Lectures on Energy in General Relativity
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comments on mistakes and misprints, and suggestions for improvements welcome

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Part I

Energy in general relativity

Chapter 1

Mass and Energy-momentum

There exist various reasons why one might be interested in the notion of energy in general relativity:

1. First, one expects that generic gravitating systems will emit gravitational waves. Detecting such waves requires a transfer of energy between the field and the detector, and to quantify such effects it is clearly useful to have a device that measures the energy carried away by the gravitational field.

2. Next, in Lagrangean field theories with Lagrange function $\mathscr{L}(\phi^A, \partial_\mu \phi^A)$ there is a well defined notion of energy-density, discussed shortly in Section 1.2. There is no known geometrically defined notion of energy in general relativity, which makes the question more interesting.

3. Now, it turns out that there is a well defined notion of global energy and momentum for isolated gravitating systems. The positivity of this total energy has proved surprisingly difficult to establish, a problem eventually settled by Schoen and Yau [163, 164] in space-dimension lower than or equal to seven [162], and by completely different methods by Witten [187] under the hypothesis that the manifold is spin; the general case remaining open.

4. The positive energy theorem has a surprising application in the proof by Schoen [161] in 1984 of the *Yamabe conjecture*: for every metric g on a compact manifold there exists a strictly positive function ϕ such that the metric ϕg has constant scalar curvature. See [131]. (A solution of the Yamabe problem has also be given by Bahri [9] by completely different methods in 1993.)

5. Yet another surprising application of the positive of energy is the proof, by Bunting and Masood-ul-Alam [43], that event horizons in regular, static, vacuum black holes are connected.

6. The hunting season for an optimal definition of "quasi-local" energy is still open!

1.1 The mass of asymptotically Euclidean manifolds

There exist various approaches to the definition of mass in general relativity, the first one being due to Einstein [83] himself. In Section 1.2 below we will outline two geometric Hamiltonian approaches to that question. However, those approaches require some background knowledge in symplectic field theory, and it appears useful to present an elementary approach which quickly leads to the correct definition for asymptotically flat Riemannian manifolds without any prerequisites.

In the remainder in this chapter we will restrict ourselves to dimensions greater than or equal to three, as the situation turns out to be completely different in dimension two: Indeed, it should be clear from the considerations below that the mass is an object which is related to the integral of the scalar curvature over the manifold. Now, in dimension two, that integral is a topological invariant for compact manifolds, while it is related to a *"deficit angle"* in the non-compact case. This angle, to which we return in Remark 1.1.3, appears to be the natural two-dimensional equivalent of the notion of mass.

The Newtonian approximation provides the simplest situation in which it is natural to assign a mass to a Riemannian metric: recall that in this case the space-part of the metric takes the form

$$g_{ij} = (1+2\phi)\delta_{ij} \ , \tag{1.1.1}$$

where ϕ is the Newtonian potential,

$$\Delta_\delta \phi = -4\pi\mu \ , \tag{1.1.2}$$

with μ – the energy density. (In the Newtonian approximation we also have $g_{0i} = 0$, $g_{00} = -1 + 2\phi$, but this is irrelevant for what follows.) When μ has compact support $\operatorname{supp}\mu \subset B(0,R)$ we have, at large distances,

$$\phi = \frac{M}{r} + O(r^{-2}) \ , \tag{1.1.3}$$

where M is the total Newtonian mass of the sources:

$$\begin{aligned} M &= \int_{\mathbb{R}^3} \mu \, d^3x \\ &= -\frac{1}{4\pi}\int_{\mathbb{R}^3} \Delta_\delta \phi \\ &= -\frac{1}{4\pi}\int_{B(0,R)} \Delta_\delta \phi \\ &= -\frac{1}{4\pi}\int_{S(0,R)} \nabla^i \phi \, dS_i \\ &= -\lim_{R\to\infty} \frac{1}{4\pi}\int_{S(0,R)} \nabla^i \phi \, dS_i \ . \end{aligned} \tag{1.1.4}$$

Here dS_i denotes the usual coordinate surface element,

$$dS_i = \partial_i \rfloor dx \wedge dy \wedge dz \ , \tag{1.1.5}$$

with $\rfloor$ denoting contraction. Then the number M appearing in (1.1.3) or, equivalently, given by (1.1.4), will be called the mass of the metric (1.1.1).

In Newtonian theory it is natural to suppose that $\mu \geq 0$. We then obtain the simplest possible version of the *positive mass theorem*:

THEOREM 1.1.1 (Conformally flat positive mass theorem) *Consider a C^2 metric on $\mathbb{R}^3$ of the form (1.1.1) with a strictly positive function $1+2\phi$ satisfying*

$$-4\pi\mu := \Delta_\delta \phi \leq 0 \ , \qquad \phi \to_{r\to\infty} 0 \ .$$

Then

$$0 \leq m := -\lim_{R\to\infty} \frac{1}{4\pi} \int_{S(0,R)} \nabla^i \phi \, dS_i \leq \infty \ ,$$

with m vanishing if and only if g_{ij} is flat.

PROOF: The result follows from (1.1.4); we simply note that m will be finite if and only if μ is in $L^1(\mathbb{R}^3)$. $\square$

Somewhat more generally, suppose that

$$g_{ij} = \psi\delta_{ij} + o(r^{-1}) \ , \quad \partial_k(g_{ij} - \psi\delta_{ij}) = o(r^{-2}) \ , \tag{1.1.6}$$

with ψ tending to 1 as r tends to infinity. Then a natural generalisation of (1.1.4) is

$$m := -\lim_{R\to\infty} \frac{1}{8\pi} \int_{S(0,R)} \nabla^i \psi \, dS_i \ , \tag{1.1.7}$$

provided that the limit exists.

Let us see whether Definition (1.1.7) can be applied to the Schwarzschild metric:

$${}^4g = -(1-2m/r)dt^2 + \frac{dr^2}{1-2m/r} + r^2 d\Omega^2 \ , \tag{1.1.8}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2 \ . \tag{1.1.9}$$

Here we have decorated ${}^4g$ with a subscript four, emphasising its four dimensional character, and we shall be using the symbol g for its three dimensional space-part. Now, every spherically symmetric metric is conformally flat, so that the space-part of the Schwarzschild metric can be brought to the form (1.1.6) *without the error term*, as follows: We want to find ρ such that

$$g := \frac{dr^2}{1-2m/r} + r^2 d\Omega^2 = \psi\left(d\rho^2 + \rho^2 d\Omega^2\right) \ . \tag{1.1.10}$$

Let us check that the answer is

$$\psi = \left(1 + \frac{m}{2\rho}\right)^4 \ .$$

Comparing the coefficients in front of $d\theta^2$, or in front of $d\varphi^2$, in (1.1.10) yields

$$r = \left(1 + \frac{m}{2\rho}\right)^2 \rho \ . \tag{1.1.11}$$

To finish verifying (1.1.10) it suffices to check the g_{rr} term. Differentiating we have

$$dr = \left(1+\frac{m}{2\rho}\right)\left(2\times\left(-\frac{m}{2\rho^2}\right)\times\rho+1+\frac{m}{2\rho}\right)d\rho = \left(1+\frac{m}{2\rho}\right)\left(1-\frac{m}{2\rho}\right)d\rho\;, \tag{1.1.12}$$

while

$$\begin{aligned} 1-\frac{2m}{r} &= 1-\frac{2m}{(1+\frac{m}{2\rho})^2\rho} \\ &= \frac{(1+\frac{m}{2\rho})^2-\frac{2m}{\rho}}{(1+\frac{m}{2\rho})^2} \\ &= \frac{1+\frac{m}{\rho}+\left(\frac{m}{2\rho}\right)^2-\frac{2m}{\rho}}{(1+\frac{m}{2\rho})^2} \\ &= \frac{(1-\frac{m}{2\rho})^2}{(1+\frac{m}{2\rho})^2}\;, \end{aligned}$$

and (1.1.10) readily follows. Hence

$$g = \left(1+\frac{m}{2|\vec{y}|}\right)^4\delta\;, \tag{1.1.13}$$

where δ denotes the flat Euclidean metric in the coordinate system (y^i). From the asymptotic development

$$\left(1+\frac{m}{2|\vec{y}|}\right)^4 = 1+\frac{2m}{|\vec{y}|}+O(|\vec{y}|^{-2})$$

we find that the space-part of the Schwarzschild metric has mass m, as desired. More precisely, one finds a mass m in the coordinate system in which g takes the form (1.1.13). This raises immediately the question, whether the number so obtained does, or does not, depend upon the coordinate system chosen to calculate it. We will shortly see that m is coordinate-independent, and indeed a geometric invariant.

For further reference we note that we have also obtained

$${}^4g = -\frac{(1-\frac{m}{2\rho})^2}{(1+\frac{m}{2\rho})^2}dt^2+\left(1+\frac{m}{2\rho}\right)^4\left(d\rho^2+\rho^2(d\theta^2+\sin^2\theta d\varphi^2)\right)\;. \tag{1.1.14}$$

Actually, (1.1.12) shows that the map which to ρ assigns r is not a diffeomorphism, since $dr/d\rho$ vanishes at

$$r = 2m \Longleftrightarrow \rho = m/2\;.$$

This is related to the fact that the left-hand-side of (1.1.10) is singular at $r = 2m$, while the right-hand side of (1.1.10) extends smoothly across $\rho = m/2$. This shows that the apparent singularity of the metric at the left-hand-side of (1.1.10) is only a coordinate artefact. This is called a *coordinate singularity* by physicists.

A closer inspection of (1.1.11)-(1.1.12) shows that the manifold $\mathbb{R}_t\times\{\rho>0\}\times S^2$ contains the Schwarzschild manifold $\mathbb{R}_t\times\{r>2m\}\times S^2$ twice, once for $\rho>m/2$, and one more copy for $\rho<m/2$.

A hint how to proceed in general is given by the conformally flat positive energy theorem 1.1.1, where we have used positivity properties of the "mass density $\mu := \Delta_\delta \phi/(-4\pi)$" to obtain information about the asymptotic behavior of the metric. Recall that the general relativistic correspondent of the mass density μ is the energy density ρ. Thus, we need an equation which involves ρ. A candidate here is the *scalar constraint equation*,

$$R(g) = 16\pi\rho + |K|^2 - (\mathrm{tr}_g K)^2 \;, \qquad \rho := T_{\mu\nu} n^\mu n^\nu. \tag{1.1.15}$$

(Recall that we are working in the asymptotically flat context here, which requires $\Lambda = 0$.) Here n^μ is the field of unit normals to the spacelike initial data hypersurface $\mathscr{S} \subset \mathscr{M}$, with space metric g induced from the space-time metric ${}^4g$. Further, $T_{\mu\nu}$ is the energy-momentum tensor, so that ρ has the interpretation of energy-per-unit-volume of matter fields on $\mathscr{S}$. Finally, $K = K_{ij} dx^i dx^j$ is the extrinsic curvature tensor of $\mathscr{S}$ in $\mathscr{M}$: in adapted coordinates in which $\mathscr{S} = \{x^0 = \text{const}\}$ we have

$$K_{ij} = \frac{1}{2}(\nabla_i n_j + \nabla_j n_i) \;.$$

(K is thus the space-part of half of the Lie derivative of the metric in direction normal to $\mathscr{S}$, and thus measures how the metric changes when moved in that direction.)

Now, R contains a linear combination of second derivatives of g, which is vaguely reminiscent of (1.1.2), however there are also terms which are quadratic in the Christoffel symbols, and it is not completely clear that this is the right equation. We shall, however, hope for the best, manipulate the equation involving $R(g)$, and see what comes out of that. Thus, we isolate all the second derivatives terms in $R(g)$ and we reexpress them as the divergence of a certain object:

$$\begin{aligned} R(g) &= g^{ij}\mathrm{Ric}_{ij} = g^{ij} R^k{}_{ikj} \\ &= g^{ij}\left(\partial_k \Gamma^k{}_{ij} - \partial_j \Gamma^k{}_{ik} + q\right) \;, \end{aligned}$$

where q denotes an object which is quadratic in the first derivatives of g_{ij} with coefficients which are rational functions of g_{kl}. Now,

$$\Gamma^k{}_{ij} = \frac{1}{2} g^{k\ell}\left(\partial_j g_{\ell i} + \partial_i g_{\ell j} - \partial_\ell g_{ij}\right) \;,$$

hence

$$\Gamma^k{}_{ik} = \frac{1}{2} g^{k\ell}\left(\partial_k g_{\ell i} + \partial_i g_{\ell k} - \partial_\ell g_{ik}\right) = \frac{1}{2} g^{k\ell} \partial_i g_{\ell k} \;.$$

It follows that

$$\begin{aligned} R(g) &= \frac{1}{2} g^{ij} g^{k\ell}\left(\partial_k\partial_j g_{\ell i} + \partial_k \partial_i g_{\ell j} - \partial_k \partial_\ell g_{ij} - \partial_j \partial_i g_{\ell k} + q\right) \\ &= g^{ij} g^{k\ell}\left(\partial_k \partial_j g_{\ell i} - \partial_j \partial_i g_{\ell k}\right) + \frac{q}{2} \\ &= \partial_j\left(g^{ij} g^{k\ell}\left(\partial_k g_{\ell i} - \partial_i g_{\ell k}\right)\right) + q' \;, \end{aligned}$$

with a different quadratic remainder term. We will need to integrate this expression, so we multiply everything by $\sqrt{\det g}$, obtaining finally

$$\sqrt{\det g} R(g) = \partial_j \mathbb{U}^j + q'' , \qquad (1.1.16)$$

with q'' yet another quadratic expression in ∂g, and

$$\boxed{\mathbb{U}^j := \sqrt{\det g}\; g^{ij} g^{k\ell} \left(\partial_k g_{\ell i} - \partial_i g_{\ell k}\right) \ .} \qquad (1.1.17)$$

This is the object needed for the definition of mass:

DEFINITION 1.1.2 *Let g be a $W^{1,\infty}_{\rm loc}$ metric defined on $\mathbb{R}^n \setminus B(0,R_0)$, we set*

$$\begin{aligned} m &:= \lim_{R\to\infty} \frac{1}{16\pi} \int_{S(0,R)} \mathbb{U}^j dS_j \\ &= \lim_{R\to\infty} \frac{1}{16\pi} \int_{S(0,R)} g^{ij} g^{k\ell} \left(\partial_k g_{\ell i} - \partial_i g_{\ell k}\right) \sqrt{\det g}\; dS_j \ , \end{aligned} \qquad (1.1.18)$$

whenever the limit exists.

We emphasize that we do not assume that the metric is defined on $\mathbb{R}^n$, as that would exclude many cases of interest, including the Schwarzschild metric.

The normalisation in (1.1.18) has been tailored to $n = 3$, and a different normalisation could perhaps be more convenient in higher dimension. As this is irrelevant for most of our purposes we will always use the above normalisation unless explicitly indicated otherwise.

REMARK 1.1.3 In dimension two the scalar curvature is always, locally, a total divergence, which considerably simplifies the subsequent analysis. This will be discussed in Section 1.1.1 below.

Let us consider the question of convergence of the integral (1.1.18):

PROPOSITION 1.1.4 ([10, 55, 142]) *Let g be a $W^{1,\infty}_{\rm loc}$ metric defined on $\mathbb{R}^n \setminus B(0,R_0)$ such that*

$$\forall\ i,j,k,\ell \qquad g_{ij}\ ,\ g^{k\ell} \in L^\infty\ , \qquad \partial_k g_{ij} \in L^2\ . \qquad (1.1.19)$$

1. *If*

$$R(g) \in L^1\ ,$$

then m exists, and is finite.

2. *[INFINITE POSITIVE ENERGY THEOREM] If $R(g)$ is a non-negative measurable function which is* not *in L^1, then the limit in (1.1.18) exists with*

$$m = \infty\ .$$

PROOF: The result follows immediately from the divergence theorem: we write

$$\begin{aligned}\int_{S(0,R)} \mathbb{U}^j dS_j - \int_{S(0,R_0)} \mathbb{U}^j dS_j &= \int_{B(0,R)\setminus B(0,R_0)} \partial_j \mathbb{U}^j d^3x \\ &= \int_{B(0,R)\setminus B(0,R_0)} (\sqrt{\det g} R - q'') d^3x \ ,\end{aligned}$$

with $q'' \in L^1$ since the $\partial_k g_{ij}$'s are in L^2. If $R(g)$ is in L^1, or if $R(g)$ is measurable and positive, the monotone convergence theorem gives

$$\begin{aligned}\lim_{R\to\infty} \int_{S(0,R)} \mathbb{U}^j dS_j &= \int_{\mathbb{R}^n\setminus B(0,R_0)} \sqrt{\det g} R \ d^3x \\ &\quad - \int_{\mathbb{R}^n\setminus B(0,R_0)} q'' d^3x + \int_{S(0,R_0)} \mathbb{U}^j dS_j \ , \end{aligned} \tag{1.1.20}$$

with the last two terms being finite, and the result follows. □

Since the arguments of this section have a purely Riemannian character, the extrinsic curvature tensor K, which would be present if a whole initial data set were considered, is irrelevant for the current purposes. However, it is worthwhile pointing out that similar manipulations can be done with the vector constraint equation, leading to the definition of the *ADM momentum* of an initial data set, as follows: For notational convenience let us set

$$P^{ij} := \mathrm{tr}_g K g^{ij} - K^{ij} \ , \tag{1.1.21}$$

$$J^j := T^j{}_\mu n^\mu \ , \tag{1.1.22}$$

so that the vector constraint equation can be rewritten as

$$D_i P^i{}_j = 8\pi J_j \ . \tag{1.1.23}$$

The vector field J is usually called the matter momentum vector. Similarly to (1.1.16), we want to obtain a divergence identity involving J. Now, divergence identities involve vector fields, while (1.1.23) involves the divergence of a tensor; this is easily taken care of by choosing some arbitrary vector field X and writing

$$D_i(P^i{}_j X^j) = D_i P^i{}_j X^i + P^i{}_j D_i X^j = 8\pi J_i X^i + P^i{}_j D_i X^j \ . \tag{1.1.24}$$

Integrating over large spheres gives

$$\begin{aligned}\int_{S^\infty} P^i{}_j X^j dS_i &= \lim_{R\to\infty} \int_{S(R)} P^i{}_j X^j dS_i \\ &= \int_M \left(8\pi J^i X_i + P^i{}_j D_i X^j\right) \ , \end{aligned} \tag{1.1.25}$$

provided that the last integral converges. Let X^i_∞ be any set of constants, the ADM momentum vector p is the set of numbers p_i defined using the boundary integrand above:

$$p_i X^i_\infty := \frac{1}{8\pi} \int_{S^\infty} P^i{}_j X^j_\infty dS_i \ . \tag{1.1.26}$$

To analyse convergence, let X be any differentiable vector field which coincides with X_∞ for r large, and which is zero outside of the asymptotic region. It is natural to assume that the total momentum of the fields other than the gravitational one is finite:

$$J \in L^1(M_{\mathrm{ext}}) ,$$

this ensures convergence of the J integral in (1.1.25). The convergence of the second term there is usually taken care of by requiring that

$$P^{ij} \ , \ \partial_k g_{ij} \in L^2(M_{\mathrm{ext}}) . \tag{1.1.27}$$

For then we have, for r large,

$$P^{ij}D_iX_j = P^i{}_jD_iX^j = P^i{}_j(\underbrace{\partial_i X^j_\infty}_{=0} + \Gamma^j{}_{ik}X^k_\infty) \le C|P|\sum_{i,j,k}|\partial_i g_{jk}| .$$

Integrating over M and using $2ab \le a^2 + b^2$ gives

$$\left|\int_{M_{\mathrm{ext}}} P^{ij}D_iX_j\right| = \left|\int_{M_{\mathrm{ext}}} P^{ij}D_iX_j\right| \le C\int_{M_{\mathrm{ext}}}\Big(|P|^2 + \sum_{i,j,k}|\partial_i g_{jk}|^2\Big) ,$$

and convergence follows. We have thus proved

PROPOSITION 1.1.5 *Suppose that*

$$J \in L^1(M_{\mathrm{ext}}) , \qquad P^{ij} \ , \ \partial_k g_{ij} \in L^2(M_{\mathrm{ext}}) .$$

Then the ADM momentum (1.1.26) is finite.

It seems sensible to test our definition on a few examples. First, if g is the flat Euclidean metric on $\mathbb{R}^n$, and we use the standard Euclidean coordinates, then $m = 0$, which appears quite reasonable. Consider, next, the space-part of the (four-dimensional) Schwarzschild metric: whether in the form (1.1.10) or (1.1.13) it can be written as

$$g_{ij} = \delta_{ij} + O(r^{-1}) , \quad \text{with } \ \partial_k g_{ij} = O(r^{-2}) \tag{1.1.28}$$

(for (1.1.13) this is straightforward; for (1.1.10) one should introduce the obvious pseudo-Euclidean coordinates x^i associated to the spherical coordinates (r, θ, φ). We will use the scalar constraint equation to calculate $R(g)$; this requires calculating the extrinsic curvature tensor K_{ij}. Recall that

$$K(X, Y) := g(P(\nabla_X n), Y) ,$$

where P is the orthogonal projection on the space tangent to the hypersurface in consideration; in our case these are the hypersurfaces $t = \text{const}$. From (1.1.8) the field of unit conormals $n_\mu dx^\mu$ to those hypersurfaces takes the form

$$n_\mu dx^\mu = \sqrt{1 - 2m/r}\,dt .$$

Further,

$$P(X^\mu\partial_\mu) = X^i\partial_i .$$

Let $X = P(X)$ so that $X = X^i\partial_i$, we calculate

$$\begin{aligned} \nabla_X n_k &= X(\underbrace{n_k}_{=0}) - \Gamma^\nu_{\alpha k}n_\nu X^\alpha \\ &= -n_0\Gamma^0_{ik}X^i . \end{aligned} \tag{1.1.29}$$

Further

$$\begin{aligned}\Gamma^0_{ik} &= \frac{1}{2}{}^4g^{00}(\partial_i \underbrace{{}^4g_{0k}}_{=0} + \partial_k \underbrace{{}^4g_{0i}}_{=0} - \partial_0{}^4g_{ik}) \qquad (1.1.30a)\\ &= -\frac{1}{2}{}^4g^{00}\partial_0{}^4g_{ik} = 0\ , \qquad (1.1.30b)\end{aligned}$$

hence

$$K_{ij} = 0\ .$$

The scalar constraint equation (1.1.15) gives now

$$R(g) = 0\ .$$

This is obviously in L^1, while r^{-2} is in L^2 on $\mathbb{R}^3 \setminus B(0,1)$ (since r^{-4} is in $L^1(\mathbb{R}^3 \setminus B(0,1))$), and convergence of m follows from Proposition 1.1.4. In order to calculate the value of m it is convenient to derive a somewhat simpler form of (1.1.18): generalising somewhat (1.1.28), suppose that

$$g_{ij} = \delta_{ij} + o(r^{-1/2})\ , \qquad \text{with } \partial_k g_{ij} = O(r^{-3/2})\ . \qquad (1.1.31)$$

This choice of powers is motivated by the fact that the power $r^{-3/2}$ is the borderline power to be in $L^2(\mathbb{R}^3 \setminus B(0,1))$: the function $r^{-\sigma}$ with $\sigma > 3/2$ will be in L^2, while if $\sigma = 3/2$ it will not. Under (1.1.31) we have

$$\begin{aligned}16\pi m(R) &:= \int_{S(0,R)} g^{ij}g^{k\ell}\left(\partial_k g_{\ell i} - \partial_i g_{\ell k}\right)\sqrt{\det g}\ dS_j\\ &= \int_{S(0,R)} \left(\delta^{ij} + o(r^{-1/2})\right)\left(\delta^{k\ell} + o(r^{-1/2})\right)\underbrace{\left(\partial_k g_{\ell i} - \partial_i g_{\ell k}\right)}_{O(r^{-3/2})}\ \underbrace{\sqrt{\det g}}_{1+o(r^{-1/2})}\ dS_j\\ &= \int_{S(0,R)} \delta^{ij}\delta^{k\ell}\left(\partial_k g_{\ell i} - \partial_i g_{\ell k}\right)\ dS_j + o(1)\ ,\end{aligned}$$

so that

$$m = m_{ADM} := \lim_{R\to\infty} \frac{1}{16\pi}\int_{S(0,R)} \left(\partial_\ell g_{\ell i} - \partial_i g_{\ell\ell}\right)\ dS_i\ . \qquad (1.1.32)$$

This formula is known as the *Arnowitt–Deser–Misner* (ADM) expression for the mass of the gravitational field at spatial infinity.

Exercice 1.1.6 Check that an identical calculation applies in space-dimension $n \geq 4$ provided that the the decay rates $o(r^{-1/2})$ and $O(r^{-3/2})$ in (1.1.31) are replaced by $o(r^{-(n-2)/2})$ and $O(r^{-(n-2)/2-1})$.

Returning to the Schwarzschild metric consider, first, (1.1.13), or — more generally – metrics which are conformally flat:

$$g_{ij} = (1+2\phi)\delta_{ij} \implies \partial_\ell g_{\ell i} - \partial_i g_{\ell\ell} = 2(\partial_\ell\phi\,\delta_{\ell i} - \partial_i\phi\,\underbrace{\delta_{\ell\ell}}_{=3}) = -4\partial_i\phi\ , \quad (1.1.33)$$

and (1.1.32) reduces to (1.1.4), as desired. The original form given by the left-hand-side of (1.1.10) requires some more work. Again generalising somewhat, we consider general spherically symmetric metrics

$$g = \phi(r)dr^2 + \chi(r)r^2 d\Omega^2 \ , \tag{1.1.34}$$

with ϕ, χ differentiable, tending to one as r goes to infinity at rates compatible with (1.1.31):

$$\phi - 1 = o(r^{-1/2}) \ , \quad \chi - 1 = o(r^{-1/2}) \ , \quad \partial_r \phi = O(r^{-3/2}) \ , \quad \partial_r \chi = O(r^{-3/2}) \ . \tag{1.1.35}$$

We need to reexpress the metric in the pseudo-Cartesian coordinate system associated to the spherical coordinate system (r, θ, φ):

$$x = r \sin\theta \cos\varphi \ , \ y = r \sin\theta \sin\varphi \ , \ z = r\cos\theta \ . \tag{1.1.36}$$

We have

$$\begin{aligned} g &= \phi \, dr^2 + \chi(dr^2 + r^2 d\Omega^2) - \chi \, dr^2 \\ &= \chi \, \delta + (\phi - \chi)dr^2 \\ &= \chi \, \delta + (\phi - \chi)(\sum_i \frac{x^i}{r} dx^i)^2 \ , \end{aligned}$$

so that

$$g_{ij} = \chi \delta_{ij} + \frac{(\phi - \chi)x^i x^j}{r^2} \ .$$

The contribution of the first term to the ADM integral (1.1.32) is obtained from the calculation in (1.1.33), while the second one gives

$$\begin{aligned} &\left[\partial_\ell \left(\frac{(\phi-\chi)x^\ell x^i}{r^2}\right) - \partial_i \left(\frac{(\phi-\chi)x^\ell x^\ell}{r^2}\right)\right] \frac{x^i}{r} \\ &\quad = (\phi' - \chi') + \left[(\phi - \chi)\partial_\ell \left(\frac{x^\ell x^i}{r^2}\right) - \underbrace{\partial_i(\phi - \chi)\Big] \frac{x^i}{r}}_{=\phi' - \chi'} \\ &\quad = (\phi - \chi)\left(\frac{3x^i + x^i - 2x^i}{r^2}\right) \frac{x^i}{r} = 2\frac{\phi - \chi}{r} \ . \end{aligned}$$

Summing it all up, we obtain the following expression for the ADM mass of a spherically symmetric metric (1.1.34) satisfying (1.1.35):

$$\begin{aligned} m &= \lim_{R\to\infty} \frac{1}{16\pi} \int_{S(0,R)} \left(-2r^2\chi' + 2r(\phi - \chi)\right) \, d^2 S \\ &= \lim_{r\to\infty} \frac{1}{2} \left(-r^2 \chi' + r(\phi - \chi)\right) \ . \end{aligned} \tag{1.1.37}$$

For the original form of the Schwarzschild metric we have $\chi \equiv 1$ and $\phi = 1/(1 - 2m/r)$, yielding again the value m for the ADM mass of g.

EXERCICE 1.1.7 Derive the n-dimensional equivalent of (1.1.37):

$$m = \lim_{r\to\infty} \frac{(n-1)\omega_n r^{n-2}}{16\pi} \left(-r\chi' + \phi - \chi\right) , \qquad (1.1.38)$$

where $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the area of a sphere S^{n-1}.

As another example of calculation of the ADM mass, consider the Kasner metrics on $\{t > 0\} \times \mathbb{R}^3$:

$$^4g = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2 . \qquad (1.1.39)$$

The metric (1.1.39) is vacuum provided that

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1 . \qquad (1.1.40)$$

All slices $t =$ const are flat, each of them has thus vanishing ADM mass. This seems to be extremely counter-intuitive, because the metric is highly dynamical. In fact, one would be tempted to say that it has infinite kinetic energy: Indeed, let us calculate the extrinsic curvature tensor of the $t =$ const slices: from (1.1.29)–(1.1.30a) we have

$$\begin{aligned} K &= \nabla_i n_k \, dx^i dx^k \\ &= \frac{1}{2}\partial_t g_{ik} \, dx^i dx^k \\ &= p_1 t^{2p_1-1}dx^2 + p_2 t^{2p_2-1}dy^2 + p_3 t^{2p_3-1}dz^2 . \end{aligned} \qquad (1.1.41)$$

At each value of t we obtain thus a tensor field with entries which are constant in space. The problem here is that while the space slices of the Kasner space-time are asymptotically Euclidean, the space-time metric itself is not asymptotically flat in any sensible way. This example suggests that a physically meaningful notion of total mass can only be obtained for metrics which satisfy asymptotic flatness conditions in a space-time sense; we will return to this question in Section 1.1.4.

1.1.1 Mass in two dimensions

In our approach above we have tied the notion of global mass to the second derivatives that appear in the scalar curvature. Now, there is an essential difference between dimension two and higher, in that a vanishing scalar curvature in dimension two implies local flatness of the metric, while this is not the case anymore in higher dimensions. This is at the origin of the need of a different treatment of two dimensional manifolds.

To begin, it is useful to review some simple two-dimensional manifolds with non-negative scalar curvature. The simplest is of course the flat two-dimensional metric,

$$g = dx^2 + dy^2 = dr^2 + r^2 d\varphi^2 . \qquad (1.1.42)$$

The next simplest flat models are provided by the cylinder $\mathbb{R} \times S^1$, where the x-coordinate in (1.1.42) has been periodically identified, or by the simplest torus,

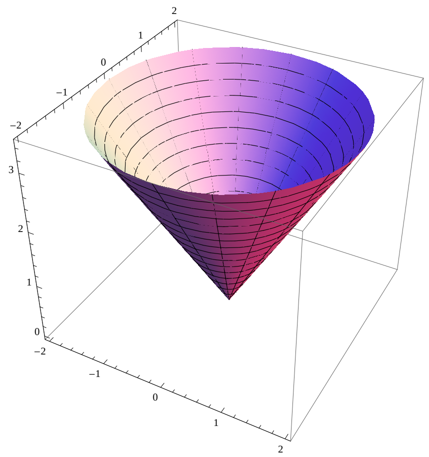

Figure 1.1.1: A flat cone in R^3 with opening angle $\pi/6$.

where x and y have been periodically identified. More sophisticated tori arise by choosing two linearly independent vectors $\vec{\ell}$ and $\vec{m}$ in the plane, and identifying $\vec{x} = (x, y)$ with $\vec{x} + n_1\vec{\ell} + n_2\vec{m}$, for all $n_1, n_2 \in \mathbb{Z}$.

Yet another family of flat examples is provided by two-dimensional axi-symmetric cones in three dimensional Euclidean space. These are defined as

$$C_\alpha = \{z = \cot(\alpha)\rho\}, \quad \text{where} \quad \rho = \sqrt{x^2 + y^2}\ ,$$

see Figure 1.1.1. Then C_α is invariant under rotations around the z-axis, is invariant under positive scaling, and has an opening angle 2α at the singular tip $\vec{x} = 0$. The metric, say g_α, induced on C_α by the Euclidean metric reads

$$\begin{aligned} g_\alpha &= (dz^2 + d\rho^2 + \rho^2 d\varphi^2)|_{C_\alpha} \\ &= (\cot\alpha^2 + 1)d\rho^2 + \rho^2 d\varphi^2\ . \end{aligned} \tag{1.1.43}$$

This can be brought to a manifestly flat form by first replacing ρ by

$$\hat{\rho} := (\sqrt{\cot\alpha^2 + 1})\rho\ ,$$

so that

$$g_\alpha = d\hat{\rho}^2 + \frac{\hat{\rho}^2\, d\varphi^2}{\cot\alpha^2 + 1}\ ,$$

and then replacing φ by

$$\hat{\varphi} := \frac{\varphi}{\sqrt{\cot\alpha^2 + 1}}\ .$$

This leads to a flat metric in polar coordinates

$$g_\alpha = d\hat{\rho}^2 + \hat{\rho}^2 d\hat{\varphi}^2\ . \tag{1.1.44}$$

Keeping in mind that φ was 2π periodic, the new variable

$$\hat{\varphi} \text{ is } \frac{2\pi}{\sqrt{\cot\alpha^2 + 1}}\text{-periodic.}$$

The difference between 2π and the new value of the period is called the *deficit angle*, which is positive for the C_α-cones. More generally, one can consider metrics (1.1.44) where $\hat\varphi$ has any period, the deficit angle can then be negative. The case where no periodicity conditions are imposed on $\hat\varphi$ corresponds to the flat metric on the universal cover of $\mathbb{R}^2 \setminus \{0\}$. The metric (1.1.44) is smooth at $\hat\rho = 0$ if and only if the deficit angle is zero.

An example of positively curved two-dimensional metric is provided by the round sphere,

$$g = dr^2 + \sin^2 r d\varphi^2 \ ,$$

where $r \in [0, \pi]$. Using the sphere metric one can construct a C^1 metric on $\mathbb{R}^2$, with scalar curvature $R \geq 0$, by setting

$$g = \begin{cases} dr^2 + \sin^2 r d\varphi^2, & r \in [0, \pi/2]; \\ dr^2 + d\varphi^2, & r \in [\pi/2, \infty). \end{cases}$$

The geometry is that of a flat cylinder for $r \geq \pi/2$. This is a “cylinder with a a spherical cap”. Note that the metric is not C^2 at the junction $r = \pi/2$, but can be smoothed there while maintaining positive scalar curvature.

Further interesting examples of similar nature, with $R \geq 0$, can be obtained by choosing $r_0 \in (0, \pi/2)$ and setting

$$g = \begin{cases} dr^2 + \sin^2 r \, d\varphi^2, & r \in [0, r_0]; \\ dr^2 + ((r - r_0)\cos r_0 + \sin r_0)^2 d\varphi^2, & r \in [r_0, \infty). \end{cases} \tag{1.1.45}$$

This is a piecewise-smooth C^1 metric, with scalar curvature equal to two for $r \leq r_0$, and zero otherwise. Keeping in mind that φ is 2π-periodic, the geometry for $r \geq r_0$ is that of a flat cone with deficit angle $2\pi(1 - \cos r_0)$. This provides a family of flat cones capped-off by a sphere. To smooth out the metric at $r = r_0$, note that by (A.12.5) below the scalar curvature of the metric

$$g = dr^2 + e^{2f(r)} d\varphi^2 \tag{1.1.46}$$

equals

$$R = -2e^{-f}(e^f)'' \ . \tag{1.1.47}$$

Given $R(r)$, this can be viewed as a linear equation for e^f. Integrating this equation from zero to infinity with $R \geq 0$ obtained as a small smoothing, localised near $r = r_0$, of the scalar curvature of the metric (1.1.45) leads to a smooth metric with positive scalar curvature which coincides with the metric on a cone for sufficiently large r.

We continue with some definitions. We shall say that a two-dimensional manifold M is *finitely connected* if M is diffeomorphic to a compact boundaryless manifold N from which a finite non-zero number of points has been removed. Equivalently, M is diffeomorphic to the union of a compact set with a finite number of exterior regions diffeomorphic to $\mathbb{R}^2 \setminus B(0, R_i)$. Let p be any point in M and let $S_p(t)$ and $B_p(t)$ be the geodesic sphere and ball around p:

$$S_p(t) := \{q \in M : d_g(p, q) = t\} \ , \quad B_p(t) = \{q \in M : d_g(p, q) < t\} \ .$$

We will denote by $L_p(t)$ the length of $S_p(t)$ and by $A_p(t)$ the area of $B_p(t)$. We have the following theorem of Shiohama [168]:

THEOREM 1.1.8 *Let (M,g) be a complete, non-compact, finitely connected two dimensional manifold. If*

$$R(g) \in L^1(M) \ ,$$

then

$$\lim_{t\to\infty} \frac{L_p(t)}{t} = \lim_{t\to\infty} \frac{2A_p(t)}{t^2} = 2\pi\chi(M) - \frac{1}{2}\int_M R\, d\mu_g \ . \tag{1.1.48}$$

There are several interesting consequences of this result. First, one notices that the right-hand-side of (1.1.48) does not depend upon p, so that the first two terms are also p–independent. Next, since the left-hand-side (1.1.48) is non-negative, if g is a complete metric on $\mathbb{R}^2$ we obtain the *Cohn-Vossen inequality*

$$\int_M R\, d\mu_g \le 4\pi \ , \tag{1.1.49}$$

with equality if and only if the metric is flat.

