omega^{\frac{n-1}{2}}y_\mu\hat h) + Q(\Omega^{\frac{n-1}{2}}\hat h, \Omega^{\frac{n+1}{2}}\frac{\partial\hat h}{\partial y^\mu}) + Q(\Omega^{\frac{n-1}{2}}\hat h, \Omega^{\frac{n-1}{2}}y_\mu y^\alpha\frac{\partial}{\partial y^\alpha}\hat h)\ .\end{aligned}$$

Thus R_2 reads:

$$\begin{aligned}R_2 &= \frac{1}{2}\eta^{\alpha\beta}\Omega^{\frac{n-1}{2}}\widehat H^{\mu\nu}\left\{\Omega^{\frac{n-1}{2}}\right\}\left\{(n-1)y_\mu + \Omega\frac{\partial}{\partial y^\mu} + 2y_\mu y^\alpha\frac{\partial}{\partial y^\alpha}\right\}\hat h_{\alpha\beta} \\ &\quad +\Omega^{\frac{n-1}{2}}\widehat H^{\mu\nu}\left\{Q(\Omega^{\frac{n-1}{2}}\hat h, \Omega^{\frac{n-1}{2}}y_\mu\hat h) + Q(\Omega^{\frac{n-1}{2}}\hat h, \Omega^{\frac{n+1}{2}}\partial_\mu\hat h)\right. \\ &\quad \left. +Q(\Omega^{\frac{n-1}{2}}\hat h, \Omega^{\frac{n-1}{2}}y_\mu y^\alpha\frac{\partial}{\partial y^\alpha}\hat h)\right\}\ . \\ &=: Q(\Omega^{\frac{n-1}{2}}\hat h, \Omega^{\frac{n-1}{2}}y_\mu\widehat H^{\mu\nu}) + Q(\Omega^{\frac{n-1}{2}}\widehat H^{\mu\nu}, \Omega^{\frac{n+1}{2}}\partial_\mu\hat h) + Q(\Omega^{\frac{n-1}{2}}y_\mu\widehat H^{\mu\nu}, \Omega^{\frac{n-1}{2}}y^\alpha\frac{\partial}{\partial y^\alpha}\hat h)\ .\end{aligned} \tag{4.1.8}$$

Next

$$\begin{aligned}R_3 &= -\eta^{\mu\nu}\partial_\mu Q(h) \\ &= \eta^{\mu\nu}\left\{Q(\Omega^{\frac{n-1}{2}}\hat h, \Omega^{\frac{n-1}{2}}y_\mu\hat h) + Q(\Omega^{\frac{n-1}{2}}\hat h, \Omega^{\frac{n+1}{2}}\partial_\mu\hat h) + Q(\Omega^{\frac{n-1}{2}}\hat h, \Omega^{\frac{n-1}{2}}y_\mu y^\alpha\frac{\partial}{\partial y^\alpha}\hat h)\right\}\end{aligned} \tag{4.1.9}$$

From this, we obtain the following form of the gauge condition (4.1.3):

$$\begin{aligned}y_\mu\widehat H^{\mu\nu} + \frac{1}{2}y^\nu\eta^{\alpha\beta}\hat h_{\alpha\beta} &= \frac{1}{1-n}\Big\{\Omega\frac{\partial}{\partial y^\mu} + 2y_\mu y^\alpha\frac{\partial}{\partial y^\alpha}\Big\}\Big(\widehat H^{\mu\nu} + \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\hat h_{\alpha\beta}\Big) \\ &\quad + \Omega^{-\frac{n-1}{2}}(R_1 + R_2 + R_3)\ .\end{aligned} \tag{4.1.10}$$

Now we recall that

$$H^{\mu\nu} := g^{\mu\nu} - \eta^{\mu\nu} = -h^{\mu\nu} + \widetilde Q^{\mu\nu}(h)\ ,$$

where $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$. Therefore

$$\eta^{\alpha\beta}\hat h_{\alpha\beta} = -\eta_{\alpha\beta}\widehat H^{\alpha\beta} + \Omega^{-\frac{n-1}{2}}\widetilde Q(\Omega^{\frac{n-1}{2}}\widehat H).$$

Equations (4.1.7)-(4.1.10) lead finally to the following form of the gauge *condition (4.1.3):*

$$\begin{aligned}
y_\mu \widehat{H}^{\mu\nu} - \frac{1}{2} y^\nu tr_\eta(\widehat{H}) &= \frac{1}{1-n}\Big\{\Omega \frac{\partial}{\partial y^\mu} + 2 y_\mu y^\alpha \frac{\partial}{\partial y^\alpha}\Big\}\Big(\widehat{H}^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} tr_\eta \widehat{H}\Big) \\
&\quad + \Omega^{-\frac{n-1}{2}} Q(\Omega^{\frac{n-1}{2}}\widehat{H}, \Omega^{\frac{n-1}{2}}\widehat{H}) \\
&\quad + \Omega^{-\frac{n-1}{2}} Q(\Omega^{\frac{n-1}{2}}\widehat{H}, \Omega^{\frac{n+1}{2}}\partial\widehat{H}) \\
&\quad + \Omega^{-\frac{n-1}{2}} Q(\Omega^{\frac{n-1}{2}}\widehat{H}, \Omega^{\frac{n-1}{2}} y^\alpha \frac{\partial}{\partial y^\alpha}\widehat{H}) . \qquad (4.1.11)
\end{aligned}$$

We will need the following consequence of this equation: multiplying by y_ν and commuting derivatives one is led to

$$\begin{aligned}
(n-5) y_\nu y_\mu \widehat{H}^{\mu\nu} &= 2 y^\alpha \frac{\partial}{\partial y^\alpha}\left(y_\mu y_\nu \widehat{H}^{\mu\nu} + \frac{1}{2}\Omega tr_\eta \widehat{H}\right) \\
&\quad + \Omega\Big(\frac{n-5}{2} tr_\eta(\widehat{H}) + y_\nu \frac{\partial}{\partial y^\mu}\big(\widehat{H}^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} tr_\eta(\widehat{H})\big)\Big) \\
&\quad + \Omega^{-\frac{n-1}{2}} Q(\Omega^{\frac{n-1}{2}}\widehat{H}, \Omega^{\frac{n-1}{2}}\widehat{H}) \\
&\quad + \Omega^{-\frac{n-1}{2}} Q(\Omega^{\frac{n-1}{2}}\widehat{H}, \Omega^{\frac{n+1}{2}}\partial\widehat{H}) \\
&\quad + \Omega^{-\frac{n-1}{2}} Q(\Omega^{\frac{n-1}{2}}\widehat{H}, \Omega^{\frac{n-1}{2}} y^\alpha \frac{\partial}{\partial y^\alpha}\widehat{H}) . \qquad (4.1.12)
\end{aligned}$$

4.1.2 On the wave equation

In wave coordinates (x^μ), we consider the following wave equation

$$\eta^{\alpha\beta}\frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} + H^{\alpha\beta}(f,\partial f)\frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} = F(f,\partial f) . \qquad (4.1.13)$$

In order to check all the hypotheses made on components of the metric in our theorem on the energy estimate, we have to rewrite this equation with respect the system of coordinates (τ, x, v^A) used there. According to our previous calculations, equation (4.1.13) can be written as

$$\eta^{\lambda\mu}\frac{\partial^2 \hat{f}}{\partial y^\lambda \partial y^\mu} + \Omega^{-\frac{n+3}{2}} H^{\lambda\mu}(f,\partial f)\frac{\partial^2 f}{\partial x^\lambda \partial x^\mu} = \Omega^{-\frac{n+3}{2}} F(f,\partial f) , \qquad (4.1.14)$$

where

$$\hat{f} = \Omega^{-\frac{n-1}{2}} f .$$

So, let us express the second term of the above equation in terms of coordinates y^ν. We already know the identity:

$$\frac{\partial^2 f}{\partial x^\lambda \partial x^\mu}\circ\phi^{-1} = \frac{\partial^2(f\circ\phi^{-1})}{\partial y^\alpha \partial y^\beta} A^\alpha_\mu A^\beta_\lambda + \frac{\partial(f\circ\phi^{-1})}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\lambda}\circ\phi^{-1} =: K_{\lambda\mu} + V_{\lambda\mu} , \qquad (4.1.15)$$

with

$$\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\lambda} \circ \phi^{-1} = 2\Omega \delta^\alpha_\mu \eta_{\lambda\sigma} y^\sigma + 2\Omega \delta^\alpha_\lambda \eta_{\mu\tau} y^\tau + 2\Omega \eta_{\mu\lambda} y^\alpha + 8\eta_{\lambda\sigma}\eta_{\mu\theta} y^\sigma y^\alpha y^\theta$$

and

$$A^\alpha_\mu A^\beta_\lambda = \Omega^2 \delta^\alpha_\mu \delta^\beta_\lambda + 4 y_\lambda y_\mu y^\alpha y^\beta + 2\Omega(\delta^\alpha_\mu y_\lambda y^\beta + \delta^\beta_\lambda y_\mu y^\alpha)\ .$$

These identities lead to

$$H^{\lambda\mu} V_{\lambda\mu} = H^{\lambda\mu} \left\{ 2\Omega \delta^\alpha_\mu \eta_{\lambda\sigma} y^\sigma + 2\Omega \delta^\alpha_\lambda \eta_{\mu\theta} y^\theta + 2\Omega \eta_{\mu\lambda} y^\alpha + 8\eta_{\lambda\sigma}\eta_{\mu\theta} y^\sigma y^\alpha y^\theta \right\} \frac{\partial f}{\partial y^\alpha}\ . \tag{4.1.16}$$

Now we also know that

$$\frac{\partial f}{\partial y^\alpha} = \Omega^{\frac{n-3}{2}} \left\{ \Omega \frac{\partial \hat{f}}{\partial y^\alpha} - (n-1) y_\alpha \hat{f} \right\}\ . \tag{4.1.17}$$

This implies that (note that in this equation, the term $y_\mu y_\lambda H^{\mu\lambda}$ is the one which has the the smallest multiplicative power of Ω):

$$H^{\lambda\mu} V_{\lambda\mu} = 2\Omega^{\frac{n-1}{2}} H^{\lambda\mu} \left\{ (n-1) \left\{ \Omega \eta_{\lambda\mu} + 2 y_\mu y_\lambda \right\} \hat{f} + \left(2\Omega \delta^\alpha_\mu y_\lambda + \Omega \eta_{\lambda\mu} y^\alpha + 4 y_\mu y_\lambda y^\alpha \right) \frac{\partial \hat{f}}{\partial y^\alpha} \right\}\ . \tag{4.1.18}$$

On the other hand we have

$$\begin{aligned} \frac{\partial^2 (f \circ \phi^{-1})}{\partial y^\alpha \partial y^\beta} \quad &= \quad \Omega^{\frac{n-5}{2}} \Bigg\{ \Omega^2 \frac{\partial^2 \hat{f}}{\partial y^\alpha \partial y^\beta} - (n-1)\Omega \left(y_\beta \frac{\partial \hat{f}}{\partial y^\alpha} + y_\alpha \frac{\partial \hat{f}}{\partial y^\beta} \right) \\ &\qquad + (n-1) \left[(n-3) y_\alpha y_\beta - \Omega \eta_{\alpha\beta} \right] \hat{f} \Bigg\}\ , \end{aligned}$$

which leads to the following expression of $H^{\lambda\mu} K_{\lambda\mu}$:

$$\begin{aligned} H^{\lambda\mu} K_{\lambda\mu} \quad &= \quad \Omega^{\frac{n-5}{2}} H^{\lambda\mu} \left\{ \Omega^2 \delta^\alpha_\mu \delta^\beta_\lambda + 4 y_\mu y_\lambda y^\alpha y^\beta + 2\Omega y^\theta (\eta_{\lambda\theta} \delta^\alpha_\mu y^\beta + \eta_{\mu\theta} \delta^\beta_\lambda y^\alpha) \right\} \\ &\times \quad \left\{ \Omega^2 \frac{\partial^2 \hat{f}}{\partial y^\alpha \partial y^\beta} - (n-1)\Omega \left(y_\beta \frac{\partial \hat{f}}{\partial y^\alpha} + y_\alpha \frac{\partial \hat{f}}{\partial y^\beta} \right) + (n-1) \left[(n-3) y_\alpha y_\beta - \Omega \eta_{\alpha\beta} \right] \hat{f} \right\}, \end{aligned}$$

and after simplifications, we find that

$$\begin{aligned} H^{\lambda\mu} K_{\lambda\mu} \quad &= \quad \Omega^{\frac{n-1}{2}} H^{\lambda\mu} \left\{ \Omega^2 \delta^\alpha_\mu \delta^\beta_\lambda + 4 y_\mu y_\lambda y^\alpha y^\beta + 2\Omega(\delta^\alpha_\mu y_\lambda y^\beta + \delta^\beta_\lambda y_\mu y^\alpha) \right\} \frac{\partial^2 \hat{f}}{\partial y^\alpha \partial y^\beta} \\ &+ \quad (n-1)\Omega^{\frac{n-1}{2}} H^{\lambda\mu} \left\{ 2 \left(2 y_\lambda y_\mu y^\alpha + \Omega \delta^\alpha_\lambda y_\mu \right) \frac{\partial \hat{f}}{\partial y^\alpha} + \left[(n-3) y_\mu y_\lambda - \Omega \eta_{\lambda\mu} \right] \hat{f} \right\}\ . \end{aligned} \tag{4.1.19}$$

With the expressions (4.1.18) and (4.1.19) and writing $H^{\lambda\mu} = \Omega^{\frac{n-1}{2}}\widehat{H}^{\lambda\mu}$, *equation (4.1.14) reads after simplifications*

$$\begin{aligned}
&\Big\{\eta^{\alpha\beta} + \Omega^{\frac{n-5}{2}}\widehat{H}^{\lambda\mu}\Big[\Omega^2\delta^\alpha_\mu\delta^\beta_\lambda + 4y_\mu y_\lambda y^\alpha y^\beta + 2\Omega(\delta^\alpha_\mu y_\lambda y^\beta + \delta^\beta_\lambda y_\mu y^\alpha)\Big]\Big\}\frac{\partial^2\hat f}{\partial y^\alpha\partial y^\beta}\\
+ \quad & 2\Omega^{\frac{n-5}{2}}\widehat{H}^{\lambda\mu}\Bigg\{\{2(n+1)y_\mu y_\lambda y^\alpha + (n+1)\Omega\delta^\alpha_\mu y_\lambda + \eta_{\lambda\mu}\Omega y^\alpha\}\frac{\partial\hat f}{\partial y^\alpha}\\
&\qquad + (n-1)\{(n+1)y_\mu y_\lambda + \Omega\eta_{\lambda\mu}\}\hat f\Bigg\}\\
= \quad & \Omega^{-\frac{n+3}{2}}F\left(\Omega^{\frac{n-1}{2}}\hat f, \Omega^{(n-1)/2}\{-\Omega\frac{\partial}{\partial y^\nu} - 2y_\nu y^\alpha\frac{\partial}{\partial y^\alpha} - (n-1)y_\nu\}\hat f\right)\\
=: \quad & \Omega^{-\frac{n+3}{2}}\widetilde{F}\left(\Omega^{\frac{n-1}{2}}\hat f, \Omega^{\frac{n-1}{2}}\frac{\partial\hat f}{\partial y^\nu}\right) . \qquad (4.1.20)
\end{aligned}$$

We want to apply the energy estimates of Section 3.2.3 to the equation considered here. So for consistency of notation in that section, we write the above equation in the form (recall that $\Omega = x(\rho-\tau)$*):*

$$\Box_{\mathfrak{g}} u = \mathcal{F}(u, \partial u) , \qquad (4.1.21)$$

with

$$u = \hat f , \qquad (4.1.22)$$

$$\begin{aligned}
\mathfrak{g}^{\alpha\beta} \quad = \quad & \eta^{\alpha\beta} + \{x(\rho-\tau)\}^{\frac{n-5}{2}}\widehat{H}^{\lambda\mu}\times\\
& \underbrace{\Big\{\{x(\rho-\tau)\}^2\delta^\alpha_\mu\delta^\beta_\lambda + 4y_\mu y_\lambda y^\alpha y^\beta + 2\{x(\rho-\tau)\}(\delta^\alpha_\mu y_\lambda y^\beta + \delta^\beta_\lambda y_\mu y^\alpha)\Big\}}_{:=\psi^{\alpha\beta}{}_{\lambda\mu}} ,
\end{aligned} \qquad (4.1.23)$$

(in order to reduce the typographical length of formulae we will sometimes write $\psi^{\alpha\beta}_{\mu\nu}$ *for* $\psi^{\alpha\beta}{}_{\mu\nu}$*) and*

$$\begin{aligned}
\mathcal{F}\left(u, \frac{\partial u}{\partial y^\nu}\right) \quad = \quad & \Omega^{-\frac{n+3}{2}}\widetilde{F}\left(\Omega^{\frac{n-1}{2}}u, \Omega^{\frac{n-1}{2}}\frac{\partial u}{\partial y^\nu}\right)\\
& + \Bigg\{\Upsilon^\alpha - 2\Omega^{\frac{n-5}{2}}\widehat{H}^{\lambda\mu}\{2(n+1)y_\mu y_\lambda y^\alpha + (n+1)\Omega\delta^\alpha_\mu y_\lambda + \eta_{\lambda\mu}\Omega y^\alpha\}\Bigg\}\frac{\partial u}{\partial y^\alpha}\\
& - 2(n-1)\Omega^{\frac{n-5}{2}}\widehat{H}^{\lambda\mu}\{(n+1)y_\mu y_\lambda + \Omega\eta_{\lambda\mu}\}u . \qquad (4.1.24)
\end{aligned}$$

So, we have to check that the metric $\mathfrak{g}$ *defined by (4.1.23) and the harmonicity functions*

$$\Upsilon^\mu = \frac{1}{\sqrt{|\det \mathfrak{g}|}} \partial_\nu \left\{ \sqrt{|\det \mathfrak{g}|} \mathfrak{g}^{\mu\nu} \right\} \tag{4.1.25}$$

satisfy the hypotheses of our theorem.

The tensor $\psi^{\alpha\beta}{}_{\mu\nu}$ *defined in (4.1.23) has the property*

$$\eta_{\alpha\beta}\psi^{\alpha\beta}{}_{\mu\nu} = \Omega^2 \eta_{\mu\nu} , \tag{4.1.26}$$

which implies that the contraction

$$\eta_{\alpha\beta}(\mathfrak{g}^{\alpha\beta} - \eta^{\alpha\beta}) = \Omega^{\frac{n-1}{2}} tr_\eta \widehat{H}$$

gains two powers of Ω*, as compared to a direct power-counting based on (4.1.23). Furthermore, the structure* $y^\alpha y^\beta y_\mu y_\nu$ *of the term without powers of* Ω *in* $\psi^{\alpha\beta}{}_{\mu\nu}$ *implies that any contraction of the form* $\psi^{\alpha\beta}{}_{\mu\nu}\eta_{\alpha\rho}\psi^{\rho\sigma}{}_{\gamma\delta}$ *acquires an overall multiplicative factor of* Ω*. So if we set*

$$\delta\mathfrak{g}^\alpha{}_\beta := \mathfrak{g}^{\alpha\mu}\eta_{\mu\beta} - \delta^\alpha_\beta ,$$

it follows that for $k \geq 2$ *we have*

$$\left((\delta\mathfrak{g})^k\right)^\alpha{}_\beta := \delta\mathfrak{g}^\alpha{}_{\alpha_1}\delta\mathfrak{g}^{\alpha_1}{}_{\alpha_2}\cdots\delta\mathfrak{g}^{\alpha_{k-1}}{}_\beta = \Omega^{k-1}Q_k(\Omega^{\frac{n-5}{2}}\widehat{H}) ,$$

where we use the symbol Q_k *to denote a smooth function (in this case, a polynomial) with a uniform zero of order* k*, and which may change from line to line. A similar analysis shows that, again for* $k \geq 2$*, the trace*

$$p_k(\delta\mathfrak{g}) := tr(\delta\mathfrak{g})^k = \delta\mathfrak{g}^\alpha{}_{\alpha_1}\delta\mathfrak{g}^{\alpha_1}{}_{\alpha_2}\cdots\delta\mathfrak{g}^{\alpha_{k-1}}{}_\alpha = \Omega^k Q_k(\Omega^{\frac{n-5}{2}}\widehat{H}) \tag{4.1.27}$$

(no summation over k*) gains one more power of* Ω*.*

Set

$$A^\alpha{}_\beta := \delta^\alpha_\beta + \delta\mathfrak{g}^\alpha{}_\beta . \tag{4.1.28}$$

Equation (4.1.27) implies

$$p_i(A) = tr(I+\delta\mathfrak{g})^i = \sum_{j=0}^n C_i^j p_j(\delta\mathfrak{g}) = n+1+i tr\delta\mathfrak{g}+\Omega^2 Q_2(\Omega^{\frac{n-5}{2}}\widehat{H}) . \tag{4.1.29}$$

Let $W(\lambda)$ *denote the characteristic polynomial of* A*,*

$$W(\lambda) = \det(A - \lambda I) = \det A + w_1\lambda + \ldots + w_n\lambda^n + (-\lambda)^{n+1} .$$

Then the coefficients w_i are homogeneous polynomials of order $n+1-i$ in the entries of $A = I + \delta\mathfrak{g}$, with $w_n = (-1)^n trA = (-1)^n(n+1+tr\delta\mathfrak{g})$. It is a well known consequence of the Cayley-Hamilton theorem (see, e.g., [52, Theorem 1]) that both $\det A$ and the w_i's can be written as polynomials in the p_i's, and since each $p_i(A)$ has a factor Ω^2 in front of the Q_2 terms, we find that the w_i's take the form

$$w_i(A) = w_i(I) + \ell_i(tr\delta\mathfrak{g}) + \Omega^2 Q_2(\Omega^{\frac{n-5}{2}}\widehat{H}) , \tag{4.1.30}$$

where $\ell_i(tr\delta\mathfrak{g})$ is linear in $tr\delta\mathfrak{g}$.

Now

$$\mathfrak{g}^{\alpha\beta} = \mathfrak{g}^{\alpha\sigma}\eta_{\sigma\rho}\eta^{\rho\beta} = \left(\delta^\alpha_\rho + \delta\mathfrak{g}^\alpha{}_\rho\right)\eta^{\rho\beta} = A^\alpha{}_\rho\eta^{\rho\beta} , \tag{4.1.31}$$

hence

$$\det\mathfrak{g}^\sharp = -\det(A) ,$$

which shows that

$$\det\mathfrak{g}^\sharp = -1 + \Omega^2\left(-\Omega^{\frac{n-5}{2}} tr_\eta\widehat{H} + Q_2(\Omega^{\frac{n-5}{2}}\widehat{H})\right) = -1 + \Omega^2 Q_1(\Omega^{\frac{n-5}{2}}\widehat{H}) . \tag{4.1.32}$$

From the Cayley-Hamilton theorem we have

$$A^{-1} = -\frac{1}{\det A}\left(w_1 I + \cdots + w_n A^{n-1} + (-1)^{n+1}A^n\right) ,$$

and we conclude that $\mathfrak{g}_{\alpha\beta} = (\eta^{-1}A^{-1})_{\alpha\beta}$ takes the form

$$\begin{aligned}
\mathfrak{g}_{\alpha\beta} &= \frac{1}{1+\Omega^2 Q_1(\Omega^{\frac{n-5}{2}})}\left(\eta_{\alpha\beta} - \Omega^{\frac{n-5}{2}}\widehat{H}^{\mu\nu}\psi_{\alpha\beta\mu\nu} + \Omega^2 Q_2(\Omega^{\frac{n-5}{2}}\widehat{H})\right) \\
&= \eta_{\alpha\beta} - \Omega^{\frac{n-5}{2}}\widehat{H}^{\mu\nu}y_\mu y_\nu y^\alpha y^\beta + \Omega Q_1(\Omega^{\frac{n-5}{2}}\widehat{H}) \\
&\quad + \Omega^2 Q_2(\Omega^{\frac{n-5}{2}}\widehat{H}) ,
\end{aligned} \tag{4.1.33}$$

where the indices on $\psi_{\alpha\beta\mu\nu}$ have been lowered with the metric $\eta_{\alpha\beta}$.

4.1.3 On the components of the metric

Recall that, to obtain energy inequalities, our hypotheses on certain components of the metric were

$$\mathfrak{g}^{00} = -1 + x\mathfrak{h}^0; \quad \mathfrak{g}^{0\rho} = -x\mathfrak{h}^1; \quad \mathfrak{g}^{0A} + \mathfrak{g}^{\rho A} = -x\mathfrak{h}^A \quad \textit{and} \quad \mathfrak{g}^{\rho\rho} = 1 + x\mathfrak{h} , \tag{4.1.34}$$

where the functions $\mathfrak{h}$, $\mathfrak{h}^0$, $\mathfrak{h}^A$ *are bounded on bounded sets. Since (compare (4.1.4))*

$$\mathfrak{g}^{0\rho} = \mathfrak{g}^{0i}\omega_i, \quad \mathfrak{g}^{0A} = \mathfrak{g}^{0i}\frac{\partial v^A}{\partial y^i}, \quad \mathfrak{g}^{\rho A} = \mathfrak{g}^{ij}\omega_j\frac{\partial v^A}{\partial y^i} \quad \text{and} \quad \mathfrak{g}^{\rho\rho} = \mathfrak{g}^{ij}\omega_i\omega_j \ ,$$

from (4.1.23) we have (note that $y^i\omega_i = \rho$, $\rho\omega_i\delta^i_\mu = y_\mu + \tau\delta^\tau_\mu$*):*

$$\mathfrak{h}^0 = x^{\frac{n-7}{2}}(\rho-\tau)^{\frac{n-5}{2}}\widehat{H}^{\lambda\mu}\left\{\{x(\rho-\tau)\}^2\delta^\tau_\mu\delta^\tau_\lambda + 4\tau^2 y_\mu y_\lambda + 4\tau\{x(\rho-\tau)\}\delta^\tau_\mu y_\lambda\right\} \ , \tag{4.1.35}$$

$$\mathfrak{h}^1 = -x^{\frac{n-7}{2}}(\rho-\tau)^{\frac{n-5}{2}}\widehat{H}^{\lambda\mu}\left\{\{x(\rho-\tau)\}^2\delta^\tau_\mu\delta^i_\lambda\omega_i + 4\tau\rho y_\mu y_\lambda + 2\rho\{x(\rho-\tau)\}y_\lambda(\delta^0_\mu\rho + \tau\delta^i_\mu\omega_i)\right\} \ , \tag{4.1.36}$$

$$\mathfrak{h} = x^{\frac{n-7}{2}}(\rho-\tau)^{\frac{n-5}{2}}\widehat{H}^{\lambda\mu}\Big\{\{x(\rho-\tau)\}^2\delta^i_\mu\delta^j_\lambda\omega_i\omega_j + 4\rho^2 y_\mu y_\lambda + 4\{x(\rho-\tau)\}y_\lambda\rho\delta^i_\mu y_\lambda\omega_i\Big\} \ , \tag{4.1.37}$$

$$\mathfrak{h}^A = -x^{\frac{n-3}{2}}(\rho-\tau)^{\frac{n-3}{2}}\Big\{(\rho-\tau)\Big(\widehat{H}^{0i} + \omega_j\widehat{H}^{ij}\Big) - 2y_\lambda\widehat{H}^{\lambda i}\Big\}\frac{\partial v^A}{\partial y^i} \ . \tag{4.1.38}$$

We see that the components of the metric (4.1.23) have the right structure (4.1.34) if the space dimension n *is greater then or equal to 7. We will see in Section 4.2.1 (see* (4.1.12)*) that this can be lowered to* $n \geq 6$ *using the harmonic coordinates condition.*

We note the identities,

$$\eta^{ij}\omega_j\frac{\partial v^A}{\partial y^i} = \sum_{j=1}^n \omega_j\frac{\partial v^A}{\partial y^j} = \frac{\partial v^A}{\partial r} = 0 \ ,$$

which justify that $\mathfrak{g}^{0A} + \mathfrak{g}^{\rho A}$ *has the right structure. In particular, for this component the condition* $n \geq 4$ *suffices to fulfill the structure condition.*

We will also need

$$\mathfrak{g}^{\tau\tau} = -1 + O(x^{\frac{n-5}{2}}), \quad \mathfrak{g}^{\tau x} = 1 + O(x^{\frac{n-3}{2}}), \quad \mathfrak{g}^{xx} = O(x^{\frac{n-1}{2}}), \tag{4.1.39}$$
$$\mathfrak{g}^{xA} = O(x^{\frac{n-3}{2}}), \quad \mathfrak{g}^{AB} = \eta^{AB} + O(x^{\frac{n-5}{2}}) \ .$$

4.1.4 On the harmonicity functions

Now let us look at the harmonicity functions, defined as

$$\Upsilon^\mu := \frac{1}{\sqrt{|\det\mathfrak{g}|}}\partial_\nu\left\{\sqrt{|\det\mathfrak{g}|}\mathfrak{g}^{\mu\nu}\right\} \ .$$

Since our energy estimates have been established using the coordinate system (x,τ,v^A) *as defined in (2.3.3), we need to calculate* Υ^μ *in that coordinate* *system. But so far we only have the expression of the metric in the y^μ–coordinate system. To avoid confusion let us write ${}^{(2)}\Upsilon$ for Υ associated to the coordinates (τ, x, v^A) and ${}^{(1)}\Upsilon$ for that associated to the coordinates y^μ. To understand the behaviour of Υ under coordinate changes, it is useful to write the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$ of the metric $\mathfrak{g}$ in the form*

$$\Gamma^\alpha_{\beta\gamma} = \mathring{\Gamma}^\alpha_{\beta\gamma} + C^\alpha_{\beta\gamma} \ ,$$

where the $\mathring{\Gamma}^\alpha_{\beta\gamma}$'s are the Christoffel symbols of the Minkowski metric η, and $C^\alpha_{\beta\gamma}$ is a tensor. Then, in the coordinate system y^μ we have

$$ {}^{(1)}\Upsilon^\alpha = -\underbrace{\mathfrak{g}^{\beta\gamma}C^\alpha_{\beta\gamma}}_{=:C^\alpha} \ , \tag{4.1.40}$$

since the $\mathring{\Gamma}^\alpha_{\beta\gamma}$'s vanish in the y^μ–coordinates. Note that C^α as defined in (4.1.40) is a vector field, being the contraction of two tensors. In the coordinates (τ, x, v^A) we have

$$ {}^{(2)}\Upsilon^\alpha = -\mathfrak{g}^{\beta\gamma}\left(\mathring{\Gamma}^\alpha_{\beta\gamma} + C^\alpha_{\beta\gamma}\right) = -\mathfrak{g}^{\beta\gamma}\mathring{\Gamma}^\alpha_{\beta\gamma} - C^\alpha \ . \tag{4.1.41}$$

Thus, to calculate ${}^{(2)}\Upsilon$ we need to vector-transform C^α to the (τ, x, v^A) coordinates, and calculate the missing term $\mathfrak{g}^{\beta\gamma}\mathring{\Gamma}^\alpha_{\beta\gamma}$ above. We start by calculating the vector field C^μ. We set

$$\mathfrak{g}^{\alpha\beta} =: \eta^{\alpha\beta} + \Omega^{\frac{n-5}{2}} K^{\alpha\beta} \ , \tag{4.1.42}$$

thus

$$K^{\alpha\beta} = \widehat{H}^{\mu\nu}\psi^{\alpha\beta}_{\mu\nu}$$

as in (4.1.23); we hope that the clash of notation with the completely different $K_{\alpha\beta}$ appearing in (4.1.15) will not confuse the reader.

From (4.1.32) we have (recall that Q means Q_2)

$$\left(\sqrt{|\det \mathfrak{g}|}\right)^{\mp 1} = 1 \pm \frac{1}{2}\Omega^{\frac{n-1}{2}} tr_\eta(\widehat{H}) + \Omega^2 Q(\Omega^{\frac{n-5}{2}}\widehat{H}) \ .$$

Thus in the coordinate system y^μ,

$$\begin{aligned} \mathfrak{g}^{\mu\nu}\sqrt{|\det \mathfrak{g}|} &= \eta^{\mu\nu}\left(1 - \frac{1}{2}\Omega^{\frac{n-1}{2}} tr_\eta \widehat{H}\right) + \Omega^{\frac{n-5}{2}} K^{\mu\nu} \\ &\quad + \Omega^2 Q^{\mu\nu}(\Omega^{\frac{n-5}{2}}\widehat{H}) \ , \end{aligned} \tag{4.1.43}$$

$$\partial_\nu\left(\mathfrak{g}^{\mu\nu}\sqrt{|\det\mathfrak{g}|}\right) = \frac{1}{2}\Omega^{\frac{n-3}{2}}\left\{(n-1)y^\mu tr_\eta\widehat{H} - \eta^{\mu\nu}\Omega\partial_\nu tr_\eta\widehat{H}\right\} + \Omega^{\frac{n-7}{2}}\left\{(5-n)y_\nu K^{\mu\nu} + \Omega\partial_\nu K^{\mu\nu}\right\} + \partial_\nu\left\{\Omega^2 Q^{\mu\nu}(\Omega^{\frac{n-5}{2}}\widehat{H})\right\}$$

and since

$$\partial_\nu\Omega K^{\mu\nu} \sim y_\nu K^{\mu\nu} = -\Omega y_\beta \widehat{H}^{\alpha\beta}\left\{\Omega\delta^\mu_\alpha + 2y_\alpha y^\mu\right\}\ , \qquad (4.1.44)$$

and

$$\partial_\nu K^{\mu\nu} = \partial_\nu\widehat{H}^{\alpha\beta}\psi^{\mu\nu}_{\alpha\beta} + 2(n+3)y_\beta\widehat{H}^{\alpha\beta}\left\{\Omega\delta^\mu_\alpha + 2y_\alpha y^\mu\right\} + 2\Omega y^\mu tr_\eta\widehat{H}\ , \quad (4.1.45)$$

we obtain

$$\partial_\nu\left(\mathfrak{g}^{\mu\nu}\sqrt{|\det\mathfrak{g}|}\right) = \frac{1}{2}\Omega^{\frac{n-3}{2}}\left\{(n+3)y^\mu tr_\eta\widehat{H} - \eta^{\mu\nu}\Omega\partial_\nu tr_\eta\widehat{H}\right\} + \Omega^{\frac{n-5}{2}}\left\{\partial_\nu\widehat{H}^{\alpha\beta}\psi^{\mu\nu}_{\alpha\beta} + (3n+1)y_\beta\widehat{H}^{\alpha\beta}(\Omega\delta^\mu_\alpha + 2y_\alpha y^\beta)\right\} + \Omega^2 Q^\mu_2(\Omega^{\frac{n-5}{2}}\widehat{H}) + \Omega^2 Q^{\mu\nu}_2(\Omega^{\frac{n-5}{2}}\widehat{H}, \Omega^{\frac{n-5}{2}}\partial_\nu\widehat{H})\ .$$

Multiplying this last identity with $\left(\sqrt{|\det\mathfrak{g}|}\right)^{-1}$ *we then obtain the following expression for the vector field* C^μ*:*

$$C^\mu = {}^{(1)}\Upsilon^\mu = \frac{1}{2}\Omega^{\frac{n-3}{2}}\left\{(n+3)y^\mu tr_\eta\widehat{H} - \eta^{\mu\nu}\Omega\partial_\nu tr_\eta\widehat{H}\right\} + \Omega^{\frac{n-5}{2}}\left\{\partial_\nu\widehat{H}^{\alpha\beta}\psi^{\mu\nu}_{\alpha\beta} + (3n+1)y_\beta\widehat{H}^{\alpha\beta}\left\{\Omega\delta^\mu_\alpha + 2y_\alpha y^\mu\right\}\right\} + \Omega^2 Q^\mu(\Omega^{\frac{n-5}{2}}\widehat{H}) + \Omega^2 Q^{\mu\nu}(\Omega^{\frac{n-5}{2}}\widehat{H}, \Omega^{\frac{n-5}{2}}\partial_\nu\widehat{H})\ . \quad (4.1.46)$$

Now writing the vector field C *as*

$$C = C^\mu\partial_\mu =: C^\tau\partial_\tau + C^x\partial_x + C^A\partial_A\ ,$$

one is led to:

$$C^\tau = C^0\ , \quad C^\tau + C^x = -\omega_i(v)C^i\ , \quad C^A = \frac{\partial v^A}{\partial y^i}C^i\ .$$

In order to have all the harmonicity functions in the (τ, x, v^A)*-coordinates, it remains to calculate the term* $\mathfrak{g}^{\beta\gamma}\mathring{\Gamma}^\alpha_{\beta\gamma}$ *of the formula (4.1.41). In these coordinates the Christoffell's symbol of the Minkowski metric* $\mathring{\Gamma}^\alpha_{\beta\gamma}$ *read:*

$$\begin{aligned}
\mathring{\Gamma}^\tau_{\alpha\beta} &= 0,\\
\mathring{\Gamma}^x_{\tau\mu} &= \mathring{\Gamma}^x_{x\mu} = 0,\ \ \mathring{\Gamma}^x_{AB} = \rho\chi_{AB}\\
\mathring{\Gamma}^A_{\tau\tau} &= \Gamma^A_{\tau x} = \mathring{\Gamma}^A_{xx} = 0,\ \ \mathring{\Gamma}^A_{\tau B} = \mathring{\Gamma}^A_{xB} = -\frac{1}{\rho}\delta^A_B,\ \ \mathring{\Gamma}^A_{BC} = \gamma^A_{BC}\ ,
\end{aligned}$$

where we have denoted the round metric on the sphere by χ, and its corresponding Christoffel symbols γ^A_{BC}. These identities lead to the following (see identity (4.1.42)):

$$\mathfrak{g}^{\beta\gamma}\mathring{\Gamma}^\tau_{\beta\gamma} = 0 \tag{4.1.47a}$$

$$\mathfrak{g}^{\beta\gamma}\mathring{\Gamma}^x_{\beta\gamma} = \rho\mathfrak{g}^{AB}\chi_{AB} = \frac{n-1}{\rho} + \rho\Omega^{\frac{n-5}{2}}\widehat{H}^{\mu\nu}\psi^{AB}_{\mu\nu}\chi_{AB} \tag{4.1.47b}$$

$$\mathfrak{g}^{\beta\gamma}\mathring{\Gamma}^A_{\beta\gamma} = -\frac{2}{\rho}(\mathfrak{g}^{\tau A} + \mathfrak{g}^{xA}) + \mathfrak{g}^{BC}\gamma^A_{BC} \tag{4.1.47c}$$

$$= \frac{1}{\rho^2}\mathring{C}^A + \Omega^{\frac{n-5}{2}}\widehat{H}^{\mu\nu}\left(2\psi^{iA}_{\mu\nu}\frac{\omega_i}{\rho} + \psi^{BC}_{\mu\nu}\gamma^A_{BC}\right) ; \tag{4.1.47d}$$

where $\mathring{C}^A = \chi^{BC}\gamma^A_{BC}$ is minus the harmonicity function on the unit sphere. Finally, we obtain that the harmonicity functions of the metric $\mathfrak{g}$ in the (τ, x, v^A)-coordinates read:

$$^{(2)}\Upsilon^\tau = -C^0 \tag{4.1.48a}$$

$$^{(2)}\Upsilon^\tau + {}^{(2)}\Upsilon^x = \omega_i(v)C^i - \frac{n-1}{\rho} - \rho\Omega^{\frac{n-5}{2}}\widehat{H}^{\mu\nu}\psi^{AB}_{\mu\nu} \tag{4.1.48b}$$

$$^{(2)}\Upsilon^A = -\frac{\partial v^A}{\partial y^i}C^i - \frac{1}{\rho^2}\mathring{C}^A - \Omega^{\frac{n-5}{2}}\widehat{H}^{\mu\nu}\left(2\psi^{iA}_{\mu\nu}\frac{\omega_i}{\rho} + \psi^{BC}_{\mu\nu}\gamma^A_{BC}\right) . \tag{4.1.48c}$$

We revert now to the notation Υ for what was denoted by $^{(2)}\Upsilon$ above.

4.1.5 The source term $\mathcal{F}$

Recall that the source term in y^μ-coordinates reads:

$$\mathcal{F}\left(u, \frac{\partial u}{\partial y^\nu}\right) = \Omega^{-\frac{n+3}{2}}\widetilde{F}\left(\Omega^{\frac{n-1}{2}}u, \Omega^{\frac{n-1}{2}}\frac{\partial u}{\partial y^\nu}\right) + \left\{{}^{(1)}\Upsilon^\alpha - 2\Omega^{\frac{n-5}{2}}\widehat{H}^{\lambda\mu}\left\{2(n+1)y_\mu y_\lambda y^\alpha + (n+1)\Omega\delta^\alpha_\mu y_\lambda + \eta_{\lambda\mu}\Omega y^\alpha\right\}\right\}\frac{\partial u}{\partial y^\alpha} - 2(n-1)\Omega^{\frac{n-5}{2}}\widehat{H}^{\lambda\mu}\left\{(n+1)y_\mu y_\lambda + \Omega\eta_{\lambda\mu}\right\}u . \tag{4.1.49}$$

From (4.1.46) we have

$$
\begin{aligned}
&{}^{(1)}\Upsilon^\alpha - 2\Omega^{\frac{n-5}{2}}\widehat{H}^{\lambda\mu}\left\{2(n+1)y_\mu y_\lambda y^\alpha + (n+1)\Omega\delta^\alpha_\mu y_\lambda + \eta_{\lambda\mu}\Omega y^\alpha\right\}\\
= \;& \frac{1}{2}\Omega^{\frac{n-3}{2}}\left\{(n-1)y^\alpha tr_\eta \widehat{H} - \eta^{\alpha\nu}\Omega\partial_\nu tr_\eta\widehat{H}\right\}\\
&+\Omega^{\frac{n-5}{2}}\left\{\psi^{\alpha\nu}_{\mu\lambda}\partial_\nu\widehat{H}^{\lambda\mu} + (n-1)y_\lambda\widehat{H}^{\mu\lambda}\left\{\Omega\delta^\alpha_\mu + 2y_\mu y^\alpha\right\}\right\}\\
&+\Omega^2 Q^\alpha(\Omega^{\frac{n-5}{2}}\widehat{H},\Omega^{\frac{n-5}{2}}\widehat{H}) + \Omega^2 Q^{\alpha\beta}(\Omega^{\frac{n-5}{2}}\widehat{H},\Omega^{\frac{n-5}{2}}\partial_\beta\widehat{H})\ .
\end{aligned}
$$

This shows that the source term takes the following form:

$$
\begin{aligned}
\mathcal{F}\left(u,\frac{\partial u}{\partial y^\nu}\right) \;=\;& \Omega^{-\frac{n+3}{2}}\widetilde{F}\left(\Omega^{\frac{n-1}{2}}u,\Omega^{\frac{n-1}{2}}\frac{\partial u}{\partial y^\nu}\right)\\
&-2(n-1)\Omega^{\frac{n-5}{2}}\widehat{H}^{\lambda\mu}\left\{(n+1)y_\mu y_\lambda + \Omega\eta_{\lambda\mu}\right\}u\\
&+\frac{1}{2}\Omega^{\frac{n-3}{2}}\left\{(n-1)y^\alpha tr_\eta\widehat{H} - \eta^{\alpha\nu}\Omega\partial_\nu tr_\eta\widehat{H}\right\}\frac{\partial u}{\partial y^\alpha}\\
&+\Omega^{\frac{n-5}{2}}\left\{\psi^{\alpha\nu}_{\mu\lambda}\partial_\nu\widehat{H}^{\lambda\mu} + (n-1)y_\lambda\widehat{H}^{\mu\lambda}\left\{\Omega\delta^\alpha_\mu + 2y_\mu y^\alpha\right\}\right\}\frac{\partial u}{\partial y^\alpha}\\
&+\left\{\Omega^2 Q^\alpha(\Omega^{\frac{n-5}{2}}\widehat{H},\Omega^{\frac{n-5}{2}}\widehat{H}) + \Omega^2 Q^{\alpha\beta}(\Omega^{\frac{n-5}{2}}\widehat{H},\Omega^{\frac{n-5}{2}}\partial_\beta\widehat{H})\right\}\frac{\partial u}{\partial y^\alpha}\ .
\end{aligned}
\tag{4.1.50}
$$

4.2 The Einstein-Maxwell case

4.2.1 Existence of a solution

The Einstein-Maxwell equations, in harmonic and Lorenz gauge, take the form (4.1.13) (see [9, 37, 39]) with the following replacements there:

$$
f = (\underbrace{g_{\mu\nu} - \eta_{\mu\nu}}_{:=\, h_{\mu\nu}}, A_\mu) \quad \textit{and} \quad H^{\alpha\beta} = g^{\alpha\beta} - \eta^{\alpha\beta}\ . \tag{4.2.1}
$$

Recall that, if v is an arbitrary function, then

$$
\hat{v} = \Omega^{-\frac{n-1}{2}} v\ . \tag{4.2.2}
$$

Therefore, we have

$$
\hat{f} = (\hat{h}_{\mu\nu},\ \widehat{A}_\mu) := (\Omega^{-\frac{n-1}{2}}h_{\mu\nu},\ \Omega^{-\frac{n-1}{2}}A_\mu) \quad \textit{and} \quad \widehat{H}^{\alpha\beta} = \Omega^{-\frac{n-1}{2}}H^{\alpha\beta}\ .
$$

For consistency of notation with Section 3.2.3 we set

$$\hat{f} \equiv u \; .$$

In this notation

$$\|\hat{h}_{\mu\nu}\|_{\mathscr{H}_k^\theta} \le \|u\|_{\mathscr{H}_k^\theta} \; ,$$

and, since

$$\widehat{H}^{\alpha\beta} = -\eta^{\alpha\mu}\eta^{\beta\nu}\hat{h}_{\mu\nu} + \Omega^{-(n-1)/2}Q^{\alpha\beta}\left(\Omega^{(n-1)/2}\hat{h}_{\mu\nu}\right) \; ,$$

where $Q^{\alpha\beta}$ has a uniform zero of order two, from Proposition B.2.2 Appendix B.2 we obtain that

$$\begin{aligned}
\|\widehat{H}^{\alpha\beta}\|_{\mathscr{H}_k^\theta} &\le \|\eta^{\alpha\mu}\eta^{\beta\nu}\hat{h}_{\mu\nu}\|_{\mathscr{H}_k^\theta} + \|\Omega^{-\frac{n-1}{2}}Q^{\alpha\beta}\left(\Omega^{\frac{n-1}{2}}\hat{h}_{\mu\nu}\right)\|_{\mathscr{H}_k^\theta} \\
&\le C\left(\|\hat{h}_{\mu\nu}\|_{L^\infty}\right)\|u\|_{\mathscr{H}_k^{\theta-(n-1)/2}} \\
&\le C\left(\|\hat{h}_{\mu\nu}\|_{L^\infty}\right)\|u\|_{\mathscr{H}_k^\theta} \; . \qquad (4.2.3)
\end{aligned}$$

We define the energy $E^\alpha_{k,\lambda}[u(\tau)]$ as in Equation (3.2.28) of Section (3.2.3), the metric being defined by (4.1.23). Recall (see Equation (3.2.30) of Section (3.2.3)) that this quantity controls the $\mathscr{H}_k^\alpha$-norms of $\partial\hat{f}$. Now,

$$\|\partial\widehat{H}^{\alpha\beta}\|^2_{\mathscr{H}_k^\theta} \le \|\partial(\eta^{\alpha\mu}\eta^{\beta\nu}\hat{h}_{\mu\nu})\|^2_{\mathscr{H}_k^\theta} + \|\partial\left(\Omega^{-\frac{n-1}{2}}Q^{\alpha\beta}(\Omega^{(n-1)/2}\hat{h}_{\mu\nu})\right)\|^2_{\mathscr{H}_k^\theta} \; .$$

Since

$$\begin{aligned}
\partial\left(\Omega^{-\frac{n-1}{2}}Q^{\alpha\beta}(\Omega^{(n-1)/2}\hat{h}_{\mu\nu})\right) &= \Omega^{-\frac{n+1}{2}}Q^{\alpha\beta}(\Omega^{\frac{n-1}{2}}\hat{h}) + \Omega^{-\frac{n-1}{2}}Q^{\alpha\beta}(\Omega^{\frac{n-1}{2}}\hat{h},\Omega^{\frac{n-1}{2}}\partial\hat{h}) \\
&= \Omega^{-\frac{n+1}{2}}Q^{\alpha\beta}(\Omega^{\frac{n-1}{2}}\hat{h}) + \Omega^{-\frac{n-1}{2}+\alpha}Q^{\alpha\beta}(\Omega^{\frac{n-1}{2}}(\hat{h},x^{-\alpha}\partial\hat{h})),
\end{aligned}$$

we have the estimate:

$$\begin{aligned}
\|\partial\left(\Omega^{-\frac{n-1}{2}}Q^{\alpha\beta}(\Omega^{(n-1)/2}\hat{h}_{\mu\nu})\right)\|^2_{\mathscr{H}_k^\theta} &\le \|\Omega^{-\frac{n+1}{2}}Q^{\alpha\beta}(\Omega^{\frac{n-1}{2}}\hat{h})\|_{\mathscr{H}_k^\theta} \\
&\quad + \|\Omega^{-\frac{n-1}{2}+\alpha}Q^{\alpha\beta}(\Omega^{\frac{n-1}{2}}(\hat{h},x^{-\alpha}\partial\hat{h}))\|_{\mathscr{H}_k^\theta} \\
&\le C(\|\hat{h}\|_{L^\infty})\|\hat{h}\|_{\mathscr{H}_k^{\theta-(n-1)/2}} \\
&\quad + C(\|\hat{h},\ x^{-\alpha}\partial\hat{h})\|_{L^\infty}\|(\hat{h},\ x^{-\alpha}\partial\hat{h})\|_{\mathscr{H}_k^{\theta-\alpha-(n-1)/2}} \\
&\le C(\|\hat{h},\ x^{-\alpha}\partial\hat{h})\|_{L^\infty}\left(\|\hat{h}\|_{\mathscr{H}_k^\theta} + \|\hat{h}\|_{\mathscr{H}_k^{\theta-\alpha}} + \|\partial\hat{h}\|_{\mathscr{H}_k^\theta}\right) \\
&\le C(\|\hat{h},\ x^{-\alpha}\partial\hat{h})\|_{L^\infty}\left(\|\hat{h}\|_{\mathscr{H}_k^{\theta-\alpha}} + \|\partial\hat{h}\|_{\mathscr{H}_k^\theta}\right) \; .
\end{aligned}$$

Thus,

$$\|\partial \widehat{H}^{\alpha\beta}\|^2_{\mathscr{H}^\theta_k} \le C\left(\|\hat{h},\ x^{-\alpha}\partial\hat{h})\|_{L^\infty}\right)\left(\|u\|_{\mathscr{H}^{\theta-\alpha}_k} + \|\partial u\|_{\mathscr{H}^\theta_k}\right) \ .$$

To continue, we suppose that at $x = x_1 > 0$ the maximal globally hyperbolic development of the data exists for $\tau \in [\tau_0, \tau_1]$, with

$$M_1 := \|\hat{f}|_{\{x=x_1\}}\|_{L^\infty} < \infty \ .$$

We define (compare (3.2.32))

$$\begin{aligned}\hat{M}(\tau) \ &:= \ \|\mathcal{F}\|^2_{\mathscr{B}^\alpha_0(\mathbf{H}_\tau)} + \|(\mathfrak{g}, (\partial_\tau - \partial_x)\mathfrak{g}^\sharp)\|^2_{L^\infty(\mathbf{H}_\tau)} + \|(\mathfrak{g}^\sharp, \mathfrak{h}^\sharp, \Upsilon)\|^2_{\mathscr{C}^0_{\{x=0\},1}(\mathbf{H}_\tau)} \\ &\quad + \|\big((\partial_\tau - \partial_x)\hat{f}, \partial_x\hat{f}, \partial_A\hat{f}\big)\|^2_{\mathscr{B}^\alpha_1(\mathbf{H}_\tau)} + \|\hat{f}(\tau)|_{\{x=x_1\}}\|_{L^\infty} \ , \qquad (4.2.4)\end{aligned}$$

with the functions $\mathfrak{g}^\sharp$, $\mathfrak{h}^\sharp$, $\Upsilon^\mu \equiv {}^{(2)}\Upsilon$ and $\mathcal{F}$ defined by equations (4.1.23), (4.1.35)-(4.1.38), (4.1.48) and (4.1.50).

For any positive function $N(\tau)$ we set

$$\underline{N(\tau)} := \sup_{s\in[\tau_0,\tau]} N(s) \ . \qquad (4.2.5)$$

We then have the following:

Proposition 4.2.1 *Let $k \in \mathbb{N}$, $\alpha \in (-1, -1/2]$. Consider the Einstein-Maxwell equations (4.1.13) in space-time dimension $1 + n \ge 7$ if $\alpha = -\frac{1}{2}$, and $1 + n \ge 8$ otherwise. Let f be defined in (4.2.1), suppose that $t_0 > 0$ and assume that the initial data, given on the hyperboloid*

$$\mathscr{S}_0 = \left\{(x^\mu) : x^0 - t_0 = \sqrt{t_0^2 + |\vec{x}|^2}\ \right\} \qquad (4.2.6)$$

in Minkowski space-time, are such that:

$$\hat{f}\,|_{\phi(\mathscr{S}_0)} \in \left(\mathscr{H}^\alpha_{k+1} \cap L^\infty\right)(\phi(\mathscr{S}_0)) \ , \quad \text{and} \quad \left((\partial_\tau - \partial_x)\hat{f}, \partial_x\hat{f}, \partial_A\hat{f}\right)|_{\phi(\mathscr{S}_0)} \in \mathscr{H}^\alpha_k(\phi(\mathscr{S}_0)) \ . \qquad (4.2.7)$$

There exists functions $\hat{C}_3(n, k, \epsilon_0, C_0, \alpha, \hat{M})$ and $\hat{C}_4(n, k, \epsilon_0, C_0, \alpha, \hat{M})$, monotonously increasing in $\hat{M}$, which we write as $C_3(\hat{M})$ and $C_4(\hat{M})$, such that the energy of the system as defined in (3.2.28), Section 3.2.3 satisfies the inequality

$$\begin{aligned}\|\hat{f}(\tau)\|^2_{L^\infty} + E^\alpha_k[\hat{f}(\tau)] \ &\le \ 2\Big\{M_1^2 + E^\alpha_k[\hat{f}(\tau_0)] \\ &\quad + \int_{\tau_0}^\tau \hat{C}_3(\hat{M}(s))E^\alpha_k[\hat{f}(s)]ds\Big\} \ , \qquad (4.2.8)\end{aligned}$$

where $\tau_0 = -\frac{1}{2t_0}$. *Furthermore, for* $n+1 \geq 7$ *and* $\alpha = -1/2$ *one has*

$$\begin{aligned}\underline{\|\hat f(\tau)\|^2_{L^\infty}} + \underline{E^\alpha_k[\hat f(\tau)]} \ &\leq\ 2\Big\{M_1^2 + E^\alpha_k[\hat f(\tau_0)] + \|x^{(n-7)/2}\widehat{H}^{\mu\nu}y_\mu y_\nu(\tau_0)\|^2_{\mathscr{G}^0_k} \\ &\quad + \int_{\tau_0}^{\tau} \hat C_4(\underline{\hat M(s)})\underline{E^\alpha_k[\hat f(s)]}ds\Big\} . \qquad (4.2.9)\end{aligned}$$

Remark 4.2.2 For $n \geq 7$, a prefactor $\Omega^{\frac{n-7}{2}}$ in the fourth line (the fall-off of the component of this term with the lowest power of Ω can be improved using the gauge condition) of the nonlinear term in (4.1.50) still leads to the estimates here. This remark is important for the estimation of the time derivatives in Section 4.2.2 below.

Proof: *For all* $0 < x < x_1$ *the trivial identity*

$$\hat f(\tau, x) = \hat f(\tau, x_1) - \int_x^{x_1} \partial_x \hat f(\tau, s)ds$$

leads to the estimate (recall that $\alpha > -1$*)*

$$\begin{aligned}\|\hat f(\tau)\|_{L^\infty} \ &\leq\ M_1 + \int_x^{x_1} \|\partial_x \hat f(\tau)\|_{\mathscr{C}^\alpha_{\{x=0\},k}} s^\alpha ds \\ &\leq\ M_1 + \|\partial_x \hat f(\tau)\|_{\mathscr{G}^\alpha_k} .\end{aligned}$$

From this one easily concludes

$$\|\hat f(\tau)\|_{\mathscr{G}^0_k} \leq C\big(M_1 + \|(\partial_x \hat f, \partial_A \hat f)(\tau)\|_{\mathscr{G}^\alpha_{k-1}}\big) . \qquad (4.2.10)$$

Now we apply Proposition 3.2.8 of Section 3.2.3. To obtain (4.2.8) we will show first that, in the Einstein-Maxwell case, the $\mathscr{H}^\alpha_k$*-norm of the source term, the* $\mathscr{G}^0_k$*-norms of* $\mathfrak{g}^\sharp$*,* $\mathfrak{h}^\sharp$ *and* Υ^μ *are controlled by the energy. Let us start with the* $\mathscr{G}^0_k$*-norm of* $\mathfrak{g}^\sharp$*. From the expression of* $\mathfrak{g}$ *given by (4.1.23) and the estimate (4.2.10), if* $n \geq 5$ *then*

$$\begin{aligned}\|\mathfrak{g}^\sharp(\tau)\|^2_{\mathscr{G}^0_k} \ &\leq\ C\left(M_1 + \|\partial \widehat{H}\|^2_{\mathscr{G}^\alpha_k}\right) \\ &\leq\ C\left(M_1 + E^\alpha_{k,\lambda}[u(\tau)]\right) . \qquad (4.2.11)\end{aligned}$$

The same holds for $\mathfrak{h}^\sharp$ *but with the constraint that the space dimension* n *is larger than or equal to 7. We will return later to the question how to improve on the dimension on this term when* $\alpha = -1/2$.