Of particular interest to us are metrics of non-negative scalar curvature. As this imposes restrictions on the topology of the manifold, such theorems belong to the family of "topological censorship theorems". One has:

THEOREM 1.1.9 ("Two-dimensional topological censorship") *Let (M,g) be a complete Riemannian manifold with $R \ge 0$, then M is diffeomorphic to $\mathbb{R}^2$, S^2, or a quotient thereof.*

Some comments are in order: The result under the hypothesis that M is non-compact is attributed in [47] to Cohn-Vossen [76]. There remains the compact case, which follows from the usual Gauss-Bonnet theorem together with the classification of two-dimensional compact manifolds in terms of their Euler characteristic. We note that a complete two-dimensional manifold with $R \ge \delta > 0$ must be compact by a classical theorem of Myers in any case. Under the supplementary conditions of Theorem 1.1.8, one can also argue from there, using the fact that the left-hand side of (1.1.48) is non-negative, but note that the conditions of Theorem 1.1.9 allow *a priori* more general manifolds.

As another application of (1.1.48), consider a manifold which is the union of a compact set with a finite number of ends M_i, $i = 1, \dots, I$, diffeomorphic to $[R_i, \infty) \times S^1$, and with the metric asymptotically approaching a flat metric on a cone on M_i:

$$g(\omega_i) = dr^2 + r^2 \left(\frac{\omega_i}{2\pi}\right)^2 d\varphi^2 \tag{1.1.50}$$

for some positive constant ω_i. Here we parameterize S^1 by an angular variable $\varphi \in [0, 2\pi]$, so that the circles $r =$ const. have $g(\omega_i)$–length equal to $\omega_i r$. Such metrics will be called *asymptotically conical.* Under very mild conditions on the convergence of g to $g(\omega_i)$ we will have

$$A(B(t) \cap M_i) = \frac{1}{2}\omega_i t^2 + o(t^2) \ ,$$

for t large. In the simplest case $M = \mathbb{R}^2$ we then obtain

$$\frac{1}{2}\int_M R\, d\mu_g = 2\pi - \omega \ ,$$

with $\omega = \omega_1$. Hence, the integral of R equals the *deficit angle* $2\pi - \omega$. This leads to the:

THEOREM 1.1.10 (Two-dimensional positive energy theorem) *For asymptotically conical complete metrics on $\mathbb{R}^2$ with $L^1 \ni R \geq 0$ the deficit angle is non-negative, vanishing if and only if the metric is flat.*

PROOF: The result follows of course from Theorem 1.1.8, but in the current restrictive setting the proof is simple enough to be given here.

To simplify the argument we will assume that there exists a global coordinate system in which the metric g is exactly flat near the origin, and coincides with (1.1.50) for large distances. Under the asymptotic fall-off conditions (1.1.53) below the proof for the more general case proceeds in an identical way, except that one has to keep track of annoying error terms. Near the origin one should then use coordinates in which $g_{ij} = \delta_{ij} + O(|x|^2)$; such coordinates always exist. The fact that the last coordinate system does not have to coincide with the coordinate system in the asymptotic region, in which the metric takes the form (1.1.50), also needs to be addressed.

Thus, let e_2 be any nowhere vanishing unit vector field on $\mathbb{R}^2 \setminus \{0\}$ such that $e_2 = r^{-1}\partial_\varphi$ in local coordinates near the origin in which $g_{ij} = \delta_{ij}$, and which equals $(2\pi/\omega r)\partial_\varphi$ for large distances in the region where (1.1.50) holds. Let $\{e_a\}$ be the associated orthonormal frame, hence $e_1 = \partial_r$ both for small and large distances in the relevant coordinates. By (A.17.22) we have

$$\begin{aligned}\int_{B(R)\setminus B(\epsilon)} R\, d\mu_g &= 2\int_{B(R)\setminus B(\epsilon)} d\omega^1{}_2 \\ &= 2\Big(\int_{S(R)} \omega^1{}_2 - \int_{S(\epsilon)} \omega^1{}_2\Big)\,, \end{aligned} \tag{1.1.51}$$

where $B(\epsilon)$ denotes an open coordinate ball centered at the origin in the coordinates where $e_1 = \partial_r$, similarly for $B(R)$, and we need to calculate the connection forms $\omega^1{}_2$. Let $\theta^1 = dr$, $\theta^2 = (\omega r/2\pi)d\varphi$ be the coframe dual to $\{e_a\}$, the structure equations

$$0 = d\theta^1 + \omega^1{}_a \wedge \theta^a = d(dr) + \omega^1{}_2 \wedge \theta^2 = r\omega^1{}_2 \wedge d\varphi$$

and

$$0 = d\theta^2 + \omega^2{}_b \wedge \theta^b = \frac{\omega}{2\pi} d(r d\varphi) + \omega^2{}_1 \wedge dr = \frac{\omega}{2\pi} dr \wedge d\varphi + \omega^2{}_1 \wedge dr\ ,$$

are solved by

$$\omega^1{}_2 = -\omega^2{}_1 = -\frac{\omega}{2\pi} d\varphi \tag{1.1.52}$$

for large $|x|$. An identical formula with $\omega = 2\pi$ holds near the origin. The integrals over the circles in the last line of (1.1.51) are thus straightforward to evaluate, giving

$$\int_{\mathbb{R}^2} R\, d\mu_g = \lim_{\epsilon\to 0\,,\, R\to\infty} \int_{B(R)\setminus B(\epsilon)} R\, d\mu_g \quad = \quad 2(2\pi - \omega)\ ,$$

proving non-negativity of the right-hand side. Further, the vanishing of the deficit angle together with $R \geq 0$ implies $R \equiv 0$, leading to a flat metric, as desired. □

We note that the last calculations go through with error terms which give a vanishing contribution in the limit if one assumes that in the asymptotic region the derivatives

$$|\mathring{D}_k g_{ij}|_g \tag{1.1.53}$$

decay sufficiently fast, where $\mathring{D}$ is the covariant derivative operator of the metric (1.1.50), and $|\cdot|_g$ denotes the norm of a tensor with respect to the metric g. But, again, this assumption is not needed for the current conclusion in view of Theorem 1.1.8.

1.1.2 Coordinate independence

The next example is due to Denissov and Solovyev [78]: let δ be the Euclidean metric on $\mathbb{R}^3$ and introduce a new coordinate system (ρ, θ, φ) by changing the radial variable r to

$$r = \rho + c\rho^{1-\alpha} \;, \tag{1.1.54}$$

with some constants $\alpha > 0$, $c \in \mathbb{R}$. This gives

$$dr^2 + r^2 d\Omega^2 = (1 + (1-\alpha)c\rho^{-\alpha})^2 d\rho^2 + (1 + c\rho^{-\alpha})^2 \rho^2 d\Omega^2 \;.$$

This is of the form (1.1.34) with r replaced by ρ and

$$\phi(\rho) = (1 + (1-\alpha)c\rho^{-\alpha})^2 \;, \quad \chi(\rho) = (1 + c\rho^{-\alpha})^2 \;,$$

so we can apply (1.1.37):

$$\begin{aligned}
-\rho^2\chi' + \rho(\phi - \chi) &= 2c\alpha\rho^{-\alpha+1}(1 + c\rho^{-\alpha}) + \rho\left((1 + (1-\alpha)c\rho^{-\alpha})^2 - (1 + c\rho^{-\alpha})^2\right) \\
&= 2c\alpha\rho^{-\alpha+1}(1 + c\rho^{-\alpha}) + \rho\left((1 + c\rho^{-\alpha} - \alpha c\rho^{-\alpha})^2 - (1 + c\rho^{-\alpha})^2\right) \\
&= 2c\alpha\rho^{-\alpha+1}(1 + c\rho^{-\alpha}) + \rho\left(-2\alpha c\rho^{-\alpha}(1 + c\rho^{-\alpha}) + \alpha^2 c^2 \rho^{-2\alpha}\right) \\
&= \alpha^2 c^2 \rho^{1-2\alpha}.
\end{aligned}$$

It follows that

$$\begin{aligned}
m_{ADM} &= \lim_{\rho\to\infty} \frac{1}{2}\left(-\rho^2\chi' + \rho(\phi - \chi)\right) \\
&= \begin{cases} \infty \;, & \alpha < 1/2 \;, \\ c^2/8 \;, & \alpha = 1/2 \;, \\ 0 \;, & \alpha > 1/2 \;. \end{cases}
\end{aligned} \tag{1.1.55}$$

Let y^i denote the coordinate system associated to the angular variables (ρ, θ, φ) by replacing r with ρ in (1.1.36). Then the exponent α in (1.1.54) dictates the rate at which the metric components approach δ_{ij}:

$$\delta_{ij} dx^i dx^j = g_{ij} dy^i dy^j \;, \text{ with } g_{ij} - \delta_{ij} = O(\rho^{-\alpha}) \;, \; \partial_k g_{ij} = O(\rho^{-\alpha-1}) \;.$$

Note that above we have calculated the ADM mass integral (1.1.32), rather than the original integral (1.1.18). We have already seen that both integrals coincide if $\alpha > 1/2$ (compare (1.1.31)), but they do not necessarily do that for $\alpha \leq 1/2$. One can similarly calculate the mass m of (1.1.18) obtaining an identical conclusion: the mass m of the flat metric in the coordinate system y^i is infinite if $\alpha < 1/2$, can have an arbitrary positive value depending upon c if $\alpha = 1/2$, and vanishes for $\alpha > 1/2$. The lesson of this is that the mass appears to depend upon the coordinate system chosen, even within the class of coordinate systems in which the metric tends to a constant coefficients matrix as r tends to infinity.

The reader will notice that for $\alpha = 1/2$ the metric does not satisfy the conditions of Proposition 1.1.4, as the derivatives of g_{ij} in the new coordinate system will not be in L^2. It follows that the conditions of Proposition 1.1.4 are not necessary for the existence of those limits, though they seem to be very close to be optimal, since — as shown above — allowing α's smaller than 1/2 leads to infinite mass representations for Euclidean space.

Exercice 1.1.11 Check that in dimensions $n > 3$ the coordinate transformation (1.1.54) leads to

$$m_{ADM} = \lim_{\rho\to\infty} \frac{(n-1)\omega_n\alpha^2c^2\rho^{n-2-2\alpha}}{16\pi} = \begin{cases} \infty \ , & \alpha < (n-2)/2 \ , \\ \frac{(n-1)\,\omega_n\alpha^2c^2}{16\pi} \ , & \alpha = (n-2)/2 \ , \\ 0 \ , & \alpha > (n-2)/2 \ , \end{cases} \tag{1.1.56}$$

so that the borderline decay exponent is now $\alpha = (n-2)/2$.

In order to clarify the question of dependence of the mass upon coordinates it is useful to include those coordinate systems explicitly in the notation. Consider, thus, a pair (g, ϕ), where

1. g is a Riemannian metric on an n–dimensional manifold N, N diffeomorphic to $\mathbb{R}^n \setminus B(R)$, where $B(R)$ is a closed ball. N should be thought of as one of (possible many) asymptotically flat ends of M.

2. ϕ is a coordinate system on the complement of a compact set K of N such that, in local coordinates $\phi^i(p) = x^i$ the metric takes the following form:
$$g_{ij} = \delta_{ij} + h_{ij} \ , \tag{1.1.57}$$
with h_{ij} satisfying
$$\forall_{i,j,k} \qquad |h_{ij}| \leq \ c(r+1)^{-\alpha} \ , \quad |\frac{\partial h_{ij}}{\partial x^k}| \leq \ c(r+1)^{-\alpha-1} \ , \tag{1.1.58}$$
for some constant $c \in \mathbb{R}$, where $r(x) = (\sum (x^i)^2 \ )^{1/2}$.

3. Finally, g_{ij} is uniformly equivalent to the flat metric δ: there exists a constant C such that
$$\forall X^i \in \mathbb{R}^n \qquad C^{-1}\sum (X^i)^2 \leq g_{ij} \ X^iX^j \leq C\sum (X^i)^2 \ . \tag{1.1.59}$$

Such a pair (g, ϕ) will be called α–admissible.

We note that (1.1.59) is equivalent to the requirement that all the g_{ij}'s and g^{ij}'s are uniformly bounded: indeed, at any point we can diagonalise g_{ij} using a rotation; arranging the resulting eigenvalues λ_i in increasing order we have

$$\lambda_1 \sum (X^{\hat{i}})^2 \le \underbrace{\lambda_1 (X^{\hat{1}})^2 + \ldots + \lambda_n (X^{\hat{n}})^2}_{=g_{ij}\ X^i X^j} \le \lambda_n \sum (X^{\hat{i}})^2 , \tag{1.1.60}$$

where we have used the symbol $X^{\hat{i}}$ to denote the components of X in the diagonalising frame. Since the X^i's differ from the $X^{\hat{i}}$'s by a rotation we have

$$\sum (X^{\hat{i}})^2 = \sum (X^i)^2 ,$$

leading to

$$C = \max(\lambda_1^{-1}, \lambda_n) .$$

In order to prove that uniform boundedness of g_{ij}'s leads to the second inequality in (1.1.59) we note that in an arbitrary, not necessarily diagonalising, frame we have

$$\begin{aligned} g_{ij} X^i X^j &\le \sup_{i,j,x} |g_{ij}(x)| \sum_{i,j} |X^i X^j| \\ &= \sup_{i,j,x} |g_{ij}(x)| \Big((X^1)^2 + \ldots + (X^n)^2 + \sum_{i<j} \underbrace{2|X^i X^j|}_{\le (X^i)^2 + (X^j)^2} \Big) \\ &\le \Big(1 + \frac{(n-1)}{2} \Big) \sup_{i,j,x} |g_{ij}(x)| ((X^1)^2 + \ldots + (X^n)^2) , \end{aligned}$$

with a similar calculation for g^{ij}, leading to (recall that, after diagonalisation, the largest eigenvalue of g^{ij} is λ_1^{-1})

$$\lambda_n \le \frac{n+1}{2} \sup_{i,j,x} |g_{ij}(x)| , \qquad \lambda_1^{-1} \le \frac{n+1}{2} \sup_{i,j,x} |g^{ij}(x)| . \tag{1.1.61}$$

We thus have the following estimate for the constant C in (1.1.59):

$$C \le C(n) \max(\sup_{i,j,x} |g_{ij}(x)|, \sup_{i,j,x} |g^{ij}(x)|) . \tag{1.1.62}$$

To finish the proof of equivalence, we note that (1.1.59) gives directly

$$|g_{ij}| = |g(\partial_i, \partial_j)| \le 2\lambda_n \le 2C , \qquad \text{similarly} \ \ |g^{ij}| \le 2C . \tag{1.1.63}$$

We have the following result, we follow the proof in [55]; an independent, completely different proof, under slightly different conditions, can be found in [10]:

THEOREM 1.1.12 (Coordinate-independence of the mass [10, 55]) *Consider two α–admissible coordinate systems ϕ_1 and ϕ_2, with some*

$$\alpha > (n-2)/2 , \tag{1.1.64}$$

and suppose that

$$R(g) \in L^1(N) .$$

Let $S(R)$ be any one-parameter family of differentiable spheres, such that $r(S(R)) = \min_{x\in S(R)} r(x)$ tends to infinity, as R does. For $\phi=\phi_1$ and $\phi=\phi_2$ define

$$m(g,\phi) \quad = \quad \lim_{R\to\infty}\frac{1}{16\pi}\int_{S(R)}(g_{ik,i}-g_{ii,k})dS_k\ , \tag{1.1.65}$$

with each of the integrals calculated in the respective local α–admissible coordinates ϕ_a. Then

$$m(g,\phi_1)=m(g,\phi_2)\ .$$

The example of Denissov and Solovyev presented above shows that the condition $\alpha>(n-2)/2$ in Theorem 1.1.12 is sharp.

Proof: We start with a lemma:

Lemma 1.1.13 (Asymptotic symmetries of asymptotically Euclidean manifolds) *Let (g,ϕ_1) and (g,ϕ_2) be α_1 and α_2–admissible, respectively, with $\alpha_a>0$. Let $\phi_1\circ\phi_2^{-1}:\mathbb{R}^n\backslash K_2\to\mathbb{R}^n\backslash K_1$ be a twice differentiable diffeomorphism, for some compact sets K_1 and $K_2\subset\mathbb{R}^n$. Set*

$$\alpha=\min(\alpha_1,\alpha_2)\ .$$

Then there exists a matrix ${\omega^i}_j\in O(n)$ such that, in local coordinates

$$\phi_1^i(p)=x^i\ ,\qquad \phi_2^i(p)=y^i\ ,$$

the diffeomorphisms $\phi_1\circ\phi_2^{-1}$ and $\phi_2\circ\phi_1^{-1}$ take the form

$$x^i(y)={\omega^i}_j\ y^i+\eta^i(y)\ ,\qquad y^i(x)={(\omega^{-1})^i}_j\ x^i\ +\zeta^i(x)\ .$$

When $\alpha<1$ the fields ζ^i and η^i satisfy, for some constant $C\in\mathbb{R}$,

$$|{\zeta^i}_{,j}(x)|\le C(r(x)+1)^{-\alpha}\ ,\quad |\zeta^i(x)|\le\begin{cases}C(\ln r(x)+1)\ , & \alpha=1,\\ C(r(x)+1)^{1-\alpha}\ , & \textit{otherwise},\end{cases}$$

$$|{\eta^i}_{,j}(y)|\le C(r(y)+1)^{-\alpha}\ ,\quad |\eta^i(y)|\le\begin{cases}C(\ln r(y)+1)\ , & \alpha=1,\\ C(r(y)+1)^{1-\alpha}\ , & \textit{otherwise},\end{cases}$$

$$r(x)=(\textstyle\sum(x^i)^2)^{1/2}\ ,\qquad r(y)=(\textstyle\sum(y^i)^2)^{1/2}\ .$$

On the other hand, for $\alpha>1$ there exist constants $\mathring{A}^i$ and a constant C such that

$$|{\zeta^i}_{,j}(x)|\le C(r(x)+1)^{-\alpha}\ ,\quad |\zeta^i(x)-\mathring{A}^i|\le Cr^{1-\alpha}\ ,$$

with an analogous statement for η.

Proof: Let us first note that both (g,ϕ_1) and (g,ϕ_2) are α–admissible, so that we do not have to worry about two constants α_1 and α_2. Let g^1_{ij} and g^2_{ij} be the representatives of g in local coordinates ϕ_1 and ϕ_2:

$$g=g^1_{ij}(x)dx^idx^j=g^2_{k\ell}(y)dy^kdy^\ell\ .$$

In the proof that follows the letters C,C', *etc.*, will denote constants which may vary from line to line, their exact values can be estimated at each step but are irrelevant for further purposes. Let us write down the equations following from the transformation properties of the metric

$$g^2_{ij}(y) = g^1_{k\ell}(x(y))\frac{\partial x^k}{\partial y^i}\frac{\partial x^\ell}{\partial y^j}\,, \tag{1.1.66a}$$

$$g^1_{ij}(x) = g^2_{k\ell}(y(x))\frac{\partial y^k}{\partial x^i}\frac{\partial y^\ell}{\partial x^j}\,. \tag{1.1.66b}$$

Contracting (1.1.66a) with $g_1^{ij}(x(y))$, where g_1^{ij} denotes the inverse matrix to g^1_{ij}, one obtains

$$g_1^{ij}(x(y))g^2_{ij}(y) = g_1^{ij}(x(y))g^1_{k\ell}(x(y))\frac{\partial x^k}{\partial y^i}\frac{\partial x^\ell}{\partial y^j}\,. \tag{1.1.67}$$

Now, the function appearing on the right-hand-side above is a strictly positive quadratic form in $\partial x^i/\partial y^j$, and uniform ellipticity of g_1^{ij} gives

$$C^{-1}\sum_{k,i}(\frac{\partial x^k}{\partial y^i})^2 \le g_1^{ij}(x(y))g^1_{k\ell}(x(y))\frac{\partial x^k}{\partial y^i}\frac{\partial x^\ell}{\partial y^j} \le C\sum_{k,i}(\frac{\partial x^k}{\partial y^i})^2\,.$$

In order to see this, we let $A^i{}_j$ be the tensor field $\partial x^i/\partial x^j$; in a frame diagonalising g^1_{ij}, as in (1.1.60), we have

$$g_1^{ij}(x(y))g^1_{k\ell}(x(y))A^k{}_iA^\ell{}_j = \sum_{i,j}\lambda_i^{-1}\lambda_j(A^j{}_i)^2$$

and we conclude with (1.1.61)

Since the function appearing at the left-hand-side of (1.1.67) is uniformly bounded we obtain

$$\sum_{k,i}|\frac{\partial x^k}{\partial y^i}| \le C\,. \tag{1.1.68}$$

Similar manipulations using (1.1.66b) give

$$\sum_{k,i}|\frac{\partial y^k}{\partial x^i}| \le C\,. \tag{1.1.69}$$

Inequalities (1.1.68)–(1.1.69) show that all the derivatives of $x(y)$ and $y(x)$ are uniformly bounded. Let Γ_x be the ray joining x and K_1, and let $y_0^i(x)$ be the image by $\phi_2\circ\phi_1^{-1}$ of the intersection point of K_1 with Γ_x (if there is more than one, choose the one which is closest to x). We have, in virtue of (1.1.69),

$$|y^i(x)-y_0^i(x)| = \left|\int_{\Gamma_x}\frac{\partial y^i}{\partial x^k}dx^k\right| \le C\ r(x)\,,$$

so that

$$r(y(x)) \le C\ r(x) + C_1\,. \tag{1.1.70}$$

A similar reasoning shows

$$r(x(y)) \le C\ r(y) + C_1\,. \tag{1.1.71}$$

Equations (1.1.70) and (1.1.71) can be combined into a single inequality

$$r(y(x))/C - C_1 \le r(x) \le C\ r(y(x)) + C_1\ . \tag{1.1.72}$$

This equation shows that any quantity which is[1] $O(r(x)^{-\beta})$ ($O(r(y)^{-\beta})$) is also $O(r(y)^{-\beta})$ ($O(r(x)^{-\beta})$), when composed with $\phi_2 \circ \phi_1^{-1}$ ($\phi_1 \circ \phi_2^{-1}$).

We continue using the transformation law of the connection coefficients under changes of coordinates. If we write ${}^1\Gamma^i{}_{jk}$ for the Christoffel symbols of g^1_{ij}, and ${}^2\Gamma^i{}_{jk}$ for those of g^2_{ij}, Equation (A.9.13) of Appendix (A.9.3) gives

$$ {}^1\Gamma^i{}_{jk} \quad = \quad \frac{\partial x^i}{\partial y^s}\frac{\partial^2 y^s}{\partial x^k \partial x^j} + \frac{\partial x^i}{\partial y^s}\frac{\partial y^\ell}{\partial x^j}\frac{\partial y^r}{\partial x^k}{}^2\Gamma^s{}_{r\ell}\ . \tag{1.1.73}$$

This can be rewritten as an equation for the second derivatives of y with respect to x:

$$\begin{aligned} \frac{\partial^2 y^p}{\partial x^k \partial x^j} &= \frac{\partial y^p}{\partial x^i}\Big({}^1\Gamma^i{}_{jk} - \frac{\partial x^i}{\partial y^s}\frac{\partial y^\ell}{\partial x^j}\frac{\partial y^r}{\partial x^k}{}^2\Gamma^s{}_{r\ell}\Big) \\ &= \frac{\partial y^p}{\partial x^i}{}^1\Gamma^i{}_{jk} - \frac{\partial y^\ell}{\partial x^j}\frac{\partial y^r}{\partial x^k}{}^2\Gamma^p{}_{r\ell}\ . \end{aligned} \tag{1.1.74}$$

The decay rate of the connection coefficients, and the fact that $r(x)$ is equivalent to $r(y)$, gives

$$\frac{\partial^2 y^i}{\partial x^j \partial x^k} = O(r^{-\alpha-1})\ . \tag{1.1.75}$$

In a similar way one establishes

$$\frac{\partial^2 x^i}{\partial y^j \partial y^k} = O(r^{-\alpha-1})\ . \tag{1.1.76}$$

An alternative, direct way is to inspect the equation obtained by differentiating (1.1.66). This involves several terms, so in order to simplify the notations we introduce

$$\begin{aligned} &A^i{}_j = \frac{\partial y^i}{\partial x^j}\ , \qquad B^i{}_j = \frac{\partial x^i}{\partial y^j}\ , \\ &C_{ijk} = A^m{}_i g^2_{m\ell}\frac{\partial A^\ell{}_j}{\partial x^k} = g^2_{m\ell}\frac{\partial y^m}{\partial x^i}\frac{\partial^2 y^\ell}{\partial x^j \partial x^k}\ , \\ &D_{ijk} = B^m{}_i g^1_{m\ell}\frac{\partial B^\ell{}_j}{\partial y^k}\ . \end{aligned}$$

Differentiating (1.1.66b) with respect to x, taking into account (1.1.58), (1.1.69) and (1.1.72) leads to

$$C_{ijk} + C_{jik} = O(r^{-\alpha-1})\ .$$

We perform the usual cyclic permutation calculation:

$$\begin{aligned} C_{ijk} + C_{jik} &= O(r^{-\alpha-1})\ , \\ -C_{jki} - C_{kji} &= O(r^{-\alpha-1})\ , \end{aligned}$$

[1] $f(s) = O(s^\gamma)$ is used here to denote a function satisfying $|f(s)| \le C(|s|+1)^\gamma$ for some positive constant C.

$$C_{kij} + C_{ikj} = O(r^{-\alpha-1}) .$$

Adding the three equations and using the symmetry of C_{ijk} in the last two indices yields

$$C_{ijk} = O(r^{-\alpha-1}) .$$

This equality together with (1.1.68) and the definition of C_{ijk} gives (1.1.75)

We need a lemma:

LEMMA 1.1.14 *Let $\sigma > 0$ and let $f \in C^1(\mathbb{R}^n \setminus \overline{B(R)})$ satisfy*

$$\partial_i f = O(r^{-\sigma-1}) .$$

Then there exists a constant f_∞ such that

$$f - f_\infty = O(r^{-\sigma}) .$$

PROOF: Integrating along a ray we have

$$f(r_1\vec{n}) - f(r_2\vec{n}) = \int_{r_2}^{r_1} \frac{\partial f}{\partial r}(r\vec{n})dr = \int_{r_2}^{r_1} O(r^{-\sigma-1})dr = O(r_2^{-\sigma}) . \qquad (1.1.77)$$

It follows that the sequence $\{f(i\vec{n})\}_{i\in\mathbb{N}}$ is Cauchy, therefore the limit

$$f_\infty(\vec{n}) = \lim_{i\to\infty} f(i\vec{n})$$

exists. Letting $r_1 = i$ in (1.1.77) and passing with i to infinity we obtain

$$f(\vec{x}) - f_\infty(\frac{\vec{x}}{r}) = O(r^{-\sigma}) .$$

Integrating over an arc of circle Γ connecting the vectors $r\vec{n}_1$ and $r\vec{n}_1$ we have

$$|f(r\vec{n}_1) - f(r\vec{n}_1)| = \left|\int_\Gamma df\right| \leq \sup_\Gamma |df||\Gamma| ,$$

where $|\Gamma|$ denotes the Euclidean length of Γ. Since $|\Gamma| \leq 2\pi r$ we obtain

$$|f(r\vec{n}_1) - f(r\vec{n}_1)| \leq 2\pi C r^{-\sigma} .$$

Passing with r to infinity we find

$$f_\infty(\vec{n}_1) = f_\infty(\vec{n}_1) ,$$

so that f_∞ is $\vec{n}$–independent, as desired. □

Lemma 1.1.14 shows that the limits

$$\mathring{A}^i{}_j = \lim_{r\to\infty} A^i{}_j(r\vec{n}) ,$$

$$\mathring{B}^i{}_j = \lim_{r\to\infty} B^i{}_j(r\vec{n}) ,$$

($\vec{n}$ — any vector satisfying $\sum(n^i)^2 = 1$) exist and are n^i independent matrices, with $A = B^{-1}$. Define

$$\zeta^i(x) = y^i(x) - \mathring{A}^i{}_j\, x^j\ , \qquad \eta^i(y) = x^i(y) - \mathring{B}^i{}_j\, y^j\ .$$

Equation (1.1.76) leads to

$$A^i{}_j(r_2\vec{n}) - A^i{}_j(r_1\vec{n}) = \int_{r_1}^{r_2} \frac{\partial^2 x^i}{\partial x^j \partial x^k}(r\vec{n})\, n^k\, dr = O(r_1^{-\alpha})$$

for $r_2 > r_1$. We have $A^i{}_j = \mathring{A}^i{}_j + \zeta^i{}_{,j}$, so that passing with r_2 to infinity one finds

$$\zeta^i{}_{,j}(x) = O(r^{-\alpha})\ .$$

Integrating along rays one obtains

$$\zeta^i(x) = \begin{cases} O(r^{1-\alpha})\ , & 0 < \alpha < 1, \\ O(\ln r)\ , & \alpha = 1, \end{cases}$$

with a similar calculation for η. Finally, for $\alpha > 1$ we find that there exist constants $\mathring{A}^i$ such that

$$\zeta^i(x) = \mathring{A}^i + O(r^{1-\alpha})\ .$$

Equations (1.1.58) and (1.1.72) allow us to write (1.1.66) in the following form

$$\sum_k \frac{\partial y^k}{\partial x^i}\,\frac{\partial y^k}{\partial x^j} \;=\; \delta_{ij} + O(r^{-\alpha})\ , \tag{1.1.78a}$$

$$\sum_k \frac{\partial x^k}{\partial y^i}\,\frac{\partial x^k}{\partial y^j} \;=\; \delta_{ij} + O(r^{-\alpha})\ . \tag{1.1.78b}$$

Passing to the limit $r \to \infty$ one obtains that $\mathring{A}^i{}_j$ and $\mathring{B}^i{}_j$ are rotation matrices, which finishes the proof. □

Let us return to the proof of Theorem 1.1.12. We start by noting that the limit in (1.1.65) does not depend upon the family of spheres chosen — this follows immediately from the identity (1.1.20).

Next, let us show that the integrand of the mass has tensorial properties under rotations: if $y^i = \omega^i{}_j x^j$, then

$$g^1_{ij}(x) = g^2_{k\ell}(y(x))\frac{\partial y^k}{\partial x^i}\,\frac{\partial y^\ell}{\partial x^j} = g^2_{k\ell}(\omega x)\omega^k{}_i\,\omega^\ell{}_j\ ,$$

so that

$$\frac{\partial g^1_{ij}(x)}{\partial x^j} - \frac{\partial g^1_{jj}(x)}{\partial x^i} \;=\; \frac{\partial g^2_{k\ell}(\omega x)}{\partial y^r}\omega^r{}_j\omega^k{}_i\,\omega^\ell{}_j - \frac{\partial g^2_{k\ell}(\omega x)}{\partial y^r}\omega^r{}_i\omega^k{}_j\,\omega^\ell{}_j\ . \tag{1.1.79}$$

Now, a rotation matrix satisfies

$$\omega^r{}_i\omega^s{}_i = \delta^r_s\ , \tag{1.1.80}$$

so that (1.1.79) can be rewritten as

$$\begin{aligned}\frac{\partial g^1_{ij}(x)}{\partial x^j}-\frac{\partial g^1_{jj}(x)}{\partial x^i} &= \frac{\partial g^2_{k\ell}(\omega x)}{\partial y^\ell}\omega^k{}_i-\frac{\partial g^2_{\ell\ell}(\omega x)}{\partial y^r}\omega^r{}_i\\ &= \left(\frac{\partial g^2_{k\ell}(\omega x)}{\partial y^\ell}-\frac{\partial g^2_{\ell\ell}(\omega x)}{\partial y^k}\right)\omega^k{}_i\,. \end{aligned}\tag{1.1.81}$$

Finally, the surface forms dS_j also undergo a rotation:

$$\frac{\partial}{\partial x^i}\rfloor dx^1\wedge\ldots\wedge dx^n=\omega^s{}_i\frac{\partial}{\partial y^s}\rfloor\underbrace{\left(\det\frac{\partial x}{\partial y}\right)}_{=1}dy^1\wedge\ldots\wedge dy^n=\omega^s{}_i\frac{\partial}{\partial y^s}\rfloor dy^1\wedge\ldots\wedge dy^n\,.$$

This, together with (1.1.81) and (1.1.80) leads to

$$\begin{aligned}&\left(\frac{\partial g^1_{ij}(x)}{\partial x^j}-\frac{\partial g^1_{jj}(x)}{\partial x^i}\right)\frac{\partial}{\partial x^j}\rfloor dx^1\wedge\ldots\wedge dx^n\\ &\quad=\left(\frac{\partial g^2_{k\ell}(\omega x)}{\partial y^\ell}-\frac{\partial g^2_{\ell\ell}(\omega x)}{\partial y^k}\right)\omega^k{}_i\omega^s{}_i\frac{\partial}{\partial y^s}\rfloor dy^1\wedge\ldots\wedge dy^n\\ &\quad=\left(\frac{\partial g^2_{k\ell}(\omega x)}{\partial y^\ell}-\frac{\partial g^2_{\ell\ell}(\omega x)}{\partial y^k}\right)\frac{\partial}{\partial y^k}\rfloor dy^1\wedge\ldots\wedge dy^n\,.\end{aligned}$$

It follows that the mass will not change if a rigid coordinate rotation is performed.

In particular, replacing the coordinate y^i by $(\omega^{-1})^i{}_jy^j$ will preserve the mass, and to finish the proof it remains to consider coordinate transformations such that the matrix ω in Lemma 1.1.13 is the identity. We then have

$$h^2_{ij}=g^2_{ij}-\delta_{ij}=h^1_{ij}(x(y))+\eta^k{}_{,i}(y)+\eta^i{}_{,j}(y)+O(r^{-2\alpha})\tag{1.1.82}$$

where

$$h^1_{ij}=g^1_{ij}-\delta_{ij}\,.$$

Therefore

$$\begin{aligned}\frac{\partial g^2_{ij}(y)}{\partial y^j}-\frac{\partial g^2_{jj}(y)}{\partial y^i} &= \frac{\partial h^1_{ij}(x(y))}{\partial x^j}-\frac{\partial h^1_{jj}(x(y))}{\partial x^i}\\ &\quad+\frac{\partial}{\partial x^j}\left(\frac{\partial \eta^i}{\partial x^j}-\frac{\partial \eta^j}{\partial x^i}\right)+O(r^{-2\alpha-1})\,.\end{aligned}\tag{1.1.83}$$

While integrated over the sphere $r(y)=$ const, the last term in (1.1.83) will give no contribution in the limit $r(y)\to\infty$ since $2\alpha+1>n-1$ by hypothesis. The next to last term in (1.1.83) will give no contribution being the divergence of an antisymmetric quantity: indeed, we have

$$\frac{\partial}{\partial x^j}\left(\frac{\partial \eta^i}{\partial x^j}-\frac{\partial \eta^j}{\partial x^i}\right)\frac{\partial}{\partial x^i}\rfloor dx^1\wedge\ldots\wedge dx^n=d\left(\frac{\partial \eta^i}{\partial x^j}\frac{\partial}{\partial x^j}\rfloor\frac{\partial}{\partial x^j}\rfloor dx^1\wedge\ldots\wedge dx^n\right)\,,$$

and Stokes' theorem shows that the integral of that term over $S(R)$ vanishes. Finally, the first term in (1.1.83) reproduces the ADM mass of the metric g^1_{ij}. $\square$

1.1.3 Moving hypersurfaces in space-time

By way of example, we consider a family of hypersurfaces $\mathscr{S}_\tau$ in Minkowski space-time given by the equation

$$\mathscr{S}_\tau = \{t = f(\tau, \vec{x})\} ,$$

with

$$f(\tau) = \tau + ar^{1/2}$$

for r large. We then have

$$\begin{aligned}\eta &= -dt^2 + dr^2 + r^2 d\Omega^2 \\ &= (d\tau + \frac{a}{2} r^{-1/2} dr)^2 + dr^2 + r^2 d\Omega^2 \\ &= d\tau^2 + ar^{-1/2} d\tau\, dr + (1 - \frac{a^2}{4r}) dr^2 + r^2 d\Omega^2 .\end{aligned}$$

This shows that, for r large, the metric induced on the $\mathscr{S}_\tau$'s reads

$$(1 - \frac{a^2}{4r}) dr^2 + r^2 d\Omega^2 . \qquad (1.1.84)$$

At leading order this is the same as a Schwarzschild metric with mass parameter $-a^2/8$, so that the ADM mass of the slices $\mathscr{S}_\tau$ is negative and equals

$$m_{ADM} = -\frac{a^2}{8} .$$

This example shows that deforming a hypersurface in space-time might lead to a change of mass. The fact that this can happen should already have been clear from the Kasner example (1.1.39), where the space-time itself does not satisfy any asymptotic flatness conditions. But this might seem a little more surprising in Minkowski space-time, which is flat. It should be emphasised that the (strictly negative) mass of the $\mathscr{S}_\tau$'s is not an artifact of a funny coordinate system chosen on $\mathscr{S}_\tau$: indeed, Theorem 1.1.12 shows that m is a geometric invariant of the geometry of $\mathscr{S}_\tau$. Further, one could suspect that negativity of m arises from singularities of the $\mathscr{S}_\tau$'s arising from the singular behaviour of $r^{-1/2}$ at $r = 0$. However, this is not the case, since we are free to modify f at will for r smaller than some constant R to obtain globally smooth spacelike hypersurfaces.

A somewhat similar behavior can be seen when Lemaître coordinates $(\tau, \rho, \theta, \varphi)$ are used in Schwarzschild space-time: in this coordinate system the Schwarzschild metric takes the form [172]

$$ds^2 = -d\tau^2 + \left(\frac{\partial Y}{\partial \rho} d\rho\right)^2 + Y^2 d\Omega^2 , \qquad (1.1.85)$$

with

$$Y = \left(3\sqrt{\frac{m}{2}}\tau + \rho^{3/2}\right)^{2/3} .$$

On any fixed hypersurface $\tau = \text{const}$ we can replace ρ by a new radial coordinate Y; Equation (1.1.85) shows then that the slices $\tau = \text{const}$ are flat, hence have zero mass.

All the foliations we have been considering above have mass which does not depend upon the slice $\mathscr{S}_\tau$. This is not true in general, consider a new time τ in Minkowski space-time which, for r large, is given by the formula

$$\tau = \frac{t}{1+ar^{1/2}}\ .$$

We then have

$$\begin{aligned}\eta &= -d\Big(\tau(1+ar^{1/2})\Big)^2 + dr^2 + r^2 d\Omega^2\\ &= -(1+ar^{1/2})^2 dt^2 - (a+r^{-1/2})a\tau\, d\tau dr + \Big(1-\frac{\tau^2a^2}{4r}\Big)dr^2 + r^2d\Omega^2\ .\end{aligned}$$

It follows that the ADM mass of the slices $\tau = \text{const}$ is well defined, equal to

$$m_{ADM} = -\frac{\tau^2a^2}{8}\ ,$$

which clearly changes when going from one slice to another.

1.1.4 Alternative expressions

In this section we review some space-time expressions for the total energy. Let X^μ_∞ be a set of constants, and let X be any vector field on M such that

$$X = \Big(X^\mu_\infty + O(r^{-\alpha})\Big)\partial_\mu$$

in a coordinate system x^μ on the asymptotic region such that $\mathscr{S}$ is given by the equation $x^0 = 0$, and such that on $\mathscr{S}$ we have

$$g_{\mu\nu} = \eta_{\mu\nu} + O(r^{-\alpha})\ , \quad \partial_\sigma g_{\mu\nu} = O(r^{-\alpha-1})\ , \quad \partial_\sigma\partial_\rho g_{\mu\nu} = O(r^{-\alpha-2})\ , \tag{1.1.86}$$

with, as usual, $\alpha > (n-2)/2$. In dimension $n = 3$, a calculation leads to [54]

$$p_\mu X^\mu_\infty = \lim_{R\to\infty}\frac{3}{16\pi}\int_{S(R)} \delta^{\alpha\beta\gamma}_{\lambda\mu\nu} X^\nu \eta^{\lambda\rho}\eta_{\gamma\sigma}\partial_\rho g^{\sigma\mu} dS_{\alpha\beta}\ . \tag{1.1.87}$$

Here

$$\delta^{\alpha\beta\gamma}_{\lambda\mu\nu} := \delta^\alpha_{[\lambda}\delta^\beta_\mu\delta^\gamma_{\nu]}\ , \qquad dS_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}dx^\gamma\wedge dx^\delta\ ,$$

with $\epsilon_{0123} = \sqrt{|\det g|}$, *etc.*

Expression (1.1.87) is well suited for the proof that p_μ is invariant under a certain class of coordinate transformations, closely related to those considered in Section 1.1.2: Indeed, suppose that the x^μ's have been replaced by new coordinates y^μ such that

$$y^\mu = x^\mu + \zeta^\mu\ ,$$

with ζ^μ satisfying fall-off conditions analogous to, though somewhat stronger than those of Lemma 1.1.13:

$$|\zeta^\mu{}_{,\nu}(x)| \le C(r(x)+1)^{-\alpha}\ , \quad |\zeta^\mu{}_{,\nu\rho}(x)| \le C(r(x)+1)^{-\alpha-1}\ , \tag{1.1.88}$$

$$|\zeta^\mu(x) - \mathring{A}^\mu| \le \begin{cases} C(\ln r(x)+1)\ , & \alpha = 1,\\ C(r(x)+1)^{1-\alpha}\ , & \text{otherwise,}\end{cases} \tag{1.1.89}$$

for some constants $\mathring{A}^\mu$. This leads to a change of the metric as in (1.1.82),

$$g_{\mu\nu} \longrightarrow g_{\mu\nu} + \zeta_{\mu,\nu} + \zeta_{\nu,\mu} + O(r^{-2\alpha-1}) ,$$

with $\zeta_\mu = \eta_{\mu\nu}\zeta^\nu$. Further, up to terms which obviously do not contribute in the limit,

$$\Delta(p_\mu X^\mu_\infty) = \lim_{R\to\infty} \frac{3}{16\pi} \int_{S(R)} (\delta^{\alpha\beta\gamma}_{\lambda\mu\nu} X^\nu \eta^{\lambda\rho} \zeta^\mu{}_{,\rho})_{,\gamma} dS_{\alpha\beta} = 0 , \qquad (1.1.90)$$

as the integral of a total divergence integrates out to zero.

The calculations that follow have been carried-out in dimension three, and we have not checked their validity in higher dimensions. With some effort one finds the identity, essentially due to Ashtekar and Hansen [7] (compare [54])

$$\begin{aligned} p_\mu X^\mu_\infty &= \lim_{R\to\infty} \frac{1}{32\pi} \Big(\int_{S(R)} \epsilon_{\mu\nu\alpha\beta} X^\mu x^\nu R^{\alpha\beta}{}_{\rho\sigma} dx^\rho \wedge dx^\sigma \\ &\qquad + 2 \int_{S(R)} d(\epsilon_{\mu\nu\alpha\beta} X^\mu x^\nu g^{\alpha\gamma} \Gamma^\beta{}_{\gamma\rho} dx^\rho) \Big) \\ &= \lim_{R\to\infty} \frac{1}{32\pi} \int_{S(R)} \epsilon_{\mu\nu\alpha\beta} X^\mu x^\nu R^{\alpha\beta}{}_{\rho\sigma} dx^\rho \wedge dx^\sigma , \qquad (1.1.91) \end{aligned}$$

since the integral of the exterior derivative of a one form gives zero by Stokes' theorem.

The expression (1.1.91) looks somewhat more geometric than the more familiar ADM formula (1.1.32). However it should be remembered that the coordinate functions x^ν appearing in (1.1.91) do not transform as a vector field under coordinate changes so that one still needs to appeal to Lemma 1.1.13 to establish the geometric character of (1.1.91). On the other hand, the proof of coordinate invariance under transformations (1.1.88)-(1.1.89) is immediate, as the error introduced by the non-vectorial character of x^ν gives directly a vanishing contribution in the limit.

Formula (1.1.91) can be rewritten in a $3+1$ form, as follows: let, first $X^\mu_\infty = \delta^\mu_0$. Recalling that $\mathscr{S}$ is given by the formula $x^0 = 0$ and that p_0 is the ADM mass m we find

$$m = \lim_{R\to\infty} \frac{1}{32\pi} \int_{S(R)} \epsilon_{ijk} x^i \, {}^{(4)}R^{jk}{}_{\ell m} \, dx^\ell \wedge dx^m . \qquad (1.1.92)$$

Here we have decorated the space-time Riemann curvature tensor with a subscript four to emphasise its four-dimensional nature. Supposing that the extrinsic curvature tensor K_{ij} falls–off as $r^{-\alpha-1}$, with $\alpha > 1/2$ (which will be the case under (1.1.86)), we then find by the Gauss-Codazzi equation

$${}^{(4)}R^{jk}{}_{\ell m} = {}^{(3)}R^{jk}{}_{\ell m} + O(r^{-2\alpha-2}) , \qquad (1.1.93)$$

where ${}^{(3)}R^{jk}{}_{\ell m}$ denotes the curvature tensor of the space-metric h induced on $\mathscr{S}$ by g. Since $\alpha > 1/2$ the error terms in (1.1.93) will give no contribution in the limit $r \to \infty$ so that we finally obtain the purely three-dimensional formula

$$m = \lim_{R\to\infty} \frac{1}{32\pi} \int_{S(R)} \epsilon_{ijk} x^i \, {}^{(3)}R^{jk}{}_{\ell m} dx^\ell \wedge dx^m . \qquad (1.1.94)$$

1.1.5 Energy in stationary space-times, Komar mass

Yet another way of rewriting (1.1.91) is given by

$$p_\mu X^\mu_\infty = \lim_{R\to\infty} \frac{1}{16\pi} \int_{S(R)} X^\mu x^\nu R_{\mu\nu\alpha\beta} dS^{\alpha\beta} \,. \tag{1.1.95}$$

This is particularly convenient when X is a Killing vector, as then we have the identity (A.16.5)

$$R_{\mu\nu\alpha\beta} X^\mu = \nabla_\nu \nabla_\alpha X_\beta \,.$$

Inserting this into (1.1.95) one obtains

$$\begin{aligned} p_\mu X^\mu_\infty &= \lim_{R\to\infty} \Big(\frac{1}{16\pi} \int_{S(R)} X^{[\beta;\alpha]}{}_{;\gamma} x^\gamma dS_{\alpha\beta} \\ &= \lim_{R\to\infty} \frac{1}{16\pi} \Big(2 \int_{S(R)} X^{[\alpha;\beta]} dS_{\alpha\beta} + 3 \int_{S(R)} (X^{[\alpha;\beta} x^{\gamma]})_{;\gamma} dS_{\alpha\beta} \Big) \\ &= \lim_{R\to\infty} \frac{1}{8\pi} \int_{S(R)} X^{[\alpha;\beta]} dS_{\alpha\beta} \,. \end{aligned} \tag{1.1.96}$$

This last integral is known as the *Komar integral*. A general argument for equality of the ADM mass and of the Komar mass for stationary space-times has been first proved by Beig [19] in space-dimension three. In higher dimensions this can be established by showing that any asymptotically flat stationary metric has the same leading order behavior as the higher-dimensional Schwarzschild metric, for which the equality is straightforward.

1.2 A space-time formulation

The aim of the following two sections is to sketch a Hamiltonian derivation of the ADM energy-momentum, within the geometric Hamiltonian formalism of Kijowski and Tulczyjew [127].

1.2.1 Hamiltonian dynamics

As a motivation, consider a mechanical system of N bodies with positions q^i, $i = 1, \dots, 3N$, with Lagrange function $L(q^i, \dot q^i)$. By definition, the equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^i}\right) = \frac{\partial L}{\partial q^i} \,. \tag{1.2.1}$$

The *canonical momenta* are defined as

$$p_i := \frac{\partial L}{\partial \dot q^i} \,. \tag{1.2.2}$$

Since we have assumed that the Lagrange function has no explicit time dependence, the energy

$$H = \sum_i p_i \dot q^i - L \tag{1.2.3}$$

is conserved, and therefore provides useful information about the dynamics:

$$\begin{aligned}\dot H &= \sum(\dot p_i \dot q^i + p_i \ddot q^i) - \dot L\\ &= \sum\Big(\underbrace{\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot q^i}\Big)}_{=\frac{\partial L}{\partial q^i}}\dot q^i + \frac{\partial L}{\partial \dot q^i}\ddot q^i\Big) - \frac{\partial L}{\partial \dot q^i}\ddot q^i - \frac{\partial L}{\partial q^i}\dot q^i\\ &= 0\,. \end{aligned} \tag{1.2.4}$$

The energy also has a *Hamiltonian* interpretation, this proceeds as follows: The *configuration space* Q is defined as the collection of all positions $\{(q^i)\}$. In the simplest case above this is $\mathbb{R}^{3N}$, but more generally it is natural to assume that Q is a manifold; this is useful e.g. when constraints are present. The *phase space* is the collection of all q^i's and p_i's. This is thus $\mathbb{R}^{6N}$ for an unconstrained mechanical system of N bodies, or more generally the cotangent bundle T^*Q of the configuration space Q. The phase space carries a *canonical symplectic form*

$$\Omega := dp_i \wedge dq^i\ .$$

Assuming that the relation (1.2.2) defining the momenta p_i can be inverted to express the velocities $\dot q^i$ as functions of p_i and q^j, the energy H can then be viewed as a function on phase space. The equations of motion of the system define a vector field T on the phase space

$$T := \dot q^i \partial_{q^i} + \dot p^i \partial_{p_i}\ ,$$

where $\dot q^i$ is calculated as a function of q^j and p_i from (1.2.2), and $\dot p^i$ is calculated using (1.2.1)-(1.2.2). Thus, the flow of T provides solutions of the equations of motion. The Hamilton equations of motion

$$\dot p_i = \frac{\partial H}{\partial q^i}\ , \quad \dot q^i = -\frac{\partial H}{\partial p_i}\ ,$$

can be rewritten as an equation that ties T, H and Ω:

$$\begin{aligned} T\rfloor\Omega &= \dot p_i dq^i - \dot q^i dp_i\\ &= dH\ . \end{aligned} \tag{1.2.5}$$

We say that H *generates* T *with respect to the symplectic form* Ω. Equivalently, H is a *generating function* for the dynamics T.

Consider, now, a Lagrangean field theory with Lagrange function $L(\varphi^A, {\varphi^A}_\mu)$, where

$${\varphi^A}_\mu := \partial_\mu \varphi^A\ .$$

The field equations are

$$\partial_\mu {p_A}^\mu = \frac{\partial L}{\partial \varphi^A}\ , \text{ where } {p_A}^\mu := \frac{\partial L}{\partial {\varphi^A}_\mu}\ . \tag{1.2.6}$$

To define energy we need a replacement of the notion of time-derivative, this is provided by the Lie derivative of the fields with respect to a vector field

X. Next, the sum over the index i in (1.2.3) is replaced by a sum over the index A carried by the fields *and* integration over a hypersurface $\mathscr{S}$: Indeed, given a spacelike hypersurface $\mathscr{S}$ and a vector field X there is a natural notion of Hamiltonian generating the flow of X, given by the formula [127]

$$H(X,\mathscr{S}) = \int_{\mathscr{S}} \underbrace{(p_A{}^\mu \pounds_X \varphi^A - X^\mu L)}_{=:H^\mu(X)} \, dS_\mu \,. \tag{1.2.7}$$

The conservation law (1.2.4) of a Hamiltonian from mechanics is replaced by the vanishing of the divergence of $H^\mu(X)$ *provided that the theory is invariant under changes of coordinates.* More precisely, away from zeros of X we can always find adapted coordinates so that $X^\mu\partial_\mu = \partial_0$ (in the argument below, the zeros of X can be handled by continuity, or by taking linear combinations of vector fields). The requirement of *coordinate invariance* means that in the new coordinate system the Lagrangean depends again upon the fields only; in other words, no explicit dependence upon the coordinates is introduced when coordinate changes are performed. Now, for most fields occurring in theoretical physics, in adapted coordinates the Lie derivatives are just partial derivatives with respect to x^0, and then

$$\begin{aligned}
\partial_\mu H^\mu &= \partial_\mu (p_A{}^\mu \pounds_X \varphi^A - X^\mu L) \\
&= \partial_\mu p_A{}^\mu \partial_0 \varphi^A + p_A{}^\mu \partial_0 \partial_\mu \varphi^A - \partial_0 L \\
&= \frac{\partial L}{\partial \varphi^A} \partial_0 \varphi^A + \frac{\partial L}{\partial {\varphi^A}_\mu} \partial_0 \partial_\mu \varphi^A - \frac{\partial L}{\partial \varphi^A} \partial_0 \varphi^A - \frac{\partial L}{\partial {\varphi^A}_\mu} \partial_0 {\varphi^A}_\mu \\
&= 0 \,,
\end{aligned}$$

as claimed.

There is also a canonical symplectic form $\Omega_{\mathscr{S}}$ associated to the hypersurface $\mathscr{S}$, defined as

$$\Omega_{\mathscr{S}}((\delta_1 p_A{}^\lambda, \delta_1 \varphi^B), (\delta_2 p_C{}^\lambda, \delta_2 \varphi^D)) = \int_{\mathscr{S}} (\delta_1 p_A{}^\mu \delta_2 \varphi^A - \delta_2 p_A{}^\mu \delta_1 \varphi^A) \, dS_\mu \,. \tag{1.2.8}$$

The function $H(X,\mathscr{S})$ generates the field equations with respect to $\Omega_{\mathscr{S}}$, in a sense which is made precise in [127].

1.2.2 Hamiltonian general relativity

Let us show how those ideas can be applied to the gravitational field, described by a metric tensor $g_{\mu\nu}$, of signature $(-1,+1,\dots,+1)$ on a space-time M. For notational simplicity only the vacuum Einstein equations will be considered,

$$R_{\mu\nu}(g) = 0 \,; \tag{1.2.9}$$

the inclusion of matter fields presents no difficulties but complicates the notation.

There exist several variational principles which produce (1.2.9) [27, 75, 79, 82, 83, 106, 115, 116, 124, 151], as well as several canonical formulations of the associated theory [5, 8, 20, 24, 107, 110, 144, 158]. Our presentation is borrowed from [64], where a generalization of the background field formulation used in [53], based on a first order variational principle "covariantized" by the introduction of a background connection[2] is presented. (see also [73, Section 5] or [69, Appendix A]).

Recall that in the original variational principle of Einstein [83] one removes from the Hilbert Lagrangian [115] a coordinate-dependent divergence, obtaining thus a first order variational principle for the metric. In [53] the Hilbert Lagrangian was modified by removing a coordinate-independent divergence, which, however, did depend upon a background metric. This proceeds as follows: Consider the Ricci tensor,

$$R_{\mu\nu} = \partial_\alpha \left[\Gamma^\alpha_{\mu\nu} - \delta^\alpha_{(\mu}\Gamma^\kappa_{\nu)\kappa}\right] - \left[\Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\alpha\nu} - \Gamma^\alpha_{\mu\nu}\Gamma^\sigma_{\alpha\sigma}\right] \ , \tag{1.2.10}$$

where the Γ's are the Christoffel symbols of g. Contracting $R_{\mu\nu}$ with the contravariant density of the metric,

$$\mathfrak{g}^{\mu\nu} := \frac{1}{16\pi}\sqrt{-\det g}\ g^{\mu\nu} \ , \tag{1.2.11}$$

one obtains the following expression for the Hilbert Lagrangian density:

$$\begin{aligned}\tilde{L} &= \frac{1}{16\pi}\sqrt{-\det g} R = \mathfrak{g}^{\mu\nu}R_{\mu\nu} \\ &= \partial_\alpha\left[\mathfrak{g}^{\mu\nu}\left(\Gamma^\alpha_{\mu\nu} - \delta^\alpha_{(\mu}\Gamma^\kappa_{\nu)\kappa}\right)\right] + \mathfrak{g}^{\mu\nu}\left[\Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\alpha\nu} - \Gamma^\alpha_{\mu\nu}\Gamma^\sigma_{\alpha\sigma}\right] \ . \end{aligned} \tag{1.2.12}$$

Here we have used the metricity condition of Γ, which is equivalent to the following identity:

$$\mathfrak{g}^{\mu\nu}{}_{,\alpha} := \partial_\alpha \mathfrak{g}^{\mu\nu} = \mathfrak{g}^{\mu\nu}\Gamma^\sigma_{\alpha\sigma} - \mathfrak{g}^{\mu\sigma}\Gamma^\nu_{\sigma\alpha} - \mathfrak{g}^{\nu\sigma}\Gamma^\mu_{\sigma\alpha} \ . \tag{1.2.13}$$

Suppose now, that $B^\alpha_{\sigma\mu}$ is another symmetric connection in M, which will be used as a "background" (or "reference") connection. Denote by $r_{\mu\nu}$ its Ricci tensor. From the metricity condition (1.2.13) we similarly obtain

$$\begin{aligned}\mathfrak{g}^{\mu\nu}r_{\mu\nu} &= \partial_\alpha\left[\mathfrak{g}^{\mu\nu}\left(B^\alpha_{\mu\nu} - \delta^\alpha_{(\mu}B^\kappa_{\nu)\kappa}\right)\right] - \mathfrak{g}^{\mu\nu}\left[B^\alpha_{\sigma\mu}B^\sigma_{\alpha\nu} - B^\alpha_{\mu\nu}B^\sigma_{\alpha\sigma}\right] \\ &\quad + \mathfrak{g}^{\mu\nu}\left[\Gamma^\alpha_{\sigma\mu}B^\sigma_{\alpha\nu} + B^\alpha_{\sigma\mu}\Gamma^\sigma_{\alpha\nu} - \Gamma^\alpha_{\mu\nu}B^\sigma_{\alpha\sigma} - B^\alpha_{\mu\nu}\Gamma^\sigma_{\alpha\sigma}\right] \ . \end{aligned} \tag{1.2.14}$$

It is useful to introduce the tensor field

$$p^\alpha_{\mu\nu} := \left(B^\alpha_{\mu\nu} - \delta^\alpha_{(\mu}B^\kappa_{\nu)\kappa}\right) - \left(\Gamma^\alpha_{\mu\nu} - \delta^\alpha_{(\mu}\Gamma^\kappa_{\nu)\kappa}\right) \ . \tag{1.2.15}$$

Once the reference connection $B^\alpha_{\mu\nu}$ is given, the tensor $p^\alpha_{\mu\nu}$ encodes the entire information about the connection $\Gamma^\alpha_{\mu\nu}$:

$$\Gamma^\alpha_{\mu\nu} = B^\alpha_{\mu\nu} - p^\alpha_{\mu\nu} + \frac{2}{3}\delta^\alpha_{(\mu}p^\kappa_{\nu)\kappa} \ .$$

[2]The idea of using a background connection instead of a background metric has been advocated in [171]. However, the framework we use here is closer in spirit to the one in [53].

Subtracting Equation (1.2.14) from (1.2.12), and using the definition of $p^\alpha_{\mu\nu}$, we arrive at the equation

$$\mathfrak{g}^{\mu\nu}R_{\mu\nu} = -\partial_\alpha\left(\mathfrak{g}^{\mu\nu}p^\alpha_{\mu\nu}\right) + L\ ,$$

where

$$L := \mathfrak{g}^{\mu\nu}\left[\left(\Gamma^\alpha_{\sigma\mu} - B^\alpha_{\sigma\mu}\right)\left(\Gamma^\sigma_{\alpha\nu} - B^\sigma_{\alpha\nu}\right) - \left(\Gamma^\alpha_{\mu\nu} - B^\alpha_{\mu\nu}\right)\left(\Gamma^\sigma_{\alpha\sigma} - B^\sigma_{\alpha\sigma}\right) + r_{\mu\nu}\right]\ .$$

Since the quantity L differs by a total divergence from the Hilbert Lagrangian, the associated variational principle leads to the same equations of motion. Further, the metricity condition (1.2.13) enables us to rewrite L in terms of the first derivatives of $\mathfrak{g}^{\mu\nu}$: Indeed, replacing in (1.2.13) the partial derivatives $\mathfrak{g}^{\mu\nu}{}_{,\alpha}$ by the covariant derivatives $\mathfrak{g}^{\mu\nu}{}_{;\alpha}$, calculated with respect to the background connection B,

$$\mathfrak{g}^{\mu\nu}{}_{;\alpha} := \mathfrak{g}^{\mu\nu}\left(\Gamma^\sigma_{\alpha\sigma} - B^\sigma_{\alpha\sigma}\right) - \mathfrak{g}^{\mu\sigma}\left(\Gamma^\nu_{\sigma\alpha} - B^\nu_{\sigma\alpha}\right) - \mathfrak{g}^{\nu\sigma}\left(\Gamma^\mu_{\sigma\alpha} - B^\mu_{\sigma\alpha}\right)\ , \qquad (1.2.16)$$

we may calculate $p^\alpha_{\mu\nu}$ in terms of the latter derivatives. The final result is:

$$\begin{aligned} p^\lambda_{\mu\nu} &= \frac{1}{2}\mathfrak{g}_{\mu\alpha}\mathfrak{g}^{\lambda\alpha}{}_{;\nu} + \frac{1}{2}\mathfrak{g}_{\nu\alpha}\mathfrak{g}^{\lambda\alpha}{}_{;\mu} - \frac{1}{2}\mathfrak{g}^{\lambda\alpha}\mathfrak{g}_{\sigma\mu}\mathfrak{g}_{\rho\nu}\mathfrak{g}^{\sigma\rho}{}_{;\alpha} \\ &\quad + \frac{1}{4}\mathfrak{g}^{\lambda\alpha}\mathfrak{g}_{\mu\nu}\mathfrak{g}_{\sigma\rho}\mathfrak{g}^{\sigma\rho}{}_{;\alpha}\ , \end{aligned} \qquad (1.2.17)$$

where by $\mathfrak{g}_{\mu\nu}$ we denote the matrix inverse to $\mathfrak{g}^{\mu\nu}$. Inserting these results into the definition of L, we obtain:

$$\begin{aligned} L &= \frac{1}{2}\mathfrak{g}_{\mu\alpha}\mathfrak{g}^{\mu\nu}{}_{;\lambda}\mathfrak{g}^{\lambda\alpha}{}_{;\nu} - \frac{1}{4}\mathfrak{g}^{\lambda\alpha}\mathfrak{g}_{\sigma\mu}\mathfrak{g}_{\rho\nu}\mathfrak{g}^{\mu\nu}{}_{;\lambda}\mathfrak{g}^{\sigma\rho}{}_{;\alpha} \\ &\quad + \frac{1}{8}\mathfrak{g}^{\lambda\alpha}\mathfrak{g}_{\mu\nu}\mathfrak{g}^{\mu\nu}{}_{;\lambda}\mathfrak{g}_{\sigma\rho}\mathfrak{g}^{\sigma\rho}{}_{;\alpha} + \mathfrak{g}^{\mu\nu}r_{\mu\nu}\ . \end{aligned} \qquad (1.2.18)$$

We note the identity

$$\frac{\partial L}{\partial \mathfrak{g}^{\mu\nu}{}_{,\lambda}} = \frac{\partial L}{\partial \mathfrak{g}^{\mu\nu}{}_{;\lambda}} = p^\lambda_{\mu\nu}\ , \qquad (1.2.19)$$

which shows that the tensor field $p^\lambda_{\mu\nu}$ is the momentum canonically conjugate to the contravariant tensor density $\mathfrak{g}^{\mu\nu}$; prescribing this last object is of course equivalent to prescribing the metric. From this point of view, gravitational fields on a manifold M are sections of the bundle

$$\begin{array}{c} F = S_0T^2M \otimes \tilde{\Lambda}^{n+1}M\ \ , \\ \Big\downarrow{\scriptstyle pr_{F\to M}} \\ M \end{array} \qquad (1.2.20)$$

where S_0T^2M denotes the bundle of non-degenerate symmetric contravariant tensors over M and $\tilde{\Lambda}^{n+1}M$ is the bundle of densities. Given a background symmetric connection B on M, we take L given by Equation (1.2.18) as the Lagrangian for the theory. The canonical momentum $p^\lambda_{\mu\nu}$ is defined by Equation (1.2.17). If $\mathscr{S}$ is any piecewise smooth hypersurface in M, and if $(\delta_a p^\lambda_{\mu\nu}, \delta_a \mathfrak{g}^{\alpha\beta})$, $a = 1, 2$, are two sections over $\mathscr{S}$ of the bundle of vertical vectors tangent to the space-time phase bundle, one sets

$$\Omega_{\mathscr{S}}((\delta_1 p^\lambda_{\mu\nu}, \delta_1 \mathfrak{g}^{\alpha\beta}), (\delta_2 p^\lambda_{\mu\nu}, \delta_2 \mathfrak{g}^{\alpha\beta})) = \int_{\mathscr{S}} (\delta_1 p^\mu_{\alpha\beta} \delta_2 \mathfrak{g}^{\alpha\beta} - \delta_2 p^\mu_{\alpha\beta} \delta_1 \mathfrak{g}^{\alpha\beta})\, dS_\mu\,. \tag{1.2.21}$$

Here (see [64, 127] for details) only such variations $(\delta_a p^\lambda_{\mu\nu}, \delta_a \mathfrak{g}^{\alpha\beta})$ are allowed which arise from one-parameter families of solutions of the vacuum Einstein equations.

Under suitable boundary conditions, the dynamics of the gravitational field obtained generated by flowing, in space-time, along a vector field X is Hamiltonian with respect to this symplectic structure, with

$$H(X, \mathscr{S}) = \int_{\mathscr{S}} (p^\mu_{\alpha\beta} \pounds_X \mathfrak{g}^{\alpha\beta} - X^\mu L)\, dS_\mu\,. \tag{1.2.22}$$

This follows from the variational formula

$$\begin{aligned} -\delta H \;&=\; \int_{\mathscr{S}} \left(\pounds_X p^\lambda{}_{\mu\nu} \delta \mathfrak{g}^{\mu\nu} - \pounds_X \mathfrak{g}^{\mu\nu} \delta p^\lambda{}_{\mu\nu} \right) dS_\lambda \\ &\quad + \int_{\partial\mathscr{S}} X^{[\mu} p^{\nu]}{}_{\alpha\beta} \delta \mathfrak{g}^{\alpha\beta}\, dS_{\mu\nu}\,. \end{aligned} \tag{1.2.23}$$

This identity reduces to the desired Hamilton equations of motion in spaces of fields where the boundary integral vanishes as a result of boundary conditions. The vanishing of those integrals requires a careful case-by-case analysis: indeed, the analysis will be different for asymptotically flat space-times [53, 64], or for asymptotically anti-de Sitter space-times [70], or for boundaries at finite distance. We note that this last case has not received adequate Hamiltonian treatment so far.

Consider (1.2.22) when B is the metric connection of a given background metric $b_{\mu\nu}$, and when X is a Killing vector field of $b_{\mu\nu}$. Under those restrictions it was shown in [53] that the integrand in (1.2.22) is equal to the divergence of a "Freud-type superpotential" [87]:

$$H^\mu \;\equiv\; p^\mu_{\alpha\beta} \pounds_X \mathfrak{g}^{\alpha\beta} - X^\mu L = \partial_\alpha \mathbb{U}^{\mu\alpha}\,, \tag{1.2.24}$$

$$\mathbb{U}^{\nu\lambda} \;=\; \mathbb{U}^{\nu\lambda}{}_\beta X^\beta - \frac{1}{8\pi}\sqrt{|\det g_{\rho\sigma}|}\, g^{\alpha[\nu} \delta^{\lambda]}_\beta X^\beta{}_{;\alpha}\,, \tag{1.2.25}$$

$$\begin{aligned} \mathbb{U}^{\nu\lambda}{}_\beta \;&=\; \frac{2|\det b_{\mu\nu}|}{16\pi\sqrt{|\det g_{\rho\sigma}|}} g_{\beta\gamma} (e^2 g^{\gamma[\lambda} g^{\nu]\kappa})_{;\kappa} \\ &=\; 2\mathfrak{g}_{\beta\gamma} \left(\mathfrak{g}^{\gamma[\lambda} \mathfrak{g}^{\nu]\kappa} \right)_{;\kappa} \\ &=\; 2\mathfrak{g}^{\mu[\nu} p^{\lambda]}_{\mu\beta} - 2\delta^{[\nu}_\beta p^{\lambda]}_{\mu\sigma} \mathfrak{g}^{\mu\sigma} - \frac{2}{3} \mathfrak{g}^{\mu[\nu} \delta^{\lambda]}_\beta p^\sigma_{\mu\sigma}\,, \end{aligned} \tag{1.2.26}$$

where a semi-column denotes the covariant derivative of the metric b, square brackets denote antisymmetrization (with a factor of 1/2 when two indices are involved), as before $\mathfrak{g}_{\beta\gamma} \equiv (\mathfrak{g}^{\alpha\sigma})^{-1} = 16\pi g_{\beta\gamma}/\sqrt{|\det g_{\rho\sigma}|}$, and

$$e \equiv \frac{\sqrt{|\det g_{\rho\sigma}|}}{\sqrt{|\det b_{\mu\nu}|}} .$$

Now, the hypothesis of metricity of the background connection B is rather natural when metrics, which asymptote to a prescribed background metric $b_{\mu\nu}$, are considered. Further, the assumption in [53] that X is a Killing vector field of the background metric seems to be natural, and not overly restrictive, when X is thought of as representing time translations in the asymptotic region. Nevertheless, that last hypothesis is not adequate when one wishes to obtain simultaneously Hamiltonians for several vector fields X. Consider, for example, the problem of assigning a four-momentum to an asymptotically flat space-time — in that case four vector fields X, which asymptote to four linearly independent translations, are needed. The condition of invariance of the background metric $b_{\mu\nu}$ under a four-parameter family of flows generated by those vector fields puts then undesirable topological constraints on M. The situation is even worse when considering vector fields X, which asymptote to BMS supertranslations: in that case there are no background metrics, which are asymptotically flat and for which X is a Killing vector. This leads to the necessity of finding formulae in which neither the condition that $B^\mu_{\alpha\beta}$ is the Levi–Civita connection of some metric $b_{\alpha\beta}$, nor the condition that X is a Killing vector field of the background are imposed. It may be checked that the following generalization of Equations (1.2.24)–(1.2.25) holds[3]:

$$p^\mu_{\alpha\beta}\pounds_X\mathfrak{g}^{\alpha\beta} - X^\mu L = \partial_\alpha\mathbb{U}^{\mu\alpha} - 2\mathfrak{g}^{\beta[\gamma}\delta^{\mu]}_\sigma(X^\sigma{}_{;\beta\gamma} - B^\sigma{}_{\beta\gamma\kappa}X^\kappa) , \quad (1.2.27)$$

$$\mathbb{U}^{\nu\lambda} = \left(2\mathfrak{g}^{\mu[\nu}p^{\lambda]}_{\mu\beta} - 2\delta^{[\nu}_\beta p^{\lambda]}_{\mu\sigma}\mathfrak{g}^{\mu\sigma} - \tfrac{2}{3}\mathfrak{g}^{\mu[\nu}\delta^{\lambda]}_\beta p^\sigma_{\mu\sigma}\right)X^\beta - 2\mathfrak{g}^{\alpha[\nu}\delta^{\lambda]}_\beta X^\beta{}_{;\alpha} . \quad (1.2.28)$$

Here $B^\sigma{}_{\beta\gamma\kappa}$ is the curvature tensor of the connection $B^\sigma{}_{\beta\gamma}$.

We close this section by noting that the background connection can be completely eliminated from the relevant volume integrals. (On the other hand, we emphasize that the formula with the background metric is very convenient for most practical calculations.) For this purpose we introduce the quantity

$$A^\alpha_{\mu\nu} := \Gamma^\alpha_{\mu\nu} - \delta^\alpha_{(\mu}\Gamma^\kappa_{\nu)\kappa} , \quad (1.2.29)$$

together with its counterpart for the background metric:

$$\mathfrak{A}^\alpha_{\mu\nu} := B^\alpha_{\mu\nu} - \delta^\alpha_{(\mu}B^\kappa_{\nu)\kappa} . \quad (1.2.30)$$

It follows that $p^\alpha_{\mu\nu} = \mathfrak{A}^\alpha_{\mu\nu} - A^\alpha_{\mu\nu}$. Now, we use the formula for the Lie derivative of a connection:

$$\pounds_X B^\lambda{}_{\mu\nu} = X^\lambda{}_{;\mu\nu} - X^\sigma B^\lambda{}_{\mu\nu\sigma} .$$

[3] The calculations of [53, Appendix] are actually done without any hypotheses on the vector field X; it is only at the end that it is assumed that X is a Killing vector field of the background metric. See also [123].

Consequently, we have:

$$\mathfrak{g}^{\mu\nu}\pounds_X\, \overline{\mathcal{A}}^\lambda{}_{\mu\nu} = 2\mathfrak{g}^{\beta[\gamma}\delta^{\lambda]}_\sigma(X^\sigma{}_{;\beta\gamma} - B^\sigma{}_{\beta\gamma\kappa}X^\kappa)\ .$$

This, together with (1.2.27), shows that $H(X,\mathscr{S})$ given by (1.2.22) takes the form

$$\begin{aligned} H(X,\mathscr{S}) &= H_{\text{boundary}}(X,\mathscr{S}) + H_{\text{volume}}(X,\mathscr{S})\ , && (1.2.31)\\ H_{\text{boundary}}(X,\mathscr{S}) &= \frac{1}{2}\int_{\partial\mathscr{S}} \mathbb{U}^{\alpha\mu}\, dS_{\alpha\mu}\ , && (1.2.32)\\ H_{\text{volume}}(X,\mathscr{S}) &= -\int_{\mathscr{S}} \mathfrak{g}^{\mu\nu}\pounds_X\, \overline{\mathcal{A}}^\lambda{}_{\mu\nu}\, dS_\lambda \\ &= -\int_{\mathscr{S}} 2\mathfrak{g}^{\beta[\gamma}\delta^{\lambda]}_\sigma(X^\sigma{}_{;\beta\gamma} - B^\sigma{}_{\beta\gamma\kappa}X^\kappa)\, dS_\lambda\ . && (1.2.33) \end{aligned}$$

It follows that (1.2.23) can be rewritten as

$$\begin{aligned} -\delta H_{\text{boundary}}(X,\mathscr{S}) \ &= \ \int_{\mathscr{S}}\Big(\pounds_X\mathfrak{g}^{\mu\nu}\delta A^\lambda{}_{\mu\nu} - \pounds_X A^\lambda{}_{\mu\nu}\delta\mathfrak{g}^{\mu\nu}\Big)\, dS_\lambda \\ &\quad + \int_{\partial\mathscr{S}} X^{[\mu}p_{\alpha\beta}{}^{\nu]}\delta\mathfrak{g}^{\alpha\beta}\, dS_{\mu\nu}\ . \qquad (1.2.34) \end{aligned}$$

In this formula the Hamiltonian is a boundary integral, and the only background dependence in the right hand side of (1.2.34) is through the boundary terms.