To estimate the harmonicity functions ${}^{(2)}\Upsilon$ *given by (4.1.48), we start by estimating the functions* C^μ*. We decompose* $C^\mu = C^\mu_1 + C^\mu_2 + C^\mu_3$*, each* *corresponding to a line in (4.1.46). The first and second terms are estimated as we did for* $\mathfrak{g}^\sharp$ *and* $\mathfrak{h}^\sharp$*:*

$$\begin{aligned}\|C_1^\mu\|^2_{\mathscr{G}_k^0} &\le C(\|x^{\frac{n-3}{2}}\widehat{H}\|^2_{\mathscr{G}_k^0} + \|x^{\frac{n-1}{2}}\partial\widehat{H}\|^2_{\mathscr{G}^0}) \\ &\le C\big(M_1 + E^\alpha_{k,\lambda}[u(\tau)]\big) \quad for \quad n \ge 3\,, \end{aligned} \tag{4.2.12}$$

and

$$\begin{aligned}\|C_2^\mu\|^2_{\mathscr{G}_k^0} &\le C\big(\|x^{\frac{n-5}{2}}\widehat{H}\|^2_{\mathscr{G}_k^0} + \|x^{\frac{n-5}{2}}\partial\widehat{H}\|^2_{\mathscr{G}_k^0}\big) \\ &\le C\big(M_1 + E^\alpha_{k,\lambda}[u(\tau)]\big) \quad for \quad n \ge 5-2\alpha\,. \end{aligned} \tag{4.2.13}$$

To estimate C_3^μ *we recall that its components have a uniform zero of order two in* $\widehat{H}$ *and* $(\widehat{H}, x^{-\alpha}\partial\widehat{H})$ *respectively, with the second term linear in* $\partial\widehat{H}$*, thus we can apply Inequality B.2.8 of Appendix B.2 on the* $\mathscr{G}$*-norm with* $\ell = 2,\ \beta = \frac{n-5}{2}$*. We obtain:*

$$\begin{aligned}\|C_3^\mu\|^2_{\mathscr{G}_k^0} &\le \|Q^\mu(\Omega^{\frac{n-5}{2}}\widehat{H})\|^2_{\mathscr{G}_k^{-2}} + \|Q^{\mu\nu}(\Omega^{\frac{n-5}{2}}(\widehat{H}, x^{-\alpha}\partial_\nu\widehat{H}))\|^2_{\mathscr{G}_k^{-2-\alpha}} \\ &\le C(\|(\widehat{H}\|_{L^\infty})\|\widehat{H}\|^2_{\mathscr{G}_k^{3-n}} + C(\|(\widehat{H}, x^{-\alpha}\partial\widehat{H})\|_{L^\infty})\|(\widehat{H}, x^{-\alpha}\partial\widehat{H})\|^2_{\mathscr{G}_k^{3-\alpha-n}} \\ &\le C\left(\|\widehat{H}, x^{-\alpha}\partial\widehat{H})\|_{L^\infty}\right)\left(\|\widehat{H}\|^2_{\mathscr{G}_k^0} + \|\partial\widehat{H}\|^2_{\mathscr{G}_k^\alpha}\right) \quad for \quad n \ge 3-\alpha\,. \end{aligned}$$

Thus we have:

$$\|C_3^\mu\|^2_{\mathscr{G}_k^0} \le C\left(\|\widehat{H}, x^{-\alpha}\partial\widehat{H})\|_{L^\infty}\right)\left(\|u\|^2_{L^\infty} + E^\alpha_{k,\lambda}[u(\tau)]\right) \quad for \quad n \ge 4\,. \tag{4.2.14}$$

Note that the function $C\left(\|\widehat{H}, x^{-\alpha}\partial\widehat{H})\|_{L^\infty}\right)$ *will give a contribution to the function* $C_2(M(s))$ *of (4.2.8). The remaining terms of* ${}^{(2)}\Upsilon$ *as given by (4.1.47) are estimated in a similar way. They are controlled by*

$$C\left(1 + E^\alpha_{k,\lambda}[u(\tau)]\right) \qquad for \quad n \ge 5\,. \tag{4.2.15}$$

We continue by writing the source term $\mathcal{F}$ *(see (4.1.50)) as a sum of terms, each of the following form*

$$x^{p_i}\mathcal{F}_i\left(\ .\ , x^{q_i}(\hat{f},\ x^{-\alpha}\partial\hat{f})\right)\,. \tag{4.2.16}$$

Note that all terms are polynomial in ∂f*, at most quadratic in* ∂f*. For instance, the first term* $\tilde{F}$ *arises from products of the Christoffels in the Ricci tensor, and from the products of the derivatives* ∂A *of the vector potential*

$\mathcal{F}_i$	p_i	q_i	ℓ_i	constraint	$n \geq$
$\mathcal{F}_1$	$-\frac{n+3}{2}$	$\frac{n-1}{2} + \alpha$	2	$n > 5 - 2\alpha$	7[6]
$\mathcal{F}_2$	$-\frac{n-5}{2}$	$\frac{n-5}{2}$	2	$n > 5$	6
$\mathcal{F}_3$	$\frac{n-3}{2} + \alpha$	0	2	$n > 3$	4
$\mathcal{F}_4$	$\frac{n-5}{2} + 2\alpha$	0	2	$n > 5 - 2\alpha$	7[6]
$\mathcal{F}_5$	$2 - \frac{n-5}{2} + \alpha$	$\frac{n-5}{2}$	3	$n > 3$	4
	$2 - \frac{n-5}{2} + 2\alpha$	$\frac{n-5}{2}$	3	$n > 3 - \alpha$	4

Table 4.1: Restrictions on the dimension from the source terms.

A in the energy-momentum tensor. We then write, for example, in the x^μ coordinates,

$$\Gamma^2 \sim (g^\sharp \partial g)^2 \sim F(g^\sharp)\partial g \partial g = x^{2\alpha} F(g^\sharp)(x^{-\alpha}\partial g)(x^{-\alpha}\partial g) \ ;$$

we then express this in term of $h_{\mu\nu}$, transform the whole expression to the y^μ–coordinates, and finally reexpress $h_{\mu\nu}$ in term of $\hat{H}_{\mu\nu}$. This formula shows that the Γ^2 in the Einstein equations have a uniform zero of order two in $(\hat{f}, x^{-\alpha}\hat{f})$. A similar analysis applies to the contribution of the Maxwell fields to the Einstein-Maxwell equations.

We use the following estimate to show that the $\mathscr{H}^\alpha_k$-norm of $\mathcal{F}$ is controlled by the energy of the system: Suppose that $\mathcal{F}_i$ has a uniform zero of order ℓ_i in $(u,\ x^{-\alpha}\partial u)$, then applying to this function the second part of Lemma B.2.2 Appendix B.2, for

$$p_i + \ell_i q_i > \alpha \ . \tag{4.2.17}$$

We choose $\epsilon > 0$ so that $p_i + \ell_i q_i > \alpha + \epsilon$, and write

$$\begin{aligned}
&\|x^{p_i}\mathcal{F}_i\big(\ .\ , x^{q_i}(u,\ x^{-\alpha}\partial u)\big)\,\|^2_{\mathscr{H}^\alpha_k} \\
&= \ \|\mathcal{F}_i\big(\ .\ , x^{q_i}(u, x^{-\alpha}\partial u)\big)\,\|^2_{\mathscr{H}^{\alpha-p_i}_k} \\
&\leq \ C\big(\|(u,\ x^{-\alpha}\partial u)\|_{L^\infty}\big)\|(u,\ x^{-\alpha}\partial u)\|^2_{\mathscr{H}^{\alpha-p_i-\ell_i q_i}_k} \\
&\leq \ C\big(\|(u,\ x^{-\alpha}\partial u)\|_{L^\infty}\big)\left(\|u\|^2_{\mathscr{H}^{-\epsilon}_k} + \|x^{-\alpha}\partial u\|^2_{\mathscr{H}^{-\epsilon}_k}\right) \\
&\leq \ C\big(\|(u,\ x^{-\alpha}\partial u)\|_{L^\infty}\big)\left(\|u\|^2_{L^\infty} + E^\alpha_{k,\lambda}[u(\tau)]\right) \ .
\end{aligned}$$

The analysis of the nonlinear terms (4.1.50) along those lines gives the following table: Here the $\mathcal{F}_i$'s, $i = 1, \ldots, 4$, correspond to the i-th line of (4.1.50), while the two rows for $\mathcal{F}_5$ correspond to the two respective terms *in the last line of (4.1.50). In the last column the number in square bracket is obtained by estimating below the non-linearity in a more efficient way.*

It turns out that the threshold on the space dimension n can be lowered to $n = 6$ for the components $\mathcal{F}_1$ and $\mathcal{F}_4$ of the source term $\mathcal{F}$. The quadratic terms in those expressions with the lowest powers of Ω are of the form $\Omega^{\frac{n-5}{2}} G(\Omega^{\frac{n-1}{2}} \hat{f}) \partial \hat{f} \partial \hat{f}$ for $\mathcal{F}_1$ and $\Omega^{\frac{n-5}{2}} \widehat{H} \partial u$ and $\Omega^{\frac{n-5}{2}} \partial \widehat{H} \partial u$ for $\mathcal{F}_4$. One can estimate the $\mathscr{H}_k^\alpha$-norm of $\Omega^{\frac{n-5}{2}} \widehat{H} \partial u$ using instead (B.2.9):

$$\begin{aligned}
\|\Omega^{\frac{n-5}{2}} \widehat{H} \partial u\|^2_{\mathscr{H}_k^\alpha} &\leq \|\widehat{H} \partial u\|^2_{\mathscr{H}_k^{\alpha - \frac{n-5}{2}}} \\
&\leq C \left(\|\widehat{H}\|^2_{\mathscr{C}_0^0} \|\partial u\|^2_{\mathscr{H}_k^{\alpha-\frac{n-5}{2}}} + \|x^{\frac{n-5}{2}} \widehat{H}\|^2_{\mathscr{G}_k^0} \|\partial u\|^2_{\mathscr{B}_0^\alpha} \right) \\
&\leq C(\|u\|^2_{L^\infty} + \|\partial u\|^2_{\mathscr{B}_0^\alpha}) \left(\|u\|^2_{\mathscr{G}_k^0} + \|\partial u\|^2_{\mathscr{H}_k^\alpha} \right) \quad \text{if} \quad n \geq 5 \\
&\underbrace{\leq}_{\text{see (4.2.10)}} C(\|u\|^2_{L^\infty} + \|\partial u\|^2_{\mathscr{B}_0^\alpha}) \left(1 + \|\partial u\|^2_{\mathscr{H}_k^\alpha} \right) \\
&\leq C(\|u\|^2_{L^\infty} + \|\partial u\|^2_{\mathscr{H}_k^\alpha}) \left(1 + \|\partial u\|^2_{\mathscr{H}_k^\alpha} \right) , \qquad (4.2.18)
\end{aligned}$$

for $k > n/2$. Next,

$$\begin{aligned}
\|\Omega^{\frac{n-5}{2}} \partial \widehat{H} \partial u\|^2_{\mathscr{H}_k^\alpha} &\leq \|\partial \widehat{H} \partial u\|^2_{\mathscr{H}_k^{\alpha-\frac{n-5}{2}}} \\
&\leq C \left(\|\partial \widehat{H}\|^2_{\mathscr{C}_0^0} \|\partial u\|^2_{\mathscr{H}_k^{-\frac{n-5}{2}}} + \|\partial \widehat{H}\|^2_{\mathscr{H}_k^\alpha} \|\partial u\|^2_{\mathscr{C}_0^{-\frac{n-5}{2}}} \right) \\
&\leq C \|\partial u\|^2_{\mathscr{C}_0^\alpha} \|\partial u\|^2_{\mathscr{H}_k^\alpha} \quad \text{if} \quad -\frac{n-5}{2} - \alpha \leq 0 \quad \text{i.e.} \quad n \geq 5 - 2\alpha \\
&\leq C \|\partial \widehat{H}\|^2_{\mathscr{C}_0^\alpha} E^\alpha_{\lambda,k}[u(\tau)] ,
\end{aligned}$$

and so the last inequality will be true provided that

$$\begin{cases} n \geq 6 & \text{if} \quad \alpha = -\frac{1}{2} \\ n \geq 7 & \text{if} \ -1 < \alpha < -\frac{1}{2} \end{cases} .$$

A similar calculation applies to $\mathcal{F}_1$.

These estimates and the table show that

$$\|\mathcal{F}(u, \partial u)\|^2_{\mathscr{H}_k^\alpha} \leq C(\|u\|_{L^\infty}, \|\partial u\|_{\mathscr{C}_0^\alpha}) \left(1 + E^\alpha_{k,\lambda}[u(\tau)]\right) \qquad (4.2.19)$$

for

$$\begin{cases} n \geq 6 & \text{if} \quad \alpha = -\frac{1}{2} \\ n \geq 7 & \text{if} \ -1 < \alpha < -\frac{1}{2} \end{cases} .$$

Inserting inequalities (4.2.11)-(4.2.13) and (4.2.14)-(4.2.19) in (3.2.33) of Section 3.2.3 gives (4.2.8).

Now, at several places of the calculations above the term

$$\psi := y_\alpha y_\beta \hat{H}^{\alpha\beta}$$

is the one that occurs with the lowest power of Ω. *It follows from the wave-coordinates conditions that this term solves equation (4.1.12), which can be written in the form*

$$-y^\alpha \partial_\alpha \psi + \frac{n-5}{2}\psi = \zeta \; , \tag{4.2.20}$$

where

$$\begin{aligned}\zeta \; &:= \; \Omega\Big(\frac{n-1}{2} tr_\eta(\widehat{H}) + y_\nu \frac{\partial}{\partial y^\mu}\big(\widehat{H}^{\mu\nu} + \frac{1}{2}\eta^{\mu\nu} tr_\eta(\widehat{H})\big)\Big) \\ &\quad + \Omega^{-\frac{n-1}{2}} Q(\Omega^{\frac{n-1}{2}}\widehat{H}, \Omega^{\frac{n-1}{2}}\widehat{H}) \\ &\quad + \Omega^{-\frac{n-1}{2}} Q(\Omega^{\frac{n-1}{2}}\widehat{H}, \Omega^{\frac{n+1}{2}}\partial\widehat{H}) \\ &\quad + \Omega^{-\frac{n-1}{2}} Q(\Omega^{\frac{n-1}{2}}\widehat{H}, \Omega^{\frac{n-1}{2}} y^\alpha \frac{\partial}{\partial y^\alpha}\widehat{H}) \\ &=: \; \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 \; ,\end{aligned} \tag{4.2.21}$$

where ζ_i *corresponds to the i-the line. The point is that all terms in* ζ *contain effectively multiplicative powers of* Ω.

Solutions of (4.2.20) take the form, for $\tau_0 \le \tau \le \tau_1 < 0$,

$$\psi(\tau, x) = (-\tau)^{-(n-5)/2}\left(\int_{\tau_0}^{\tau} (-s)^{(n-7)/2}\zeta\left(s, \frac{sx}{\tau}\right) ds + (-\tau_0)^{(n-5)/2}\psi\left(\tau_0, \frac{x\tau_0}{\tau}\right)\right) . \tag{4.2.22}$$

This gives immediately, for any γ,

$$\|\psi(\tau)\|_{\mathscr{G}_k^\gamma} \le \|\psi(\tau_0)\|_{\mathscr{G}_k^\gamma} + C(\tau_0, \tau_1)\int_{\tau_0}^{\tau} \|\zeta(s)\|_{\mathscr{G}_k^\gamma} ds \; , \tag{4.2.23}$$

similarly for $\mathscr{H}^\gamma$*- or* $\mathscr{C}^\gamma$*-norms. In the notation of (4.2.5) one thus finds*

$$\begin{aligned}\underline{\|\psi(\tau)\|_{\mathscr{G}_k^\gamma}} \;&\le\; \|\psi(\tau_0)\|_{\mathscr{G}_k^\gamma} + C(\tau_0,\tau_1)\int_{\tau_0}^{\tau} \underline{\|\zeta(s)\|_{\mathscr{G}_k^\gamma}} ds \\ &\le\; \|\psi(\tau_0)\|_{\mathscr{G}_k^\gamma} + C(\tau_0,\tau_1)(\tau_1 - \tau_0)\underline{\|\zeta(\tau)\|_{\mathscr{G}_k^\gamma}} \; .\end{aligned}$$

Using this to estimate $\mathfrak{h}^0$ *we obtain*

$$\begin{aligned}\underline{\|\mathfrak{h}^0(\tau)\|_{\mathscr{G}_k^0}} \;&\le\; C\big(\underline{\|x^{(n-7)/2}\psi(\tau)\|_{\mathscr{G}_k^0}} + \underline{\|x^{(n-5)/2}\widehat{H}(\tau)\|_{\mathscr{G}_k^0}} \\ &\le\; C\big(\|x^{(n-7)/2}\psi(\tau_0)\|_{\mathscr{G}_k^0} + \underline{\|x^{(n-7)/2}\zeta(\tau)\|_{\mathscr{G}_k^0}} + \underline{\|x^{(n-5)/2}\widehat{H}(\tau)\|_{\mathscr{G}_k^0}}\big) \; .\end{aligned}$$

We have, for example,

$$\underline{\|x^{(n-7)/2}\zeta_1(\tau)\|_{\mathscr{G}^0_k}} \quad \le \quad C\left(\underline{\|x^{(n-5)/2}\widehat{H}(\tau)\|_{\mathscr{G}^0_k}} + \underline{\|x^{(n-5)/2}\partial\widehat{H}(\tau)\|_{\mathscr{G}^0_k}}\right) ,$$

which, for $n-5 \ge -2\alpha$*, can be controlled by* $\underline{\|\widehat{H}(\tau)\|_{L^\infty}}$ *and* $\underline{E^\alpha_k[u(\tau)]}$ *in view of (4.2.10). This requires* $n \ge 6$ *if* $\alpha = -1/2$*, or* $n \ge 7$ *if* $\alpha \in (-1, -1/2)$*. An estimation of the remaining* ζ_i*'s along the lines of those already done above presents no difficulties.*

The functions $\mathfrak{h}^1$ *and* $\mathfrak{h}$ *have the same structure and so the same estimate applies; the function* $\mathfrak{h}^A$ *has a higher multiplicative power of* Ω *so that the original straightforward estimate applies.*

The final inequality (4.2.9) follows immediately from this and from an obvious version of the estimate (4.2.8) for the remaining terms in the equation.

We finish this proof by noting that the above treatment of $y_\alpha y_\beta \hat{H}^{\alpha\beta}$ *can be used to improve the threshold on dimension for some of the entries of Table 4.1; this will, however, not improve the threshold on* n *of the theorem.* □

We are now ready to prove existence of solutions in weighted Sobolev spaces. For $s > 0$ *consider the family of hyperboloids:*

$$\mathscr{S}_s = \left\{(x^\mu) : x^0 - s = \sqrt{s^2 + |\vec{x}|^2}\right\} . \qquad (4.2.24)$$

Let ϕ *be defined in* (1.2.2)*. We have the following*

Theorem 4.2.3 (Propagation of weighted Sobolev regularity) *Suppose that* $k > \left[\frac{n}{2}\right] + 1$*, with* $n = 6$ *and* $\alpha = -1/2$*, or* $n \ge 7$ *with* $\alpha \in (-1, -1/2]$*, and let* $t_0 > 0$*. Suppose that*

$$\hat{f}\,|_{\phi(\mathscr{S}_0)} \in \left(\,\mathscr{H}^\alpha_{k+1} \cap L^\infty\right)(\phi(\mathscr{S}_0)) , \quad \left(\partial_\tau \hat{f}, \partial_x \hat{f}, \partial_A \hat{f}\right)|_{\phi(\mathscr{S}_0)} \in \mathscr{H}^\alpha_k(\phi(\mathscr{S}_0)) , \qquad (4.2.25)$$

where f *and* $\hat{f}$ *are defined by (4.2.1)-(4.2.2). In the case* $\alpha = -1/2$ *and* $n = 6$ *assume moreover that*

$$x^{-1/2} y_\alpha y_\beta \widehat{H}^{\alpha\beta}\big|_{\phi(\mathscr{S}_0)} \in \mathscr{G}^0_k . \qquad (4.2.26)$$

Then there exists $t_* > t_0$ *and a solution of (4.1.13) defined on* $\underset{s\in[t_0,\, t_*]}{\cup} \mathscr{S}_s$ *such that,* $\forall \tau \in [-\frac{1}{2t_0}, -\frac{1}{2t_*}] =: [\tau_0, \tau_*]$ *we have:*

$$\hat{f} \in L^\infty\Big([\tau_0, \tau_*],\ \mathscr{H}^\alpha_k(\mathbf{H}_\tau) \cap L^\infty(\mathbf{H}_\tau)\Big) , \qquad (4.2.27)$$

$$\Big(\partial_\tau \hat f,\ \partial_x \hat f,\ \partial_A \hat f\Big) \in L^\infty\Big([\tau_0,\tau_*],\ \mathscr{H}^\alpha_k(\mathbf{H}_\tau)\Big)\ . \tag{4.2.28}$$

Moreover, any solution for which $\hat M(\tau)$, as defined in (4.2.4), is bounded on $[\tau_0,\tau_1]$ satisfies (4.2.27)-(4.2.28) with $\tau_* = \tau_1$.

Remark 4.2.4 Using the weighted Sobolev embedding theorem we conclude

$$\hat f(\tau) \in \Big(\mathscr{C}^\alpha_{k-[\frac{n}{2}]-1} \cap L^\infty\Big)(\mathbf{H}_\tau)\ , \tag{4.2.29}$$

$$\Big(\partial_\tau \hat f(\tau), \partial_x \hat f(\tau), \partial_A \hat f(\tau)\Big) \in \mathscr{C}^\alpha_{k-[\frac{n}{2}]-1}(\mathbf{H}_\tau)\ . \tag{4.2.30}$$

when the prescribed data are as in Theorem 4.2.3.

Proof: *In order to apply the Gronwall-type Lemma 5.2 of [20], we need to prove that all the norms in $\hat M$ (see (4.2.4) and (4.2.8)) are controlled by the energy or the L^∞-norm of u. Since $k > \left[\frac{n}{2}\right]+1$, from the weighted Sobolev's inequality, we have:*

$$\|(\partial_\tau-\partial_x,\partial_x,\partial_A)\hat f\|^2_{\mathscr{B}^\alpha_1} \le \|(\partial_\tau-\partial_x,\partial_x,\partial_A)\hat f\|^2_{\mathscr{H}^\alpha_k} \le E^\alpha_{k,\lambda}[u(\tau)]\ . \tag{4.2.31}$$

Let us look at the L^∞-norm of $(\partial_\tau-\partial_x)\mathfrak{g}^\sharp$. Recall that the expression of $\mathfrak{g}^\sharp$ is given by (4.1.23). We estimate here only its worse term which is of the form $\Omega^{\frac{n-5}{2}}\widehat H$. We have:

$$\begin{aligned}\|(\partial_\tau-\partial_x)(\Omega^{\frac{n-5}{2}}\widehat H)\|^2_{L^\infty} &\le C\Big(\|\Omega^{\frac{n-5}{2}}(\partial_\tau-\partial_x)\widehat H\|^2_{L^\infty} + \|\Omega^{\frac{n-7}{2}}\widehat H\|^2_{L^\infty}\Big)\\ &\le C\left(\|u\|^2_{L^\infty} + E^\alpha_{k,\lambda}[u(\tau)]\right) \qquad \textit{for} \quad n\ge 7\ .\end{aligned}$$

Thus,

$$\|(\partial_\tau-\partial_x)\mathfrak{g}^\sharp\| \le C\left(\|u\|^2_{L^\infty} + E^\alpha_{k,\lambda}[u(\tau)]\right) \qquad \textit{for} \quad n\ge 7\ . \tag{4.2.32}$$

$$\begin{aligned}\|\mathfrak{g}^\sharp\|^2_{\mathscr{C}^0_1} &\le \|\mathfrak{g}^\sharp\|^2_{\mathscr{G}^0_k}\\ &\le C\left(M_1 + E^\alpha_{k,\lambda}[u(\tau)]\right)\ , \quad \textit{for} \quad n\ge 5\end{aligned} \tag{4.2.33}$$

Similarly,

$$\|\mathfrak{h}^\sharp\|^2_{\mathscr{C}^0_1} \le \|\mathfrak{h}^\sharp\|^2_{\mathscr{G}^0_k} \le C\left(M_1 + E^\alpha_{k,\lambda}[u(\tau)]\right) \quad \textit{for} \quad n\ge 7\ . \tag{4.2.34}$$

If $\alpha=-1/2$ the threshold $n=7$ in (4.2.32) *and* (4.2.34) *can be lowered to $n=6$ by using the estimate* (4.2.23) *on the slowest decaying term ψ.*

To estimate the $\mathscr{C}^0_1$-norms of the harmonicity functions, we use again as in the previous estimate the Sobolev inequality and obtain a control of these norms by the energy with the same constrains as in (4.2.12)-(4.2.15). Let us estimate now the L^∞-norm of u. Integrating backward along the integral curve of the vector field $Y^\nu\partial_\nu = \partial_\tau - \partial_x$ we can write the identity (here we omit the variable v^A)

$$u(\tau,x) - u(\tau_0, \tau-\tau_0+x) = \int_{\tau_0}^{\tau} (\partial_\tau - \partial_x)\, u(s, \tau - s + x) ds \; . \qquad (4.2.35)$$

Thus we have

$$\begin{aligned}
|u(\tau,x)| &\leq |u(\tau_0,\tau-\tau_0+x)| + \int_{\tau_0}^{\tau} |\,(\tau-s+x)^{-\alpha}\,(\partial_\tau-\partial_x)\,u(s,\tau-s+x)|\,(\tau-s+x)^{\alpha}\,ds \\
&\leq |u(\tau_0,\tau-\tau_0+x)| + \int_{\tau_0}^{\tau} \|\,(\partial_\tau-\partial_x)\,u(s)\|_{\mathscr{C}^\alpha_0}\,(\tau-s+x)^{\alpha}\,ds \\
&\leq \|u(\tau_0)\|_{L^\infty} + \int_{\tau_0}^{\tau} \|\,(\partial_\tau-\partial_x)\,u(s)\|_{\mathscr{C}^\alpha_0}\,(\tau-s)^{\alpha}\,ds \; .
\end{aligned}$$

Since $k > \frac{n}{2}$ we can now write ($-1 < \alpha \leq -1/2$):

$$\begin{aligned}
\|u(\tau)\|_{L^\infty} &\leq \|u(\tau_0)\|_{L^\infty} + \int_{\tau_0}^{\tau} \|\,(\partial_\tau-\partial_x)\,u(s)\|_{\mathscr{H}^\alpha_k}\,(\tau-s)^{\alpha}\,ds \\
&\leq \|u(\tau_0)\|_{L^\infty} + \int_{\tau_0}^{\tau} \sqrt{E^\alpha_{k,\lambda}[u(s)]}\,(\tau-s)^{\alpha}\,ds \; . \qquad (4.2.36)
\end{aligned}$$

Inequalities (4.2.31)-(4.2.36) show that from (4.2.8) we have the following:

$$\begin{aligned}
\|u(\tau)\|^2_{L^\infty} + E^\alpha_{k,\lambda}[u(\tau)] &\leq C\left(\|u(\tau_0)\|_{L^\infty} + E^\alpha_{k,\lambda}[u(\tau_0)]\right) \\
&\quad + \int_{\tau_0}^{\tau} \Phi\left(E^\alpha_{k,\lambda}[u(s)],\ \|u(s)\|_{L^\infty}\right)(1+(\tau-s)^{\alpha})\,ds; \qquad (4.2.37)
\end{aligned}$$

where Φ is bounded on bounded sets. Setting

$$\chi(s) \equiv E^\alpha_{k,\lambda}[u(s)] + \|u(s)\|_{L^\infty} \; ,$$

(4.2.37) reads

$$\chi(\tau) \leq C\left(\chi(\tau_0)\right) + \int_{\tau_0}^{\tau} \Phi\left(\chi(s)\right)(1+(\tau-s)^{\alpha})\,ds. \qquad (4.2.38)$$

We have the following:

Lemma 4.2.5 *There exists a time $\tau_0 < \tau_* < 0$, depending only upon C, $F(\tau_0)$, and the function Φ, such that any positive continuous function $F : [\tau_0, \tau_*) \to \mathbb{R}$ satisfying the inequality (4.2.38) with $\alpha > -1$ is bounded from above by $CF(T_0) + 1$ on $[\tau_0, \tau_*)$.*

Proof: *Let*

$$M = \sup_{0 \le \xi \le C\chi(\tau_0)+1} |\Phi(\xi)| \ ;$$

if $M = 0$ the result is obviously true, so assume that $M \neq 0$. From Equation (4.2.38) we obtain that on any interval $[\tau_0, \tau)$ on which $\chi \le C\chi(\tau_0) + 1$ we have

$$\chi(\tau) \le C\chi(\tau_0) + \int_0^\tau M\left(1 + (\tau - \sigma)^\alpha\right) \, d\sigma = C\chi(\tau_0) + M\left(\tau + \frac{\tau^{\alpha+1}}{\alpha+1}\right) .$$

(Equation (4.2.38) with $\tau = \tau_0$ shows that $C\chi(\tau_0) \ge \chi(\tau_0)$, and continuity of χ implies that the set of such intervals is non-empty.) The result is established by choosing

$$\tau_* = \min\left(\frac{1}{2M}, \left[\frac{\alpha+1}{2M}\right]^{1/(\alpha+1)}\right) .$$

□

By this Lemma, there exists a time $\tau_0 < \tau_* < 0$ depending on $\|u(\tau_0)\|_{L^\infty} + E^\alpha_{k,\lambda}[u(\tau_0)]$ and on the function Φ such that $\forall \tau \in [\tau_0, \tau_*]$,

$$\|u(\tau)\|_{L^\infty} + E^\alpha_{k,\lambda}[u(\tau)] \le 1 + C\left(\|u(\tau_0)\|_{L^\infty} + E^\alpha_{k,\lambda}[u(\tau_0)]\right) , \qquad (4.2.39)$$

which provides the desired bounds.

If one knows a priori that $\hat{M}(\tau)$ is bounded, (4.2.37) *becomes effectively a linear inequality, and the claimed global bound immediately follows.*

Actually, the solution constructed here is defined on $\mathscr{U}_{\tau_*}$ (see Figure 4.1). In order to obtain a solution in a whole neighborhood of the hyperboloid $\mathscr{S}_0$, we proceed as follows: Let $R > 0$ be a real positive number such that the level set $r = R$ lies in the region where the energy estimates above apply. We consider the Cauchy problem for (4.1.13) with initial data obtained by restriction on

$$\mathscr{S}_0(R) = \mathscr{S}_0 \cap \{(x^\mu) : 0 \le |\vec{x}| \le R\} .$$

We thus obtain a Cauchy problem on a compact region. We can now apply to this problem the conclusion of Proposition 3.2, p. 378 of [50]: there exists a time $\tau_+ \in]\tau_0, 0[$ and a smooth solution on (see Figure 4.1)

$$\mathscr{V}_+ = \bigcup_{t \in [t_0, -\frac{1}{2\tau_+}]} \phi(\mathscr{S}_t(R)) \cap \mathscr{D}^+(\phi(\mathscr{S}_0(R))) ,$$

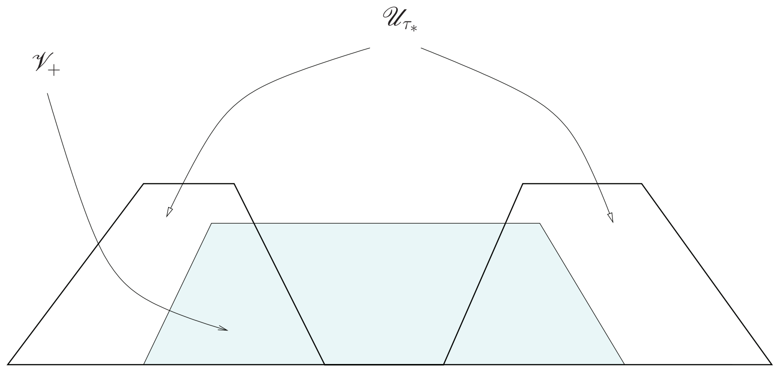

Figure 4.1: The sets $\mathscr{V}_+$ and $\mathscr{U}_{\tau_*}$.

where $\mathscr{D}^+$ denotes the domain of dependence, and where

$$\mathscr{S}_t(R) = \mathscr{S}_t \cap \{(x^\mu) : 0 \le |\vec{x}| \le R\} \ .$$

From uniqueness in Proposition 3.2, p. 378 of [50], we conclude that the solutions constructed on $\mathscr{V}_+$ and $\mathscr{U}_{\tau_*}$ coincide on $\mathscr{V}_+ \cap \mathscr{U}_{\tau_*}$ which is not empty for R large enough. We thus obtain a solution of (4.1.13) with (4.2.1) in a whole neighborhood of $\mathscr{S}_0$. □

Space-regularity of the solution

For smooth initial data the solution constructed in the previous section is in $C^\infty(\mathscr{V}_+ \cup \mathscr{U}_{\tau_*})$. In this section we want to show that, for data given in the space $\bigcap_{k\in\mathbb{N}} \mathscr{H}^\alpha_k$, we can control the growth, near $x = 0$, of all space derivatives of the corresponding solution. We have the following:

Theorem 4.2.6 *Under the hypotheses of Theorem 4.2.3, suppose moreover that the initial data given on the hyperboloid $\mathscr{S}_0$ satisfy*

$$\hat{f}\,\big|_{\phi(\mathscr{S}_0)} \in (\mathscr{H}^\alpha_\infty \cap L^\infty)\,(\mathbf{H}_{\tau_0}) \ \ \text{and}\ \partial\hat{f}\,\big|_{\phi(\mathscr{S}_0)} \in \mathscr{H}^\alpha_\infty(\mathbf{H}_{\tau_0}) \ . \qquad (4.2.40)$$

If $\alpha = -1/2$ and $n = 6$ we also suppose that (4.2.26) holds for all k. Let τ_* be as in that theorem with $k = k_0$, where k_0 is the smallest integer larger than $[n/2] + 1$. Then

$$\forall \tau \in [\tau_0, \tau_*] \qquad \hat{f}(\tau) \in (\mathscr{H}^\alpha_\infty \cap L^\infty)(\mathbf{H}_\tau) \ , \quad \partial\hat{f}(\tau) \in \mathscr{H}^\alpha_\infty(\mathbf{H}_\tau) \ . \quad (4.2.41)$$

Furthermore, any solution with smooth initial data as above for which $\hat{M}(\tau)$, as defined in (4.2.4), is bounded on $[\tau_0, \tau_1]$ satisfies (4.2.41) with $\tau_* = \tau_1$.

Proof: *We provide the details for $n > 6$; the treatment of the case $n = 6$ is similar. From Theorem 4.2.3 there exists a time τ_* and a constant C^* depending on k_0 such that $\forall \tau \in [\tau_0,\ \tau_*[$,*

$$\|u(\tau)\|^2_{L^\infty} + E^\alpha_{k_0,\lambda}[u(\tau)] \leq C^* \ . \tag{4.2.42}$$

Now let $k \in \mathbb{N},\ k \geq k_0$, since $\hat{f}\,|_{\phi(\mathscr{S}_0)} \in (\mathscr{H}^\alpha_k \cap L^\infty)\,(\mathbf{H}_{\tau_0})$ inequality (4.2.8) holds. Now the function $C_3(\hat{M}(s))$ appearing in this inequality is controlled by $E^\alpha_{k_0,\lambda}[u(\tau)]$ and thus by C^*, therefore, from (4.2.42) we have:

$$E^\alpha_{k,\lambda}[u(\tau)] \leq C(C^*)\left(1 + \int_{\tau_0}^{\tau} E^\alpha_{k,\lambda}[u(s)]ds\right) \ .$$

Applying Gronwall's inequality we obtain:

$$E^\alpha_{k,\lambda}[u(\tau)] \leq Ce^{C\tau_*} \ .$$

This inequality shows that, for all k,

$$\partial u \in \mathscr{H}^\alpha_k \ , \tag{4.2.43}$$

as desired. □

4.2.2 Estimates on time derivatives of the solution

In order to estimate the time derivatives of the solution, we introduce a new set of variables $(y, \tilde{x})$ (compare Figure 4.2):

$$\begin{cases} \tau = \frac{y-\tilde{x}}{2} + \tau_0 \\ x = \tilde{x} \end{cases} \qquad \textit{which implies that} \qquad \begin{cases} \partial_y = \frac{1}{2}\partial_\tau \\ \partial_{\tilde{x}} = \partial_x - \frac{1}{2}\partial_\tau \end{cases} .$$

Note that in these new coordinates, the hyperboloid $\mathscr{S}_0$ is represented by the set $\{y = \tilde{x}\}$. Since we are interested in the behavior of solution in a neighborhood of the set $\{x = 0\}$, as in [19] we restrict our attention on the subset $\mathscr{U}$ of $\mathscr{U}_{\tau_*}$ defined by:

$$\mathscr{U} = \left\{(y, \tilde{x}, v^A) : 0 < x < y,\ v \in \mathscr{O}\ , 0 < y < 2(\tau_* - \tau_0)\right\} \ .$$

Recall that the definitions of the spaces

$$\mathscr{C}^\alpha_{\{x=0\},k}(\mathscr{U}), \quad \mathscr{C}^\sigma_{\{y=0\},k}(\mathscr{U}), \quad \mathscr{C}^\alpha_{\{0\leq x\leq y\},k}(\mathscr{U}), \quad \textit{and} \quad \mathscr{C}^{\alpha,\sigma}_{\{0\leq x\leq y\},k}(\Omega) \ ,$$

can be found in Appendix A.2 page 191 with ∂_x there corresponding to $\partial_{\tilde{x}}$ here.

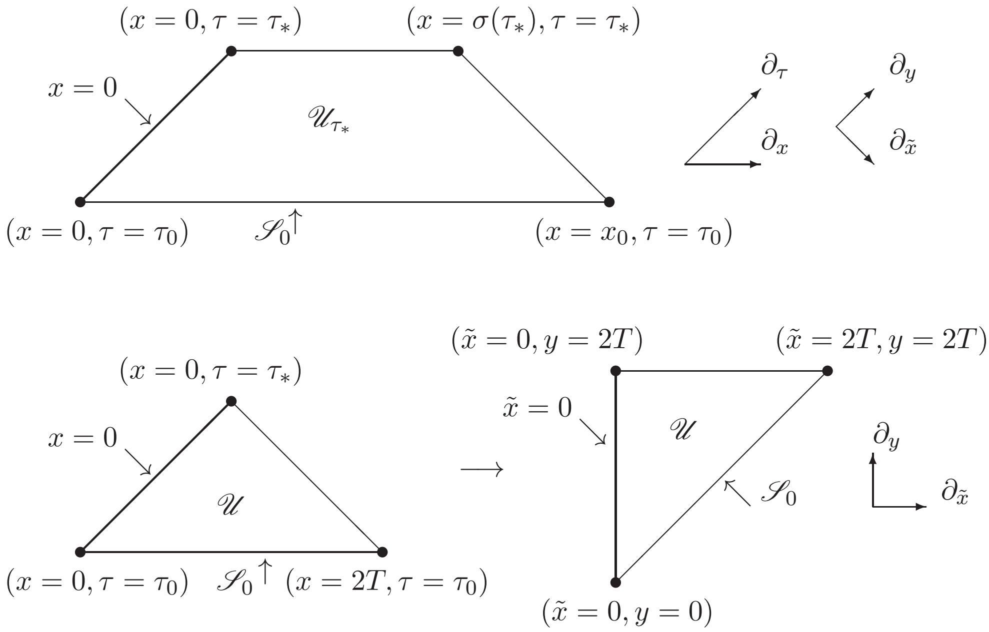

Figure 4.2: The variables (x,τ) and $(\tilde x,y)$, with $T:=\tau_*-\tau_0$. The function σ has been introduced in (3.1.16). We hope that the reader will not get confused by the fact that the boundary $x=0$, at the left-hand sides of the figures here, is depicted at the right-hand side of Figure 3.1.

Remark 4.2.7 In the coordinates $(y,\tilde x)$ the components of the inverse of the metric read (compare 4.1.39):

$$\mathfrak{g}^{yy}=4\left(\mathfrak{g}^{\tau\tau}+\mathfrak{g}^{x\tau}\right)+\mathfrak{g}^{xx}=\mathrm{O}(x^{\frac{n-5}{2}}) \tag{4.2.44}$$

$$\mathfrak{g}^{y\tilde x}=2\mathfrak{g}^{x\tau}+\mathfrak{g}^{xx} \tag{4.2.45}$$

$$\mathfrak{g}^{yA}=2\mathfrak{g}^{\tau A}+\mathfrak{g}^{xA} \tag{4.2.46}$$

$$\mathfrak{g}^{\tilde x\tilde x}=\mathfrak{g}^{xx}=\mathrm{O}(x^{\frac{n-1}{2}}) \tag{4.2.47}$$

$$\mathfrak{g}^{\tilde xA}=\mathfrak{g}^{xA}\,. \tag{4.2.48}$$

Recall that the hypersurfaces $\mathscr{S}_s$ have been defined in (4.2.24). As a first step towards proving propagation of polyhomogeneity, we obtain some information about the ∂_y-derivatives of the fields:

Theorem 4.2.8 *Suppose that $k > \left[\frac{n}{2}\right] + 1$. Under the hypotheses of Proposition 4.2.1, there exists $t_* > t_0$ and a solution of (4.1.13) defined on $\underset{s\in[t_0,\, t_*]}{\cup} \mathscr{S}_s$ such that:*

$$\hat{f} \in \left(\mathscr{C}^{\alpha}_{\{0\le x\le y\},k-\left[\frac{n}{2}\right]-1} \cap L^{\infty}\right)(\mathscr{U}) \ . \tag{4.2.49}$$

$$\left(\partial_\tau \hat{f}, \partial_x \hat{f}, \partial_A \hat{f}\right) \in \mathscr{C}^{\alpha}_{\{0\le x\le y\},k-\left[\frac{n}{2}\right]-1}(\mathscr{U}) \ , \tag{4.2.50}$$

where f and $\hat{f}$ are defined by (4.2.1)-(4.2.2).

Proof: *The proof of existence is given by Theorem 4.2.3 and we have $\hat{f} \in L^{\infty}(\mathscr{U})$, $\partial\hat{f} \in \mathscr{C}^{\alpha}_{\{\tilde{x}=0\},k-\left[\frac{n}{2}\right]-1}(\mathscr{U})$. We note that from (4.1.4) and (4.1.5) we have:*

$$\Omega = \tilde{x}(-y - 2\tau_0), \qquad y\partial_y\Omega = -\tilde{x}y, \qquad \tilde{x}\partial_{\tilde{x}}\Omega = \Omega \qquad \text{and} \qquad \partial_A\Omega = 0 \ . \tag{4.2.51}$$

Identities (4.2.51) show that if we apply to (4.1.21) the operator $(\partial_A, \tilde{x}\partial_{\tilde{x}}, y\partial_y)$, then we obtain a wave equation with $(u, \partial_A u, y\partial_y u, \tilde{x}\partial_{\tilde{x}} u\ )$ as the new unknown functions in which the coefficients have the same powers of x as in the original equation, and the source term the same structure. More precisely, set

$$U = \begin{pmatrix} u \\ \partial_A u \\ \tilde{x}\partial_{\tilde{x}} u \\ y\partial_y u \end{pmatrix}, \qquad \text{we thus obtain} \qquad \begin{pmatrix} U \\ \partial U \end{pmatrix} = \begin{pmatrix} u \\ \partial_A u \\ \tilde{x}\partial_{\tilde{x}} u \\ y\partial_y u \\ \partial u \\ \partial(\partial_A u) \\ \partial(\tilde{x}\partial_{\tilde{x}} u) \\ \partial\left(y\partial_y u\right) \end{pmatrix}, \tag{4.2.52}$$

and let us derive a wave equation on U. Straightforward calculations lead to the following identity (here we write the source term as a function of variables p_1 and p_2^σ, that is $\mathcal{F} = \mathcal{F}(\cdot,\ p_1, p_2^\sigma)$):

$$\begin{aligned} \Box_{\mathfrak{g}}(y\partial_y u) \ &= \ -(y\partial_y \mathfrak{g}^{\alpha\beta})\partial^2_{\alpha\beta} u + 2\mathfrak{g}^{\alpha y}\partial_\alpha\partial_y u - (y\partial_y \Upsilon^\alpha)\partial_\alpha u + \Upsilon^y \partial_y u \\ &\quad + (y\partial_y \mathcal{F})(\cdot, u, \partial u) + (y\partial_y u)\frac{\partial \mathcal{F}}{\partial p_1}(\cdot, u, \partial u) \\ &\quad + \left(\partial_y(y\partial_\sigma u) - \delta^y_\sigma \partial_y u\right)\frac{\partial \mathcal{F}}{\partial p_2^\sigma}(\cdot, u, \partial u) \ . \end{aligned} \tag{4.2.53}$$

We write

$$(y\partial_y\mathfrak{g}^{yy})\partial_y^2 u = \partial_y\mathfrak{g}^{yy}\left(\partial_y(y\partial_y u) - \partial_y u\right) \sim \Omega^{\frac{n-5}{2}}(\partial U + U)\partial U \ ,$$

$$(y\partial_y\mathfrak{g}^{\tilde{x}y})\partial_y\partial_{\tilde{x}} u = \partial_y\mathfrak{g}^{yy}\partial_y(\tilde{x}\partial_{\tilde{x}} u) \sim \Omega^{\frac{n-5}{2}}\partial U\partial U \ ,$$

$$(y\partial_y\mathfrak{g}^{\tilde{x}\tilde{x}})\partial_{\tilde{x}}^2 u = O(\tilde{x}^{\frac{n-3}{2}})\left(\partial_{\tilde{x}}(\tilde{x}\partial_{\tilde{x}} u) - \partial_{\tilde{x}} u\right) \sim \Omega^{\frac{n-3}{2}}U\partial U \quad \text{see (4.1.39)},$$

$$\mathfrak{g}^{yy}\partial_y^2 u = O(\tilde{x}^{\frac{n-7}{2}})\frac{\tilde{x}}{y}\left(\partial_y(y\partial_y u) - \partial_y u\right) \sim \Omega^{\frac{n-7}{2}}U\partial U \ ,$$

and

$$2\mathfrak{g}^{\alpha y}\partial_\alpha\partial_y u = \mathfrak{g}^{yy}\partial_y^2 u - \left\{\mathfrak{g}^{\tilde{x}\tilde{x}}\partial_{\tilde{x}}^2 u + 2\mathfrak{g}^{\tilde{x}A}\partial_{\tilde{x}}\partial_A u + \mathfrak{g}^{AB}\partial_A\partial_B u + \Upsilon^\sigma\partial_\sigma u - \mathcal{F}(u,\partial u)\right\} \ .$$

All the terms arising above have a structure similar to (4.1.50). A similar comparison of the remaining terms shows that we have

$$\Box_{\mathfrak{g}}(y\partial_y u) = \mathcal{F}_1(U,\partial U) \ , \tag{4.2.54}$$

where the source term $\mathcal{F}_1$ is of the general form as in (4.1.50) with the difference that it has a term $\Omega^{\frac{n-7}{2}}U\partial U$ with a multiplicative $\Omega^{\frac{n-7}{2}}$; this term can be estimated as in (4.2.18) as long as $n \geq 7$. Moreover, it is easily checked that this remains compatible with the estimate of Proposition 4.2.1 (see Remark 4.2.2). Note that the procedure above introduces into the coefficients of the source terms the function $(y,\tilde{x}) \longmapsto \frac{\tilde{x}}{y}$, which is bounded on $\mathscr{U}$; furthermore, $\tilde{x}\partial_{\tilde{x}}\frac{\tilde{x}}{y} = -y\partial_y\frac{\tilde{x}}{y} = \frac{\tilde{x}}{y}$, which implies that we will not loose the regularity of the source terms, as needed for the problem at hand, when iterating the process.

From the identities,

$$\begin{aligned}\Box_{\mathfrak{g}}(\tilde{x}\partial_{\tilde{x}} u) \ = \ & -(\tilde{x}\partial_{\tilde{x}}\mathfrak{g}^{\alpha\beta})\partial^2_{\alpha\beta}u + 2\mathfrak{g}^{\alpha\tilde{x}}\partial_\alpha\partial_{\tilde{x}}u - (\tilde{x}\partial_{\tilde{x}}\Upsilon^\alpha)\partial_\alpha u + \Upsilon^{\tilde{x}}\partial_{\tilde{x}}u \\ & +(\tilde{x}\partial_{\tilde{x}}\mathcal{F})(\cdot,u,\partial u) + \partial_{\tilde{x}}(\tilde{x}\partial_{\tilde{x}}u)\frac{\partial\mathcal{F}}{\partial p_1}(\cdot,u,\partial u) \\ & + \left(\partial_{\tilde{x}}(\tilde{x}\partial_\sigma u) - \delta^{\tilde{x}}_\sigma\partial_{\tilde{x}}u\right)\frac{\partial\mathcal{F}}{\partial p_2^\sigma}(\cdot,u,\partial u) \ ,\end{aligned} \tag{4.2.55}$$

$$\Box_{\mathfrak{g}}(\partial_A u) = -(\partial_A\mathfrak{g}^{\alpha\beta})\partial^2_{\alpha\beta}u - (\partial_A\Upsilon^\alpha)\partial_\alpha u + \partial_A u\frac{\partial\mathcal{F}}{\partial p_1}(u,\partial u) + \partial\partial_A u\frac{\partial\mathcal{F}}{\partial p_2}(u,\partial u) \ , \tag{4.2.56}$$

we deduce that the same analysis holds for $\Box_{\mathfrak{g}}(\partial_A u)$ and $\Box_{\mathfrak{g}}(\tilde{x}\partial_{\tilde{x}}u)$. Therefore we have derived for the new unknown function U a wave equation of the form (4.1.21), i.e.:

$$\Box_{\mathfrak{g}}U = \mathfrak{F}(U,\ \partial U) \ . \tag{4.2.57}$$

In order to apply to this equation Theorem 4.2.3, we have to check that the initial data for U are in the right spaces. Note that the initial data are prescribed on the subset $\{x = y\}$ of $\mathscr{U}$. We denote this hypersurface by Σ_0, thus $\Sigma_0 = \phi(\mathscr{S}_0) \cap \mathscr{U}$, and we set

$$\Sigma_s = \phi(\mathscr{S}_s) \cap \mathscr{U} \subset \mathbf{H}_{-1/2s} . \tag{4.2.58}$$

We want to prove the following.

Lemma 4.2.9 *Under the hypotheses of Proposition 4.2.1 we have:*

$$(u, \partial_A u, \tilde{x}\partial_{\tilde{x}} u, y\partial_y u)|_{\Sigma_0} \in (\mathscr{H}^\alpha_k \cap L^\infty)(\Sigma_0) , \tag{4.2.59}$$

$$(\partial u, \partial\partial_A u, \partial(\tilde{x}\partial_{\tilde{x}} u), \partial(y\partial_y u))|_{\Sigma_0} \in \mathscr{H}^\alpha_{k-1}(\Sigma_0) . \tag{4.2.60}$$

Proof: *By assumption, we have*

$$u|_{\Sigma_0} \in (\mathscr{H}^\alpha_k \cap L^\infty)(\Sigma_0), \quad \text{and} \quad (\partial_A u, \partial_{\tilde{x}} u , \partial_y u)|_{\Sigma_0} \in \mathscr{H}^\alpha_k(\Sigma_0) . \tag{4.2.61}$$

Now, using Sobolev's embedding theorem, we have

$$\tilde{x}^{-\alpha} (\partial_A u, \partial_{\tilde{x}} u , \partial_y u)\big|_{\Sigma_0} \in L^\infty(\Sigma_0) . \tag{4.2.62}$$

This leads to the following estimates:

$$\begin{aligned}
|\tilde{x}\partial_{\tilde{x}} u|_{\Sigma_0}| &= \tilde{x}^{1+\alpha} \left|\tilde{x}^{-\alpha}\partial_{\tilde{x}} u|_{\Sigma_0}\right| < \infty, \\
|y\partial_y u|_{\Sigma_0}| &= |\tilde{x}\partial_y u|_{\Sigma_0}| = \tilde{x}^{1+\alpha} \left|\tilde{x}^{-\alpha}\partial_y u|_{\Sigma_0}\right| < \infty .
\end{aligned}$$

To see that $\partial_A u(\tau_0)$ is in $L^\infty(\mathscr{S}_0)$, we proceed as follows: integrating $\partial_A u(\tau_0)$ in x until x_0 gives the inequality

$$\partial_A u(\tau_0, x_0, v^A) - \partial_A u(\tau_0, \tilde{x}, v^A) = \int_{\tilde{x}}^{x_0} \partial_{\tilde{x}}\partial_A u(\tau_0, s, v^A) ds ,$$

which leads to the estimate

$$\begin{aligned}
|\partial_A u(\tau_0, \tilde{x}, v^A)| &\le |\partial_A u(\tau_0, x_0, v^A)| + \|\partial_{\tilde{x}} u(\tau_0)\|_{\mathscr{C}^\alpha_{\{\tilde{x}=0\},1}} \int_{\tilde{x}}^{x_0} s^\alpha ds \\
&\le |\partial_A u(\tau_0, x_0, v^A)| + \|\partial_{\tilde{x}} u(\tau_0)\|_{\mathscr{H}^\alpha_k} \int_{\tilde{x}}^{x_0} s^\alpha ds
\end{aligned}$$

(recall $k - 1 > \frac{n}{2}$). Since

$$\|\partial_A u(\tau_0, x_0, v^A)\|_{L^\infty(\mathscr{O})} < \infty, \qquad \|\partial_{\tilde{x}} u(\tau_0)\|_{\mathscr{H}^\alpha_k} \le E^\alpha_k[u(\tau_0)] < \infty ,$$

and

$$\int_{\tilde{x}}^{x_0} s^\alpha ds = \frac{1}{\alpha+1}\left(x_0^{\alpha+1} - \tilde{x}^{\alpha+1}\right) < \infty ,$$

we conclude that $\|\partial_A u(\tau_0)\|_{L^\infty} < \infty$*. Thus* $(\partial_A u)\,|_{\Sigma_0} \in L^\infty(\Sigma_0)$ *and we then obtain (4.2.59). On the other hand we have*

$$\begin{aligned} \|\partial_\nu(\tilde{x}\partial_{\tilde{x}}u)|_{\Sigma_0}\|_{\mathscr{H}^\alpha_{k-1}(\Sigma_0)} &\le \|\tilde{x}\partial_{\tilde{x}}(\partial_\nu u)|_{\Sigma_0}\|_{\mathscr{H}^\alpha_{k-1}(\Sigma_0)} + \|\delta^{\tilde{x}}_\nu \partial_{\tilde{x}} u|_{\Sigma_0}\|_{\mathscr{H}^\alpha_{k-1}(\Sigma_0)} \\ &\le \|\partial_\nu u|_{\Sigma_0}\|_{\mathscr{H}^\alpha_k(\Sigma_0)} \\ &< \infty \qquad \text{see (4.2.61)} . \end{aligned}$$

Similarly, we have $\partial(y\partial_y u)|_{\Sigma_0},\ \partial\partial_A u|_{\Sigma_0} \in \mathscr{H}^\alpha_{k-1}(\Sigma_0)$*. We thus obtain (4.2.59) and the proof of the lemma is complete.* □

Now, we apply Theorem 4.2.3 to (4.2.57) and obtain that

$$(u, \partial_A u, \tilde{x}\partial_{\tilde{x}}u, y\partial_y u) \in L^\infty\Big([0, 2(\tau_* - \tau_0)],\ (\mathscr{H}^\alpha_k \cap L^\infty)(\Sigma_\epsilon)\Big) , \qquad (4.2.63)$$

$$(\partial u, \partial\partial_A u, \partial(\tilde{x}\partial_{\tilde{x}}u), \partial(y\partial_y u)) \in L^\infty\Big([0, 2(\tau_* - \tau_0)],\ \mathscr{H}^\alpha_k(\Sigma_\epsilon)\Big) . \qquad (4.2.64)$$

Using once more the Sobolev embedding theorem, we obtain that $\forall \epsilon \in [0, 2(\tau_* - \tau_0)]$

$$(u, \partial_A u, \tilde{x}\partial_{\tilde{x}}u, y\partial_y u)|_{\Sigma_\epsilon} \in \mathscr{C}^\alpha_{\{\tilde{x}=0\}, k-[\frac{n}{2}]-1}(\Sigma_\epsilon) .$$

$$\begin{aligned} \|(u, \partial_A u, \tilde{x}\partial_{\tilde{x}}u, y\partial_y u)\|_{L^\infty(\mathscr{U})} &= \sup_{\tau\in[\tau_0,\tau_*]} \|\,(u, \partial_A u, \tilde{x}\partial_{\tilde{x}}u, y\partial_y u)|_{\mathscr{S}_\tau}\,\|_{L^\infty(\mathscr{S}_\tau)} \\ &\le \sup_{\tau\in[\tau_0,\tau_*]} \|\,(u, \partial_A u, \tilde{x}\partial_{\tilde{x}}u, y\partial_y u)|_{\mathscr{S}_\tau}\,\|_{\mathscr{H}^\alpha_k(\mathscr{S}_\tau)} \\ &\underbrace{<}_{\text{see (4.2.63)}} \infty . \end{aligned}$$

Using now (4.2.64) instead of (4.2.63) we have

$$\|(u, \partial_A \partial u, \tilde{x}\partial_{\tilde{x}}\partial u, y\partial_y \partial u)\|_{L^\infty(\mathscr{U})} < \infty .$$

This allows us to conclude that $(u,\ \partial u)$ *is in* $\mathscr{C}^\alpha_{\{0\le\tilde{x}\le y\},1}(\mathscr{U})$*. Now, if we repeat this process j times with* $j = k - \left[\frac{n}{2}\right] - 1$ *then we obtain that u is in* $\mathscr{C}^\alpha_{\{0\le\tilde{x}\le y\},k-\frac{n}{2}-1}(\mathscr{U})$*. This completes the proof of Theorem 4.2.8.* □

Corollary 4.2.10 *Under the hypotheses of Theorem 4.2.6 we have the following:*

$$\hat{f} \in \Big(\mathscr{C}^\alpha_{\{0\le\tilde{x}\le y\},\infty}(\mathscr{U}) \cap L^\infty\Big)(\mathscr{U}) \qquad \text{and} \qquad \partial\hat{f} \in \mathscr{C}^\alpha_{\{0\le\tilde{x}\le y\},\infty}(\mathscr{U}) .$$

Proof: *The result is a combination of Theorems 4.2.6 and 4.2.8.* □

Chapter 5

Polyhomogeneous solutions of the Einstein-Maxwell equations

Let δ be a positive real number. We recall that the spaces of polyhomogeneous functions $\mathscr{A}_{\{x=0\}}$, $\mathscr{A}^{\delta}_{\{x=0\}}$, $\mathscr{A}_{\{0\le x\le y\}}$ and $\mathscr{A}^{\delta}_{\{0\le x\le y\}}$ are defined in Appendix A.3 Equations A.3.1-A.3.2 (see also [19, Equations (A.1)-(A.2)]). We consider the Cauchy problem for the Einstein-Maxwell equations (4.1.13) with (4.2.1) in wave coordinates (x^{μ}) and Lorenz gauge with prescribed data on the hyperboloid $\mathscr{S}_0$ (see (4.2.6)) at the interior of the future light-cone with vertex the origin of coordinates. The coordinate x in which the polyhomogeneous expansion is taken is $x = \frac{1}{t+r}$ where $t = x^0$ and $r = |\vec{x}| = \sum_{i=1}^{n}(x^i)^2)$. Indeed we have (see (4.1.4)):

$$\begin{aligned} x = -\tau - \rho &= -\frac{t}{-t^2+r^2} - \left(\sum \frac{(x^i)^2}{(-t^2+r^2)^2}\right)^{1/2} \\ &= -\frac{t}{-t^2+r^2} - \frac{r}{t^2-r^2} \\ &= \frac{1}{t+r}\ . \end{aligned}$$

We want to prove that, polyhomogeneous initial data for the above Cauchy problem lead to polyhomogeneous solution. We have the following:

Theorem 5.0.11 *Consider the Einstein-Maxwell equations on $\mathbb{R}^{1+n}$, $n \ge 8$. Let $\delta \in \mathbb{R}$ be such that $1/(2\delta) \in \mathbb{N}$ when n is even and $1/\delta \in \mathbb{N}$ when n*

is odd. Suppose that the initial data for (4.1.13) in wave coordinates and Lorenz gauge are polyhomogeneous on the hyperboloid $\mathscr{S}_0$ *:*

$$f\big|_{\mathscr{S}_0} \in x^{\frac{n-1}{2}} \mathscr{A}^{\delta}_{\{x=0\}} \cap L^\infty, \qquad \partial_\tau f\big|_{\mathscr{S}_0} \in x^{\frac{n-1}{2}} \mathscr{A}^{\delta}_{\{x=0\}} \ , \tag{5.0.1}$$

with $f = (g_{\mu\nu} - \eta_{\mu\nu}, A_\mu)$*. There exists a time* $t_+ > t_0$ *and a solution defined on* $\bigcup_{t\in[t_0,t_+]} \mathscr{S}_t$ *such that* $\forall t \in [t_0, t_+]$ *we have:*

$$f(t) = f\big|_{\mathscr{S}_t} \in x^{\frac{n-1}{2}} \mathscr{A}^{\delta}_{\{x=0\}} \qquad \text{and} \qquad \partial_\tau f(t) = \partial_\tau f\big|_{\mathscr{S}_t} \in x^{\frac{n-1}{2}-1} \mathscr{A}^{\delta}_{\{x=0\}} \ . \tag{5.0.2}$$

Moreover, the solution is polyhomogeneous at $\mathscr{I}$*, in the above polyhomogeneity class, as long as it remains in* $\mathscr{H}^\alpha_k(\mathbf{H}_\tau)$*, for some* $\alpha \in (-1, -1/2]$*.*

Proof: *Choose any* $\alpha < 0$*; we then have the inclusion* $\mathscr{A}^{\delta}_{\{x=0\}}(\overline{\phi(\mathscr{S}_0)}) \subset \mathscr{H}^\alpha_\infty(\overline{\phi(\mathscr{S}_0)})$*. It follows from (5.0.1) that we have:*

$$\hat f\,\big|_{\phi(\mathscr{S}_0)} \in \left(\mathscr{H}^\alpha_\infty \cap L^\infty\right)(\phi(\mathscr{S}_0)) \quad \text{and} \quad \partial \hat f\,\big|_{\phi(\mathscr{S}_0)} \in \mathscr{H}^\alpha_\infty(\phi(\mathscr{S}_0)) \ . \tag{5.0.3}$$

For definiteness set $\alpha = -1/2$*. From Theorem 4.2.6, there exists a time* τ_* *and a smooth solution* $\hat f$ *of (4.1.13)-(4.2.1)-(5.0.3) defined on* $\mathscr{U}_{\tau_*}$ *such that* $\forall \tau \in [\tau_0, \tau_*]$, $\hat f(\tau) \in \mathscr{C}^\alpha_j(\mathbf{H}_\tau)$*. Next, applying Corollary 4.2.10 one obtains that*

$$\hat f \in \left(\mathscr{C}^\alpha_{\{0\le x\le y\},\infty} \cap L^\infty\right)(\mathscr{U}) \qquad \text{and} \qquad \partial \hat f \in \mathscr{C}^\alpha_{\{0\le x\le y\},\infty}(\mathscr{U}) \ .$$

From Theorem 1.2.8 of Section 1.2, with

$$\psi_1 = \hat f \ , \quad \psi_2 = (\partial_y \hat f, \partial_A \hat f) \ , \quad \varphi = \partial_x f \ ,$$

we obtain (5.0.2), and the proof is completed. □

It is natural to find conditions which guarantee that solutions remain in weighted Sobolev spaces on hyperboloids, and hence remain polyhomogeneous if the initial data are. One such criterion is provided by the following:

Theorem 5.0.12 *Suppose that* $k > \left[\frac{n}{2}\right] + 1$*, with* $n = 6$ *and* $\alpha = -1/2$*, or* $n \ge 7$ *with* $\alpha \in (-1, 1/2]$*. Solutions of the Einstein-Maxwell equations remain in* $\mathscr{H}^\alpha_k$*,* $\alpha \in (-1, -1/2]$ *as long as* $\hat f$ *remains in* $\mathscr{C}^\kappa_{\{x=0\},1}$*, with*

$$\kappa > -\frac{(n-7)}{2} \ . \tag{5.0.4}$$

The same is true for

$$\kappa > -\frac{(n-5)}{2} \text{ provided that } \|x^{\frac{n-7}{2}} y_\mu y_\nu \hat{H}^{\mu\nu}(\tau_0)\|_{L^\infty} < \infty \,. \tag{5.0.5}$$

In particular, in dimensions $n+1 \geq 9$ the small data solutions of [39, 40] evolving out from data stationary outside of a compact set are polyhomogeneous.