1.2.3 Poincaré charges, Lorentz invariance

Let $\mathscr{S}$ be an n-dimensional spacelike hypersurface in a $n+1$-dimensional Lorentzian space-time $(\mathscr{M}, g)$, $n \geq 2$. Suppose that $\mathscr{M}$ contains an open set $\mathscr{U}$ with a global time coordinate t, and that $\mathscr{S}\cap\mathscr{U} = \{t = 0\}$. Assume that there exists a coordinate system x^μ covering a set which contains

$$\mathscr{S}_0 := \{x^0 = 0\ , r(x) := \sqrt{\sum (x^i)^2} > R\}\ ,$$

and assume that the tensors $g_{\mu\nu} := g(\partial_\mu, \partial_\nu)$ and $b_{\mu\nu} := b(\partial_\mu, \partial_\nu)$ satisfy along $\mathscr{S}_0$

$$b_{\mu\nu} = \eta_{\mu\nu} := \text{diag}(-1, +1, \dots, +1)\ , \qquad (1.2.35a)$$

$$|g_{\mu\nu} - b_{\mu\nu}| \leq Cr^{-\alpha}\ ,\ |\partial_\sigma g_{\mu\nu}| \leq Cr^{-\alpha-1}\ ,\ n/2 - 1 < \alpha \leq n-2\ . \qquad (1.2.35b)$$

The *ADM energy-momentum vector* is defined as

$$p_\mu(\mathscr{S}_0) := H(\partial_\mu, \mathscr{S}_0)\ . \qquad (1.2.36)$$

One checks that p_0 coincides with the ADM mass:

$$p_0 = m_{ADM} = \lim_{R\to\infty}\frac{1}{16\pi}\int_{S(R)}(g_{ik,i} - g_{ii,k})dS_k\ .$$

A further calculation shows that the p_i's can be rewritten as in (1.1.26):

$$p_i := \frac{1}{8\pi} \lim_{R\to\infty} \int_{S(R)} P_i{}^j dS_j \;, \tag{1.2.37}$$

an expression known as the *ADM momentum* of an asymptotically flat initial data set.

As pointed out in Proposition 1.1.5, the set of numbers (p_i) is well defined whenever the mass is (see Theorem 1.1.12) *provided that* one also has

$$D_j P_i{}^j \in L^1 \;. \tag{1.2.38}$$

It easily follows from Lemma 1.1.13 that the collection (p_i) transforms as a covector under asymptotic rotations.

The remaining *Lorentz charges* are defined as the Hamiltonians associated to the generators $x_\mu\partial_\nu - x_\nu\partial_\mu$, where $x_\mu := \eta_{\mu\nu}x^\nu$:

$$J_{\mu\nu}(\mathscr{S}_0) := H(x_\mu\partial_\nu - x_\nu\partial_\mu, \mathscr{S}_0) \;. \tag{1.2.39}$$

In space-time dimension four, the $J_{\mu\nu}$'s split naturally into an *angular-momentum vector* $\vec{J}$, associated with rotations, and a *center of mass vector* $\vec{c}$, associated with boosts:

$$J_i \;:=\; H(\epsilon_{ijk}x^j\partial_k, \{t=0\}) \;, \tag{1.2.40}$$

$$c_i \;:=\; H(t\partial_i + x^i\partial_t, \{t=0\}) = H(x^i\partial_t, \{t=0\}) \;. \tag{1.2.41}$$

The corresponding integrals will *not* converge without further restrictions. This is *not* unfamiliar from Newtonian theory, where finiteness of Newtonian mass,

$$\int_{\mathbb{R}^3} \rho d^3x \;.$$

does not guarantee finiteness of angular-momentum integrals:

$$\epsilon_{ijk}\int_{\mathbb{R}^3} \rho x^j v^k d^3x \;.$$

Indeed, even small velocities and masses can give arbitrarily large contributions to the angular momentum for objects located sufficiently far away from the origin. There are various ways to guarantee well defined global charges [24, 56, 158, 170], here we follow both the approach and the presentation of [61]. This is most convenient for the purposes of *initial data gluing theorems*.

Let $\Omega \subset \mathbb{R}^{1,n}$ be invariant under the transformation

$$x^\mu \to -x^\mu \;, \tag{1.2.42}$$

for any $f : \Omega \to \mathbb{R}$ we set

$$f^+(x) = \frac{1}{2}\left(f(x) + f(-x)\right) \;, \quad f^-(x) = \frac{1}{2}\left(f(x) - f(-x)\right) \;.$$

We shall henceforth only consider metrics defined on domains of coordinate systems which are invariant under (1.2.42), and we will assume that in addition to (1.2.35) we have

$$|g^-_{\mu\nu}| \le C(1+r)^{-\alpha_-}\ ,\ |\partial_\sigma(g^-_{\mu\nu})| \le C(1+r)^{-1-\alpha_-}\ , \quad \alpha_- > \alpha\ ,\ \alpha+\alpha_- > n-1\ . \tag{1.2.43}$$

We note that in dimension $n+1 = 3+1$, Equations (1.2.35) and (1.2.42) hold for the Schwarzschild metric in the usual static coordinates, with $\alpha = 1$ and α_- — as large as desired. Similarly (1.2.35), (1.2.42) hold for the Kerr metric in the Boyer-Lindquist coordinates, with $\alpha = 1$ and $\alpha_- = 2$.

Recall that a *boost-type domain* $\Omega_{R,T,\theta} \subset \mathbb{R}^{1,n}$ is defined as

$$\Omega_{R,T,\theta} := \{r > R\ ,\ |t| < \theta r + T\}\ , \tag{1.2.44}$$

with $\theta \in (0,\infty]$. We have the following:

PROPOSITION 1.2.1 *Let $g_{\mu\nu}$ be a Lorentzian metric satisfying (1.2.35) and (1.2.43) on a boost-type domain $\Omega_{R,T,\theta}$, and suppose that the coordinate components $\mathscr{T}_{\mu\nu} := \mathscr{T}(\partial_\mu, \partial_\nu)$ of the energy-momentum tensor density,*

$$\mathscr{T}_{\mu\nu} := \frac{\sqrt{|\det g_{\alpha\beta}|}}{8\pi}\left(\mathrm{Ric}_{\mu\nu} - \frac{1}{2}tr_g \mathrm{Ric}\, g_{\mu\nu}\right)\ , \tag{1.2.45}$$

satisfy

$$|\mathscr{T}_{\mu\nu}| \le C(1+r)^{-n-\epsilon}\ , \qquad |\mathscr{T}^-_{\mu\nu}| \le C(1+r)^{-n-1-\epsilon}\ , \qquad \epsilon > 0. \tag{1.2.46}$$

Let $\mathscr{S} \subset \Omega_{R,T,\theta}$ be the hypersurface $\{y^0 = 0\} \cap \Omega_{R,T,\theta}$, where the coordinates y^μ are obtained from the x^μ's by a Poincaré transformation,

$$x^\mu \to y^\mu := \Lambda^\mu{}_\nu x^\nu + a^\mu\ , \tag{1.2.47}$$

so that $\Lambda^\mu{}_\nu$ is a constant-coefficients Lorentz matrix, and a^μ is a set of constants, set $\mathscr{S}_0 := \{x^0 = 0\}$. Then:

1. *The integrals defining the "Poincaré charges" (1.2.36)-(1.2.39) of $\mathscr{S}$ and $\mathscr{S}_0$ converge.*

2. *We have*

$$\begin{aligned}(p_\mu(\mathscr{S}), J_{\mu\nu}(\mathscr{S})) \;&=\; (\Lambda_\mu{}^\alpha p_\alpha(\mathscr{S}_0), \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta J_{\alpha\beta}(\mathscr{S}_0)\\ &\quad + a_\mu \Lambda_\nu{}^\alpha p_\alpha(\mathscr{S}_0) - a_\nu \Lambda_\mu{}^\alpha p_\alpha(\mathscr{S}_0))\ .\end{aligned} \tag{1.2.48}$$

Here $\Lambda_\alpha{}^\beta := \eta_{\alpha\mu}\Lambda^\mu{}_\nu \eta^{\nu\beta}$ and

$$p_\mu(\mathscr{S}_0) = H(\partial/\partial x^\mu, \mathscr{S}_0),\ \textit{while}\ p_\mu(\mathscr{S}) = H(\partial/\partial y^\mu, \mathscr{S}),$$

similarly for $J_{\mu\nu}$.

PROOF: Recall that

$$H(X,\mathscr{S}) = \frac{1}{2}\int_{\partial\mathscr{S}} \mathbb{U}^{\alpha\beta} dS_{\alpha\beta} \ , \tag{1.2.49}$$

$$\mathbb{U}^{\nu\lambda} = \mathbb{U}^{\nu\lambda}{}_{\beta}X^{\beta} + \frac{1}{8\pi}(\sqrt{|\det g_{\rho\sigma}|}\ g^{\alpha[\nu} - \sqrt{|\det b_{\rho\sigma}|}\ b^{\alpha[\nu})X^{\lambda]}{}_{;\alpha} \ , \tag{1.2.50}$$

$$\mathbb{U}^{\nu\lambda}{}_{\beta} = \frac{2|\det b_{\mu\nu}|}{16\pi\sqrt{|\det g_{\rho\sigma}|}} g_{\beta\gamma}(e^2 g^{\gamma[\nu}g^{\lambda]\kappa})_{;\kappa} \ , \tag{1.2.51}$$

$$e = \sqrt{|\det g_{\rho\sigma}|}/\sqrt{|\det b_{\mu\nu}|} \ . \tag{1.2.52}$$

Now,

$$\int_{\{x^0=0,r=R\}} \mathbb{U}^{\alpha\beta} dS_{\alpha\beta} = 2\int_{\{x^0=0,R_0\le r\le R\}} \mathring{\nabla}_{\beta}\mathbb{U}^{\alpha\beta} dS_{\alpha} + \int_{\{x^0=0,r=R_0\}} \mathbb{U}^{\alpha\beta} dS_{\alpha\beta} \ , \tag{1.2.53}$$

with

$$16\pi\mathring{\nabla}_{\beta}\mathbb{U}^{\alpha\beta} = \mathscr{T}^{\alpha}{}_{\beta}X^{\beta} + \sqrt{|\det b|}\ \left(Q^{\alpha}{}_{\beta}X^{\beta} + Q^{\alpha\beta}\gamma\ \mathring{\nabla}_{\beta}X^{\gamma}\right) \ , \tag{1.2.54}$$

where $Q^{\alpha}{}_{\beta}$ is a quadratic form in $\mathring{\nabla}_{\sigma}g_{\mu\nu}$, and $Q^{\alpha\beta}{}_{\gamma}$ is bilinear in $\mathring{\nabla}_{\sigma}g_{\mu\nu}$ and $g_{\mu\nu}-b_{\mu\nu}$, both with bounded coefficients which are constants plus terms $O(r^{-\alpha})$. For p_{μ} and for $R \ge R_0$ one immediately obtains

$$\begin{aligned}\int_{\{x^0=0,r=R\}} \mathbb{U}^{\alpha\beta} dS_{\alpha\beta} &= \int_{\{x^0=0,r=R_0\}} \mathbb{U}^{\alpha\beta} dS_{\alpha\beta} + O(R_0^{n-2-2\alpha}) \\ &\quad + \frac{1}{8\pi}\int_{\{x^0=0,R_0\le r\le R\}} \mathscr{T}^{\alpha}{}_{\beta}X^{\beta} dS_{\alpha} && (1.2.55) \\ &= \int_{\{x^0=0,r=R_0\}} \mathbb{U}^{\alpha\beta} dS_{\alpha\beta} + O(R_0^{n-2-2\alpha}) + O(R_0^{-\epsilon}) \ . && (1.2.56)\end{aligned}$$

For $J_{\mu\nu}$ simple parity considerations lead instead to

$$\int_{\{x^0=0,r=R\}} \mathbb{U}^{\alpha\beta} dS_{\alpha\beta} = \int_{\{x^0=0,r=R_0\}} \mathbb{U}^{\alpha\beta} dS_{\alpha\beta} + O(R_0^{n-1-\alpha-\alpha_-}) + O(R_0^{-\epsilon}) \ . \tag{1.2.57}$$

Passing to the limit $R\to\infty$ one obtains convergence of $p_{\mu}(\mathscr{S}_0)$ and of $J_{\mu\nu}(\mathscr{S}_0)$. For further reference we note the formulae

$$p_{\mu}(\mathscr{S}_0) = \int_{\{x^0=0,r=R_0\}} \mathbb{U}^{\alpha\beta} dS_{\alpha\beta} + \frac{1}{16\pi}\int_{r\ge R_0} \mathscr{T}^{\mu}{}_{\nu}X^{\nu} dS_{\mu} + O(R_0^{n-2-2\alpha}) \ , \tag{1.2.58a}$$

$$J_{\mu\nu}(\mathscr{S}_0) = \int_{\{x^0=0,r=R_0\}} \mathbb{U}^{\alpha\beta} dS_{\alpha\beta} + \frac{1}{16\pi}\int_{r\ge R_0} \mathscr{T}^{\mu}{}_{\nu}X^{\nu} dS_{\mu} + O(R_0^{n-1-\alpha-\alpha_-}) \ . \tag{1.2.58b}$$

Because Lorentz transformations commute with the antipodal map (1.2.42) the boundary conditions (1.2.35) and (1.2.43) are preserved under them, and convergence of the Poincaré charges of $\mathscr{S}$ for transformations of the form (1.2.47) with $a^\mu = 0$ follows. In order to establish point 2., still for $a^\mu = 0$, we use Stokes' theorem on a set $\mathscr{T}_R$ defined as

$$\mathscr{T}_R = \{r = R, 0 \le t \le -(\Lambda^0{}_0)^{-1}\Lambda^0{}_i x^i\} \cup \{r = R, 0 \ge t \ge -(\Lambda^0{}_0)^{-1}\Lambda^0{}_i x^i\} \,, \tag{1.2.59}$$

so that the boundary $\partial\mathscr{T}_R$ has two connected components, the set $\mathscr{S}_0 \cap \{r = R\}$ and the set $\mathscr{S} \cap \{r = R\}$. This leads to

$$\int_{\mathscr{S}\cap\{r=R\}} \mathbb{U}^{\alpha\beta} dS_{\alpha\beta} = 2\int_{\mathscr{T}_R} \mathring{\nabla}_\beta \mathbb{U}^{\alpha\beta} dS_\alpha + \int_{\mathscr{S}_0\cap\{r=R\}} \mathbb{U}^{\alpha\beta} dS_{\alpha\beta} \,, \tag{1.2.60}$$

The boundary conditions ensure that the integral over $\mathscr{T}_R$ vanishes in the limit $R \to \infty$ (for p_μ this is again straightforward, while for $J_{\mu\nu}$ this follows again by parity considerations), so that

$$H(X, \mathscr{S}) \quad = \quad H(X, \mathscr{S}_0) \,. \tag{1.2.61}$$

We consider finally a translation; Stokes' theorem on the n–dimensional region

$$\{y^\mu = x^\mu + s a^\mu \,, s \in [0,1] \,, x^\mu \in \mathscr{S} \,, r(x^\mu) = R\}$$

leads again — in the limit $R \to \infty$ — to (1.2.61), in particular $H(X, \mathscr{S})$ converges. The transformation law (1.2.48) follows now from (1.2.61) by the following calculation:

$$\begin{aligned} J_{\mu\nu}(\mathscr{S}) \quad &:= \quad H(y_\mu \tfrac{\partial}{\partial y^\nu} - y_\nu \tfrac{\partial}{\partial y^\mu}, \mathscr{S},) \\ &= \quad H(y_\mu \tfrac{\partial}{\partial y^\nu} - y_\nu \tfrac{\partial}{\partial y^\mu}, \mathscr{S}_0,) \\ &= \quad H((\Lambda_\mu{}^\alpha x_\alpha + a_\mu)\Lambda_\nu{}^\beta \tfrac{\partial}{\partial x^\beta} - (\Lambda_\nu{}^\alpha x_\alpha + a_\nu)\Lambda_\mu{}^\beta \tfrac{\partial}{\partial x^\beta}, \mathscr{S}_0) \,. \end{aligned}$$

□

It is sometimes convenient to have an $(n+1)$-decomposed version of (1.2.49), in the asymptotically flat vacuum case this is easily implemented as follows: Let $(\mathscr{S}, K, g)$ be an asymptotically flat vacuum initial data set, if the data are sufficiently differentiable there exists a vacuum development $(M, {}^{n+1}g)$ of the data so that $\mathscr{S}$ can be isometrically identified with a hypersurface $t = 0$ in M, with K corresponding to the second fundamental form of $\mathscr{S}$ in $(M, {}^{n+1}g)$. We can introduce Gauss coordinates around $\mathscr{S}$ to bring ${}^{n+1}g$ to the form

$${}^{n+1}g = -dt^2 + g_t$$

where g_t is a family of Riemannian metrics on $\mathscr{S}$ with $g_0 = g$. We then set

$$b = -dt^2 + e \,,$$

where e is the Euclidean flat metric equal to diag$(+1, \dots, +1)$ in asymptotically flat coordinates on $\mathscr{S}$. Let n_b be the future directed b-unit normal to $\mathscr{S}$ and let (Y, N) be the KID determined on $\mathscr{S}$ by the b-Killing vector X; by definition,

$$X = N n_b + Y \,, \ b(n_b, Y) = 0 \ \text{ along } \mathscr{S} \,. \tag{1.2.62}$$

Since the future pointing g-unit normal to $\mathscr{S}$, say n_g, coincides with n_b, we also have

$$X = Nn_g + Y \ , \ g(n_g, Y) = 0 \ . \tag{1.2.63}$$

We define the *Poincaré charges* Q by the formula

$$Q((Y,N),(K,g)) := H(VN + Y, \mathscr{S}) \ . \tag{1.2.64}$$

Expressing the integrand of (1.2.64) in terms of K, g, as well as the first derivatives of g, one obtains

$$\begin{aligned} \int_{\{x^0=0,r=R\}} \mathbb{U}^{\alpha\beta} dS_{\alpha\beta} &= \int_{\{x^0=0,r=R_0\}} \mathbb{U}^{\alpha\beta} dS_{\alpha\beta} \\ &\quad + \frac{1}{8\pi} \int_{\{x^0=0,R_0\le r\le R\}} \big(Y^i J_i + N\rho + q\big)\, d\mu_g \ , \end{aligned} \tag{1.2.65}$$

where q is a quadratic form in $g_{ij}-\delta_{ij}$, $\partial_k g_{ij}$, and K_{ij}, with uniformly bounded coefficients whenever g_{ij} and g^{ij} are uniformly bounded. This follows immediately from (1.2.53)-(1.2.54), together with the $n+1$ decomposition of the energy-momentum tensor density (1.2.45), and of the error term in (1.2.54). One can also work directly with the $n+1$ equivalents of the boundary integrals in (1.2.65) — *cf., e.g.*, [24] — but those are somewhat cumbersome when studying behavior of the charges under Lorentz transformations.

1.3 Mass of asymptotically anti-de Sitter space-times

We review here the Hamiltonian definition of mass in asymptotically anti-de Sitter space-times of [69, 70]; see [112] for an alternative Hamiltonian treatment of that case. Such space-times require the cosmological constant Λ to be strictly negative, which is assumed throughout this section.

Let $\mathscr{S}$ be an n-dimensional spacelike hypersurface in a $(n+1)$-dimensional Lorentzian space-time $(\mathscr{M}, g)$. Suppose that $\mathscr{M}$ contains an open set $\mathcal{U}$ which is covered by a finite number of coordinate charts (t, r, v^A), with $r \in [R, \infty)$, and with (v^A) — local coordinates on some compact $(n-1)$-dimensional manifold N, such that $\mathscr{S} \cap \mathcal{U} = \{t = 0\}$. Assume that the metric g approaches a background metric b of the form

$$b = -a^{-2}(r)dt^2 + a^2(r)dr^2 + r^2 h \ , \qquad h = h_{AB}(v^C)dv^A dv^B \ , \tag{1.3.1}$$

with $a(r) = 1/\sqrt{r^2/\ell^2 + k}$, where $k = 0, \pm 1$, h is a Riemannian Einstein metric on N with Ricci scalar $n(n-1)k$, and ℓ is a strictly positive constant related to the cosmological constant Λ by the formula $2\Lambda = -n(n-1)/\ell^2$. For example, if h is the standard round metric on S^2 and $k = 1$, then b is the anti-de Sitter metric. It seems that the most convenient way to make the approach rates precise is to introduce an orthonormal frame for b,

$$e_0 = a(r)\partial_t \ , \qquad e_1 = \frac{1}{a(r)}\partial_r \ , \quad e_A = \frac{1}{r}\beta_A \ , \tag{1.3.2}$$

with β_A — an h-orthonormal frame on (N, h), so that $b_{ab} = b(e_a, e_b) = \eta_{ab}$ — the usual Minkowski matrix diag$(-1, +1, \cdots, +1)$. We then require that the frame components g_{ab} of g with respect to the frame (1.3.2) satisfy

$$e^{ab} = O(r^{-\beta}) , \quad e_a(e^{bc}) = O(r^{-\beta}) , \quad b_{ab}e^{ab} = O(r^{-\gamma}) , \tag{1.3.3}$$

where $e^{ab} = g^{ab} - b^{ab}$, with

$$\beta > n/2 , \qquad \gamma > n . \tag{1.3.4}$$

(The $(n + 1)$-dimensional generalizations of the Kottler metrics (sometimes referred to as "Schwarzschild-anti de Sitter" metrics) are of the form (1.3.1) with

$$a(r) = 1/\sqrt{r^2/\ell^2 + k - 2\eta/r}$$

for a constant η, and thus satisfy (1.3.3) with $\beta = n$, and with $\gamma = 2n$.) One can check (*cf.* [70]) that we have the following asymptotic behaviour of the frame components of the b-Killing vector fields,

$$X^a = O(r) , \quad \mathring{\nabla}_a X^b = O(r) .$$

Assuming that $\mathscr{L}_X p^\lambda{}_{\mu\nu}$ and $\mathscr{L}_X \mathfrak{g}^{\mu\nu}$ have the same asymptotic behaviour as $\delta p^\lambda{}_{\mu\nu}$ and $\delta\mathfrak{g}^{\mu\nu}$ (which is equivalent to requiring that the dynamics preserves the phase space), it is then easily seen that under the asymptotic conditions (1.3.3)-(1.3.4) the volume integrals appearing in Equations (1.2.22)-(1.2.23) are convergent, the undesirable boundary integral in the variational formula (1.2.23) vanishes, so that the integrals (1.2.49) do indeed provide Hamiltonians on the space of fields satisfying (1.3.3)-(1.3.4). (Assuming (1.3.3)-(1.3.4) and $X = \partial_t$, the numerical value of the integral (1.2.49) coincides with that of an expression proposed by Abbott and Deser [1]). This singles out the charges (1.2.49) amongst various alternative expressions because Hamiltonians are uniquely defined, up to the addition of a constant, on each path connected component of the phase space. The key advantage of the Hamiltonian approach is precisely this uniqueness property, which does not seem to have a counterpart in the Noether charge analysis [176] (*cf.*, however [121, 181]), or in Hamilton-Jacobi type arguments [41].

To define the integrals (1.2.49) we have fixed a model background metric b, as well as an orthonormal frame as in (1.3.2); this last equation required the corresponding coordinate system (t, r, v^A) as in (1.3.1). Hence, the background structure necessary for our analysis consists of a *background metric* and a *background coordinate system*. This leads to a *potential coordinate dependence* of the integrals (1.2.49): let g be any metric such that its frame components g^{ab} tend to η^{ab} as r tends to infinity, in such a way that the integrals $H(\mathscr{S}, X, b)$ given by (1.2.49) converge. Consider another coordinate system $(\hat{t}, \hat{r}, \hat{v}^A)$ with the associated background metric $\hat{b}$:

$$\hat{b} = -a^{-2}(\hat{r})d\hat{t}^2 + a^2(\hat{r})d\hat{r}^2 + \hat{r}^2\hat{h} , \qquad \hat{h} = h_{AB}(\hat{v}^C)d\hat{v}^A d\hat{v}^B ,$$

together with an associated frame $\hat{e}^a$,

$$\hat{e}_0 = a(\hat{r})\partial_{\hat{t}} , \qquad \hat{e}_1 = \frac{1}{a(\hat{r})}\partial_{\hat{r}} , \quad \hat{e}_A = \frac{1}{\hat{r}}\hat{\beta}_A , \tag{1.3.5}$$

and suppose that in the new hatted coordinates the integrals defining the Hamiltonians $H(\widehat{\mathscr{S}}, \hat{X}, \hat{b})$ converge again. An obvious way of obtaining such coordinate systems is to make a coordinate transformation

$$t \to \hat{t} = t + \delta t \ , \ r \to \hat{r} = r + \delta r \ , \ v^A \to \hat{v}^A = v^A + \delta v^A \ , \tag{1.3.6}$$

with $(\delta t, \delta r, \delta v^A)$ decaying sufficiently fast:

$$\begin{aligned} \hat{t} &= t + O(r^{-1-\beta}) \ , & e_a(\hat{t}) &= \ell\, \delta^0_a + O(r^{-1-\beta}) \ , \\ \hat{r} &= r + O(r^{1-\beta}) \ , & e_a(\hat{r}) &= \tfrac{\delta^1_a}{\ell} + O(r^{1-\beta}) \ , \\ \hat{v}^A &= v^A + O(r^{-1-\beta}) \ , & e_a(\hat{v}^A) &= \delta^A_a + O(r^{-1-\beta}) \ , \end{aligned} \tag{1.3.7}$$

and with analogous conditions on second derivatives; this guarantees that the hatted analogue of Equations (1.3.3) and (1.3.4) will also hold. In [70] the following is proved:

- All backgrounds satisfying the requirements above and preserving $\mathscr{S}$ (so that $\hat{t} = t$) differ from each other by a coordinate transformation of the form (1.3.7). Equivalently, coordinate transformations compatible with our fall-off conditions are compositions of (1.3.7) with an isometry of the background. (This is the most difficult part of the work in [70].)

- Under the coordinate transformations (1.3.7) the integrals (1.2.49) remain unchanged:
$$H(\mathscr{S}, X, b) = H(\widehat{\mathscr{S}}, \hat{X}, \hat{b}) \ .$$
Here, if $X = X^\mu(t, r, v^A)\partial_\mu$, then the vector field $\hat{X}$ is defined using the *same* functions X^μ of the *hatted* variables.

- The conditions (1.3.4) are optimal[4], in the sense that allowing $\beta = n/2$ leads to a background-dependent numerical value of the Hamiltonian.

- For some topologies of N, isometries of b lead to interesting, non-trivial transformation properties of the mass integrals $H(\mathscr{S}, X, b)$, which have to be accounted for when defining a single number called mass. More precisely, if N is negatively curved, a geometric invariant is obtained by setting
$$m = H(\mathscr{S}, \partial_t, b) \ . \tag{1.3.8}$$
If N is a flat torus, then any choice of normalization of the volume of N leads again to an invariant via (1.3.8). If $N = S^{n-1}$, then the group G of isometries of b preserving $\{t = 0\}$ is the Lorentz group $O(n, 1)$, which acts on the space $\mathscr{K}^\perp$ of b-Killing vectors normal to $\{t = 0\}$ through its usual defining representation, in particular $\mathscr{K}^\perp$ is equipped in a natural way with a G-invariant Lorentzian scalar product $\eta^{(\mu)(\nu)}$. Choosing a basis $X_{(\mu)}$ of $\mathscr{K}^\perp$ and setting
$$m_{(\mu)} = H(\mathscr{S}, X_{(\mu)}, b) \ , \tag{1.3.9}$$
the invariant mass is obtained by calculating the Lorentzian norm of $m_{(\mu)}$:
$$m^2 := |\eta^{(\mu)(\nu)} m_{(\mu)} m_{(\nu)}| \ . \tag{1.3.10}$$

[4]Strictly speaking, it is the Riemannian counterpart of (1.3.4) that is optimal, see [63].

1.4 The mass of asymptotically hyperboloidal Riemannian manifolds

As seen above, in the asymptotically flat case the mass is an object that can be defined purely in Riemannian terms, *i.e.*, without making any reference to a space-time. This remains true in the asymptotically hyperboloidal case. However, the situation is somewhat more delicate there, because the transcription of the notion of a *space-time background Killing vector field* in a purely Riemannian setting requires more care. The Riemannian information carried by space-time Killing vector fields of the form $X = Ve_0$, where e_0 is a unit normal to the hypersurface $\mathscr{S}$, is encoded in the function V, which for static vacuum backgrounds satisfies the set of equations

$$\Delta_b V + \lambda V = 0 \, , \tag{1.4.1}$$

$$\mathring{D}_i \mathring{D}_j V = V(\mathrm{Ric}(b)_{ij} - \lambda b_{ij}) \, , \tag{1.4.2}$$

where $\mathring{D}$ is the Levi-Civita covariant derivative of b and λ is a constant. We can forget now that $\mathscr{S}$ is a hypersurface in some space-time, and consider an n-dimensional Riemannian manifold $(\mathscr{S}, g)$ together with the set, denoted by $\mathscr{N}_b$, of solutions of (1.4.1)-(1.4.2); we shall assume that $\mathscr{N}_b \neq \emptyset$. If one imposes boundary conditions in the spirit of Equations (1.3.2)-(1.3.4) on the Riemannian metric g, *except that the condition there on the space-time trace* $b_{ab}g^{ab}$ *is not needed any more*, then well defined global geometric invariants can be extracted — in a way similar to that discussed at the end of the previous section — from the integrals

$$H(V,b) := \lim_{R\to\infty} \int_{r=R} \mathbb{U}^i(V) dS_i \tag{1.4.3}$$

where $V \in \mathscr{N}_b$ and [63]

$$\mathbb{U}^i(V) := 2\sqrt{\det g} \, \left(V g^{i[k} g^{j]l} \mathring{D}_j g_{kl} + D^{[i} V g^{j]k} (g_{jk} - b_{jk}) \right) \, . \tag{1.4.4}$$

If N is an $(n-1)$-dimensional sphere, and if the manifold $\mathscr{S}$ admits a spin structure, then a positive energy theorem holds [2, 63, 102, 185, 189]; this isn't true anymore for general N's, *cf.*, *e.g.*, [117].

1.5 Quasi-local mass

The purpose of this section is to give a very succint review of the question of localisation of mass in general relativity. This has a long history, with no unanimously accepted candidate emerging so far, see [173, 176] and references therein. There are at least two strategies which one might adopt here: trying to isolate a mathematically interesting object, or trying to find a physically relevant one. In the best of the worlds the same quantity would result, but no such thing has been found yet.

1.5.1 Hawking's quasi-local mass

An early proposal for a definition of quasi-local mass is due to Hawking. This definition turned out to play a key role in the Geroch-Huisken-Ilmanen proof of the Penrose inequality [99, 119, 120]. The definition can be described as follows: Consider a spacelike two dimensional surfaces S in a four-dimensional Lorentzian manifold (M, g). At each point $p \in S$ there exist exactly two null directions orthogonal to S, spanned on null future directed vectors n^+ and n^-. Both n^+ and n^- are defined only up to a multiplicative factor, but half of this freedom can be gotten rid of by requiring that

$$g(n^+, n^-) = -2 \,. \tag{1.5.1}$$

Let $\theta^\pm$ denote the divergences of the null hypersurfaces emanating from S tangentially to $n^\pm$. When n^+ is replaced by fn^+, for a function f, then θ^+ is multiplied by f; similarly for θ^-. However, the product $\theta^+\theta^-$ remains invariant under such rescalings as long as the normalisation (1.5.1) is maintained. It follows that the *Hawking mass* $m_H(S)$,

$$m_H(S) \;=\; \sqrt{\frac{A}{16\pi}}\left(1 - \frac{1}{16\pi}\int_S \theta^-\theta^+ d^2\mu\right) \,, \tag{1.5.2}$$

is a well defined geometric invariant of S.

In the special case when S lies within a spacelike hypersurface $\mathscr{S}$ with vanishing extrinsic curvature, (1.5.2) reduces to

$$m_H(S) \;=\; \sqrt{\frac{A}{16\pi}}\left(1 - \frac{1}{16\pi}\int_S H^2 d^2\mu\right) \,, \tag{1.5.3}$$

with H the mean extrinsic curvature of S within $\mathscr{S}$. When $S = S_r$ is a coordinate sphere of radius r in an asymptotically flat region, u passing with r to infinity one recovers the ADM mass.

1.5.2 Kijowski's quasi-local mass

From a physical point of view the strongest case can be made for definitions obtained by Hamiltonian methods. Recall that the geometric symplectic framework of Kijowski and Tulczyjew [127], briefly described in Section 1.2.2, has been applied to general relativity by Kijowski and collaborators [53, 64, 124–126]. The framework allows a systematic treatment of boundary terms, together with associated Hamiltonians, at least at a formal level[5]. One of the Hamiltonians that emerges in this way is the following [126]: Consider a three dimensional initial data set (M, g, K) in a four-dimensional space-time $(\mathscr{M}, {}^4g)$.

[5]Kijowski's analysis leads to symplectic structures on spaces of fields with prescribed boundary data. To obtain a bona fide Hamiltonian system one should prove that the resulting initial-boundary value problems are well posed, which has not been done so far for boundaries at finite distance. It would be of interest to analyse how the Friedrich-Nagy [90] initial-boundary value problems fits into this framework.

Let Σ be a two dimensional surface within $\mathscr{M}$ and suppose that the mean extrinsic curvature vector κ of Σ is *spacelike*. Let

$$\lambda := \sqrt{{}^4g(\kappa, \kappa)}$$

be the ${}^4g$-length of κ. Assuming that the dominant energy condition holds in $(\mathscr{M}, {}^4g)$, it follows from the embedding equations that the Gauss curvature of the metric induced by ${}^4g$ on Σ is positive. One can then invoke the Weyl embedding theorem [148, 156] to isometrically embed $(\Sigma, {}^4g|_\Sigma)$ into $\mathbb{R}^3$. We shall denote by λ_0 the associated λ as calculated using the flat metric in $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$. Let m_{K} be the *Kijowski mass* of Σ;

$$m_{\mathrm{K}} = \frac{1}{8\pi}\int_\Sigma (\lambda_0 - \lambda) d^2\mu \; . \tag{1.5.4}$$

A surprising theorem of Liu and Yau [134] asserts that

$$m_{\mathrm{K}} \geq 0 \; ,$$

with equality if and only if (M, g, K) *is a subset of Euclidean* $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$. The key to the proof is a similar result by Shi and Tam [166], which is the Riemannian analogue of this statement: Shi and Tam prove that for manifolds of positive scalar curvature, the mean curvature H of a convex surface bounding a compact set satisfies

$$m_{\mathrm{BY}} = \frac{1}{8\pi}\int_\Sigma (H_0 - H) d^2\mu \geq 0 \; . \tag{1.5.5}$$

Here H_0 is the mean curvature of an isometric embedding of ∂M into $\mathbb{R}^3$, thus H_0 coincides with λ_0. (This "quasi-local mass" has been introduced by Brown and York [40, 41].) Liu and Yau show that positivity of Kijowski's mass (1.5.4) can be reduced to the Shi-Tam inequality using Jang's equation, in a way somewhat similar to the transition from the "Riemannian" to the "full" Schoen-Yau positive mass theorems [163, 164].

In [143] O'Murchadha, Szabados and Tod show that $m_{\mathrm{K}} \neq 0$ for some surfaces in Minkowski space-time, showing that the normalisation in (1.5.4) is not optimal. This issue has been adressed in [182, 183], the reader is referred to those works as well as [48, 184] for further information and results.

1.5.3 Bartnik's quasi-local mass

Yet another definition of quasi-local mass $m_B(\Omega)$ has been given by Bartnik in [11], a variation of this definition due to Huisken-Ilmanen [120] proceeds as follows: Consider a compact manifold (Ω, g) with smooth boundary and with nonnegative scalar curvature. Let $\mathcal{PM}_o$ be the set of complete asymptotically flat manifolds (M, g) with nonnegative scalar curvature such that M contains no compact minimal surfaces except perhaps at its boundary. For $(M, g) \in \mathcal{PM}_o$, let us write $\Omega \subset\subset M$ to mean that Ω is isometrically embedded in M. Then the Bartnik quasi-local mass $m_B(\Omega)$ of Ω is defined as:

$$m_B(\Omega) = \inf_{M \in \mathcal{PM}_o} \{ m_{\mathrm{ADM}}(M) | \; \Omega \subset\subset M \} \; , \tag{1.5.6}$$

where $m_{\mathrm{ADM}}(M)$ is the ADM mass of (M, g). (Note that non-existence of minimal surfaces in M guarantees that M has at most one asymptotically flat end.)

Bartnik has suggested a list of properties that should be satisfied by a good definition m_{ql} of quasi-local mass:

1. *Positivity:* $m_{\mathrm{ql}} \geq 0$.
2. *Rigidity:* $m_{\mathrm{ql}}(\Omega) = 0$ if and only if (Ω, g) is flat.
3. *Monotonicity:* $m_{\mathrm{ql}}(\Omega) \leq m_{\mathrm{ql}}(\Omega')$ whenever $\Omega \subset\subset \Omega'$.
4. *Spherical mass:* m_{ql} should agree with the spherical mass $m(r)$, as defined in Section 2.1 (see (2.1.5) and (2.1.12)), on spherically symmetric balls or annuli.
5. *ADM limit:* m_{ql} should be asymptotic to the ADM mass.

For simplicity let us assume that $n \leq 7$, so that the positive energy theorem, in its version with boundary, holds. Then points 1 and 3 for m_B follows immediately from the definition. It is highly non-trivial to show rigidity for m_B, this has been done in [120]. Similarly points 4 and 5 follow from the work there.

A lower bound on $m_B(\Omega)$ in terms of other quantities of geometric interest has been proved in [167].

It is an open question whether the infimum in (1.5.6) is realised; this is the contents of a conjecture by Bartnik (see [12]):

Conjecture 1.5.1 (Bartnik's conjecture) *Consider (Ω, g) for which there exists at least one $(\mathring{M}, \mathring{g}) \in \mathcal{PM}_o$ satisfying $\Omega \subset\subset \mathring{M}$. Then there exists $(\hat{M}, \hat{g}) \in \mathcal{PM}_o$ on which the infimum is attained. The metric $\hat{g}$ is static outside of Ω, Lipschitz continuous across $\partial\Omega$, with inner and outer mean curvatures of $\partial\Omega$ coinciding.*

Chapter 2

Non-spinorial positive energy theorems

In this chapter we will prove positivity of energy under various restrictive conditions — spherical symmetry, axial symmetry, small data, *etc.* We will also review several positivity proofs under less restrictive conditions. Witten's positive energy proof, which requires spinors, is deferred to the next chapter.

2.1 Spherically symmetric positive energy theorem

We suppose that we are given a three-dimensional Riemannian manifold with a metric with positive scalar curvature; this will be the case if, *e.g.,* the matter energy density is positive and the trace of the extrinsic curvature tensor vanishes.

We start with the simplest case possible — that of spherical symmetry. By definition, the metric is invariant under an effective action of $G = SO(3)$, with two-dimensional principal orbits. The orbit space M/G is diffeomorphic to $\mathbb{R}$, or $[0,\infty)$, or $[0,1]$. The last case is excluded if we restrict attention to asymptotically flat manifolds. If $M/G = \mathbb{R}$, we have two asymptotic ends, with M diffeomorphic to $\mathbb{R}\times S^2$. (The alternative possibility $\mathbb{R}\times P^2$, where P^2 is the two-dimensional projective space, is excluded by the requirement of existence of an asymptotically flat region.) The model metric is the space part of the Schwarzschild metric with $m > 0$; in "isotropic coordinates":

$$g = \left(1+\frac{m}{2|x|}\right)^4 \left(\sum_{1=1}^{n}(dx^i)^2\right) , \quad |x| > 0 \, . \tag{2.1.1}$$

If M has no boundary, and if $M/G = [0,\infty)$ there are two possible topologies: the first is $\mathbb{R}^3$, with the usual action of $SO(3)$ by rotations, the model metric being the standard flat metric. The second is $([0,\infty) \times S^2)/\sim$, where the equivalence relation $\sim$ identifies $(0,p)$, $p \in S^2$, with $(0,Pp)$, where P is the antipodal map from S^2 into itself. In this case a geodesic segment $\gamma(s)$ normal to the orbits of the isometry group of the form, say, $\gamma(s) = (s,p)$ with s decreasing from one to zero, is smoothly continued by a geodesic segment (s,Pp), with s

now increasing from, say zero to one. A model metric is provided again by (2.1.1) with $|x| \geq m/2$: the reader might wish to check that the set $|x| = m/2$ is totally geodesic, and that the metric g defines a smooth metric on the quotient manifold.

If M has a boundary, the only possible asymptotically flat topology is $[0, \infty) \times S^2$.

One can always use the distance ρ between the orbits of $SO(3)$ as a coordinate on M. Because there are no non-trivial rotation-invariant vector fields on S^2, the metric is then necessarily of the form

$$g = d\rho^2 + f(\rho)d\Omega^2 , \tag{2.1.2}$$

where $d\Omega^2$ is the unit round metric on S^2.

It is a common abuse of terminology to say that a hypersurface S is *minimal* if the trace of the extrinsic curvature of S vanishes; with this terminology, S could actually be a maximum, or a saddle point, of the area functional. As an example, an orbit $\rho = \rho_0$ is minimal in the metric (2.1.2) if and only if

$$f'(\rho_0) = 0$$

(compare (2.1.10) below). We have the following:

THEOREM 2.1.1 *Consider a complete, asymptotically flat, spherically symmetric, boundaryless Riemannian manifold (M, g) with*

$$R(g) \geq 0 ,$$

and with ADM mass m. Then

$$m \geq 0 , \tag{2.1.3}$$

with equality if and only if $M = \mathbb{R}^3$ with g – the Euclidean metric. Furthermore, if M contains a spherically symmetric minimal sphere then (2.1.3) can be strengthened to

$$m \geq \sqrt{\frac{|S_{r_0}|}{16\pi}} , \tag{2.1.4}$$

where $|S_{r_0}|$ denotes the area of the outermost[1] *minimal sphere $S_{r_0} := \{r = r_0\}$. Equality in (2.1.4) holds if and only if the metric is the space Schwarzschild metric (2.1.1) in the region enclosing S_{r_0}.*

Equation (2.1.4) is the spherically symmetric case of the *Penrose inequality*.

The Schwarzschild metric with $m < 0$ shows that the hypothesis of *completeness* is necessary in the theorem.

REMARK 2.1.2 The proof below works directly when S_{r_0} is an outermost minimal boundary. Alternatively, the case of a complete manifold with a minimal boundary S can be reduced to the boundaryless case by doubling M across S; spherical symmetry ensures that the resulting metric is at least C^2, which is enough for the result at hand.

[1]This means that there are no minimal spherically symmetric spheres enclosing S_{r_0}.

PROOF: It is convenient to start with a form of the metric that is somewhat more flexible than (2.1.2):

$$g = e^{2\beta(r)}dr^2 + e^{2\gamma(r)}(d\theta^2 + \sin^2\theta d\varphi^2) \; . \tag{2.1.5}$$

The curvature scalar $R = R(g)$ is calculated in Appendix A.17 (see (A.17.28) there), with the result

$$R = -4(\gamma' e^{-\beta+\gamma})' e^{-\beta-\gamma} + 2(e^{-2\gamma} - (\gamma')^2 e^{-2\beta}) \; . \tag{2.1.6}$$

Now, by definition, a sphere is minimal if its area increases under deformations. In particular its area should be a minimum under radial deformations, which is equivalent to

$$\gamma' = 0 \; , \tag{2.1.7}$$

provided the coordinate system of (2.1.5) is well behaved. Now, for the problem at hand, it is convenient to use coordinate systems which are singular precisely at the minimal sphere, so a more geometric version of this equation is needed. Note, first that the area of the symmetry orbits is $4\pi e^\gamma$, which shows that e^γ is a smooth function on the manifold for smooth actions on the open dense set covered by orbits of principal type. Then a geometric version of the equation $\gamma' = 0$ is that

$$n^i\partial_i\gamma = 0 \; , \tag{2.1.8}$$

where $n = n^i\partial_i$ is the field of unit normals to the orbits of symmetry group. In our case

$$n^i\partial_i = e^{-\beta}\partial_r \; ,$$

and so minimality is equivalent to

$$e^{-\beta}\partial_r\gamma = 0 \; . \tag{2.1.9}$$

It follows that S_{r_0} is minimal if and only if $\gamma'(r_0) = 0$ *or* $e^{-\beta} = 0$. We emphasise that the second possibility can occur only if the coordinate r becomes singular at S_{r_0}.

Another common definition of a minimal sphere S_{r_0} is that the trace of the extrinsic curvature tensor of S_{r_0} vanishes. This last trace equals

$$D_i n^i = \frac{1}{\sqrt{\det g_{\ell m}}}\partial_i(\sqrt{\det g_{\ell m}}\, n^i) \; ,$$

where n^i is the field of unit normals to the level sets of r. To see that, recall that in adapted coordinates $x^i = (x^1, x^A)$ such that a family of submanifolds is given by the equations $\{x^1 = \text{const}\}$, the extrinsic curvature tensor is defined as

$$K_{AB} = \frac{1}{2}(D_A n_B + D_B n_A) \; ,$$

where $n^i\partial_i$ is the field of unit normals. Its trace, with respect to the induced metric $\gamma_{AB} = g_{AB} \equiv g_{AB} - n_A n_B$, is

$$\text{tr}K = \gamma^{AB}K_{AB} = (g^{ij} - n^i n^j)K_{ij} = (g^{ij} - n^i n^j)D_i n_j = D_i n^i - \frac{1}{2}\underbrace{n^i D_i(\underbrace{n^j n_j}_{1})}_{0} \; ,$$

hence $\mathrm{tr}K = D_i n^i$, as desired. For the metric (2.1.5) we have

$$n^i\partial_i = e^{-\beta}\partial_r \;, \quad \sqrt{\det g_{\ell m}} = e^{2\gamma+\beta}\sin\theta \;,$$

so that

$$D_i n^i = e^{-2\gamma-\beta}(e^{2\gamma})' = 2e^{-\beta}\gamma' \;, \tag{2.1.10}$$

and one recovers (2.1.9).

Now, for large distances, it follows from asymptotic flatness that the area of the orbits behaves as in Euclidean space-time, and thus γ' has no zeros there. Since the manifold is complete, we can shoot inwards geodesics normal to the orbits, and either reach an outermost minimal sphere at $r = r_0$, or the centre of symmetry at $r = r_0$. In either case γ' has no zeros for $r \geq r_0$. We conclude that either globally, or at least for $r \geq r_0$, we can choose a new radial variable ρ so that

$$e^{\gamma(r)} = \rho \;. \tag{2.1.11}$$

The coordinate so defined is often called *the area coordinate*, since the area of the isometry-invariant spheres equals $4\pi e^{2\gamma(r)} = 4\pi\rho^2$.

We rewrite the metric in the new coordinate system (ρ, θ, φ), and change the name of the new variable ρ to r, keeping the old symbol β for the new function β appearing in the metric. It is then convenient to define yet another function $m(r)$ by the equation

$$e^{-\beta(r)} = \sqrt{1 - \frac{2m(r)}{r}} \quad \Longleftrightarrow \quad m(r) := \frac{r}{2}(1 - e^{-2\beta(r)}) \;. \tag{2.1.12}$$

In other words, we have brought the metric to the form

$$g = \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2 d\Omega^2 \;.$$

Note that from (2.1.12) we necessarily have

$$m(r) \leq \frac{r}{2} \;, \tag{2.1.13}$$

with equality if and only if S_r is minimal by (2.1.10). Further

$$r \geq 0 \;,$$

by (2.1.12), with equality possible only if $M = \mathbb{R}^3$, and $r = 0$ corresponding to the fixed point of the action of $SO(3)$. At the center of symmetry, elementary regularity considerations show that we have $e^\beta = 1$, hence $m(r) = o(r)$ for r near zero. (In fact, one must have $m(r) = O(r^3)$ for small r when the metric is C^2, but this is irrelevant for the current considerations.)

It is remarkable that the seemingly complicated formula (2.1.12) together with (2.1.6) lead to a very simple form of R:

$$R = \frac{4m'}{r^2} \;. \tag{2.1.14}$$

If there are no minimal surfaces we set $r_0 = \inf_{p\in M} r(p)$, otherwise we let r_0 be the value of r corresponding to the outermost minimal surface S_{r_0}. Viewing (2.1.14) as defining the derivative of m, one obtains

$$m(r) = m(r_0) + \frac{1}{4}\int_{r_0}^{r} Rr^2 dr \ . \tag{2.1.15}$$

Passing to the limit $r \to \infty$ we therefore conclude that, for $R \geq 0$,

$$m = m(r_0) + \frac{1}{4}\int_{r_0}^{\infty} Rr^2 dr \geq m(r_0) \ . \tag{2.1.16}$$

Now, $r_0 > 0$ is only possible if there are minimal spheres: this is due to the fact that in the absence of minimal spheres the function e^β is uniformly bounded on any compact set $[r_0, r_1]$, so that the geodesics normal to the orbits of $SO(3)$ reach S_{r_0} in a finite distance. For a complete boundaryless manifold this occurs only if S_{r_0} is a totally geodesic P^2, hence minimal. Thus, without minimal spheres we must have $r_0 = 0$, but then $m(r_0) = 0$. Further, (2.1.16) together with $R \geq 0$ show that $m = 0$ if and only if $R \equiv 0$, then $m(r) = 0$ by (2.1.15) for all r, so that $\beta \equiv 0$, and g is the Euclidean metric, as claimed.

Suppose, finally, that S_{r_0} is minimal, Equations (2.1.10)-(2.1.12) show that this is possible with $e^\gamma(r) = r$ only if $0 < r_0 = 2m(r_0)$, so that $m(r_0) > 0$ and

$$|S_{r_0}| = 4\pi r_0^2 = 16\pi (m(r_0))^2 \ ,$$

proving (2.1.4). If equality holds in (2.1.4) then R vanishes, and $m(r) = m(r_0) = m$ for all $r \in [r_0, \infty)$ by (2.1.15). We have thus proved that for $r \geq r_0$ the metric g takes the Schwarzschild form

$$g = \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\Omega^2 \ , \quad m' = 0 \ ,$$

compare (1.1.10), and the argument is complete. □

Remark 2.1.3 In dimension n, consider a metric of the form

$$g = \frac{dr^2}{\lambda r^2 + k - \frac{2m(r)}{r^{n-2}}} + r^2 h \ , \tag{2.1.17}$$

where $k, \lambda \in \{-1, 0, 1\}$ and h is a (r–independent) metric with Ricci scalar satisfying $R(h) = k(n-1)(n-2)$. As before, the obstruction to the introduction of this coordinate system is the existence of minimal surfaces. Vincent Bonini pointed out to me that Equation (2.1.14) becomes now

$$R(g) + \lambda n(n-1) = \frac{(n-1)m'}{r^{n-1}} \ . \tag{2.1.18}$$

This leads to mass inequalities, and rigidity statements, as in Theorem 2.1.1, provided that $R(g) + \lambda n(n-1) \geq 0$. This last condition is precisely the positivity condition for the energy density of matter fields in the presence of a cosmological constant $\Lambda = -n(n-1)\lambda/2$.

2.2 Axi-symmetry

In [38] Brill proved a positive energy theorem for a *certain* class of *maximal, axi-symmetric* initial data sets on $\mathbb{R}^3$. Brill's analysis has been extended independently by Moncrief (unpublished), Dain (unpublished), and Gibbons and Holzegel [104] to the following class of metrics:

$$g = e^{-2U+2\alpha}\left(d\rho^2 + dz^2\right) + \rho^2 e^{-2U}\left(d\varphi + \rho B_\rho d\rho + A_z dz\right)^2 . \qquad (2.2.1)$$

All the functions are assumed to be φ–independent.

The above form of the metric, together with Brill's formula for the mass, are the starting points of the work of Dain [77], who proves an upper bound for angular momentum in terms of the mass for a class of maximal, vacuum, axi-symmetric initial data sets with a metric of the form above.

In this section we prove the energy positivity for a class of axi-symmetric metrics, following [59]. We start by proving that any sufficiently differentiable axially symmetric metric on a simply connected manifold with a finite number of asymptotically flat ends can be written in the form (2.2.1). In general the functions appearing in (2.2.1) will not satisfy the fall-off conditions imposed in [77, 104], but we verify that the proof extends to the more general situation.

It is conceivable that, regardless of simple-connectedness and isotropy subgroups conditions, axi-symmetric metrics on manifolds obtained by blowing-up a finite number of points in a compact manifold can be represented as in (2.2.1), with the coordinates (ρ, z) ranging over a subset Ω of $\mathbb{R}^2$, and with identifications on $\partial\Omega$, but this remains to be seen; in any case it is not clear how to adapt the arguments leading to the mass and angular-momentum inequalities to such situations.

2.2.1 Axi-symmetric metrics on simply connected asymptotically flat three dimensional manifolds

Let us start with a general discussion of Riemannian manifolds (M, g) with a Killing vector η with periodic orbits; without loss of generality we can assume that the period of principal orbits is 2π.

Let $M/\mathrm{U}(1)$ denote the collection of the orbits of the group of isometries generated by η, and let $\pi : M \to M/\mathrm{U}(1)$ be the canonical projection. An orbit $p \in M/\mathrm{U}(1)$ will be called *non-degenerate* if it is *not* a point in M. Recall that near any $p \in M/\mathrm{U}(1)$ which lifts to an orbit of principal type there exists a canonical metric q defined as follows: let $X, Y \in T_p(M/\mathrm{U}(1))$, let $\hat{p} \in M$ be any point such that $\pi\hat{p} = p$, and let $\hat{X}, \hat{Y} \in T_{\hat{p}}M$ be the unique vectors orthogonal to η such that $\pi_*\hat{X} = X$ and $\pi_*\hat{Y} = Y$. Then

$$q(X, Y) := g(\hat{X}, \hat{Y}) . \qquad (2.2.2)$$

(The reader will easily check that the right-hand-side of (2.2.2) is independent of the choice of $\hat{p} \in \pi^{-1}(\{p\})$.)

There exists an open dense set of the quotient manifold $M/\mathrm{U}(1)$ which can, at least locally, be conveniently modeled on smooth submanifolds (perhaps with boundary), say N, of M, which meet orbits of η precisely once; these are called *cross-sections* of the group action. (For metrics of the form (2.2.1) there actually exists a *global* cross-section N, meeting *all* orbits precisely once.) The manifold structure of $M/\mathrm{U}(1)$ near p is then, by definition, the one arising from N. For

$$p \in \mathring{N} := N \setminus \{\eta = 0\}$$

and for $X, Y \in T_p\mathring{N}$ set

$$q(X,Y) = g(X,Y) - \frac{g(\eta,X)g(\eta,Y)}{g(\eta,\eta)} . \tag{2.2.3}$$

One easily checks that this coincides with our previous definition of q.

The advantage of (2.2.3) is that it allows us to read-off properties of q directly from those of g near N. On the other hand, the abstract definition (2.2.2) makes clear the Riemannian character of q, and does not require any specific transverse submanifold. This allows to use different N's, adapted to different problems at hand, to draw conclusions about $M/\mathrm{U}(1)$; this freedom will be made use of in what follows.

Clearly all the information about g is contained in q and in the one-form field

$$\eta^\flat := g(\eta, \cdot) ,$$

since we can invert (2.2.3) using the formula, valid for any $X, Y \in TM$,

$$g(X,Y) = q(P_\eta X, P_\eta Y) + \frac{g(\eta,X)g(\eta,Y)}{g(\eta,\eta)} , \tag{2.2.4}$$

where $P_\eta : TM \to T\mathring{N}$ is the projection from TM to $T\mathring{N}$ along η. (Recall that P_η is defined as follows: since η is transverse to $T\mathring{N}$, every vector $X \in TM$ can be uniquely written as $X = \alpha\eta + Y$, where $Y \in T\mathring{N}$, then one sets $P_\eta X := Y$.) In order to establish (2.2.4) note, first, that this is only a rewriting of (2.2.3) when both X and Y are tangent to $\mathring{N}$. Next, (2.2.4) is an identity if either X or Y is proportional to η, and the result easily follows.

Let x^A, $A = 1, 2$ be any local coordinates on $\mathring{N}$, propagate them off $\mathring{N}$ by requiring that $\mathscr{L}_\eta x^A = 0$, and let φ be a coordinate that vanishes on $\mathring{N}$ and satisfies $\mathscr{L}_\eta \varphi = 1$. Then $\eta = \partial_\varphi$, and $P_\eta(X^A\partial_A + X^\varphi\partial_\varphi) = X^A\partial_A$, so that (2.2.4) can be rewritten as

$$g = \underbrace{q_{AB}\, dx^A dx^B}_{q} + g(\eta,\eta)(d\varphi + \underbrace{\tilde\theta_A dx^A}_{=:\tilde\theta})^2 , \tag{2.2.5}$$

with

$$\partial_\varphi q_{AB} = \partial_\varphi \tilde\theta_A = \partial_\varphi(g(\eta,\eta)) = 0 .$$

2.2.2 Global considerations

So far our considerations were completely general, but local. Suppose, however, that M is *simply connected*, with or without boundary, and satisfies the usual condition that it is the union of a compact set and of a finite number of asymptotically flat ends. Then every asymptotic end can be compactified by adding a point, with the action of U(1) extending to the compactified manifold in the obvious way. Similarly every boundary component has to be a sphere [111, Lemma 4.9], which can be filled in by a ball, with the action of U(1) extending in the obvious way, reducing the analysis of the group action to the boundaryless case. Existence of asymptotically flat regions implies (see, e.g., [22]) that the set of fixed points of the action is non-empty. It is then shown in [157] that, after the addition of a ball to every boundary component if necessary, M is homeomorphic to $\mathbb{R}^3$, with the action of U(1) conjugate, by a homeomorphism, to the usual rotations of $\mathbb{R}^3$. On the other hand, it is shown in [150] that the actions are classified, up to smooth conjugation, by topological invariants. It follows that the action is in fact smoothly conjugate to the usual rotations of $\mathbb{R}^3$. In particular there exists a global cross-section $\mathring{N}$ for the action of U(1) away from the set of fixed points $\mathscr{A}$, with $\mathring{N}$ diffeomorphic to an open half-plane, with all isotropy groups trivial or equal to U(1), and with $\mathscr{A}$ diffeomorphic to $\mathbb{R}$.[2]

Somewhat more generally, the above analysis applies whenever M can be compactified by adding a finite number of points or balls. A nontrivial example is provided by manifolds with a finite number of asymptotically flat and asymptotically cylindrical ends, as is the case for the Cauchy surfaces for the domain of outer communication of the extreme Kerr solution.

2.2.3 Regularity at the axis

In the coordinates of (2.2.1) the *rotation axis*

$$\mathscr{A} := \{g(\eta,\eta) = 0\}$$

corresponds to the set $\rho = 0$, which for asymptotically flat metrics is never empty, see, *e.g.,* the proof of Proposition 2.4 in [22].

In order to study the properties of q near $\mathscr{A}/\mathrm{U}(1) \approx \mathscr{A}$, recall that $\mathscr{A}$ is a geodesic in M. It is convenient to introduce normal coordinates $(\hat{x},\hat{y},\hat{z}) : \mathscr{U} \to \mathbb{R}^3$ defined on an open neighborhood $\mathscr{U}$ of $\mathscr{A}$, where $\hat{z}$ is a unit-normalized affine parameter on $\mathscr{A}$, and $(\hat{x},\hat{y})$ are geodesic coordinates on $\exp((T\mathscr{A})^\perp)$. Without loss of generality we can assume that $\mathscr{U}$ is invariant under the flow of η.

As is well known, we have (recalling that orbits of principal type form an open and dense set of M, as well as our normalization of 2π–periodicity of the principal orbits)

$$\eta = \hat{x}\partial_{\hat{y}} - \hat{y}\partial_{\hat{x}}\ .$$

If we denote by ϕ_t the flow of η, on $\mathscr{U}$ the map ϕ_π is therefore the symmetry across the axis $\mathscr{A}$:

$$\phi_\pi(\hat{x},\hat{y},\hat{z}) = (-\hat{x},-\hat{y},\hat{z})\ .$$

[2] I am grateful to João Lopez Costa and Allen Hatcher for discussions and comments on the classification of U(1) actions.

This formula has several useful consequences. First, it follows that the manifold with boundary

$$N := \{\hat{x} \geq 0, \hat{y} = 0\} \subset \mathscr{U}$$

is a cross-section for the action of U(1) on $\mathscr{U}$. This shows that near zeros of η the quotient space $M/\mathrm{U}(1)$ can be equipped with the structure of a smooth manifold with boundary. The analysis of the behavior of q near $\partial N \approx \mathscr{A}$ requires some work because of the factor $1/g(\eta,\eta)$ appearing in (2.2.3).

For further use we note that the manifold

$$\tilde{N} := \{\hat{y} = 0\} \subset \mathscr{U} \tag{2.2.6}$$

provides, near $\mathscr{A}$, a natural doubling of N across its boundary $\mathscr{A}$.

In order to understand the smoothness of q on N and $\tilde{N}$, we start by considering the function

$$f(\hat{x},\hat{z}) := g(\eta,\eta)(\hat{x},0,\hat{z}) \ .$$

Then $f(-\hat{x},\hat{z}) = f(\hat{x},\hat{z})$ because $g(\eta,\eta)\circ\phi_\pi = g(\eta,\eta)$. It follows that all odd x–derivatives of f vanish at $\hat{x}=0$. It is then standard to show, using Borel's summation lemma (*cf.*, *e.g.,* [58, Proposition C1, Appendix C]), that there exists a smooth function $h(s,\hat{z})$ such that

$$f(\hat{x},\hat{z}) = \hat{x}^2 h(\hat{x}^2,\hat{z}) \ .$$

Letting $\hat{\rho} = \sqrt{\hat{x}^2+\hat{y}^2}$, invariance of g under ϕ_t allows us to conclude that

$$g(\eta,\eta)(\hat{x},\hat{y},\hat{z}) = g(\eta,\eta)(\hat{\rho},0,\hat{z}) = \hat{\rho}^2 h(\hat{\rho}^2,\hat{z}) \ . \tag{2.2.7}$$

Define $\hat{\varphi}$ via the equations

$$\hat{x} = \hat{\rho}\cos\hat{\varphi} \ , \quad \hat{y} = \hat{\rho}\sin\hat{\varphi} \ ,$$

so that

$$\eta = \partial_{\hat{\varphi}} \ .$$

Considerations similar to those leading to (2.2.7) (see Lemma 5.1 of [58]) show that there exist functions α, β, γ, δ, μ and $g_{\hat{z}\hat{z}}$, which are smooth *with respect to the arguments* $\hat{\rho}^2$ *and* $\hat{z}$,[3] with

$$\mu(0,\hat{z}) = 1 \ , \quad g_{\hat{z}\hat{z}}(0,\hat{z}) = 1 \ ,$$

such that

$$\begin{aligned} g &= g_{\hat{z}\hat{z}}d\hat{z}^2 + 2\alpha\hat{\rho}d\hat{z}d\hat{\rho} + 2\beta\hat{\rho}^2 d\hat{z}d\hat{\varphi} + \gamma\hat{\rho}^2 d\hat{\rho}^2 + 2\delta\hat{\rho}^3 d\hat{\rho}d\hat{\varphi} + \mu(d\hat{\rho}^2 + \hat{\rho}^2 d\hat{\varphi}^2) \\ &= \underbrace{\Big(g_{\hat{z}\hat{z}} - \frac{\beta^2\hat{\rho}^2}{\mu}\Big)d\hat{z}^2 + 2\Big(\alpha - \frac{\delta\beta\hat{\rho}^2}{\mu}\Big)\hat{\rho}d\hat{z}d\hat{\rho} + \Big(\mu + \gamma\hat{\rho}^2 - \frac{\delta^2\hat{\rho}^2}{\mu}\Big)d\hat{\rho}^2}_{q} \\ &\quad + \mu\hat{\rho}^2\Big(\underbrace{d\hat{\varphi} + \frac{\delta}{\mu}\hat{\rho}d\hat{\rho} + \frac{\beta}{\mu}d\hat{z}}_{\tilde{\theta}}\Big)^2 \ . \end{aligned} \tag{2.2.8}$$

[3]By this we mean that $\alpha(s,\hat{z})$ is a smooth function of its arguments, and enters (2.2.8) in the form $\alpha(\hat{\rho}^2,\hat{z})$, *etc.*

We say that $\hat{N}$ is a *doubling* of a manifold N across a boundary $\dot{N}$ if $\hat{N}$ consists of two copies of N with points on $\dot{N}$ identified in the obvious way. From what has been said, by inspection of (2.2.8) it follows that:

PROPOSITION 2.2.1 *The quotient space $M/\mathrm{U}(1)$ has a natural structure of manifold with boundary near $\mathscr{A}$. The metric q and the one-form $\tilde{\theta}$ are smooth up-to-boundary, and extend smoothly across $\mathscr{A}$ by continuity to themselves when $M/\mathrm{U}(1)$ is doubled at $\mathscr{A}$.*

For further use we note the formula

$$g(\eta,\eta) = \hat{\rho}^2 + O(\hat{\rho}^4) , \tag{2.2.9}$$

for small $\hat{\rho}$, which follows from (2.2.8), where $\hat{\rho}$ is either the geodesic distance from $\mathscr{A}$, or the geodesic distance from $\mathscr{A}$ on $\exp((T\mathscr{A})^\perp)$ (the latter being, for small $\hat{\rho}$, the restriction to $\exp((T\mathscr{A})^\perp)$ of the former).

2.2.4 Axisymmetry and asymptotic flatness

We will consider Riemannian manifolds (M,g) with asymptotically flat ends, in the usual sense that there exists a region $M_{\mathrm{ext}} \subset M$ diffeomorphic to $\mathbb{R}^3 \setminus B(R)$, where $B(R)$ is a coordinate ball of radius R, such that in local coordinates on M_{ext} obtained from $\mathbb{R}^3 \setminus B(R)$ the metric satisfies the fall-off conditions, for some $k \geq 1$,

$$g_{ij} - \delta_{ij} = o_k(r^{-1/2}) , \tag{2.2.10}$$
$$\partial_k g_{ij} \in L^2(M_{\mathrm{ext}}) , \tag{2.2.11}$$
$$R^i{}_{jk\ell} = o(r^{-5/2}) , \tag{2.2.12}$$

where we write $f = o_k(r^\alpha)$ if f satisfies

$$\partial_{k_1} \ldots \partial_{k_\ell} f = o(r^{\alpha-\ell}) , \quad 0 \leq \ell \leq k .$$

As shown in Theorem 1.1.12, (2.2.10)-(2.2.11) together with $R(g) \geq 0$ or $R(g) \in L^1$, where $R(g)$ is the Ricci scalar of g, guarantees a well-defined ADM mass (perhaps infinite). On the other hand, the condition (2.2.12) (which follows in any case from (2.2.10) for $k \geq 2$) is useful when analyzing the asymptotic behavior of Killing vector fields.

We will use (2.2.10)-(2.2.12) to construct the coordinate system of (2.2.4), and also to derive the asymptotic behavior of the fields appearing in (2.2.4). We start by noting that the arguments of [21, Appendix C] with $N \equiv 0$ there show that there exists a rotation matrix ω such that in local coordinates on M_{ext} we have

$$\eta^i = \omega^i{}_j x^j + o_k(r^{1/2}) , \tag{2.2.13}$$

where $\omega^i{}_j$ is anti-symmetric. It will be clear from the proof below (see (2.2.24)) that this equation provides the information needed in the region

$$x^2 + y^2 \geq z^2 , \quad x^2 + y^2 + z^2 \geq R^2 . \tag{2.2.14}$$

However, near the axis a more precise result is required, and we continue by constructing new asymptotically flat coordinates which are better adapted to the problem at hand. The difficulties arise from the need to obtain decay estimates on $q-\delta$, where δ is the Euclidean metric on $\mathbb{R}^2$, and on $\tilde\theta$ as defined in (2.2.8), which are *uniform in r up to the axis $\mathscr{A}$*.

Let $(\hat x^i)\equiv(\hat x,\hat y,\hat z)$ be coordinates on $\mathbb{R}^3\setminus B(R)$, obtained by a rigid rotation of x^i, such that $\omega^i{}_j\hat x^j=\hat y\partial_{\hat x}-\hat x\partial_{\hat y}$. Set

$$x:=\frac{\hat x-\hat x\circ\phi_\pi}{2}\;,\quad y:=\frac{\hat y-\hat y\circ\phi_\pi}{2}\;,\quad z:=\frac{1}{2\pi}\int_0^{2\pi}\hat z\circ\phi_s\,ds\;. \tag{2.2.15}$$

Using the techniques in [21, 22] one finds

$$\phi_s(\hat x^i)=(\cos(s)\hat x-\sin(s)\hat y+z^{\hat x}(s,\hat x^i),\sin(s)\hat x+\cos(s)\hat y+z^{\hat y}(s,\hat x^i),\hat z+z^{\hat z}(s,\hat x^i))\;,$$

with z^i satisfying

$$z^i=o_{k+1}(r^{1/2})\;.$$

We then have

$$\frac{\partial z}{\partial\hat z}\;=\;1+\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial z^{\hat z}(\phi_s(\hat x^i))}{\partial\hat z}\,ds=1+o_k(r^{-1/2})\;,$$

Further,

$$\frac{\partial z}{\partial\hat x}\;=\;\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial z^{\hat z}(\phi_s(\hat x^i))}{\partial\hat x}\,ds=o_k(r^{-1/2})\;,$$

similarly

$$\frac{\partial z}{\partial\hat y}\;=\;o_k(r^{-1/2})\;.$$

The estimates for the derivatives of x and y are straightforward, and we conclude that

$$\frac{\partial x^i}{\partial\hat x^i}=\delta^i_j+o_k(r^{-1/2})\;,$$

where, by an abuse of notation, we write again x^i for the functions (x,y,z). Standard considerations based on the implicit function theorem show that, increasing R if necessary, the x^i's form a coordinate system on $\mathbb{R}^3\setminus B(R)$ in which (2.2.10)-(2.2.12) hold. Subsequently, (2.2.13) holds again.

From (2.2.15) one clearly has

$$\forall\; s\in\mathbb{R}\qquad z\circ\phi_s=z\;,$$

which shows that the planes

$$\mathscr{P}_\tau:=\{z=\tau\}\;,\quad \tau\in\mathbb{R}\;,\;|\tau|\geq R\;,$$

are invariant under the flow of η; equivalently,

$$\eta^z=0\;.$$

Moreover,

$$x \circ \phi_\pi = -x \ , \quad y \circ \phi_\pi = -y \ , \tag{2.2.16}$$

so that all points with coordinates $x = y = 0$ are fixed points of ϕ_π, and that *these are the only such points in* M_{ext}. Equation (2.2.16) further implies that ϕ_π maps the surfaces $\{x = 0\}$ and $\{y = 0\}$ into themselves. Since ϕ_π is an isometry, we obtain

$$\begin{aligned} g_{ab}(0, y, z) = g_{ab}(0, -y, z) \ , \quad g_{zz}(0, y, z) = g_{zz}(0, -y, z) \ , \\ g_{za}(0, y, z) = -g_{za}(0, -y, z) \ ; \end{aligned} \tag{2.2.17}$$

similarly

$$\begin{aligned} g_{ab}(x, 0, z) = g_{ab}(-x, 0, z) \ , \quad g_{zz}(x, 0, z) = g_{zz}(-x, 0, z) \ , \\ g_{za}(x, 0, z) = -g_{za}(-x, 0, z) \ . \end{aligned} \tag{2.2.18}$$

Equation (2.2.17) leads to

$$\frac{\partial^{2\ell+1} g_{ab}}{\partial y^{2\ell+1}}(0, 0, z) = 0 \ , \quad \frac{\partial^{2\ell+1} g_{zz}}{\partial y^{2\ell+1}}(0, 0, z) = 0 \ , \quad \frac{\partial^{2\ell} g_{az}}{\partial y^{2\ell}}(0, 0, z) = 0 \tag{2.2.19}$$

for $\ell \in \mathbb{N}$ (or at least as far as the differentiability of the metric allows). The analogous implication of (2.2.18) allows us to conclude that

$$\frac{\partial g_{ab}}{\partial x^c}(0, 0, z) = 0 \ , \quad \frac{\partial g_{zz}}{\partial x^a}(0, 0, z) = 0 \ , \quad g_{az}(0, 0, z) = 0 \ . \tag{2.2.20}$$

Incidentally, the last two equations in (2.2.20) show that $\{x = y = 0\}$ is a geodesic; this follows in any case from the well-known fact that the set of fixed points of an isometry is totally geodesic.

Consider a point p lying on the axis of rotation $\mathscr{A}$, then $\phi_t(p) = p$ for all t, in particular $\phi_\pi(p) = p$. From what has been said we obtain that

$$\mathscr{A} \cap M_{\text{ext}} \subset \{x = y = 0\} \ . \tag{2.2.21}$$

Recall, again, that every connected component of the axis of rotation $\mathscr{A}$ is *an inextendible geodesic* in (M, g). Since the set at the right-hand-side of (2.2.21) is a geodesic ray, we conclude that equality holds in (2.2.21). Hence

$$\eta^i(0, 0, z) = 0 \tag{2.2.22}$$

and, for $|z| \geq R$, the origin is the only point within the plane $\mathscr{P}_z$ at which η vanishes.

We are ready now to pass to the problem at hand, namely an asymptotic analysis of the fields $g(\eta, \eta)$, q and $\tilde{\theta}$ as in (2.2.5); we start with q. For ρ sufficiently large the hypersurface $\{y = 0\}$ is transverse to η (for small ρ we will return to this issue shortly) and therefore the coordinates

$$(x^A) := (x, z)$$

on this hypersurface, with $x \geq 0$, can be used as local coordinates on $M/\mathrm{U}(1)$. The contribution of g_{AB} to q_{AB} is of the form $g_{AB} = \delta_{AB} + o_k(r^{-1/2})$, which is manifestly asymptotically flat in the usual sense. Next, from (2.2.10) and (2.2.13) we obtain

$$g(\eta,\eta) = \rho^2 + o_k(r^{3/2}) \; ; \tag{2.2.23}$$

here, as elsewhere, $\rho^2 = x^2 + y^2$. Further

$$\begin{aligned}\frac{g_{Ai}\eta^i g_{Bj}\eta^j}{g(\eta,\eta)}dx^A dx^B &= \Big(\delta_{Ai} + o_k(r^{-1/2})\Big)\Big({\omega^i}_a x^a + o_k(r^{1/2})\Big) \times \\ &\quad \frac{\Big(\delta_{Bj} + o_k(r^{-1/2})\Big)\Big({\omega^j}_b x^b + o_k(r^{1/2})\Big)}{\rho^2 + o_k(r^{3/2})} dx^A dx^B \\ &= \frac{o_k(r^{1/2}) dx^A dx^B}{\rho^2 + o_k(r^{3/2})} \; ,\end{aligned} \tag{2.2.24}$$

because ${\omega^i}_a x^a {\omega^j}_b x^b dx^i dx^j = (xdy - ydx)^2$, which vanishes when pulled-back to $\{y = 0\}$. In the region (2.2.14) we thus obtain

$$q_{AB} = \delta_{AB} + o_k(r^{-1/2}) \; , \tag{2.2.25}$$

which is the desired estimate. However, near the zeros of η this calculation is not precise enough to obtain uniform estimates on q and its derivatives.