Proof: *We want to use Proposition 3.2.9 to show that solutions as above remain in $\mathscr{H}^\alpha_k$, $\alpha \in (-1, -1/2]$. For this, consider first the right-hand side of (3.2.85). For $\kappa \geq -(n-5)/2$ one immediately finds that $\|\delta \mathfrak{g}^\sharp\|_{\mathscr{C}^0_{\{x=0\},1}}$ is finite, similarly for $(\partial_x - \partial_\tau)\delta\mathfrak{g}^\sharp$ when $\kappa \geq -(n-7)/2$. Finiteness of $\|\delta\mathfrak{h}^\sharp\|_{\mathscr{C}^0_{\{x=0\},1}}$ is straightforward for $\kappa \geq -(n-7)/2$ from (4.1.35)-(4.1.38). The estimate on $\delta\Upsilon$ follows from (4.1.46) and (4.1.48) provided again that $\kappa \geq (n-7)/2$.*

For $\kappa \geq -(n-5)/2$ the slowest decaying terms in $\mathfrak{h}$, Υ, and in $(\partial_x - \partial_\tau)\mathfrak{g}^\sharp$ are handled by the $\mathscr{C}^0_{\{x=0\},1}$-spaces equivalent of (4.2.23),

$$\begin{aligned}\|x^{\frac{(n-7)}{2}}\psi(\tau)\|_{\mathscr{C}^0_{\{x=0\},1}} \\ \leq \quad \|x^{\frac{(n-7)}{2}}\psi(\tau_0)\|_{\mathscr{C}^0_{\{x=0\},1}} + C(\tau_0,\tau_1)\int_{\tau_0}^{\tau} \|x^{\frac{(n-7)}{2}}\zeta(s)\|_{\mathscr{C}^0_{\{x=0\},1}} ds \,,\end{aligned} \tag{5.0.6}$$

under the supplementary condition that $\|x^{\frac{(n-7)}{2}}\psi(\tau_0)\|_{\mathscr{C}^0_{\{x=0\},1}}$ is finite.

For any σ such that

$$\sigma < \kappa \tag{5.0.7}$$

we have

$$\mathscr{C}^\kappa_{\{x=0\},1} \subset \mathscr{B}^\sigma_1 \,.$$

Hence the right-hand side of (3.2.86) is finite for all such σ's, and so (3.2.87) applies. It remains to show that the integrand in the second line of (3.2.87) can be bounded by a multiple of the energy:

$$\|(\delta\mathfrak{g}^\sharp, \delta\mathfrak{h}^\sharp, \delta\Upsilon)\|^2_{\mathscr{G}^{\alpha-\sigma}_k(\mathbf{H}_\tau)} \leq C E^\alpha_k[u(s)] \,.$$

This is easily checked to hold under (5.0.4) or (5.0.5) if we choose σ so that

$$\sigma > -\frac{n-7}{2} \,.$$

This, together with (5.0.7), explains (5.0.4).

The property that the solutions of the Einstein-Maxwell equations constructed by Loizelet are in $\mathscr{C}^\kappa_{\{x=0\},1}$ on all hyperboloidal slices has been verified in (2.3.15). There $-\kappa = \delta \in (0, 1/4)$ □

Conclusion of the first part

The results which are established in this first part of the thesis join within the framework of a mathematical program the ultimate stage of which would be to prove that hyperboloidal polyhomogeneous initial data lead to polyhomogeneous solutions of the coupled vacuum Einstein-Maxwell equations in space-time dimension $n+1 \geq 4$. *This program was initiated by Piotr T. Chruściel and his collaborators. As a first step towards the solutions of this problem, they proved existence of polyhomogeneous solutions for hyperboloidal Cauchy problem for semi-linear wave equations and waves maps. See [19, 20]. Inspired by these works, we have proved propagation of weighted Sobolev regularity with uniform time of existence near the conformal null infinity for solutions of the hyperboloidal Cauchy problem for a class of quasi-linear symmetric hyperbolic systems, under structure conditions compatible with the Einstein-Maxwell equations in space-time dimensions* $n + 1 \geq 7$. *Similarly, for these equations, we have proved propagation of polyhomogeneity at null infinity of solutions in space-time dimensions* $n+1 \geq 9$. *In those dimensions we obtained that the global solutions of the Einstein-Maxwell equations for small data which are stationary outside of a compact set obtained in [39, 40] are polyhomogeneous. In the process we also proved a theorem of existence of a solution within the class of polyhomogeneous solutions for the Einstein-Maxwell equations in even or odd dimension of space* $n \geq 8$, *complementing the result known so far (see [9]) only when the space dimension* n *is odd and greater or equal to* 5.

The fact that our results are valid only in high space dimension is, in our opinion, due to the choice of the conformal transformation we used and/or to the choice of the gauges. We thus think that, if one wants to improve the threshold on the space dimension n, *one could for example think to a different conformal transformation and/or to keep this transformation, but use different gauges so as to get rid of the dangerous terms which impose to the space dimension to be so large. For example, in [34], H. Friedrich gave a conformal representation of the Einstein equations in a conformally invari-*

ant gauge as a system of first order partial differential equations with smooth coefficients. We expect that in the case $n = 3$*, using this representation of Einstein equations and the energy estimates obtained by O. Lengard in the second part of his thesis, one should be able to establish propagation near* $\mathscr{I}^+$ *of polyhomogenity of solution of Einstein equations in this dimension.*

It would be interesting in view of its physical applications, (see [27,29,47] and the references therein) to obtain a characteristic version of the results obtained so far. In other words, one can enquire wether polyhomogeneous initial data prescribed on one or several intersecting characteristic hypersurfaces can be evolved to obtain polyhomogeneous solutions of the vacuum Einstein-Maxwell equations. We think that this can be overcome with a good combination of the techniques developed by A. Cabet in her thesis, the corresponding techniques of conformal compactification which is used here and the results of M. Dossa [25–28]. The second part of the thesis is our contribution towards the construction of solutions of this problem.

Part II

Solutions with a uniform time of existence of a class of Characteristic semi-linear wave equations near $\mathscr{I}^+$

Introduction of the second part

Let $(\mathbb{R}^{n+1}_x, \eta_x)$ be the usual Minkowski space time with the global canonical coordinates system (x^μ). We denote by $\mathcal{C}^+_{a,x}$ the translated half cone of equation $x^0 = r + a$ where $a > 0,\ r^2 = \sum\limits_{i=1}^{n}(x^i)^2,\ r \geq 0$ and by $\mathcal{Y}^+_{a,x}$ the interior of $\mathcal{C}^+_{a,x}$, that is the set of points (x^μ) such that $x^0 > r + a$ (see Figure 5.1). In this work, we are interested with the following characteristic semi-linear Cauchy problem

$$\begin{cases} \Box_{x,\eta_x} f &= & F(\cdot, f, \partial f) & \text{in } \ \mathcal{Y}^+_{a,x} \\ f &= & \varphi & \text{on } \ \mathcal{C}^+_{a,x} \end{cases} \tag{5.0.8}$$

where $\eta_x = (\eta^{\alpha\beta})$ is the Minkowski metric on $\mathbb{R}^{n+1}_x$, $\eta = diag(-1, +1, \ldots, +1)$, $\Box_{x,\eta_x}$ the flat wave operator,

$$f = (f^I), \quad \partial f = \left(\frac{\partial f^I}{\partial x^\alpha}\right), \quad F = (F^I), \quad \alpha = 0, 1 \ldots, n,\ I = 1, \ldots, N\ ,$$

and

$$\varphi = (\varphi^I), \quad \text{the initial data prescribed on} \quad \mathcal{C}^+_{a,x}\ .$$

There exists in the literature a complete study (even in the quasi-linear case) of problem (5.0.8) near the tip of the cone $\mathcal{C}^+_{a,x}$, see the series of papers [8,25,27,28] and the references therein; compare [35,44,45] for a very general treatment of Lipschitz initial data hypersurfaces for the linear wave equation. Under suitable conditions on the source term and/or on the initial data, in these papers, it is shown that, in the semi-linear or quasi-linear case, there exists a neighborhood of the tip of the initial cone in $\mathcal{Y}^+_{a,x}$ on which one can find a unique solution. As far as the global solution of (5.0.8) is concerned, a lot remains to be done. It is well known that for an arbitrary nonlinear *function F, in general it is not possible to solve globally or semi-globally this problem, that is, without restriction on F and/or the space dimension n, it is not possible to find a neighborhood of the whole half cone $\mathcal{C}^+_{a,x}$ on which we can get existence and uniqueness of solution of such problem. In [5], A. Cabet gave some example of nonlinearities for which the solution develops singularities in finite time regardless the smallness and/or the smoothness of the initial data in the case $n = 1$. To the best of our knowledge, three types of nonlinearities have been considered so far, leading to global or semi-global solution of (5.0.8):*

- *In [32], M. Dossa and F. Touadera assume that the space dimension n is odd and greater than or equal to 3, that the source term $F = F(f, \partial f)$ is such that $F(0,0) = F'(0,0) = 0$ and $F^{(2)}$ satisfies the null condition of S. Klainerman when $n = 3$. With these conditions on the nonlinear term and the space dimension n, it is shown that if the initial data prescribed on the light cone are small in some appropriate norms, then (5.0.8) has a global solution in the whole interior of the initial cone.*

- *In [30], the authors suppose that, the restriction to the initial cone of the functions $F(x^\mu, f(x^\mu), \partial f(x^\mu))$ is a linear function with respect to the restriction to same cone of the derivatives of the unknown function $f(x^\mu)$ with respect to x^0. With this hypothesis, they proved that there exists a neighborhood of the entire initial cone on which problem (5.0.8) has a unique solution. We notice that this result does not guarantee that the thickness of the obtained neighborhood does not shrink to zero as one approaches infinity.*

- *In [5, 31, 36], analogous characteristic Cauchy problem are considered with initial data specify on two intersecting smooth null hypersurfaces under some suitable null condition on F. The results of these last references combined with local existence results on a neighborhood of the tip of the cone $\mathcal{C}^+_{a,x}$ of [26, 27] can also permit to study problem (5.0.8) under the condition that F is linear with respect to the derivatives of the unknown function in the normal direction of the initial cone $\mathcal{C}^+_{a,x}$. Indeed, assuming this, one succeed in concluding as in the previous case. We should point here that in the reference [5], it remains to fix a problem of regularity of initial data and of dependance of some constants used in the iterative scheme on λ. In that reference, the definition of the surface element $dS' = e^{-\lambda\Psi_+}dS$ on the slices $N^-_u|_V \equiv [0, V] \times Y$ in the unnumbered equation after equation 4.2 page 2115, implies that the Sobolev constant c' of equation 4.4 page 2116, depends on λ. In* *fact as it is said there, $c' = c_s e^{\lambda V}$ where c_s is an universal Sobolev constant coming from the embedding $H^m(U) \hookrightarrow C^1(U)$, U subset of $\mathbb{R}^n$ and $m > \frac{n}{2} + 1$. The consequence is that the constant $\tilde{c}_3(\rho)$ might depends exponentially on λ and it will not be possible to choose λ such that $\tilde{c}_3(\rho) - \lambda\tilde{c} \leq 0$ as stated there.*

The difficulty here is due to the fact that, in the processus of solving such problem, one needs to estimate the outgoing derivatives of the unknown function on the initial cone. The null property of this cone does not allow to choose arbitrarily the first of these derivatives as it is the case in the classical Cauchy problem. In order to obtain global solution, we need to solve globally a nonlinear ordinary differential equation with a nonlinear part which is exactly F. In the third case we mentioned above, this equation is linear and thus can be globally solved on $\mathcal{C}^+_{a,x}$. We intend in this paper to show that there exists a future neighborhood not only of the entire null cone $\mathcal{C}^+_{a,x}$ but also by guaranteeing that the thickness of this neighborhood does not nullify when one reaches infinity, on which there exists a unique solution of (5.0.8) . To do this, we shall impose on the function F a hypothesis of nullity of the kind of [20], see hypothesis 4.21 of this reference. More precisely, we shall suppose that the function F has a uniform zero in $(f, \partial f) = 0$ *of order r which is related to the space dimension (regarless the fact that n is odd or not) by the condition*

$$n \geq 1 + \frac{4}{r-1} - 2\alpha \ ,$$

and that the initial data φ are in some weighted Sobolev spaces near the conformal infinity. The strategy here will be based on the techniques of conformal method used in [11, 12] by P.T. Chruściel and R. T. Wafo in the case of classical Cauchy problem, the method of iterative scheme introduced in [42] by A. Majda and repeated by A. Cabet in [5] and R. Racke in [46] and finally the method of local solution developed by M. Dossa in [25–28].

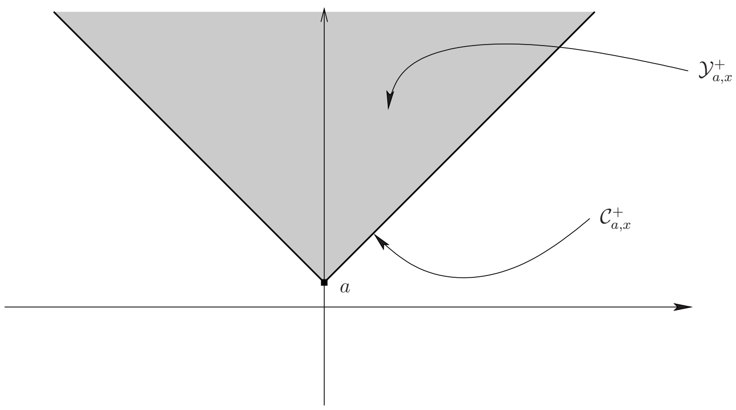

Figure 5.1: Characteristic cone $\mathcal{C}^+_{a,x}$ and its interior.

Chapter 6

Transformation of the system

6.1 Conformal transformation

Let $\mathcal{C}_{0,x}$ be the light cone of Minkowski space $\mathbb{R}^{n+1}_x$ of equation $(x^0)^2 = r^2$. We will denote by $\mathcal{C}^+_{0,x}$ and $\mathcal{C}^-_{0,x}$ the future and past light cone of the origin of coordinate respectively, by $\mathcal{Y}^+_{0,x}$ the interior of $\mathcal{C}^+_{0,x}$, by $\mathcal{Y}^-_{0,x}$ the interior of $\mathcal{C}^-_{0,x}$ and by $\mathcal{V}^+_{a,x}$ the set $\{(x^i) \in \mathbb{R}^n_x, (r+a, x^i) \in \mathcal{C}^+_{a,x}\}$, which is the projection of the cone $\mathcal{C}^+_{a,x}$ on the space variables. As in the first part of the thesis, we consider the map ϕ defined as:

$$\phi : \mathbb{R}^{n+1}_x \setminus \mathcal{C}_{0,x} \to \mathbb{R}^{n+1}_y \ \ by \ \ x^\alpha \mapsto y^\alpha := \frac{x^\alpha}{\eta_{\lambda\mu} x^\lambda x^\mu} \ , \ \alpha = 0, 1, \ldots, n \ . \quad (6.1.1)$$

Note that $\phi(\mathbb{R}^{n+1}_x \setminus \mathcal{C}_{0,x}) \subset \mathbb{R}^{n+1}_y$. Any of the sets defined above in $\mathbb{R}^{n+1}_x$ has its counterpart in $\mathbb{R}^{n+1}_y$, we keep the same notations. The indices x or y will be used to indicate if the set under consideration is a subset of $\mathbb{R}^{n+1}_x$ or $\mathbb{R}^{n+1}_y$. As an example, the set $\mathcal{C}_{0,y}$ is the light cone with vertex the origin of coordinates in $\mathbb{R}^{n+1}_y$, its equation is given by $(y^0)^2 = \rho^2$ where $\rho^2 = \sum_{i=1}^{n} (y^i)^2$. We have the following

Proposition 6.1.1 *The map ϕ is a bijection from $\mathcal{Y}^+_{0,x}$ onto $\phi(\mathcal{Y}^+_{0,x}) = \mathcal{Y}^-_{0,y}$, with inverse*

$$\phi^{-1} : y^\alpha \mapsto x^\alpha \quad by \quad x^\alpha := \frac{y^\alpha}{\eta_{\lambda\mu} y^\lambda y^\mu} \ . \quad (6.1.2)$$

ϕ is also a bijection from $\mathcal{Y}^+_{a,x}$ onto the relatively compact domain $\phi(\mathcal{Y}^+_{a,x}) = \mathcal{Y}^+_{-\frac{1}{a},y} \cap \mathcal{Y}^-_{0,y}$ (see Figure 6.1) with the same inverse as in (6.1.2).

Proof: *Let* $(x^\alpha) \in \mathcal{Y}^+_{0,x}$, *if* $(y^\alpha) = \phi(x^\alpha)$ *then we have*

$$(\eta_{\alpha\beta}x^\alpha x^\beta)(\eta_{\mu\nu}y^\mu y^\nu) = 1$$

and thus $y^\alpha = \frac{x^\alpha}{\eta_{\lambda\mu}x^\lambda x^\mu}$ *implies that* $x^\alpha = y^\alpha(\eta_{\lambda\mu}x^\lambda x^\mu) = \frac{y^\alpha}{\eta_{\lambda\mu}y\lambda y^\mu}$. *Therefore,* ϕ *is a bijection from* $\mathcal{Y}^+_{0,x}$ *onto* $\phi(\mathcal{Y}^+_{0,x})$ *with inverse given by (6.1.2). On the other hand, let* $(x^\alpha) \in \mathcal{Y}^+_{a,x}$ *and suppose* $(y^\alpha) = \phi(x^\alpha)$, *then*

$$\begin{aligned}
(x^\alpha) \in \mathcal{Y}^+_{a,x} \quad &\textit{if and only if} \quad \begin{cases} \eta_{\alpha\beta}x^\alpha x^\beta < 0 \\ x^0 > 0 \end{cases} \\
&\textit{if and only if} \quad \begin{cases} \eta_{\alpha\beta}\frac{y^\alpha y^\beta}{(\eta_{\sigma\lambda}y^\sigma y^\lambda)^2} < 0 \\ \frac{y^0}{\eta_{\sigma\lambda}y^\sigma y^\lambda} > 0 \end{cases} \\
&\textit{if and only if} \quad \begin{cases} \eta_{\sigma\lambda}y^\sigma y^\lambda < 0 \\ y^0 < 0 \end{cases} \\
&\textit{if and only if} \quad (y^\alpha) = \phi(x^\alpha) \in \mathcal{Y}^-_{0,x} .
\end{aligned}$$

thus, $\phi(\mathcal{Y}^+_{0,x}) = \mathcal{Y}^-_{0,y}$. *Similar calculations establish the second part of the proposition.* □

6.2 Transformed wave equation

In this section, we want to show how the wave equation (5.0.8) transform under the change of coordinates (6.1.1). For this purpose, we set

$$\Omega = -\eta_{\alpha\beta}y^\alpha y^\beta \qquad \textit{and} \qquad \hat{f} = \Omega^{-\frac{n-1}{2}} f \circ \phi^{-1} . \tag{6.2.1}$$

We have the following

Proposition 6.2.1 *The identity*

$$\Box_{x,\eta_x} f = \Omega^{\frac{n+3}{2}} \Box_{y,\eta_y} \hat{f} \tag{6.2.2}$$

holds.

Proof: *We have:*

$$\begin{aligned}
\frac{\partial f}{\partial x^\mu} &= A^\alpha_\mu \frac{\partial f \circ \phi^{-1}}{\partial y^\alpha}, & (6.2.3)\\
\frac{\partial^2 f}{\partial x^\lambda \partial x^\mu} &= \frac{\partial^2 (f\circ\phi^{-1})}{\partial y^\alpha \partial y^\beta} A^\alpha_\mu A^\beta_\lambda + \frac{\partial (f\circ\phi^{-1})}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\lambda} & (6.2.4)
\end{aligned}$$

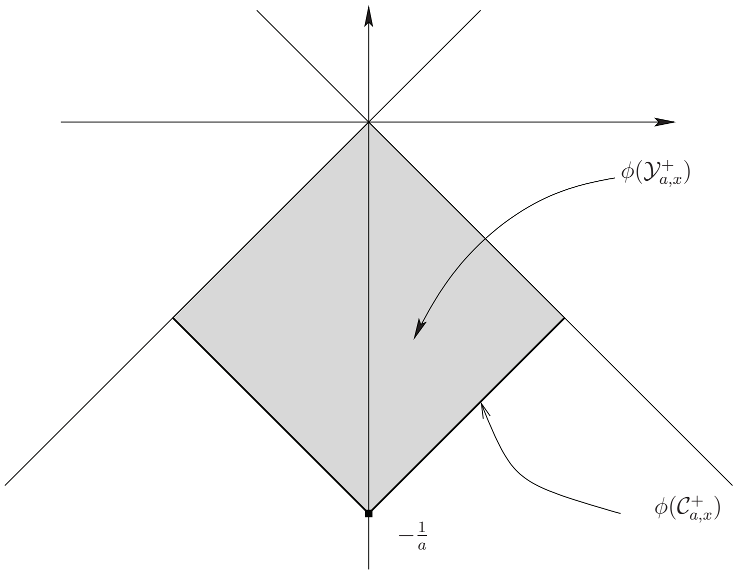

Figure 6.1: Images of the unbounded domain $\mathcal{Y}^+_{a,x}$ and the cone $\mathcal{C}^+_{a,x}$ with respect to the conformal map ϕ .

with

$$\begin{aligned} A^\alpha_\mu &= \frac{\partial y^\alpha}{\partial x^\mu} = -\delta^\alpha_\mu \Omega - 2y^\alpha y_\mu \quad \text{and} \\ A^\alpha_\mu A^\beta_\lambda &= \Omega^2 \delta^\alpha_\mu \delta^\beta_\lambda + 4y_\lambda y_\mu y^\alpha y^\beta + 2\Omega(\delta^\alpha_\mu y_\lambda y^\beta + \delta^\beta_\lambda y_\mu y^\alpha) \\ \frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\lambda} &= 2\Omega \delta^\alpha_\mu y_\lambda + 2\Omega \delta^\alpha_\lambda y_\mu + 2\Omega \eta_{\mu\lambda} y^\alpha + 8y_\lambda y_\mu y^\alpha \ . \end{aligned}$$

From these identities, we have:

$$\eta^{\lambda\mu} A^\alpha_\mu A^\beta_\lambda = \Omega^2 \eta^{\alpha\beta} \quad \text{and} \quad \eta^{\lambda\mu} \frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\lambda} = 2(n-1) y^\alpha \ .$$

It then follows that:

$$\Box_{x,\eta} f = \Omega^2 \Box_{y,\eta}(f \circ \phi^{-1}) + 2(n-1) y^\alpha \frac{\partial (f \circ \phi^{-1})}{\partial y^\alpha} \ . \tag{6.2.5}$$

Now, $\hat{f} = \Omega^{-\frac{n-1}{2}} f \circ \phi^{-1}$ *implies that* $f \circ \phi^{-1} = \Omega^{\frac{n-1}{2}} \hat{f}$ *and using identity* $\frac{\partial \Omega}{\partial y^\alpha} = -2y_\alpha$*, one is led to*

$$\frac{\partial (f \circ \phi^{-1})}{\partial y^\alpha} = \Omega^{\frac{n-3}{2}} \left\{ \Omega \frac{\partial \hat{f}}{\partial y^\alpha} - (n-1) y_\alpha \hat{f} \right\} \tag{6.2.6}$$

and thus,

$$y^\alpha \frac{\partial (f\circ\phi^{-1})}{\partial y^\alpha} = \Omega^{\frac{n-1}{2}} \left\{ y^\alpha \frac{\partial \hat f}{\partial y^\alpha} + (n-1)\hat f \right\} . \qquad (6.2.7)$$

On the other hand, we have:

$$\begin{aligned} \frac{\partial^2 (f\circ\phi^{-1})}{\partial y^\alpha \partial y^\beta} &= \Omega^{\frac{n-1}{2}} \frac{\partial^2 \hat f}{\partial y^\alpha \partial y^\beta} + (1-n)\Omega^{\frac{n-3}{2}} \left\{ y_\alpha \frac{\partial \hat f}{\partial y^\beta} + y_\beta \frac{\partial \hat f}{\partial y^\alpha} \right\} \\ &\quad + (1-n)\Omega^{\frac{n-5}{2}} \left\{ (3-n) y_\alpha y_\beta + \Omega \eta_{\alpha\beta} \right\} \hat f . \end{aligned}$$

From this last identity and from identities (6.2.6) and (6.2.7) we deduce that:

$$\Box_{y,\eta}(f\circ\phi^{-1}) = \Omega^{\frac{n-1}{2}} \Box_{y,\eta}\hat f - 2(n-1)^2 \Omega^{\frac{n-3}{2}} \hat f + 2(1-n)\Omega^{\frac{n-3}{2}} y^\mu \frac{\partial \hat f}{\partial y^\mu} .$$

Replacing now this expression of $\Box_{y,\eta}(f\circ\phi^{-1})$ in (6.2.5) and simplifying using identity (6.2.7), one obtains:

$$\Box_{x,\eta} f = \Omega^{\frac{n+3}{2}} \Box_{y,\eta} \hat f .$$

This complete the proof. $\Box$

If we use expression (6.2.6) in (6.2.3) we obtain:

$$\frac{\partial f}{\partial x^\mu} = \Omega^{\frac{n-1}{2}} \left\{ (1-n) y_\mu \hat f - \Omega \frac{\partial \hat f}{\partial y^\mu} - 2 y_\mu y^\alpha \frac{\partial \hat f}{\partial y^\alpha} \right\} ; \qquad (6.2.8)$$

thus the right-hand side of equation (5.0.8) reads:

$$\begin{aligned} F\left(x^\nu, f(x^\nu), \partial_\mu f(x^\nu)\right) &= F\left(\phi^{-1}(y^\nu), f\circ\phi^{-1}(y^\nu), \partial_\mu f(x^\nu)\right) \\ &= F\left(\phi^{-1}(y^\nu), \Omega^{\frac{n-1}{2}} \hat f, \Omega^{\frac{n-1}{2}} \left\{ (1-n) y_\mu \hat f - \Omega \frac{\partial \hat f}{\partial y^\mu} - 2 y_\mu y^\alpha \frac{\partial \hat f}{\partial y^\alpha} \right\}\right) \\ &\equiv \widetilde F\left(y^\nu, \Omega^{\frac{n-1}{2}} \hat f, \Omega^{\frac{n-1}{2}} \frac{\partial \hat f}{\partial y^\mu}\right) . \end{aligned}$$

We obtain that under the coordination transformation (6.1.1) and the rescaling (6.2.1), the wave equation (5.0.8) read:

$$(E_0) \quad \left\{ \begin{array}{l} \Box_{y,\eta} \hat f = \Omega^{-\frac{n+3}{2}} \widetilde F\left(y^\nu, \Omega^{\frac{n-1}{2}} \hat f, \Omega^{\frac{n-1}{2}} \frac{\partial \hat f}{\partial y^\mu}\right) \quad in \quad \phi(\mathcal{Y}^+_{a,x}) \\ \hat f = \hat\varphi \quad on \quad \phi(\mathcal{C}^+_{a,x}) \end{array} \right. \qquad (6.2.9)$$

where

$$\hat{\varphi} = \left(\Omega^{-\frac{n-1}{2}} f \circ \phi^{-1}\right)\Big|_{\phi(\mathcal{C}^+_{a,x})}.$$

REMARK 6.2.2 1. On the behavior of $\hat{\varphi}$: Since

$$\Omega = (y^0)^2 - \rho^2 \text{ and } \phi(\mathcal{C}^+_{a,x}) \subset \{(y^\alpha) \in \mathcal{Y}^-_{-\frac{1}{a},y} : y^0 = \rho - \frac{1}{a}\},$$

we have $\Omega|_{\phi(\mathcal{C}^+_{a,x})} = \frac{1}{a}\left(\frac{1}{a} - 2\rho\right), \ 0 \le \rho < \frac{1}{2a}$. It then follows that, for any $(y^i) \in \mathbb{R}^{n+1}_y$ such that $(\rho - \frac{1}{a}, y^i) \in \phi(\mathcal{C}^+_{a,x})$,

$$\hat{\varphi}(y^i) = \left(\frac{1}{a}\left(\frac{1}{a} - 2\rho\right)\right)^{-\frac{n-1}{2}} \varphi(x^i),$$

(x^i) being such that $(\rho - \frac{1}{a}, y^i) = \phi(r + a, x^i)$. These calculations show that it will be possible to choose the initial data φ such that the initial data $\hat{\varphi}$ of the transformed equations (6.2.9) are smooth on the whole $\phi(\mathcal{C}^+_{a,x})$ as long as $\{\rho < \frac{1}{2a}\}$. In general these data will be singular at $\{\rho = \frac{1}{2a}\}$.

2. To the system 6.2.9 we can apply the results of [28] to obtain that there exists a neighborhood which will be denoted by $V_{0,y}$ (see Figure 6.2) of the tip of the cone $\phi(\mathcal{C}^+_{a,x})$ on which (6.2.9) has a unique smooth solution. We denote this local solution by $\hat{f}_0$.

6.3 Goursat problem associated to the transformed system

As in [7], we consider the Cauchy problem associated to the wave equation (6.2.9) with prescribed data on two truncated (such as to get rid of the tips) intersecting cones $\mathcal{C}^+ \subset \mathcal{C}^+_{-\frac{1}{a},y} \cap \mathcal{Y}^-_{0,y}$ *and* $\mathcal{C}^- \subset \mathcal{C}^-_{\lambda,y} \cap \mathcal{Y}^+_{-\frac{1}{a},y}$*, where* λ *is a fixed parameter belonging to the interval* $]-\frac{1}{a}, 0[$ *sufficently close to* $-\frac{1}{a}$ *such that* $\mathcal{C}^-_{\lambda,y} \cap \mathcal{Y}^+_{-\frac{1}{a},y}$ *intercepts* $V_{0,y}$ *(see Figures 6.2 and 6.3):*

$$\Box_{y,\eta}\hat{f} = \Omega^{-\frac{n+3}{2}} \widetilde{F}\left(y^\nu, \Omega^{\frac{n-1}{2}}\hat{f}, \Omega^{\frac{n-1}{2}}\frac{\partial \hat{f}}{\partial y^\mu}\right) \quad ; \tag{6.3.1}$$

in the future neighborhood of $\mathcal{C}^+ \cup \mathcal{C}^-$ *with initial data*

$$\hat{f} = \hat{\varphi} \quad on \quad \mathcal{C}^+ \quad and \quad \hat{f} = \hat{f}_0 \quad sur \quad \mathcal{C}^- \; ; \tag{6.3.2}$$

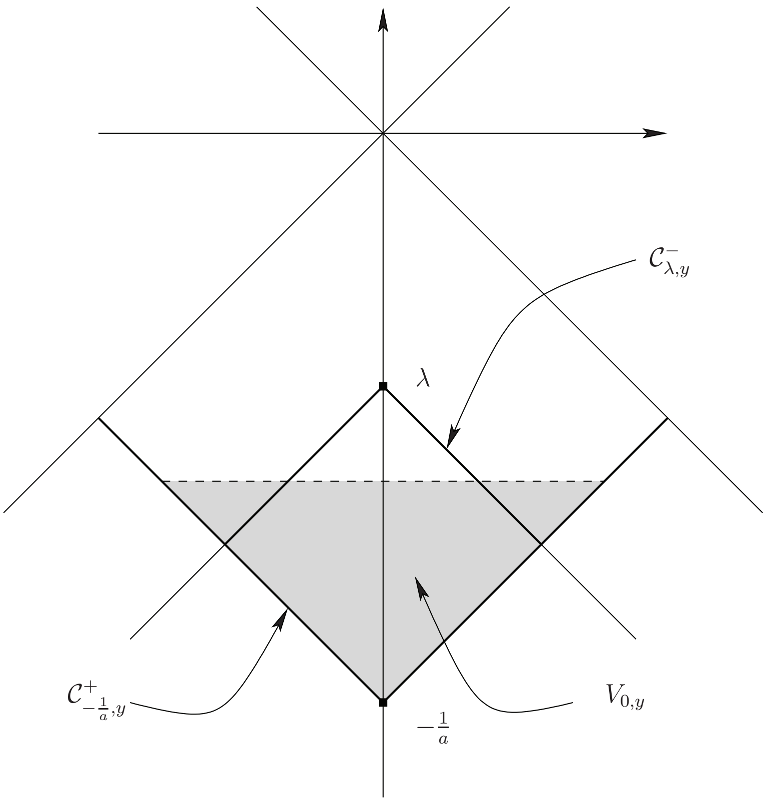

Figure 6.2: Neighborhood $V_{0,y}$ of the tip of the cone $\mathcal{C}^+_{-\frac{1}{a},y}$ and the cone $\mathcal{C}^-_{\lambda,y}$.

where $\hat{f}_0$ is the smooth function given by the second item of Remark 6.2.2 in the neighborhood $V_{0,y}$ of the tip $(-\frac{1}{a}, 0)$. We will be concerned now in deriving a global process which solves (6.3.1)-(6.3.2). The next chapter is devoted to this goal.

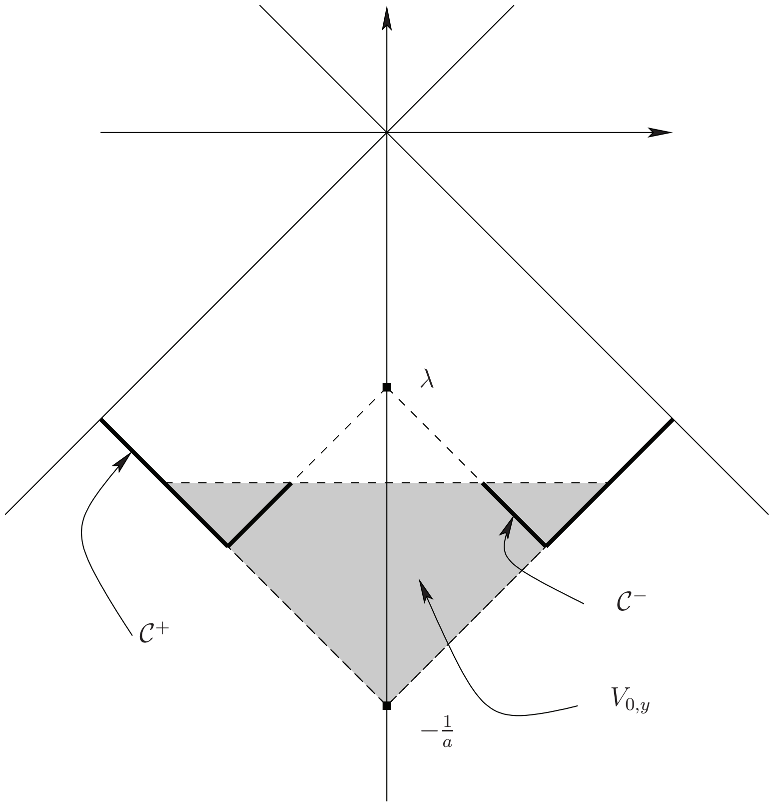

Figure 6.3: Truncated cones $\mathcal{C}^+$ et $\mathcal{C}^-$.

Chapter 7

Existence and uniqueness theorem

7.1 Second transformation

In the space $\mathbb{R}^{n+1}_y$ *we consider now the spherical coordinates* (τ, ρ, θ) *defined as:*

$$\begin{cases} \tau = y^0 \ , \\ \rho = \big(\sum\limits_{i=1}^{n} (y^i)^2 \big)^{1/2} \ , \\ y^i = \rho \omega^i(\theta), \ i = 1, \dots, n \end{cases} \qquad \text{with} \qquad \begin{cases} \omega^1 = \cos\theta^1 \\ \omega^2 = \sin\theta^1 \cos\theta^2 \\ \omega^3 = \sin\theta^1 \sin\theta^2 \cos\theta^3 \\ \dots \quad \dots \\ \omega^{n-1} = \sin\theta^1 \sin\theta^2 \dots \sin\theta^{n-2} \cos\theta^{n-1} \\ \omega^n = \sin\theta^1 \sin\theta^2 \dots \sin\theta^{n-2} \sin\theta^{n-1} \end{cases}$$

where $0 < \theta^{n-1} < 2\pi$ *and* $0 < \theta^i < \pi, \ i = 1, 2, \dots, n-2$. *We set:*

$$\begin{cases} x = \tau + \rho \le 0 \\ y = \tau - \rho + \frac{1}{a} \ge 0 \end{cases} \qquad i.e. \qquad \begin{cases} \tau = \frac{1}{2}(y + x - \frac{1}{a}) \\ \rho = \frac{1}{2}(\frac{1}{a} + x - y) \end{cases} \ . \tag{7.1.1}$$

We have the following

Proposition 7.1.1 *In the new coordinate system* (y, x, θ), *we have the identity*

$$\Box_{y,\eta} = -4\partial_x\partial_y + \frac{n-1}{\rho}(\partial_x - \partial_y) + \frac{\Delta_{S^{n-1}}}{\rho^2} \tag{7.1.2}$$

where $\Delta_{S^{n-1}}$ *is the Laplace-Beltrami operator on the sphere* S^{n-1} *endowed with its canonical metric.*

Proof: *From equation (7.1.1) we have*

$$dy^0 = \frac{1}{2}(dx+dy) \quad \textit{and} \quad dy^i = \frac{1}{2}\omega^i(dy-dx)+\rho d\omega^i$$

thus,

$$(dy^0)^2 = \frac{1}{4}\big((dy)^2+(dx)^2+dx\otimes dy+dy\otimes dx\big)$$

and

$$\begin{aligned}(dy^i)^2 &= \frac{1}{4}(\omega^i)^2\big((dy)^2+(dx)^2-dx\otimes dy-dy\otimes dx\big)\\ &\quad +\rho^2 d(\omega^i)^2+\frac{1}{2}\omega^i\rho\left\{(dx-dy)\otimes(d\omega^i)+(d\omega^i)\otimes(dx-dy)\right\}\ .\end{aligned}$$

We then obtain (recall that $\sum_{i=1}^{n}\omega^i d\omega^i = 0$*),*

$$-(dy^0)^2+\sum_{i=1}^{n}(dy^i)^2 = -\frac{1}{2}(dx\otimes dy+dy\otimes dx)+\rho^2\sum_{i=1}^{n}d(\omega^i)^2. \qquad (7.1.3)$$

If we denote by h *the round metric on the sphere, that is the metric induced on* $\mathbb{S}^{n-1}$ *by the Euclidean metric of* $\mathbb{R}^n$ *then, identity (7.1.3) takes the form:*

$$\eta_y = -\frac{1}{2}(dx\otimes dy+dy\otimes dx)+\rho^2 h\ .$$

The matrix of the metric η_y *thus has the form:*

$$\Big(\eta_y\Big)_{\alpha\beta} = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & \dots & 0\\ -\frac{1}{2} & 0 & 0 & \dots & 0\\ 0 & 0 & & & \\ \vdots & \vdots & & \rho^2 h_{AB} & \\ 0 & 0 & & & \end{pmatrix} \qquad (7.1.4)$$

where the h_{AB}*'s are the components of the metric* h *of* S^{n-1} *in the coordinates* θ^A, $A=1,2,\dots,n-1$. *It then follows that:*

$$\sqrt{|\det\eta|} = \frac{1}{2}\rho^{n-1}\sqrt{|\det h|}\ .$$

We then obtain the matrix form of the inverse metric $\eta_y^\sharp$

$$\Big(\eta_y\Big)^{\alpha\beta} = \begin{pmatrix} 0 & -2 & 0 & \dots & 0\\ -2 & 0 & 0 & \dots & 0\\ 0 & 0 & & & \\ \vdots & \vdots & & \frac{1}{\rho^2}h^{AB} & \\ 0 & 0 & & & \end{pmatrix}\ . \qquad (7.1.5)$$

Now we have:

$$\begin{aligned}\Box_{\eta,y} &= \frac{1}{\sqrt{|\widetilde{\eta}_{\alpha\beta}|}}\partial_\mu\big(\sqrt{|\widetilde{\eta}_{\alpha\beta}|}\,\widetilde{\eta}^{\mu\nu}\partial_\nu\big)\\ &= \frac{2}{\rho^{n-1}\sqrt{|h_{AB}|}}\partial_\mu\Big(\frac{1}{2}\rho^{n-1}\sqrt{|h_{AB}|}\,\widetilde{\eta}^{\mu\nu}\partial_\nu\Big)\\ &= \frac{1}{\rho^{n-1}\sqrt{|h_{AB}|}}\partial_\mu\Big(\rho^{n-1}\sqrt{|h_{AB}|}\,\widetilde{\eta}^{\mu\nu}\partial_\nu\Big)\\ &= -4\partial_x\partial_y+\frac{n-1}{\rho}\Big(\partial_x-\partial_y\Big)+\frac{\Delta_{S^{n-1}}}{\rho^2}\ , \end{aligned}\qquad (7.1.6)$$

where $\Delta_{S^{n-1}}$ *is the canonical Laplace-Beltrami operator on* $\mathbb{S}^{n-1}$. □

From this proposition, we deduce the new form of the transformed equation (6.2.9) with respect to the new coordinates system $z := (y, x, \theta)$:

$$\begin{cases} -4\partial_x\partial_y\hat{f}+\frac{n-1}{\rho}\Big(\partial_x-\partial_y\Big)\hat{f}+\frac{\Delta_{S^{n-1}}\hat{f}}{\rho^2}=\Omega^{-\frac{n+3}{2}}\widetilde{F}\left(z,\Omega^{\frac{n-1}{2}}\hat{f},\Omega^{\frac{n-1}{2}}\frac{\partial\hat{f}}{\partial y^\mu}\right) \ \textit{in}\ \phi(\mathcal{Y}^+_{a,x})\\ \hat{f}=\hat{\varphi}\quad \textit{on}\quad \phi(\mathcal{C}^+_{a,x})\end{cases}\qquad (7.1.7)$$

Remark 7.1.2 We emphasise the fact that $\Omega = -x(1/a - y)$ and $y^\mu\frac{\partial}{\partial y^\mu} = x\partial_x + (y-1/a)\partial_y$. Thus by identity (6.2.8), we will suppose without further restriction on F that when replacing the first order derivatives $\frac{\partial\hat{f}}{\partial y^\mu}$ in $\widetilde{F}$ by their value in terms of $\partial_y,\ \partial_x,\ \partial_A$, any derivative $\partial_x\hat{f}$ comes with a pre-factor x.

Remark 7.1.3 In the coordinates system (τ, ρ, θ) the Minkowski metric reads:

$$\eta = -(d\tau)^2+\sum_{i=1}^{n}(dy^i)^2 = -(d\tau)^2+(d\rho)^2+\rho^2 ds^2$$

where

$$\begin{aligned} ds^2 &= (d\theta^1)^2+\sin^2\theta^1(d\theta^2)^2+\sin^2\theta^1\sin^2\theta^2(d\theta^3)^2+\dots\\ &+\ \sin^2\theta^1\sin^2\theta^2\dots\sin^2\theta^{n-2}(d\theta^{n-1})^2 \end{aligned}\qquad (7.1.8)$$

thus

$$\eta = -(d\tau)^2+(d\rho)^2+\rho^2h^2_{AB}d\theta^A d\theta^B,$$

with $h_{AB} = 0$ if $A \neq B$ and $h_{AA},\ A = 1,\dots,n-1$ being defined by equation (7.1.8). The inverse metric is then given by

$$\eta^\sharp = -(\partial_\tau)^2+(\partial_\rho)^2+h^{AB}\partial_{\theta^A}\partial_{\theta^B}\quad\text{with}\quad h^{AB}=\begin{cases}0 & \text{if}\quad A\neq B\\ \frac{1}{\rho^2 h_{AA}} & \text{if}\quad A=B\end{cases}.$$

Remark 7.1.4

- $\phi(\mathcal{Y}^+_{a,x}) = \mathcal{Y}^+_{-\frac{1}{a},y} \cap \mathcal{Y}^-_{0,y} = \{(y,x,\theta) : -\frac{\sqrt{2}}{2a} \le x \le 0;\ 0 \le y \le \frac{\sqrt{2}}{2a}\}$;

$$\phi(\mathcal{C}^+_{a,x}) = \mathcal{C}^+_{-\frac{1}{a},y} \cap \mathcal{Y}^-_{0,y} = \{(y,x,\theta) : y = 0;\ -\frac{\sqrt{2}}{2a} \le x \le 0\} .$$

- $\frac{1}{a} + x - y = 0$ is equivalent to $\rho = 0$, thus the function $x \longmapsto \frac{1}{\frac{1}{a} - x - y}$ is well defined as far as one does not reach $\{\rho = 0\}$ (which will be the case in the domain of interest).

- Setting $x_0 = \sqrt{2}/2\lambda$, $y_0 = \sqrt{2}/2(\lambda + 1/a)$ we have:

$$\mathcal{C}^- = \mathcal{C}^-_{\lambda,y} \cap \mathcal{Y}^+_{-\frac{1}{a},y} = \{(y,x,\theta) : x = x_0,\ 0 \le y \le y_0\} .$$

and

$$\mathcal{C}^+ = \{(y,x,\theta) : y = 0,\ x_0 \le x < 0\} .$$

7.2 Functional spaces

We intend in this section to describe the slices (see Figure 7.1) on which we will get our energy estimates. Let $z := (y,x,\theta)$, *be a generic point and denote by* $\mathcal{D}$ *the set defined by* $\mathcal{D} = [0,y_0] \times [x_0, 0[\times \mathscr{O}$, *where* $\mathscr{O}$ *is a subset of the unit sphere* $\mathbb{S}^{n-1}$ *of* $\mathbb{R}^n$. *For any* $(u,v) \in [0,y_0] \times [x_0,0[$, *we set*

$$\mathcal{D}_{u,v} = [0,u] \times [x_0,v] \times \mathscr{O} ,$$

$$\mathcal{C}^+_{u,v} = \{u\} \times [x_0,v] \times \mathscr{O} = \{(y,x,\theta) :\ y = u;\ x_0 \le x \le v\}$$

and

$$\mathcal{C}^-_{u,v} = [0,u] \times \{v\} \times \mathscr{O} = \{(y,x,\theta) :\ 0 \le y \le u;\ x = v\} .$$

Thus,

$$\mathcal{D}_{u,v} = \underset{0 \le y \le u}{\cup} \mathcal{C}^+_{y,v} = \underset{x_0 \le x \le v}{\cup} \mathcal{C}^-_{u,x} .$$

For $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, *we set* $\partial^\beta = \frac{\partial^{|\beta|}}{(\partial x)^{\beta_1}(\partial\theta^1)^{\beta_2} \ldots (\partial\theta^{n-1})^{\beta_n}}$; *for* $\beta = (\beta_0, \beta_1, \ldots, \beta_n) \in \mathbb{N}^{1+n}$, *we set* $D^\beta = \frac{\partial^{|\beta|}}{(\partial y)^{\beta_0}(\partial x)^{\beta_1}(\partial\theta^1)^{\beta_2} \ldots (\partial\theta^{n-1})^{\beta_n}}$ *and* $\partial_\mu = \frac{\partial}{\partial y^\mu}$. *We recall that from (7.1.1), we have:*

$$\left\{ \begin{array}{l} \frac{\partial}{\partial x} = \frac{1}{2}\left(\frac{\partial}{\partial\tau} + \sum\limits_{i=1}^{n} \frac{y^i}{\rho}\frac{\partial}{\partial y^i}\right) = \frac{1}{2}\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\rho}\right) \\ \frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial\tau} - \sum\limits_{i=1}^{n} \frac{y^i}{\rho}\frac{\partial}{\partial y^i}\right) = \frac{1}{2}\left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\rho}\right) \end{array} \right. . \tag{7.2.1}$$

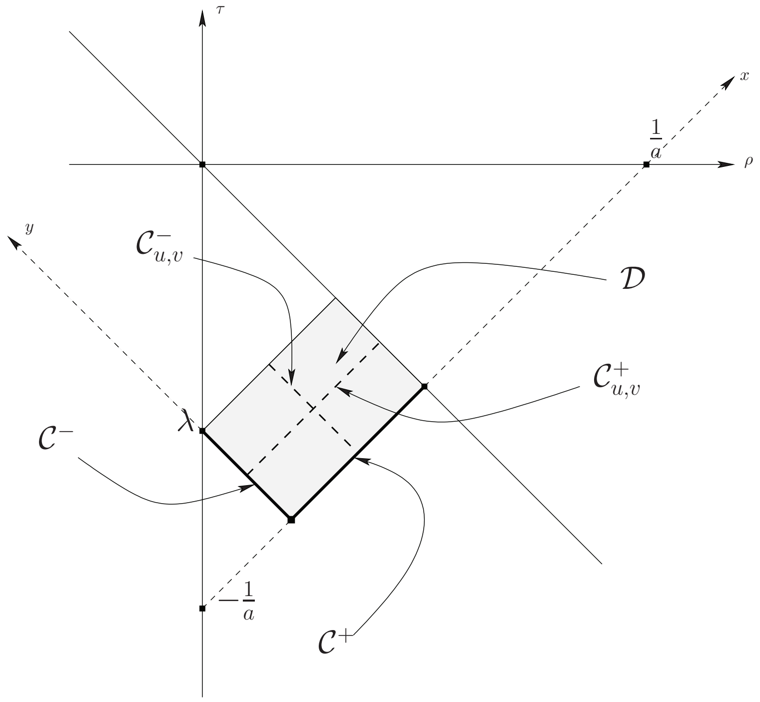

Figure 7.1: Future neighborhood $\mathcal{D}$ of the union of truncated cones $\mathcal{C}^+$ and $\mathcal{C}^-$.

Let $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and U a subset of $\mathbb{R}^{n+1}$. We will denote $H^m(U)$ the usual Sobolev space on U. Further for $U = \mathcal{C}^+$ or $\mathcal{C}^+_{u,v}$ we denote by

- *$\mathscr{C}^\alpha_0(U)$ the set of continuous functions f on U for which the quantity $\|f\|_{\mathscr{C}^\alpha_0(U)} := \sup_{p\in U} |x|^{-\alpha}|f(p)|$ is finite,*
- *$\mathscr{C}^\alpha_k(U)$ the set of $k-$times continuously differentiable functions f on U such that the quantity $\|f\|_{\mathscr{C}^\alpha_k(U)} := \sum_{0\le|\beta|\le k} \||x|^{\beta_1}\partial^\beta f\|_{\mathscr{C}^\alpha_0(U)}$ is finite,*
- *$\mathscr{H}^\alpha_k(U)$ the space of those functions f in $H^k_{loc}(U)$ for which the norm*

$$\|f\|^2_{\mathscr{H}^\alpha_k(U)} := \sum_{0\le|\beta|\le k} \int_U (|x|^{-\alpha+\beta_1}\partial^\beta f)^2 \frac{dx}{|x|} d\nu$$

  is finite. Here $d\nu$ is a measure on $\mathscr{O}$ arising from a smooth Riemannian metric on $\mathbb{S}^{n-1}$.

7.3 Existence and uniqueness for a Goursat problem

Let $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$. Let ω_0^-, be a defined function on $\mathcal{C}^-$ and ω_0^+, be defined on $\mathcal{C}^+$ such that

$$\omega_0^- \in H^{m+2}(\mathcal{C}^-); \quad \omega_0^+ \in \mathscr{H}^\alpha_{m+1}(\mathcal{C}^+) \quad and \quad \partial_x\omega_0^+ \in \mathscr{H}^\alpha_{m+1}(\mathcal{C}^+), \tag{7.3.1}$$

and satisfying the compatibility condition

$$\omega_0^- = \omega_0^+ \quad sur \quad \mathcal{C}^+ \cap \mathcal{C}^- = \mathscr{O} \ . \tag{7.3.2}$$

Actually, $\omega_0^+ \in \mathscr{H}^\alpha_{m+1}(\mathcal{C}^+)$ implies that $\partial_x\omega_0^+ \in \mathscr{H}^{\alpha-1/2}_m(\mathcal{C}^+)$, which will not be sufficient to obtain the control of some of our constants, we thus assume that $\partial_x\omega_0^+ \in \mathscr{H}^\alpha_{m+1}(\mathcal{C}^+)$. The purpose of this section is to state and prove an existence and uniqueness theorem for the following characteristic Cauchy problem ($z = (y, x, \theta)$):

$$\begin{cases} \Box_{y,\eta}\omega = x^{-\frac{n+3}{2}} G\left(z, x^{\frac{n-1}{2}}\omega, x^{\frac{n-1}{2}}(\partial_y\omega, x\partial_x\omega, \partial_A\omega)\right) \ in \ \mathcal{D} \\ \omega = \omega_0^+ \quad on \quad \mathcal{C}^+ \quad and \quad \omega = \omega_0^- \quad on \quad \mathcal{C}^- \end{cases} . \tag{7.3.3}$$

7.3.1 Hypothesis on the non linear term

In analogy with the procedure used in [20], we make the following assumption on the non linear source term G:

($\mathcal{H}$) *We suppose that the function $G = G(y^\mu, p, q)$, is of C^m class in all its variables and that the restriction $G(y, x)$ on every slice $\{y = const\} \cap \{x = const\}$ has a* uniform zero of order $r \geq 1$ *at $p = q = 0$ in the sense that, for all $B > 0$ there exists a constant $\hat{C}(B)$ such that for $0 \leq j + \ell + i_1 + i_2 \leq \min(r, m)$ and $|(p, q)| \leq B$ one has:*

$$\left\| \frac{|x|^{i_1}\partial^{j+\ell+i_1+i_2}G(y,\ x,\ .\ ,p,\ q)}{(\partial x)^{i_1}(\partial y)^{i_2}\partial p^j\partial q^\ell} \right\|_{C^{m-(j+\ell+i_1+i_2)}(\{y\}\times\{x\}\times\mathscr{O})} \leq \hat{C}(B)\,\|(p,q)\|^{r-j-\ell} \ . \tag{7.3.4}$$

We point out for later use that this hypothesis implies that for all $\sigma \geq 0$, there exists a constant $C(\hat{C}, r, m, \sigma, B)$ such that for all $f \in H^m(\mathscr{O})$, with $\|f\|_{L^\infty(\mathscr{O})} \leq B$, we have

$$\|G(y, x, \cdot, |x|^\sigma f)\|_{H^m(\mathscr{O})} \leq C|x|^{r\sigma}\|f\|_{H^m(\mathscr{O})} \ . \tag{7.3.5}$$

7.3.2 First inequality

As a first step towards an existence theorem of the characteristic Cauchy problem (7.3.3), we prove now the equivalent of Lemma 7.1.2. of [51]. Let $\ell,\ \Lambda \in \mathbb{R},\ \Lambda > 0$ *and* ω *a sufficiently differentiable function defined on* $\mathcal{Y}^+_{-\frac{1}{a},y}$*, set*

$$L^\ell[\omega] = H(y,x)(\partial_x\omega + \partial_y\omega)\Box_{\eta,y}\omega \quad \textit{with} \quad H(x,y) = (-x)^\ell e^{-\Lambda(y+x)},$$

and

$$\nabla\omega = (\partial_x\omega,\ \partial_y\omega,\ \partial_\theta\omega),\ \nabla_x\omega = (\partial_x\omega,\ \partial_\theta\omega),\ \nabla_y = (\partial_y\omega, \partial_\theta\omega)$$

where $\partial_\theta\omega = (\partial_{\theta^1}\omega, \ldots, \partial_{\theta^{n-1}}\omega)$ *. Assume that* c_0 *and* $\bar{c}_0$ *are two positive constants such that:*

$$c_0|X|^2 \leq X_x^2 + \sum_{A,B} \frac{h^{AB}X_AX_B}{2\rho^2} \leq \bar{c}_0|X|^2 \ . \tag{7.3.6}$$

we have the following proposition:

Proposition 7.3.1 *There exists a positive constant* c_1 *depending upon* $h, c_0, \bar{c}_0$ *and* n *such that for all* $\ell \geq 0,\ u \in [0, y_0],\ v \in [x_0, 0[,$ *and for any function* ω *defined and at least of class* C^2 *on* $\phi(\mathcal{Y}^+_{a,x}) = \mathcal{Y}^+_{-\frac{1}{a},y} \cap \mathcal{Y}^-_{0,y}$*, we have:*

$$\begin{aligned}
&\int_{x_0}^v H(u,x)\|(\omega,\nabla_x\omega)(u,x)\|^2_{L^2(\mathscr{O})}dx + \int_0^u H(y,v)\|(\omega,\nabla_y\omega)(y,v)\|^2_{L^2(\mathscr{O})}dy \leq \\
&\quad \int_{x_0}^v H(0,x)\|(\omega,\nabla_x\omega)(0,x)\|^2_{L^2(\mathscr{O})}dx + \int_0^u H(y,x_0)\|(\omega,\nabla_y\omega)(y,x_0)\|^2_{L^2(\mathscr{O})}dy \\
&\quad +(c_1(c_0,\bar{c}_0,n,h) - 2\Lambda)\int_0^u\int_{x_0}^v H(x,y)\|(\omega,\nabla\omega)(y,x)\|^2_{L^2(\mathscr{O})}dxdy \\
&\quad +\frac{1}{c_0}\int_{\mathcal{D}_{u,v}} \left|L^\ell[\omega]\right| dydx\, d\nu \ .
\end{aligned} \tag{7.3.7}$$

Proof: *Recall that*

$$\begin{aligned}
\Box_{\eta,y}\omega &= -4\partial_x\partial_y\omega + \frac{n-1}{\rho}(\partial_x - \partial_y)\omega + \frac{\Delta_{S^{n-1}}\omega}{\rho^2} \\
&= -4\partial_x\partial_y\omega + \frac{1}{\rho^2}h^{AB}\partial_A\partial_B\omega + \frac{n-1}{\rho}(\partial_x - \partial_y)\omega - \frac{1}{\rho^2}\Gamma^A\partial_A\omega \ ,
\end{aligned}$$

where $\Gamma^A = h^{BC}\Gamma^A_{BC}$*, the* Γ^A_{BC}*'s being the Christoffel symbols on the unit sphere* $\mathbb{S}^{n-1}$ *of* $\mathbb{R}^n$*. We point out the following trivial identities*

$$-4H\partial_x\omega\partial_x\partial_y\omega = -2\partial_y(H\partial_x\omega)^2 + (2\partial_y H)(\partial_x\omega)^2 \ ,$$

$$-4H\partial_y\omega\partial_x\partial_y\omega = -2\partial_x(H\partial_y\omega)^2 + (2\partial_x H)(\partial_y\omega)^2 \ ,$$

$$\begin{aligned}\frac{1}{\rho^2}Hh^{AB}\partial_x\omega\partial^2_{AB}\omega &= \partial_A\left(\frac{1}{\rho^2}Hh^{AB}\partial_x\omega\partial_B\omega\right) - \frac{1}{\rho^2}H\partial_A(h^{AB})\partial_x\omega\partial_B\omega \\ &\quad -\frac{1}{\rho^2}Hh^{AB}\partial_x\partial_A\omega\partial_B\omega \ .\end{aligned}$$

The last term of the last identity can be written as

$$\begin{aligned}\frac{1}{\rho^2}Hh^{AB}\partial_x\partial_A\omega\partial_B\omega &= \frac{1}{2\rho^2}Hh^{AB}(\partial_x\partial_A\omega\partial_B\omega + \partial_A\omega\partial_x\partial_B\omega) \\ &= \frac{1}{2\rho^2}Hh^{AB}\partial_x(\partial_A\omega\partial_B\omega) \\ &= \partial_x\left(\frac{1}{2\rho^2}Hh^{AB}\partial_A\omega\partial_B\omega\right) - \partial_x\left(\frac{1}{2\rho^2}H\right)h^{AB}\partial_A\omega\partial_B\omega \ .\end{aligned}$$

It then follows that

$$\begin{aligned}\frac{1}{\rho^2}Hh^{AB}\partial_x\omega\partial^2_{AB}\omega &= \partial_A\left(\frac{1}{\rho^2}Hh^{AB}\partial_x\omega\partial_B\omega\right) - \frac{1}{\rho^2}H\partial_A(h^{AB})\partial_x\omega\partial_B\omega \\ &\quad -\partial_x\left(\frac{1}{2\rho^2}Hh^{AB}\partial_A\omega\partial_B\omega\right) + \partial_x\left(\frac{1}{2\rho^2}H\right)h^{AB}\partial_A\omega\partial_B\omega \ .\end{aligned}$$

Similar calculations lead to

$$\begin{aligned}\frac{1}{\rho^2}Hh^{AB}\partial_y\omega\partial^2_{AB}\omega &= \partial_A\left(\frac{1}{\rho^2}Hh^{AB}\partial_y\omega\partial_B\omega\right) - \frac{1}{\rho^2}H\partial_A(h^{AB})\partial_y\omega\partial_B\omega \\ &\quad -\partial_y\left(\frac{1}{2\rho^2}Hh^{AB}\partial_A\omega\partial_B\omega\right) + \partial_y\left(\frac{1}{2\rho^2}H\right)h^{AB}\partial_A\omega\partial_B\omega \ .\end{aligned}$$

We then obtain the following expression of $L^\ell[\omega]$ *:*

$$\begin{aligned}L^\ell[\omega] &= -\partial_y\left(2H(\partial_x\omega)^2 + \frac{1}{2\rho^2}Hh^{AB}\partial_A\omega\partial_B\omega\right) - \partial_x\left(2H(\partial_y\omega)^2 + \frac{1}{2\rho^2}Hh^{AB}\partial_A\omega\partial_B\omega\right) \\ &\quad +\partial_A\left(\frac{1}{\rho^2}Hh^{AB}(\partial_B\omega)(\partial_x\omega + \partial_y\omega)\right) + \frac{n-1}{\rho}H\left((\partial_x\omega)^2 - (\partial_y\omega)^2\right) \\ &\quad +2\partial_y H(\partial_x\omega)^2) + 2\partial_x H(\partial_y\omega)^2) - \frac{1}{\rho^2}H\partial_A h^{AB}(\partial_B\omega)(\partial_x\omega + \partial_y\omega) \\ &\quad +\frac{1}{2}h^{AB}\partial_A\omega\partial_B\omega(\partial_x + \partial_y)(H/\rho^2)) - \frac{1}{\rho^2}H\Gamma^A(\partial_x\omega + \partial_y\omega)\partial_A\omega \ .\end{aligned}$$

Since $x = \tau + \rho$ *and* $y = \tau - \rho + \frac{1}{a}$ *we have*

$$\partial_x H = -\ell x^{-1} H - \Lambda H, \quad \partial_y H = -\Lambda H \quad \textit{and} \quad (\partial_x + \partial_y)(H/\rho^2) = \frac{1}{\rho^2}(-\ell x^{-1} H - 2\Lambda H)\,. \tag{7.3.8}$$

Thus we have

$$\begin{aligned} L^\ell[\omega] &= -\partial_y \left(2H(\partial_x\omega)^2 + \frac{1}{2\rho^2} H h^{AB} \partial_A\omega\partial_B\omega \right) - \partial_x \left(2H(\partial_y\omega)^2 + \frac{1}{2\rho^2} H h^{AB} \partial_A\omega\partial_B\omega \right) \\ &\quad + \partial_A \left(\frac{1}{\rho^2} H h^{AB} (\partial_B\omega)(\partial_x\omega + \partial_y\omega) \right) + \frac{n-1}{\rho} H \left((\partial_x\omega)^2 - (\partial_y\omega)^2 \right) \\ &\quad - \frac{1}{\rho^2} H \left(\partial_A h^{AB} + \Gamma^B \right) (\partial_B\omega)(\partial_x\omega + \partial_y\omega) \\ &\quad - 2\Lambda H \left((\partial_x\omega)^2 + (\partial_y\omega)^2 + \frac{h^{AB}}{2\rho^2} \partial_A\omega\partial_B\omega \right) \\ &\quad - \ell x^{-1} H \left((\partial_y\omega)^2 + \frac{h^{AB}}{2\rho^2} \partial_A\omega\partial_B\omega \right)\,. \end{aligned}$$

Integrating this identity on $\mathcal{D}_{u,v} = [0,u] \times [x_0, v] \times \mathscr{O}$ *and using Stokes theorem, one is led to*

$$\begin{aligned} \int_{\mathcal{D}_{u,v}} L^\ell[\omega] dy dx\, d\nu &= -\int_{\partial\mathcal{D}_{u,v}} \left\{ 2(\partial_x\omega)^2 + \frac{h^{AB}}{2\rho^2} \partial_A\omega\partial_B\omega \right\} n_y H d\sigma \\ &\quad - \int_{\partial\mathcal{D}_{u,v}} \left\{ 2(\partial_y\omega)^2 + \frac{h^{AB}}{2\rho^2} \partial_A\omega\partial_B\omega \right\} n_x H d\sigma \\ &\quad + \int_{\partial\mathcal{D}_{u,v}} \frac{h^{AB}}{\rho^2} (\partial_B\omega)(\partial_x\omega + \partial_y\omega) n_A H d\sigma \\ &\quad - \ell \int_{\mathcal{D}_{u,v}} x^{-1} \left\{ (\partial_y\omega)^2 + \frac{h^{AB}}{2\rho^2} \partial_A\omega\partial_B\omega \right\} H dy dx\, d\nu \\ &\quad - 2\Lambda \int_{\mathcal{D}_{u,v}} \left\{ (\partial_x\omega)^2 + (\partial_y\omega)^2 + \frac{h^{AB}}{2\rho^2} \partial_A\omega\partial_B\omega \right\} H dy dx\, d\nu \\ &\quad + (n-1) \int_{\mathcal{D}_{u,v}} \frac{1}{\rho} \left\{ (\partial_x\omega)^2 - (\partial_y\omega)^2 \right\} H dy dx\, d\nu \\ &\quad - \int_{\mathcal{D}_{u,v}} \frac{1}{\rho^2} \left(\partial_A h^{AB} + \Gamma^B \right) (\partial_B\omega)(\partial_x\omega + \partial_y\omega) H dy dx\, d\nu. \end{aligned} \tag{7.3.9}$$

In equation (7.3.9),

- *$d\nu$ is the surface element on $\mathbb{S}^{n-1}$ defined by the induced metric on $\mathbb{S}^{n-1}$ by the Euclidean metric on $\mathbb{R}^n$,*
- *$n = n_y\partial_y + n_x\partial_x + \sum_{A=1}^{n} n_\theta\partial_\theta$ is the unit outward normal of the boundary $\partial\mathcal{D}_{u,v}$,*
- *and $d\sigma$ is the surface element on $\partial\mathcal{D}_{u,v}$ induced by the volume element $dydx\, d\nu$.*

The right-hand side of equation (7.3.9) is made of seven terms which will be labeled A, B , C, D, E, F and G where A is the terms of the first line, B those of the second line and so on.

Remark 7.3.2 On the Riemannian manifold $\mathbb{R}^{n+1}$ endowed with the Euclidean metric, the family of vectors $\{\partial_\tau, \partial_\rho, \partial_\theta\}$ is an orthogonal frame and then we deduce that (note that $\partial\mathcal{D}_{u,v}$ is made of four pieces: $\partial\mathcal{D}_{u,v} = \mathcal{C}^+_{0,v} \cup \mathcal{C}^-_{u,x_0} \cup \mathcal{C}^+_{u,v} \cup \mathcal{C}^-_{u,v}$):

- on $\mathcal{C}^+_{0,v}$ the unit outward normal is $n = -\frac{1}{\sqrt{2}}\partial_y$, thus $n_y = -\frac{1}{\sqrt{2}},\ n_x = 0,\ n_A = 0,\ A = 1, \ldots, n-1$;
- on $\mathcal{C}^+_{u,v}$ the outward unit normal is given by $n = \frac{1}{\sqrt{2}}\partial_y$, i.e. $n_y = \frac{1}{\sqrt{2}},\ n_x = 0,\ n_A = 0,\ A = 1, \ldots, n-1$.
- on $\mathcal{C}^-_{u,x_0}$ the unit outward normal $n = -\frac{1}{\sqrt{2}}\partial_x$, i.e. $n_y = 0,\ n_x = -\frac{1}{\sqrt{2}},\ n_A = 0,\ A = 1, \ldots, n-1$.
- on $\mathcal{C}^-_{u,v}$ we have $n = \frac{1}{\sqrt{2}}\partial_x$, thus $n_y = 0,\ n_x = \frac{1}{\sqrt{2}},\ n_A = 0,\ A = 1, \ldots, n-1$.

In (7.3.9) we replace $\partial\mathcal{D}_{u,v}$ by $\mathcal{C}^+_{0,v} \cup \mathcal{C}^-_{u,x_0} \cup \mathcal{C}^+_{u,v} \cup \mathcal{C}^-_{u,v}$ and after using on each piece of $\partial\mathcal{D}_{u,v}$ the corresponding value of the outward unit normal, we find that:

$$\begin{aligned}
A + B + C &= \frac{1}{\sqrt{2}}\int_{\mathcal{C}^+_{0,v}} \Big(2(\partial_x\omega)^2 + \frac{h^{AB}}{2\rho^2}\partial_A\omega\partial_B\omega\Big) H d\sigma \\
&\quad -\frac{1}{\sqrt{2}}\int_{\mathcal{C}^+_{u,v}} \Big(2(\partial_x\omega)^2 + \frac{h^{AB}}{2\rho^2}\partial_A\omega\partial_B\omega\Big) H d\sigma \\
&\quad +\frac{1}{\sqrt{2}}\int_{\mathcal{C}^-_{u,x_0}} \Big(2(\partial_y\omega)^2 + \frac{h^{AB}}{2\rho^2}\partial_A\omega\partial_B\omega\Big) H d\sigma \\
&\quad -\frac{1}{\sqrt{2}}\int_{\mathcal{C}^-_{u,v}} \Big(2(\partial_y\omega)^2 + \frac{h^{AB}}{2\rho^2}\partial_A\omega\partial_B\omega\Big) H d\sigma\ .
\end{aligned}$$

Identity (7.3.9) then takes the form:

$$\frac{1}{\sqrt{2}}\int_{\mathcal{C}^+_{u,v}}\Big(2(\partial_x\omega)^2+\frac{h^{AB}}{2\rho^2}\partial_A\omega\partial_B\omega\Big)H d\sigma+\frac{1}{\sqrt{2}}\int_{\mathcal{C}^-_{u,v}}\Big(2(\partial_y\omega)^2+\frac{h^{AB}}{2\rho^2}\partial_A\omega\partial_B\omega\Big)H d\sigma$$
$$=\frac{1}{\sqrt{2}}\int_{\mathcal{C}^+_{0,v}}\Big(2(\partial_x\omega)^2+\frac{h^{AB}}{2\rho^2}\partial_A\omega\partial_B\omega\Big)H d\sigma+\frac{1}{\sqrt{2}}\int_{\mathcal{C}^-_{u,x_0}}\Big(2(\partial_y\omega)^2+\frac{h^{AB}}{2\rho^2}\partial_A\omega\partial_B\omega\Big)H d\sigma$$
$$-\int_{\mathcal{D}_{u,v}}L^\ell[\omega]dydx\,d\nu-\ell\int_{\mathcal{D}_{u,v}}x^{-1}\Big((\partial_y\omega)^2+\frac{h^{AB}}{2\rho^2}\partial_A\omega\partial_B\omega\Big)H dydx\,d\nu$$
$$-2\Lambda\int_{\mathcal{D}_{u,v}}\Big((\partial_x\omega)^2+(\partial_y\omega)^2+\frac{h^{AB}}{2\rho^2}\partial_A\omega\partial_B\omega\Big)H dydx\,d\nu$$
$$+(n-1)\int_{\mathcal{D}_{u,v}}\frac{1}{\rho}\Big((\partial_x\omega)^2-(\partial_y\omega)^2\Big)H dydx\,d\nu$$
$$-\int_{\mathcal{D}_{u,v}}\frac{1}{\rho^2}\left(\partial_A h^{AB}+\Gamma^B\right)(\partial_B\omega)(\partial_x\omega+\partial_y\omega)H dydx\,d\nu\ . \tag{7.3.10}$$

We then obtain the following estimate:

$$\int_{x_0}^{v}H(u,x)\|\nabla_x\omega(u,x)\|^2_{L^2(\mathscr{O})}dx+\int_0^u H(y,v)\|\nabla_y\omega(y,v)\|^2_{L^2(\mathscr{O})}dy$$
$$\leq\int_{x_0}^{v}H(0,x)\|\nabla_x\omega(0,x)\|^2_{L^2(\mathscr{O})}dx+\int_0^u H(y,x_0)\|\nabla_y\omega(y,x_0)\|^2_{L^2(\mathscr{O})}dy$$
$$+\big(c(c_0,\bar c_0,n,\rho)-2\Lambda\big)\int_0^u\int_{x_0}^{v}H(x,y)\|\nabla\omega(y,x)\|^2_{L^2(\mathscr{O})}dxdy$$
$$+\frac{1}{c_0}\int_{\mathcal{D}_{u,v}}\Big|L^\ell[\omega]\Big|\,dydx\,d\nu\ . \tag{7.3.11}$$

On the other hand, we have:

$$\frac{1}{2}(\partial_x+\partial_y)(H\omega^2)=H\omega(\partial_x v+\partial_y)\omega-\frac{\ell}{2}x^{-1}H\omega^{\ 2}-\Lambda H\omega^2\ ;$$

which implies that

$$\begin{aligned}(\partial_x+\partial_y)(H\omega^2)&\leq 2H\omega(\partial_x+\partial_y)\omega-2\Lambda H\omega^2\\&\leq\Big((1-2\Lambda)\omega^2+2|\partial_x\omega|^2+2|\partial_y\omega|^2\Big)H\ .\end{aligned}$$

If we integrate once more on $\mathcal{D}_{u,v}$ then we obtain via Stokes formula the following inequality:

$$\int_{\partial\mathcal{D}_{u,v}}\omega^2 n_x H d\sigma+\int_{\partial\mathcal{D}_{u,v}}\omega^2 n_y H d\sigma\leq\int_{\mathcal{D}_{u,v}}\Big((1-2\Lambda)\omega^2+2|\partial_x\omega|^2+2|\partial_y\omega|^2\Big)H dydx\,d\nu\ ,$$

which is equivalent to

$$\int_{\mathcal{C}^+_{u,v}} \omega^2 H d\sigma + \int_{\mathcal{C}^-_{u,v}} \omega^2 H d\sigma \quad \leq \quad \int_{\mathcal{C}^+_{0,v}} \omega^2 H d\sigma + \int_{\mathcal{C}^-_{u,x_0}} \omega^2 H d\sigma$$
$$+\sqrt{2}\int_{\mathcal{D}_{u,v}} \Big((1-2\Lambda)\omega^2 + 2|\partial_x\omega|^2 + 2|\partial_y\omega|^2\Big) H dydx\, d\nu \ .$$

The estimate thus follows

$$\int_{x_0}^{v} H(u,x)\|\omega(u,x)\|^2_{L^2(\mathscr{O})} dx + \int_0^u H(y,v)\|\omega(y,v)\|^2_{L^2(\mathscr{O})} dy \leq$$
$$\int_{x_0}^{v} H(0,x)\|\omega(0,x)\|^2_{L^2(\mathscr{O})} dx \int_0^{y_0} H(x_0,y)\|\omega(y,x_0)\|^2_{L^2(\mathscr{O})} dy$$
$$+\int_{\mathcal{D}_{\varepsilon,u}} \Big((1-2\Lambda)\omega^2 + 2|\partial_x\omega|^2 + 2|\partial_y\omega|^2\Big) H dydx\, d\nu \ . \qquad (7.3.12)$$

Finally, adding side by side inequalities (7.3.11) and (7.3.12) leads to the stated inequality:

$$\int_{x_0}^{v} H(u,x)\|(\omega,\nabla_x\omega)(u,x)\|^2_{L^2(\mathscr{O})} dx + \int_0^u H(y,v)\|(\omega,\nabla_y\omega)(y,v)\|^2_{L^2(\mathscr{O})} dy \leq$$
$$\int_{x_0}^{v} H(0,x)\|(\omega,\nabla_x\omega)(0,x)\|^2_{L^2(\mathscr{O})} dx + \int_0^u H(y,x_0)\|(\omega,\nabla_y\omega)(y,x_0)\|^2_{L^2(\mathscr{O})} dy$$
$$+\Big(c_1(c_0,\bar{c}_0,n,h) - 2\Lambda\Big)\int_0^u\int_{x_0}^{v} H(x,y)\|(\omega,\nabla\omega)(y,x)\|^2_{L^2(\mathscr{O})} dxdy$$
$$+\frac{1}{c_0}\int_{\mathcal{D}_{u,v}} \Big|L^\ell[\omega]\Big|\, dydx\, d\nu \ .$$

□

7.3.3 Iterative scheme

Our aim now, is to show that there exists a real number $u_* \in]0,y_0]$ *and a sequence of smooth functions* $(\omega^k)_{k\in\mathbb{N}}$ *which converges towards a solution* ω *of (7.3.3) on the set* $\mathcal{D}_* := [0,u_*]\times[x_0,0[\times\mathscr{O}$*. In order to use the* C^∞ *results of [47], first, we need to approximate the data* ω_0^+ *and* ω_0^- *with sequences of smooth functions* $(\omega_0^{+,k})_{k\in\mathbb{N}}$ *and* $(\omega_0^{-,k})_{k\in\mathbb{N}}$ *for which the compatibility condition*

$$\omega_0^{+,k}(x_0,\theta) = \omega_0^{-,k}(0,\theta) \qquad (7.3.13)$$

holds at every step of the iteration. These sequences are constructed as follows: Denote by $(\bar{\omega}_0^{+,k})_{k\in\mathbb{N}}$ *an arbitrary sequence of smooth functions which converges towards* $\partial_x\omega_0^+$ *in* $\mathscr{H}^\alpha_{m+1}(\mathcal{C}^+)$ *and by* $(\omega_0^{-,k})_{k\in\mathbb{N}}$ *an arbitrary sequence of smooth functions on* $\mathcal{C}^-$ *which converges to* ω_0^- *in the Sobolev spaces* $H^{m+2}(\mathcal{C}^-)$*. Then, for all* $(x,\theta)\in[x_0,0[\times\mathscr{O}$ *and* $k\in\mathbb{N}$*, set*

$$\omega_0^{+,k}(x,\theta)=\omega_0^{-,k}(0,\theta)+\int_{x_0}^{x}\bar{\omega}_0^{+,k}(s,\theta)ds\ . \tag{7.3.14}$$

For later use we point out in the following Lemma some properties of the sequences $(\omega_0^{+,k})_{k\in\mathbb{N}}$ *and* $(\omega_0^{-,k})_{k\in\mathbb{N}}$*.*

Lemma 7.3.3 *Suppose that* $-1<\alpha\le -1/2$*. Then, the sequences* $(\omega_0^{-,k})_{k\in\mathbb{N}}$ *and* $(\omega_0^{+,k})_{k\in\mathbb{N}}$ *given respectively by (7.3.14) satisfy the following:*

1. $\forall\ \theta\in\mathscr{O},\ \omega_0^{+,k}(x_0,\theta)=\omega_0^{-,k}(0,\theta)$;
2. $\omega_0^{+,k}\longrightarrow\omega_0^+$ *in* $\mathscr{H}^\alpha_{m+1}(\mathcal{C}^+)$ *and* $\partial_x\omega_0^{+,k}\longrightarrow\partial_x\omega_0^+$ *in* $\mathscr{H}^\alpha_{m+1}(\mathcal{C}^+)$;
3. $\sup\limits_{k\in\mathbb{N},\ x\in[x_0,0[}(-x)^{-\alpha}\|\partial_x\omega_0^{+,k}(x)\|_{H^{m-1}(\mathscr{O})}<\infty$.

Proof: *The first statement is obvious. As far as the second statement is concerned, we write*

$$\begin{aligned}\|\omega_0^{+,k}-\omega_0^+\|^2_{\mathscr{H}^\alpha_{m+1}(\mathcal{C}^+)} &= \|\omega_0^{+,k}-\omega_0^+\|^2_{\mathscr{H}^\alpha_0(\mathcal{C}^+)}+\|x\partial_x(\omega_0^{+,k}-\omega_0^+)\|^2_{\mathscr{H}^\alpha_m(\mathcal{C}^+)}\\ &\quad+\|\partial_A(\omega_0^{+,k}-\omega_0^+)\|^2_{\mathscr{H}^\alpha_m(\mathcal{C}^+)}\ .\end{aligned}$$

We have

$$\begin{aligned}\|x\partial_x(\omega_0^{+,k}-\omega_0^+)\|^2_{\mathscr{H}^\alpha_m(\mathcal{C}^+)} &= \|\bar{\omega}_0^{+,k}-\partial_x\omega_0^+\|^2_{\mathscr{H}^{\alpha-1/2}_m(\mathcal{C}^+)}\\ &\le c\|\bar{\omega}_0^{+,k}-\partial_x\omega_0^+\|^2_{\mathscr{H}^\alpha_m(\mathcal{C}^+)}\underset{k\longrightarrow\infty}{\longrightarrow}0\end{aligned} \tag{7.3.15}$$

On the other hand,

$$\begin{aligned}\omega_0^{+,k}(x,\theta)-\omega_0^+(x,\theta) &= \omega_0^{-,k}(0,\theta)-\omega_0^+(x,\theta)+\int_{x_0}^{x}\bar{\omega}_0^{+,k}(s,\theta)ds\\ &= \omega_0^{-,k}(0,\theta)-\omega_0^+(x_0,\theta)+\omega^+(x_0,\theta)-\omega^+(x,\theta)+\int_{x_0}^{x}\bar{\omega}_0^{+,k}(s,\theta)ds\\ &= \omega_0^{-,k}(0,\theta)-\omega_0^-(0,\theta)+\int_{x_0}^{x}\Big(\bar{\omega}_0^{+,k}(s,\theta)-\partial_x\omega_0^+(s,\theta)\Big)\,ds\ ;\end{aligned}$$

thus (recall $-1 < \alpha \leq -\frac{1}{2}$ implies $(-x)^{-2\alpha-1} \leq (-s)^{-2\alpha-1}$)

$$\begin{aligned}
&(-x)^{-2\alpha-1}|\omega_0^{+,k}(x,\theta) - \omega_0^+(x,\theta)|^2 \\
&\quad \leq \quad c(x_0)\left(|\omega_0^{-,k}(0,\theta) - \omega_0^-(0,\theta)|^2 + \int_{x_0}^{x}(-s)^{-2\alpha-1}|\bar{\omega}_0^{+,k} - \partial_x\omega_0^+|^2(s,\theta)ds\right) \\
&\quad \leq \quad c(x_0)\left(|\omega_0^{-,k}(0,\theta) - \omega_0^-(0,\theta)|^2 + \int_{x_0}^{0}(-s)^{-2\alpha-1}|\bar{\omega}_0^{+,k} - \partial_x\omega_0^+|^2(s,\theta)ds\right) .
\end{aligned}$$

Now integrating this inequality on $\mathcal{C}^+$ gives (the second inequality is obtained by trace theorem):

$$\begin{aligned}
\|\omega_0^{+,k} - \omega_0^+\|^2_{\mathscr{H}_0^\alpha(\mathcal{C}^+)} &\leq c(x_0)\left(\|\omega_0^{-,k}(0) - \omega_0^-(0)\|^2_{L^2(\mathscr{O})} + \|\bar{\omega}_0^{+,k} - \partial_x\omega_0^+\|^2_{\mathscr{H}_0^\alpha(\mathcal{C}^+)}\right) \\
&\leq c(x_0)\left(\|\omega_0^{-,k} - \omega_0^-\|^2_{H^1(\mathcal{C}^-)} + \|\bar{\omega}_0^{+,k} - \partial_x\omega_0^+\|^2_{\mathscr{H}_0^\alpha(\mathcal{C}^+)}\right) \underset{k\longrightarrow\infty}{\longrightarrow} 0 \quad (7.3.16)
\end{aligned}$$

Now let $\beta \in \mathbb{N}^n$ such that $|\beta| \leq m$. Similarly to the previous calculations, we have

$$\begin{aligned}
&\partial^\beta\partial_A\left(\omega_0^{+,k}(x,\theta) - \omega_0^+(x,\theta)\right) \\
&\qquad = \partial^\beta\partial_A\left(\omega_0^{-,k}(0,\theta) - \omega_0^-(0,\theta) + \int_{x_0}^{x}\bar{\omega}_0^{+,k}(s,\theta) - \partial_x\omega_0^+(s,\theta)ds\right) .
\end{aligned}$$

If $\beta_1 = 0$ then,

$$\begin{aligned}
&\partial^\beta\partial_A\left(\omega_0^{+,k}(x,\theta) - \omega_0^+(x,\theta)\right) \\
&\qquad = \partial^\beta\partial_A\left(\omega_0^{-,k}(0,\theta) - \omega_0^-(0,\theta)\right) + \int_{x_0}^{x}\partial^\beta\partial_A\left(\bar{\omega}_0^{+,k}(s,\theta) - \partial_x\omega_0^+(s,\theta)\right)ds ,
\end{aligned}$$

thus (recall $-1 < \alpha \leq -\frac{1}{2}$ implies $(-x)^{-2\alpha-1} \leq (-s)^{-2\alpha-1}$)

$$\begin{aligned}
&(-x)^{-2\alpha-1}|\partial^\beta\partial_A\left(\omega_0^{+,k}(x,\theta) - \omega_0^+(x,\theta)\right)|^2 \\
&\quad \leq \quad c(x_0)\left(|\partial^\beta\partial_A\left(\omega_0^{-,k}(0,\theta) - \omega_0^-(0,\theta)\right)|^2 + \int_{x_0}^{x}(-s)^{-2\alpha-1}|\partial^\beta\partial_A(\bar{\omega}_0^{+,k} - \partial_x\omega_0^+)|^2(s,\theta)ds\right) \\
&\quad \leq \quad c(x_0)\left(|\partial^\beta\partial_A\left(\omega_0^{-,k}(0,\theta) - \omega_0^-(0,\theta)\right)|^2 + \int_{x_0}^{0}(-s)^{-2\alpha-1}|\partial^\beta\partial_A(\bar{\omega}_0^{+,k} - \partial_x\omega_0^+)|^2(s,\theta)ds\right) .
\end{aligned}$$

Integrating on $\mathcal{C}^+$, we have:

$$\|\partial^\beta\partial_A(\omega_0^{+,k} - \omega_0^+)\|^2_{\mathscr{H}_0^\alpha(\mathcal{C}^+)} \leq c(x_0)\left(\|\omega_0^{-,k}(0) - \omega_0^-(0)\|^2_{H^{m+1}(\mathscr{O})} + \|\partial^\beta\partial_A(\bar{\omega}_0^{+,k} - \partial_x\omega_0^+)\|^2_{\mathscr{H}_0^\alpha(\mathcal{C}^+)}\right) . \tag{7.3.17}$$

Suppose now $\beta_1 \geq 1$ and set $\tilde{\beta} = (\beta_1 - 1, \beta_2, \dots, \beta_n)$. We have

$$\partial^\beta \partial_A \left(\omega_0^{+,k}(x,\theta) - \omega_0^+(x,\theta)\right) = \partial^{\tilde{\beta}} \partial_A \left(\bar{\omega}_0^{+,k}(s,\theta) - \partial_x \omega_0^+(s,\theta)\right) ,$$

thus,

$$\|(-x)^{\beta_1} \partial^\beta \partial_A \left(\omega_0^{+,k} - \omega_0^+\right)\|^2_{\mathscr{H}_0^\alpha} = \|(-x)^{\beta_1 - 1} \partial^{\tilde{\beta}} \partial_A \left(\bar{\omega}_0^{+,k} - \partial_x \omega_0^+\right)\|^2_{\mathscr{H}_0^{\alpha-1}} . \tag{7.3.18}$$

By (7.3.17) and (7.3.18), we have

$$\begin{aligned}&\|\partial_A \left(\omega_0^{+,k}(x,\theta) - \omega_0^+\right)\|^2_{\mathscr{H}_m^\alpha} \\ &\quad \leq c(x_0)\left(\|\omega_0^{-,k}(0) - \omega_0^-(0)\|^2_{H^{m+1}(\mathscr{O})} + \|\bar{\omega}_0^{+,k} - \partial_x \omega_0^+\|^2_{\mathscr{H}_{m+1}^\alpha}\right) \underset{k \longrightarrow \infty}{\longrightarrow} 0\end{aligned} \tag{7.3.19}$$

From (7.3.15), (7.3.16) and (7.3.19) it follows that $\omega_0^{+,k} \longrightarrow \omega_0^+$ in $\mathscr{H}_{m+1}^\alpha$. This proves that the sequence $(\omega_0^{+,k})_{k\in\mathbb{N}}$ is such that

- $\omega_0^{+,k} \longrightarrow \omega_0^+$ *in* $\mathscr{H}_{m+1}^\alpha(\mathcal{C}^+)$,
- $\partial_x \omega_0^{+,k} \longrightarrow \partial_x \omega_0^+$ *in* $\mathscr{H}_{m+1}^\alpha(\mathcal{C}^+)$.

Let now prove that the quantity $\sup\limits_{k\in\mathbb{N},\ x\in[x_0,0[} (-x)^{-\alpha} \|\partial_x \omega_0^{+,k}(x)\|_{H^{m-1}(\mathscr{O})}$ is finite. We know that $[x_0, 0[= \underset{n\in\mathbb{N}^*}{\cup} [\frac{x_0}{2^{n-1}}, \frac{x_0}{2^n}]$ and $s = \frac{2^n x}{x_0} \in [1,2]$ if and only if $x = \frac{s x_0}{2^n} \in [x_0, 0[$. For any function f defined on $[x_0, 0[\times \mathscr{O}$, set $f_n(s,\theta) = f(x = \frac{s x_0}{2^n}, \theta)$. The $\mathscr{H}_m^\alpha(\mathcal{C}^+)-$ norm of f can be rewritten as (see Equation (B.1.7) of Appendix B.1):

$$\|f\|_{\mathscr{H}_m^\alpha(\mathcal{C}^+)} \approx (-x_0)^{-2\alpha} \sum_{n\geq 1} 2^{2n\alpha} \|f_n\|^2_{H^m([1,2]\times\mathscr{O})} . \tag{7.3.20}$$

Here we write $A \approx B$ if there exist constant $C_1, C_2 > 0$ such that $C_1 A \leq B \leq C_2 A$. We have the following

$$\begin{aligned}\sup_{x\in[x_0,0[} (-x)^{-2\alpha} \|f(x)\|^2_{H^{m-1}(\mathscr{O})} &= \sup_{n\geq 1} \sup_{\frac{x_0}{2^{n-1}} \leq x \leq \frac{x_0}{2^n}} (-x)^{-2\alpha} \|f(x)\|^2_{H^{m-1}(\mathscr{O})} \\ &= \sup_{n\geq 1} \sup_{s\in[1,2]} \left(\frac{s x_0}{2^n}\right)^{-2\alpha} \|f(\frac{s x_0}{2^n})\|^2_{H^{m-1}(\mathscr{O})} \\ &= (-x_0)^{-2\alpha} \sup_{n\geq 1} \left\{ 2^{2n\alpha} \sup_{s\in[1,2]} (s)^{-2\alpha} \|f_n(s)\|^2_{H^{m-1}(\mathscr{O})} \right\} \\ &\leq c(-x_0)^{-2\alpha} \sum_{n\geq 1} 2^{2n\alpha} \sup_{s\in[1,2]} \|f_n(s)\|^2_{H^{m-1}(\mathscr{O})} .\end{aligned}$$

Now writing

$$\partial_\theta^\gamma f_n(s,\theta) = \partial_\theta^\gamma f_n(1,\theta) + \int_1^s \partial_s \partial_\theta^\gamma f_n(\tau,\theta) d\tau \ ,$$

implies that

$$|\partial_\theta^\gamma f_n(s,\theta)|^2 \le |\partial_\theta^\gamma f_n(1,\theta)|^2 + c \int_1^s |\partial_s \partial_\theta^\gamma f_n(\tau,\theta)|^2 d\tau \ .$$

Integrating this estimate on $\mathscr{O}$ gives,

$$\begin{aligned} \|f_n(s)\|^2_{H^{m-1}(\mathscr{O})} &\le \|f_n(1)\|^2_{H^{m-1}(\mathscr{O})} + c \int_1^s \|\partial_s f_n(\tau)\|^2_{H^{m-1}(\mathscr{O})} d\tau \\ &\le \|f_n(1)\|^2_{H^{m-1}(\mathscr{O})} + \|f_n\|^2_{H^m([1,2]\times\mathscr{O})} \\ &\le c\|f_n\|^2_{H^m([1,2]\times\mathscr{O})} \quad \textit{by trace theorem} \ . \end{aligned}$$

Therefore,

$$\begin{aligned} \sup_{x\in[x_0,0[} (-x)^{-2\alpha} \|f(x)\|^2_{H^{m-1}(\mathscr{O})} &\le c(-x_0)^{-2\alpha} \sum_{n\ge 1} 2^{2n\alpha} \|f_n\|^2_{H^m([1,2]\times\mathscr{O})} \\ &\approx \|f_n\|_{\mathscr{H}^\alpha_m(\mathcal{C}^+)} \ \textit{(see (7.3.20))} \ . \end{aligned}$$

Now choosing $f = \partial_x \omega_0^{+,k}$ in the previous estimate leads to

$$\sup_{x\in[x_0,0[} (-x)^{-2\alpha} \|\partial_x \omega^{+,k}(x)\|^2_{H^{m-1}(\mathscr{O})} \le \|\partial_x \omega^{+,k}\|_{\mathscr{H}^\alpha_m(\mathcal{C}^+)} \ .$$

Since convergent sequences are bounded, we conclude that

$$\sup_{k\in\mathbb{N},\ x\in[x_0,0[} (-x)^{-2\alpha} \|\partial_x \omega^{+,k}(x)\|^2_{H^{m-1}(\mathscr{O})} < \infty \ .$$

This completes the proof of the Lemma. □

We denote by ω_0^k the continuous functions defined on $\mathcal{C}^+ \cup \mathcal{C}^-$ which coincide with $\omega_0^{+,k}$ on $\mathcal{C}^+$ and with $\omega_0^{-,k}$ on $\mathcal{C}^-$. The sequence $(\omega^k)_{k\in\mathbb{N}}$ is then constructed by induction.

- *Set $\omega^0 = \omega_0$ where ω_0 is a smooth function defined on $\mathcal{D}$ and which coincides with ω_0^0 on $\mathcal{C}^+ \cup \mathcal{C}^-$.*
- *Then, let ω^{k+1} be defined by iteration as the solution of the linear characteristic Cauchy problem*

$$\left\{ \begin{array}{l} \Box_{y,\eta} \omega^{k+1} = x^{-\frac{n+3}{2}} G\left(z, (-x)^{\frac{n-1}{2}}(\omega^k, \nabla\omega^k)\right) \quad \textit{in} \ \ \mathcal{D} \\ \omega^{k+1} = \omega_0^{k+1} \quad \textit{on} \quad \mathcal{C}^+ \cup \mathcal{C}^- \end{array} \right. . \qquad (7.3.21)$$

In order to prove existence of the sequence $\left(\omega^k\right)_{k\in\mathbb{N}}$, first we have to prove existence of the function ω_0 used in the above iterative scheme. We define ω_0 for any $(y,x,\theta)\in\mathbb{R}^{n+1}$ by setting

$$\omega_0(y,x,\theta)=\omega_0^{+,0}(x,\theta)+\omega_0^{-,0}(y,\theta)-\omega_0^{-,0}(0,\theta). \tag{7.3.22}$$

Next we have to justify existence of a smooth solution of (7.3.21). We quote Theorem 1 of [47]. Actually that reference gives a local solution on a neighborhood of the intersecting hypersurfaces, but in the case of the linear problem (7.3.21), we will obtain a global solution on $\mathcal{D}$.

7.3.4 Boundedness properties of $\left(\omega^k\right)_{k\in\mathbb{N}}$

Set

$$\begin{aligned} C_0 \;=\; & \sup_{k\in\mathbb{N},\, x\in\mathcal{C}^+} |x|^{-\alpha}\Big\{\|(\omega_0^{+,k},\nabla_x\omega_0^{+,k})(x)\|_{W^{1,\infty}(\mathscr{O})}+\|\partial_y\omega^k(0,x)\|_{W^{1,\infty}(\mathscr{O})}\Big\} \\ & +\sup_{k\in\mathbb{N},\, y\in\mathcal{C}^-}\Big\{(-x_0)^{-\alpha}\|\partial_y\omega_0^{-,k}(y)\|_{W^{1,\infty}(\mathscr{O})}\Big\} \end{aligned}$$

We will show later that $C_0<\infty$. We have the following lemma:

Lemma 7.3.4 *Assume (7.3.1) and (7.3.2) with $-1<\alpha\le -1/2$ and $m>\frac{n+7}{2}$. If the source term G satisfies hypothesis $(\mathcal{H})$ page 135 with a* zero order *r such that*

$$n\ge 1+\frac{4}{r-1}-2\alpha\ , \tag{7.3.23}$$

then there exists a real number $u_*\in]0,y_0]$ for which we have

$$\sup_{k\in\mathbb{N},\,(y,x)\in[0,u_*]\times[x_0,0[}|x|^{-\alpha}\|(\omega^k,\nabla\omega^k)(y,x)\|_{W^{1,\infty}(\mathscr{O})}<2C_0. \tag{7.3.24}$$

Proof: *The proof will be made by induction on the integer k. Let us show that the statement holds when $k=0$ i.e*

$$\sup_{(y,x)\in[0,u_*]\times[x_0,0[}|x|^{-\alpha}\|(\omega^0,\nabla\omega^0)(y,x)\|_{W^{1,\infty}(\mathscr{O})}<2C_0\ .$$

From definition (7.3.22) of ω_0, we have

$$\nabla\omega_0(y,x,\theta)=\Big(\partial_y\omega_0^{-,0}(y,\theta),\partial_x\omega_0^{+,0}(x,\theta),\partial_\theta\omega_0^{+,0}(x,\theta)+\partial_\theta\big(\omega_0^{-,0}(y,\theta)-\omega^{-,0}(0,\theta)\big)\Big)\ ,$$

thus,

$$\begin{aligned}(-x)^{-\alpha}\|(\omega^0,\nabla\omega^0)(y,x)\|_{W^{1,\infty}(\mathscr{O})} &\leq (-x)^{-\alpha}\|(\omega_0^{+,0},\nabla_x\omega_0^{+,0})(x)\|_{W^{1,\infty}(\mathscr{O})}\\ &\quad +(-x)^{-\alpha}\|(\omega_0^{-,0}(y)-\omega_0^{-,0}(0)\|_{W^{1,\infty}(\mathscr{O})}\\ &\quad +(-x)^{-\alpha}\|\partial_y\omega_0^{-,0}(y)\|_{W^{1,\infty}(\mathscr{O})}\\ &\quad +(-x)^{-\alpha}\|\partial_\theta(\omega_0^{-,0}(y)-\omega_0^{-,0}(0))\|_{W^{1,\infty}(\mathscr{O})}\\ &\leq C_0+(-x_0)^{-\alpha}\|(\omega_0^{-,0}(y)-\omega_0^{-,0}(0)\|_{W^{1,\infty}(\mathscr{O})}\\ &\quad +(-x_0)^{-\alpha}\|\partial_\theta(\omega_0^{-,0}(y)-\omega_0^{-,0}(0))\|_{W^{1,\infty}(\mathscr{O})}\end{aligned}$$

Now recall that $\omega_0^{-,0}\in C^\infty\big([0,y_0]\times\mathscr{O}\big)$, *thus*

$$(-x_0)^{-\alpha}\|(\omega_0^{-,0}(y)-\omega_0^{-,0}(0)\|_{W^{1,\infty}(\mathscr{O})}+(-x_0)^{-\alpha}\|\partial_\theta(\omega_0^{-,0}(y)-\omega_0^{-,0}(0))\|_{W^{1,\infty}(\mathscr{O})}\underset{y\longrightarrow 0}{\longrightarrow}0\ .$$

It then follows that there exists a real number $u_0\in[0,y_0]$ *such that,* $\forall y\in[0,u_0]$,

$$(-x_0)^{-\alpha}\|(\omega_0^{-,0}(y)-\omega_0^{-,0}(0)\|_{W^{1,\infty}(\mathscr{O})}+(-x_0)^{-\alpha}\|\partial_\theta(\omega_0^{-,0}(y)-\omega_0^{-,0}(0))\|_{W^{1,\infty}(\mathscr{O})}<C_0\ .$$

Therefore,

$$\sup_{(y,x)\in[0,u_0]\times[x_0,0[}|x|^{-\alpha}\|(\omega^0,\nabla\omega^0)(y,x)\|_{W^{1,\infty}(\mathscr{O})}<2C_0\ ,$$

and the property holds for $k=0$. *Note that the real* u_* *will be determined later from the induction scheme and will be less than or equal to* u_0. *Let* j *be an integer greater than or equal to* 1, *and suppose that for any integer* $k\leq j$ *the following holds:*

$$\sup_{(y,x)\in[0,u_*]\times[x_0,0[}|x|^{-\alpha}\|(\omega^k,\nabla\omega^k)(y,x)\|_{W^{1,\infty}}<2C_0\ . \tag{7.3.25}$$

We want to prove that (7.3.25) holds with $k=j+1$. *Let* $\gamma\in\mathbb{N}^{n-1}$ *be such that* $|\gamma|\leq m$. *If in Proposition 7.3.1 page 136 we choose* $\omega=\partial_\theta^\gamma\omega^{k+1}$ *and*

$\ell = -2\alpha$ then we obtain the following inequality:

$$\int_{x_0}^{v} H(u,x)\|(\partial_\theta^\gamma \omega^{k+1}, \nabla_x \partial_\theta^\gamma \omega^{k+1})(u,x)\|^2_{L^2(\mathscr{O})} dx$$
$$+ \int_0^u H(y,v)\|(\partial_\theta^\gamma \omega^{k+1}, \nabla_y \partial_\theta^\gamma \omega^{k+1})(y,v)\|^2_{L^2(\mathscr{O})} dy \leq$$
$$\int_{x_0}^{v} H(0,x)\|(\partial_\theta^\gamma \omega^{k+1}, \nabla_x \partial_\theta^\gamma \omega^{k+1})(0,x)\|^2_{L^2(\mathscr{O})} dx$$
$$+ \int_0^u H(y,x_0)\|(\partial_\theta^\gamma \omega^{k+1}, \nabla_y \partial_\theta^\gamma \omega^{k+1})(y,x_0)\|^2_{L^2(\mathscr{O})} dy$$
$$+(c_1(c_0,\bar{c}_0,n,h) - 2\Lambda) \int_0^u \int_{x_0}^{v} H(x,y)\|(\partial_\theta^\gamma \omega^{k+1}, \nabla \partial_\theta^\gamma \omega^{k+1})(y,x)\|^2_{L^2(\mathscr{O})} dxdy$$
$$+\frac{1}{c_0} \int_{\mathcal{D}_{u,v}} \left| L^\ell[\partial_\theta^\gamma \omega^{k+1}] \right| dydx\, d\nu \ .$$

Summing up the above identities for all multi-indices γ such that $|\gamma| \leq m$, one is led to:

$$\int_{x_0}^{v} H(u,x)\|(\omega^{k+1}, \nabla_x \omega^{k+1})(u,x)\|^2_{H^m(\mathscr{O})} dx + \int_0^u H(y,v)\|(\omega^{k+1}, \nabla_y \omega^{k+1})(y,v)\|^2_{H^m(\mathscr{O})} dy \leq$$
$$\int_{x_0}^{v} H(0,x)\|(\omega^{k+1}, \nabla_x \omega^{k+1})(0,x)\|^2_{H^m(\mathscr{O})} dx + \int_0^u H(y,x_0)\|(\omega^{k+1}, \nabla_y \omega^{k+1})(y,x_0)\|^2_{H^m(\mathscr{O})} dy$$
$$+(c(c_0,\bar{c}_0,n,\rho) - 2\Lambda) \int_0^u \int_{x_0}^{v} H(x,y)\|(\omega^{k+1}, \nabla \omega^{k+1})(y,x)\|^2_{H^m(\mathscr{O})} dxdy$$
$$+\frac{1}{c_0} \sum_{|\gamma|\leq m} \int_{\mathcal{D}_{u,v}} \left| L^\ell[\partial_\theta^\gamma \omega^{k+1}] \right| dydx\, d\nu \ . \tag{7.3.26}$$

Let us control the terms with $L^\ell[\partial_\theta^\gamma \omega^{k+1}]$. In all the remaining of this section we will use the symbol $G^k(\dots)$ to denote quantity $G\left(z, x^{-\frac{n-1}{2}}(\omega^k, \nabla\omega^k)\right)$. We have:

$$\begin{aligned}
L^\ell[\partial_\theta^\gamma \omega^{k+1}] &= H(x,y)(\partial_x \partial_\theta^\gamma \omega^{k+1} + \partial_y \partial_\theta^\gamma \omega^{k+1}) \Box_{\eta,y} \partial_\theta^\gamma \omega^{k+1} \\
&= H(x,y)(\partial_x \partial_\theta^\gamma \omega^{k+1} + \partial_y \partial_\theta^\gamma \omega^{k+1}) \left(\partial_\theta^\gamma \Box_{\eta,y} \omega^{k+1} + [\Box_{\eta,y}, \partial_\theta^\gamma] \omega^{k+1} \right) \\
&= H(x,y)(\partial_x \partial_\theta^\gamma \omega^{k+1} + \partial_y \partial_\theta^\gamma \omega^{k+1}) \left(x^{-\frac{n+3}{2}} \partial_\theta^\gamma G^k(\dots) + [\Box_{\eta,y}, \partial_\theta^\gamma] \omega^{k+1} \right) \\
&=: A + B + C + D \ .
\end{aligned}$$

We will use at many places the inequality $ab \le a^2/(4\epsilon) + \epsilon b^2$. The term A is controlled as follows:

$$\begin{aligned} A &= x^{-\frac{n+3}{2}} H(x,y)\partial_\theta^\gamma \partial_x \omega^{k+1} \partial_\theta^\gamma G^k(\ldots) \\ &\le c(\epsilon) H |\partial_\theta^\gamma \partial_x \omega^{k+1}|^2 + \epsilon H x^{-(n+3)} |\partial_\theta^\gamma G^k(\ldots)|^2 \ , \end{aligned}$$

which implies

$$\begin{aligned} \sum_{|\gamma|\le m} \int_{\mathcal{D}_{u,v}} A d\nu dx dy &\le c(\epsilon) \int_0^u \int_{x_0}^v H(x,y) \|\partial_x \omega^{k+1}(y,x)\|^2_{H^m(\mathscr{O})} dx dy \\ &\quad + \epsilon \int_0^u \int_{x_0}^v x^{-(n+3)} H(x,y) \|G^k(\ldots)\|^2_{H^m(\mathscr{O})} dx dy \ . \end{aligned}$$

Now, recall that from the induction hypothesis (7.3.25) we know that

$$\sup_{y,x} |x|^{-\alpha} \|(\omega^k, \nabla \omega^k)(y,x)\|_{L^\infty(\mathscr{O})} < 2C_0,$$

thus, one can use (7.3.5) to control the $H^m(\mathscr{O})-$norme of $G^k(\ldots)$:

$$\begin{aligned} \|G(\ldots)\|_{H^m(\mathscr{O})} &= \|G\left(y, x, \theta, |x|^{\frac{n-1}{2}+\alpha}.|x|^{-\alpha}(w^k, \nabla_x w^k)\right)\|_{H^m(\mathscr{O})} \\ &\le C(C_0)|x|^{r\left(\frac{n-1}{2}+\alpha\right)} \| |x|^{-\alpha} (\omega^k, \nabla \omega^k)(y,x)\|_{H^m(\mathscr{O})} \ . \end{aligned}$$

Now, $-(n+3) + 2r\left(\frac{n-1}{2} + \alpha\right) - 2\alpha \ge 0$ if and only if $n \ge 1 + \frac{4}{r-1} - 2\alpha$. The constraint $n \ge 1 + \frac{4}{r-1} - 2\alpha$ ensures that $-(n+3) + 2r\left(\frac{n-1}{2} + \alpha\right) - 2\alpha \ge 0$, and $(-x)^{-(n+3)+2r\left(\frac{n-1}{2}+\alpha\right)-2\alpha}$ is a bounded quantity in the range of coordinates we are concerned with. We then obtain

$$\begin{aligned} \sum_{|\gamma|\le m} \int_{\mathcal{D}_{u,v}} A d\nu dx dy &\le c(\epsilon) \int_0^u \int_{x_0}^v H(x,y) \|\partial_x \omega^{k+1}(y,x)\|^2_{H^m(\mathscr{O})} dx dy \\ &\quad + \epsilon C(C_0) \int_0^u \int_{x_0}^v H(x,y) \|(\omega^k, \nabla \omega^k)(y,x)\|^2_{H^m(\mathscr{O})} dx dy \\ &\quad \text{for} \quad n \ge 1 + \frac{4}{r-1} - 2\alpha \ . \end{aligned} \tag{7.3.27}$$

Similarly,

$$\begin{aligned} \sum_{|\gamma|\le m} \int_{\mathcal{D}_{u,v}} B d\nu dx dy &\le c(\epsilon) \int_0^u \int_{x_0}^v H(x,y) \|\partial_y \omega^{k+1}(y,x)\|^2_{H^m(\mathscr{O})} dx dy \\ &\quad + \epsilon C(C_0) \int_0^u \int_{x_0}^v H(x,y) \|(\omega^k, \nabla \omega^k)(y,x)\|^2_{H^m(\mathscr{O})} dx dy \\ &\quad \text{for} \quad n \ge 1 + \frac{4}{r-1} - 2\alpha \ . \end{aligned} \tag{7.3.28}$$

As far as the terms C and D are concerned, we recall that the commutators read

$$
\begin{aligned}
[\Box_{\eta,y}, \partial_\theta^\gamma]\omega^{k+1} &= \frac{h^{AB}}{\rho^2}\partial_A\partial_B\partial_\theta^\gamma\omega^{k+1} - \Gamma^B\partial_B\partial_\theta^\gamma\omega^{k+1} \\
&\quad -\partial_\theta^\gamma\left(\frac{h^{AB}}{\rho^2}\partial_A\partial_B\omega^{k+1}\right) + \partial_\theta^\gamma\left(\Gamma^B\partial_B\partial_\theta^\gamma\omega^{k+1}\right) \\
&= -\sum_{\gamma_1\neq 0,\ \gamma_1+\gamma_2=\gamma} c(\gamma,\rho)\partial_\theta^{\gamma_1}h^{AB}\partial_\theta^{\gamma_2}\partial_A\partial_B\omega^{k+1} \\
&\quad +\sum_{\gamma_1\neq 0,\ \gamma_1+\gamma_2=\gamma} c(\gamma,\rho)\partial_\theta^{\gamma_1}\Gamma^B\partial_\theta^{\gamma_2}\partial_B\omega^{k+1}\ ;
\end{aligned}
$$

whence, using inequality $ab \le a^2+b^2$ one has:

$$
\sum_{|\gamma|\le m}\int_{\mathcal{D}_{u,v}} C d\nu dx dy \le C(h,\rho)\int_0^u\int_{x_0}^v H(x,y)\|\nabla_x\omega^{k+1}(y,x)\|^2_{H^m(\mathscr{O})}dxdy \tag{7.3.29}
$$

and

$$
\sum_{|\gamma|\le m}\int_{\mathcal{D}_{u,v}} D d\nu dx dy \le C(h,\rho)\int_0^u\int_{x_0}^v H(x,y)\|\nabla_y\omega^{k+1}(y,x)\|^2_{H^m(\mathscr{O})}dxdy\ . \tag{7.3.30}
$$

Summing inequalities (7.3.27)-(7.3.30) gives:

$$
\begin{aligned}
\sum_{|\gamma|\le m}\int_{\mathcal{D}_{u,v}}\left|L^\ell[\partial_\theta^\gamma\omega^{k+1}]\right| dydx\, d\nu &\le c(\epsilon)\int_0^u\int_{x_0}^v H(x,y)\|\nabla\omega^{k+1}(y,x)\|^2_{H^m(\mathscr{O})}dxdy \\
&\quad +\epsilon C(C_0)\int_0^u\int_{x_0}^v H(x,y)\|(\omega^k,\nabla\omega^k)(y,x)\|^2_{H^m(\mathscr{O})}dxdy \\
&\qquad \text{for}\quad n\ge 1+\frac{4}{r-1}-2\alpha\ .
\end{aligned}
$$

We can then rewrite (7.3.26) as:

$$\int_{x_0}^{v} H(u,x)\|(\omega^{k+1},\nabla_x\omega^{k+1})(u,x)\|^2_{H^m(\mathscr{O})}dx + \int_0^u H(y,v)\|(\omega^{k+1},\nabla_y\omega^{k+1})(y,v)\|^2_{H^m(\mathscr{O})}dy \leq$$
$$\int_{x_0}^{v} H(0,x)\|(\omega^{k+1},\nabla_x\omega^{k+1})(0,x)\|^2_{H^m(\mathscr{O})}dx + \int_0^u H(y,x_0)\|(\omega^{k+1},\nabla_y\omega^{k+1})(y,x_0)\|^2_{H^m(\mathscr{O})}dy$$
$$+(c(c_0,\bar{c}_0,n,\rho)+c(c_0,\epsilon)-2\Lambda)\int_0^u\int_{x_0}^v H(x,y)\|(\omega^{k+1},\nabla\omega^{k+1})(y,x)\|^2_{H^m(\mathscr{O})}dxdy$$
$$+\epsilon C(C_0)\int_0^u\int_{x_0}^v H(x,y)\|(\omega^{k},\nabla\omega^{k})(y,x)\|^2_{H^m(\mathscr{O})}dxdy$$
$$\textit{for} \quad n \geq 1+\frac{4}{r-1}-2\alpha\ .$$