In fact, it will be seen in the remainder of the proof that we need uniform estimates for derivatives up to second order. Since $g(\eta,\eta)$ vanishes quadratically at the origin we need uniform control of the numerator of (2.2.24) up to terms $O(\rho^4)$, in a form which allows the division to be performed without losing uniformity.

So in the region $\{\rho \le |z|\} \cap M_{\rm ext}$, in which $|z|$ is comparable with r, we proceed as follows: Let

$$ {\lambda^a}_b \equiv {\lambda^a}_b(z) := \frac{\partial \eta^a}{\partial x^b}(0,0,z) \; , \quad \lambda_{ab} := g_{ac}(0,0,z){\lambda^c}_b \; ;$$

note that ${\lambda^a}_b = {\omega^a}_b + o_{k-1}(|z|^{-1/2}) = {\omega^a}_b + o_{k-1}(r^{-1/2})$, similarly for λ_{ab}. The Killing equations imply that λ_{ab} is anti-symmetric, hence

$$\lambda_{xx} = \lambda_{yy} = 0 \; , \qquad \lambda_{xy} = -\lambda_{yx} = 1 + o_{k-1}(|z|^{-1/2}) = 1 + o_{k-1}(r^{-1/2}) \; .$$

From (2.2.22) we further obtain

$$\partial_i \eta^z = 0 \quad \Longrightarrow \quad \nabla_i \eta^z|_{\mathscr{A}} = 0 \quad \Longrightarrow \quad \nabla_i \eta_z|_{\mathscr{A}} = \nabla_z \eta_i|_{\mathscr{A}} = \nabla_z \eta^i|_{\mathscr{A}} = 0 \; .$$

Recall the well known consequence of the Killing equations (see (A.16.5), p. 231 below),

$$\nabla_i \nabla_j \eta_k = {R^\ell}_{ijk}\eta_\ell \; ,$$

which implies, at $\mathscr{A}$,

$$0 = \nabla_a \nabla_b \eta_c = \partial_a \partial_b \eta_c \; , \tag{2.2.26}$$

$$0 = \nabla_a \nabla_b \eta_z = \partial_a \nabla_b \eta_z - {\Gamma^c}_{az}\lambda_{bc} = \partial_a \partial_b \eta_z - 2{\Gamma^c}_{az}\lambda_{bc} \; . \tag{2.2.27}$$

From (2.2.13) we obtain, by integration of third derivatives of η_a along rays from the origin $x = y = 0$ within the planes $z = \text{const}$,

$$\frac{\partial^2 \eta_a}{\partial x^b \partial x^c} = o_{k-3}(|z|^{-5/2})x^c = o_{k-3}(r^{-5/2})x^c ,$$

and then successive such integrations give

$$\frac{\partial \eta_a}{\partial x^b} = \lambda_{ab} + o_{k-3}(|z|^{-5/2})x^c x^d = \lambda_{ab} + o_{k-3}(r^{-5/2})x^c x^d ,$$

$$\eta_a = \lambda_{ab}x^b + o_{k-3}(r^{-5/2})x^c x^d x^e . \tag{2.2.28}$$

At $y = 0$ we conclude that

$$\eta_x = o_{k-3}(r^{-5/2})x^c x^d x^e .$$

Similarly we have $\nabla_i \nabla_j \eta^k = R^\ell{}_{ij}{}^k \eta_\ell$, hence $\nabla_a \nabla_b \eta^c = \partial_a \partial_b \eta^c = 0$ at $\mathscr{A}$, and we conclude that

$$\eta^a = \lambda^a{}_b x^b + o_{k-3}(r^{-5/2})x^c x^d x^e . \tag{2.2.29}$$

This allows us to prove transversality of η to the plane $\{y = 0\}$. Indeed, from (2.2.29) at $y = 0$ we have

$$\eta^y = (1 + o(r^{-1/2}))x + o(r^{-5/2})x^3 = (1 + o(r^{-1/2}))x$$

which has no zeros for $x \neq 0$ and $r \geq R$ if R is large enough. Recall that we have been assuming that $|x| \leq |z|$ in the current calculation; however, we already know that η is transverse for $|z| \geq |x|$, and transversality follows. Increasing the value of the radius R defining M_{ext} if necessary, we conclude that $\{y = 0, x \geq 0\} \cap M_{\text{ext}}$ provides a global cross-section for the action of U(1) in M_{ext}.

Using (2.2.27), a similar analysis of η_z gives

$$\eta_z = -\underbrace{\Gamma^c{}_{az}|_{\mathscr{A}}}_{o_{k-1}(r^{-3/2})} \lambda_{bc} x^a x^b + o_{k-3}(r^{-5/2})x^c x^d x^e .$$

We are now ready to return to (2.2.23),

$$g(\eta, \eta) = \eta_i \eta^i = \eta_a \eta^a = \hat{\rho}^2 + o_{k-3}(r^{-5/2})x^a x^b x^c x^d , \tag{2.2.30}$$

where, at $y = 0$,

$$\hat{\rho}^2 := \mathring{g}_{ab}\lambda^a{}_c x^c \lambda^b{}_d x^d = (1 + o_{k-1}(r^{-1/2}))x^2 ;$$

it follows that the last equality also holds for $g(\eta, \eta)$ with $k - 1$ replaced by $k - 3$. Instead of (2.2.24) we write

$$\begin{aligned}
\frac{g_{Ai}\eta^i g_{Bj}\eta^j}{g(\eta,\eta)} dx^A dx^B &= \frac{\eta_A \eta_B dx^A dx^B}{(1 + o_{k-3}(r^{-1/2}))x^2} \\
&= \frac{\eta_x^2\, dx^2 + 2\eta_x \eta_z dx\, dz + \eta_z^2\, dz^2}{(1 + o_{k-3}(r^{-1/2}))x^2} \\
&= \frac{o_{k-3}(r^{-3})x^2 dx^A dx^B}{(1 + o_{k-3}(r^{-1/2}))} \\
&= o_{k-3}(r^{-1}) dx^A dx^B .
\end{aligned} \tag{2.2.31}$$

We conclude that (2.2.25) holds throughout $\{y=0\}\cap M_{\rm ext}$ with k replaced by $k-3$.

To analyse the fall-off of B_ρ and A_z, note first that the discussion in the paragraph before (2.2.5) shows that it suffices to do this at one single surface transverse to the flow of the Killing vector field η; unsurprisingly, we choose

$$N:=\{y=0\,,\ x>0\,,\ x^2+z^2\geq R^2\}\ ,$$

with R sufficiently large to guarantee transversality. Next, from (2.2.1) we find

$$\eta_i dx^i = g(\eta,\cdot)=g(\partial_\varphi,\cdot)=g(\eta,\eta)(d\varphi+\rho B_\rho d\rho+A_z dz)\ ,$$

which will allow us to relate B_ρ and A_z to η_i if we determine, say $\partial_i\varphi$ and $\partial_i\rho$ on N. For the sake of clarity of intermediate calculations it is convenient to denote by $\bar z$ the coordinate z appearing in (2.2.1), we thus seek a coordinate transformation

$$(x,y,z)\to(\rho,\varphi,\bar z)\ ,\ \text{with } \bar z = z \text{ everywhere and } \rho = x \text{ on } N,$$

which brings the metric to the form (2.2.1), with z there replaced by $\bar z$. We wish to show that, on N,

$$J:=\begin{pmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\varphi} & \frac{\partial x}{\partial\bar z}\\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\varphi} & \frac{\partial y}{\partial\bar z}\\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\varphi} & \frac{\partial z}{\partial\bar z}\end{pmatrix} = \begin{pmatrix} 1 & \eta^x & 0\\ 0 & \eta^y & 0\\ 0 & 0 & 1\end{pmatrix}\ . \tag{2.2.32}$$

The second column is immediate from

$$\eta^x\partial_x+\eta^y\partial_y+\eta^z\partial_z=\eta=\partial_\varphi=\frac{\partial x}{\partial\varphi}\partial_x+\frac{\partial y}{\partial\varphi}\partial_y+\frac{\partial z}{\partial\varphi}\partial_z\ .$$

Similarly the third row follows immediately from $dz=d\bar z$. It seems that the remaining entries require considering J^{-1}. Now, φ is a coordinate that vanishes on N, so that $\partial_x\varphi=\partial_z\varphi=0$ there. From $\eta^i\partial_i\varphi=1$ we thus obtain $\partial_y\varphi=1/\eta^y$. Next, $\rho=x$ on N, giving $\partial_x\rho=1$ and $\partial_z\rho=0$ there. The equation $\eta^i\partial_i\rho=0$ gives then $\eta^x+\eta^y\partial_y\rho=0$, so that $\partial_y\rho=-\eta^x/\eta^y$. The derivatives of $\bar z$ are straightforward, leading to

$$J^{-1}=\begin{pmatrix} \frac{\partial \rho}{\partial x} & \frac{\partial \rho}{\partial y} & \frac{\partial \rho}{\partial z}\\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z}\\ \frac{\partial \bar z}{\partial x} & \frac{\partial \bar z}{\partial y} & \frac{\partial \bar z}{\partial z}\end{pmatrix} = \begin{pmatrix} 1 & -\eta^x/\eta^y & 0\\ 0 & 1/\eta^y & 0\\ 0 & 0 & 1\end{pmatrix}\ .$$

Inverting J^{-1} leads to (2.2.32).

From now on we drop the bar on $\bar z$. From (2.2.32) one immediately has on N

$$\begin{aligned} A_z &= \frac{\eta_z}{g(\eta,\eta)}=\begin{cases} o_{k-1}(r^{-3/2})+o_{k-3}(r^{-5/2})x, & |x|\leq|z|,\\ o_k(r^{-3/2}), & \text{otherwise,}\end{cases}\\ &= o_{k-3}(r^{-3/2})\ . \end{aligned} \tag{2.2.33}$$

Similarly, again on N,

$$\begin{aligned} B_\rho &= \frac{\eta_i}{\rho g(\eta,\eta)}\frac{\partial x^i}{\partial \rho} = \frac{\eta_x}{x g(\eta,\eta)} = \begin{cases} o_{k-3}(r^{-5/2}), & |x|\le |z|, \\ o_k(r^{-5/2}), & \text{otherwise}, \end{cases} \\ &= o_{k-3}(r^{-5/2}) \ . \end{aligned} \tag{2.2.34}$$

Finally, we note that

$$\begin{aligned} e^{-2U} &:= \frac{g(\eta,\eta)}{\rho^2} = \begin{cases} 1+o_{k-1}(r^{-1/2})+o_{k-3}(r^{-5/2})x^2, & |x|\le |z|, \\ 1+o_k(r^{-1/2}), & \text{otherwise}, \end{cases} \\ &= 1+o_{k-3}(r^{-1/2}) \ . \end{aligned} \tag{2.2.35}$$

In summary:

Proposition 2.2.2 *Under (2.2.10) with $k\ge 3$ the metric q is asymptotically flat. In fact, there exist coordinates (x,y,z) satisfying (2.2.10) and a constant $R\ge 0$ such that the plane $\{y=0\}\cap\{r\ge R\}$ is transverse to η except at $x=z=0$ where η vanishes and, setting $x^A=(x,z)$ we have*

$$q_{AB}-\delta_{AB}=o_{k-3}(r^{-1/2}) \ . \tag{2.2.36}$$

Furthermore (2.2.33)-(2.2.35) hold.

2.2.5 Isothermal coordinates

We will use the same symbol q for the metric on the manifold obtained by doubling $M/\mathrm{U}(1)$ across the axis.

We start by noting the following:

Proposition 2.2.3 *Let q be an asymptotically flat metric on $\mathbb{R}^2$ in the sense of (2.2.36) with $k\ge 5$. Then q has a global representation*

$$q=e^{2u}(dv^2+dw^2) \ , \quad \text{with } u \longrightarrow_{\sqrt{v^2+w^2}\to\infty} 0 \ . \tag{2.2.37}$$

In fact, $u=o_{k-4}(r^{-1/2})$.

Remark 2.2.4 The classical justification of the existence of global isothermal coordinates proceeds by constructing the coordinate v of (2.2.37) as a solution of the equation $\Delta_q v=0$. A more careful version of the arguments in the spirit of [190, Lemma 2.3] shows that v has no critical points. However, the approach here appears to be simpler.

Proof: Let $\tilde q_{AB}=e^{-2u}q_{AB}$, then $\tilde q$ is flat if and only if u satisfies the equation

$$\Delta_q u = -\frac{R(q)}{2} \ , \tag{2.2.38}$$

where $R(q)$ is the scalar curvature of q. For asymptotically flat metrics q, with asymptotically Euclidean coordinates (x,z), this equation always has a solution such that

$$u+\mu\ln(\sqrt{x^2+z^2}) \longrightarrow_{\sqrt{x^2+z^2}\to\infty} 0 \ , \quad \text{where } \mu=\frac{1}{4\pi}\int_{\mathbb{R}^2} R(q)\, d\mu_q \ , \tag{2.2.39}$$

where $d\mu_q$ is the volume form of q. More precisely, we have the following:

LEMMA 2.2.5 *Consider a metric q on $\mathbb{R}^2$ satisfying*

$$q_{AB} - \delta_{AB} = o_\ell(r^{-1/2})$$

for some $\ell \geq 2$, with $(x^A) = (x, z)$. For any continuous function $R = o_{\ell-2}(r^{-5/2})$ there exists $\hat{u} = o_{\ell-1}(r^{-1/2})$ and a solution of (2.2.38) such that

$$u = \hat{u} - \mu \ln(\sqrt{x^2 + z^2}) \ ,$$

with μ as in (2.2.39).

PROOF: We start by showing that (2.2.38) can be solved for $|x|$ large. Indeed, consider the sequence v_i of solutions of (2.2.38) on the annulus

$$\Gamma(\rho, \rho + i) := D(0, \rho + i) \setminus D(0, \rho) \ ,$$

with zero boundary values. Here ρ is a constant chosen large enough so that the functions $\pm C|x|^{-1/2}$, with $C = 8\|R|x|^{5/2}\|_{L^\infty}$, are sub- and super-solutions of (2.2.38). Shifting by a constant if necessary, the usual elliptic estimates (compare [57]) show that a subsequence can be chosen which converges, uniformly on compact sets, to a solution $v = O_{\ell-1}(r^{-1/2})$ of (2.2.38) on $\mathbb{R}^2 \setminus D(0, \rho)$. In the notation of [57] we have in fact $v \in C^{\ell-1,\lambda}_{-1/2,0}$ for any $\lambda \in (0, 1)$. Furthermore, using the techniques in [57] one checks that $v = o_{\ell-1}(r^{-1/2})$.

We extend v in any way to a $C^{\ell-1,\lambda}$ function on $\mathbb{R}^2$, still denoted by v. Let $\hat{q} := e^{-2v}q$, then $\hat{q}$ is flat for $|x| \geq \rho$. Let $\hat{e}^A$ be any $\hat{q}$–parallel orthonormal coframe on $\mathbb{R}^2 \setminus D(0, \rho)$, performing a rigid rotation of the coordinates if necessary we will have $\hat{e}^A = dx^A + \sum_B o_{\ell-1}(|x|^{-1/2})dx^B$ for $|x|$ large. Let $\hat{x}^A$ be any solutions of the set of equations $d\hat{x}^A = \hat{e}^A$. By the implicit function theorem the functions $\hat{x}^A$ cover $\mathbb{R}^2 \setminus D(0, \hat{\rho})$, for some $\hat{\rho}$, and form a coordinate system there, in which $\hat{q}_{AB} = \delta_{AB}$.

Since (2.2.38) is conformally covariant, we have reduced the problem to one where R has compact support, and q is a $C^{\ell-1,\lambda}$ metric which is flat outside of a compact set. This will be assumed in what follows.

Let us use the stereographic projection, say ψ, to map $\mathbb{R}^2$ to a sphere, then (2.2.38) becomes an equation for $\hat{u} := (u - v) \circ \psi^{-1}$ on $S^2 \setminus \{i^0\}$, where i^0 is the north pole of S^2, of the form

$$\Delta_h \hat{u} = |x|^4 f \ , \tag{2.2.40}$$

where $h_{AB} := |x|^{-4} q_{AB}$ is a $C^{\ell-1,\lambda}$ metric on S^2, similarly f is a $C^{\ell-2}$ function on S^2 supported away from the north pole. In fact, in a coordinate system

$$y^A = x^A/|x|^2 \tag{2.2.41}$$

near $i^0 = \{y^A = 0\}$, where the x^A's are the explicitly flat coordinates on $\mathbb{R}^2 \setminus D(0, R)$ for the metric q, we have

$$h_{AB} = \delta_{AB} \ .$$

Let $H_k(S^2)$ be the usual L^2-type Sobolev space of functions on S^2 and set

$$\mathcal{H}_k = \left\{ \chi \in H_k(S^2) \mid \int_{S^2} \chi \, d\mu_h = 0 \right\} , \tag{2.2.42}$$

where $d\mu_h$ is the measure associated with the metric h. We have

PROPOSITION 2.2.6 *Let h be a twice-differentiable metric on S^2, then $\Delta_h : \mathcal{H}_2 \to \mathcal{H}_0$ is an isomorphism.*

PROOF: Injectivity is straightforward. To show surjectivity, let $X \subset L^2$ be the image of $\mathcal{H}_2$ by Δ_h, by elliptic estimates X is a closed subspace of $L^2(S^2)$. Let $\varphi \in L^2$ be orthogonal to X, then

$$\forall \chi \in \mathcal{H}_2 \qquad \int \varphi \Delta_h \chi \, d\mu_h = 0 \, .$$

Thus φ is a weak solution of $\Delta_h \varphi = 0$, by elliptic estimates $\varphi \in \mathcal{H}_2$. But setting $\chi = \varphi$ and integrating by parts one obtains $d\varphi = 0$, hence φ is constant, which shows that $X = \mathcal{H}_0$. □

Returning to the proof of Lemma 2.2.5, we have seen that (2.2.38) can be reduced to solving the problem

$$\Delta_{\bar{h}} \bar{u} = \bar{f} \, , \tag{2.2.43}$$

where $\bar{h}$ is flat outside of a compact set. Let

$$\mu := -\frac{1}{2\pi} \int_{\mathbb{R}^2} \bar{f} \, d\mu_{\bar{h}} \, ,$$

then

$$\begin{aligned} \int_{\mathbb{R}^2} \Delta_{\bar{h}} \Big(\mu \ln \sqrt{1 + x^2 + z^2} \Big) \, d\mu_{\bar{h}} &= \lim_{\rho \to \infty} \mu \oint_{C(0,\rho)} D \Big(\ln \sqrt{1 + x^2 + z^2} \Big) \cdot n \\ &= 2\pi\mu = - \int_{\mathbb{R}^2} \bar{f} \, d\mu_{\bar{h}} \end{aligned}$$

Thus (2.2.43) is equivalent to the following equation for the function $\tilde{u} := \bar{u} + \mu \ln \sqrt{1 + x^2 + z^2}$:

$$\Delta_{\bar{h}} \tilde{u} = \bar{f} + \Delta_{\bar{h}} \Big(\mu \ln \sqrt{1 + x^2 + z^2} \Big) \, ,$$

and the right-hand-side has vanishing average. Transforming to a problem on S^2 as in (2.2.40), we can solve the resulting equation by Proposition 2.2.6. Transforming back to $\mathbb{R}^2$, and shifting u by a constant if necessary, the result follows. □

Returning to the proof of Proposition 2.2.3, we claim that $\mu = 0$; that is,

$$\int_{\mathbb{R}^2} R(q) \, d\mu_q = 0 \, . \tag{2.2.44}$$

This is the simplest version of the Gauss-Bonnet theorem, we give the proof for completeness: consider any metric on $\mathbb{R}^2$ satisfying

$$q_{AB} - \delta_{AB} = o_1(1) , \qquad R(q) \in L^1 .$$

Let $\tilde\theta^a$, $a = 1, 2$, be an orthonormal co-frame for q obtained by a Gram–Schmidt procedure starting from (dx^1, dx^2), with connection coefficients $\omega^a{}_b$. Then $\omega^a{}_b = o(r^{-1})$. It is well known that, for a metric defined on an open subset of $\mathbb{R}^2$,

$$R(q)\, d\mu_q = 2d\omega^1{}_2 , \tag{2.2.45}$$

as follows from Cartan's second structure equation (A.17.20); see (A.17.22)). Equation (2.2.44) immediately follows by integration on $B(R)$, using Stokes' theorem, and passing to the limit $R \to \infty$.

To finish the proof, note that the metric $\tilde q$ is a complete flat metric on $\mathbb{R}^2$, and the Hadamard–Cartan theorem shows the existence of global manifestly flat coordinates, say (v, w) so that q can be written as in (2.2.37). $\Box$

Returning to the problem at hand, recall that the metric q on $\mathbb{R}^2$ has been obtained by doubling $M/\mathrm{U}(1)$ across $\mathscr{A}$. Let us denote by ϕ the corresponding isometry; note that in $M_{\mathrm{ext}}/\mathrm{U}(1)$, in the coordinates (x, z) constructed in Section 2.2.4, the isometry ϕ is the symmetry around the z-axis: $\phi(x, z) = (-x, z)$. Similarly, in geodesic coordinates centred on $\mathscr{A}$, $\phi(x, z) = (-x, z)$.

As ϕ is an isometry of q, preserving the boundary conditions satisfied by u, uniqueness of solutions of (2.2.38) implies that $u \circ \phi = u$. This, together with smoothness of u on the doubled manifold, shows that on $\mathscr{A}$ the gradient ∇u has only components tangential to $\mathscr{A}$. This implies that $\mathscr{A}$ is totally geodesic both for q and $\tilde q$.

Choose any point p on $\mathscr{A}$. By a shift of (v, w) we can arrange to have $(v(p), w(p)) = (0, 0)$. Let (ρ, z) be coordinates obtained by a rigid rotation of (v, w) around the origin so that the vector tangent to $\mathscr{A}$ at p coincides with ∂_z. Then the axis $\{(0, z)\}_{z\in\mathbb{R}}$ is a geodesic of $\tilde q$, sharing a common direction at p with $\mathscr{A}$, hence

$$\mathscr{A} \equiv \{(0, z)\}_{z\in\mathbb{R}} .$$

Since ϕ is an isometry of $\tilde q$ which is the identity on $\mathscr{A}$, it easily follows that

$$\phi(\rho, z) = (-\rho, z) ,$$

so that $M/\mathrm{U}(1) = \{\rho \geq 0\}$. We have thus obtained the representation (2.2.1) of g.

The reader might have noticed that the function u constructed in this section is a solution of a Neumann problem with vanishing Neumann data on the axis.

For further use, we note that from (2.2.1), on $\exp((T\mathscr{A})^\perp)$ the geodesic distance $\hat\rho$ from the origin equals

$$\hat\rho = e^{-(U-\alpha)(0,z)}\rho + O(\rho^3) ,$$

and comparing with (2.2.9) we obtain

$$\alpha(0, z) = 0 . \tag{2.2.46}$$

Now, the function $u = o_{k-4}(r^{-1/2})$ of Proposition 2.2.3 equals $u = 2(\alpha - U)$ (compare (2.2.1)). By (2.2.46) and an analysis of Taylor expansions as in Section 2.2.4 we infer that, at $\{y = 0\}$,

$$\alpha = o_{k-5}(r^{-3/2})x \; . \tag{2.2.47}$$

From Proposition 2.2.2 we conclude:

THEOREM 2.2.7 *Let $k \geq 5$. Any Riemannian metric on $\mathbb{R}^3$ invariant under rotations around a coordinate axis and satisfying*

$$g_{ij} - \delta_{ij} = o_k(r^{-1/2}) \tag{2.2.48}$$

admits a global representation of the form (2.2.1), with the functions U, α, B_ρ and A_z satisfying

$$A_z = o_{k-3}(r^{-3/2}) \; ; \quad B_\rho = o_{k-3}(r^{-5/2}) \; ; \quad U = o_{k-3}(r^{-1/2}) \; ; \quad \alpha = o_{k-4}(r^{-1/2}) \; . \tag{2.2.49}$$

Furthermore (2.2.47) holds.

REMARK 2.2.8 The decay rate $o(r^{-1/2})$ in (2.2.48) has been tailored to the requirement of a well-defined ADM mass; the result remains true with decay rates $o(r^{-\alpha})$ or $O(r^{-\alpha})$ for any $\alpha \in (0, 1)$, with the decay rate carrying over to the functions appearing in (2.2.1) in the obvious way, as in (2.2.49).

Several asymptotically flat ends

The above considerations generalize to several asymptotically flat ends:

THEOREM 2.2.9 *Let $k \geq 5$, and consider a simply connected three-dimensional Riemannian manifold (M, g) which is the union of a compact set and of N asymptotically flat ends, and let M_{ext} denote the first such end. If g is invariant under an action of $\mathrm{U}(1)$, then g admits a global representation of the form (2.2.1), where the coordinates (z, ρ) cover $(\mathbb{R} \times \mathbb{R}^+) \setminus \{\vec{a}_i\}_{i=2}^N$, with the punctures $\vec{a}_i = (0, a_i)$ lying on the z-axis, each $\vec{a}_i$ representing "a point at infinity" of the remaining asymptotically flat regions. The functions U, α, B_ρ and A_z satisfy (2.2.49) in M_{ext}.*

If we set

$$r_i = \sqrt{\rho^2 + (z - a_i)^2} \; ,$$

then we have the following asymptotic behavior near each of the punctures

$$U = 2 \ln r_i + o_{k-4}(r_i^{1/2}) \; , \quad \alpha = o_{k-4}(r_i^{1/2}) \; , \tag{2.2.50}$$

where $f = o_\ell(r_i^{1/2})$ means that $\partial_{A_1} \ldots \partial_{A_j} f = o_{\ell-j}(r_i^{1/2-j})$ for $0 \leq j \leq \ell$. Finally, (2.2.47) holds.

PROOF: As discussed in Section 2.2.2, M is diffeomorphic to $\mathbb{R}^3$ minus a finite set of points and, after perfoming a diffeomorphism if necessary, the action of the group is that by rotations around a coordinate axis of $\mathbb{R}^3$. As in the proof of

Theorem 2.2.7 there exists a function $v = o_{k-4}(r^{-1/2})$ so that the metric $e^{-2v}q$ is flat for $|x|$ large enough in each of the asymptotic regions. Equation (2.2.38) is then equivalent to the following equation for $u - v$,

$$\Delta_{e^{-2v}q}(u-v) = -e^{2v}\Big(\frac{R(q)}{2} + \Delta_q v\Big) \,, \tag{2.2.51}$$

where the right-hand-side is compactly supported on $M/\mathrm{U}(1)$. Let $M_{\mathrm{ext}}/\mathrm{U}(1)$ be the orbit space associated to the first asymptotically flat region and let ψ be any smooth strictly positive function on $M/\mathrm{U}(1)$ which coincides with $|\vec{y}|^{-4}$ in each of the remaining asymptotically flat regions of $M/\mathrm{U}(1)$, where the y^A's are the manifestly flat coordinates there, with ψ equal to one in $M_{\mathrm{ext}}/\mathrm{U}(1)$. Then (2.2.51) is equivalent to

$$\Delta_{\psi e^{-2v}q}(u-v) = -\psi^{-1}e^{2v}\Big(\frac{R(q)}{2} + \Delta_q v\Big) \,. \tag{2.2.52}$$

Both the metric $\psi e^{-2v}q$ and the source term extend smoothly through the origins, say i^0_j, $2 = 1, \ldots, N$, of each of the local coordinate systems $x^A := y^A/|\vec{y}|^2$. Simple connectedness of the two-dimensional manifold

$$\bar{\mathscr{N}} := M/\mathrm{U}(1) \cup \{i^0_j\}_{j=2}^N$$

implies that $\bar{\mathscr{N}} \approx \mathbb{R}^2$, so that (2.2.52) is an equation to which Lemma 2.2.5 applies. We thus obtain a solution, say w, of (2.2.52), and subsequently a solution $v+w$ of (2.2.38) which tends to a constant in each of the asymptotically flat regions (possibly different constants in different ends), except (as will be seen shortly) in M_{ext} where it diverges logarithmically. Note that at large distances in each of the asymptotically flat regions the function w is harmonic with respect to the Euclidean metric, hence approaches its asymptotic value as $|y|^{-1}$, with gradient falling-off one order faster. Similarly v has controlled asymptotics there, as in the proof of Lemma 2.2.5. Integrating (2.2.38) over $M/\mathrm{U}(1)$ one finds that the coefficient of the logarithmic term is again as in (2.2.39).

In order to determine that coefficient, we note that since $\bar{\mathscr{N}} \approx \mathbb{R}^2$ there exists a global orthonormal coframe for g, e.g. obtained by a Gram–Schmidt procedure from a global trivialization of $T^*\mathbb{R}^2$. As a starting point for this procedure one can, and we will do so, use a holonomic basis dx^A with the coordinate functions x^A equal to the manifestly flat coordinates in $M_{\mathrm{ext}}/\mathrm{U}(1)$. Furthermore, after a rigid rotation of the y^A's if necessary, where the y^A's are the manifestly flat coordinates for the metric $e^{-2(w+v)}q$ in the asymptotically flat regions other than $M_{\mathrm{ext}}/\mathrm{U}(1)$, we can also assume that the dx^A's coincide with $d(y^A/|\vec{y}|^2)$ near each i^0_j. By (2.2.45) and by what is said in the paragraph following that equation we have

$$\mu = \frac{1}{4\pi}\int_{M/\mathrm{U}(1)} R(q)d\mu_q = \sum_{j=2}^N \lim_{\epsilon\to 0}\frac{1}{2\pi}\oint_{C(i^0_j,\epsilon)} \omega^1{}_2 \,.$$

where the $C(i^0_j, \epsilon)$'s are circles of radius ϵ centred at the i^0_j's. Near each i^0_j the metric q takes the form $e^{2(v+w)}\delta_{AB}dy^A dy^B = e^{2(v+w)}|\vec{x}|^{-4}\delta_{AB}dx^A dx^B$. The co-frame $\tilde{\theta}^A$ is given by $\tilde{\theta}^A = e^{(v+w)}|\vec{x}|^{-2}dx^A$, leading to

$$\omega^1{}_2 = \frac{2}{|\vec{x}|^2}(x^1dx^2 - x^2dx^1) + o(|\vec{x}|^{-1/2})dx^A \ ,$$

so that

$$\lim_{\epsilon\to 0}\oint_{C(i^0_j,\epsilon)} \omega^1{}_2 = 4\pi \ .$$

We note that we have proved:

Proposition 2.2.10 *Let q be a Riemannian metric on a simply connected two-dimensional manifold which is the union of a compact set and N ends which are asymptotically flat in the sense of (2.2.36), then*

$$\mu := \frac{1}{4\pi}\int R(q)d\mu_q = 2(N-1) \ .$$

□

Since $\mu \neq 0$, the function $v+w$ obtained so far needs to be modified to get rid of the logarithmic divergence. In order to do that for $j = 2, \ldots, N$ we construct functions u_j, q-harmonic on $M/\mathrm{U}(1)$, such that, in coordinates x^A which are manifestly conformally flat in each of the asymptotic regions,

$$u_j = \begin{cases} \ln|\vec{x}| + o(1), & \text{in } M_{\mathrm{ext}}/\mathrm{U}(1); \\ -\ln|\vec{x}| + O(1), & \text{in the AF coordinates in the } j\text{'th asymptotic region;} \\ O(1), & \text{in the remaining asymptotic regions.} \end{cases} \tag{2.2.53}$$

This can be done as follows: let $\hat{u}_j$ be any smooth function which in local manifestly conformally flat coordinates both near i^0_j and on $M_{\mathrm{ext}}/\mathrm{U}(1)$ equals $\ln|\vec{x}|$, and which equals one at large distances in the remaining asymptotically flat regions. Let ψ be as in (2.2.52), then $\Delta_{\psi e^{-2(v+w)}q}\hat{u}_j$ is compactly supported in $M/\mathrm{U}(1)$. Further

$$\begin{aligned} &\int_{M/\mathrm{U}(1)} \Delta_{\psi e^{-2(v+w)}q}\hat{u}_j \, d\mu_{\psi e^{-2(v+w)}q} \\ &\quad = \int_{M/\mathrm{U}(1)} \Delta_{\psi e^{-2v}q}\hat{u}_j \, d\mu_{\psi e^{-2v}q} \\ &\quad = \lim_{R\to\infty}\int_{C(0,\rho)} D\ln|\vec{x}|\cdot n - \lim_{\epsilon\to 0}\int_{C(0,\epsilon)} D\ln|\vec{x}|\cdot n \\ &\quad = 0 \ . \end{aligned}$$

We can therefore invoke Lemma 2.2.5 to conclude that there exists a uniformly bounded function $\hat{v}$, approaching zero as one recedes to infinity in $M_{\mathrm{ext}}/\mathrm{U}(1)$, such that

$$\Delta_{\psi e^{-2(v+w)}q}\hat{v} = -\Delta_{\psi e^{-2(v+w)}q}\hat{u}_j \ .$$

Subsequently the function $u_j := \hat{u}_j + \hat{v}$ is q–harmonic and satisfies (2.2.53).

The function

$$u := v + w + 2\sum_{j=2}^{N} u_j + \alpha \ ,$$

where α is an appropriately chosen constant (compare [57]), defines the desired conformal factor approaching one as one tends to infinity in $M_{\rm ext}/\mathrm{U}(1)$ so that $e^{-2u}q$ is flat. This conformal factor further compactifies each of the asymptotic infinities except the first one to a point, so that $e^{-2u}q$ extends by continuity to a flat complete metric on the simply connected manifold $\overline{\mathscr{N}}$. By the Hadamard–Cartan theorem there exists on $\overline{\mathscr{N}}$ a global manifestly flat coordinate system for $e^{-2u}q$. The axis of rotation can be made to coincide with a coordinate axis as in the proof of Theorem 2.2.7. It should be clear that the points at infinity i^0_j lie on that axis.

In order to prove (2.2.50), note that the construction above gives directly.

$$U - \alpha = u = C_i + 2\ln r_i + o_{k-4}(r_i^{1/2}) \ ,$$

Next, U can be determined by applying an inversion

$$y^A \mapsto (\rho, z - a_i) = (x^A) = (y^A/|\vec{y}|^2) \qquad (2.2.54)$$

to (2.2.35),

$$\rho^2 e^{-U} = g(\eta,\eta) = \frac{\rho^2}{(\rho^2 + (z-a_i)^2)^2}\Big(1 + o_{k-3}((\rho^2 + (z-a_i)^2)^{1/4})\Big) \ .$$

Since α vanishes on the axis $(y^1)^2 + (y^2)^2 = 0$ in each of the asymptotic regions, we conclude that $C_i = 0$, and (2.2.50) follows. □

2.2.6 ADM mass

Let m be the ADM mass of g,

$$m := \lim_{R\to\infty} \frac{1}{16\pi}\int_{S_R} (g_{ij,j} - g_{jj,i}) dS_i \ ,$$

where $dS_i = \partial_i \rfloor (dx \wedge dy \wedge dz)$. This has to be calculated in a coordinate system satisfying (2.2.10). Typically one takes S_R to be a coordinate sphere $S(R)$ of radius R; however, as is well-known, under (2.2.10) S_R can be taken to be any piecewise differentiable surface homologous to $S(R)$ such that

$$\inf\{r(p) | p \in S_R\} \to_{R\to\infty} \infty \ . \qquad (2.2.55)$$

We will exploit this freedom in our calculation to follow.