All the derivatives appearing in the first term of the second line of the above equation are interior derivatives to the hypersurface $\{y=0\}$ and those of the second term are interior to $\{x=x_0\}$, therefore we can rewrite this last estimate using the initial data of the Cauchy problem (7.3.21):

$$\int_{x_0}^{v} H(u,x)\|(\omega^{k+1},\nabla_x\omega^{k+1})(u,x)\|^2_{H^m(\mathscr{O})}dx$$
$$+\int_0^u H(y,v)\|(\omega^{k+1},\nabla_y\omega^{k+1})(y,v)\|^2_{H^m(\mathscr{O})}dy$$
$$\leq \int_{x_0}^{v} H(0,x)\|(\omega_0^{+,k+1},\nabla_x\omega_0^{+,k+1})(0,x)\|^2_{H^m(\mathscr{O})}dx$$
$$+\int_0^u H(y,x_0)\|(\omega_0^{-,k+1},\nabla_y\omega_0^{-,k+1})(y,x_0)\|^2_{H^m(\mathscr{O})}dy$$
$$+(c(c_0,\bar{c}_0,n,\rho)+c(c_0,\epsilon)-2\Lambda)\int_0^u\int_{x_0}^v H(x,y)\|(\omega^{k+1},\nabla\omega^{k+1})(y,x)\|^2_{H^m(\mathscr{O})}dxdy$$
$$+\epsilon C(C_0)\int_0^u\int_{x_0}^v H(x,y)\|(\omega^{k},\nabla\omega^{k})(y,x)\|^2_{H^m(\mathscr{O})}dxdy$$
$$\textit{for} \quad n \geq 1+\frac{4}{r-1}-2\alpha\ .$$

We choose now in the above estimate the parameter $\Lambda=\Lambda_0$ large enough so that $c(c_0,\bar{c}_0,n,\rho)+c(c_0,\epsilon)-2\Lambda_0<0$ and we have prove the following

Lemma 7.3.5 *Suppose $n \geq 1+\frac{4}{r-1}-2\alpha$. $\forall\epsilon\in]0,1]$, there exists $\Lambda_0 =$*

$\Lambda_0(c, c_0, \bar{c}_0, \epsilon, h) > 0$ *such that* $\forall (u,v) \in [0, y_0] \times [x_0, 0[$ *and* $k \leq j$*, we have:*

$$\int_{x_0}^{v} H(u,x) \|(\omega^{k+1}, \nabla_x \omega^{k+1})(u,x)\|^2_{H^m(\mathscr{O})} dx + \int_0^u H(y,v)\|(\omega^{k+1}, \nabla_y \omega^{k+1})(y,v)\|^2_{H^m(\mathscr{O})} dy \leq$$
$$\int_{x_0}^{v} H(0,x)\|(\omega_0^{+,k+1}, \nabla_x \omega_0^{+,k+1})(x)\|^2_{H^m(\mathscr{O})} dx + \int_0^u H(y,x_0)\|(\omega_0^{-,k+1}, \nabla_y \omega_0^{-,k+1})(y)\|^2_{H^m(\mathscr{O})} dy$$
$$+\epsilon C(C_0) \int_0^u \int_{x_0}^{v} H(x,y) \|(\omega^k, \nabla \omega^k)(y,x)\|^2_{H^m(\mathscr{O})} dx dy \, . \qquad (7.3.31)$$

One would like to get rid of the dependence of k *in the right-hand side of the above estimate. We proceed as follows. Set*

$$\begin{aligned}\hat{C}(u,v) &= |x_0|^{-2\alpha} e^{-\Lambda_0 x_0} \sup_{k\in\mathbb{N}} \int_0^u e^{-\Lambda_0 y} \|(\omega_0^{-,k}, \nabla_y \omega_0^{-,k})(y)\|^2_{H^m(\mathscr{O})} dy \\ &\quad + \sup_{k\in\mathbb{N}} \int_{x_0}^{v} |x|^{-2\alpha} e^{-\Lambda_0 x} \|(\omega_0^{+,k}, \nabla_x \omega_0^{+,k})(x)\|^2_{H^m(\mathscr{O})} dx \\ &\quad + \frac{1}{2(y_0 + |x_0|)} \int_0^u \int_{x_0}^{v} H(y,x) \|(\omega^0, \nabla\omega^0)\|^2_{H^m(\mathscr{O})} \, . \qquad (7.3.32)\end{aligned}$$

We notice that $\forall (u,v) \in [0,y_0] \times [x_0, 0]$*, the quantity* $\hat{C}(u,v)$ *is finite. Indeed we have*

$$\begin{aligned}\hat{C}(u,v) &\leq c(x_0, y_0, \Lambda_0) \left(\sup_{k\in\mathbb{N}} \|\omega_0^{-,k}\|^2_{H^{m+1}(\mathcal{C}^-)} + \sup_{k\in\mathbb{N}} \|(\omega_0^{+,k}, \nabla_x \omega_0^{+,k})\|^2_{\mathscr{H}^\alpha_m(\mathcal{C}^+)} \right) \\ &\quad + c(x_0, y_0, \Lambda_0) \int_0^u \int_{x_0}^{v} |x|^{-2\alpha} \|(\omega^0, \nabla \omega^0)\|^2_{H^m(\mathscr{O})} \, .\end{aligned}$$

The two terms in the first line of this estimate are bounded because convergent sequences (see Lemma 7.3.3), are bounded. From (7.3.22) we have

$$\begin{aligned}&\int_0^u \int_{x_0}^{v} |x|^{-2\alpha} \|(\omega^0, \nabla\omega^0)\|^2_{H^m(\mathscr{O})} \leq \\ &\quad C\Big(u \|(\omega_0^{0,+}, \nabla_x \omega_0^{0,+})\|^2_{\mathscr{H}^\alpha_m(\mathcal{C}^+)} + (v - x_0) \|(\omega_0^{0,-}, \nabla_y \omega_0^{0,-})\|^2_{H^m(\mathcal{C}^-)} \\ &\quad + u(v - x_0) \|\omega_0^{-,0}(0)\|^2_{H^{m+1}(\mathscr{O})} \Big) < \infty \, .\end{aligned}$$

This proves that (7.3.32) defines a finite quantity. Now by the definition of this constant, (7.3.31) implies:

$$\begin{aligned}&\int_{x_0}^{v} H(u,x) \|(\omega^{k+1}, \nabla_x \omega^{k+1})(u,x)\|^2_{H^m(\mathscr{O})} dx \leq \hat{C}(u,v) \\ &\qquad + \epsilon C(C_0, c_0) \int_0^u \int_{x_0}^{v} H(x,y) \|(\omega^k, \nabla\omega^k)(y,x)\|^2_{H^m(\mathscr{O})} dx dy \qquad (7.3.33)\end{aligned}$$

Suppose that for all $u, v \in [0, y_0] \times [x_0, 0[$

$$\int_0^u \int_{x_0}^v H(x,y) \|(\omega^k, \nabla\omega^k)(y,x)\|^2_{H^m(\mathscr{O})} dxdy \le 2\hat{C}(y_0, x_0)(y_0 + |x_0|) \,. \tag{7.3.34}$$

After integration with respect to y *on* $[0, u]$*, inequality (7.3.33) gives*

$$\begin{aligned}\int_0^u \int_{x_0}^v H(u,x) \|(\omega^{k+1}, \nabla_x\omega^{k+1})(u,x)\|^2_{H^m(\mathscr{O})} dxdy &\le \hat{C}(y_0, 0)y_0 + 2\epsilon C(C_0, c_0)\hat{C}(y_0 + |x_0|)y_0 \\ &\le 2\hat{C}(y_0, 0)y_0 \,,\end{aligned}$$

if ϵ *small enough. Using now this inequality in (7.3.31) leads to:*

$$\int_0^u H(y,v) \|(\omega^{k+1}, \nabla_y\omega^{k+1})(y,v)\|^2_{H^m(\mathscr{O})} dy \le \hat{C}(y_0, 0) + 2\epsilon C(C_0)\hat{C}(y_0, 0)(y_0 + |x_0|) \,.$$

Again by integration, we have

$$\begin{aligned}\int_0^u \int_{x_0}^v H(y,v) \|(\omega^{k+1}, \nabla_y\omega^{k+1})(y,v)\|^2_{H^m(\mathscr{O})} dy &\le \Big(\hat{C}(y_0, 0) + 2\epsilon C(C_0)\hat{C}(y_0 + |x_0|)\Big)|x_0| \\ &\le 2\hat{C}(y_0, 0)|x_0| \quad \text{if } \epsilon \text{ is small enough} \,.\end{aligned}$$

Therefore, assuming that (7.3.34) is true, we have proved that

$$\int_0^u \int_{x_0}^v H(x,y) \|(\omega^{k+1}, \nabla\omega^{k+1})(y,x)\|^2_{H^m(\mathscr{O})} dxdy \le 2\hat{C}(y_0, 0)(y_0 + |x_0|) \,.$$

Considering the definition of the constant $\hat{C}(u,v)$ *(see (7.3.32)), we then obtain that (7.3.34) holds for* $k = 0$*, and one can conclude that for any* $k \in \mathbb{N}$ *inequality (7.3.34) is satisfied. We have proved*

Lemma 7.3.6 *Suppose that the constant* $\hat{C}$ *is defined by (7.3.32). One can choose* $\epsilon = \epsilon_0(c_0, \bar{c}_0, x_0, y_0, C_0, h)$ *such that*

$$\sup_{k\in\mathbb{N},\, (u,v)\in[0,y_0]\times[x_0,0[} \int_0^u \int_{x_0}^v H(x,y) \|(\omega^k, \nabla\omega^k)(y,x)\|^2_{H^m(\mathscr{O})} dxdy \le 2\hat{C}(y_0, 0)(y_0 + |x_0|) \,, \tag{7.3.35}$$

and for any $\Lambda \ge \Lambda_0$,

$$\begin{aligned}&\int_{x_0}^v H(u,x) \|(\omega^{k+1}, \nabla_x\omega^{k+1})(u,x)\|^2_{H^m(\mathscr{O})} dx \\ &\quad + \int_0^u H(y,v) \|(\omega^{k+1}, \nabla_y\omega^{k+1})(y,v)\|^2_{H^m(\mathscr{O})} dy \le 2\hat{C}(y_0, 0)\end{aligned} \tag{7.3.36}$$

REMARK 7.3.7 Note that as we assume that the induction hypothesis holds for any $k \leq j$, inequality (7.3.36) hold for any $k \leq j$.

Recall $\gamma \in \mathbb{N}^{n-1}$, such that $|\gamma| \leq m-1$. To proceed further we apply ∂_θ^γ to the differential equation satisfies by ω^{k+1} and then multiply the differentiated equation by $H\partial_\theta^\gamma \partial_y \omega^{k+1}$. As in the proof of the Proposition 7.3.1, we obtain:

$$\begin{aligned}\partial_x \left(H(\partial_y \partial_\theta^\gamma \omega^{k+1})^2 \right) &= (\partial_x H - H\frac{n-1}{2\rho})(\partial_y \partial_\theta^\gamma \omega^{k+1})^2 + H\frac{n-1}{2\rho}\partial_y \partial_\theta^\gamma \omega^{k+1} \partial_x \partial_\theta^\gamma \omega^{k+1} \\ &+\partial_y \partial_\theta^\gamma \omega^{k+1} \sum_{\gamma_1+\gamma_2=\gamma} H\frac{\partial_\theta^{\gamma_1} h^{AB}}{2\rho^2} \partial^{\gamma_2} \partial_A \partial_B \omega^{k+1} \\ &-\partial_y \partial_\theta^\gamma \omega^{k+1} \sum_{\gamma_1+\gamma_2=\gamma} H\frac{\partial_\theta^{\gamma_1} \Gamma^B}{2\rho^2} \partial_\theta^{\gamma_2} \partial_B \omega^{k+1} \\ &-\frac{1}{2}|x|^{-\frac{n+3}{2}} H\partial_y \partial_\theta^\gamma \omega^{k+1} \partial_\theta^\gamma G^k(\ldots) \, .\end{aligned} \tag{7.3.37}$$

We integrate this identity on the set $\{y\} \times [x_0, x] \times \mathscr{O}$. From Stokes' theorem we have for $n \geq 1 + \frac{4}{r-1} - 2\alpha$:

$$\begin{aligned}H(y,x)\|\partial_y \omega^{k+1}(y,x)\|^2_{H^{m-1}(\mathscr{O})} &\leq H(y,x_0)\|\partial_y \omega^{k+1}(y,x_0)\|^2_{H^{m-1}(\mathscr{O})} \\ &+(c_2(h,c_0,\bar{c}_0) + c(\epsilon) - \Lambda) \int_{x_0}^x H(y,s)\|\partial_y \omega^{k+1}(y,s)\|^2_{H^{m-1}(\mathscr{O})} ds \\ &+c_3(h,c_0,\bar{c}_0) \int_{x_0}^x H(y,x)\|\nabla_x \omega^{k+1}(y,s)\|^2_{H^m(\mathscr{O})} ds \\ &+\epsilon C(C_0) \int_{x_0}^x H(y,s)\|(\omega^k, \nabla \omega^k)(y,s)\|^2_{H^{m-1}(\mathscr{O})} ds.\end{aligned}$$

As we did before, we choose in the above inequality $\Lambda = \Lambda_1(h, c_0, \bar{c}_0, \epsilon)$ large enough so as to get rid of the terms containing $\|\partial_y \omega^{k+1}(y,s)\|^2_{H^{m-1}(\mathscr{O})}$. We then obtain

$$\begin{aligned}H(y,x)\|\partial_y \omega^{k+1}(y,x)\|^2_{H^{m-1}(\mathscr{O})} &\leq H(y,x_0)\|\partial_y \omega_0^{-,k+1}(y)\|^2_{H^{m-1}(\mathscr{O})} \\ &+c_3(h,c_0,\bar{c}_0) \int_{x_0}^x H(y,s)\|\nabla_x \omega^{k+1}(y,s)\|^2_{H^m(\mathscr{O})} ds \\ &+\epsilon C(C_0) \int_{x_0}^x H(y,s)\|(\omega^k, \nabla \omega^k)(y,s)\|^2_{H^{m-1}(\mathscr{O})} ds \, .\end{aligned} \tag{7.3.38}$$

Then according to Lemma 7.3.6, we estimates the terms containing $\|\nabla_x \omega^{k+1}(y,s)\|^2_{H^m(\mathscr{O})}$ *and* $\|(\omega^k, \nabla_x \omega^k)(y,s)\|^2_{H^{m-1}(\mathscr{O})}$ *by using inequality (7.3.36) twice: first as it is written and secondly by replacing in that inequality* k *with* $k-1$ *(which remains true according to Remark 7.3.7):*

$$\begin{aligned} H(y,x)\|\partial_y \omega^{k+1}(y,x)\|^2_{H^{m-1}(\mathscr{O})} \quad \leq \quad & H(y,x_0)\|\partial_y \omega_0^{-,k+1}(y)\|^2_{H^{m-1}(\mathscr{O})} + 2\epsilon C(C_0)\hat{C}(y,x) \\ & + 2c_3(h,c_0,\bar{c}_0)\hat{C}(y,x) \\ & + \epsilon C(C_0)\int_{x_0}^{x} H(y,s)\|\partial_y \omega^k(y,s)\|^2_{H^{m-1}(\mathscr{O})} ds \ . \qquad (7.3.39) \end{aligned}$$

We then integrate (7.3.39) with respect to x *on* $[x_0, v]$ *for any* $v \in [x_0, 0[$

$$\begin{aligned} & \int_{x_0}^{v} \overline{H}(x)\|\partial_y \omega^{k+1}(y,x)\|^2_{H^{m-1}(\mathscr{O})} dx \leq \\ & \quad |x_0| \left(\overline{H}(x_0)\|\partial_y \omega_0^{-,k+1}(y)\|^2_{H^{m-1}(\mathscr{O})} + 2e^{\lambda y}\epsilon C(C_0)\hat{C}(y,0) \right) \\ & \quad + 2c_2(h,c_0,\bar{c}_0)\hat{C}(y,0)|x_0|e^{\lambda y} \\ & \quad + \epsilon C(C_0)\int_{x_0}^{v}\int_{x_0}^{x} \overline{H}(s)\|\partial_x \omega^k(y,s)\|^2_{H^{m-1}(\mathscr{O})} ds dx \ ; \end{aligned}$$

where $\overline{H}(x) = |x|^{-2\alpha}e^{-\Lambda x}$. *Let*

$$\begin{aligned} \widetilde{C}(y_0,0) \quad = \quad & \sup_{k\in\mathbb{N},\ y\in[0,y_0]} \Big\{ |x_0| \Big(\overline{H}(x_0)\|\partial_y \omega_0^{-,k+1}(y)\|^2_{H^{m-1}(\mathscr{O})} \\ & + 2e^{\lambda y}\epsilon C(C_0)\hat{C}(y,0) \Big) + 2c_2(h,c_0,\bar{c}_0)\hat{C}(y_0,0)|x_0|e^{\Lambda y} \Big\} \\ & + \frac{1}{2}\sup_{y\in[0,y_0]} \int_{x_0}^{0} \overline{H}(x)\|\partial_y \omega^0(y,x)\|^2_{H^{m-1}(\mathscr{O})} dx \ . \qquad (7.3.40) \end{aligned}$$

Again we need to prove that $\widetilde{C}(y_0,0)$ *is a finite quantity. For all* $\gamma \in \mathbb{N}^{n-1}$ *such that* $|\gamma| \leq m-1$, *we have the following trivial identity*

$$\partial_y |\partial_\theta^\gamma \partial_y \omega_0^{-,k+1}|^2 = 2\partial_y \partial_\theta^\gamma \partial_y^2 \omega_0^{-,k+1}.\partial_\theta^\gamma \partial_y \omega_0^{-,k+1} \leq |\partial_\theta^\gamma \partial_y^2 \omega_0^{-,k+1}|^2 + |\partial_\theta^\gamma \partial_y \omega_0^{-,k+1}|^2 \ .$$

Integrating with respect to y *on the interval* $[0, y_0]$ *gives*

$$\begin{aligned} |\partial_\theta^\gamma \partial_y \omega_0^{-,k+1}(y,\theta)|^2 \quad = \quad & |\partial_\theta^\gamma \partial_y \omega_0^{-,k+1}(0,\theta)|^2 \\ & + \int_0^{y_0} \left(|\partial_\theta^\gamma \partial_y^2 \omega_0^{-,k+1}(s,\theta)|^2 + |\partial_\theta^\gamma \partial_y \omega_0^{-,k+1}(s,\theta)|^2 \right) ds. \end{aligned}$$

We integrate this new identity now with respect to the angular variables on $\mathscr{O}$ and obtain that

$$\|\partial_y\omega_0^{-,k+1}(y)\|^2_{H^{m-1}(\mathscr{O})} \leq \|\partial_y\omega_0^{-,k+1}(0)\|^2_{H^{m-1}(\mathscr{O})} + 2\|\omega_0^{-,k+1}\|^2_{H^{m+1}(\mathcal{C}^-)} \,.$$

By the trace theorem (recall $\partial\mathcal{C}^- = (\{0\}\times\mathscr{O})\cup(\{y_0\}\times\mathscr{O})$):

$$\|\partial_y\omega_0^{-,k+1}(0)\|^2_{H^{m-1}(\mathscr{O})} \leq c\|\partial_y\omega_0^{-,k+1}\|^2_{H^{m}(\mathcal{C}^-)} \leq c\|\omega_0^{-,k+1}\|^2_{H^{m+1}(\mathcal{C}^-)} \,.$$

It then follows that

$$\|\partial_y\omega_0^{-,k+1}(y)\|^2_{H^{m-1}(\mathscr{O})} \leq c\|\omega_0^{-,k+1}\|^2_{H^{m+1}(\mathcal{C}^-)} < \infty \,. \tag{7.3.41}$$

On the other hand, by the Equation 7.3.22 page 146 which defined ω_0, we have

$$\begin{aligned}\int_{x_0}^0 \overline{H}(x)\|\partial_y\omega^0(y,x)\|^2_{H^{m-1}(\mathscr{O})}dx &\leq \|\partial_y\omega_0^{-,0}(y)\|^2_{H^{m-1}(\mathscr{O})}\int_{x_0}^0 |x|^{-2\alpha}e^{-\Lambda x}dx \\ &\leq C \quad \text{independently of } y \,. \end{aligned}\tag{7.3.42}$$

The estimates (7.3.41) and (7.3.42) prove that (7.3.40) defines a finite quantity. By the definition of $\widetilde{C}(y_0,0)$ we have the following form of inequality (7.3.39)

$$\begin{aligned}\int_{x_0}^v \overline{H}(x)\|\partial_y\omega^{k+1}(y,x)\|^2_{H^{m-1}(\mathscr{O})}dx &\leq \widetilde{C}(y_0,0) \\ &+\epsilon C(C_0)\int_{x_0}^v\int_{x_0}^x H(y,s)\|\partial_y\omega^k(y,s)\|^2_{H^{m-1}(\mathscr{O})}dsdx \,.\end{aligned}\tag{7.3.43}$$

Suppose that $\forall v\in[x_0,0[$,

$$\int_{x_0}^v \overline{H}(x)\|\partial_y\omega^k(y,x)\|^2_{H^{m-1}(\mathscr{O})}dx \leq 2\widetilde{C}(y_0,0). \tag{7.3.44}$$

Then inequality (7.3.43) gives:

$$\begin{aligned}\int_{x_0}^v \overline{H}(x)\|\partial_y\omega^{k+1}(y,x)\|^2_{H^{m-1}(\mathscr{O})}dx &\leq \widetilde{C}(y_0,0) + 2\epsilon C(C_0)\widetilde{C}(u_0,0)|x_0| \\ &\leq 2\widetilde{C}(y_0,0), \quad \text{for } \epsilon \text{ small enough.}\end{aligned}$$

Note that from the definition of the constant $\widetilde{C}(y_0,0)$, inequality (7.3.44) remains true when $k=0$ and then one can conclude that it holds for any integer $k\in\mathbb{N}$. Inequality (7.3.39) then implies:

$$|x|^{-2\alpha}\|\partial_y\omega^{k+1}(y,x)\|^2_{H^{m-1}(\mathscr{O})} \leq C_1(c_0,\bar{c}_0,h,C_0,\Lambda_0,\Lambda_1,\epsilon,\hat{c},\widetilde{C}) \,. \tag{7.3.45}$$

In order to obtain the analog of (7.3.45) with instead $\partial_x \omega^{k+1}$*, we repeat the previous argument. Once more we differentiate with* ∂_θ^γ *the equation satisfies by* ω^{k+1} *and multiply the resulting equation by* $H\partial_\theta^\gamma \partial_x \omega^{k+1}$ *and obtain*

$$\begin{aligned}
\partial_y \left(H(\partial_x \partial_\theta^\gamma \omega^{k+1})^2 \right) &= (\partial_y H + H\frac{n-1}{2\rho})(\partial_x \partial_\theta^\gamma \omega^{k+1})^2 - H\frac{n-1}{2\rho}\partial_y \partial_\theta^\gamma \omega^{k+1} \partial_x \partial_\theta^\gamma \omega^{k+1} \\
&\quad + \partial_x \partial_\theta^\gamma \omega^{k+1} \sum_{\gamma_1+\gamma_2=\gamma} H\frac{\partial_\theta^{\gamma_1} h^{AB}}{2\rho^2} \partial_\theta^{\gamma_2} \partial_A \partial_B \omega^{k+1} \\
&\quad \partial_x \partial_\theta^\gamma \omega^{k+1} \sum_{\gamma_1+\gamma_2=\gamma} H\frac{\partial_\theta^{\gamma_1} \Gamma^B}{2\rho^2} \partial_\theta^{\gamma_2} \partial_B \omega^{k+1} \\
&\quad -\frac{1}{2}|x|^{-\frac{n+3}{2}} H \partial_x \partial_\theta^\gamma \omega^{k+1} \partial_\theta^\gamma G^k(\ldots)\ .
\end{aligned}$$

Then, we integrate on $[0,y] \times \{x\} \times \mathscr{O}$*, and obtain for* $n \geq 1 + \frac{4}{r-1} - 2\alpha$ *via Stokes theorem*

$$\begin{aligned}
H(y,x)\|\partial_x \omega^{k+1}(y,x)\|^2_{H^{m-1}(\mathscr{O})} &\leq H(0,x)\|\partial_x \omega_0^{+,k+1}(x)\|^2_{H^{m-1}(\mathscr{O})} \\
&\quad +(c_4(h,c_0,\bar{c}_0) + c(\epsilon) - \Lambda) \int_0^y H(s,x)\|\partial_x \omega^{k+1}(s,x)\|^2_{H^{m-1}(\mathscr{O})} ds \\
&\quad +c_5(h,c_0,\bar{c}_0) \int_0^y H(s,x)\|\nabla_y \omega^{k+1}(s,x)\|^2_{H^m(\mathscr{O})} ds \\
&\quad +\epsilon C(C_0) \int_0^y H(s,x)\|(\omega^k, \nabla\omega^k)(s,x)\|^2_{H^{m-1}(\mathscr{O})} ds
\end{aligned}$$

As we did previously, we choose in this inequality $\Lambda = \Lambda_2(h, c_0, \bar{c}_0, \epsilon)$ *large enough so as to get rid of the terms with* $\|\partial_x \omega^{k+1}(y,s)\|^2_{H^{m-1}(\mathscr{O})}$ *and we obtain:*

$$\begin{aligned}
H(y,x)\|\partial_x \omega^{k+1}(y,x)\|^2_{H^{m-1}(\mathscr{O})} &\leq H(0,x)\|\partial_x \omega_0^{+,k+1}(x)\|^2_{H^{m-1}(\mathscr{O})} \\
&\quad +c_5(h,c_0,\bar{c}_0) \int_0^y H(s,x)\|\nabla_y \omega^{k+1}(s,x)\|^2_{H^m(\mathscr{O})} ds \\
&\quad +\epsilon C(C_0) \int_0^y H(s,x)\|(\omega^k, \nabla\omega^k)(s,x)\|^2_{H^{m-1}(\mathscr{O})} ds\ .
\end{aligned} \tag{7.3.46}$$

By Lemma 7.3.6, we estimate the quantities $\|\nabla_y\omega^{k+1}(y,s)\|^2_{H^m}$ *and* $\|(\omega^k,\nabla_y\omega^k)(y,s)\|^2_{H^{m-1}(\mathscr{O})}$ *using inequality (7.3.36):*

$$\begin{aligned} H(y,x)\|\partial_x\omega^{k+1}(y,x)\|^2_{H^{m-1}(\mathscr{O})} \le\;& H(0,x)\|\partial_x\omega_0^{+,k+1}(x)\|^2_{H^{m-1}(\mathscr{O})}+2\epsilon C(C_0)\hat{C}(y,x)\\ &+2c_5(h,c_0,\bar{c}_0)\hat{C}(y,x)\\ &+\epsilon C(C_0)\int_0^y H(s,x)\|\partial_x\omega^k(s,x)\|^2_{H^{m-1}(\mathscr{O})}ds\ . \end{aligned} \tag{7.3.47}$$

We integrate in y*, and obtain that for any* $u\in[0,y_0]$,

$$\begin{aligned} \int_0^u \widetilde{H}(y,x)\|\partial_x\omega^{k+1}(y,x)\|^2_{H^{m-1}(\mathscr{O})}dy \le\;& y_0\widetilde{H}(0,x)\|\partial_x\omega_0^{+,k+1}(x)\|^2_{H^{m-1}(\mathscr{O})}\\ &+2y_0e^{\Lambda x}\epsilon C(C_0)\hat{C}(y_0,x)+2c_5(h,c_0,\bar{c}_0)\hat{C}(y_0,x)y_0e^{\Lambda x}\\ &+\epsilon C(C_0)\int_0^u\int_0^y \widetilde{H}(s,x)\|\partial_x\omega^k(y,s)\|^2_{H^{m-1}(\mathscr{O})}dsdy\ , \end{aligned} \tag{7.3.48}$$

where $\widetilde{H}(y,x)=|x|^{-2\alpha}e^{-\Lambda y}$*. Now, we define a new constant* $\check{C}(y_0,0)$ *as*

$$\begin{aligned} \check{C}(y_0,0) =& \sup_{k\in\mathbb{N},\,x\in[x_0,0[}\Big\{y_0\Big(\widetilde{H}(0,x)\|\partial_x\omega_0^{+,k+1}(x)\|^2_{H^{m-1}(\mathscr{O})}+2e^{\Lambda x}\epsilon C(C_0)\hat{C}(y_0,x)\Big)\\ &+2c_5(h,c_0,\bar{c}_0)\hat{C}(y_0,0)|x_0|e^{\Lambda x}\Big\}\\ &+\frac{1}{2}\sup_{x\in[x_0,0[}\int_0^{y_0}\widetilde{H}(y,x)\|\partial_x\omega^0(s,x)\|_{H^{m-1}(\mathscr{O})}. \end{aligned} \tag{7.3.49}$$

As before let us prove that (7.3.49) is finite. By the Lemma 7.3.3 page 142

$$\sup_{k\in\mathbb{N},\,x\in[x_0,0[}\widetilde{H}(0,x)\|\partial_x\omega_0^{+,k+1}(x)\|^2_{H^{m-1}(\mathscr{O})}<\infty\ . \tag{7.3.50}$$

Next, by the definition of ω_0 *given by (7.3.22) page 146, we have*

$$\begin{aligned} \int_0^{y_0}\widetilde{H}(s,x)\|\partial_x\omega^0(s,x)\|^2_{H^{m-1}(\mathscr{O})}ds \le\;& \|\partial_x\omega_0^{+,0}(x)\|^2_{H^{m-1}(\mathscr{O})}|x|^{-2\alpha}\int_0^{y_0}e^{-\Lambda s}ds\\ \le\;& C\quad \textit{independently of } x\ . \end{aligned} \tag{7.3.51}$$

From the estimates (7.3.50) and (7.3.51) we obtain that (7.3.49) define a finite quantity. We thus obtain the following form of (7.3.48):

$$\int_0^u\widetilde{H}(y,x)\|\partial_x\omega^{k+1}(y,x)\|^2_{H^{m-1}(\mathscr{O})}dy\le\check{C}+\epsilon C(C_0)\int_0^u\int_0^y\widetilde{H}(s,x)\|\partial_x\omega^k(s,x)\|^2_{H^{m-1}(\mathscr{O})}dsdy\ . \tag{7.3.52}$$

Suppose again that $\forall u \in [0, y_0]$,

$$\int_0^u \widetilde{H}(y,x)\|\partial_x \omega^k(y,x)\|^2_{H^{m-1}(\mathscr{O})} dy \leq 2\check{C}(y_0, 0). \tag{7.3.53}$$

Then inequality (7.3.52) gives:

$$\begin{aligned}\int_0^u \widetilde{H}(y,x)\|\partial_x \omega^{k+1}(y,x)\|^2_{H^{m-1}(\mathscr{O})} dy &\leq \check{C}(y_0,0) + 2\epsilon C(C_0)\check{C}(y_0,0)y_0 \\ &\leq 2\check{C}(y_0,0), \quad \textit{if } \epsilon \textit{ small enough.}\end{aligned}$$

By the definition of the constant $\check{C}(y_0, 0)$, *(7.3.53) is satisfied for* $k = 0$ *and so it is for any integer* $k \in \mathbb{N}$. *Inequality (7.3.47) implies:*

$$|x|^{-2\alpha}\|\partial_x \omega^{k+1}(y,x)\|^2_{H^{m-1}(\mathscr{O})} \leq C_2(c_0, \bar{c}_0, h, C_0, \Lambda_0, \Lambda_2, \epsilon, \hat{C}, \check{C}) \; . \tag{7.3.54}$$

It remains to control the $H^m(\mathscr{O})$ *norms of* ω^{k+1}, *that is to control its angular derivatives. Let* $(y,x) \in [0, y_0] \times [x_0, 0[$, $\gamma \in \mathbb{N}^{n-1}$ *such that* $|\gamma| \leq m$

$$|x|^{-\alpha}e^{-\frac{\Lambda}{2}y}|\partial_\theta^\gamma \omega^{k+1}(y,x)| \quad \leq \quad |x|^{-\alpha}|\partial_\theta^\gamma \omega_0^{+,k+1}(x)| + \int_0^y |x|^{-\alpha}|e^{-\frac{\Lambda}{2}s}\partial_y \partial_\theta^\gamma \omega^{k+1}(s,x)|ds.$$

It then follows that:

$$|x|^{-2\alpha}|e^{-\Lambda y}\partial_\theta^\gamma \omega^{k+1}(y,x)|^2 \leq 2\left(|x|^{-2\alpha}\partial_\theta^\gamma \omega_0^{+,k+1}(x)|^2 + y_0\int_0^y |x|^{-2\alpha}|e^{-\Lambda s}\partial_\theta^\gamma \partial_y \omega^{k+1}(s,x)|^2 ds\right) .$$

By integration we then obtain:

$$\begin{aligned}&|x|^{-2\alpha}e^{-\Lambda y}\|\omega^{k+1}(y,x)\|^2_{H^{m-1}(\mathscr{O})} \leq \\ &\quad 2\left(|x|^{-2\alpha}\|\omega_0^{+,k+1}(x)\|^2_{H^{m-1}(\mathscr{O})} + y_0\int_0^y |x|^{-2\alpha}e^{-\Lambda s}\|\partial_y \omega^{k+1}(s,x)\|^2_{H^{m-1}}(\mathscr{O})ds\right) \\ &\quad \leq C_3(\check{C}) \; .\end{aligned}$$

We have proved the following Lemma:

Lemma 7.3.8 *Let* $m \in \mathbb{N}^*$. *If* $n \geq 1 + \frac{4}{r-1} - 2\alpha$, *then there exists a positive constant* $C_4 = C_4(c_0, \bar{c}_0, h, \Lambda_0, \Lambda_1, \Lambda_2, \epsilon_0)$ *such that:*

$$\sup_{(y,x)\in[0,y_0]\times[x_0,0[} x^{-\alpha}\|(\omega^{k+1}, \nabla\omega^{k+1})\|_{H^{m-1}(\mathscr{O})} \leq C_4 \; . \tag{7.3.55}$$

Now to prove that (7.3.25) holds for $k = j+1$ we are going to show that it suffices to replace in (7.3.55) y_0 with a certain u_* sufficiently small. Let $j_0 \in \mathbb{N}^*$. if

$$m-1 > \frac{n-1}{2} + j_0 \ ,$$

then from (7.3.55) and the Sobolev embedding theorem, for all $(y,x) \in [0,y_0] \times [x_0, 0[$, we have:

$$|x|^{-\alpha}\|(\omega^{k+1}, \nabla\omega^{k+1})(y,x)\|_{C^{j_0}(\mathscr{O})} \leq C \ . \tag{7.3.56}$$

It then follows from the differential equation satisfies by ω^{k+1} that

$$\|\partial_y(|x|^{-\alpha}\partial_x\omega^{k+1})(y,x)\|_{C^{j_0-1}(\mathscr{O})} \leq C \ , \tag{7.3.57}$$

$$\|x\partial_x(|x|^{-\alpha}\partial_y\omega^{k+1})(y,x)\|_{C^{j_0-1}(\mathscr{O})} \leq C \ . \tag{7.3.58}$$

Integrating (7.3.57) in y from $(0,x)$ to (y,x) we find that for $j_0 \geq 2$,

$$\begin{aligned} |x|^{-\alpha}\|\partial_x\omega^{k+1}(y,x)\|_{C^1(\mathscr{O})} &\leq |x|^{-\alpha}\|\partial_x\omega_0^{+,k+1}(0,x)\|_{C^1(\mathscr{O})} + Cy \\ &\leq C_0 + Cy \\ &\leq 2C_0 \quad \text{if} \quad y \leq u_1 \ . \end{aligned} \tag{7.3.59}$$

Note that inequality (7.3.59) shall be read as a first condition in the determination of u_*. Further, to control $|x|^{-\alpha}\|\partial_y\omega^{k+1}(y,x)\|_{C^1(\mathscr{O})}$, we $y-$differentiate the differential equation satisfied by ω^{k+1}. We write here

$$G(z, x^{\frac{n-1}{2}}(\omega^k, \partial_x\omega^k, \nabla_y\omega^k) = G(z, x^{\frac{n-1}{2}}(p_1, p_2, p_3))$$

so that $\partial_y G^k(\ldots)$ reads

$$\begin{aligned} \partial_y G^k(\ldots) &= (\partial_y G)^k(\ldots) + |x|^{\frac{n-1}{2}}\partial_y\omega^k \left(\frac{\partial G}{\partial p_1}\right)^k(\ldots) \\ &\quad + |x|^{\frac{n-1}{2}}\partial_y\partial_x\omega^k \left(\frac{\partial G}{\partial p_2}\right)^k(\ldots) + |x|^{\frac{n-1}{2}}\partial_y\nabla_y\omega^k \left(\frac{\partial G}{\partial p_3}\right)^k(\ldots) \ . \end{aligned}$$

The differentiated equation reads

$$\partial_x(\partial_y\nabla_y\omega^{k+1}) = \xi\partial_y\nabla_y\omega^{k+1} + \psi^k\partial_y\nabla_y\omega^k + \varphi^k \ , \tag{7.3.60}$$

where we have set

$$4\xi = \left(-\tfrac{n-1}{\rho} \quad , \quad 0\right), \qquad 4\psi^k = -|x|^{-2}\left(\frac{\partial G}{\partial p_3}\right)^k(\ldots) \ ,$$

and where the components of φ^k are given by

$$\begin{aligned}
4\varphi^k_y(y,x) &= \frac{n-1}{\rho}\partial_x\partial_y\omega^{k+1}(y,x) + \frac{h^{AB}}{\rho^3}\partial_A\partial_B\omega^{k+1}(y,x) + \frac{h^{AB}}{\rho^2}\partial_y\partial_A\partial_B\omega^{k+1}(y,x) \\
&\quad + \frac{n-1}{2\rho^2}(\partial_x - \partial_y)\omega^{k+1} - \frac{\Gamma^B}{\rho^2}\partial_y\partial_B\omega^{k+1} - \frac{\Gamma^B}{\rho^3}\partial_B\omega^{k+1} - (-x)^{\frac{n+3}{2}}(\partial_y G)^k(\ldots) \\
&\quad - x^{-2}\left(\partial_y\omega^k\left(\frac{\partial G}{\partial p_1}\right)^k(\ldots) + \partial_y\partial_x\omega^k\left(\frac{\partial G}{\partial p_2}\right)^k(\ldots)\right) , \\
\varphi^k_A(y,x) &= \partial_y\partial_x\partial_A\omega^{k+1} .
\end{aligned}$$

By hypothesis (7.3.4)) and the induction assumption (7.3.25), we have the following estimate on $\psi^k(y,x)$ for all $(y,x) \in [0,y_0]\times[x_0,0[$:

$$\begin{aligned}
\|\psi^k(y,x)\|_{C^{j_0}} &\le C(|x|^{-\alpha}\|(\omega^k,\nabla\omega^k)(y,x)\|_{L^\infty(\mathscr{O})})|x|^{-2+(r-1)(\frac{n-1}{2}+\alpha)} \\
&\le C \qquad \text{if} \quad n \ge 1 + \frac{4}{r-1} - 2\alpha . \qquad (7.3.61)
\end{aligned}$$

By using simultaneously (7.3.56) and (7.3.57), for all $(y,x)\in[0,y_0]\times[x_0,0[$ we have:

$$|x|^{-\alpha}\|\varphi^k(y,x)\|_{C^{j_0-2}(\mathscr{O})} < C \quad \text{if} \quad n \ge 1 + \frac{4}{r-1} - 2\alpha . \qquad (7.3.62)$$

On the other hand we have:

$$\begin{aligned}
\partial_x\left(\overline{H}(x)|\partial_y\nabla_y\omega^{k+1}|^2\right) &= (2\alpha|x|^{-1} - \Lambda)\overline{H}(x)|\partial_y\nabla_y\omega^{k+1}|^2 + 2\overline{H}(x)\partial_x\partial_y\nabla_y\omega^{k+1}.\partial_y\nabla_y\omega^{k+1} \\
&\le -\Lambda\overline{H}(x)|\partial_y\nabla_y\omega^{k+1}|^2 + 2\overline{H}(x)\partial_x\partial_y\nabla_y\omega^{k+1}.\partial_y\nabla_y\omega^{k+1} .
\end{aligned}$$

Considering now inequalities (7.3.61) and (7.3.62), we deduce that for $j_0 \ge 2$,

$$\begin{aligned}
\partial_x\left(\overline{H}(x)|\partial_y\nabla_y\omega^{k+1}|^2\right) &\le -\Lambda\overline{H}(x)|\partial_y\nabla_y\omega^{k+1}|^2 \\
&\quad + 2\overline{H}(x)\partial_y\nabla_y\omega^{k+1}\left(\xi\partial_y\nabla_y\omega^{k+1} + \psi^k\partial_y\nabla_y\omega^k + \varphi^k\right) \\
&\le \overline{H}(x)\left((c + \frac{C}{\epsilon} - \Lambda)|\partial_y\nabla_y\omega^{k+1}|^2 + \epsilon C|\partial_y\nabla_y\omega^k|^2\right) + C|x|^{-\alpha}e^{-\Lambda x} .
\end{aligned}$$

Choosing Λ large enough, we have thus proved the following inequality

$$\partial_x\left(\overline{H}(x)|\partial_y\nabla_y\omega^{k+1}|^2\right) \le \epsilon\overline{H}(x)C|\partial_y\nabla_y\omega^k|^2 + C|x|^{-\alpha}e^{-\Lambda x} , \qquad (7.3.63)$$

which is then integrated in x to obtain:

$$\overline{H}(x)|\partial_y \nabla_y \omega^{k+1}|^2(y,x) \leq \overline{H}(x_0)|\partial_y \nabla_y \omega^{k+1}|^2(y,x_0)+C\int_{x_0}^{x} e^{-\Lambda s}\left(\epsilon|s|^{-2\alpha}|\partial_y \nabla_y \omega^{k}|^2+|s|^{-\alpha}\right)ds.$$

Equivalently this reads

$$\overline{H}(x)|\partial_y \nabla_y \omega^{k+1}|^2(y,x) \leq \overline{H}(x_0)|\partial_y \nabla_y \omega_0^{-,k+1}|^2(y,x_0)+C\int_{x_0}^{x} e^{-\Lambda s}\left(\epsilon|s|^{-2\alpha}|\partial_y \nabla_y \omega^{k}|^2+|s|^{-\alpha}\right)ds$$

As we did many times before, for a convenient choice of ϵ, we obtain the estimate:

$$\begin{aligned}|x|^{-2\alpha}|\partial_y \nabla_y \omega^{k+1}|^2(y,x) &\leq 2\left(C\int_{x_0}^{0}|s|^{-\alpha}e^{-\Lambda s}ds+\sup_{k\in\mathbb{N},\, y\in[0,y_0]}\overline{H}(x_0)|\partial_y \nabla_y \omega_0^{-,k}|^2(y)\right)\\ &\leq c(x_0,\alpha)\left(1+\sup_{k\in\mathbb{N}}\|\omega_0^{-,k+1}\|^2_{C^2([0,y_0]\times\mathscr{O})}\right)\\ &\leq c(x_0,\alpha)\left(1+\sup_{k\in\mathbb{N}}\|\omega_0^{-,k+1}\|^2_{H^{m+1}(\mathcal{C}^-)}\right)<\infty\end{aligned}$$

i.e.

$$|x|^{-\alpha}|\partial_y \nabla_y \omega^{k+1}|(y,x) < C\ . \tag{7.3.64}$$

The same procedure can exactly be repeated by using instead the angular derivatives of ω^{k+1} leading to

$$|x|^{-\alpha}\|\partial_y \nabla_y \omega^{k+1}(y,x)\|_{C^{j_0-2}} < C\ . \tag{7.3.65}$$

Remark 7.3.9 The bound (7.3.65) follows from a control on the coefficients ξ, ψ^k and φ^k in (7.3.60) after have y-differentiated the equation satisfied by ω^{k+1}. We notice that, if instead we have $x\partial_x-$ differentiated the same equation we should have been led to the bound:

$$|x|^{-\alpha}\|(x\partial_x)\nabla_x \omega^{k+1})(y,x)\|_{C^{j_0-2}} < C\ . \tag{7.3.66}$$

Now we integrate once more in y Equation 7.3.65 and obtain that for $j_0 \geq 3$:

$$\begin{aligned}|x|^{-\alpha}\|\nabla_y \omega^{k+1}(y,x)\|_{C^1(\mathscr{O})} &\leq |x|^{-\alpha}\|\nabla_y \omega^{k+1}(0,x)\|_{C^1}+Cy\\ &\leq 2C_0 \qquad \textit{pour} \quad y<u_2\ .\end{aligned} \tag{7.3.67}$$

In order to finish the proof of Lemma 7.3.4, we write:

$$\begin{aligned}|x|^{-\alpha}\|\omega^{k+1}(y,x)\|_{C^1(\mathscr{O})} &\leq |x|^{-\alpha}\|\omega_0^{+,k+1}(x)\|_{C^1(\mathscr{O})}+\int_0^y|x|^{-\alpha}\|\partial_y\omega^{k+1}(s,x)\|_{C^1(\mathscr{O})}ds\\ &\leq C_0+2C_0y\\ &\leq 2C_0 \qquad \textit{pour} \quad y<u_3\ .\end{aligned} \tag{7.3.68}$$

We define u_* as

$$u_* = \min\{u_0, u_1, u_2, u_3\}$$

and inequalities (7.3.59), (7.3.67), (7.3.68) allow us to write:

$$\sup_{(y,x)\in[0,u_*]\times[x_0,0[} |x|^{-\alpha}\|(\omega^{k+1}, \nabla\omega^{k+1})(y,x)\|_{W^{1,\infty}} < 2C_0 \ .$$

This completes the proof of Lemma 7.3.4. □

The previous lemma will be useful only if we prove that the constant C_0 is finite. We thus have to prove that the quantity $\sup_{k\in\mathbb{N},\, x\in\mathcal{C}^+} |x|^{-\alpha}\|\partial_y\omega^k(0,x)\|_{W^{1,\infty}(\mathscr{O})}$ is finite. This will be a consequence of the next Lemma. Set (recall that $\overline{H}(x) = e^{-\Lambda x}|x|^{-2\alpha}$)

$$\hat{C}_0 := \sup_{k\in\mathbb{N},\, x\in[x_0,0[} \overline{H}^{\frac{1}{2}}(x)\|(\omega_0^{+,k}, \nabla_x\omega_0^{+,k})(x)\|_{H^m(\mathscr{O})} \lesssim \check{C}(y_0,0) < \infty$$

$$\widetilde{C}_0 := \sup_{k\in\mathbb{N}} \overline{H}^{\frac{1}{2}}(x_0)\|\partial_y\omega_0^{-,k}(0)\|_{H^{m-1}(\mathscr{O})} + \sup_{x\in[x_0,0[} \overline{H}^{\frac{1}{2}}(x)\|\partial_y\omega^0(0,x)\|_{H^{m-1}(\mathscr{O})} < \infty \ .$$

Here and elsewhere we write $A \lesssim B$ if and only if there exists a constant $c > 0$ such that $A \leq cB$. We have the following:

Lemma 7.3.10 *Under the hypotheses of Lemma 7.3.4, we have:*

$$\sup_{k\in\mathbb{N},\, x\in[x_0,0[} \overline{H}^{\frac{1}{2}}(x)\|\partial_y\omega^k(0,x)\|_{H^{m-1}(\mathscr{O})} < 2(\hat{C}_0 + \widetilde{C}_0). \tag{7.3.69}$$

Proof: *The proof will be carried out by induction on the integer k. By definition of the constants $\hat{C}_0$ and $\widetilde{C}_0$ the assumption is fulfilled when $k = 0$. Suppose that*

$$\sup_{x\in[x_0,0[} \overline{H}^{\frac{1}{2}}(x)\|\partial_y\omega^k)(0,x)\|_{H^{m-1}(\mathscr{O})} < 2(C_0 + \widetilde{C}_0) \ .$$

We shall prove that this inequality remains true if we replace k with $k+1$. If in Inequality 7.3.38 page 154 we choose $\{y = 0\}$ we have (note that in (7.3.38) there is no ϵ in the second line but things can be arranged from (7.3.37) so as to get an ϵ there):

$$\begin{aligned} H(0,x)\|\partial_y\omega^{k+1}(0,x)\|^2_{H^{m-1}(\mathscr{O})} \quad \leq \quad & H(0,x_0)\|\partial_y\omega^{k+1}(0,x_0)\|^2_{H^{m-1}(\mathscr{O})} \\ & +\epsilon c_3(h,c_0,\bar{c}_0)\int_{x_0}^{x} H(0,s)\|\nabla_x\omega^{k+1}(0,s)\|^2_{H^m(\mathscr{O})}ds \\ & +\epsilon C(C_0)\int_{x_0}^{x} H(0,s)\|(\omega^k, \nabla\omega^k)(0,s)\|^2_{H^{m-1}(\mathscr{O})}ds \ . \end{aligned}$$

This can be rewritten as

$$\begin{aligned}
\overline{H}(x)\|\partial_y\omega^{k+1}(0,x)\|^2_{H^{m-1}(\mathscr{O})} &\leq \overline{H}(x_0)\|\partial_y\omega_0^{-,k+1}(0)\|^2_{H^{m-1}(\mathscr{O})}\\
&+\epsilon c_3(h,c_0,\bar{c}_0)\int_{x_0}^{x}\overline{H}(s)\|\nabla_x\omega_0^{+,k+1}(s)\|^2_{H^m(\mathscr{O})}ds\\
&+\epsilon C(C_0)\int_{x_0}^{x}\overline{H}(s)\|(\omega_0^{+,k},\nabla_x\omega_0^{+,k})(s)\|^2_{H^{m-1}(\mathscr{O})}ds\\
&+\epsilon C(C_0)\int_{x_0}^{x}\overline{H}(s)\|\partial_y\omega^{k}(0,s)\|^2_{H^{m-1}(\mathscr{O})}ds\ ,
\end{aligned}$$

which implies that:

$$\begin{aligned}
\overline{H}(x)\|\partial_y\omega^{k+1}(0,x)\|^2_{H^{m-1}(\mathscr{O})} &\leq \widetilde{C}_0^2+\epsilon c_3(h,c_0,\bar{c}_0)|x_0|\hat{C}_0^2+\epsilon C(C_0)|x_0|\hat{C}_0^2\\
&+4\epsilon C(C_0)|x_0|(\hat{C}_0+\widetilde{C}_0)^2\\
&\leq 4(\hat{C}_0+\widetilde{C}_0)^2 \quad \textit{since } \epsilon \textit{ is sufficiently small}\,.
\end{aligned}$$

We then obtain

$$\sup_{x\in[x_0,0[}\overline{H}^{\frac{1}{2}}(x)\|\partial_y\omega^{k+1}(0,x)\|^2_{H^{m-1}(\mathscr{O})}\leq 2(\hat{C}_0+\widetilde{C}_0)\ ,$$

and the proof is complete. □

Lemma 7.3.11 *Under the hypotheses of Lemma 7.3.4, there exists a constant $M_0>0$ such that:*

$$\sup_{k\in\mathbb{N},\ (y,x)\in[0,u_*]\times[x_0,0[}\|\omega^k(y,x)\|_{W^{1,\infty}(\mathscr{O})}<M_0\ . \tag{7.3.70}$$

Proof: *let $\gamma\in\mathbb{N}^{n-1}$, such that $|\gamma|\in\{0,1\}$. By Lemma 7.3.4, for all $(y,x)\in[0,u_*]\times[x_0,0[$, we have:*

$$\begin{aligned}
|\partial_\theta^\gamma\omega^k(y,x)| &\leq |\partial_\theta^\gamma\omega_0^{-,k}(y)|+\int_{x_0}^{x}|\partial_x\partial_\theta^\gamma\omega^k(y,s)|ds\\
&\leq |\partial_\theta^\gamma\omega_0^{-,k}(y)|+\int_{x_0}^{x}|s|^\alpha|s|^{-\alpha}\|\partial_x\omega^k(s,x)\|_{C^1(\mathscr{O})}ds\\
&\leq \underbrace{\sup_{|\gamma|\in\{0,1\},\ k\in\mathbb{N},\ y\in[0,u_*]}|\partial_\theta^\gamma\omega_0^{-,k}(y)|+2C_0\int_{x_0}^{0}|s|^\alpha ds}_{:=M_0}\ .
\end{aligned}$$

□

We also have the following:

Lemma 7.3.12 *Under the hypotheses of the previous lemma, there exists two constants $M_1 > 0$ and $M_2 > 0$ such that:*

$$\sup_{k\in\mathbb{N},\ (y,x)\in[0,u_*]\times[x_0,0[} \|\omega^k(y,x)\|_{H^{m-1}(\mathscr{O})} < M_1 \tag{7.3.71}$$

and

$$\sup_{k\in\mathbb{N},\ (y,x)\in[0,u_*]\times[x_0,0[} |x|^{-\alpha}\|(\partial_x\omega^k, \partial_y\omega^k)(y,x)\|_{H^{m-1}(\mathscr{O})} < M_2\ . \tag{7.3.72}$$

Proof: *First we will prove (7.3.72) and secondly, we will show that (7.3.71) is actually a consequence of (7.3.72). We proceed by induction on k. Set*

$$\begin{aligned}
\bar{C}_0 &:= \sup_{(y,x)\in[0,u_*]\times[x_0,0[} |x|^{-\alpha}\|(\partial_x\omega^0, \partial_y\omega^0)(y,x)\|_{H^{m-1}(\mathscr{O})} < \infty\ ,\\
\bar{C}_1 &:= \sup_{k\in\mathbb{N},\ (y,x)\in[0,u_*]\times[x_0,0[} \Big\{H(y,x_0)\|\partial_y\omega_0^{-,k+1}(y)\|^2_{H^{m-1}(\mathscr{O})}\\
&\qquad\qquad + H(0,x)\|\partial_x\omega_0^{+,k+1}(x)\|^2_{H^{m-1}(\mathscr{O})}\Big\}\\
&\quad +2\epsilon C(C_0)\hat{C}(y_0,0) + 2c_3(h,c_0,\bar{c}_0)\hat{C}(y_0,0) + 2\epsilon C(C_0)\hat{C}(y_0,0)\\
&\quad +2c_5(h,c_0,\bar{c}_0)\hat{C}(y_0,0) < \infty\ ,
\end{aligned} \tag{7.3.73}$$

and suppose that

$$\sup_{(y,x)\in[0,u_*]\times[x_0,0[} H^{\frac{1}{2}}(y,x)\|(\partial_x\omega^k, \partial_y\omega^k)(y,x)\|_{H^{m-1}(\mathscr{O})} < 2(\bar{C}_0 + \bar{C}_1)\ .$$

Let us show that this remains true when we replace k with $k+1$. Adding inequalities (7.3.39) and (7.3.47) leads to:

$$\begin{aligned}
H(y,x)&\|(\partial_y\omega^{k+1}, \partial_x\omega^{k+1})(y,x)\|^2_{H^{m-1}(\mathscr{O})} \leq\\
&\bar{C}_1 + \epsilon C(C_0)\int_{x_0}^{x} H(y,s)\|\partial_y\omega^k(y,s)\|^2_{H^{m-1}(\mathscr{O})}ds\\
&+\epsilon C(C_0)\int_0^y H(s,x)\|\partial_x\omega^k(s,x)\|^2_{H^{m-1}(\mathscr{O})}ds\\
\leq\ & \bar{C}_1 + 2\epsilon C(C_0)(\bar{C}_0 + \bar{C}_1)(y_0 + |x_0|)\\
\leq\ & 2(\bar{C}_0 + \bar{C}_1)\ , \quad \textit{even if it means redefining } \epsilon\ .
\end{aligned} \tag{7.3.74}$$

Since $\forall (y,x) \in [0,y_0]\times[x_0,0[,\ e^{-\Lambda y_0} \leq e^{-\Lambda(y+x)}$, it then suffices to set $M_2 := (2e^{\Lambda y_0}(\bar{C}_0 + \bar{C}_1))^{1/2}$.

In order to obtain the uniform control (7.3.71), we repeat the argument leading to the proof of Lemma 7.3.11. For all $(y,x)\in[0,u_*]\times[x_0,0[$, we have:

$$\begin{aligned}\|\omega^k(y,x)\|_{H^{m-1}(\mathscr{O})} &\leq \|\omega_0^{-,k}(y)\|_{H^{m-1}(\mathscr{O})}+\int_{x_0}^{x}\|\partial_x\omega^k(y,s)\|_{H^{m-1}(\mathscr{O})}ds\\ &\leq \|\omega_0^{-,k}(y)\|_{H^{m-1}(\mathscr{O})}+\int_{x_0}^{x}|s|^{\alpha}|s|^{-\alpha}\|\partial_x\omega^k(s,x)\|_{H^{m-1}(\mathscr{O})}ds\\ &\leq \underbrace{\sup_{k\in\mathbb{N},\ y\in[0,u_*]}\|\omega_0^{-,k}(y)\|_{H^{m-1}(\mathscr{O})}+M_2\int_{x_0}^{0}|s|^{\alpha}ds}_{:=M_1}\ . \qquad (7.3.75)\end{aligned}$$

□

7.3.5 Convergence of the sequence $(\omega^k)_{k\in\mathbb{N}}$ and existence

Set $\delta\omega^k=\omega^{k+1}-\omega^k$ and $\delta\nabla\omega^k=\nabla\omega^{k+1}-\nabla\omega^k$. We have the following (recall that $\mathcal{D}_*=[0,u_*]\times[x_0,0[\times\mathscr{O}$):

Lemma 7.3.13 *Under the hypotheses of Lemma 7.3.11, even if it means to replace $(\omega^k)_{k\in\mathbb{N}}$ by one of its subsequences, there exist two real numbers $\sigma\in]0,1[$ and $\varsigma>0$ such that:*

$$\|(-x)^{-\alpha}(\delta\omega^k,\delta(\nabla\omega^k))\|_{L^2(\mathcal{D}_*)}\leq\frac{\varsigma}{2^k}+\sigma\|(-x)^{-\alpha}(\delta\omega^{k-1},\delta(\nabla\omega^{k-1}))\|_{L^2(\mathcal{D}_*)}\ . \qquad (7.3.76)$$

Proof: *We apply Proposition 7.3.1 with $\omega=\delta\omega^k$, $u\in[0,u_*]$, and $v\in[x_0,0[$. We have:*

$$\begin{aligned}&e^{-\Lambda u}\|(\delta\omega^k,\delta(\nabla_x\omega^k))\overline{H}^{\frac12}\|^2_{L^2([x_0,0[\times\mathscr{O})}+e^{-\Lambda v}\|(\delta\omega^k,\delta(\nabla_y\omega^k))\widetilde{H}^{\frac12}\|^2_{L^2([0,u]\times\mathscr{O})}\leq\\ &\quad\|(\delta\omega^k,\delta(\nabla_x\omega^k))\overline{H}^{\frac12}\|^2_{L^2([x_0,0[\times\mathscr{O})}+e^{-\Lambda x_0}\|(\delta\omega^k,\delta(\nabla_y\omega^k))\widetilde{H}^{\frac12}\|^2_{L^2([0,u]\times\mathscr{O})}\\ &\quad+(c_1(c_0,\bar c_0,n,h)-2\Lambda)\int_0^u\int_{x_0}^{v}H(y,x)\|(\delta\omega^k,\delta\nabla\omega^k)\|^2_{L^2(\mathscr{O})}dxdy\\ &\quad+\frac{1}{c_0}\int_{D_{u,v}}\left|L^{\ell}[\delta\omega^k]\right|dsdyd\nu\ . \qquad (7.3.77)\end{aligned}$$

Recall that:

$$L^{\ell}[\delta\omega^k]=|x|^{-\frac{n+3}{2}}H(\delta\partial_x\omega^k+\delta\partial_y\omega^k)\left(G^k(\ldots)-G_2^{k-1}(\ldots)\right)\ .$$

We have:

$$|x|^{-\frac{n+3}{2}} H \left(G^k(\dots) - G^{k-1}(\dots) \right) = H \int_0^1 |x|^{-2} \xi^k(t,y,x) dt$$

with

$$\begin{aligned} \xi^k(t,y,x) &= \delta\omega^{k-1} \frac{\partial G}{\partial p}(z, t|x|^{\frac{n-1}{2}}(\omega^k, \nabla\omega^k) + (1-t)|x|^{\frac{n-1}{2}}(\omega^{k-1}, \nabla\omega^{k-1})) \\ &\quad + \delta\nabla\omega^{k-1} \frac{\partial G}{\partial q}(z, t|x|^{\frac{n-1}{2}}(\omega^k, \nabla\omega^k) + (1-t)|x|^{\frac{n-1}{2}}(\omega^{k-1}, \nabla\omega^{k-1})) \; . \end{aligned}$$

Using once more hypothesis (7.3.4), one is led to the following estimate

$$\begin{aligned} |x|^{-2}|\xi(t,y,x)| &\leq C_5 \, |x|^{-2+(r-1)(\alpha + \frac{n-1}{2})} \left(|\delta\omega^{k-1}| + |\delta\nabla\omega^{k-1}| \right) \\ &\leq C_5 \left(|\delta\omega^{k-1}| + |\delta\nabla\omega^{k-1}| \right) ; \quad \textit{if} \quad n \geq 1 + \frac{4}{r-1} - 2\alpha \; . \end{aligned}$$

We should point out that the constant C_5 depends on the quantity

$$\sup_{k\in\mathbb{N}, \; (y,x)\in[0,u_*]\times[x_0,0[} |x|^{-\alpha} \|(\omega^k, \nabla\omega^k)\|_{L^\infty(\mathscr{O})} \; ,$$

which does neither depend upon Λ nor on k. We then obtain that if $n \geq 1 + \frac{4}{r-1} - 2\alpha$,

$$\begin{aligned} \int_{D_{u,v}} \left| L^\ell[\delta\omega^k] \right| &\leq C_5 \int_0^u \int_{x_0}^v \Bigg(\|\partial_x \delta\omega^k(y,x)\|^2_{L^2(\mathscr{O})} + \|\partial_y \delta\omega^k\|^2_{L^2(\mathscr{O})} \\ &\qquad + \|(\delta\omega^{k-1}, \nabla\delta\omega^{k-1})\|^2_{L^2(\mathscr{O})} \Bigg) H(y,x) dx dy \; ; \end{aligned} \tag{7.3.78}$$

and inequality (7.3.77) implies

$$\begin{aligned} & e^{-\Lambda u} \|(\delta\omega^k, \delta(\nabla_x \omega^k)) \overline{H}^{\frac{1}{2}}\|^2_{L^2([x_0,0[\times\mathscr{O})} + e^{-\Lambda v} \|(\delta\omega^k, \delta(\nabla_y \omega^k)) \widetilde{H}^{\frac{1}{2}}\|^2_{L^2([0,u]\times\mathscr{O})} \leq \\ &\quad \|(\delta\omega^k, \delta(\nabla_x \omega^k)) \overline{H}^{\frac{1}{2}}\|^2_{L^2([x_0,0[\times\mathscr{O})} + e^{-\Lambda x_0} \|(\delta\omega^k, \delta(\nabla_y \omega^k)) \widetilde{H}^{\frac{1}{2}}\|^2_{L^2([0,u]\times\mathscr{O})} \\ &\quad + (c_1(c_0, \bar{c}_0, n, h) + C_5 - 2\Lambda) \int_0^u \int_{x_0}^v H(y,x) \|(\delta\omega^k, \delta\nabla\omega^k)\|^2_{L^2(\mathscr{O})} dx dy \\ &\quad + C_5 \int_0^u \int_{x_0}^v H(y,x) \|(\delta\omega^{k-1}, \nabla_x \delta\omega^{k-1})\|^2_{L^2(\mathscr{O})} dx dy \; . \end{aligned} \tag{7.3.79}$$

From this inequality and by a convenient choice of Λ *we obtain:*

$$
\begin{aligned}
&e^{-\Lambda u}\|(\delta\omega^k,\delta(\nabla_x\omega^k))\overline{H}^{\frac{1}{2}}\|^2_{L^2([x_0,0[\times\mathscr{O})}+e^{-\Lambda v}\|(\delta\omega^k,\delta(\nabla_y\omega^k))\widetilde{H}^{\frac{1}{2}}\|^2_{L^2([0,u]\times\mathscr{O})}\leq\\
&\quad\|(\delta\omega_0^{+,k},\delta(\nabla_x\omega_0^{+,k}))\overline{H}^{\frac{1}{2}}\|^2_{L^2([x_0,0[\times\mathscr{O})}+e^{-\Lambda x_0}\|(\delta\omega_0^{-,k},\delta(\nabla_y\omega_0^{-,k}))\widetilde{H}^{\frac{1}{2}}\|^2_{L^2([0,u]\times\mathscr{O})}\\
&\quad+C_5\int_0^u\int_{x_0}^v H(y,x)\|(\delta\omega^{k-1},\nabla\delta\omega^{k-1})\|^2_{L^2(\mathscr{O})}dxdy\,. \qquad (7.3.80)
\end{aligned}
$$

We have:

$$
\begin{aligned}
&\|(\delta\omega_0^{+,k},\delta(\nabla_x\omega_0^{+,k}))\overline{H}^{\frac{1}{2}}\|^2_{L^2([x_0,0[\times\mathscr{O})}+e^{-\Lambda x_0}\|(\delta\omega_0^{-,k},\delta(\nabla_y\omega_0^{-,k}))\widetilde{H}^{\frac{1}{2}}\|^2_{L^2([0,u]\times\mathscr{O})}=\\
&\qquad\|e^{-\frac{\Lambda x}{2}}|x|^{-\alpha}(\delta\omega_0^{+,k},\delta(\nabla_x\omega_0^{+,k}))\|^2_{L^2([x_0,0[\times\mathscr{O})}\\
&\qquad+|x_0|^{-\alpha}e^{-\Lambda x_0}\|e^{-\frac{1}{2}\Lambda y}(\delta\omega_0^{-,k},\delta(\nabla_y\omega_0^{-,k}))\|^2_{L^2([0,u]\times\mathscr{O})}\\
&\quad\leq\;c(\Lambda,x_0)\left(\||x|^{-\alpha}(\delta\omega_0^{+,k},\delta(\nabla_x\omega_0^{+,k}))\|^2_{L^2([x_0,0[\times\mathscr{O})}+\|(\delta\omega_0^{-,k},\delta(\nabla_y\omega_0^{-,k}))\|^2_{L^2([0,y_0]\times\mathscr{O})}\right)\,.
\end{aligned}
$$

Since the sequences $\left(|x|^{-\alpha}(\omega_0^{+,k},\nabla_x\omega_0^{+,k})\right)_{k\in\mathbb{N}}$ *and* $\left(\omega_0^{-,k},\nabla_y\omega_0^{-,k}\right)_{k\in\mathbb{N}}$ *are convergent respectively in the spaces* $L^2([x_0,0[\times\mathscr{O})$ *and* $L^2([0,y_0]\times\mathscr{O})$, *we know that*

$$
\lim_{k\longrightarrow\infty}\left(\||x|^{-\alpha}(\delta\omega_0^{+,k},\delta(\nabla_x\omega_0^{+,k}))\|^2_{L^2([x_0,0[\times\mathscr{O})}+\|(\delta\omega_0^{-,k},\delta(\nabla_y\omega_0^{-,k}))\|^2_{L^2([0,y_0]\times\mathscr{O})}\right)=0\,.
$$

Therefore, $\forall i\in\mathbb{N},\ \exists k_i\in\mathbb{N}$, *such that*

$$
c(\Lambda,x_0)\left(\||x|^{-\alpha}(\delta\omega_0^{+,k_i},\delta(\nabla_x\omega_0^{+,k_i}))\|^2_{L^2([x_0,0[\times\mathscr{O})}+\|(\delta\omega_0^{-,k_i},\delta(\nabla_y\omega_0^{-,k_i}))\|^2_{L^2([0,y_0]\times\mathscr{O})}\right)\leq\frac{1}{2^i}. \qquad (7.3.81)
$$

We then write inequality (7.3.80) with instead the subsequence $(\omega^{k_i})_{i\in\mathbb{N}}$ *which will be denoted again* $(\omega^k)_{k\in\mathbb{N}}$ *and one obtains:*

$$
\begin{aligned}
&e^{-\Lambda u}\|(\delta\omega^k,\delta(\nabla_x\omega^k))\overline{H}^{\frac{1}{2}}\|^2_{L^2([x_0,0[\times\mathscr{O})}+e^{-\Lambda v}\|(\delta\omega^k,\delta(\nabla_y\omega^k))\widetilde{H}^{\frac{1}{2}}\|^2_{L^2([0,u]\times\mathscr{O})}\leq\\
&\quad\frac{1}{2^k}+C_5\int_0^u\int_{x_0}^v H(y,x)\|(\delta\omega^{k-1},\nabla\delta\omega^{k-1})\|^2_{L^2(\mathscr{O})}dxdy\,.
\end{aligned}
$$

This leads to the following inequalities:

$$
\begin{aligned}
&\forall u\in[0,u_*],\ e^{-\Lambda u}\||x|^{-\alpha}(\delta\omega^k,\delta(\nabla_x\omega^k))(u)\|^2_{L^2([x_0,0[\times\mathscr{O})}\\
&\qquad\leq\frac{1}{2^k}+C_5\int_0^u\int_{x_0}^v H(y,x)\|(\delta\omega^{k-1},\nabla\delta\omega^{k-1})\|^2_{L^2(\mathscr{O})}dxdy\,; \qquad (7.3.82)
\end{aligned}
$$

$$\forall v \in [x_0, 0[,\ |v|^{-\alpha} e^{-\Lambda v} \|(\delta\omega^k, \delta(\nabla_y \omega^k))(v)\|^2_{L^2([0,u]\times\mathscr{O})}$$
$$\leq \frac{1}{2^k} + C_5 \int_0^u \int_{x_0}^v H(y,x) \|(\delta\omega^{k-1}, \nabla\delta\omega^{k-1})\|^2_{L^2(\mathscr{O})} dx dy \ ; \tag{7.3.83}$$

$$\int_0^u \int_{x_0}^v H(y,x) \|(\delta\omega^k, \nabla\delta\omega^k)\|^2_{L^2(\mathscr{O})} \leq \frac{1}{2^k a(\Lambda)} + \sigma^2 \int_0^u \int_{x_0}^v H(y,x) \|(\delta\omega^{k-1}, \nabla\delta\omega^{k-1})\|^2_{L^2(\mathscr{O})} \ . \tag{7.3.84}$$

where

$$a(\Lambda) = 2\Lambda - C_5 - c_1(c_0, \bar{c}_0, n, h) \quad \textit{and} \quad 0 < \sigma^2 < \frac{C_5}{a(\Lambda)} < \frac{1}{2} \ ,$$

provided that Λ *is large enough.* □

Now we have all we need to show that the sequence $(\omega^k)_{k\in\mathbb{N}}$ *converges towards a function* ω *of class* C^2 *on* $\mathcal{D}_* = [0, u_*] \times [x_0, 0[\times \mathscr{O}$ *which is a solution of the characteristic initial value problem (7.3.3). We have the following consequence of the previous Lemma.*

Corollary 7.3.14 *There exists a continuous and bounded function* ω *on* $\mathcal{D}_*$ *such that* $(\omega^k)_{k\in\mathbb{N}}$ *converges to* ω *uniformly on any compact subset of* $\mathcal{D}_*$.

Proof: *We point out the elementary implication: If* $(U_n)_{n\in\mathbb{N}}$ *is a sequence of positive real numbers satisfying* $U_{n+1} \leq \alpha U_n + \frac{1}{2^n}$ *, then*

$$U_n \leq \alpha^n U_0 + 2 \left(\frac{(1/2)^n - (\alpha)^n}{1 - 2\alpha} \right) \ .$$

Therefore, the series $\sum U_n$ *will converge if* $0 \leq \alpha < 1$. *This remark and inequality (7.3.84) show that the function series* $\sum e^{-\Lambda(y+x)/2} |x|^{-\alpha} (\delta\omega^k, \nabla\delta\omega^k)$ *converges in the space* $L^2(\mathcal{D}_*)$. *Since the sequence of partial sums of this series write* $S_k = e^{-\Lambda(y+x)/2} |x|^{-\alpha} \left((\omega^k, \nabla\omega^k) - (\omega^0, \nabla\omega^0) \right)$, *the sequence* $((\omega^k, \nabla\omega^k))_{k\in\mathbb{N}}$ *converges to a function* $(\omega_\varepsilon, \tilde{\omega}_\varepsilon)$ *in the space* $L^2(\mathcal{D}_{*,\varepsilon})$, *with* $\mathcal{D}_{*,\varepsilon} = [0, u_*] \times [x_0, -\varepsilon] \times \mathscr{O}$, *for any* $0 < \varepsilon < -x_0$. *Note that the continuous embedding* $L^2(\mathcal{D}_{*,\varepsilon}) \hookrightarrow \mathscr{D}'(\mathcal{D}_{*,\varepsilon})$ *implies that* $\tilde{\omega}_\varepsilon = \nabla\omega_\varepsilon$. *We define* ω *by setting for any* $(y, x, \theta) \in \mathcal{D}_*$, $\omega(y, x, \theta) = \omega_\varepsilon(y, x, \theta)$ *if* $(y, x, \theta) \in \mathcal{D}_{*,\varepsilon}$. *First of all we need to prove that* ω *is a well defined function. Let* $\varepsilon_1, \varepsilon_2 \in [0, -x_0[$ *such that* $\varepsilon_1 > \varepsilon_2$. *Since* $\mathcal{D}_{*,\varepsilon_1} \subset \mathcal{D}_{*,\varepsilon_2}$, $L^2(\mathcal{D}_{*,\varepsilon_2})$ *embeds continuously in* $L^2(\mathcal{D}_{*,\varepsilon_1})$ *and then, the convergence of the sequence* $(\omega^k, \nabla\omega^k)_{k\in\mathbb{N}}$ *towards the function* $(\omega_{\varepsilon_2}, \nabla\omega_{\varepsilon_2})$ *in* $L^2(\mathcal{D}_{*,\varepsilon_2})$, *also holds in* $L^2(\mathcal{D}_{*,\varepsilon_1})$. *By uniqueness of limits of sequences in this space one is led to*

$$(\omega_{\varepsilon_1}, \nabla\omega_{\varepsilon_1}) = (\omega_{\varepsilon_2}, \nabla\omega_{\varepsilon_2}) \ \textit{almost everywhere on} \ \mathcal{D}_{*,\varepsilon_1} \ . \tag{7.3.85}$$

Let $\varepsilon \in [0, -x_0[$. *By Lemma 7.3.11, the sequence* $(\omega^k)_{k\in\mathbb{N}}$ *is uniformly bounded on* $\mathcal{D}_*$ *and therefore is uniformly bounded on* $\mathcal{D}_{*,\varepsilon}$*, by Lemma 7.3.4 page 146 there exists a contant* $C = C(C_0, x_0, \varepsilon)$ *such that*

$$\sup_{k\in\mathbb{N}} \|\nabla\omega^k\|_{L^\infty(\mathcal{D}_{*,\varepsilon})} < C(C_0, x_0, \varepsilon) ,$$

thus the sequence $(\omega^k)_{k\in\mathbb{N}}$ *is uniformly equicontinuous on* $\mathcal{D}_{*,\varepsilon}$*. Then, By Arzela-Ascoli theorem, there exists a subsequence* $(\omega^{k_j})_{j\in\mathbb{N}}$ *of the sequence* $(\omega^k)_{k\in\mathbb{N}}$ *which converges uniformly on the compact set* $\mathcal{D}_{*,\varepsilon}$ *to a continuous function* ω'_ε*. The embedding* $C^0(\mathcal{D}_{*,\varepsilon}) \hookrightarrow L^2(\mathcal{D}_{*,\varepsilon})$ *proves that this convergence also holds in* $L^2(\mathcal{D}_{*,\varepsilon})$ *and by uniqueness of limits in* $L^2(\mathcal{D}_{*,\varepsilon})$ *we conclude that the equality in (7.3.85) holds everywhere and that:*

- ω *is a continuous function on* $\mathcal{D}_*$,
- *the sequence* $(\omega^{k_j})_{j\in\mathbb{N}}$ *uniformly converges to* ω *on any compact subset of* $\mathcal{D}_* = [0, u_*] \times [x_0, 0[\times\mathscr{O}$.

It remains to prove that ω *is bounded on* $\mathcal{D}_*$*. From the Sobolev embedding theorem (recall* $m - 1 > \frac{n-1}{2} + 2$*), by (7.3.71) we have:*

$$\sup_{(y,x)\in[0,u_*]\times[x_0,0[} \|\omega^k(y, x)\|_{C^2(\mathscr{O})} < M_1 \quad \forall k \in \mathbb{N} .$$

By taking the limits in this estimate we obtain that ω *is a bounded function as well as its angular derivatives up to order two on* $\mathcal{D}_*$*.* □

Lemma 7.3.15 $\forall s \in [0, m-2] \cap \mathbb{N}$, $(\omega^{k_j}(y, x))_{j\in\mathbb{N}}$ *converges towards* $\omega(y, x)$ *in* $H^s(\mathscr{O})$ *uniformly in* (y, x) *on* $[0, u_*] \times [x_0, 0[$ *and*

$$\omega \in \bigcap_{0\le s\le m-2} C^0([0, u_*] \times [x_0, 0[;\ H^s(\mathscr{O})) .$$

Proof: *By the previous corollary, for all* $(y, x) \in [0, u_*] \times [x_0, 0[$*, the sequence* $(\omega^{k_j}(y, x))_{j\in\mathbb{N}}$ *converges to* $\omega(y, x)$ *in* $C^0(\mathscr{O})$ *and since* $C^0(\mathscr{O}) \hookrightarrow L^2(\mathscr{O})$*, this convergence also holds in the space* $L^2(\mathscr{O})$*. On the other hand, by Lemma 7.3.12 the sequence* $(\omega^{k_j}(y, x))_{j\in\mathbb{N}}$ *is bounded in the Hilbert space* $H^{m-1}(\mathscr{O})$*, uniformly in* $(y, x) \in [0, u_*] \times [x_0, 0[$*. By weak compactness there exists a subsequence of* $(\omega^{k_j}(y, x))_{j\in\mathbb{N}}$ *denoted again by the same symbol which converges weakly to a function* $\bar\omega \in H^{m-1}(\mathscr{O})$*. This weak convergence also holds in* $L^2(\mathscr{O})$*, and by uniqueness of the weak limits, we obtain that* $\omega(y, x) = \bar\omega(y, x) \in H^{m-1}(\mathscr{O})$*. Now, we use the interpolation Theorem C.0.9* *with* $p = r = 2,\ s = m-1,\ u = \omega^{k_{j_1}} - \omega^{k_{j_2}},\ j_1, j_2 \in \mathbb{N}$. *We then obtain:* $q = 2$ *and that for all* $i \in \{0, 1, \ldots, m-1\}$,

$$\begin{aligned}
&\sum_{|\gamma|=i} \|\partial_\theta^\gamma(\omega^{k_{j_1}} - \omega^{k_{j_2}})\|^2_{L^2(\mathscr{O})} \\
&\leq\ c \left(\sum_{|\gamma|=m-1} \|\partial_\theta^\gamma(\omega^{k_{j_1}} - \omega^{k_{j_2}})\|_{L^2(\mathscr{O})} \right)^{\frac{2i}{m-1}} \|\omega^{k_{j_1}} - \omega^{k_{j_2}}\|^{2(1-\frac{i}{m-1})}_{L^2(\mathscr{O})} \\
&\leq\ C(M_1)\|\omega^{k_{j_1}} - \omega^{k_{j_2}}\|^{2(1-\frac{i}{m-1})}_{L^2(\mathscr{O})} .
\end{aligned}$$

This estimate implies that, if $s < m-1$, *then the sequence* $(\omega^{k_j}(y,x))_{j\in\mathbb{N}}$ *is a Cauchy sequence in the Hilbert space* $H^s(\mathscr{O})$ *uniformly in* (y,x). □

Corollary 7.3.16 *The following holds:*

- $\omega \in C^2([0,u_*] \times [x_0, 0[\times \mathscr{O})$,
- ω *solves the characteristic initial value problem (7.3.3).*

Proof: *Let* $\varepsilon \in]0, -x_0[$. *Recall* $\mathcal{D}_{*,\varepsilon} = [0,u_*] \times [x_0, -\varepsilon] \times \mathscr{O}$. *In order to show that* $\omega \in C^1(\mathcal{D}_*)$ *we will show that* $\omega \in C^1(\mathcal{D}_{*,\varepsilon})$ *for any epsilon. We repeat what we did before to obtain that* ω *is continuous. Since*

$$\sup_{k\in\mathbb{N}} \|\nabla\omega^k\|_{L^\infty(\mathcal{D}_{*,\varepsilon})} < C(C_0, x_0, \varepsilon) ,$$

we only need to show that the sequence of second order derivatives $(\nabla^2\omega^k)_{k\in\mathbb{N}}$ *is bounded on* $\mathcal{D}_{*,\varepsilon}$. *This follows from (7.3.65), (7.3.66), (7.3.71) and (7.3.72) (recall* $m-2 > \frac{n-1}{2} + 2$*). Thus again by Arzela-Ascoli theorem, the weak compactness and the interpolation theorem, one obtains that:*

- *the sequence (or a subsequence of it)* $(\nabla\omega^k)_{k\in\mathbb{N}}$ *converges uniformly towards* $\nabla\omega$ *on* $\mathcal{D}_{*,\varepsilon}$,
- $\nabla\omega$ *is a continuous function on* $\mathcal{D}_{*,\varepsilon}$ *and then on* $\mathcal{D}_*$,
- $\forall s \in [0, m-2] \cap \mathbb{N},\ \nabla\omega^k(y,x) \longrightarrow \nabla\omega(y,x)$ *in* $H^s(\mathscr{O})$ *uniformly in* (y,x) *on the compact* $[0,u_*] \times [x_0, -\varepsilon]$ *and that*

$$\forall (y,x) \in [0,u_*] \times [x_0, 0[,\ \nabla\omega(u,v) \in C^2(\mathscr{O}) .$$

Let us show that $\omega \in C^2(\mathcal{D}_*)$. Again, we repeat the previous argument. Let $\varepsilon \in]0, -x_0]$ be fixed. We already know that the sequence of second order derivatives $(\nabla^2\omega^k)_{k\in\mathbb{N}}$ is uniformly bounded on $\mathcal{D}_{*,\varepsilon}$. Thus, it remains to show that the sequence of third order derivatives $(\nabla^3\omega^k)_{k\in\mathbb{N}}$ is also uniformly bounded on $\mathcal{D}_{*,\varepsilon}$. From this property, it will follow that the sequence of second order derivatives is uniformly equicontinuous and then the theorem of Arzela-Ascoli applies. From some inequalities obtained so far, we see that the sequences $(\partial^3_{\mu\nu\beta}\omega^k)_{k\in\mathbb{N}}$ are uniformly bounded on $\mathcal{D}_{*,\varepsilon}$ for $\mu\nu\beta \neq xxx$ and $\mu\nu\beta \neq yyy$:

- *By choosing $j_0 \geq 3$ (which is the case since $m-1 > 3+\frac{n-1}{2}$), inequality (7.3.57) shows that the sequence $(\partial^3_{\mu\nu\beta}\omega^k)_{k\in\mathbb{N}}$ is uniformly bounded on $\mathcal{D}_{*,\varepsilon}$ for $\mu\nu\beta = xyA$.*

- *From inequalities (7.3.65) and (7.3.66) with $j_0 = 3$, we obtain that this sequence is uniformly bounded on $\mathcal{D}_{*,\varepsilon}$ for $\mu\nu\beta \in \{yyA, yAB, xxA, xAB\}$.*

- *The case $\mu\nu\beta = ABC$ will follows from inequality (7.3.71).*

- *The analysis of the right hand side of identity (7.3.60) gives the desired control in the case $\mu\nu\beta = xyy$ whereas $x\partial_x-$differentiating the partial differential equation satisfied by ω^{k+1} gives the result in the case $\mu\nu\beta = yxx$.*

It thus remains to show that the sequences $(\partial^3_{yyy}\omega^k)_{k\in\mathbb{N}}$ and $(\partial^3_{xxx}\omega^k)_{k\in\mathbb{N}}$ are uniformly bounded on $\mathcal{D}_{*,\varepsilon}$. We start with $(\partial^3_{yyy}\omega^k)_{k\in\mathbb{N}}$. If we ∂^2_y-differentiate the differential equation satisfied by ω^{k+1} we obtain:

$$4\partial_x(\partial^3_y\omega^{k+1}) + \frac{n-1}{\rho}\partial^3_y\omega^{k+1} + (-x)^{-2}\partial^3_y\omega^k\left(\frac{\partial G}{\partial p_3}\right)^k(\ldots) = \Phi^k \quad (7.3.86)$$

where

$$
\begin{aligned}
\Phi^k = \; = \;& \frac{n-1}{\rho^2}(\partial_x - \partial_y)\partial_y \omega^{k+1}(y,x) + \frac{n-1}{2\rho^3}(\partial_x - \partial_y)\omega^{k+1} + \frac{n-1}{\rho}\partial_x\partial_y^2\omega^{k+1} \\
& + \frac{h^{AB}}{\rho^2}\partial_A\partial_B\partial_y^2\omega^{k+1} + \frac{3h^{AB}}{2\rho^4}\partial_A\partial_B\omega^{k+1}(y,x) + \frac{2h^{AB}}{\rho^3}\partial_y\partial_A\partial_B\omega^{k+1}(y,x) \\
& - \frac{2\Gamma^B}{\rho^3}\partial_y\partial_B\omega^{k+1} - \frac{3\Gamma^B}{\rho^4}\partial_B\omega^{k+1} - \frac{\Gamma^B}{\rho^2}\partial_B\partial_y^2\omega^{k+1} \\
& - (-x)^{\frac{n+3}{2}}(\partial_y^2 G)^k(\ldots) - x^{-2}\partial_y^2\omega^k\left(\frac{\partial G}{\partial p_1}\right)^k(\ldots) - x^{-2}\partial_y\omega^k\partial_y\left(\left(\frac{\partial G}{\partial p_1}\right)^k(\ldots)\right) \\
& - x^{-2}\partial_y^2(x\partial_x\omega^k)\left(\frac{\partial G}{\partial p_2}\right)^k(\ldots) - x^{-2}\partial_y(x\partial_x\omega^k)\partial_y\left(\left(\frac{\partial G}{\partial p_2}\right)^k(\ldots)\right) \\
& - x^{-2}\partial_y^2\omega^k\partial_y\left(\left(\frac{\partial G}{\partial p_3}\right)^k(\ldots)\right) \\
& - x^{-2}\partial_y^2\partial_A\omega^k\left(\frac{\partial G}{\partial p_4}\right)^k(\ldots) + x^{-2}\partial_y\partial_A\omega^k\partial_y\left(\left(\frac{\partial G}{\partial p_4}\right)^k(\ldots)\right) .
\end{aligned}
$$

From what has been said so far, we deduce that the coefficients of Equation (7.3.86) are uniformly bounded on $\mathcal{D}_{*,\varepsilon}$. Namely, we have $\|\frac{n-1}{\rho}\|_{L^\infty(\mathcal{D}_{*,\varepsilon})} < C$ and

$$
\sup_{k\in\mathbb{N}}\|(-x)^{-2}\left(\frac{\partial G}{\partial p_3}\right)^k\|_{L^\infty(\mathcal{D}_{*,\varepsilon})} < C(x_0,\varepsilon),\; \sup_{k\in\mathbb{N}}\|\Phi^k\|_{L^\infty(\mathcal{D}_{*,\varepsilon})} < C(x_0,\varepsilon) . \tag{7.3.87}
$$

As we did before, we have:

$$
\begin{aligned}
\partial_x\left(\overline{H}(x)|\partial_y^3\omega^{k+1}|^2\right) &= (2\alpha|x|^{-1} - \Lambda)\overline{H}(x)|\partial_y^3\omega^{k+1}|^2 + 2\overline{H}(x)\partial_x\partial_y^3\omega^{k+1}.\partial_y^3\omega^{k+1} \\
&\le -\Lambda\overline{H}(x)|\partial_y^3\omega^{k+1}|^2 + 2\overline{H}(x)\partial_x\partial_y^3\omega^{k+1}.\partial_y^3\omega^{k+1} .
\end{aligned}
$$

From (7.3.87), we deduce that

$$
\begin{aligned}
\partial_x\left(\overline{H}(x)|\partial_y^3\omega^{k+1}|^2\right) \le\;& -\Lambda\overline{H}(x)|\partial_y\nabla_y\omega^{k+1}|^2 \\
& + 2\overline{H}(x)\partial_y^3\omega^{k+1}\left(-\frac{n-1}{\rho}\partial_y^3\omega^{k+1} - (-x)^{-2}\partial_y^3\omega^k\left(\frac{\partial G}{\partial p_3}\right)^k(\ldots) + \Phi^k\right) \\
\le\;& \overline{H}(x)\left((c + \frac{C}{\delta} - \Lambda)|\partial_y^3\omega^{k+1}|^2 + \delta C|\partial_y^3\omega^k|^2\right) + C|x|^{-\alpha}e^{-\Lambda x} .
\end{aligned}
$$

By choosing Λ large enough, we have:

$$\partial_x \left(\overline{H}(x)|\partial_y^3\omega^{k+1}|^2\right) \leq \delta\overline{H}(x)C|\partial_y^3\omega^k|^2 + C|x|^{-\alpha}e^{-\Lambda x} \ , \tag{7.3.88}$$

which is then integrated in x to obtain:

$$\overline{H}(x)|\partial_y^3\omega^{k+1}|^2(y,x) \leq \overline{H}(x_0)|\partial_y^3\omega^{k+1}|^2(y,x_0)+C\int_{x_0}^{x} e^{-\Lambda s}\left(\delta|s|^{-2\alpha}|\partial_y^3\omega^k|^2+|s|^{-\alpha}\right)ds.$$

Equivalently this reads

$$\overline{H}(x)|\partial_y^3\omega^{k+1}|^2(y,x) \leq \overline{H}(x_0)|\partial_y^3\omega_0^{-,k+1}|^2(y,x_0)+C\int_{x_0}^{x} e^{-\Lambda s}\left(\delta|s|^{-2\alpha}|\partial_y^3\omega^k|^2+|s|^{-\alpha}\right)ds$$

As we did many times before, for a convenient choice of δ, we obtain the estimate:

$$\begin{aligned}
|x|^{-2\alpha}|\partial_y^3\omega^{k+1}|^2(y,x) &\leq 2\left(C\int_{x_0}^{0}|s|^{-\alpha}e^{-\Lambda s}ds + \sup_{k\in\mathbb{N},\, y\in[0,y_0]}\overline{H}(x_0)|\partial_y^3\omega_0^{-,k}|^2(y)\right)\\
&\leq c(x_0,\alpha)\left(1+\sup_{k\in\mathbb{N}}\|\omega_0^{-,k+1}\|^2_{C^3([0,y_0]\times\mathscr{O})}\right)\\
&\leq c(x_0,\alpha)\left(1+\sup_{k\in\mathbb{N}}\|\omega_0^{-,k+1}\|^2_{H^{m+1}(\mathcal{C}^-)}\right) < \infty
\end{aligned}$$

i.e.

$$|x|^{-\alpha}|\partial_y^3\omega^{k+1}|(y,x) < C \ . \tag{7.3.89}$$

The same holds for $|x|^{-\alpha}|\partial_x^3\omega^{k+1}$ if instead we ∂_x^2-differentiate the differential equation satisfied by ω^{k+1}: $|x|^{-\alpha}|\partial_x^3\omega^{k+1}|(y,x) < C$. Therefore we have proved that

$$\sup_{k\in\mathbb{N}}\|\nabla_y^3\omega^k\|_{L^\infty(\mathcal{D}_{*,\varepsilon})} < C \ . \tag{7.3.90}$$

It then follows that the family of functions $\{\nabla^2\omega^{k_j},\ j\in\mathbb{N}\}$ is uniformly equicontinuous and from the Arzela-Ascoli theorem there exists a subsequence of $(\nabla^2\omega^{k_j})_{j\in\mathbb{N}}$ denoted again by the same symbol which converges uniformly to a continuous function $\tilde{\omega}$ on $\mathcal{D}_{*,\varepsilon}$. Since the sequence $(\nabla\omega^{k_j})_{j\in\mathbb{N}}$ converges uniformly to $\nabla\omega$ on $\mathcal{D}_{*,\varepsilon}$ we conclude that $\nabla\omega$ is differentiable on $\mathcal{D}_{*,\varepsilon}$ and $\nabla^2\omega = \tilde{\omega}$ which proves that $\omega \in C^2(\mathcal{D}_{*,\varepsilon})$ for all $\varepsilon \in]0,-x_0]$ i.e. $\omega \in C^2(\mathcal{D}_*)$. To end the proof, it remains to show that ω solves the characteristic Cauchy problem (7.3.3). Recall that the partial differential equation satisfied by ω^{k+1} reads

$$\partial_x\partial_y\omega^{k+1} = \Psi^k \tag{7.3.91}$$

where

$$\Psi^k = \frac{(n-1)}{4\rho}\Big(\partial_x - \partial_y\Big)\omega^{k+1} + \frac{h^{AB}\partial_A\partial_B\omega^{k+1}}{4\rho^2}$$
$$-\frac{1}{4}\Gamma^B\partial_B\omega^{k+1} - \frac{1}{4}|x|^{-\frac{n+3}{2}}G(z, |x|^{\frac{n-1}{2}}(\omega^k, \nabla\omega^k))$$

being a continuous function on $\mathcal{D}_*$. To conclude we consider the limits pointwise in (7.3.91) and we are led to

$$\Box_{y,\eta}\omega = |x|^{-\frac{n+3}{2}}G(z, |x|^{\frac{n-1}{2}}(\omega, \nabla\omega))$$

thus ω is a classical solution of the characteristic initial value problem (7.3.3). □

7.3.6 Uniqueness and statement of the results

We are now going to show that the solution of (7.3.3) constructed in the previous section is the unique C^2 solution. Let ω_1, ω_2 be two functions of differentiability class C^2 on $\mathcal{D}_*$ both solution of (7.3.3). Set $\delta\omega = \omega_2 - \omega_1$ and $\delta G(z) = G(z, |x|^{-\frac{n-1}{2}}(\omega_2, \nabla\omega_2)) - G(z, |x|^{-\frac{n-1}{2}}(\omega_1, \nabla\omega_1))$. It follows that $\delta\omega$ solves the characteristic initial value problem with vanishing data

$$\begin{cases} \Box_{y,\eta}\delta w = x^{-\frac{n+3}{2}}\delta G \quad \textit{dans } \mathcal{D}_* \\ \delta\omega = 0 \quad \textit{sur} \quad \mathcal{C}^+ \cup \mathcal{C}^- \end{cases} . \tag{7.3.92}$$

We repeat the proof of Lemma 7.3.13 with instead $\delta\omega$ and obtain the following inequality which is the equivalent of (7.3.84) there:

$$\|H^{\frac{1}{2}}(y,x)(\delta\omega, \nabla\delta\omega)\|^2_{L^2(\mathcal{D}_*)} \le \sigma^2\|H^{\frac{1}{2}}(y,x)(\delta\omega, \nabla\delta\omega)\|^2_{L^2(\mathcal{D}_*)} .$$

This proves that $\omega_1 = \omega_2$ almost everywhere and since these functions are continuous functions they are equal everywhere. We have thus proved

Theorem 7.3.17 *Consider the characteristic initial value problem (7.3.3) on the subset $\mathcal{D} = [0, y_0] \times [x_0, 0[\times \mathscr{O}$ of $\mathbb{R}^{n+1}_y$. Suppose that the initial data ω_0^+ and ω_0^- satisfy (7.3.1) and (7.3.2) with $m \ge \frac{n+7}{2}$ and $-1 < \alpha \le -1/2$. Moreover suppose that the nonlinear source term G satisfies the nullity property $(\mathcal{H})$ page 135 with a* uniform zero *of order r being such that*

$$n \ge 1 + \frac{4}{r-1} - 2\alpha . \tag{7.3.93}$$

Then there exists a positive real number $u_* \in]0, y_0]$ and a unique function ω of differentiability class C^2 on $\mathcal{D}_* = [0, u_*] \times [x_0, 0[\times \mathscr{O}$, solution of (7.3.3) with the following properties:

- $\sup_{z \in \mathcal{D}_*} |\omega(z)| < \infty$,

- $\sup_{z \in \mathcal{D}_*} |x|^{-\alpha} |\nabla \omega(z)| < \infty$,

- $\forall s \in [0, m-2] \cap \mathbb{N},\ \omega \in C^0([0, u_*] \times [x_0, 0[;\ H^s(\mathscr{O}))$.

A direct consequence of Theorem 7.3.17 is an existence and uniqueness result for the Cauchy problem (5.0.8) on the light cone. We want to solve this problem on a neighborhood of the entire cone. For this purpose, we need to make sure the data are such that, the problem at hand can be solved locally on a neighborhood $V_{0,x}$ *of the tip of the initial cone and that the restriction of this local solution on any incoming cone intersecting this neighborhood is of* H^{m+2}*−regularity class (as in* (7.3.1)*). In order to obtain this local solution, we will use the result of [27] (see Théorème 2, page 47 of this reference). For any* $\tau > 0$ *set*

$$Y^\tau = \{(y^\mu) \in \phi(\mathcal{Y}^+_{a,x}),\ 0 \le y^0 \le \tau\}\ , \tag{7.3.94a}$$

$$C^\tau = \{(y^\mu) \in \phi(\mathcal{C}^+_{a,x}),\ 0 \le y^0 \le \tau\}\ , \tag{7.3.94b}$$

$$\forall \varepsilon > 0, \quad \mathcal{C}^+(\varepsilon) = \phi\left(\mathcal{C}^+_{a,x}\right) \setminus C^\varepsilon\ . \tag{7.3.94c}$$

we have the following

Theorem 7.3.18 *Let* $m \in \mathbb{N}$*. Consider the characteristic initial value problem (5.0.8) on the light cone in the unbounded domain* $\mathcal{Y}^+_{a,x}$ *of* $\mathbb{R}^{n+1}_x$*. Assume that the source term* F *is a smooth function of all its variables and that the initial data* φ *are such that:*

- *there exists a real number* $0 < \varepsilon_0 < \frac{1}{2a}$ *such that* $\hat{\varphi} = \left(\Omega^{-\frac{n-1}{2}} \varphi \circ \phi^{-1}\right)\Big|_{C^{\varepsilon_0}}$ *satisfies the last hypothesis* $\mathcal{H}_{m+2}$ *of [27, Théorème 2, p. 47],*

- *and* $\forall\ \varepsilon\ \in\ ]0, \varepsilon_0]$,

$$\Omega^{-\frac{n-1}{2}} \varphi \circ \phi^{-1}\Big|_{\mathcal{C}^+(\varepsilon)}\ ;\ \partial_x(\Omega^{-\frac{n-1}{2}} \varphi \circ \phi^{-1})\Big|_{\mathcal{C}^+(\varepsilon)} \in \mathscr{H}^\alpha_{m+1}(\mathcal{C}^+(\varepsilon)), \tag{7.3.95}$$

where $m \ge \frac{n+7}{2}$ *and* $-1 < \alpha \le -1/2$*. Further assume that the function* $\widetilde{F}$ *of equation(7.1.7) has a* uniform zero *of order* ℓ *satisfying*

$$n \ge 1 + \frac{4}{\ell - 1} - 2\alpha\ . \tag{7.3.96}$$

Then, there exist three real numbers a_0, C, R such that $a_0 > a$, $C, R > 0$ and a unique function f of class C^2 on the future neighborhood $\mathscr{V}$ of $\mathcal{C}^+_{a,x}$ defined by $\mathscr{V} = \underset{a \leq b \leq a_0}{\cup} \mathcal{C}^+_{b,x}$ solution of (5.0.8) with the following decay property

$$\forall (t, x^i) \in \mathscr{V}, \qquad |f(t, x^i)| \leq C(t+r)^{-\frac{n-1}{2}}, \quad r = \sqrt{\sum_{i=1}^{n} (x^i)^2} > R$$

$$\forall (t, x^i) \in \mathscr{V}, \qquad |\partial_t f(t, x^i)| \leq C(t+r)^{-\frac{n-1}{2} - \alpha}, \quad r > R$$

$$\forall (t, x^i) \in \mathscr{V}, \qquad |\partial_r f(t, x^i)| \leq C(t+r)^{-\frac{n-1}{2} - \alpha}, \quad r > R\,.$$

Proof: *Since $m + 2 > \frac{n}{2} + 1$, the last statement of hypotheses $\mathcal{H}_{m+2}$ of [27, Théorème 2, page 47], assumes that $\hat{\varphi}$ can be decompose as $\hat{\varphi} = \bar{\varphi}\big|_{\phi(\mathcal{C}^+_{a,x})} + \hat{\varphi}_1$ where $\bar{\varphi}$ is a polynomial function of degree $2(m+1)$ on Y^τ, $\tau > 0$, and where φ_1 belongs to a weighted Sobolev space of differentiability class $2m + 3$ on C^τ, the weight being choose so as to control the singularities at the tip of the cone. The results of this reference yield a neighborhood $V_{0,y}$ of the tip of the cone $\phi(\mathcal{C}^+_{a,x})$ in $\phi(\mathcal{Y}^+_{a,x})$ and a local solution $\hat{f}_0$ which restriction on any incoming cone intersecting $V_{0,y}$ is in the usual Sobolev space H^{m+2}. We then apply Theorem 7.3.17 to the Goursat problem (6.3.1)-(6.3.2) with $\omega_0^- = \hat{f}_0\Big|_{\mathcal{C}^-}$ and $\omega_0^+ = \hat{\varphi}|_{\mathcal{C}^+}$ and obtain a bounded solution $\hat{f}$ of this problem on the future neighborhood*

$$\mathcal{D}_* = [0, u_*] \times [x_0, 0[\times \mathscr{O} = \underset{0 \leq u \leq u_*}{\cup} \mathcal{C}^+_{u,0}$$

of $\mathcal{C}^+ \cup \mathcal{C}^+$. Note that by uniqueness $\hat{f}$ and $\hat{f}_0$ coincide on the intersection of $\mathcal{D}_*$ with $V_{0,y}$ the future neighborhood of the tip of the initial cone $\mathcal{C}^+_{x,a}$ on which we obtain from Dossa's results [27] the local solution $\hat{f}_0$. Therefore, there exists a constant $c > 0$ such that for all $(y^\mu) \in \mathcal{D}_*$, $|\hat{f}(y^\mu)| < c$. This estimate can be rewritten as $|\hat{f} \circ \phi(x^\mu)| < c$, for all $(x^\mu) \in \phi^{-1}(D_*)$, i.e (see (6.2.1)) $|f(x^\mu)| < c\,|\Omega \circ \phi|^{-\frac{n-1}{2}}$. By the definition Ω, (see (6.2.1)) we have

$$|\Omega| = | - \eta_{\alpha\beta} y^\alpha y^\beta| = | - (y^0)^2 + \rho^2| = \frac{1}{(t+r)(t-r)} \leq \frac{\tilde{c}}{t+r}\,.$$

This proves that for all $(t, x^i) \in \phi^{-1}(D_*)$, $|f(t, x^i)| \leq c(t+r)^{-\frac{n-1}{2}}$. Now according to some of our previous calculations, we have:

$$\begin{aligned} \frac{\partial}{\partial x^\mu} &= -\Omega \frac{\partial}{\partial y^\mu} - 2 y_\mu y^\alpha \frac{\partial}{\partial y^\alpha} \\ &= x(\frac{1}{a} - y)\frac{\partial}{\partial y^\mu} - 2 y_\mu \Big(x\partial_x + (y - \frac{1}{a})\partial_y \Big)\,. \end{aligned}$$

This identity implies that (recall $t = x^0$ and $\tau = y^0 = -y_0$)

$$\frac{\partial}{\partial t} = x(\frac{1}{a} - y)\partial_\tau + 2\tau\Big(x\partial_x + (y - \frac{1}{a})\partial_y\Big) .$$

Using identities (7.1.1) and (7.2.1) leads to:

$$\frac{\partial}{\partial t} = x^2\partial_x + \left(y - \frac{1}{a}\right)^2 \partial_y . \tag{7.3.98}$$

On the other hand, we have $\frac{\partial}{\partial r} = \frac{x^i}{r}\frac{\partial}{\partial x^i}$ and $\frac{x^i}{r} = -\frac{y^i}{\rho}$ thus,

$$\begin{aligned}\frac{\partial}{\partial r} &= \Omega\frac{y^i}{\rho}\frac{\partial}{\partial y^i} + 2y_i\frac{y^i}{\rho}\Big(x\partial_x + (y - \frac{1}{a})\partial_y\Big) \\ &= -x(\frac{1}{a} - y)\frac{\partial}{\partial \rho} + 2\rho\Big(x\partial_x + (y - \frac{1}{a})\partial_y\Big) .\end{aligned}$$

Again from identities (7.1.1) and (7.2.1) we obtain

$$\frac{\partial}{\partial r} = x^2\partial_x - \left(y - \frac{1}{a}\right)^2 \partial_y . \tag{7.3.99}$$

For all (t, x^i) and (τ, y^i) such that $(\tau, y^i) = \phi(t, x^i)$, we have (recall $x\partial_x\Omega = \left(y - \frac{1}{a}\right)\partial_y\Omega = \Omega$)

$$\begin{aligned}\partial_t f(t, x^i) &= x^2\partial_x f\circ\phi^{-1}(\tau, y^i) + \left(y - \frac{1}{a}\right)^2 \partial_y f\circ\phi^{-1}(\tau, y^i) \\ &= x^2\partial_x\left(\Omega^{\frac{n-1}{2}}\hat{f}(\tau, y^i)\right) + \left(y - \frac{1}{a}\right)^2 \partial_y\left(\Omega^{\frac{n-1}{2}}\hat{f}(\tau, y^i)\right) \\ &= \frac{n-1}{2}\left(x + y - \frac{1}{a}\right)\Omega^{\frac{n-1}{2}}\hat{f}(\tau, y^i) + x^2\Omega^{\frac{n-1}{2}}\partial_x\hat{f}(\tau, y^i) + \left(y - \frac{1}{a}\right)^2\Omega^{\frac{n-1}{2}}\partial_y\hat{f}(\tau, y^i) .\end{aligned}$$

From Theorem 7.3.17 we know that for $r > R$,

$$|\hat{f}| \lesssim 1 \ , \quad (-x)^{-\alpha}|\partial_x\hat{f}| \lesssim 1 \quad \textit{and} \quad (-x)^{-\alpha}|\partial_y\hat{f}| \lesssim 1 .$$

Thus for all (t, x^i) such that $r > R$, we have (recall $|\Omega| \lesssim \frac{1}{t+r}$)

$$\begin{aligned}|\partial_t f(t, x^i)| &\lesssim (t+r)^{-\frac{n-1}{2}} + (t+r)^{-\frac{n-1}{2}-2-\alpha} + (t+r)^{-\frac{n-1}{2}-\alpha} \\ &\lesssim (t+r)^{-\frac{n-1}{2}-\alpha}.\end{aligned}$$

The same holds for $|\partial_r f(t, x^i)|$. This proves that in general, the decay at infinity of the derivatives of the solution is not as fast as the decay of the solution itself and complete the proof. □

7.4 Application to wave maps

The aim of this section is to show that Theorem 7.3.18 applies to wave maps with source manifold the Minkowski space-time. Let $(\mathscr{N}, g)$ *be a smooth Riemannian manifold with finite dimension* N*, we wish to find a map* $f : (\mathbb{R}^{n+1}_x, \eta) \to (\mathscr{N}, g)$ *solving the Cauchy problem for the wave map equation. As in [20], we will be interested in maps* f *which have the property that* f *approaches a constant map* f_0 *as* r *tends to infinity along lightlike directions,* $f_0(x^\mu) = p_0 \in \mathscr{N}$ *for* $x^\mu \in \mathbb{R}^{n+1}_x$*. Introducing normal coordinate around* p_0*, we can write* $f = f^a$*,* $a = 1, \ldots, N$*, with the functions* f^a *satisfying the following system of semi-linear partial differential equations*

$$\Box_{\eta_x} f^a = F^a(f, \partial f) \ ; \tag{7.4.1}$$

with

$$F^a(f, \partial f) := -\eta^{\alpha\beta}\Gamma^a_{bc}(f)\frac{\partial f^b}{\partial x^\alpha}\frac{\partial f^c}{\partial x^\beta} \ ;$$

and where the Γ^a_{bc}*'s are the Christoffel symbols of the metric* g*. Using as before the conformal transformation*

$$\phi : \mathbb{R}^{n+1}_x \setminus \mathcal{C}_{0,x} \to \mathbb{R}^{n+1}_y \ \text{by} \ x^\alpha \mapsto y^\alpha := \frac{x^\alpha}{\eta_{\lambda\mu}x^\lambda x^\mu} \ , \ \alpha = 0, 1, \ldots, n \ .$$

and setting again $\Omega = -\eta_{\alpha\beta}y^\alpha y^\beta$*;* $\hat{f} = \Omega^{-\frac{n-1}{2}} f \circ \phi^{-1}$*, (7.4.1) reads (see (6.2.8), page 126):*

$$\Box_{\eta_y} \hat{f}^a = \Omega^{-\frac{n+3}{2}} \widetilde{F}^a(\hat{f}, \partial_{y^\mu}\hat{f}), \tag{7.4.2}$$

with

$$\begin{aligned} \widetilde{F}^a(\hat{f}, \partial_{y^\mu}\hat{f}) \ &= \ -\Omega\Gamma^a_{bc}(\Omega^{\frac{n-1}{2}}\hat{f})\Big\{\Omega\eta^{\alpha\beta}(\Omega^{\frac{n-1}{2}}\partial_{y^\alpha}\hat{f}^b)(\Omega^{\frac{n-1}{2}}\partial_{y^\beta}\hat{f}^c) \\ &\quad -(1-n)^2(\Omega^{\frac{n-1}{2}}\hat{f}^b)(\Omega^{\frac{n-1}{2}}\hat{f}^c) + 2(1-n)(\Omega^{\frac{n-1}{2}}\hat{f}^b)y^\mu(\Omega^{\frac{n-1}{2}}\partial_{y^\mu}\hat{f}^c)\Big\} \ . \end{aligned}$$

This expression shows that when transforming (7.4.2) with data on a null cone into a Goursat problem as in (6.3.1) we will instead have a pre-factor $\Omega^{-\frac{n+1}{2}}$*. On the other hand, from the assumption on* f*, we know that* $\widetilde{F}$ *here has a* uniform zero *of order* $r = 3$*, thus in the case of wave maps condition (7.3.23) reads:*

$$n \geq 2 - 2\alpha \ .$$

We have proved the following:

Theorem 7.4.1 *Let $a > 0$, $n, m \in \mathbb{N}$, $n \geq 3$. Consider Equation (7.4.1) on the Minkowski space-time $\mathbb{R}_x^{n+1}$ with initial data given on the translated cone $\mathcal{C}^+_{a,x}$ and are such that:*

- *there exists a real number $0 < \varepsilon_0 < \frac{1}{2a}$ such that $\hat{\varphi} = \left(\Omega^{-\frac{n-1}{2}} f \circ \phi^{-1}\right)\Big|_{C^{\varepsilon_0}}$ satisfies the last hypothesis $\mathcal{H}_{m+2}$ of [27, Théorème 2, p. 47],*
- *and $\forall\ \varepsilon \in]0, \varepsilon_0]$,*

$$\Omega^{-\frac{n-1}{2}} f \circ \phi^{-1}\big|_{\mathcal{C}^+(\varepsilon)}\ ;\ \partial_x\left(\Omega^{-\frac{n-1}{2}} f \circ \phi^{-1}\big|_{\mathcal{C}^+(\varepsilon)}\right) \in \mathscr{H}^{-1/2}_{m+1}(\mathcal{C}^+(\varepsilon))\ , \tag{7.4.3}$$

with $m \geq \frac{n+7}{2}$. Then, there exists three real numbers a_0, C, R such that $a_0 > a$, $C, R > 0$ and a unique function f of class C^2 on the future neighborhood $\mathscr{V}$ of $\mathcal{C}^+_{a,x}$ defined by $\mathscr{V} = \underset{a\leq b\leq a_0}{\cup} \mathcal{C}^+_{b,x}$ solution of (7.4.2) with the following decay property

$$\begin{aligned}
&\forall (t, x^i) \in \mathscr{V}, \qquad |f(t, x^i)| \leq C(t+r)^{-\frac{n-1}{2}}, \quad r > R\\
&\forall (t, x^i) \in \mathscr{V}, \qquad |\partial_t f(t, x^i)| \leq C(t+r)^{-\frac{n-1}{2}-\alpha}, \quad r > R\\
&\forall (t, x^i) \in \mathscr{V}, \qquad |\partial_r f(t, x^i)| \leq C(t+r)^{-\frac{n-1}{2}-\alpha}, \quad r > R\ .
\end{aligned}$$

7.5 High regularity of the solution

In order to prove higher regularity theorem for a solution of (5.0.8), we restrict our attention to the case where the function F does not depend on the normal (with respect to the initial cone) derivative of the solution. This implies that the function $\widetilde{F}$ in (6.2.9) does not depend on $\partial_y\omega$. We thus suppose that the characteristic Cauchy problem (7.3.3) takes the form:

$$\begin{cases} \Box_{y,\eta}\omega = x^{-\frac{n+3}{2}} G\left(z, x^{\frac{n-1}{2}}(\omega, x\partial_x\omega, \partial_\theta\omega)\right) & \text{in } \mathcal{D} \\ \omega = \omega_0^+ \quad \text{on} \quad \mathcal{C}^+ \quad \text{and} \quad \omega = \omega_0^- \quad \text{on} \quad \mathcal{C}^- \end{cases} . \tag{7.5.1}$$

We need first to show that for any solution of (7.5.1), we control its outgoing derivatives on the surface $\{x = x_0\}$. We have the following

Lemma 7.5.1 *Suppose that ω is the solution of the Cauchy problem (7.5.1) with data satisfying*

$$\omega_0^- \in C^\infty(\mathcal{C}^-)\ ;\ \ \omega_0^+ \in \mathscr{H}^\alpha_\infty(\mathcal{C}^+)\ .$$

Then, there exists a real number $u_{**} \in]0, y_0]$ such that $\forall j, m \in \mathbb{N}$, one can find a positive constant $C^* = C(y_0, j, m)$ satisfying

$$\sup_{y\in[0,u_{**}]} \|\partial_x^j(\omega, \partial_y\omega)(y, x_0)\|_{H^m(\mathscr{O})} < C^* \; . \tag{7.5.2}$$

Proof: *The proof will be carried out by induction on j. The case $j = 0$ is given by hypotheses. Let us handle the case $j = 1$. Again we differentiate the partial differential equation satisfied by ω with ∂_θ^γ, and multiply the resulting equation by $\partial_\theta^\gamma \partial_x \omega$ and obtain*

$$\begin{aligned}
\partial_y(\partial_x\partial_\theta^\gamma\omega)^2 &= \frac{n-1}{2\rho}(\partial_x\partial_\theta^\gamma\omega)^2 - \frac{n-1}{2\rho}\partial_y\partial_\theta^\gamma\omega\partial_x\partial_\theta^\gamma\omega \\
&\quad +\partial_x\partial_\theta^\gamma\omega \sum_{\gamma_1+\gamma_2=\gamma} \frac{\partial_\theta^{\gamma_1} h^{AB}}{2\rho^2}\partial_\theta^{\gamma_2}\partial_A\partial_B\omega \\
&\quad +\partial_x\partial_\theta^\gamma\omega \sum_{\gamma_1+\gamma_2=\gamma} \frac{\partial_\theta^{\gamma_1} \Gamma^{B}}{2\rho^2}\partial_\theta^{\gamma_2}\partial_B\omega \\
&\quad -\frac{1}{2}|x|^{-\frac{n+3}{2}}\partial_x\partial_\theta^\gamma\omega\partial_\theta^\gamma G(\ldots) \; .
\end{aligned}$$

Then, we integrate on $[0, y] \times \{x_0\} \times \mathscr{O}$, and obtain via Stokes theorem

$$\begin{aligned}
\|\partial_x\omega(y, x_0)\|^2_{H^m(\mathscr{O})} &\leq \|\partial_x\omega_0^+(x_0)\|^2_{H^m(\mathscr{O})} + c(h, c_0, \bar{c}_0)\|\omega_0^-\|_{H^{m+2}(\mathcal{C}^-)} \\
&\quad +c(h, c_0, \bar{c}_0)\int_0^y \|\partial_x\omega(s, x_0)\|^2_{H^m(\mathscr{O})}ds \\
&\quad +c(x_0)\int_0^y \|G(\ldots)(s, x_0)\|^2_{H^m(\mathscr{O})}ds \; .
\end{aligned}$$