We introduce new coordinates x and y so that ρ and φ in (2.2.1) become the usual polar coordinates on $\mathbb{R}^2$:

$$x = \rho\cos\varphi \ , \quad y = \rho\sin\varphi \ .$$

This implies

$$\begin{aligned} \rho d\rho = \tfrac{1}{2}d(\rho^2) = xdx + ydy \ , \\ \rho^2 d\varphi = xdy - ydx \ , \\ \rho^2 d\varphi^2 = dx^2 + dy^2 - d\rho^2 \ . \end{aligned}$$

Inserting the above in (2.2.1) one obtains

$$\begin{aligned} g &= e^{-2U}\underbrace{(dx^2+dy^2)}_{d\rho^2+\rho^2 d\varphi^2} + \frac{e^{-2U}(e^{2\alpha}-1)}{\rho^2}\underbrace{(xdx+ydy)^2}_{\rho^2 d\rho^2} + e^{-2U+2\alpha}dz^2 \\ &\quad +2e^{-2U}(xdy-ydx)\Big(B_\rho(xdx+ydy)+A_z dz\Big) \\ &\quad + \text{terms quadratic in } (B_\rho, A_z)\,. \end{aligned} \tag{2.2.56}$$

This will satisfy (2.2.10) if we assume that

$$U \ , \ \frac{(e^{2\alpha}-1)x^2}{\rho^2} \ , \ \frac{(e^{2\alpha}-1)xy}{\rho^2} \ , \ \frac{(e^{2\alpha}-1)y^2}{\rho^2} \ = \ o_1(r^{-1/2}) \ , \tag{2.2.57}$$

$$B_\rho x^2 \ , \ B_\rho xy \ , \ B_\rho y^2 \ , \ A_z x \ , \ A_z y \ = \ o_1(r^{-1/2}) \ , \tag{2.2.58}$$

consistently with Theorem 2.2.7. Then the terms occurring in the last line of (2.2.56) will not give any contribution to the mass integral. We rewrite g as

$$\begin{aligned} g &= \underbrace{e^{-2U}\left(dx^2+dy^2\right)}_{(a)} + \underbrace{\frac{e^{2\alpha}-1}{\rho^2}(xdx+ydy)^2}_{(b)} + \underbrace{e^{-2U+2\alpha}dz^2}_{(c)} \\ &\quad + \underbrace{2(xdy-ydx)\Big(B_\rho(xdx+ydy)+A_z dz\Big)}_{(d)} \\ &\quad + o_1(r^{-1})dx^i dx^j\,. \end{aligned} \tag{2.2.59}$$

Let us denote by x^a the variables x, y. As an example, consider the contribution of (c) to the mass integrand:

$$(c) \ \longrightarrow \ g_{zz,z}dS_z - g_{zz,i}dS_i = -g_{zz,a}dS_a = \Big(2(U-\alpha)_{,a} + o(r^{-2})\Big)dS_a \ .$$

A similar calculation of (a) easily leads to

$$(a)+(c) \ \longrightarrow \ (4U_{,i} + o(r^{-2}))dS_i - 2\alpha_{,a}dS_a \ .$$

The contribution of (b) to the mass integrand looks rather uninviting at first sight:

$$\begin{aligned} (b) \ \longrightarrow \ & \Big[\Big(\frac{e^{2\alpha}-1}{\rho^2}\Big)_{,y}xy - \Big(\frac{e^{2\alpha}-1}{\rho^2}\Big)_{,x}y^2 + \frac{e^{2\alpha}-1}{\rho^2}x\Big]dS_x \\ & +\Big[\Big(\frac{e^{2\alpha}-1}{\rho^2}\Big)_{,x}xy - \Big(\frac{e^{2\alpha}-1}{\rho^2}\Big)_{,y}x^2 + \frac{e^{2\alpha}-1}{\rho^2}y\Big]dS_y \\ & -\Big(\frac{e^{2\alpha}-1}{\rho^2}\Big)_{,z}(x^2+y^2)dS_z \ . \end{aligned}$$

Fortunately, things simplify nicely if S_R is chosen to be the boundary of the solid cylinder

$$C_R := \{-R \le z \le R\ ,\ 0 \le \rho \le R\}\ . \tag{2.2.60}$$

Then S_R is the union of the bottom $B_R = \{z = -R\ ,\ 0 \le \rho \le R\}$, the lid $L_R = \{z = R\ ,\ 0 \le \rho \le R\}$, and the wall $W_R = \{-R \le z \le R\ ,\ \rho = R\}$. On the bottom and on the lid we only have a contribution from dS_z, which equals

$$-\Big(2\alpha_{,z} + o(r^{-2})\Big)dx \wedge dy$$

on the lid, and minus this expression on the bottom. On the wall dS_z gives no contribution, while

$$dS_x|_{W_R} = (dy{\wedge}dz)|_{W_R} = x|_{W_R} d\varphi{\wedge}dz\ , \quad dS_y|_{W_R} = -(dx{\wedge}dz)|_{W_R} = y|_{W_R} d\varphi{\wedge}dz\ .$$

Surprisingly, the terms in $(b)|_{W_R}$ containing derivatives of α drop out, leading to

$$(b)|_{W_R} \ \longrightarrow \Big(2\alpha + o(r^{-2})\Big)d\varphi \wedge dz\ .$$

We continue with the contribution of B_ρ to (d):

$$\Big[\underbrace{\Big((x^2-y^2)B_\rho\Big)_{,y}}_{(1)} - \underbrace{(2xyB_\rho)_{,x}}_{(2)}\Big]dS_x + \Big[\underbrace{\Big((x^2-y^2)B_\rho\Big)_{,x}}_{(3)} + \underbrace{(2xyB_\rho)_{,y}}_{(4)}\Big]dS_y\ .$$

It only contributes on the wall W_R, giving however a zero contribution there:

$$\begin{aligned}&\Big[\Big(\underbrace{(x^2-y^2)(x\partial_y + y\partial_x)}_{(1)+(3)} + \underbrace{2xy(y\partial_y - x\partial_x)}_{(4)+(2)}\Big)B_\rho\Big]d\varphi \wedge dz \\ &\qquad = \Big[(x^2+y^2)\underbrace{(x\partial_y - y\partial_x)B_\rho}_{=0}\Big]d\varphi \wedge dz = 0\ .\end{aligned}$$

Finally, A_z produces the following boundary integrand:

$$-y\partial_z A_z dS_x + x\partial_z A_z dS_y + \Big[\underbrace{(x\partial_y - y\partial_x)A_z}_{=0}\Big]dS_z\ ,$$

and one easily checks that the dS_x and dS_y terms cancel out when integrated upon W_R, while giving no contribution on the bottom and the lid.

Collecting all this we obtain

$$\begin{aligned} m &= \lim_{R\to\infty}\frac{1}{16\pi}\Big[4\int_{S_R}\partial_i U dS_i + 2\int_{W_R}(\alpha - \frac{x^a}{\rho}\partial_a\alpha)\,d\varphi\,dz \\ &\quad -2\int_{L_R}\partial_z\alpha\,dx\,dy + 2\int_{B_R}\partial_z\alpha\,dx\,dy\Big] \\ &= \lim_{R\to\infty}\frac{1}{4\pi}\Big[\int_{S_R}\partial_i(U - \frac{1}{2}\alpha)dS_i + \frac{1}{2}\int_{W_R}\alpha\,d\varphi\,dz\Big]\ .\end{aligned}$$

We have the following formula for the Ricci scalar ${}^{(3)}R$ of the metric (2.2.1) (the details of the calculation can be found in [104]):[4]

$$-\frac{e^{-2U+2\alpha}}{4}\,{}^{(3)}R = -\Delta_\delta(U-\frac{1}{2}\alpha)+\frac{1}{2}\,(DU)^2-\frac{1}{2\rho}\frac{\partial\alpha}{\partial\rho}+\frac{\rho^2e^{-2\alpha}}{8}\,(\rho B_{\rho,z}-A_{z,\rho})^2\ . \tag{2.2.61}$$

The Laplacian Δ_δ and the gradient D are taken with respect to the flat metric δ on $\mathbb{R}^3$.

Exercice 2.2.11 Let us verify (2.2.61) in the special case

$$g = e^{-2(U-\alpha)}\left(\mathrm{d}\rho^2+\mathrm{d}z^2\right)+\rho^2e^{-2U}\mathrm{d}\varphi^2\ .$$

We choose the obvious coframe,

$$\theta^\rho = F\,\mathrm{d}\rho\ ,\qquad \theta^z = F\,\mathrm{d}z\ ,\qquad \theta^\varphi = G\,\mathrm{d}\varphi\ ,$$

with $F := e^{-(U-\alpha)}$ and $G := \rho e^{-U}$. Recall Cartan's first structure equation, with $\omega_{ab}=\omega_{[ab]}$:

$$\mathrm{d}\theta^a = -\omega^a{}_b\wedge\theta^b\ .$$

This gives

$$\begin{aligned}
\frac{\partial_zF}{F^2}\,\theta^\rho\wedge\theta^z &= \omega^\rho{}_\varphi\wedge\theta^\varphi+\omega^\rho{}_z\wedge\theta^z\ ,\\
\frac{\partial_\rho F}{F^2}\,\theta^\rho\wedge\theta^z &= \omega^\rho{}_z\wedge\theta^\rho+\omega^\varphi{}_z\wedge\theta^\varphi\ ,\\
\frac{1}{FG}\Big((\partial_\rho G)\,\theta^\rho\wedge\theta^\varphi-(\partial_zG)\,\theta^\varphi\wedge\theta^z\Big) &= \omega^\rho{}_\varphi\wedge\theta^\rho-\omega^\varphi{}_z\wedge\theta^z\ .
\end{aligned}$$

The following form of the connection one-forms is compatible with the above:

$$\omega^\rho{}_\varphi = \bar\alpha\,\theta^\varphi\ ,\qquad \omega^\rho{}_z=\bar\delta\,\theta^\rho+\bar\gamma\,\theta^z\ ,\qquad \omega^\varphi{}_z=\bar\beta\,\theta^\varphi\ .$$

A comparison of the coefficients gives

$$\begin{aligned}
\bar\alpha &= -\frac{\partial_\rho G}{FG}=\frac{\partial_\rho U-\frac{1}{\rho}}{F}\ ,\\
\bar\beta &= \frac{\partial_z G}{FG}=-\frac{\partial_z U}{F}\ ,\\
\bar\gamma &= -\frac{\partial_\rho F}{F^2}=\frac{\partial_\rho(U-\alpha)}{F}\ ,\\
\bar\delta &= \frac{\partial_z F}{F^2}=-\frac{\partial_z(U-\alpha)}{F}\ .
\end{aligned}$$

The curvature two-forms are calculated using the second structure equation,

$$\Omega^a{}_b=\mathrm{d}\omega^a{}_b+\omega^a{}_c\wedge\omega^c{}_b\ .$$

We have

$$\Omega^\rho{}_z = d\omega^\rho{}_z+\omega^\rho{}_\varphi\wedge\omega^\varphi{}_z$$

[4]In the time-symmetric case (2.2.61) can be viewed as a PDE for U given the remaining functions and the matter density. Assuming that this equation can indeed be solved, this allows us to prescribe freely the functions α, B_ρ and A_z. In such a rough analysis there does not seem to be any constraints on α, B_ρ and A_z (in particular they can be chosen to satisfy (2.2.57)-(2.2.58)), while U, and hence its asymptotic behavior, is determined by (2.2.61).

$$
\begin{aligned}
&= \mathrm{d}\left(\bar\delta F \mathrm{d}\rho + \bar\gamma F \mathrm{d}z\right) + \bar\alpha\,\theta^\varphi \wedge \bar\beta\,\theta^\varphi \\
&= \partial_z(\bar\delta F)\,\mathrm{d}z \wedge \mathrm{d}\rho + \partial_\rho(\bar\gamma F)\,\mathrm{d}\rho \wedge \mathrm{d}z \\
&= \frac{1}{F}\Big(\partial_\rho(\bar\gamma F) - \partial_z(\bar\delta F)\Big)\theta^\rho \wedge \theta^z \ .
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\Omega^\rho{}_\varphi &= \left(\frac{\partial_\rho(G\bar\alpha)}{GF} - \bar\delta\bar\beta\right)\theta^\rho \wedge \theta^\varphi + \left(\bar\gamma\bar\beta - \frac{\partial_z(G\bar\alpha)}{GF}\right)\theta^\varphi \wedge \theta^z \ , \\
\Omega^\varphi{}_z &= \left(\frac{\partial_\rho(G\bar\beta)}{GF} + \bar\alpha\bar\delta\right)\theta^\rho \wedge \theta^\varphi - \left(\frac{\partial_z(G\bar\beta)}{GF} + \bar\alpha\bar\beta\right)\theta^\varphi \wedge \theta^z \ .
\end{aligned}
$$

To calculate the scalar curvature R we only need some components of the Riemann tensor:

$$
R = g^{ab}R_{ab} = g^{ab}R^c{}_{acb} = 2R^{\rho\varphi}{}_{\rho\varphi} + 2R^{\rho z}{}_{\rho z} + 2R^{\varphi z}{}_{\varphi z} \ .
$$

We find (in a potentially misleading notation, where all indices are frame indices)

$$
\begin{aligned}
R^\rho{}_{z\rho z} &= \frac{\partial_\rho(F\bar\gamma) - \partial_z(F\bar\delta)}{F^2} = \frac{1}{F^2}\left(\partial^2_\rho(U-\alpha) + \partial^2_z(U-\alpha)\right) \\
&= \frac{\Delta_\delta(U-\alpha) - \frac{1}{\rho}\partial_\rho(U-\alpha)}{F^2} \ , \\
R^\varphi{}_{z\varphi z} &= -\left(\frac{\partial_z\bar\beta}{F} + \bar\alpha\bar\gamma\right) = \frac{1}{FG}\partial_z\left(\frac{\partial_z G}{FG}G\right) + \frac{\partial_z G}{FG}\frac{\partial_\rho F}{F^2} \\
&= \frac{1}{F^2}\left(\partial^2_z U - (\partial_z U)(\partial_z\alpha) - (\partial_\rho U)^2 + (\partial_\rho U)(\partial_\rho\alpha) + \frac{\partial_\rho(U-\alpha)}{\rho}\right) \ , \\
R^\rho{}_{\varphi\rho\varphi} &= \frac{\partial_\rho(G\bar\alpha)}{FG} - \bar\beta\bar\delta = \frac{1}{FG}\partial_\rho\left(-\frac{\partial_\rho G}{FG}G\right) - \frac{\partial_z F}{F^2}\frac{\partial_z G}{FG} \\
&= \frac{1}{F^2}\left(\frac{\partial_\rho U}{\rho} + \partial^2_\rho U + \frac{\partial_\rho\alpha}{\rho} - (\partial_z U)^2 + (\partial_z U)(\partial_z\alpha) - (\partial_\rho U)(\partial_\rho\alpha)\right) \ .
\end{aligned}
$$

In $R^\rho{}_{\varphi\rho\varphi} + R^\varphi{}_{z\varphi z}$ most terms cancel out, and there remains

$$
\begin{aligned}
R^\rho{}_{\varphi\rho\varphi} + R^\varphi{}_{z\varphi z} &= \frac{(\partial^2_\rho + \partial^2_z)U + \frac{2}{\rho}\partial_\rho U - (\partial_\rho U)^2 - (\partial_z U)^2}{F^2} \\
&= \frac{\Delta_\delta U + \frac{1}{\rho}\partial_\rho U - (DU)^2}{F^2} \ ,
\end{aligned}
$$

keeping in mind that in cylindrical coordinates

$$
\Delta_\delta f = \frac{\partial^2 f}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2} + \frac{\partial^2 f}{\partial z^2} \ .
$$

We conclude that

$$
\begin{aligned}
\frac{F^2}{2}R &= \Delta_\delta U + \frac{1}{\rho}\partial_\rho U - (DU)^2 + \Delta_\delta(U-\alpha) - \frac{1}{\rho}\partial_\rho(U-\alpha) \\
&= \Delta_\delta(2U-\alpha) + \frac{1}{\rho}\partial_\rho\alpha - (DU)^2 \ ,
\end{aligned}
$$

which is indeed (2.2.61) when B_ρ and A_z vanish. □

Now,

$$\lim_{R\to\infty}\frac{1}{4\pi}\Big[\int_{S_R}\partial_i(U-\frac{1}{2}\alpha)dS_i+\frac{1}{2}\int_{W_R}\alpha\,d\varphi\,dz\Big]$$
$$=\lim_{R\to\infty}\Big[\frac{1}{4\pi}\int_{C_R}[\Delta_\delta(U-\frac{\alpha}{2})+\frac{1}{2\rho}\frac{\partial\alpha}{\partial\rho}]\,d^3x+\frac{1}{4}\int_{-R}^{R}\alpha(\rho=0,z)dz\Big]\,. \tag{2.2.62}$$

The last integral vanishes by (2.2.46). Equations (2.2.61)-(2.2.62) and the dominated convergence theorem yield now

$$\begin{aligned} m &= \frac{1}{16\pi}\int\Big[{}^{(3)}R+\frac{1}{2}\rho^2e^{-4\alpha+2U}\left(\rho B_{\rho,z}-A_{z,\rho}\right)^2\Big]e^{2(\alpha-U)}d^3x \\ &\quad+\frac{1}{8\pi}\int(DU)^2\,d^3x\,. \end{aligned}\tag{2.2.63}$$

Since ${}^{(3)}R=16\pi\mu+K_{ab}K^{ab}\geq 0$ for initial data sets satisfying $\mathrm{tr}_gK=0$, where μ is the energy density (not to be confused with the constant μ in (2.2.39)), this proves positivity of mass for initial data sets as considered above.

Suppose that $m=0$ with ${}^{(3)}R\geq 0$, then (2.2.63) gives

$${}^{(3)}R=\rho B_{\rho,z}-A_{z,\rho}=DU=0\,. \tag{2.2.64}$$

The last equality implies $U\equiv 0$, and from (2.2.61) we conclude that

$$\Delta_\delta\alpha-\frac{1}{2\rho}\frac{\partial\alpha}{\partial\rho}=0\,.$$

The maximum principle applied on the set

$$B(R)\setminus\{\rho\leq 1/R\}$$

gives $\alpha\equiv 0$ after passing to the limit $R\to\infty$. The before-last equality in (2.2.64) shows that the form $\rho B_\rho d\rho+A_zdz$ is closed, and simple-connectedness implies that there exists a function λ such that $\rho B_\rho d\rho+A_zdz=d\lambda$, bringing the metric (2.2.1) to the form

$$d\rho^2+dz^2+\rho^2\left(d(\varphi+\lambda)\right)^2\,. \tag{2.2.65}$$

Hence g is flat. One could now attempt to analyse $\varphi+\lambda$ near the axis of rotation to conclude that $(\rho,\varphi+\lambda,z)$ provide a new global polar coordinate system, and deduce that g is the Euclidean metric. However, it is simpler to invoke the Hadamard–Cartan theorem to achieve that conclusion.

Summarizing, we have proved:

THEOREM 2.2.12 *Consider a metric of the form (2.2.1) on $M=\mathbb{R}^3$, where (ρ,φ,z) are polar coordinates, with Killing vector ∂_φ, and suppose that the decay conditions (2.2.57)-(2.2.58) hold. If*

$${}^3R\geq 0$$

then $0 \le m \le \infty$*. Furthermore, we have* $m < \infty$ *if and only if*

$$^3R \in L^1(\mathbb{R}^3) \;, \qquad DU,\; \rho B_{\rho,z} - A_{z,\rho} \in L^2(\mathbb{R}^3) \;.$$

Finally, $m = 0$ *if and only if* g *is the Euclidean metric.* □

REMARK 2.2.13 Theorem 2.2.7 shows that the coordinates required above exist for a general asymptotically flat axisymmetric metric on $\mathbb{R}^3$ if (2.2.10) holds with $k = 6$.

2.2.7 Several asymptotically flat ends

Theorem 2.2.12 proves positivity of mass for axi-symmetric metrics on $\mathbb{R}^3$. More generally, one has the following:

THEOREM 2.2.14 *Let* (M, g) *be a simply connected three dimensional Riemannian manifold which is the union of a compact set and of a finite number of asymptotic regions* M_i, $i = 1, \ldots, N$*, which are asymptotically flat in the sense of (2.2.10)-(2.2.11) with* $k \ge 6$*. If* g *is invariant under an action of* $\mathrm{U}(1)$*, and if*

$$^3R \ge 0 \;,$$

then the ADM mass m_i *of each of the ends* M_i *satisfies* $0 < m_i \le \infty$*, with* $m_i < \infty$ *if and only if*

$$^3R \in L^1(M_i) \;, \qquad DU,\; \rho B_{\rho,z} - A_{z,\rho} \in L^2(M_i) \;.$$

PROOF: The result follows immediately from the calculations in this section together with Theorem 2.2.9: Indeed, one can integrate (2.2.61) on a set

$$\hat{C}_R := C_R \setminus C_{1/R} = \{-R \le z \le R \;,\; 1/R \le \rho \le R\} \;,$$

where C_R is as in (2.2.60). The asymptotics (2.2.50) implies that the boundary integrals over the boundary of $C_{1/R}$ give zero contribution in the limit $R \to \infty$, so that (2.2.63) remains valid by the monotone convergence theorem in spite of the (mildly) singular behavior at the punctures $\vec{a}_i$ of the functions appearing in the metric. □

2.2.8 Nondegenerate instantaneous horizons

In order to motivate the boundary conditions in this section, recall that in Weyl coordinates the Schwarzschild metric takes the form (*cf.*, *e.g.*, [172, Equation (20.12)])

$$\mathfrak{g} = -e^{2U_{\mathrm{Schw}}}dt^2 + e^{-2U_{\mathrm{Schw}}}\rho^2 d\varphi^2 + e^{2\lambda_{\mathrm{Schw}}}(d\rho^2 + dz^2) \;, \tag{2.2.66}$$

where

$$U_{\mathrm{Schw}} \;=\; \ln\rho - \ln\left(m\sin\tilde{\theta} + \sqrt{\rho^2 + m^2\sin^2\tilde{\theta}}\right) \tag{2.2.67}$$

$$\begin{aligned}
&= \frac{1}{2}\ln\left[\frac{\sqrt{(z-m)^2+\rho^2}+\sqrt{(z+m)^2+\rho^2}-2m}{\sqrt{(z-m)^2+\rho^2}+\sqrt{(z+m)^2+\rho^2}+2m}\right] , && (2.2.68)\\
\lambda_{\mathrm{Schw}} &= -\frac{1}{2}\ln\left[\frac{(r_{\mathrm{Schw}}-m)^2-m^2\cos^2\tilde\theta}{r_{\mathrm{Schw}}^2}\right] && (2.2.69)\\
&= -\frac{1}{2}\ln\left[\frac{4\sqrt{(z-m)^2+\rho^2}\sqrt{(z+m)^2+\rho^2}}{\left(2m+\sqrt{(z-m)^2+\rho^2}+\sqrt{(z+m)^2+\rho^2}\right)^2}\right] && (2.2.70)
\end{aligned}$$

In (2.2.67) the angle $\tilde\theta$ is a Schwarzschild angular variable, with the relations

$$\begin{gathered}
2m\cos\tilde\theta = \sqrt{(z+m)^2+\rho^2}-\sqrt{(z-m)^2+\rho^2}\ ,\\
2(r_{\mathrm{Schw}}-m) = \sqrt{(z+m)^2+\rho^2}+\sqrt{(z-m)^2+\rho^2}\ ,\\
\rho^2 = r_{\mathrm{Schw}}(r_{\mathrm{Schw}}-2m)\sin^2\tilde\theta\ ,\quad z=(r_{\mathrm{Schw}}-m)\cos\tilde\theta\ ,
\end{gathered}$$

where r_{Schw} is the usual Schwarzschild radial variable such that $e^{2U_{\mathrm{Schw}}} = 1 - 2m/r_{\mathrm{Schw}}$. As is well known, and in any case easily seen, U_{Schw} is smooth on $\mathbb{R}^3$ except on the set $\{\rho=0, -m\le z\le m\}$. From (2.2.67) we find, at fixed z in the interval $-m<z<m$ and for small ρ,

$$U_{\mathrm{Schw}}(\rho,z) = \ln\rho - \ln(2\sqrt{(m+z)(m-z)}) + O(\rho^2) \tag{2.2.71}$$

(with the error term *not* uniform in z). This justifies our definition: an interval $[a,b]\subset\mathscr{A}$ will be said to be a *nondegenerate instantaneous horizon* if for fixed $z\in(a,b)$ and for small ρ we have

$$U(\rho,z) = \ln\rho + \mathring{U}(z) + o(1)\ ,\quad \partial U(\rho,z) = \partial\ln\rho + \partial\mathring{U}(z) + o(1)\ , \tag{2.2.72}$$

for a smooth function $\mathring{U}$. As in the Schwarzschild case the function $U-\alpha$ is assumed to be smooth across I. Thus, to compensate for the logarithmic singularity of U, we further assume, again for fixed $z\in(a,b)$ and for small ρ, that there exists a function $\mathring{\lambda}(z)$ such that

$$\alpha(\rho,z) = U(\rho,z) + \mathring{\lambda}(z) + o(1)\ . \tag{2.2.73}$$

Under those conditions the calculation of the mass formula proceeds as follows. For $k=1,\dots,N$ let

$$I_k = [c_k,d_k]\subset\mathscr{A}$$

be pairwise disjoint intervals at which the nondegenerate instantaneous horizon boundary conditions hold. Denote by $\tilde U$ the function, harmonic on $\mathbb{R}^3\setminus\cup_k I_k$, which is the sum of Schwarzschild potentials U_{Schw} as in (2.2.68), each with mass $(d_k-c_k)/2$ and a logarithmic singularity at I_k. As in [104], the term $|DU|^2$ in (2.2.61) is rewritten as:

$$|DU|^2 = |D(U-\tilde U+\tilde U)|^2 = |D(U-\tilde U)|^2 + D_i\Big[(2U-\tilde U)D^i\tilde U\Big]\ .$$

Denote by I_ϵ the set of points which lie a distance less than or equal to ϵ from the singular set $\cup_k I_k$:

$$I_\epsilon = \{p\mid d(p,\cup_k I_k)\le\epsilon\}\ .$$

By inspection of the calculations so far one finds that (2.2.63) becomes now

$$
\begin{aligned}
m &= \frac{1}{16\pi}\int \left[{}^{(3)}R + \frac{1}{2}\rho^2 e^{-4\alpha+2U}\left(\rho B_{\rho,z} - A_{z,\rho}\right)^2\right] e^{2(\alpha-U)} d^3x \\
&\quad + \frac{1}{8\pi}\int \left(D(U-\tilde U)\right)^2 d^3x \\
&\quad + \frac{1}{8\pi}\lim_{\epsilon\to 0}\int_{\partial I_\epsilon}\left[D^i(2U-\alpha) - (2U-\tilde U)D^i\tilde U + \alpha\frac{D^i\rho}{\rho}\right] n_i d^2S\,. \quad (2.2.74)
\end{aligned}
$$

In the last line of (2.2.74) the normal n_i, taken with respect to the flat metric, has been chosen to point away from I_ϵ.

Away from the end points of the intervals I_k the logarithmic terms in U, $\tilde U$ and α cancel out, leaving a contribution

$$
\frac{1}{4}\sum_k \left(|I_k| + \int_{I_k}(\mathring\lambda + \mathring\beta)dz\right)\,,
$$

where $|I_k|$ is the length of I_k, and where we have denoted by $\mathring\beta$ the limit at $\cup_k I_k$ of $\tilde U - U$,

$$
\mathring\beta(z) := \lim_{\rho\to 0\,,\ z\in\cup_k I_k}\left(\tilde U(\rho,z) - U(\rho,z)\right)\,.
$$

As already pointed out, the error term in (2.2.71) is not uniform in z, and therefore it is not clear whether or not there will be a separate contribution from the end points of I_k to the limit as ϵ tends to zero of the integral over ∂I_ϵ. Assuming that no such contribution arises[5], we conclude that the following formula for the mass holds:

$$
\begin{aligned}
m &= \frac{1}{16\pi}\int \left[{}^{(3)}R + \frac{1}{2}\rho^2 e^{-4\alpha+2U}\left(\rho B_{\rho,z} - A_{z,\rho}\right)^2\right] e^{2(\alpha-U)} d^3x \\
&\quad + \frac{1}{8\pi}\int \left(D(U-\tilde U)\right)^2 d^3x \\
&\quad + \frac{1}{4}\sum_k \left(|I_k| + \int_{I_k}(\mathring\lambda + \mathring\beta)dz\right)\,. \quad (2.2.75)
\end{aligned}
$$

In the Schwarzschild case the volume integrals vanish, $\mathring\beta = 0$, for $z \in (-m, m)$ the function $\mathring\lambda$ equals

$$
\mathring\lambda(z) = -\frac{1}{2}\ln\left[\frac{(m-z)(z+m)}{(2m)^2}\right]\,,
$$

and one can check (2.2.75) by a direct calculation of the integral over I_1.

[5]Note that this assumption, asymptotic flatness, finiteness of the volume integral in (2.2.74), and the boundary condition (2.2.72) on U essentially enforce the boundary condition (2.2.73) on α.

2.2.9 Conical singularities

So far we have assumed that the metric is smooth across the rotation axis $\mathscr{A}$. However, in some situations this might not be the case. One of the simplest examples is the occurrence of conical singularities, when the regularity condition (2.2.46) fails to hold. It is not clear what happens with the construction of the coordinates (2.2.1) in such a case, and therefore it appears difficult to make general statements concerning such metrics. Nevertheless, there is at least one instance where conical singularities occur naturally, namely in the usual construction of stationary axisymmetric solutions: here one assumes at the outset that the space-time metric takes a form which reduces to (2.2.1) after restriction to slices of constant time; and the components of the metric are then obtained by various integrations starting from a solution of a harmonic map equation; *cf.*, *e.g.,* [84, 145, 186].

So consider a metric of the form (2.2.1) on $\mathbb{R}^3 \setminus \{\vec{a}_i\}$, where each puncture $\vec{a}_i$ corresponds to either an asymptotically flat region or to asymptotically cylindrical regions (which, typically, correspond to degenerate black holes). Assuming that $d\alpha$ is bounded at the axis and does not give any supplementary contribution at the punctures, (2.2.63) becomes instead

$$\begin{aligned} m &= \frac{1}{16\pi}\int_{\mathbb{R}^3\setminus\{\vec{a}_i\}}\Big[{}^{(3)}R+\frac{1}{2}\rho^2e^{-4\alpha+2U}\left(\rho B_{\rho,z}-A_{z,\rho}\right)^2\Big]e^{2(\alpha-U)}d^3x \\ &\quad +\frac{1}{8\pi}\int_{\mathbb{R}^3\setminus\{\vec{a}_i\}}(DU)^2\,d^3x+\frac{1}{4}\int_{\mathscr{A}\setminus\{\vec{a}_i\}}\mathring{\alpha}\,dz\,, \end{aligned} \tag{2.2.76}$$

where $\mathring{\alpha}$ denotes the restriction of α to $\mathscr{A}$.

Using (2.2.76) and (2.2.75), the reader will easily work out a mass formula when both conical singularities and nondegenerate instantaneous horizons occur.

2.2.10 A lower bound on the mass of black-holes

An alternative to Weyl coordinates, in which event horizons are *not* subsets of the axis of rotation, is provided by the following result proved in [71]:

THEOREM 2.2.15 *Let (M,g) be a three-dimensional smooth simply connected manifold with a smooth connected compact boundary ∂M and assume that (M,g) admits a Killing vector field with periodic orbits. Furthermore, assume that (M,g) has one asymptotically flat end where it satisfies (2.2.10) for some $k\geq 5$. Then there exists a unique real number $m_1>0$ such that M is diffeomorphic to $\mathbb{R}^3\setminus B(0,\frac{m_1}{2})$ and, in cylindrical-type coordinates (ρ_S,z_S,φ) on $\mathbb{R}^3$, g takes the form*

$$g=e^{-2U_S+2\alpha_S}(d\rho_S^2+dz_S^2)+\rho_S^2\,e^{-2U_S}(d\varphi+\rho_S\,\bar{B}_S\,d\rho_S+\bar{A}_S\,dz_S)^2\,, \tag{2.2.77}$$

where ∂_φ is the rotational Killing vector field, U_S, α_S, $\bar{B}_S$ and $\bar{A}_S$ are smooth functions on M which are φ-independent and satisfy $\alpha_S=0$ whenever $\rho_S=0$ and

$$U_S=o_{k-3}(r_S^{-1/2})\,,\quad \alpha_S=o_{k-4}(r_S^{-1/2})\,,\qquad \bar{B}_S=o_{k-3}(r_S^{-5/2})\,,$$

$$\bar{A}_S = o_{k-3}(r_S^{-3/2}) \quad \text{for} \quad r_S = \sqrt{\rho_S^2 + z_S^2} \to \infty \, . \tag{2.2.78}$$

□

The main result of [71] is stated as follows:

THEOREM 2.2.16 *Under the hypotheses of Theorem 2.2.15, if M has non-negative scalar curvature and if the mean curvature of ∂M with respect to the normal pointing towards M is non-positive, then the ADM mass of (M, g) satisfies*

$$m > \frac{\pi}{4} m_1 \, , \tag{2.2.79}$$

where m_1 is given by Theorem 2.2.15.

Even though the constant m_1 is uniquely determined, it should be admitted that it is not easy to determine m_1 if the metric is not given directly in the coordinate system (2.2.77). In the general case, one needs to solve a certain PDE on M, and then m_1 can be read off from the asymptotic behaviour of the solution at infinity, the reader is referred to [71] for details.

Note that the simple-connectedness of M would be a consequence of the topological censorship theorem of [96] if M were a Cauchy hypersurface for $J^+(M) \cap J^-(\mathscr{I}^+)$.

A satisfactory generalisation of the above to degenerate horizons would require a thorough understanding of the behaviour of the metric near such horizons, a problem which is widely unexplored.

The Schwarzschild metric shows that the inequality (2.2.79) is *not* sharp. One expects that the sharp inequality is

$$m \geq m_1 \, , \tag{2.2.80}$$

with equality if and only if M is a time-symmetric Cauchy hypersurface for the d.o.c. of the Schwarzschild-Kruskal-Szekeres space-time.

2.3 The Bartnik-Witten rigidity theorem

A simple proof of positivity of mass can be given when one assumes that the Ricci tensor of (M, g) is non-negative:

THEOREM 2.3.1 ("Non-existence of gravitational instantons" (Witten [187], Bartnik [10])) *Let (M, g) be a complete Riemannian manifold with an asymptotically flat end, in the sense of (1.1.57)-(1.1.58) with decay rate $\alpha > (n-2)/2$, and suppose that the Ricci tensor of g is non-negative definite:*

$$\forall \; X \qquad \mathrm{Ric}(X, X) \geq 0 \, . \tag{2.3.1}$$

Then

$$0 \leq m \leq \infty \, ,$$

with $m = 0$ if and only if $(M, g) = (\mathbb{R}^n, \delta)$.

PROOF: If $R(g) \notin L^1(M)$, the result follows from point 2 of Theorem 1.1.4. From now on we therefore assume that the Ricci scalar of g is integrable over M.

We start by deriving the so-called *Bochner* identity. Suppose that

$$\Delta f = 0 \ , \tag{2.3.2}$$

set

$$\varphi := |Df|^2 = D^k f D_l f \ .$$

We have

$$\begin{aligned} \Delta\varphi &= D^i D_i (D^k f D_k f) \\ &= 2\Big(D^i D^k f D_i D_k f + D^k f \underbrace{D^i D_i D_k f}_{= D^i D_k D_i f = D_k \Delta f + R_i{}^{ji}{}_k D_j f} \Big) \\ &= 2\Big(|\mathrm{Hess}\, f|^2 + \mathrm{Ric}(Df, Df) \Big) \ . \end{aligned} \tag{2.3.3}$$

This shows that $\Delta\varphi \geq 0$ when (2.3.1) holds.

We shall for simplicity assume that (M,g) has only one asymptotic end, the general case requires some technicalities which are not interesting from the point of view of this work. We will use (2.3.3) with $f = y^i$, where y^i is a solution of the Laplace equation (2.3.2) with the asymptotic condition

$$y^i - x^i = O(r^{1-\alpha}) \ . \tag{2.3.4}$$

The existence of such functions is plausible, but a complete proof requires some work, we refer the reader to [10] for details. The results there also show that the functions y^i form an admissible coordinate system, at least for large r, and Theorem 1.1.12 implies that we can use those coordinates to calculate the mass. We denote by φ^i the corresponding φ function, $\varphi^i = |Dy^i|^2$.

In the y–coordinate system we have

$$\varphi^i := g^{kl}\partial_k y^i \partial_l y^i = g^{ii} \quad \text{(no summation over } i\text{)},$$

so that

$$D^k\varphi^i = g^{kl}\partial_l g^{ii} = -\partial_k g_{ii} + O(r^{-2\alpha-1}) \quad \text{(no summation over } i\text{)}. \tag{2.3.5}$$

Integrating (2.3.3) with φ replaced by φ^i over (M,g) one has

$$\int_{S_\infty} D^k\varphi^i dS_k = \int_M \Delta\varphi^i = 2\int_M (|\mathrm{Hess}\, y^i| + \mathrm{Ric}(D\varphi^i, D\varphi^i)) \geq 0 \ , \tag{2.3.6}$$

and (2.3.5) gives

$$-\sum_i \int_{S_\infty} \partial_k g_{ii} dS_k \geq 0 \ . \tag{2.3.7}$$

It remains to relate this to the ADM mass. Since the coordinates y^i are harmonic we have

$$0 = \Delta y^i = \frac{1}{\sqrt{\det g}}\partial_k(\sqrt{\det g}\, g^{kl}\partial_l y^i) = \frac{1}{\sqrt{\det g}}\partial_k(\sqrt{\det g}\, g^{ki}) \ ,$$

so that

$$0 = \partial_k(\sqrt{\det g} g^{ki}) = \frac{1}{2}\partial_i g_{jj} - \partial_k g_{ki} + O(r^{-1-2\alpha}) ,$$

which leads to

$$m = \frac{1}{16\pi}\int_{S_\infty}(\partial_i g_{ik} - \partial_k g_{ii})dS_k = -\frac{3}{32\pi}\int_{S_\infty}\partial_k g_{ii} dS_k \geq 0 \qquad (2.3.8)$$

by (2.3.7). This establishes non-negativity of m. Now, if the mass vanishes, then (2.3.6) enforces $\mathrm{Hess}\, y^i = 0$ for all i. It follows that the one forms $Y^{(i)} := dy^i$ are covariantly constant,

$$DY^{(i)} = Ddy^i = \mathrm{Hess}\, y^i = 0 .$$

This implies

$$0 = D_{[i}D_{j]}Y^{(k)} = \frac{1}{2}R^\ell{}_{kij}\partial_\ell$$

so that the Riemann tensor vanishes. Let $\hat{M}$ be the universal covering space of M with the metric obtained by pull-back from the projection map, the Hadamard-Cartan theorem (see, *e.g.*, [149, Theorem 22, p. 278]) shows that the exponential map of any point $p \in \hat{M}$ is a global diffeomorphism from $\hat{M}$ to $\mathbb{R}^n$. It follows that M is a quotient of Euclidean $\mathbb{R}^n$ by a subgroup G of the Euclidean group. The existence of an asymptotically flat region in M, diffeomorphic to $\mathbb{R}^n \setminus B(R)$, shows that G must be trivial, and the result follows. □

2.4 Small data positive energy theorem

One of the results of Bartnik in [10] is the proof of positivity of energy under the hypothesis that the space-metric is near the Euclidean one; a closely related analysis has been previously carried out by Brill, Choquet-Bruhat and Deser [39, 50]. The result follows from the striking fact that, for metrics near the flat one, in harmonic coordinates the L^2 norm of the derivatives of the metric is estimated by the mass: see (2.4.3) below.