Now,

$$\|G(\ldots)(s, x_0)\| = G\left(s, x_0, \theta, \omega^-(s), \partial_x\omega(s, x_0)\right) ,$$

thus from the usual Moser inequality (see [50], Proposition 3.9, page 11) we have

$$\begin{aligned}
\|G(\ldots)(s, x_0)\|_{H^m(\mathscr{O})} &\leq C(\|\partial_x\omega(s, x_0)\|_{L^\infty(\mathscr{O})})(1 + \|\partial_x\omega(s, x_0)\|_{H^m(\mathscr{O})}) \\
&\leq \Phi(\|\partial_x\omega(s, x_0)\|_{H^m(\mathscr{O})}) \; ;
\end{aligned}$$

where Φ in an increasing real-valued function bounded on bounded set (to obtain the last inequality, we have used the Sobolev's imbedding theorem). We thus obtain the following:

$$\|\partial_x\omega(y, x_0)\|^2_{H^m(\mathscr{O})} \leq C\left(1 + \int_0^y \Phi(\|\partial_x\omega(s, x_0)\|_{H^m(\mathscr{O})})ds\right) \; . \tag{7.5.3}$$

We can now apply Lemma 5.2 of [20] and obtain that there exists a time $0 < u_{**} \leq y_0$ such that

$$\forall y \in [0, u_{**}],\ \|\partial_x \omega(y, x_0)\|^2_{H^m(\mathscr{O})} < C\ ,$$

which provides the desired bounds. Suppose now that (7.5.2) holds for a certain $j \in \mathbb{N}$ and let us show that it remains true when we replace j there by $j+1$. We $x-$differentiate j times the partial differential equation satisfied by ω and obtain an equation linear in $\partial_x^{j+1}\omega$ of the form:

$$\Box \partial_x^j \omega = \left(\partial_x^{j+1}\omega\right) G_1(z, \omega, \partial_x \partial_\theta^\gamma \omega, \dots, \partial_x^j \partial_\theta^\gamma \omega) + G_2(z, \omega, \partial_x \partial_\theta^\gamma \omega, \dots, \partial_x^j \partial_\theta^\gamma \omega)$$

where $|\gamma| \leq 2$. To this equation we apply what we did earlier in the case $j = 1$ and using the induction hypothesis, instead of (7.5.3), we are led to an linear inequality:

$$\|\partial_x^{j+1}\omega(y, x_0)\|^2_{H^m(\mathscr{O})} \leq C \left(1 + \int_0^y \|\partial_x^{j+1}\omega(s, x_0)\|^2_{H^m(\mathscr{O})}) ds\right)\ .$$

This proves that the higher derivatives are controlled on the same time interval as in the case $j = 1$. It remains to have similar estimates on $\partial_y \omega$. This will follow easily from the equation:

$$\partial_x \partial_y \omega = \underbrace{\frac{n-1}{4\rho}(\partial_x - \partial_y)\omega + \frac{h^{AB}}{4\rho^2}\partial_A \partial_B \omega - \frac{\Gamma^B}{4\rho^2}\partial_B \omega + \frac{1}{4} x^{\frac{-n+3}{2}} G(\dots)}_{:=\ \xi(y,x,\theta)}\ .$$

From the first part of this proof, $\xi(y, x_0, \theta) \in L^\infty\left([0, u_{**}]; H^m(\mathscr{O})\right),\ \forall m \in \mathbb{N}$. It then follows by induction that

$$\forall j, m \in \mathbb{N},\ \forall y \in [0, u_{**}],\ \|\partial_x^j \partial_y \omega(y, x_0)\|^2_{H^m(\mathscr{O})} < C\ .$$

□

We have the following:

Theorem 7.5.2 *Consider the characteristic initial value problem (7.5.1) in the neighborhood $\mathcal{D} = [0, y_0] \times [x_0, 0[\times \mathscr{O}$ of the truncated cones $\mathcal{C}^+ \cup \mathcal{C}^-$. Suppose that the initial data ω_0^+ and ω_0^- are such that the compatibility condition (7.3.2) holds and satisfy*

$$\omega_0^- \in C^\infty(\mathcal{C}^-)\ ; \quad \omega_0^+,\ \partial\omega_0^+ \in \mathscr{H}^\alpha_\infty(\mathcal{C}^+)\ ; \tag{7.5.4}$$

with $-1 < \alpha \leq -1/2$. *Further assume that the function* G *satisfies the nullity hypothesis* $(\mathcal{H})$ *page 135, with a uniform zero of order* $r > 1$ *such that*

$$n \geq 1 + \frac{4}{r-1} - 2\alpha \ .$$

Then there exists a real number $u_* \in]0, y_0]$ *and a unique smooth (i.e.* C^∞*) function* ω *on* $\mathcal{D}_* = [0, u_*] \times [x_0, 0[\times \mathscr{O}$ *solution of the Goursat problem (7.5.1) satisfying:*

$$\omega \in L^\infty \Big([0, u_*], (\mathscr{H}^\alpha_\infty \cap L^\infty) (\mathcal{C}^+_{u,0}) \Big) , \qquad (7.5.5a)$$

$$\partial\omega \in L^\infty \Big([0, u_*], \mathscr{H}^\alpha_\infty (\mathcal{C}^+_{u,0}) \Big) \ . \qquad (7.5.5b)$$

Moreover, we have:

$$\forall j \in \mathbb{N}, \ \ \partial^j_y \omega \in L^\infty \Big([0, u_*], \mathscr{H}^\alpha_\infty (\mathcal{C}^+_{u,0}) \Big) \ . \qquad (7.5.6)$$

Proof: *Let* m_0 *be the smallest positive integer lager than* $\frac{n+7}{2}$. *Since the hypersurface* $\mathcal{C}^-$ *is a bounded subset of* $\mathbb{R}^{n+1}$, *for all* $k \in \mathbb{N}$, *we have*

$$\omega^-_0 \in C^\infty(\mathcal{C}^-) \hookrightarrow H^{k+2}(\mathcal{C}^-) \quad \textit{and} \quad \omega^+_0 \in \mathscr{H}^\alpha_\infty(\mathcal{C}^+) \hookrightarrow \mathscr{H}^\alpha_{k+1}(\mathcal{C}^+) \ . \ (7.5.7)$$

For $k = m_0$, *the data of (7.5.1) satisfy the hypotheses of Theorem 7.3.17. By this theorem, there exists a real number* $u_* \in]0, y_0]$, *a unique function* ω *of regularity class* C^2 *on* $\mathcal{D}_*$ *such that* $\|(\omega, |x|^{-\alpha}\partial\omega)\|_{L^\infty(\mathcal{D}_*)} < \infty$. *In (7.5.5) it thus remains to prove that* $\forall u, \ (\omega, \partial\omega)(u) \in \mathscr{H}^\alpha_\infty(\mathcal{C}^+_{u,0})$. *Let* $m \in \mathbb{N}$ *and* $\beta \in \mathbb{N}^n$ *such that* $m > m_0$ *and* $|\beta| \leq m$. *We apply again Proposition 7.3.1 page 136 with* $\ell = -2\alpha - 1 + 2\beta_1 \geq 0$ *and* ω *there replaced by* $\partial^\beta\omega$. *For all* $u \in [0, u_*]$, *we have:*

$$\begin{aligned}
&\int_{\mathcal{C}^+_{u,0}} H(u,x) |\partial^\beta(\omega, \nabla_x\omega)(u,x)|^2 dx\, d\nu \leq \\
&\quad \int_{\mathcal{C}^+} H(0,x)|\partial^\beta(\omega,\nabla_x\omega)(0,x)|^2 dx\, d\nu + \int_{\mathcal{C}^-_{u,x_0}} H(y,x_0)|\partial^\beta(\omega,\nabla_y\omega)(y,x_0)|^2 dy d\nu \\
&\quad + (c_1(c_0,\bar{c}_0,n,h) - 2\Lambda) \int_0^u \int_{x_0}^0 H(x,y) \|\partial^\beta(\omega,\nabla\omega)(y,x)\|^2_{L^2(\mathscr{O})} dx dy \\
&\quad + \frac{1}{c_0} \int_{\mathcal{D}_{u,0}} \Big| L^\ell[\partial^\beta\omega] \Big| dy dx\, d\nu \ . \qquad (7.5.8)
\end{aligned}$$

The quantity in the last integral reads:

$$\begin{aligned} L^\ell[\partial^\beta\omega] &= H(x,y)(\partial^\beta\partial_x\omega+\partial^\beta\partial_y\omega)\Box_{\eta,y}\partial^\beta\omega \\ &= H(x,y)(\partial^\beta\partial_x\omega+\partial^\beta\partial_y\omega)\left(\partial^\beta\Box_{\eta,y}\omega+[\Box_{\eta,y},\partial^\beta]\omega\right) \\ &= H(x,y)(\partial^\beta\partial_x\omega+\partial^\beta\partial_y\omega)\left(x^{-\frac{n+3}{2}}\partial^\beta G(\ldots)+[\Box_{\eta,y},\partial^\beta]\omega\right) \\ &=: A+B+C+D\ . \end{aligned}$$

We estimate these quantities as we did in the proof of the Lemma 7.3.5. We repeat these estimates here as we need to consider the $\mathscr{H}^\alpha_m$−norms on $\mathcal{C}^+_u$, we have to make sure that none of the constants in front of the norm of $\partial_y\omega$ depends on Λ.

$$\begin{aligned} A &= x^{-\frac{n+3}{2}}H(x,y)\partial^\beta\partial_x\omega\partial^\beta G(\ldots) \\ &\le H|\partial^\beta\partial_x\omega|^2+Hx^{-(n+3)}|\partial^\beta G(\ldots)|^2\ , \end{aligned}$$

which implies

$$\begin{aligned} \int_{\mathcal{D}_{u,0}} A d\nu dx dy &\le \int_0^u\int_{x_0}^0 H(x,y)|\partial^\beta\partial_x\omega(y,x)|^2 dx dy d\nu \\ &\quad+\int_0^u e^{-\Lambda y}\|x^{-\frac{n+3}{2}-\alpha-\frac12+\beta_1}e^{-\frac12\Lambda x}\partial^\beta G(\ldots)\|^2_{L^2(\mathcal{C}^+_{u,0})}dy\ . \end{aligned}$$

To estimate the second term of the right-hand side of this inequality, we want to use the second part of Proposition A.2 page 53 of [20]. For this purpose, we recall that hypothesis (7.3.4) page 135 implies that for all (p,q) such that $|(p,q)|\le B$, and for all $u\in[0,y_0]$,

$$\left\|\frac{\partial^{j+\ell+i}G(u,\ \cdot,\ \cdot\ ,p,\ q)}{(\partial y)^i\partial p^j\partial q^\ell}\right\|_{\mathscr{C}^0_{m-(j+\ell+i)}(\mathcal{C}^+_{u,0})}\le\hat{C}(B)\,\|(p,q)\|^{m-j-\ell}\ ,$$

thus the conclusion of this Proposition applies. By Theorem 7.3.17 page 175 we have the a priori estimate $M:=\||x|^{-\alpha}(\omega,\nabla_x\omega)\|_{L^\infty(\mathcal{D}_*)}<\infty$. Thus,

$$\begin{aligned} \|x^{-\frac{n+3}{2}-\alpha-\frac12+\beta_1}e^{-\frac12\Lambda x}\partial^\beta G(\ldots)\|^2_{L^2(\mathcal{C}^+_{u,0})} &\le e^{-\Lambda x_0}\|G(\ldots)\|^2_{\mathscr{H}_m^{\alpha+\frac{n+3}{2}}(\mathcal{C}^+_{u,0})} \\ &\le c(M)e^{-\Lambda x_0}\|(w,\nabla_x w)\|_{\mathscr{H}_m^{2\alpha+\frac{n+3}{2}-r(\frac{n-1}{2}+\alpha)}(\mathcal{C}^+_{u,0})} \\ &\le c(M)e^{-\Lambda x_0}\|(w,\nabla_x w)\|_{\mathscr{H}^\alpha_m(\mathcal{C}^+_{u,0})} \\ &\qquad \textit{for } n\ge 1+\frac{4}{r-1}-2\alpha\ . \end{aligned}$$

We then obtain

$$\int_{\mathcal{D}_{u,0}} A d\nu dx dy \leq c(M)e^{-\Lambda x_0}\int_0^u e^{-\Lambda y}\|(w,\nabla_x w)(y)\|^2_{\mathscr{H}^\alpha_m(\mathcal{C}^+_{y,0})}dy \quad (7.5.9)$$

$$\text{for} \quad n \geq 1 + \frac{4}{r-1} - 2\alpha\ .$$

Similarly,

$$\int_{\mathcal{D}_{u,v}} B d\nu dx dy \leq \int_0^u \int_{x_0}^0 H(x,y)\|\partial^\beta \partial_y \omega(y,x)\|^2_{L^2(\mathscr{O})} dx dy$$
$$+c(M)e^{-\Lambda x_0}\int_0^u e^{-\Lambda y}\|(w,\nabla_x w)(y)\|^2_{\mathscr{H}^\alpha_m(\mathcal{C}^+_{y,0})}dy$$
$$\text{for} \quad n \geq 1 + \frac{4}{r-1} - 2\alpha\ . \qquad (7.5.10)$$

As far as the terms C and D are concerned, we recall that the commutators read

$$\begin{aligned}[\Box_{\eta,y},\partial^\beta]\omega &= \frac{n-1}{\rho}(\partial_x-\partial_y)\partial^\beta\omega + \frac{h^{AB}}{\rho^2}\partial_A\partial_B\partial^\beta\omega - \Gamma^B\partial_B\partial^\beta\omega \\ &\quad -\partial^\beta\left(\frac{n-1}{\rho}(\partial_x-\partial_y)\omega\right) - \partial^\beta\left(\frac{h^{AB}}{\rho^2}\partial_A\partial_B\omega\right) + \partial^\beta\left(\Gamma^B\partial_B\partial^\beta\omega\right) \\ &= -\sum_{|\beta^1|<|\beta|} c(\beta,\rho)\partial^{\beta^1}(\partial_x-\partial_y)\omega \\ &\quad - \sum_{|\beta^1|\neq 0,\ \beta^1+\beta^2=\beta} c(\beta,\rho)\partial^{\beta^1}h^{AB}\partial^{\beta^2}\partial_A\partial_B\omega \\ &\quad + \sum_{|\beta^1|\neq 0,\ \beta^1+\beta^2=\beta} c(\beta,\rho)\partial^{\beta^1}\Gamma^B\partial^{\beta^2}\partial_B\omega\ ;\end{aligned}$$

whence, using inequality $ab \leq a^2+b^2$ one has:

$$\int_{\mathcal{D}_{u,0}} C d\nu dx dy \leq c(h,\rho)e^{-\Lambda x_0}\int_0^u e^{-\Lambda y}\|\nabla_x\omega(y)\|^2_{\mathscr{H}^\alpha_m(\mathcal{C}^+_{u,0})}dy$$
$$+c(h,\rho)\sum_{|\mu|<|\beta|}\int_0^u\int_{x_0}^0 H(x,y)\|\partial^\mu\partial_y\omega(y,x)\|^2_{L^2(\mathscr{O})}dx dy (7.5.11)$$

and

$$\int_{\mathcal{D}_{u,0}} D d\nu dx dy \leq c(h,\rho)e^{-\Lambda x_0}\int_0^u e^{-\Lambda y}\|\nabla_x\omega(y)\|^2_{\mathscr{H}^\alpha_m(\mathcal{C}^+_{u,0})}dy$$
$$+c(h,\rho)\sum_{|\mu|\leq|\beta|}\int_0^u\int_{x_0}^0 H(x,y)\|\partial^\mu\partial_y\omega(y,x)\|^2_{L^2(\mathscr{O})}dx dy (7.5.12)$$

Summing inequalities (7.5.10)-(7.5.12) gives:

$$\begin{aligned}\int_{\mathcal{D}_{u,0}} \left|L^\ell[\partial^\beta \omega]\right| d\nu dy dx \quad &\leq \quad c(h,\rho)e^{-\Lambda x_0}\int_0^u e^{-\Lambda y}\|\nabla_x\omega(y)\|^2_{\mathscr{H}^\alpha_m(\mathcal{C}^+_{u,0})}dy \\ &+c(c_0,\bar{c}_0,h,\rho)\sum_{|\beta|\leq m}\int_0^u\int_{x_0}^0 H(x,y)\|\partial^\beta\partial_y\omega(y,x)\|^2_{L^2(\mathscr{O})}dxdy \\ &\qquad \textit{for} \quad n \geq 1+\frac{4}{r-1}-2\alpha\ .\end{aligned}$$

We can then rewrite (7.5.8) as (note that $1 \leq e^{-\Lambda x} \leq e^{-\Lambda x_0}$*):*

$$\begin{aligned}&e^{-\Lambda u}\|(\omega,\nabla_x\omega)(u)\|^2_{\mathscr{H}^\alpha_m(\mathcal{C}^+_{u,0})} \leq \\ &\quad e^{-\Lambda x_0}\|(\omega,\nabla_x\omega)\|^2_{\mathscr{H}^\alpha_m(\mathcal{C}^+)} + \int_{\mathcal{C}^-_{u,x_0}} H(y,x_0)|\partial^\beta(\omega,\nabla_y\omega)(y,x_0)|^2 dy d\nu \\ &\quad +(c(c_0,\bar{c}_0,n,\rho)+c(h,\rho)-2\Lambda)\int_{\mathcal{D}_{u,0}} H(y,x)\|\partial^\beta\partial_y\omega(y,x)\|^2_{L^2(\mathscr{O})}dxdy \\ &\quad +e^{-\Lambda x_0}c(c_0,\bar{c}_0,h,\rho)\int_0^u e^{-\Lambda y}\|\nabla_x\omega(y)\|^2_{\mathscr{H}^\alpha_m(\mathcal{C}^+_{u,0})}dy \\ &\qquad\qquad \textit{for} \quad n \geq 1+\frac{4}{r-1}-2\alpha\ .\end{aligned}$$

As we did before, we choose Λ *sufficiently large so that the term in the second line of the previous estimate is negative and for* $n \geq 1+\frac{4}{r-1}-2\alpha$*, we obtain (note that* $e^{-\Lambda u_*} \leq e^{-\Lambda y} \leq 1$*):*

$$\begin{aligned}&e^{-\Lambda u}\|(\omega,\nabla_x\omega)(u)\|^2_{\mathscr{H}^\alpha_m(\mathcal{C}^+_{u,0})} \leq \\ &\quad e^{-\Lambda x_0}\|(\omega_0^+,\nabla_x\omega_0^+)\|^2_{\mathscr{H}^\alpha_m(\mathcal{C}^+)} + c(h,\rho,\Lambda)\int_0^u \|\nabla_x\omega(y)\|^2_{\mathscr{H}^\alpha_m(\mathcal{C}^+_{u,0})}dy \\ &\quad +(-x_0)^{-2\alpha-1}e^{-\Lambda x_0}\sum_{|\beta|\leq m}\int_0^{u_*}\|\partial_x^{\beta_1}(\omega,\partial_y\omega)\|^2_{H^{m+1}(\mathscr{O})}\ .\end{aligned}$$

Note that in the last line of this estimate, it appears the outgoing derivatives of ω *on the initial null surface* $\mathcal{C}^-$*. By Lemma 7.5.1, there exists a constant* $C_0 = C(c_0,\bar{c}_0,h,n,\Lambda)$ *such that for all* $u \in u_*$ *we have:*

$$\|(\omega,\nabla_x\omega)(u)\|^2_{\mathscr{H}^\alpha_m(\mathcal{C}^+_{u,0})} \leq C_0\left(1+\int_0^u\|\nabla_x\omega(y)\|^2_{\mathscr{H}^\alpha_m(\mathcal{C}^+_{u,0})}dy\right)\ .$$

Gronwall's Lemma then gives the following estimate:

$$\forall m > m_0,\ \forall u \in [0,u_*],\ \ \|(\omega,\nabla_x\omega)(u)\|^2_{\mathscr{H}^\alpha_m(\mathcal{C}^+_{u,0})} \leq C_0 e^{C_0 u} < \infty\ .$$

Therefore,

$$(\omega, \nabla_x \omega) \in L^\infty \left([0, u_*];\ \mathscr{H}^\alpha_\infty(\mathcal{C}^+_{u,0}) \right) .$$

In (7.5.5b)*, it remains to prove that* $\forall u \in [0, u_*], \partial_y \omega(u) \in \mathscr{H}^\alpha_\infty(\mathcal{C}^+_{u,0})$*. Recall that the partial differential equation satisfied by* ω *reads*

$$\partial_y \partial_x \omega - \frac{n-1}{4\rho} \partial_x \omega + \frac{n-1}{4\rho} \partial_y \omega = \frac{h^{AB} \partial^2_{AB} \omega}{4\rho^2} - \frac{\Gamma^B}{4\rho^2} \partial_B \omega - \frac{1}{4} x^{-\frac{n+3}{2}} G(\dots) ,$$

which can be rewritten as

$$\partial_x \left(\rho^{\frac{n-1}{2}} \partial_y \omega - \frac{n-1}{4} \rho^{\frac{n-3}{2}} \omega \right) = \xi , \tag{7.5.13}$$

where

$$\xi = \rho^{\frac{n-1}{2}} \left(\frac{h^{AB} \partial^2_{AB} \omega}{4\rho^2} - \frac{\Gamma^B}{4\rho^2} \partial_B \omega - \frac{1}{4} x^{-\frac{n+3}{2}} G(\dots) - \frac{(n-1)(n-3)}{16\rho^2} \omega \right) .$$

Integrating (7.5.13) leads to

$$\begin{aligned} \partial_y \omega(y,x,\theta) &= \frac{n-1}{4\rho} \omega(y,x,\theta) \\ &+ \rho^{-\frac{n-1}{2}} \left(\rho_0^{\frac{n-1}{2}} \partial_y \omega_0^-(y,\theta) - \frac{n-1}{4} \rho_0^{\frac{n-3}{2}} \omega_0^-(y,\theta) + \int_{x_0}^x \xi(y,s,\theta) ds \right) , \end{aligned} \tag{7.5.14}$$

where $\rho_0 = \frac{1}{a} - y + x_0$ *. The above identity implies that*

$$\forall u \in [0, u_*], \partial_y \omega(u) \in \mathscr{H}^\alpha_\infty(\mathcal{C}^+_{u,0}) .$$

The last statement (7.5.6) of Theorem 7.5.2 will be proved by induction. The cases $j = 0$ *and* $j = 1$ *follow from (7.5.5). Assuming now that (7.5.6) holds for a certain* $j \geq 1$*, we* $y-$*differentiate (7.5.14)* j *times and obtain:*

$$\partial_y^{j+1} \omega(y,x,\theta) = \sum_{0 \leq i \leq j} \left(\sigma_1(y,x) \partial_y^i \omega(y,x,\theta) + \sigma_2(y,x) \partial_y^{i+1} \omega_0^-(y,\theta) + \int_{x_0}^x \sigma_3(y,s) \partial_y^i \xi(y,s,\theta) ds \right)$$

where the σ_i*'s are bounded smooth functions on the set* $[0, u_*] \times [x_0, 0]$*. From this identity it follows that (7.5.6) holds with* j *replaced by* $j+1$ *and the proof is complete.* □

Conclusion of the second part

In this part of the thesis, we obtained existence and uniqueness of solutions of a class of semi-linear characteristic Cauchy problem with data on the light cone. By assuming that the data satisfy the property of those of [27] near the tip of the cone, and that near $\{r = \infty\}$ they are in some appropriate weighted Sobolev space, we proved that these solutions exist on a neighborhood of the entire initial cone which contains a subset of the future null infinity $\mathscr{I}^+$. We showed that this result applies to wave map on Minkowski space time with target manifold an arbitrary smooth Riemannian manifold of finite dimension. Next by assuming that the source term does not depends on the normal derivative of the unknown function, we state and prove a high regularity result which might leads to polyhomogeneity of solutions of such null Cauchy problem. To obtain polyhomogeneity of solution in the case of null initial Cauchy data, it remains to check that in this case, one can prove the characteristic version of Theorem 1.1.1 of the first part of the thesis. This will be done later.

General Conclusion

At the end of this work, we have stated and proved existence and uniqueness theorems of semi-global solutions of ordinary and characteristic Cauchy problems for symmetric hyperbolic systems of second order in high space dimension. The originality of these results is the fact that on one hand, in our approach there is no need to impose:

- *the null condition of S. Klainerman to the source terms of our equations (this condition is too restrictive for Einstein equations in space dimension* $n = 3$*)*
- *smallness of the Cauchy data*
- *further restrictions on the oddness of the space dimension,*

and on the second hand, in both cases, the constructed solutions are defined on a neighborhood of the whole initial data hypersurface which thickness does not shrink to zero as one approaches future null infinity. Nevertheless, these results need to be improved: one can consider a polyhomogeneous existence of solutions result for the Einstein equations in lower space dimension for ordinary or characteristic Cauchy problem. The idea would be to use a conformal compactification which preserve the smoothness of equations at hand as the conformal transformation introduced by H. Friedrich in [34] and to prove the characteristic analog of Theorem 1.1.1 on the polyhomogeneity of solutions. We intend to focus on these questions in a forthcoming future.

Appendix A

Spaces of polyhomogeneous functions and their properties

A.1 Introduction

The aim of this Appendix is to give a detail presentation of the spaces of smooth and polyhomogeneous functions with their properties. In both cases, these are weighted spaces, the weight being choose in order to control the singular behavior near infinity of the functions involved. We notice that this is essentially the presentation made by P. Chruściel and S. Leşki in their paper [19]. As the need arises from the problem at hand, some times we have made some slight generalizations of some definitions there and sated and proved some new properties.

The spaces of polyhomogeneous functions (i.e functions which are expandable in terms $r^{-j}\log^j r$*) were first introduced by L. Anderson and P. Chruściel in [1]. In their analysis of the constraints equations of the vacuum Einstein equations for asymptotically hyperboloidal initial data, they find that log terms arise in asymptotic expansion of the solutions of the constraints. Here, we also use the formalism stated on the space of polyhomogeneous functions in this last reference. (See Appendix E of [1]).*

In what follows, $\bar{M}$ *will be a smooth compact manifold of dimension* $n+1$*. The boundary of* $\bar{M}$ *will be denoted by* ∂M *and* M *the interior of* $\bar{M}$ *as a topological space so that* ∂M *is also the boundary of* M*. The letter* x *will denote a defining function of* ∂M *in the sense that* $x \geq 0,\ \forall p \in \bar{M},\ x(p) = 0 \iff p \in \partial M,$ *and* $|dx|\big|_{\partial M} > 0$*. It turns out that, there exists* *a neighborhood K of ∂M in $\bar{M}$ on which the positive function x can be used as coordinate in any local coordinate system on K. We assume that there exists a global coordinate system on K denoted by (y, x, v^A), $A = 1, \ldots, n-1$ which gives a product decomposition of K. We are going to introduce now some spaces of functions with controlled singular behavior at $\{x = 0\}$, $\{y = 0\}$ or $\{y = x = 0\}$.*

A.2 Spaces of differential functions with weight

For $k \in \mathbb{N}$ and any open subset Ω of M, we denote by $C_k(\Omega)$ the set of all functions which are k times continuously differentiable on Ω. We denote by $C_k(\bar{\Omega})$ the set of $C_k(\Omega)$-functions which can be extended by continuity to C_k functions defined in an open neighborhood of Ω. Consider the set denoted $\mathscr{U}$ and defined by

$$\mathscr{U} = \left\{(x, v^A, y) : \ 0 < x < y \ , \ v = (v^A) \in \mathscr{O} \ , \ 0 < y < y_0 < \infty\right\} \ , \quad \text{(A.2.1)}$$

where $\mathscr{O}$ is a compact manifold without boundary. We will write z for the joint set of variables (x, y, v^A). We use the multi-index notation of Schwartz, thus if $\beta = (\beta_0, \beta_1, \ldots, \beta_n)$, then

$$\partial^\beta = \partial_z^\beta = \partial_y^{\beta_0} \partial_x^{\beta_1} \partial_{v^1}^{\beta_2} \ldots \partial_{v^{n-1}}^{\beta_n} = \partial_y^{\beta_0} \partial_x^{\beta_1} \partial_v^\gamma$$

where $\gamma = (\beta_2, \ldots, \beta_n)$.

Definition A.2.1 *Let $k \in N$, α, $\sigma \in \mathbb{R}$ and Ω an open subset of $\mathscr{U}$. We define the spaces*

1. *$\mathscr{C}^\alpha_{\{x=0\},k}(\Omega)$ as the space of all function $f \in C_k(\Omega)$ such that $\forall i, j \in \mathbb{N}, \gamma \in N^{n-1}, i + j + |\gamma| \leq k$, the quantity $\sup_\Omega |x^{-\alpha} \partial_v^\gamma [\partial_y]^i [x\partial_x]^j f|$ is finite.*

2. *$\mathscr{C}^\sigma_{\{y=0\},k}(\Omega)$ as the space of all function $f \in C_k(\Omega)$ such that $\forall i, j \in \mathbb{N}, \gamma \in N^{n-1}, i + j + |\gamma| \leq k$ the quantity $\sup_\Omega |y^{-\sigma} \partial_v^\gamma [y\partial_y]^i [\partial_x]^j f|$ is finite.*

3. *$\mathscr{C}^\alpha_{\{0 \leq x \leq y\},k}(\Omega)$ as the space of all function $f \in C_k(\Omega)$ such that $\forall i, j \in \mathbb{N}, \gamma \in N^{n-1}, i + j + |\gamma| \leq k$, the quantity $\sup_\Omega |x^{-\alpha} \partial_v^\gamma [y\partial_y]^i [x\partial_x]^j f|$ is finite.*

4. $\mathscr{C}^{\alpha,\sigma}_{\{0\le x\le y\},k}(\Omega)$ *as the space of all function* $f\in C_k(\Omega)$ *such that* $\forall i,j\in\mathbb{N},\gamma\in N^{n-1}, i+j+|\gamma|\le k$, *the quantity* $\sup_\Omega|x^{-\alpha}y^{-\sigma}\partial_v^\gamma[y\partial_y]^i[x\partial_x]^j f|$ *is finite.*

We shall write

$$\mathscr{C}^\alpha_{\{x=0\},\infty}(\Omega)=\bigcap_{k\in\mathbb{N}}\mathscr{C}^\alpha_{\{x=0\},k}(\Omega)\ ,$$

and similarly for $C_\infty(\Omega)$, $\mathscr{C}^\alpha_{\{x=0\},\infty}(\Omega)$, *etc. Not that the estimates*

$$\begin{aligned}(x\partial_x)^i(y\partial_y)^j\partial_v^\beta(x^\alpha f) &= x^\alpha\sum_{\ell=0}^i C_{\ell,\alpha}(x\partial_x)^\ell(y\partial_y)^j\partial_v^\beta f\\ &\le Cx^\alpha y^\beta = Cx^{\alpha-\delta}x^\delta y^\beta\le Cx^{\alpha-\delta}y^{\beta+\delta}\end{aligned}$$

shows that

$\forall k\in\mathbb{N},\ \alpha,\ \beta\in\mathbb{R}$ *and* $\delta\ge 0$, *we have*

$$x^\alpha\mathscr{C}^\beta_{\{y=0\},k}(\Omega)\subset\mathscr{C}^{\alpha-\delta,\beta+\delta}_{\{0\le x\le y\},k}(\Omega)\ .$$

Example A.2.2 Any finite linear combination of functions of the form $f_{p,\ell}=x^p\ln^\ell x$ where p and ℓ are nonnegative integers, belongs to the space $\mathscr{C}^{-\epsilon}_{\{x=0\},k}(\mathscr{U})$, for all $\epsilon>0$. Indeed since the operator $x\partial_x$ obeys the Leibnitz rule, we have:

$$\left|(x\partial_x)^i(x^p\ln^\ell x)\right| = \left|\sum_{m=0}^i \mathrm{C}_i^m(x\partial_x)^m(x^p)(x\partial_x)^{i-m}\ln^\ell x\right| < Cx^{-\epsilon}\ .$$

Similarly,

$$\sum_{m=0}^M C_m x^{p_m}y^{k_m}\ln^{\ell_m}x\ln^{q_m}y\in\mathscr{C}^{\epsilon,\epsilon'}_{\{0\le x\le y\},k}(\mathscr{U})\ .$$

Remark A.2.3 The estimates

$$\begin{aligned}\left|\partial_x^i\partial_y^j\partial_v^\gamma f\right| &= \left(\frac{x}{y}\right)^j x^{-i-j}\left|x^iy^j\partial_x^i\partial_y^j\partial_v^\gamma f\right|\\ &\le Cx^{-i-j}\left|(x\partial_x)^i(y\partial_y)^j\partial_v^\gamma f\right|\end{aligned}$$

shows that $\mathscr{C}^\alpha_{\{0\le x\le y\},k}(\mathscr{U})\subset C_k(\overline{\mathscr{U}})$ for $k<\alpha$.

Notation: *Let* W^α *be a family of spaces, where* α *is a decay index, e.g.* $W^\alpha=\mathscr{C}^\alpha_{\{x=0\},k}(\mathscr{U})$, *or* $W^\alpha=\mathscr{C}^\alpha_{\{0\le x\le y\},\infty}(\mathscr{U})$, *etc. We define*

$$W^{<\alpha}=\cap_{\sigma<\alpha}W^\sigma\ .$$

This notation is very useful to accommodate $\ln^n x$ factors that arise in the problem at hand: for example, in this notation we have

$$x^\alpha \ln^N x \in \mathscr{C}^{<\alpha}_{\{x=0\},\infty}(\mathscr{U}) \,.$$

We point out the following:

Lemma A.2.4 *For $0 \le x \le y \le \mathring{y}$ consider the system*

$$\partial_x \psi + b\psi = c \,,$$

and suppose that there exists $\epsilon < 1$ such that the linear map b has coefficients in $\mathscr{C}^{-\epsilon}_{\{x=0\},0}$. For $\alpha \in \mathbb{R} \setminus \{-1\}$ there exists a constant $C = C(\alpha, \epsilon, \|b\|_{\mathscr{C}^{-\epsilon}_{\{x=0\},0}}, \mathring{y})$ such that

1. *For $\alpha > -1$ we have*

$$\|\psi\|_{L^\infty} \le C \left(\|\psi|_{x=y}\|_{L^\infty} + \|c\|_{\mathscr{C}^{\alpha}_{\{x=0\},0}} \right) \,, \tag{A.2.2}$$

2. *while for $\alpha < -1$ it holds that*

$$\|\psi\|_{\mathscr{C}^{\alpha+1}_{\{x=0\},0}} \le C \left(\|\psi|_{x=y}\|_{\mathscr{C}^{\alpha+1}_{\{x=0\},0}} + \|c\|_{\mathscr{C}^{\alpha}_{\{x=0\},0}} \right) \,. \tag{A.2.3}$$

The proof of this Lemma can be found in [19] Lemma 3.12.

A.3 Spaces of polyhomogeneous functions

Definition A.3.1 *We define the space of polyhomogeneous functions at $\{x = y = 0\}$ denoted by $\mathscr{A}_{\{0\le x\le y\}}$ as the collection of functions $f \in C_\infty(\mathscr{U})$ such that there exists integers N_i, real numbers n_i, $\hat{n}_i$ and functions $f_{ijl} \in C_\infty(\overline{\mathscr{U}})$ with the property that*

$$\forall m \in \mathbb{N}, \quad \exists N(m) \in \mathbb{N}, \quad f - \sum_{i=0}^{N(m)} \sum_{j,l=0}^{N_i} f_{ijl} y^{\hat{n}_i} x^{n_i} \ln^j y \ln^\ell x \in C_m(\overline{\mathscr{U}}). \tag{A.3.1}$$

To avoid repetitions of terms with identical powers in (A.3.1) it is convenient to impose $(n_i, \hat{n}_i) \ne (n_j, \hat{n}_j)$ for $i \ne j$, and we will always assume that this condition is satisfied.

Definition A.3.2 *Let $\delta \in \mathbb{R}_*^+$ such that $1/\delta \in \mathbb{N}^*$. We define the space $\mathscr{A}^\delta_{\{0\le x\le y\}}$ as the space of functions $f \in \mathscr{A}_{\{0\le x\le y\}}$ such that the corresponding real numbers n_i and $\hat{n}_i$ in (A.3.1) satisfy $\{n_i,\ i\in\mathbb{N}\}\subset\delta\mathbb{N}$, $\{\hat{n}_i,\ i\in\mathbb{N}\}\subset\delta\mathbb{Z}$ and $\hat{n}_i > -n_i$.*

We have the following

Proposition A.3.3 *For all function $f \in \mathscr{A}^\delta_{\{0\le x\le y\}}$, there exists and integer N and a positive conctant C such that, $\quad \forall z\in\mathscr{U}, \quad |f(z)| \le C(1+|\ln x|^N)$.*

Proof: *We write (A.3.1) with $m=0$ and we obtain that there exists $N(0)\in\mathbb{N}$ and a function $r_0 \in C_0(\overline{\mathscr{U}})$ such that*

$$f = \sum_{i=0}^{N(0)}\sum_{j,l=0}^{N_i} f_{ijl} y^{\hat{n}_i} x^{n_i} \ln^j y \ln^\ell x + r_0 \,.$$

Since $\overline{\mathscr{U}}$ is a compact subset of $\bar{M}$, there exists a positive constant C_0 such that

$$\forall z\in\mathscr{U},\ |f(z)| \le C_0\left(1+\sum_{i=0}^{N(0)}\sum_{j,l=0}^{N_i} |y^{\hat{n}_i} x^{n_i}\ln^j y\ln^\ell x|\right) \,.$$

Now, since the function $y \longmapsto y^\epsilon \ln^\ell y$, $\epsilon>0$ is bounded on any neighborhood of 0, we have

$$|y^{\hat{n}_i} x^{n_i}\ln^j y\ln^\ell x| = |\left(\frac{x}{y}\right)^{n_i} y^{\hat{n}_i+n_i}\ln^j y\ln^\ell x| \le C_1|\ln^\ell x| \,.$$

This last inequality completes the proof. □

We see that the spaces of functions $\mathscr{A}^\delta_{\{0\le x\le y\}}$ are made of function f with eventually a singular behavior at $x=0$ and/or at $y=0$. The last proposition shows that this singularity can be controlled by the multiplication with any positive power of x. We introduce now the space of functions with singular behavior only at $x=0$ or only at $y=0$. We have the following

Definition A.3.4 *We define the space $\mathscr{A}_{\{x=0\}}$ as the space of all functions in $\mathscr{A}_{\{0\le x\le y\}}$ with $\hat{n}_i=0$ for all i and no non-trivial powers of $\ln y$ in (A.3.1). Thus $f\in\mathscr{A}_{\{x=0\}}$ if and only if $f\in C_\infty(\mathscr{U})$ and there exists integers N_i, real numbers n_i, and functions $f_{ij}\in C_\infty(\overline{\mathscr{U}})$ such that*

$$\forall m\in\mathbb{N},\quad \exists N(m)\in\mathbb{N},\quad f-\sum_{i=0}^{N(m)}\sum_{j=0}^{N_i} f_{ij}x^{n_i}\ln^j x \in C_m(\overline{\mathscr{U}}). \qquad \text{(A.3.2)}$$

Similarly, we define the spaces $\mathscr{A}^\delta_{\{x=0\}}$, $\mathscr{A}_{\{y=0\}}$ and $\mathscr{A}^\delta_{\{y=0\}}$.

The following proposition will be use repeatedly.

Proposition A.3.5 *1. We have the inclusion*

$$\mathscr{A}^{\delta}_{\{0\le x\le y\}} \cap L^\infty \subset \mathscr{C}^{0}_{\{0\le x\le y\},\infty} \ .$$

It follows that for any $\epsilon > 0$ we have $\mathscr{A}^{\delta}_{\{0\le x\le y\}} \subset \mathscr{C}^{0-\epsilon}_{\{0\le x\le y\},\infty}$.

2. Similarly

$$\mathscr{A}^{\delta}_{\{x=0\}} \cap L^\infty \subset \mathscr{C}^{0}_{\{x=0\},\infty} \ ,$$

and for any $\epsilon > 0$ we have $\mathscr{A}^{\delta}_{\{x=0\}} \subset \mathscr{C}^{0-\epsilon}_{\{x=0\},\infty}$.

The proof can be found in [19] Proposition A.2. We have the following characterization of the space of polyhomogeneous functions $\mathscr{A}_{\{x=0\}}$:

Proposition A.3.6 *$f \in \mathscr{A}_{\{x=0\}}$ if and only if for every $m \in \mathbb{N}$ there exist $N(m), N_i(m) \in \mathbb{N}$, $n_i(m) \in \mathbb{R}$ and functions $f_{ij} \in C_m(\overline{\mathscr{U}})$ such that*

$$f - \sum_{i=0}^{N(m)} \sum_{j=0}^{N_i(m)} f_{ij}\ x^{n_i(m)} \ln^j x \in C_m(\overline{\mathscr{U}}) \ , \tag{A.3.3}$$

with a similar property for $\mathscr{A}^{\delta}_{\{x=0\}}$, $\mathscr{A}_{\{0\le x\le y\}}$, etc.

The proof can be found in [19] Proposition A.3.

We will need the following characterisation of functions which are polyhomogeneous up to lower order terms. To avoid annoying special cases involving logarithms we assume $\sigma \notin \mathbb{N}$, though the proof gives also a corresponding statement in this case:

Proposition A.3.7 *Suppose that $\sigma \notin \mathbb{N}$, let $\mathscr{S} = \{(y, x, v^A) \in \mathscr{U} : x = y\}$,*

$$f|_{\mathscr{S}} \in x^\beta \mathscr{A}_{\{0\le x\le y\}} \ , \quad f \in x^\beta \mathscr{A}^{\delta}_{\{0\le x\le y\}} + y^\beta \mathscr{A}^{\delta}_{\{0\le x\le y\}} + \mathscr{C}^{\sigma}_{\{0\le x\le y\},k} \ ,$$

and assume that for all i, j satisfying $i + j \le k + 1$ there exists

$$g_{i,j} \in x^\beta \mathscr{A}^{\delta}_{\{0\le x\le y\}} + y^\beta \mathscr{A}^{\delta}_{\{0\le x\le y\}}$$

such that for every multi-index γ for which $i + j + |\gamma| = k + 1$ we have

$$|(x\partial_x)^i (y\partial_y)^j \partial_v^\gamma (f - g_{i,j})| \le C x^\sigma \ .$$

Then

$$f \in x^\beta \mathscr{A}^{\delta}_{\{0\le x\le y\}} + y^\beta \mathscr{A}^{\delta}_{\{0\le x\le y\}} + \mathscr{C}^{\sigma}_{\{0\le x\le y\},k+1} \ . \tag{A.3.4}$$

The proof of this Proposition can be found in [20], Propostion A.5. We will use a slight generalization of a definition of [20]:

Definition A.3.8 *We shall say that a function $H(z,w)$ is $\mathscr{A}^{\delta}_{\{0\le x\le y\}}$*-polyhomogeneous in z with a uniform zero of order l in w *if the following hold: First, H is smooth in $w\in\mathbb{R}^N$ at fixed $z\in\mathscr{U}$. Next, it is required that for all $B\in\mathbb{R}$ and $k\in\mathbb{N}$ there exists a constant $\hat{C}(B)$ such that, for all $|w|\le B$ and* $0\le i\le \min(k,l)$,

$$\left\|\frac{\partial^i H(\cdot,w)}{\partial w^i}\right\|_{\mathscr{C}^0_{\{0\le x\le y\},k-i}(\mathscr{U})} \le \hat{C}(B)|w|^{l-i}\ . \tag{A.3.5}$$

Further,

$$\forall i\in\mathbb{N}\quad \partial^i_w H(\cdot,w)\in\mathscr{A}^{\delta}_{\{0\le x\le y\}} \tag{A.3.6}$$

at fixed constant w. Finally we demand the uniform estimate for constant w's: $\forall\epsilon>0, M\ge 0, i,k\in\mathbb{N}\ \ \exists\, C(\epsilon,M,i,k)\ \forall|w|\le M$ such that

$$\|\partial^i_w H(\cdot,w)\|_{\mathscr{C}^{-\epsilon}_{\{0\le x\le y\},k}(\mathscr{U})}\le C(\epsilon,M,i,k)\ . \tag{A.3.7}$$

The qualification "in w" in "uniform zero of order l in w" will often be omitted. Similarly to [20], the small parameter ϵ has been introduced above to take into account the possible logarithmic blow-up of functions in $\mathscr{A}^{\delta}_{\{0\le x\le y\}}$ at $x=0$; for the applications to the nonlinear scalar wave equation or to the wave map equation on Minkowski space-time, the alternative simpler requirement would actually suffice: $\forall\ M\ge 0, i,k\in\mathbb{N}\ \ \exists\, C(M,i,k)\ \forall|w|\le M$

$$\|\partial^i_w H(\cdot,w)\|_{\mathscr{C}^{0}_{\{0\le x\le y\},k}(\mathscr{U})}\le C(M,i,k)\ , \tag{A.3.8}$$

again for constant w's. Functions which are smooth in (w,z), and have a zero of order l in w at $w=0$, satisfy the above conditions. We have the following

Lemma A.3.9 *If $H(x^\mu,\omega)$ is $\mathscr{A}^{\delta}_{\{0\le x\le y\}}$-polyhomogeneous in z with a uniform zero of order m in ω and $g\in\mathscr{A}^{\delta}_{\{0\le x\le y\}}\cap L^\infty$ then,*

$$H(.,x^{q\delta}g)\in x^{qm\delta}\mathscr{A}^{\delta}_{\{0\le x\le y\}}\ .$$

Proof: *If we Taylor expand H up to order $r > m$, we obtain:*

$$\begin{aligned} H(., x^{q\delta}g) &= x^{qm\delta}\Bigg\{ \sum_{\ell=max(0,m)}^{r} \sum_{i_1+\ldots+i_N=\ell} \frac{r!}{i_1!\ldots i_N!} x^{(l-m)q\delta} g_1^{i_1}\ldots g_N^{i_N} \frac{\partial^\ell H(x^\mu,0)}{\partial\omega^\ell} \\ &+ \int_0^1 \left[\frac{(1-t)^r}{r!} \sum_{i_1+\ldots+i_N=r+1} \frac{(r+1)!}{i_1!\ldots i_N!} x^{(r+1-m)q\delta} g_1^{i_1}\ldots g_N^{i_N} \frac{\partial^{r+1}H}{\partial\omega^{r+1}}(x^\mu, tx^{q\delta}g)\right] dt \Bigg\} \\ &= I + II \end{aligned}$$

Since the term I is polyhomogeneous (product of such functions), it suffices to show that for all $k \in \mathbb{N}$, the term II is in $C_k(\overline{\mathscr{U}})$, provided that $r = r(k)$ is chosen large enough. Recall that $\mathscr{C}^\alpha_{\{0\le x\le y\},\infty} \subset C_k(\overline{\mathscr{U}})$ for $k < \alpha$. See Remark A.2.3. For fixed k, we choose r large enough such that we can write $q(r+1-m)\delta = n_1 + n_2$ with $n_1, n_2 > k$, we will then obtain

$$x^{n_1} g_1^{i_1}\ldots g_N^{i_N} \in \mathscr{C}^{n_1}_{\{0\le x\le y\},\infty} \subset C_k(\bar\Omega)$$

and as $g \in L^\infty$, (1.1.10) gives

$$x^{n_2}\frac{\partial^{r+1}H}{\partial\omega^{r+1}}(x^\mu, tx^{q\delta}g) \in \mathscr{C}^{n_2-\epsilon}_{\{0\le x\le y\},\infty} \subset C_k(\bar\Omega),$$

i.e.

$$II \in C_k(\bar\Omega) \text{ and thus } H(., x^{q\delta}g) \in x^{qm\delta}\mathscr{A}^\delta_{\{0\le x\le y\}}\ .$$

□

A.4 Auxiliary spaces: The $\mathscr{F}-$ and $\mathscr{T}-$spaces

For $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$ we set

$$\mathscr{F}^\alpha_{\{0\le x\le y\},k} = \Bigg\{ f \mid \forall\, 0 \le i+j+|\gamma| \le k \quad \exists N \in \mathbb{N}: \\ |\partial_x^i\partial_y^j\partial_v^\gamma f| \le \begin{cases} Cy^{\alpha-i-j}(1+|\ln y|)^N \text{if } \alpha-i-j\ge 0 \\ Cx^{\alpha-i-j}(1+|\ln x|)^N \text{if } \alpha-i-j< 0 \end{cases} \Bigg\}. \quad \text{(A.4.1)}$$

We will also need a version of the $\mathscr{F}$-spaces where the functions involved are "almost independent of x when α is large", in the following sense:

$$\mathring{\mathscr{F}}^\alpha_{\{0\le x\le y\},k} = \Bigg\{ f \mid \forall\, 0 \le i+j+|\gamma| \le k \quad \exists N : \\ |\partial_x^i\partial_y^j\partial_v^\gamma f| \le \begin{cases} Cy^{\alpha-j}(1+|\ln y|)^N \text{if } \alpha-j\ge 0,\ i=0 \\ Cx^{\alpha-i-j}(1+|\ln x|)^N \text{otherwise} \end{cases} \Bigg\}. \quad \text{(A.4.2)}$$

Let $\alpha, \beta \in \mathbb{R}$, $k \in \mathbb{N}$. To be able to estimate in terms of powers of $|\ln x|$ rather than $1 + |\ln x|$ it is convenient to assume $0 < y_0 < 1$. We have the following

Definition A.4.1 *We say that $f \in \mathscr{T}^{\alpha,(\beta;k)}_{\{0\le x\le y\},\infty}$ if for all i, j, γ there exist constants $C > 0$ and $N \in \mathbb{N}$ such that, for $0 < x \le y \le y_0$ we have*

$$|\partial_x^i \partial_y^j \partial_v^\gamma f| \;\le\; C\Big(x^{\alpha+\beta-i-j} + x^{\alpha-i}y^{\beta-j} + x^{\alpha+\beta-i-k}y^{k-j}\Big)|\ln^N x| \,. \quad \text{(A.4.3)}$$

We will write $f \in \mathscr{T}^{\alpha,\beta}_{\{0\le x\le y\},\infty}$ for $f \in \mathscr{T}^{\alpha,(\beta;0)}_{\{0\le x\le y\},\infty}$, and we note that for $k = 0$, or for $\beta = k$, the last term in (A.4.3) is not needed, e.g.:

$$f \in \mathscr{T}^{\alpha,\beta}_{\{0\le x\le y\},\infty} \quad\Longleftrightarrow\quad |\partial_x^i \partial_y^j \partial_v^\gamma f| \le C\Big(x^{\alpha+\beta-i-j} + x^{\alpha-i}y^{\beta-j}\Big)|\ln^N x| \,. \quad \text{(A.4.4)}$$

Finally, for $\beta \le 0$ the last term in (A.4.4) can be dropped altogether.

Strictly speaking, the only space out of the $\mathscr{T}^{\alpha,(\beta;k)}_{\{0\le x\le y\},\infty}$'s which is absolutely necessary in our proofs is the one with $k = \beta = 0$. However, we have decided to include a short discussion of the other ones as well, as those spaces appear naturally in the problem at hand.

Let $\{F_i\}_{i\in\mathbb{N}}$ be any countable family of function spaces, we shall write

$$\dot{\oplus}_n F_n = \{f : \exists N \in \mathbb{N} \,,\, f_n \in F_n \,,\, 0 \le n \le N \,,\, f = \sum_{n=0}^{N} f_n\} \,.$$

The dot over the symbol $\oplus$ is meant to emphasise the fact that only finite linear combinations are considered.

For further use we note the following elementary properties:

Proposition A.4.2

1. *If $f \in \mathscr{T}^{\alpha,(\beta;k)}_{\{0\le x\le y\},\infty}$ then $\partial_x f \in \mathscr{T}^{\alpha-1,(\beta;k)}_{\{0\le x\le y\},\infty}$ and $\partial_y f \in \mathscr{T}^{\alpha,(\beta-1;\max(k-1,0))}_{\{0\le x\le y\},\infty}$*
2. *For $\alpha' \ge \alpha$ and $\beta' \ge \beta$ we have $\mathscr{T}^{\alpha',(\beta';k)}_{\{0\le x\le y\},\infty} \subset \mathscr{T}^{\alpha,(\beta;k)}_{\{0\le x\le y\},\infty}$.*
3. *For $\sigma \ge 0$ we have $\mathscr{T}^{\alpha+\sigma,(\beta;k)}_{\{0\le x\le y\},\infty} \subset \mathscr{T}^{\alpha,(\beta+\sigma;k)}_{\{0\le x\le y\},\infty}$.*
4. *If $\mathbb{N} \ni \ell < \alpha$ and $\ell \le k \le \beta$ we have $\mathscr{T}^{\alpha,(\beta;k)}_{\{0\le x\le y\},\infty} \subset \mathscr{C}^{\beta}_{\{y=0\},\ell}$.*
5. *$\mathscr{C}^{\alpha}_{\{x=0\},\infty} \subset \mathscr{C}^{\alpha}_{\{0\le x\le y\},\infty} \subset \mathscr{T}^{\alpha,(0;k)}_{\{0\le x\le y\},\infty}$, and $\mathscr{C}^{\beta}_{\{y=0\},\infty} \subset \mathscr{T}^{0,(\beta;k)}_{\{0\le x\le y\},\infty}$ for all k.*

6. *If* $f \in \mathscr{T}^{\alpha,(\beta;k)}_{\{0\le x\le y\},\infty}$ *and* $g \in C^{\infty}$*, then* $fg \in \mathscr{T}^{\alpha,(\beta;k)}_{\{0\le x\le y\},\infty}$.

7. *If* $g \in \mathscr{C}^{\alpha}_{\{x=0\},\infty}$ *and* $h \in \mathscr{C}^{\beta}_{\{y=0\},\infty}$ *then* $gh \in \mathscr{T}^{\alpha,(\beta;k)}_{\{0\le x\le y\},\infty}$ *for all* k.

8. *We have* $x^{\sigma}\mathscr{T}^{\alpha,(\beta;k)}_{\{0\le x\le y\},\infty} = \mathscr{T}^{\alpha+\sigma,(\beta;k)}_{\{0\le x\le y\},\infty}$ *for all* $k \in \mathbb{N}$ *and* $\sigma \in \mathbb{R}$.

9. *For* $f \in \mathscr{T}^{\alpha,(\beta;k)}_{\{0\le x\le y\},\infty}$ *and* $\ell \in \mathbb{N}$ *we have* $x^{\ell}f \in \mathscr{T}^{\alpha,(\beta+\ell;k+\ell)}_{\{0\le x\le y\},\infty}$ *for all* k.

The proof can be found in [19] Proposition A.6.

We have the following

Proposition A.4.3 *For all* $\alpha,\ \alpha' \in \mathbb{R},\ \epsilon,\epsilon' > 0,\ k \in \mathbb{N}$,

$$(a) \qquad If \quad \alpha' \ge \alpha \quad then \quad \mathring{\mathscr{F}}^{\alpha'}_{\{0\le x\le y\},k} \subset \mathring{\mathscr{F}}^{\alpha}_{\{0\le x\le y\},k}$$

$$(b) \qquad y^{\epsilon'}\mathring{\mathscr{F}}^{\lambda-\epsilon-\epsilon'}_{\{0\le x\le y\},\infty} \subset \mathring{\mathscr{F}}^{\lambda-\epsilon-\epsilon'}_{\{0\le x\le y\},\infty} \qquad for\ all\ \mathbb{R} \ni \lambda > 0\ and\ \lambda-\epsilon-\epsilon' > 0.$$

Proof: *(a). Let* f *be in* $\mathring{\mathscr{F}}^{\alpha'}_{\{0\le x\le y\},k}$; $(i,i,\gamma) \in \mathbb{N}^{n+1}$ *such that* $i+j+|\gamma| \le k$, *we want to show that,*

$$\partial^i_x\partial^j_y\partial^{\gamma}_v f \le \begin{cases} cy^{\alpha-j}(1+|lny|)^N\ if\ \alpha-j\ge 0,\ i=0 \\ cx^{\alpha-i-j}(1+|lnx|)^N \quad otherwise \end{cases}$$

• *If* $i=0$ *then* $\qquad \partial^j_y\partial^{\gamma}_v f \le \begin{cases} cy^{\alpha'-j}(1+|lny|)^N\ if\ \alpha'-j\ge 0 \\ cx^{\alpha'-j}(1+|lnx|)^N\ if\ \alpha'-j<0 \end{cases}$.

- *If* $\alpha-j\ge 0$ *then* $\alpha'-j\ge 0$ *and*

$$\partial^j_y\partial^{\gamma}_v f \le cy^{\alpha'-j}(1+|lny|)^N \le cy^{\alpha-j}(1+|lny|)^N\ .$$

- *If* $\alpha-j<0$ *then for* $\alpha'-j<0$ *we have*

$$\partial^j_y\partial^{\gamma}_v f \le cx^{\alpha'-j}(1+|lnx|)^N \le cx^{\alpha-j}(1+|lnx|)^N\ ,$$

and for $\alpha'-j\ge 0$ *we have*

$$\partial^j_y\partial^{\gamma}_v f \le cy^{\alpha'-j}(1+|lny|)^N \le cx^{\alpha-j}(1+|lnx|)^N\ .$$

Thus for $i=0$ *we have:*

$$\partial^j_y\partial^{\gamma}_v f \le \begin{cases} cy^{\alpha-j}(1+|lny|)^N\ if\ \alpha-j\ge 0 \\ cx^{\alpha-j}(1+|lnx|)^N\ if\ \alpha-j<0 \end{cases}\ .$$

• $If\ i > 0$ *then,*

$$\partial_x^i \partial_y^j \partial_v^\gamma f \leq c x^{\alpha' - i - j}(1 + |lnx|)^N \leq c x^{\alpha - i - j}(1 + |lnx|)^N \ .$$

Thus $\mathring{\mathscr{F}}^{\alpha'}_{\{0 \leq x \leq y\},k} \subset \mathring{\mathscr{F}}^{\alpha}_{\{0 \leq x \leq y\},k}$ *for* $\alpha' \geq \alpha$.

(b). Let f *be in* $y^{\epsilon'} \mathring{\mathscr{F}}^{\lambda-\epsilon-\epsilon'}_{\{0 \leq x \leq y\},\infty}$ *i.e.* $f = y^\epsilon g$ *with* $g \in \mathring{\mathscr{F}}^{\lambda-\epsilon-\epsilon'}_{\{0 \leq x \leq y\},\infty}$. *We want to show that* $f \in \mathring{\mathscr{F}}^{\lambda-\epsilon-\epsilon'}_{\{0 \leq x \leq y\},\infty}$. *For this purpose, we choose an arbitrary* $k \in \mathbb{N}$, $(i, j, \gamma) \in \mathbb{N}^{n+1}$ *such that* $i + j + |\gamma| \leq k$ *and we will show that*

$$|\partial_x^i \partial_y^j \partial_v^\gamma f| \leq \begin{cases} C y^{\lambda-\epsilon-\epsilon'-j}(1 + |\ln y|)^N \, if\ \lambda - \epsilon - \epsilon' - j \geq 0,\ i{=}0 \\ C x^{\lambda-\epsilon-\epsilon'-i-j}(1 + |\ln x|)^N \, otherwise \end{cases} .$$

• $If\ i = j = 0$ *then (recall* $\lambda - \epsilon - \epsilon' \geq 0$*),*

$$|\partial_v^\gamma f| = |\partial_v^\gamma y^{\epsilon'} g| = y^{\epsilon'} |\partial_v^\gamma g| \leq y^{\epsilon'} y^{\lambda-\epsilon-\epsilon'}(1 + |lny|)^N,$$

and we obtain that

$$|\partial_v^\gamma f| \leq y^{\lambda-\epsilon-\epsilon'}(1 + |lny|)^N \ .$$

• $i = 0$ *and* $j \neq 0$

$$\partial_y^j \partial_v^\gamma f = \partial_y^j \partial_v^\gamma y^{\epsilon'} g = \sum_{j_1+j_2=j} c(j_1, j_2, \epsilon') y^{\epsilon'-j_1} \partial_y^{j_2} \partial_v^\gamma g \ .$$

- If $\lambda - \epsilon - \epsilon' - j \geq 0$ *then* $\lambda - \epsilon - \epsilon' - j_2 \geq 0$ *and we have:*

$$|\partial_y^j \partial_v^\gamma f| \leq y^{\lambda-\epsilon-\epsilon'}(1 + |lny|)^N \ .$$

- Suppose that $\lambda - \epsilon - \epsilon' - j \leq 0$:

$$\begin{aligned} \partial_y^j \partial_v^\gamma f &= y^{\epsilon'} \partial_y^j \partial_v^\gamma g + \sum_{j_1+j_2=j,\ j_1 \neq 0} c(j_1, j_2, \epsilon') y^{\epsilon'-j_1} \partial_y^{j_2} \partial_v^\gamma g \\ &\leq y^{\epsilon'} x^{\lambda-\epsilon-\epsilon'}(1 + |lnx|)^N + \sum_{j_1+j_2=j,\ j_1 \neq 0} c(j_1, j_2, \epsilon') y^{\epsilon'-j_1} \partial_y^{j_2} \partial_v^\gamma g \ . \end{aligned}$$

For $j_1 \neq 0$ *we have* $|y^{\epsilon'-j_1} \partial_y^{j_2} \partial_v^\gamma g| \leq x^{\lambda-\epsilon-j}(1 + |lnx|)^N$, *if* $\lambda - \epsilon - \epsilon' - j_2 < 0$ *and*

$$\begin{aligned} |y^{\epsilon'-j_1} \partial_y^{j_2} \partial_v^\gamma g| &\leq y^{\epsilon'-j_1} y^{\lambda-\epsilon-\epsilon'-j_2}(1 + |lny|)^N,\ if\ \lambda - \epsilon - \epsilon' - j_2 \geq 0 \\ &\leq c y^{\lambda-\epsilon-j}(1 + |lny|)^N \\ &\leq c x^{\lambda-\epsilon-j}(1 + |lnx|)^N \ . \end{aligned}$$

Thus

$$|\partial_y^j \partial_v^\gamma f| \le c x^{\lambda-\epsilon-\epsilon'-j}(1+|lnx|)^N \ .$$

- *If* $i \neq 0$ *and* $j = 0$ *then*

$$\begin{aligned} |\partial_x^i \partial_v^\gamma f| &= y^{\epsilon'} |\partial_x^i \partial_v^\gamma g| \\ &\le c y^{\epsilon'} x^{\lambda-\epsilon-\epsilon'-i}(1+|lnx|)^N \\ &\le c x^{\lambda-\epsilon-\epsilon'-i}(1+|lnx|)^N \ . \end{aligned}$$

- *If* $i \neq 0$ *and* $j \neq 0$ *then,*

$$\begin{aligned} |\partial_x^i \partial_y^j \partial_v^\gamma f| &\le \sum_{j_1+j_2=j} c y^{\epsilon'-j_1} |\partial_x^i \partial_y^{j_2} \partial_v^\gamma g| \\ &\le y^{\epsilon} |\partial_x^i \partial_y^j \partial_v^\gamma g| + \sum_{j_1+j_2=j,\ j_1\neq 0} c y^{\epsilon'-j_1} \partial_x^i \partial_y^{j_2} \partial_v^\gamma g| \\ &\le c y^{\epsilon'} x^{\lambda-\epsilon-\epsilon'-i-j}(1+|lnx|)^N \\ &\quad + \sum_{j_1+j_2=j,\ j_1\neq 0} c x^{\epsilon'-j_1} x^{\lambda-\epsilon-\epsilon'-i-j}(1+|lnx|)^N \\ &\le c x^{\lambda-\epsilon-\epsilon'-i-j}(1+|lnx|)^N \ . \end{aligned}$$

Finally , we have shown that, $\forall (i,j,\gamma) \in \mathbb{N}^{n+1}$,

$$|\partial_x^i \partial_y^j \partial_v^\gamma f| \le \begin{cases} c y^{\lambda-\epsilon-\epsilon'-j}(1+|lny|)^N, \ if\ \lambda-\epsilon-\epsilon'-j \ge 0,\ i=0 \\ c x^{\lambda-\epsilon-\epsilon'-i-j}(1+|lnx|)^N, \ otherwise \end{cases} ,$$

which proves that

$$f \in \mathring{\mathscr{F}}^{\lambda-\epsilon-\epsilon'}_{\{0\le x\le y\},\infty} \ .$$

□

We point out the following

Lemma A.4.4 *Assume that* $\mathscr{O}$ *is convex, compact, with interior points. Let* $q \in \mathbb{N}^*$, $\mathbb{R} \ni \lambda \ge 0$, *and let* $H(x^\mu, w)$ *be* $\mathscr{A}^\delta_{\{0\le x\le y\}}$*-polyhomogeneous with respect to* x^μ *with a zero of order* m *in* w. *If for all* $\epsilon > 0$ *we have*

$$g \in \left(\mathscr{A}^\delta_{\{0\le x\le y\}} + \mathscr{A}^\delta_{\{x=0\},\dot{\oplus}_i x^{i\delta} \mathring{\mathscr{F}}^{\lambda-i\delta}_{\{0\le x\le y\},\infty}} + \mathscr{T}^{\lambda-\epsilon,(0;0)}_{\{0\le x\le y\},\infty} \right) , \tag{A.4.5}$$

then it also holds, for all $\epsilon > 0$,

$$H(\cdot, x^{q\delta} g) \in x^{mq\delta} \left(\mathscr{A}^\delta_{\{0\le x\le y\}} + \mathscr{A}^\delta_{\{x=0\},\dot{\oplus}_i x^{i\delta} \mathring{\mathscr{F}}^{\lambda-i\delta}_{\{0\le x\le y\},\infty}} + \mathscr{T}^{\lambda-\epsilon,(0;0)}_{\{0\le x\le y\},\infty} \right) . \tag{A.4.6}$$

If $\lambda > 0$ *and (A.4.5) holds with* $\epsilon = 0$, *then (A.4.6) also holds with* $\epsilon = 0$.

The proof of this Lemma can be found in [19] Lemma 3.13

A.5 Extensions of a class of functions

Let $0 \le \varphi \in C^\infty(\mathbb{R})$, $\operatorname{supp}\varphi \subset [-1/2, 1/2]$, $\int_{\mathbb{R}} \varphi(x)dx = 1$. *For* $0 < x \le y \le y_0$ *we set*

$$E[f](x,y,v) := \int_0^\infty \frac{\varphi(\frac{w-y}{x})}{x} f(w,v)dw \tag{A.5.1a}$$

$$= \int_{y/2}^{3y/2} \frac{\varphi(\frac{w-y}{x})}{x} f(w,v)dw \tag{A.5.1b}$$

$$= \int_{-\infty}^{\infty} \frac{\varphi(\frac{w-y}{x})}{x} f(w,v)dw \tag{A.5.1c}$$

$$= \int_{-\infty}^{\infty} \varphi(z) f(y+xz,v)dz \tag{A.5.1d}$$

$$= \int_{-1/2}^{1/2} \varphi(z) f(y+xz,v)dz \tag{A.5.1e}$$

(there is no need to know the values of f *for negative* w *when using (A.5.1c) as* $\varphi = 0$ *there; a similar comment applies to (A.5.1d)).*

The results here are an adaptation to the problem at hand of [1, Section 3.3]. In the lemma that follows one can think of μ *as belonging to* $[0,1)$*, but this restriction is not necessary for the result:*

Lemma A.5.1 *For* $k \in \mathbb{N}$ *and* $\mu \in \mathbb{R}$ *suppose that*

$$|\partial_v^\gamma \partial_y^\ell f| \le C y^{k+\mu-\ell}(1+|\ln y|)^N \quad \textit{for} \quad 0 \le \ell \le k\ , \tag{A.5.2}$$

then

$$E[f] \in y^\mu \mathscr{F}^k_{\{0\le x\le y\},\infty}\ . \tag{A.5.3}$$

If moreover there exists $\lambda > 0$ *such that*

$$|\partial_v^\gamma \partial_y^k f(y,v) - \partial_v^\gamma \partial_y^k f(y',v)| \le C y^{\mu-\lambda}(1+|\ln y|)^N |y-y'|^\lambda \quad \textit{for} \quad |y'-y| \le y/2\ , \tag{A.5.4}$$

then we also have

$$E[f](x,y,v) \in y^{\mu-\lambda} \mathscr{F}^{k+\lambda}_{\{0\le x\le y\},\infty}\ . \tag{A.5.5}$$

The proof can be found in [19] Proposition A.7.

We continue with

Lemma A.5.2 *Let* $\mu \geq 0$ *and for* $0 \leq i \leq m$ *let* f_i *satisfy (A.5.2) with* k *there replaced by* $m - i$. *There exists* $h \in y^\mu \mathscr{F}^m_{\{0\leq x\leq y\},\infty}$ *such that*

$$0 \leq i \leq m \qquad \partial^i_x h|_{x=0} = f_i \ . \tag{A.5.6}$$

If the f_i*'s satisfy (A.5.4) with* $k = m - i$ *then* $h \in y^{\mu-\lambda}\mathscr{F}^{m+\lambda}_{\{0\leq x\leq y\},\infty}$.

The proof can be found in [19] Proposition A.8.

A.6 Two important integral operators

For $0 \leq x \leq y \leq y_0 < \infty$ *set*

$$I_1(f)(x, v^A, y) = \int_x^y f(s, v^A, y)ds \ , \tag{A.6.1}$$

$$I_2(f)(x, v^A, y) = \int_x^y f(x, v^A, s)ds \ . \tag{A.6.2}$$

In our arguments we will need to understand the action of I_1 *and* I_2 *on various spaces defined above. We start with polyhomogeneous functions:*

A.6.1 Integral operators on $\mathscr{A}$−spaces

Lemma A.6.1 *Let* $f \in C_\infty(\overline{\mathscr{U}})$, $p \in \mathbb{R}, j \in \mathbb{N}$. *For every* $m \in \mathbb{N}$ *there exist an integer* N, *sequences of numbers* $k_i \in \mathbb{N}, \ell_i \in \mathbb{N}$, *a sequence of smooth functions* f_i *and a function* $r_m \in C_m(\overline{\Omega})$ *such that*

$$\int_x^y f(s, v^A, y)s^p \ln^j s \, ds = \sum_{i=1}^N f_i \left(y^{p+k_i+1} \ln^{\ell_i} y - x^{p+k_i+1} \ln^{\ell_i} x \right) + r_m \ . \tag{A.6.3}$$

The proof can be found in [19] Proposition A.12.

Proposition A.6.2 *1. Let* $g \in x^\beta y^\gamma \mathscr{A}^\delta_{\{0\leq x\leq y\}}$. *Then*

$$I_1(g) \in \ y^{\beta+\gamma+1}\mathscr{A}^\delta_{\{y=0\}} + x^{\beta+1}y^\gamma \mathscr{A}^\delta_{\{0\leq x\leq y\}} \ ,$$

$$I_2(g) \in \ x^{\beta+\gamma+1}\mathscr{A}^\delta_{\{x=0\}} + x^\beta y^{\gamma+1} \mathscr{A}^\delta_{\{0\leq x\leq y\}} \ .$$

It follows in particular that $\mathscr{A}_{\{0\leq x\leq y\}}$ *is stable under both integrations above.*

2. *Let* $g \in x^\beta y^\gamma \mathscr{A}^\delta_{\{x=0\}}$. *Then*

$$I_1(g) \in \ y^{\beta+\gamma+1}\mathscr{A}^\delta_{\{y=0\}} + x^{\beta+1}y^\gamma \mathscr{A}^\delta_{\{x=0\}} \ ,$$

$$I_2(g) \in \ x^{\beta+\gamma+1}\mathscr{A}^\delta_{\{x=0\}} + x^{\beta}y^{\gamma+1} \mathscr{A}^\delta_{\{x=0\}} \ .$$

Proof: *Applying Proposition A.3.6 repeatedly gives the result .* □

A.6.2 Integral operators on $\mathscr{C}-$spaces

We continue with a study of the action of I_1 *and* I_2 *on the* $\mathscr{C}^{\alpha,\sigma}_{\{0\le x\le y\},k}$ *spaces. Note that the action on the* $\mathscr{C}^{\alpha}_{\{0\le x\le y\},k}$ *spaces is obtained as a special case from*

$$\mathscr{C}^{\alpha}_{\{0\le x\le y\},k} = \mathscr{C}^{\alpha,0}_{\{0\le x\le y\},k} \ .$$

Lemma A.6.3 *Let* $\alpha, \sigma \in \mathbb{R}$, $k \in \mathbb{N} \cup \{\infty\}$,

1. *If* $f \in \mathscr{C}^{\alpha,\sigma}_{\{0\le x\le y\},k}$, $\alpha < -1$, *then* $I_1(f) \in \mathscr{C}^{\alpha+1,\sigma}_{\{0\le x\le y\},k}$.

2. *If* $f \in \mathscr{C}^{\alpha,\sigma}_{\{0\le x\le y\},k}$, $\alpha > -1$, *then* $I_1(f) \in \mathscr{C}^{\alpha+\sigma+1}_{\{y=0\},k} + \mathscr{C}^{\alpha+1,\sigma}_{\{0\le x\le y\},k}$.

The proof can be found in [19] Proposition A.10.

Lemma A.6.4 *Let* $\alpha, \sigma \in \mathbb{R}$, $k \in \mathbb{N} \cup \{\infty\}$,

1. *If* $f \in \mathscr{C}^{\alpha,\sigma}_{\{0\le x\le y\},k}$, $\sigma > -1$ *then* $I_2(f) \in \mathscr{C}^{\alpha,\sigma+1}_{\{0\le x\le y\},k}$.

2. *If* $f \in \mathscr{C}^{\sigma}_{\{y=0\},\infty}$, *then* $I_2(f) \in \mathscr{F}^{\sigma+1}_{\{0\le x\le y\},\infty}$.

The proof can be found in [19] Proposition A.11.

A.6.3 Integral operators on $\mathscr{T}-$ and $\mathscr{F}-$spaces

Proposition A.6.5 *Let* $\alpha > -1$, $\beta \ge k$. *For any* $\epsilon > 0$ *we have*

$$I_1(\mathscr{T}^{\alpha,(\beta;k)}_{\{0\le x\le y\},\infty}) \subset y^\epsilon \mathring{\mathscr{F}}^{\alpha+1-\epsilon+\beta}_{\{0\le x\le y\},\infty} + \mathscr{T}^{\alpha+1-\epsilon,(\beta;k)}_{\{0\le x\le y\},\infty} \ .$$

The proof can be found in [19] Proposition A.12.

Remark A.6.6 We expect the result to remain valid with $\epsilon = 0$, but the proof below fails for this value of ϵ. In any case the current result is sufficient for our purposes.

Proposition A.6.7 *Let $\alpha + p\delta > -1$. For any $\epsilon > 0$ we have*

1. $I_1(x^{p\delta}\ln^\ell x\mathring{\mathscr{F}}^{\alpha}_{\{0\le x\le y\},\infty}) \subset y^\epsilon \mathring{\mathscr{F}}^{\alpha+p\delta+1-\epsilon}_{\{0\le x\le y\},\infty} + \mathscr{A}^{\delta}_{\{x=0\},x\mathring{\mathscr{F}}^{\alpha}_{\{0\le x\le y\},\infty}}$.

2. $I_1(\mathscr{A}^{\delta}_{\{x=0\},x^{p\delta}\mathring{\mathscr{F}}^{\alpha}_{\{0\le x\le y\},\infty}}) \subset y^\epsilon \mathring{\mathscr{F}}^{\alpha+p\delta+1-\epsilon}_{\{0\le x\le y\},\infty} + \mathscr{A}^{\delta}_{\{x=0\},x^{p\delta+1}\mathring{\mathscr{F}}^{\alpha}_{\{0\le x\le y\},\infty}}$.

The proof can be found in [19] Proposition A.14.

Proposition A.6.8 1. $I_2(\mathring{\mathscr{F}}^{\alpha}_{\{0\le x\le y\},\infty}) \subset \mathring{\mathscr{F}}^{\alpha}_{\{0\le x\le y\},\infty}$.

2. $I_2(\mathscr{A}^{\delta}_{\{x=0\},x^n\mathring{\mathscr{F}}^{\alpha}_{\{0\le x\le y\},\infty}}) \subset \mathscr{A}^{\delta}_{\{x=0\},x^n\mathring{\mathscr{F}}^{\alpha}_{\{0\le x\le y\},\infty}}$.

3. $I_2(\mathscr{T}^{\alpha,(\beta;k)}_{\{0\le x\le y\},\infty}) \subset \mathscr{T}^{\alpha,(\beta+1;k+1)}_{\{0\le x\le y\},\infty}$.

The proof can be found in [19] Proposition A.15.

Appendix B

Function spaces, Embeddings, Inequalities

B.1 Definitions of some weighted spaces.

We recall that

$$\mathbf{H}_{\lambda,\tau_0} = \{(\tau, x, v^A);\ \tau = \tau_0,\ \sigma_\lambda(\tau) < x < x_0\} \ \equiv\ ]x_2, x_0[\times \mathscr{O}$$

and

$$\mathbf{H}_{\tau_0} = \{(\tau, x, v^A);\ \tau = \tau_0,\ 0 < x < x_0\} \ \equiv\ ]0, x_0[\times \mathscr{O}\ ,$$

with $x_0 = \sigma(\tau_0)$ *and* $x_2 = \sigma_\lambda(\tau_0)$*. In what follows the symbol* Ω *will generally denote one of the sets* $\mathbf{H}_{\lambda,\tau_0}$*, or* $\mathbf{H}_{\tau_0}$*. Any subset of* $\overline{\mathbf{H}}_{\tau_0}$ *can be locally coordinatized by coordinates* $y^i = (x, v^A)$*, where the* v^A*'s can be thought of as local coordinates on* $\mathscr{O}$*. We cover* $\mathscr{O}$ *by a finite number of coordinate charts* $\mathcal{O}_i$ *so that the sets* $\overline{\Omega}_i$*, where*

$$\Omega_i := (0, x_0) \times \mathcal{O}_i\ ,$$

cover $\overline{\mathbf{H}}_{\tau_0}$*. We use the usual multi-index notation for partial derivatives: for* $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ *we set* $\partial^\beta = \partial_1^{\beta_1} \ldots \partial_n^{\beta_n}$*. We will write* ∂_v^β *for derivatives of the form* $\partial_2^{\beta_2} \ldots \partial_n^{\beta_n}$*, which do not involve the* $x^1 \equiv x$ *variable.*

If $\mathscr{O}$ *is an open set, for* $k \in \mathbb{N} \cup \infty$ *we let* $C_k(\mathscr{O})$ *denote the usual space of* k*-times differentiable functions on* $\mathscr{O}$*; the symbol* $C_k(\overline{\mathscr{O}})$ *is used to denote the set of those functions in* $C_k(\mathscr{O})$ *the derivatives of which, up to order* k*, extend by continuity to* $\overline{\mathscr{O}}$*. We emphasise that no uniformity is assumed in* $C_k(\mathscr{O})$*, so that functions there could grow without bound when* *approaching the boundary . Nevertheless, the symbol* $\|\cdot\|_{C_k}$ *will denote the usual supremum norm of* f *and its derivatives up to order* k*. For* $\alpha\in\mathbb{R}$ *and* $k\in\mathbb{N}$ *, we define* $\mathscr{C}^\alpha_0(\Omega_i)$ *(respectively* $\mathscr{C}^\alpha_k(\Omega_i)$ *as the spaces of appropriately differentiable functions such that the respective norms*

$$\begin{aligned}\|f\|_{\mathscr{C}^\alpha_0(\Omega_i)} &\equiv \sup_{p\in\Omega_i}|x^{-\alpha}f(p)|\,,\\ \|f\|_{\mathscr{C}^\alpha_k(\Omega_i)} &\equiv \sum_{0\le|\beta|\le k}\|x^{\beta_1}\partial^\beta f\|_{\mathscr{C}^\alpha_0(\Omega_i)}\,,\end{aligned}\tag{B.1.1}$$

are finite. We define the spaces $\mathscr{H}^\alpha_k(\Omega_i)$ *as the spaces of those functions in* $H^{\rm loc}_k(\Omega_i)$ *for which the norms* $\|\cdot\|_{\mathscr{H}^\alpha_k(\Omega_i)}$ *are finite, where*

$$\|f\|^2_{\mathscr{H}^\alpha_k(\Omega_i)}=\sum_{0\le|\beta|\le k}\int_{\Omega_i}(x^{-\alpha+\beta_1}\partial^\beta f)^2\frac{dx}{x}\,d\nu\,.\tag{B.1.2}$$

Here $d\nu$ *is a measure on* $\mathscr{O}$ *arising from some smooth Riemannian metric on* $\mathscr{O}$*. This is equivalent to*

$$\sum_{0\le\beta_1+|\beta|\le k}\int_{\Omega_i}(x^{-\alpha}(x\partial_x)^{\beta_1}\partial_v^\beta f)^2\frac{dx}{x}\,d\nu\,,\tag{B.1.3}$$

and it will sometimes be convenient to use (B.1.3) as the definition of $\|f\|^2_{\mathscr{H}^\alpha_k(\Omega_i)}$*. We note the equivalence of norms,*

$$\|f\|_{H_0(\mathscr{O})}\approx\|f\|_{\mathscr{H}^{-1/2}_0(\mathscr{O})}\,,$$

and that $\mathscr{H}^\alpha_k(\mathbf{H}_{\lambda,\tau_0})=H_k(\mathbf{H}_{\lambda,\tau_0})$ *for all* α *and* k *whenever* $x_2>0$*, the norms being equivalent, with the constants involved depending upon* x_2 *and* x_0*, and degenerating in general when* x_2 *tends to zero. In order to have global system of coordinate on* $\mathbf{H}_{\lambda,\tau_0}$ *we use the global vectors field* $(X_i)_{1\le i\le r}$ *defined in [20] (see Appendix A) related to coordinates* (v^A) *of* $\mathscr{O}$ *by*

$$\partial_A = \sum_{i=2}^{r}f^i_A(v^B)X_i\,,\tag{B.1.4a}$$

$$X_i = \sum_{A=2}^{n}X^A_i(v^B)\partial_A\,,\tag{B.1.4b}$$

for some locally defined smooth functions f^i_A, X^A_i*; and where*

$$X_1=\partial_x\,.$$

Clearly things can be arranged so that those functions are bounded, together with all their partial derivatives. For any multi-index $\beta = (\beta_1, \beta_2, \ldots, \beta_r) \in \mathbb{N}^r$ *we set, on* $\mathbf{H}_{\tau_0}$*,*

$$\mathscr{D}^\beta f = X_1^{\beta_1} X_2^{\beta_2} \cdots X_r^{\beta_r} f = \partial_x^{\beta_1} X_2^{\beta_2} \cdots X_r^{\beta_r} f \ . \tag{B.1.5}$$

It follows that we have

$$\|f\|_{\mathscr{C}_k^\alpha(M_{x_0})} \approx \sum_{0\le|\beta|\le k} \|x^{\beta_1}\mathscr{D}^\beta f\|_{\mathscr{C}_0^\alpha(M_{x_0})} \,,$$

$$\|f\|^2_{\mathscr{H}_k^\alpha(\mathbf{H}_{\tau_0})} \approx \sum_{0\le|\beta|\le k} \int_{M_{x_0}} (x^{-\alpha+\beta_1}\mathscr{D}^\beta f)^2 \frac{dx}{x}\, d\nu$$

(where $\approx$ *denotes the fact that the norms are equivalent),* etc. *Here,* $|\beta| = \beta_1 + \ldots + \beta_r$*. From the identity*

$$]0, x_0] = \underset{n\in\mathbb{N}^*}{\cup} \left[\frac{x_0}{2^n}, \frac{x_0}{2^{n-1}}\right]$$

and the equivalence

$$s \in [1,2] \iff x_0\frac{x}{2^n} \in \left[\frac{x_0}{2^n}, \frac{x_0}{2^{n-1}}\right]$$

we see that there is a useful way of rewriting $\|\cdot\|_{\mathscr{H}_k^\alpha(\mathbf{H}_{\tau_0})}$ *which proceeds as follows: for* $f \in \mathscr{H}_k^\alpha(\mathbf{H}_{\tau_0})$*,* $s \in (1,2)$*, and* $n \in \mathbb{N}^*$ *we set*

$$f_n(s,v) = f(x = x_0\frac{s}{2^n}, v) \ ; \tag{B.1.6}$$

letting $\approx$ *denote again equivalence of norms one then has, after a change of variables,*

$$\begin{aligned}
\|f\|^2_{\mathscr{H}_k^\alpha(\mathbf{H}_{\tau_0})} &= \sum_{n\ge1}\sum_{0\le|\beta|\le k} \int_{[2^{-n}x_0, 2^{1-n}x_0]\times\mathscr{O}} |x^{-\alpha+\beta_1}\mathscr{D}^\beta f(x,v)|^2 \frac{dx}{x}\, d\nu \\
&\approx x_0^{-2\alpha}\sum_{n\ge1}\sum_{0\le|\beta|\le k} 2^{2n\alpha}\int_{[1,2]\times\mathscr{O}} |\mathscr{D}^\beta f_n(s,v)|^2 ds\, d\nu \\
&= x_0^{-2\alpha}\sum_{n\ge1} 2^{2n\alpha}\|f_n\|^2_{H_k([1,2]\times\mathscr{O})} \ .
\end{aligned} \tag{B.1.7}$$

More precisely, we write $A \approx B$ *if there exist constants* $C_1, C_2 > 0$ *such that* $C_1 A \le B \le C_2 A$*. In (B.1.7) the relevant constants depend only upon* α *and* k*.*

It turns out to be useful to have a formula similar to (B.1.7) for functions in $\mathscr{H}_k^\alpha(\mathbf{H}_{\lambda,\tau_0})$*); this can be done for any* x_0 *and* x_2*, but in order to obtain uniform control of certain constants it is convenient to require* $2x_2 \le x_0$*. For such values of* x_0 *and* x_2 *we let* $n_0(x_0,x_2) \in \mathbb{N}$ *be such that* $\frac{x_0}{2^{n_0+1}} \le x_2 \le \frac{x_0}{2^{n_0}}$*. For* $n \in \mathbb{N}$*,* $n \ge 1$*, and for any* $f : \mathbf{H}_{\lambda,\tau_0} \to \mathbb{R}^N$ *we then define* $f_n : (1,2) \times \mathscr{O} \to \mathbb{R}^N$ *by*

$$\begin{aligned} n \le n_0\,, &\qquad f_n(s,v) = f(x_0\frac{s}{2^n},v)\,,\\ n = n_0+1\,, &\qquad f_n(s,v) = f(x_2\, s,v)\,,\\ n > n_0+1\,, &\qquad f_n = 0\,. \end{aligned} \tag{B.1.8}$$

(This coincides with the definition already given for $\mathbf{H}_{\tau_0}$*), when this set is thought of as being an "*$\mathbf{H}_{\lambda,\tau_0}$ *with* $x_2 = 0$*", if we set* $n_0 = +\infty$*.) A calculation as in (B.1.7) shows that for any* $2x_2 \le x_0$*, there exist constants* C_1 *and* c_1*, independent of* x_0*, and* x_2*, such that for all* $f \in \mathscr{H}_k^\alpha(\mathbf{H}_{\lambda,\tau_0})$*,*

$$\begin{aligned} & c_1 x_0^{-2\alpha}\sum_n \{2^{n\alpha}\|f_n\|_{H_k([1,2]\times\mathscr{O})}\}^2\\ &\quad \le \|f\|^2_{\mathscr{H}_k^\alpha(\mathbf{H}_{\lambda,\tau_0})} \le C_1 x_0^{-2\alpha}\textstyle\sum_n \{2^{n\alpha}\|f_n\|_{H_k([1,2]\times\mathscr{O})}\}^2\,. \end{aligned} \tag{B.1.9}$$

Equation (B.1.7) leads one to introduce (the symbol $\mathscr{B}$ *might suggest to the reader that we specifically have Besov spaces in mind; this is not the case, and we hope that the notation will not lead to confusion) spaces* $\mathscr{B}_k^\alpha$*, that arise naturally from weighted Sobolev embeddings,* cf. *Equation (B.2.2) below: we define*

$$\|f\|^2_{\mathscr{B}_k^\alpha(\mathbf{H}_{\tau_0})} = x_0^{-2\alpha}\sum_{n\ge1} 2^{2n\alpha}\|f_n\|^2_{C_k([1,2]\times\mathscr{O})}\,, \tag{B.1.10}$$

f_n *as in (B.1.6), and we set*

$$\mathscr{B}_k^\alpha(\mathbf{H}_{\tau_0}) = \{f \in C_k(\mathbf{H}_{\tau_0}) \mid \|f\|_{\mathscr{B}_k^\alpha(\mathbf{H}_{\tau_0})} < \infty\}\,.$$

Clearly

$$\mathscr{B}_k^\alpha(\mathbf{H}_{\tau_0}) \subset \mathscr{C}_k^\alpha(\mathbf{H}_{\tau_0})\,.$$

We have the trivial inclusion,

$$\alpha' > \alpha \quad\Longrightarrow\quad \mathscr{C}_k^{\alpha'}(\mathbf{H}_{\tau_0}) \subset \mathscr{H}_k^\alpha(\mathbf{H}_{\tau_0})\,. \tag{B.1.11}$$

The fact that the inequality $\alpha' > \alpha$ *in (B.1.11) is strict has various annoying consequences, which are best avoided by introducing yet another space — the* *space $\mathscr{G}^\alpha_k$ of functions in $H^k_{\mathrm{loc}}(\mathbf{H}_{\tau_0})$) for which the norm squared*

$$\|f\|^2_{\mathscr{G}^\alpha_k(\mathbf{H}_{\tau_0}))} = \sup_{n\geq 1}\left\{\sum_{0\leq\beta\leq k}\int_{[2^{-n}x_0,2^{1-n}x_0]\times\mathscr{O}} |x^{-\alpha+\beta_1}\mathscr{D}^\beta f(x,v)|^2\,\frac{dx}{x}\,d\nu\right\} \tag{B.1.12}$$

is finite. We note that $\|f\|_{\mathscr{G}^\alpha_k(\mathbf{H}_{\tau_0})}$ is equivalent to

$$x_0^{-\alpha}\sup_{n\geq 1}\left\{2^{n\alpha}\|f_n\|_{H_k([1,2]\times\mathscr{O})}\right\}\ , \tag{B.1.13}$$

with $f_n(s,v)=f(\frac{x_0 s}{2^n},v)$, as in (B.1.6). To define the $\mathscr{G}^\alpha_k(\mathbf{H}_{\lambda,\tau_0})$'s, assuming again that $x_2\leq x_0/2$, we let $I_n(x_0,x_2)$ be defined as

$$\begin{aligned} n\leq n_0\ , &\qquad I_n=(2^{-n}x_0,2^{1-n}x_0)\ ,\\ n=n_0+1\ , &\qquad I_{n_0+1}=(x_2,2x_2)\ ,\\ n>n_0+1\ , &\qquad I_n=\emptyset\ , \end{aligned} \tag{B.1.14}$$

where n_0 is as in (B.1.8). For all $f\in H^{\mathrm{loc}}_k(\mathbf{H}_{\lambda,\tau_0})$ we set

$$\|f\|^2_{\mathscr{G}^\alpha_k(\mathbf{H}_{\lambda,\tau_0})} = \sup_n\{\sum_i\sum_{0\leq|\beta|\leq k}\int_{\Omega_i\cap\{I_n\times\mathscr{O}\}}(x^{-\alpha+\beta_1}\mathscr{D}^\beta f)^2\frac{dx}{x}\,d\nu\ \} \tag{B.1.15}$$

Similarly to (B.1.9), there exist constants c_2 and C_2, which do not *depend upon x_0 and x_2, such that for all $2x_2\leq x_0$,*

$$c_2x_0^{-\alpha}\sup_n\|f_n\|_{H_k([1,2]\times\mathscr{O})}\leq\|f\|_{\mathscr{G}^\alpha_k(\mathbf{H}_{\lambda,\tau_0})}\leq C_2x_0^{-\alpha}\sup_n\|f_n\|_{H_k([1,2]\times\mathscr{O})}\ . \tag{B.1.16}$$

We have the obvious inequality

$$\|f\|_{\mathscr{G}^\alpha_k(\Omega)}\leq\|f\|_{\mathscr{H}^\alpha_k(\Omega)}\ , \tag{B.1.17}$$

together with the modified version of (B.1.11),

$$\alpha'\geq\alpha\quad\Longrightarrow\quad\mathscr{C}^{\alpha'}_k\subset\mathscr{G}^\alpha_k\ ; \tag{B.1.18}$$

in particular the function $(x,v)\to x^\alpha$ is in $\mathscr{G}^\alpha_k(\mathbf{H}_{\tau_0})$.

B.2 Embeddings and inequalities

If S_k denotes a space of functions, where $k\in\mathbb{N}$ is a differentiability index, we set

$$S_\infty\equiv\cap_{k\in\mathbb{N}}S_k\ ,$$

e.g., $\mathscr{G}^\alpha_\infty\equiv\cap_{k\in\mathbb{N}}\mathscr{G}^\alpha_k$, etc.

We note the following:

Proposition B.2.1 *Let* $\Omega = \mathbf{H}_{\tau_0}$, *or* $\Omega = \mathbf{H}_{\lambda,\tau_0}$, $2x_2 < x_0$, *and let* $\mathscr{H}_k^\alpha = \mathscr{H}_k^\alpha(\Omega)$, etc. *For* $k' \in \mathbb{N}$, $0 \le k' \le k - n/2 \notin \mathbb{N}$ *or* $0 \le k' < k - n/2 \in \mathbb{N}$ *we have the continuous embeddings*

$$\mathscr{H}_k^\alpha \subset \mathscr{B}_{k'}^\alpha \subset \mathscr{C}_{k'}^\alpha \;, \qquad \mathscr{H}_k^\alpha \subset \mathscr{G}_k^\alpha \subset \mathscr{C}_{k'}^\alpha \;, \tag{B.2.1}$$

and there exists an x_2*-independent constant* C *such that we have*

$$\forall f \in \mathscr{H}_k^\alpha \qquad \|f\|_{\mathscr{B}_{k'}^\alpha(\Omega)} \le C\|f\|_{\mathscr{H}_k^\alpha(\Omega)} \;, \tag{B.2.2}$$

$$\forall f \in \mathscr{G}_k^\alpha \qquad \|f\|_{\mathscr{C}_{k'}^\alpha(\Omega)} \le C\|f\|_{\mathcal{G}_k^\alpha(\Omega)} \;. \tag{B.2.3}$$

Proof: *(B.2.2)-(B.2.3) follow immediately from (B.1.7) and (B.1.9), together with the standard Sobolev embedding; the remaining inclusions in (B.2.1) are trivial.* □

All other inequalities involving Sobolev spaces have their counterpart in the weighted setting; we shall in particular need various weighted versions of the Moser inequalities. The reader should note the different weights for the members of Equation (B.2.8) below — this shift of weights in this inequality is the key to our handling of nonlinear equations.

Proposition B.2.2 *Let* $\Omega = \mathbf{H}_{\tau_0}$, *or* $\Omega = \mathbf{H}_{\lambda,\tau_0}$, $2x_2 < x_0$, *and let* $\mathscr{H}_k^\alpha = \mathscr{H}_k^\alpha(\Omega)$, etc.

1. *There exists a constant* $C = C(\alpha, \alpha', \beta, k, x_0)$ *such that, for all* $f \in \mathscr{H}_k^{\alpha'} \cap \mathscr{C}_0^\alpha$ *and* $g \in \mathscr{H}_k^\beta \cap \mathscr{C}_0^{\alpha+\beta-\alpha'}$, *we have*

$$\|fg\|_{\mathscr{H}_k^{\alpha+\beta}} \le C\left(\|f\|_{\mathscr{C}_0^\alpha}\|g\|_{\mathscr{H}_k^\beta} + \|f\|_{\mathscr{H}_k^{\alpha'}}\|g\|_{\mathscr{C}_0^{\alpha+\beta-\alpha'}}\right) \;. \tag{B.2.4}$$

   Further, $\forall\ |\gamma| \le k$,

$$\|x^{\gamma_1}\mathscr{D}^\gamma(fg) - (x^{\gamma_1}\mathscr{D}^\gamma f)g\|_{\mathscr{H}_0^{\alpha+\beta}} \le C\Bigg(\|f\|_{\mathscr{C}_0^\alpha}\|g\|_{\mathscr{H}_k^\beta} + \|f\|_{\mathscr{H}_{k-1}^{\alpha'}}\left(\|x\partial_x g\|_{\mathscr{C}_0^{\alpha+\beta-\alpha'}} + \sum_{i=2}^r \|X_i g\|_{\mathscr{C}_0^{\alpha+\beta-\alpha'}}\right)\Bigg) \;, \tag{B.2.5}$$

   where the vector fields X *are defined in Equation (B.1.4).*

2. *Let* $F \in C_k(\mathbf{H}_{\tau_0} \times \mathbb{R}^N)$ *be a function such that for all* $B \in \mathbb{R}^+$ *there exists a constant* $C_1 = C_1(B)$ *so that, for all* $p \in \mathbb{R}^N$, $|p| \le B$, *we have*

$$\|F(\cdot, p)\|_{\mathscr{C}_k^0(\mathbf{H}_{\tau_0})} \le C_1 \;.$$

Then for all $\alpha < 0$, $\beta \in \mathbb{R}$, and $B \in \mathbb{R}^+$ there exists a constant $C_2(B, k, \alpha, \beta, x_0)$ such that for all $\mathbb{R}^N$-valued functions $f \in \mathscr{H}_k^{\alpha-\beta}(\Omega)$ with $\|x^\beta f\|_{L^\infty(\Omega)} \le B$ we have

$$\Big\|F(\cdot, x^\beta f)\Big\|_{\mathscr{H}_k^\alpha} \le C_2(1 + \|f\|_{\mathscr{H}_k^{\alpha-\beta}}) \ . \tag{B.2.6}$$

Further, if F has a uniform *zero of order $l > 0$ at $p = 0$, in the sense that for all $B \in \mathbb{R}$ there exists a constant $\hat{C}(B)$ such that for all $|p| \le B$ and $0 \le i \le \min(k, l)$,*

$$\left\|\frac{\partial^i F(\cdot, p)}{\partial p^i}\right\|_{\mathscr{C}^0_{k-i}(M_{x_0})} \le \hat{C}(B)|p|^{l-i} \ , \tag{B.2.7}$$

then for all $\alpha \in \mathbb{R}$, $\beta \ge 0$, there exists a constant $C_3(\hat{C}, l, k, \alpha, \beta, B)$ such that, for all $f \in \mathscr{H}_k^{\alpha-l\beta}(\Omega)$ with $\|f\|_{L^\infty(\Omega)} \le B$, we have

$$\Big\|F(\cdot, x^\beta f)\Big\|_{\mathscr{H}_k^\alpha} \le C_3 \|f\|_{\mathscr{H}_k^{\alpha-l\beta}} \ . \tag{B.2.8}$$

Remark: *The hypothesis (B.2.7) will hold if F is* e.g. *a polynomial in p with coefficients of p^j vanishing for $j < l$, and being functions belonging to $\mathscr{C}^0_k$ for $j \ge l$.*

The proof can be found in [20] Proposition A.2.

We have the following sharper version of (B.2.4)-(B.2.5):

Proposition B.2.3 *Let $\Omega = \mathbf{H}_{\tau_0}$, or $\Omega = \mathbf{H}_{\tau_0}$, $2x_2 \le x_0$, and let $\mathscr{H}_k^\alpha = \mathscr{H}_k^\alpha(\Omega)$,* etc. *There exists a constant $C_s = C_s(\alpha, \beta, k)$ such that, for all $f \in \mathscr{H}^{\alpha+\beta-\alpha'} \cap \mathscr{B}_0^\alpha$ and $g \in \mathscr{G}_k^\beta \cap \mathscr{C}^{\alpha'}_{\{x=0\},0}$ we have*

$$\|fg\|_{\mathscr{H}_k^{\alpha+\beta}} \le C_s(\|f\|_{\mathscr{B}_0^\alpha}\|g\|_{\mathscr{G}_k^\beta} + \|f\|_{\mathscr{H}^{\alpha+\beta-\alpha'}}\|g\|_{\mathscr{C}^{\alpha'}_{\{x=0\},0}}) \,, \tag{B.2.9}$$

Moreover it also holds that

$$\begin{aligned}\forall|\gamma| \le k \ , \ &\|x^{\gamma_1}\mathscr{D}^\gamma(fg) - (x^{\gamma_1}\mathscr{D}^\gamma f)g\|_{\mathscr{H}_0^{\alpha+\beta}} \\ &\le C\left(\|f\|_{\mathscr{B}_0^\alpha}\|g\|_{\mathscr{G}_k^\beta} + \|f\|_{\mathscr{H}_{k-1}^{\alpha+\beta-\alpha'}}\left(\|x\partial_x g\|_{\mathscr{C}_0^{\alpha'}} + \sum_{i=2}^r \|X_i g\|_{\mathscr{C}_0^{\alpha'}}\right)\right) \,,\end{aligned} \tag{B.2.10}$$

where the vector fields X are defined in Equation (B.1.4).

Remark B.2.4 A useful, though less elegant, inequality related to (B.2.9) is

$$\forall\, |\gamma+\sigma| \le k \qquad \|x^{\gamma_1}(\mathscr{D}^\gamma f)x^{\sigma_1}(\mathscr{D}^\sigma g)\|_{\mathscr{H}_0^{\alpha+\beta}} \le C_s(\|f\|_{\mathscr{B}_0^\alpha}\|g\|_{\mathscr{G}_k^\beta}+\|f\|_{\mathscr{H}_k^\alpha}\|g\|_{\mathscr{C}_0^\beta})\,. \tag{B.2.11}$$

The proof can be found in [20] Proposition A.3.

Similar results can be proved in weighted Hölder spaces:

Lemma B.2.5 *Let $\Omega = \mathbf{H}_{\tau_0}$, $0 < x_1 \le x_0$, or $\Omega = \mathbf{H}_{\lambda,\tau_0}$, $2x_2 \le x_0$, and let $\mathscr{C}_k^\alpha = \mathscr{C}_k^\alpha(\Omega)$. Let $f \in \mathscr{C}_k^\alpha \cap \mathscr{C}_0^\beta$ and $g \in \mathscr{C}_k^\gamma \cap \mathscr{C}_0^\delta$ with $\alpha+\delta = \gamma+\beta = \sigma$. Then we have $fg \in \mathscr{C}_k^\sigma$ and*

$$\|fg\|_{\mathscr{C}_k^\sigma} \le C_i(\|f\|_{\mathscr{C}_0^\beta}\|g\|_{\mathscr{C}_k^\gamma} + \|g\|_{\mathscr{C}_0^\delta}\|f\|_{\mathscr{C}_k^\alpha})\,, \tag{B.2.12}$$

The proof can be found in [20] Lemma A.4.

We have the following $\mathscr{C}_k^\beta$ equivalent of the second part of Proposition B.2.2, with a similar proof, based on Lemma B.2.5:

Lemma B.2.6 *Let F be a function satisfying the hypotheses of point 2 of Proposition B.2.2, with a uniform zero of order l in p in the sense of Equation (B.2.7). Then, for any $\epsilon > 0$, $\beta \in \mathbb{R}$ and $f \in \mathscr{C}_k^\beta \cap L^\infty$ we have $F(.,x^\epsilon f) \in \mathscr{C}_k^{\beta+l\epsilon}$, and there exists a constant C depending upon $\|f\|_{L^\infty}$ such that*

$$\|F(.,x^\epsilon f)\|_{\mathscr{C}_k^{\beta+l\epsilon}} \le C(\|f\|_\infty)\|f\|_{\mathscr{C}_k^\beta}\,. \tag{B.2.13}$$

Appendix C

Some classical results

Theorem C.0.7 *(Gronwall's Lemma) Let $f,\ g,\ \varphi,\ \psi$ be four positive and continuous functions on $[a,b]\subset\mathbb{R},\ a<b$, such that*

$$f(t)\le g(t)+\varphi(t)\int_a^t f(s)\psi(s)ds\ . \tag{C.0.1}$$

Then

$$f(t)\le g(t)+\varphi(t)\int_a^t \psi(u)g(u)e^{\int_u^t\varphi(s)\psi(s)ds}du\ . \tag{C.0.2}$$

Proof: Set $F(t)=\int_a^t f(s)\psi(s)ds$ and multiply (C.0.1) by ψ. We have:

$$f(t)\psi(t)\le g(t)\psi(t)+\varphi(t)\psi(t)\int_a^t f(s)\psi(s)ds$$

i.e.

$$F'(t)-\varphi(t)\psi(t)F(t)\le\psi(t)g(t)\ .$$

Multiply this last inequality with the positive function $e^{-\int_a^t\varphi(s)\psi(s)ds}$ and obtain

$$G'(t)\le\psi(t)g(t)e^{-\int_a^t\varphi(s)\psi(s)ds}\quad\text{with}\quad G(t)=F(t)e^{-\int_a^t\varphi(s)\psi(s)ds}\ .$$

Since $G(a)=F(a)=0$, by integration, we have

$$F(t)e^{-\int_a^t\varphi(s)\psi(s)ds}\le\int_a^t\psi(u)g(u)e^{-\int_a^u\varphi(s)\psi(s)ds}du\ ;$$

i.e.

$$\int_a^t f(s)\psi(s)ds\le\int_a^t\psi(u)g(u)e^{\int_u^t\varphi(s)\psi(s)ds}du,$$

which is multiply with $\varphi(t)$ afterwards adding $g(t)$ to each member of the resulting inequality to:

$$g(t)+\varphi(t)\int_a^t f(s)\psi(s)ds \le g(t)+\varphi(t)\int_a^t \psi(u)g(u)e^{\int_u^t \varphi(s)\psi(s)ds}du \; .$$

By hypothesis, the result follows. □

Theorem C.0.8 *(Arzela-Ascoli Theorem) Suppose that $(f_k)_{k\in\mathbb{N}}$ is a sequence of real-valued functions defined on $\mathbb{R}^n$ such that there exists a positive constant M satisfying*

$$\sup_{k\in\mathbb{N},\, x\in\mathbb{R}^n} |f_k(x)| < M \; .$$

Assume further that the functions $\{f_k,\ k\in\mathbb{N}\}$ are uniformly equicontinuous.Then there exists a subsequence $(f_{k_j})_{j\in\mathbb{N}} \subseteq (f_k)_{k\in\mathbb{N}}$ and a continuous function f such that $(f_{k_j})_{j\in\mathbb{N}}$ converges to f uniformly on compact subsets of $\mathbb{R}^n$.

For the proof of this theorem, we refer the reader to [] page ...

Theorem C.0.9 *(Interpolation Theorem for L^p-norms) For any function $f\in W^{s,p}$ and any $i\in[1,s]\cap\mathbb{N}$, set $\|\nabla^i f\|_p = \left(\sum_{|\alpha|=i}\|D^\alpha f\|_{L^p}^p\right)^{1/p}$. Let $1\le r,p\le\infty$, $s\in\mathbb{N}$. There exists a constant $c>0$ such that*

$$\|\nabla^j u\|_q \le c\|\nabla^s u\|_p^{j/s}\|u\|_r^{1-j/s}, \quad \textit{forall} \quad f\in W^{s,p},$$

where $j\in\{0,1,\dots,s\}$ and $\frac{1}{q}=\frac{j}{s}\frac{1}{p}+(1-\frac{j}{s})\frac{1}{r}$.

The proof of the interpolation theorem can be found in [46], page 38.

Theorem C.0.10 *(Trace Theorem) Let U be an open subset of $\mathbb{R}^n$. Assume that U is bounded and ∂U is C^1. Then there exists a bounded linear operator $T: W^{1,p}(U)\longrightarrow L^p(\partial U)$ such that:*

1. *$Tf = f|_{\partial U}$ if $f\in W^{1,p}(U)\cap C(\overline{U})$,*
2. *$\forall f\in W^{1,p}(\Omega)$, $\|Tf\|_{L^p(\partial\Omega)} \le C\|f\|_{W^{1,p}(\Omega)}$, with the constant C depending only on p and U.*

The proof of this theorem can be found in [33] page 258.

Theorem C.0.11 *(Weak compactness). Let X be a reflexive Banach space and $(x_n)_{n\in\mathbb{N}}$ a sequence of elements of X. If $(x_n)_{n\in\mathbb{N}} \subset X$ is bounded, then there exists a subsequence $(x_{n_j})_{j\in\mathbb{N}} \subset (x_n)_{n\in\mathbb{N}}$ and $x \in X$ such that*

$$x_{n_j} \underset{j\longrightarrow\infty}{\rightharpoonup} x \quad .$$

For the proof of this theorem, we refer the reader to [33] page 639

References

[1] L. Anderson and P.T. Chruściel, *On asymptotic behavior of solutions of the constraint equations in general relativity with "hyperboloidal boundary conditions*, Dissert. Math. **355** (1996), 1–100.

[2] M.T. Anderson and P.T. Chruściel, *Asymptotically simple solutions of the vacuum Einstein equations in even dimensions*, Commun. Math. Phys. **260, 557–577** (2005).

[3] R. Bartnik, *The mass of an asymptotically flat manifold*, Commun. Pure and Appl. Math. **39** (1996), 661–693.

[4] R. Beig and P.T. Chruściel, *The asymptotics of stationary electro-vacuum metrics in odd space-time dimensions*, Class. Quantum Grav. **24** (2007), 867–874.

[5] A. Cabet, *Local existence of solution of a semi-linear wave equation with gradient in a neighborhood of initial cahracteristic hypersurface of a lorentzian manifold*, Communication in Partial differential equations **33** (2008), 2105–2156.

[6] G. Caciotta and F. Nicolò, *Global characteristic problem for the Einstein vacuum equations with small initial data (I), (II)*, ArXiv **http://www.arXiv.org/abs/gr-qc/0608038v1** (2006).

[7] G. Caciotta and F. Nicolò, *Global characteristic problem for the Einstein Vacuum Equations with small initial data: (I),(II). The existence proof*, ArXiv **http://www.arXiv.org/abs/gr-qc/0608038v1** (2006).

[8] F. Cagnac, *Problème de Cauchy sur un conoïde caractéristique pour des équations quasi-linéaires*, Annalidi di Mathematica Pura (IV) **129, No 1, 13-41** (1980).

[9] Y. Choquet-Bruhat, P.T. Chruściel, and J. Loizelet, *Global solutions of the Einstein–Maxwell equations in higher dimension*, Class. Quantum Grav. (2006), 7383–7394.

[10] D. Christodoulou and Y. Choquet-Bruhat, *Elliptic systems in $H_{s,\delta}$ spaces on manifolds which are Euclidean at infinity*, Acta. Math. **146** (1981), 129–150.

[11] P. T. Chruściel and R. T. Wafo, *Solutions polyhomogènes des équations d'ondes quasi-linéaires*, C. R. Acad. Sci. Paris **Ser. I 347** (2009).

[12] ———, *Solutions of quasi-linear wave equations polyhomogeneous at null infinity in high dimensions*, ArXiv **http://arxiv4.library.cornell.edu/abs/1010.2387** (2010).

[13] P.T. Chruściel, *Asymptotic estimates in weighted Hölder spaces for a class of elliptic scale-covariant second order operators*, Ann. Fac. Sci. Toulouse Math. (5) **11** (1990), 21–37.

[14] ———, *A poor man's positive energy theorem: II. Null geodesics*, Class. Quantum. Grav. **21** (2004), 4399–4415.

[15] P.T. Chruściel, J. Corvino, and J. Isenberg, *Construction of N-body initial data sets in general relativity*, **arXiv:0909.1101 [gr-qc].** (2010).

[16] P.T. Chruściel and E. Delay, *Existence of non-trivial asymptotically simple vacuum space-times*, Class. Quantum Grav. **19** (2002), L71–L79. MR 1902228 (2003e:83024a)

[17] ———, *On mapping properties of the general relativistic constraints operator in weighted function spaces, with applications*, Mém. Soc. Math. de France **94** (2003). MR 2031583 (2005f:83008)

[18] P.T. Chruściel, J. Jezierski, and J. Kijowski, *Hamiltonian field theory in the radiating regime*, vol. m70, 2001.

[19] P.T. Chruściel and S. Łęski, *Polyhomogeneous solutions of nonlinear wave equations without corner conditions*, Jour. Hyp. PDE **3** (2006), 81–141.

[20] P.T. Chruściel and O. Lengard, *Solutions of wave equations in the radiating regime*, Bull. Soc. Math. de France **133** (2003), 1–72.

[21] P.T. Chruściel, M.A.H. MacCallum, and D. Singleton, *Gravitational waves in general relativity. XIV: Bondi expansions and the "polyhomogeneity" of Scri*, Phil. Trans. Roy. Soc. London A **350** (1995), 113–141.

[22] R. Coquereaux and A. Jadczyk, *Riemannian geometry, fiber bundles, Kaluza-Klein theories and all that*, World Sci. Lect. Notes Phys., vol. 16, World Scientific Publishing Co., Singapore, 1988.

[23] J. Corvino, *Scalar curvature deformation and a gluing construction for the Einstein constraint equations*, Commun. Math. Phys. **214, 137–189. MR MR1794269 (2002b:53050),** (2000).

[24] J. Corvino and R.M. Schoen, *On the asymptotics for the vacuum Einstein constraint equations*, Jour. Diff. Geom. **73, 185–217. arXiv:gr-qc/0301071. MR MR2225517 (2007e:58044),** (2006).

[25] M. Dossa, *Problème de Cauchy sur un conoïde caractéristique pour des équations quasi-linéaires du second ordre, Thèse de Doctorat d'Etat, (parties 1 et 2)*, Université de Yaoundé I (1992).

[26] ______, *Problème de Cauchy sur un conoïde caractéristique pour des systèmes quasi-linéaires hyperboliques*, C. R. Math. Rep. Acad. Sci. Canada **16, No 1, 17-22** (1994).

[27] ______, *Espaces de Sobolev non isotrpes, à poids et problèmes de Cauchy quasi-linéaires sur un conoïde caractéristique*, Annales de l'Institut Henri poincaré **Section A, 66, N**^{o}1 (1997), 37–107.

[28] ______, *Solutions C^∞ d'une classe de problèmes de Cauchy quasi-linéaies hyperboliques du second ordre sur un conoïde caractéristique*, Annales de la Faculté des Sciences de Toulouse **XL, N**^{o}3 (2002), 351–376.

[29] ______, *Problèmes de Cauchy sur un conoïde caractéristique pour les équations d'Einstein (conformes) du vide et pour les équations de Yang-Mills-Higgs*, Annales de l'Institut Henri poincaré **N**^{o}4 (2003), 385–411.

[30] M. Dossa and S. Bah, *Solutions de problèmes semi-linéaires hyperboliques sur un conoïde caractéristique.*, C. R. Acad. Sci. Paris **Ser. I 333** (2001).

[31] M. Dossa and D. E. Houpa, *Problèmes de Goursat pour des systèmes semi-linéaires hyperboliques*, C. R. Acad. Sci. Paris **Ser. I 341** (2005).

[32] M. Dossa and F. Touadera, *Solutions globales des systèmes hyperboliques non linéaires sur un cône caractéristiques*, C. R. Acad. Sci. Paris **Ser. I 341** (2005).

[33] L. C. Evans, *Partial differential equations*, American Mathematical Society, 1998.

[34] H. Friedrich, *Einstein equations and conformal structure: Existence of Anti-de Sitter-type space-times*, Journal of Geometry and Physics **17, p. 125-184** (1995).

[35] Lars Hörmander, *A remark on the characteristic Cauchy problem*, J. Funct. Anal. **93** (1990), no. 2, 270–277. MR 1073287 (91m:58154)

[36] D. E. Houpa, *Solutions semi-globales pour le problème de Goursat associé à des systèmes non linéaires hyperboliques et applications*, Ph.D. thesis, Université de Yaoundé I, Decembre 2006.

[37] H. Lindblad and I. Rodnianski, *The global stability of the Minkowski space-time in harmonic gauge*, arXiv:math.ap/0411109 (2004).

[38] ———, *Global existence for the Einstein vacuum equations in wave coordinates*, Commun. Math. Phys. **256** (2005), 43–110.

[39] J. Loizelet, *Solutions globales des équations d'Einstein-Maxwell en jauge harmonique et jauge de Lorenz*, Comptes Rendus Acad. Sci. Sér. I **342** (2006), 479–482.

[40] ———, *Problémes globaux en relativité générale*, Ph.D. thesis, Université de Tours, 2008.

[41] ———, *Solutions globales des équations d'Einstein-Maxwell*, An. Fac. Sci. Toulouse. (2008).

[42] A. J. Majda, *Compressible fluid flow and systems of conservation laws in several space variables*, 1984.

[43] R.C. Myers and M.J. Perry, *Black holes in higher dimensional space-times*, Ann. Phys. **304–347** (1986).

[44] J.-P. Nicolas, *On Lars Hörmander's remark on the characteristic Cauchy problem*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 3, 517–543. MR 2244222 (2008d:35114)

[45] ———, *On Lars Hörmander's remark on the characteristic Cauchy problem*, C. R. Math. Acad. Sci. Paris **344** (2007), no. 10, 621–626. MR 2334072 (2008c:35165)

[46] R. Racke, *Lectures on nonlinear evolution equations: Initial value problems*, Wieweg, Braunschweig, 1992.

[47] A. D. Rendall, *Reduction of characteristic initial value problem to the Cauchy problem and its applications to the Einstein equations*, Proceedings of the Royal Society of London **427 N**^{o}1872 (1990), 221–239.

[48] W. Simon and R. Beig, *The multipole structure of stationary spacetimes*, Jour. Math. Phys. **24** (1981), 1163–1171.

[49] L.A. Tamburino and J. Winicour, *Gravitational fields in finite and conformal Bondi frames*, Phys. Rev. **150** (1996), 1039–1053.

[50] M.E. Taylor, *Partial Differential Equations III, Nonlinear Equations*, Springer-Verlag, New York, Berlin, Heidelberg, 1996.

[51] F. Touadera, *Transformation de compactification de Penrose et résolution globale d'une classe de systèmes hyperboliques non linéaires pour des données initiales prescrites sur un cône caractéristique, Thèse de Doctorat d'Etat*, Université de Yaoundé I, Cameroun, 2004.

[52] H.-H. Zhang, W.-B. Yan, and X.-S. Li, *Trace formulae of characteristic polynomial and Cayley–Hamilton's theorem, and applications to chiral perturbation theory and general relativity*, Commun. Theor. Phys. 801, arXiv:hep-th/0701116 **49** (2008).