In order to state the result in an optimal form we need to introduce some notation. Set

$$\sigma(x) := (1 + |x|^2)^{\frac{1}{2}} .$$

The weighted Sobolev space $W^{k,q,-\tau}$ is defined using the following norm:

$$\|f\|_{W^{k,q,-\tau}} = \sum_{i=0}^{k}\Big(\int_{\mathbb{R}^n}|D^i f|^q \sigma^{(\tau+i)q-n} d^n x\Big)^{\frac{1}{q}} . \qquad (2.4.1)$$

To gain some insight into those spaces suppose that f behaves as (or equals) r^α for $r \geq 1$, then finiteness of $\|f\|_{W^{k,q,-\tau}}$ is governed by the finiteness, as R tends to infinity, of the integrals

$$\begin{aligned}\int_1^R r^{q(\alpha-i)}r^{(\tau+i)q-n}r^{n-1}dr &= \int_1^R r^{q(\alpha+\tau)-1}dr \\ &= \begin{cases}\frac{1}{q(\alpha+\tau)}(R^{q(\alpha+\tau)}-1), & q(\alpha+\tau)\neq 0; \\ \ln R, & \text{otherwise.}\end{cases}\end{aligned}$$

It follows that a function such as, *e.g.*, $f = (1+r)^\alpha$ will be in $W^{k,q,-\tau}$ if and only if $\alpha < -\tau$. So the index τ governs the decay rate of the functions in $W^{k,q,-\tau}$: Indeed, for $kq > n$ the weighted Sobolev embeddings proved in [10] show that any function $f \in W^{k,q,-\tau}$ satisfies $f = o(r^{-\tau})$ for large r.

We have the following [10]:

THEOREM 2.4.1 *Let $q > 3$, $\tau > (n-2)/2$. There exists $\epsilon = \epsilon(n,q,\tau) > 0$ such that for any metric g on $M = \mathbb{R}^n$ satisfying*

$$\|g - \delta\|_{W^{2,q,-\tau}} \leq \epsilon \tag{2.4.2}$$

there exists a global harmonic coordinate system on M. If $R(g) \in L^1(M)$ or if $R(g) \geq 0$, in any such coordinate system one has

$$\int_{\mathbb{R}^3} \left(R(g) + \frac{1}{8}|\partial g|_g^2 \right) d^n x \leq \frac{16\pi}{3} m \,. \tag{2.4.3}$$

In particular $m \geq 0$ if $R(g) \geq 0$, with equality if and only if $g_{ij} = \delta_{ij}$.

REMARK 2.4.2 Readers who do not feel at ease with weighted Sobolev spaces can, instead of (2.4.2), assume the stronger condition

$$|g_{ij} - \delta_{ij}| \leq \epsilon\,(1+r)^{-\alpha} \,, \quad |\partial_k g_{ij}| \leq \epsilon\,(1+r)^{-\alpha-1} \,, \quad |\partial_\ell \partial_k g_{ij}| \leq \epsilon\,(1+r)^{-\alpha-2} \,, \tag{2.4.4}$$

with some $\alpha > (n-1)/2$, and reach the same conclusion. These conditions guarantee existence of a harmonic coordinate system in which the first two conditions in (2.4.4) will still hold, with ϵ replaced by $C\epsilon$ for some constant C. The last inequality in (2.4.4) might fail in the new coordinates, but the first two suffice for the remaining arguments of the proof.

PROOF: We start by noting that the weighted Sobolev embedding alluded to above garantees that the conditions for a well defined mass (perhaps infinite, if $R \notin L^1$) are met. Next, it is shown in [10] that if ϵ is small enough, then one can introduce global harmonic coordinates on $\mathbb{R}^3$, with (2.4.2) holding with ϵ replaced by $C\epsilon$ for some constant C.

Let us write $|g|$ for $\det g_{ij}$. The harmonic coordinate condition,

$$\Delta x^i = \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} g^{ij}) = 0 \,,$$

can be rewritten as

$$\partial_i \ln |g| = -2 g_{ik} \partial_j g^{jk} \,. \tag{2.4.5}$$

A somewhat lengthy calculation shows that in harmonic coordinates we have

$$-\frac{1}{2}\Delta_g(\log|g|) = R(g) + \frac{1}{4}\underbrace{g^{ij}g^{k\ell}g^{pq}\partial_i g_{kp}\partial_j g_{\ell q}}_{=:|\partial g|_g^2} - \frac{1}{2} g_{k\ell}\partial_i g^{jk}\partial_j g^{i\ell} \,. \tag{2.4.6}$$

To establish (2.4.6) we start by noting that, for any function f,

$$\Delta_g f = g^{ij} D_i D_j f = g^{ij}(\partial_i \partial_j f - \Gamma^k{}_{ij}\partial_k f \ ,$$

and setting $f = x^\ell$ this gives

$$0 = \Delta_g x^\ell = -g^{ij}\Gamma^\ell{}_{ij} =: -\Gamma^\ell \ .$$

So, harmonicity is equivalent to the vanishing of the Γ^ℓ's. Further, in harmonic coordinates, we conclude that for any f it holds that

$$\Delta_g f = g^{ij}\partial_i\partial_j f - \underbrace{g^{ij}\Gamma^k{}_{ij}}_{0}\partial_k f = g^{ij}\partial_i\partial_j f \ ,$$

Next, we will use the identity

$$\Gamma^i{}_{ij} = \frac{1}{2}\partial_j \ln|g| \ .$$

The calculation proceeds now as follows: by the formula (A.12.4) for the Riemann tensor we have

$$\begin{aligned} R_{ij} &= R^k{}_{ikj} \\ &= \partial_k\Gamma^k{}_{ij} - \partial_j\Gamma^k{}_{ik} + \Gamma^k{}_{\ell k}\Gamma^\ell{}_{ij} - \Gamma^k{}_{\ell j}\Gamma^\ell{}_{ik} \ , \end{aligned} \tag{2.4.7}$$

and so

$$\begin{aligned} R &= g^{ij}R_{ij} \\ &= g^{ij}\partial_k\Gamma^k{}_{ij} - g^{ij}\partial_j\Gamma^k{}_{ik} + \Gamma^k{}_{\ell k}\underbrace{g^{ij}\Gamma^\ell{}_{ij}}_{0} - g^{ij}\Gamma^k{}_{\ell j}\Gamma^\ell{}_{ik} \\ &= \partial_k(\underbrace{g^{ij}\Gamma^k{}_{ij}}_{0}) - \Gamma^k{}_{ij}\underbrace{\partial_k g^{ij}}_{-g^{i\ell}g^{jm}\partial_k g_{\ell m}} - g^{ij}\partial_j\underbrace{\Gamma^k{}_{ik}}_{\frac{1}{2}\partial_i\ln|g|} - g^{ij}\Gamma^k{}_{\ell j}\Gamma^\ell{}_{ik} \\ &= g^{i\ell}g^{jm}\partial_k g_{\ell m}\Gamma^k{}_{ij} - \frac{1}{2}\Delta_g\ln|g| - g^{ij}\Gamma^k{}_{\ell j}\Gamma^\ell{}_{ik} \\ &= g^{ij}(g^{\ell m}\partial_k g_{mi} - \Gamma^\ell{}_{ik})\Gamma^k{}_{\ell j} - \frac{1}{2}\Delta_g\ln|g| \\ &= \frac{1}{2}g^{ij}g^{\ell m}(\partial_k g_{mi} - \partial_i g_{mk} + \partial_m g_{ik})\Gamma^k{}_{\ell j} - \frac{1}{2}\Delta_g\ln|g| \ . \end{aligned} \tag{2.4.8}$$

It remains to express $\Gamma^k{}_{\ell j}$ in terms of partial derivatives of g and carry out the product. Each of the nine resulting terms coincides with one of the last two terms appearing in (2.4.6); summing gives the result.

It might be useful to note that the norm

$$|\partial g|^2_g = g^{ij}g^{k\ell}g^{pq}\partial_i g_{kp}\partial_j g_{\ell q}$$

used in (2.4.6) is equivalent to

$$|\partial g|^2_\delta = \delta^{ij}\delta^{k\ell}\delta^{pq}\partial_i g_{kp}\partial_j g_{\ell q} = \sum_{i,k,p=1}^{n}|\partial_i g_{kp}|^2 \ .$$

In fact, since $g_{ij} - \delta_{ij} = O(\epsilon)$, possibly in an integral sense, we have for some constant c

$$(1 - c\epsilon)|\partial g|^2_g \le |\partial g|^2_\delta \le (1 + c\epsilon)|\partial g|^2_g \ ,$$

possibly in an integral sense. Similarly,

Equation (2.4.5) further implies

$$\Delta_g \ln|g| = g^{ij}\partial_i\partial_j \ln|g| = -2\partial_j\partial_k g^{jk} + \frac{1}{2}|D\ln|g||^2_g \,.$$

Inserting into (2.4.6) we obtain

$$\partial_j\partial_k g^{jk} = R(g) + \frac{1}{4}|D\ln|g||^2_g + \frac{1}{4}|\partial g|^2_g - \frac{1}{2}g_{k\ell}\partial_i g^{jk}\partial_j g^{i\ell} \,. \tag{2.4.9}$$

Similarly to (2.3.8), the integral of the left-hand-side over $\mathbb{R}^n$ (with respect to the Euclidean measure) will give a contribution equal to $16\pi m/3$, compare (2.4.5). To finish the proof one needs to do something with the last term, the sign of which is not clear. This is handled as follows:

$$\begin{aligned} g_{k\ell}\partial_i g^{jk}\partial_j g^{i\ell} &= (g_{k\ell}-\delta_{k\ell})\partial_i g^{jk}\partial_j g^{i\ell} + \partial_i g^{j\ell}\partial_j g^{i\ell} \\ &= (g_{k\ell}-\delta_{k\ell})\partial_i g^{jk}\partial_j g^{i\ell} + \partial_i(g^{j\ell}\partial_j g^{i\ell} - g^{i\ell}\partial_j g^{j\ell}) \\ &\quad +\partial_i g^{i\ell}\partial_j g^{j\ell} \,. \end{aligned}$$

The first term in the before last-line can be estimated by $C\epsilon|\partial g|^2_g/2$, for some constant C. The term in the last line can be rewritten as

$$\partial_i g^{ik}\partial_j g^{j\ell}(\delta_{k\ell} - g_{k\ell} + g_{k\ell}) = \underbrace{\partial_i g^{ik}\partial_j g^{j\ell}(\delta_{k\ell} - g_{k\ell})}_{\leq C\epsilon/2} + \frac{1}{4}|D\ln|g||^2_g \,.$$

Hence

$$|g_{k\ell}\partial_i g^{jk}\partial_j g^{i\ell} - \partial_i(g^{j\ell}\partial_j g^{i\ell} - g^{i\ell}\partial_j g^{j\ell})| \;\leq\; C\epsilon|\partial g|^2_g + \frac{1}{4}|D\ln|g||^2_g \,.$$

Now,

$$\partial_i\Big(g^{j\ell}\partial_j g^{i\ell} - g^{i\ell}\partial_j g^{j\ell}\Big) = \partial_i\Big((g^{j\ell}-\delta^{j\ell})\partial_j g^{i\ell} - (g^{i\ell}-\delta^{i\ell})\partial_j g^{j\ell}\Big) = \partial_i\Big(O(r^{-1-2\tau})\Big) \,,$$

which, after integration, will give a vanishing boundary contribution since $\tau > (n-2)/2$. If ϵ is such that

$$\frac{C\epsilon}{2} \leq \frac{1}{8} \,,$$

(2.4.3) follows by straightforward algebra. □

2.4.1 Negative mass metrics

So far we have been studying manifolds with asymptotic regions which resemble that of Euclidean space. For some purposes it might be of interest to consider *asymptotically locally Euclidean* (ALE) manifolds, where the asymptotic end has the topology of a discrete quotient of $\mathbb{R}^n \setminus B(R)$ by a subgroup of the orthogonal group, with the metric asymptotically approaching the quotient of the Euclidean metric. For one, such manifolds provide examples of complete metrics with a well-defined negative mass.

Indeed, in this setting, LeBrun [129] presented the first known examples of well-behaved scalar-flat ALE spaces of negative mass. The metrics take the form

$$g = \frac{dr^2}{1 + Ar^{-2} + Br^{-4}} + r^2\Big[\sigma_1^2 + \sigma_2^2 + (1 + Ar^{-2} + Br^{-4})\sigma_3^2\Big], \qquad (2.4.10)$$

where r is a radial coordinate, $\{\sigma_1, \sigma_2, \sigma_3\}$ is a left-invariant coframe on $S^3 = \mathrm{SU}(2)$, and $A = n - 2$, $B = 1 - n$, with $n \in \mathbb{N}$. After redefining the radial coordinate to be $\hat{r}^2 = r^2 - 1$, and taking a quotient by $\mathbb{Z}_n$, the metric extends smoothly across $\hat{r} = 0$, and is ALE at infinity. When suitably normalised, the mass is computed to be $-4\pi^2(n-2)$, which is non-positive when $n > 2$. For $n = 1$, this construction yields the Burns metric. For $n = 2$, this space is Ricci-flat, and is exactly the metric of Eguchi-Hanson. There is a close connection with the hyperbolic monopole metrics of [130], compare [180].

2.5 Schoen and Yau's positive energy theorem

In [164] Schoen and Yau gave the first general proof of positivity of energy. There they consider initial data sets on a manifold M which is the union of a compact set and a finite number of ends $M_i \approx \mathbb{R}^3 \setminus B(R_i)$ on which the metric behaves as

$$g_{ij} = \delta_{ij} + O_3(r^{-1}) , \qquad (2.5.1)$$

where we use the symbol $f = O_k(r^\alpha)$ if $\partial^j f = O(r^{\alpha - j})$ for $0 \leq j \leq k$. The extrinsic curvature tensor K_{ij} is further assumed to satisfy

$$K_{ij} = \delta_{ij} + O_2(r^{-1}) , \qquad (2.5.2)$$

but its trace is assumed to fall-off faster:

$$\mathrm{tr}_g K = O(r^{-3}) . \qquad (2.5.3)$$

This condition can be removed when, say, vacuum space-times are considered, by deforming the initial data surface in space-time [16], but this is a restrictive condition in general.

We then have the following:

THEOREM 2.5.1 (Schoen and Yau [164]) *Under the conditions above, suppose further that*

$$|J|_g \leq R - |K|^2 + (tr_g K)^2 = O_1(r^{-4}) , \qquad (2.5.4)$$

where

$$J^i := 2D_j(K^{ij} - trK g^{ij}) .$$

Then the ADM mass of each asymptotic end N_k is non-negative. If one of the masses is zero and if (2.5.1) is strengthened to

$$g_{ij} = \delta_{ij} + O_5(r^{-1}) , \qquad (2.5.5)$$

then the initial data set is vacuum, and $(\mathscr{S}, g)$ can be isometrically embedded into Minkowski space-time, with K_{ij} corresponding to the extrinsic curvature tensor of the embedding.

As emphasised in [162], the proof generalises to asymptotically flat manifolds of dimension $n \leq 7$. The obstruction in higher dimensions arises because of the singularities of minimal surfaces that arise for $n \geq 8$. Christ and Lohkamp have announced a proof in all dimensions [51], but details have not been made available by the time of our writing. An alternative strategy for a proof in all dimensions has been presented by Schoen at the Mittag-Leffler Institute in November 2008, but no written account of that work exists so far.

A strengthening of the positivity theorem to $m \geq |\vec{p}|$, using techniques in the spirit of [164], again in dimensions $n \leq 8$, has been presented in [80, 81].

2.6 A Lorentzian proof

The positivity results proved so far do not appear to have anything to do with Lorentzian geometry. In this section, based on [62], we prove energy positivity using purely Lorentzian techniques, albeit for a rather restricted class of geometries; it seems that in practice our proof only applies to stationary (with or without black holes) space-times. This is a much weaker statement than the theorems in [164, 187] and their various extensions [14, 28, 102, 113, 120], but the proof below is of interest because the techniques involved are completely different and of a quite elementary nature. Using arguments rather similar in spirit to those of the classical singularity theorems [109], the proof here is a very simple reduction of the problem to the Lorentzian splitting theorem [94].

In lieu of the Lorentzian splitting theorem, one can impose the *"generic condition"* [109, p. 101], thereby making the proof completely elementary. However, it is not clear how "generic" the generic condition is, when, *e.g.*, vacuum equations are imposed, so it is desirable to have results without that condition.

The approach taken here bares some relation to the Penrose-Sorkin-Woolgar [155] argument for positivity of mass. One can only regret that the attractive argument in [155] has never materialized into a full proof under the generality claimed.

2.6.1 Galloway's timelike splitting theorem

The results presented in this section will play a key role in the proof of energy positivity, as presented in Section 2.6.2 below.

In Riemannian geometry *a line* is defined to be a complete geodesic which is minimising between each pair of its points. A milestone theorem of Cheeger and Gromoll [46] asserts that *a complete Riemannian manifold* (M, g) *with non-negative Ricci curvature which contains a line splits as a metric product*

$$M = \mathbb{R} \times N \;, \qquad g = dx^2 + h \;, \tag{2.6.1}$$

where x is a coordinate along the $\mathbb{R}$ factor, while h is an (x-independent) complete metric on N. This result is known under the name of *Cheeger-Gromoll splitting theorem.*

It turns out that there is a corresponding result in Lorentzian geometry, with obvious modifications: First, a line is defined by changing "minimising" to "maximising" in the definition above. Next, in the definition of "splitting" one replaces (2.6.1) with

$$M = \mathbb{R} \times N \;, \qquad g = -dt^2 + h \;, \tag{2.6.2}$$

where we use now t to denote the coordinate along the $\mathbb{R}$ factor. One has the following:

THEOREM 2.6.1 *Let $(\mathscr{M}, g)$ be a space-time satisfying the* timelike focusing condition*,*

$$\mathrm{Ric}(X,X) \geq 0 \quad \forall\ X \ \textit{timelike}\,. \tag{2.6.3}$$

Suppose that $\mathscr{M}$ contains a timelike line and either

1. *$(\mathscr{M}, g)$ is globally hyperbolic, or*
2. *$(\mathscr{M}, g)$ is timelike geodesically complete.*

Then $(\mathscr{M}, g)$ splits as in (2.6.2), for some complete metric h on N.

REMARK 2.6.2 The "geodesically complete version" of Theorem 2.6.1 was known as *Yau's splitting conjecture* before its proof by Newman [147]. The globally hyperbolic version was proved by Galloway [93]. The result assuming both timelike geodesic completeness and global hyperbolicity had previously been established by Eschenburg [85], see also [92, 94, 95].

2.6.2 The proof of positivity

For $m \in \mathbb{R}$, let g_m denote the $n+1$ dimensional, $n \geq 3$, Schwarzschild metric with mass parameter m; in isotropic coordinates [152],

$$g_m = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \left(\sum_{1=1}^{n} dx_i^2\right) - \left(\frac{1 - m/2|x|^{n-2}}{1 + m/2|x|^{n-2}}\right)^2 dt^2 \,. \tag{2.6.4}$$

We shall say that a metric g on $\mathbb{R} \times (\mathbb{R}^n \setminus B(0,R))$, $R^{n-2} > m/2$, is *uniformly Schwarzchildian* if, in the coordinates of (2.6.4),

$$g - g_m = o(|m| r^{-(n-2)}) \;, \quad \partial_\mu (g - g_m) = o(|m| r^{-(n-1)}) \,. \tag{2.6.5}$$

(Here o is meant at fixed g and m, uniformly in t and in angular variables, with r going to infinity.) It is a flagrant abuse of terminology to allow $m = 0$ in this definition, and we will happily abuse; what is meant in this case is that $g = g_0$, i.e., g is flat [6], for $r > R$.

Some comments about this notion are in order. First, metrics as above have constant Trautman-Bondi mass and therefore do not contain gravitational radiation; one expects such metrics to be stationary if physically reasonable field equations are imposed. Next, every metric in space-time dimension four which is stationary, asymptotically flat and vacuum or electro-vacuum in the

[6] The asymptotic conditions for the case $m = 0$ of our theorem are way too strong for a rigidity statement of real interest, even within a stationary context. So it is fair to say that our result only excludes $m < 0$ for stationary space-times.

asymptotically flat region is uniformly Schwarzschildian there when $m \neq 0$ (cf., e.g., [169]).

The hypotheses here are compatible with stationary black hole space-times with *non-degenerate* Killing horizons.

We say that the matter fields satisfy the *timelike convergence condition* if the Ricci tensor $R_{\mu\nu}$, as expressed in terms of the matter energy-momentum tensor $T_{\mu\nu}$, satisfies the condition

$$R_{\mu\nu}X^\mu X^\nu \geq 0 \text{ for all timelike vectors } X^\mu. \tag{2.6.6}$$

We define the *domain of outer communications* of $\mathscr{M}$ as the intersection of the past $J^-(\mathscr{M}_{\rm ext})$ of the asymptotic region $\mathscr{M}_{\rm ext} = \mathbb{R}\times(\mathbb{R}^n \setminus B(0,R))$ with its future $J^+(\mathscr{M}_{\rm ext})$.

We need a version of Hawking's *weak asymptotic simplicity* [109] for uniformly Schwarzschildian spacetimes. We shall say that a spacetime $(\mathscr{M},g)$ is *weakly asymptotically regular* if every null line starting in the domain of outer communications (d.o.c.) either crosses an event horizon (if any), or reaches arbitrarily large values of r in the asymptotically flat region. By definition, a null line in $(\mathscr{M},g)$ is an inextendible null geodesic that is globally achronal; a timelike line is an inextendible timelike geodesic, each segment of which is maximal. Finally, we shall say that the d.o.c. is *timelike geodesically regular* if every timelike line in $\mathscr{M}$ which is entirely contained in the d.o.c., and along which r is bounded, is complete.

The main result of this section is the following:

THEOREM 2.6.3 *Let $(\mathscr{M}^{n+1} = \mathscr{M}, g)$ be an $(n+1)$-dimensional space-time with matter fields satisfying the timelike convergence condition (2.6.6), and suppose that $\mathscr{M}$ contains a uniformly Schwarzschildian region*

$$\mathscr{M}_{\rm ext} = \mathbb{R}\times(\mathbb{R}^n \setminus B(0,R)) \ . \tag{2.6.7}$$

Assume that $(\mathscr{M},g)$ is weakly asymptotically regular and that the domain of outer communications is timelike geodesically regular. If the domain of outer communications of $\mathscr{M}$ has a Cauchy surface $\mathscr{S}$, the closure of which is the union of one asymptotic end and of a compact interior region (with a smooth boundary lying at the intersection of the future and past event horizons, if any), then

$$m > 0$$

unless $(\mathscr{M},g)$ isometrically splits as $\mathbb{R}\times\mathscr{S}$ with metric $g = -d\tau^2+\gamma$, $\mathscr{L}_{\partial_\tau}\gamma = 0$, and $(\mathscr{S},\gamma)$ geodesically complete. Furthermore, the last case does not occur if event horizons are present.

Before passing to the proof, we note the following Corollary:

COROLLARY 2.6.4 *In addition to the hypotheses of Theorem 2.6.3, assume that*

$$T_{\mu\nu} \in L^1(\mathbb{R}^n \setminus B(0,R)) \ , \qquad \partial_\nu\partial_\mu g = O(r^{-\alpha}) \ , \quad \alpha > 1+\frac{n}{2} \ . \tag{2.6.8}$$

Then $m > 0$ unless $\mathscr{M}$ is the Minkowski space-time.

Proof of Theorem 2.6.3: The idea is to show that for $m \le 0$ the domain of outer communications contains a timelike line, and the result then follows from Galloway's splitting theorem 2.6.1, Section 2.6.1.

From (2.6.4) and (2.6.5) we have $\Gamma^\mu_{\nu\rho} = o(|m|r^{-(n-1)})$ except for

$$\Gamma^k_{00} = \Gamma^0_{k0} = \Gamma^0_{0k} = \tfrac{(n-2)m}{r^{n-1}}\tfrac{x^k}{r} + o(|m|r^{-(n-1)}) \,, \tag{2.6.9}$$

$$\Gamma^k_{ij} = \tfrac{m}{r^{n-1}}\left(\delta_{ij}\tfrac{x^k}{r} - \delta_{jk}\tfrac{x^i}{r} - \delta_{ik}\tfrac{x^j}{r}\right) + o(|m|r^{-(n-1)}) \,. \tag{2.6.10}$$

This shows that the Hessian $\mathrm{Hess}\, r = \nabla dr$ of r is given by

$$\mathrm{Hess}\, r = -\frac{m}{r^{n-1}}\big((n-2)dt^2 - dr^2 + r^2 h\big) + r\,h + o(r^{-(n-1)}) \,, \tag{2.6.11}$$

where h is the canonical metric on S^{n-1}, and the size of the error terms refers to the components of the metric in the coordinates of (2.6.4). Note that when $m < 0$, $\mathrm{Hess}\, r$, when restricted to the hypersurfaces of constant r, is strictly positive definite for $r \ge R_1$, for some sufficiently large R_1. Increasing R_1 if necessary, we can obtain that ∂_t is timelike for $r \ge R_1$. If $m = 0$ we set $R_1 = R$. Let $p_{\pm k}$ denote the points $t = \pm k$, $\vec{x} = (0, 0, R_1)$; by global hyperbolicity there exists a maximal future directed timelike geodesic segment σ_k from p_{-k} to p_{+k}. We note, first, that the σ_k's are obviously contained in the domain of outer communications and therefore cannot cross the event horizons, if any. If $m = 0$ then σ_k clearly cannot enter $\{r > R_1\}$, since timelike geodesics in that region are straight lines which never leave that region once they entered. It turns out that the same occurs for $m < 0$: suppose that σ_k enters $\{r > R_1\}$, then the function $r \circ \sigma_k$ has a maximum. However, if s is an affine parameter along σ_k we have

$$\frac{d^2(r\circ\sigma_k)}{ds^2} = \mathrm{Hess}\, r(\dot\sigma_k, \dot\sigma_k) > 0$$

at the maximum if $m < 0$, since $dr(\dot\sigma_k) = 0$ there, which is impossible. It follows that all the σ_k's (for k sufficiently large) intersect the Cauchy surface $\mathscr{S}$ in the compact set $\overline{\mathscr{S}} \setminus \{r > R_1\}$. A standard argument shows then that the σ_k's accumulate to a timelike or null line σ through a point $p \in \overline{\mathscr{S}}$. Let $\{p_k\} = \sigma_k \cap \mathscr{S}$; suppose that $p \in \partial\mathscr{S}$, then the portions of σ_k to the past of p_k accumulate at a generator of the past event horizon $\dot J^+(\mathscr{M}_{\mathrm{ext}})$, and the portions of σ_k to the future of p_k accumulate at a generator of the future event horizon $\dot J^-(\mathscr{M}_{\mathrm{ext}})$. This would result in σ being non-differentiable at p, contradicting the fact that σ is a geodesic. Thus the p_k's stay away from $\partial\mathscr{S}$, and $p \in \mathscr{S}$. By our "weak asymptotic regularity" hypothesis σ cannot be null (as it does not cross the event horizons, nor does it extend arbitrarily far into the asymptotic region). It follows that σ is a timelike line in $\mathscr{M}$ entirely contained in the globally hyperbolic domain of outer communications $\mathscr{D}$, with $r \circ \sigma$ bounded, and hence is complete by the assumed timelike geodesic regularity of $\mathscr{D}$. Thus, one may apply Galloway's splitting theorem 2.6.1 to conclude that $(\mathscr{D}, g|_{\mathscr{D}})$ is a metric product,

$$g = -d\tau^2 + \gamma \,, \tag{2.6.12}$$

for some τ–independent complete Riemannian metric γ. The completeness of this metric product implies $\mathscr{D} = \mathscr{M}$ (and in particular excludes the existence of event horizons). □

PROOF OF COROLLARY 2.6.4: The vector field ∂_τ is a static Killing vector in $\mathscr{M}_{\rm ext}$, and the usual analysis of groups of isometries of asymptotically flat space-times shows that the metric γ in (2.6.12) is asymptotically flat.

The lapse function N associated with a Killing vector field on a totally geodesic hypersurface $\mathscr{S}$ with induced metric γ and unit normal n satisfies the elliptic equation

$$\Delta_\gamma N - \mathrm{Ric}(n,n)N = 0 \ .$$

From (2.6.12) we have $N = 1$ hence $\mathrm{Ric}(n,n) = 0$, and the Komar mass of $\mathscr{S}$ vanishes. By a theorem of Beig [19] (originally proved in dimension four, but the result generalises to any dimensions under (2.6.8)) this implies the vanishing of the ADM mass. Let e_a, $a = 0,\dots,n$, be an orthonormal frame with $e_0 = \partial_\tau$. The metric product structure implies that $R_{0i} = 0$. Thus, by the energy condition, for any fixed i we have

$$0 \le \mathrm{Ric}(e_0+e_i, e_0+e_i) = R_{00} + R_{ii} = R_{ii} \ .$$

But again by the product structure, the components R_{ii} of the space-time Ricci tensor equal those of the Ricci tensor $\mathrm{Ric}_{\mathscr{S}}$ of γ. It follows that $\mathrm{Ric}_{\mathscr{S}} \ge 0$. A generalisation by Bartnik [10] of an argument of Witten [187] shows that $(\mathscr{S},\gamma)$ is isometric to Euclidean space, see Section 2.3; we repeat the proof in a nutshell here, to make clear its elementary character: Let y^i be global harmonic functions forming an asymptotically rectangular coordinate system near infinity. Let $K^i = \nabla y^i$; then by Bochner's formula,

$$\Delta |K^i|^2 = 2|\nabla K^i|^2 + 2\mathrm{Ric}_{\mathscr{S}}(K^i,K^i) \ .$$

Integrating the sum over $i = 1,\dots,n$ of this gives the ADM mass as boundary term at infinity. But this mass is zero, so we conclude that the ∇y^i's are all parallel. Since $\mathscr{S}$ is simply connected at infinity, it must be Euclidean space. $\Box$

We close this section by showing that the conditions on geodesics in Theorem 2.6.3 are always satisfied in stationary domains of outer communications.

PROPOSITION 2.6.5 *Let the domain of outer communications $\mathscr{D}$ of $(\mathscr{M},g)$ be globally hyperbolic, with a Cauchy surface $\mathscr{S}$ such that $\overline{\mathscr{S}}$ is the union of a finite number of asymptotically flat regions and of a compact set (with a boundary lying at the intersection of the future and past event horizons, if any). Suppose that there exists on $\mathscr{M}$ a Killing vector field X with complete orbits which is timelike, or stationary-rotating*[7] *in the asymptotically flat regions. Then the weak asymptotic regularity and the timelike regularity conditions hold.*

REMARK 2.6.6 We note that there might exist maximally extended null geodesics in $(\mathscr{D},g)$ which are trapped in space within a compact set (as happens for the Schwarzschild metric), but those geodesics will not be achronal.

[7]See [74] for the definition.

PROOF: By [74, Propositions 4.1 and 4.2] we have $\mathscr{D} = \mathbb{R} \times \mathscr{S}$, with the flow of X consisting of translations along the $\mathbb{R}$ axis:

$$g = \alpha d\tau^2 + 2\beta d\tau + \gamma \ , \quad X = \partial_\tau \ , \tag{2.6.13}$$

where γ is a Riemannian metric on $\mathscr{S}$ and β is a one-form on $\mathscr{S}$. (We emphasise that we do not assume X to be timelike, so that $\alpha = g(X,X)$ can change sign.) Let ϕ_t denote the flow of X and let $\sigma(s) = (\tau(s), p(s)) \in \mathbb{R} \times \mathscr{S}$ be an affinely parameterized causal line in $\mathscr{D}$, then for each $t \in \mathbb{R}$ the curve $\phi_t(\sigma(s)) = (\tau(s)+t, p(s))$ is also an affinely parameterized causal line in $\mathscr{D}$. Suppose that there exists a sequence s_i such that $p(s_i) \to \partial\mathscr{S}$, setting $t_i = -\tau(s_i)$ we have $\tau(\phi_{t_i}(\sigma(s_i))) = 0$, then the points $\{p_{k_i}\} = \phi_{t_i}(\sigma) \cap \mathscr{S}$ accumulate at $\partial\mathscr{S}$, which is not possible as in the proof of Theorem 2.6.3. Therefore there exists an open neighborhood $\mathscr{K}$ of $\partial\mathscr{S}$ such that $\sigma \cap (\mathbb{R} \times \mathscr{K}) = \emptyset$. This implies in turn that σ meets all the level sets of τ. Standard considerations using the fact that $\mathscr{D}$ is a stationary, or stationary-rotating domain of outer communications (cf., e.g., [74]) show that for every $p, q \in \mathscr{S}$ there exists $T > 0$ and a timelike curve from $(0,p)$ to (T,q). The constant T can be chosen independently of p and q within the compact set $\overline{\mathscr{S}} \setminus (\mathscr{K} \cup \{r > R_1\})$, with $R_1 = \sup_\sigma r$. It follows that an inextendible null geodesic which is bounded in space within a compact set cannot be achronal, so that σ has to reach arbitrarily large values of r, and weak asymptotic regularity follows. Similarly, if σ is a timelike line bounded in space within a compact set, then there exists $s_1 > 0$ such that for any point $(\tau(s), p(s))$ with $s = s_1 + u$, $u > 0$ one can find a timelike curve from $(0, p(0))$ to $(\tau(s), p(s))$ by going to the asymptotic region, staying there for a time u, and coming back. The resulting curve will have Lorentzian length larger than $u/2$ if one went sufficiently far into the asymptotic region, and since σ is length-maximising it must be complete. □

The key point in the proof of Proposition 2.6.5 is non-existence of observer horizons contained in the d.o.c. Somewhat more generally, we have the following result, which does not assume existence of a Killing vector:

PROPOSITION 2.6.7 *Suppose that causal lines σ, with $r \circ \sigma$ bounded, and which are contained entirely in $\mathscr{D}$, do not have* observer horizons *extending to the asymptotic region $\mathscr{M}_{\text{ext}}$ (see (2.6.7)):*

$$\dot{J}^{\pm}(\sigma; \mathscr{D}) \cap \mathscr{M}_{\text{ext}} = \emptyset \ . \tag{2.6.14}$$

Then the weak asymptotic regularity and the timelike regularity conditions hold.

PROOF: It follows from (2.6.14) that for any $u > 0$ and for any s_1 there exists s_2 and a timelike curve Γ_{u,s_1} from $\sigma(s_1)$ to $\sigma(s_2)$ which is obtained by following a timelike curve from $\sigma(s_1)$ to the asymptotic region, then staying there at fixed space coordinate for a coordinate time u, and returning back to σ along a timelike curve. One concludes as in the proof of Proposition 2.6.5. □

2.7 The Riemannian Penrose Inequality

An important generalization of the Positive Mass Theorem is given by the Riemannian Penrose Inequality:

THEOREM 2.7.1 *Let* $(\mathscr{S}, g)$ *be a complete, smooth, asymptotically flat* 3*-manifold with nonnegative scalar curvature with total mass* m *and which has an outermost minimal surface* Σ_0 *of area* A_0*. Then*

$$m \geq \sqrt{\frac{A_0}{16\pi}}\,, \qquad (2.7.1)$$

with equality if and only if $(\mathscr{S}, g)$ *is isometric to the Schwarzschild metric* $(\mathbb{R}^3 \setminus \{0\}, (1 + \frac{m}{2|x|})^4 \delta)$ *outside their respective outermost minimal surfaces.*

Theorem 2.7.1 was first proved by Huisken and Ilmanen [120] defining instead A_0 to be the area of the largest connected component of Σ_0. The proof of the version above, using completely different methods, is due to Bray [29]. The proofs are beautiful applications of geometric flows to a fundamental problem in relativity. A number of accessible reviews has been written on these important results, to which we refer the interested reader [30–32, 37, 137]. A generalization of Theorem 2.7.1 to dimensions $n \leq 7$ has been established in [36].

One expects that some form of (2.7.1) holds for general relativistic initial data sets (g, K) satisfying the dominant energy condition. A suggestion how one could prove this has been put forward by Bray and Khuri in [34, 35], compare [33, 45, 86, 138]. A Riemannian inequality in the spirit of (2.7.1), but involving some further geometric constants, has been proved by Herzlich [113].

Chapter 3

Spinors, and Witten's positive energy theorem

The aim of this chapter is to present Witten's proof [187] of positivity of the ADM mass. This makes use of spinors, which we introduce in the first section. As it turns out, the notion of a spinor field on a manifold is somewhat involved, and a proper understanding requires a certain amount of background material. (See [178] for a historical overview of the subject.) The approach in Section 3.1.1 is aimed at a reader interested in a minimal amount of information, as needed to be able to proceed with the calculations. This reader can skip the remaining material in Section 3.1 and proceed directly to the positivity proof in section 3.2. We refer to [25, 42, 91, 128, 154, 178] for extensive treatments of spinors.

3.1 Spinors: a working approach

Our goal in this section is to explain what spinor fields are, and to present the formula for the canonical covariant derivative operator for spinors.

3.1.1 Introductory remarks

Spinor fields are, by definition, "*sections of a vector bundle associated to a spin-principal bundle over $\mathscr{M}$, which moreover* [1] *carries a representation of the Clifford algebra*". We will make those notions precise in the subsequent sections. However, none of this is needed if one is mainly interested in gaining working computational skills with spinors. Then, one can view spinor fields as certain vector-valued functions *associated to orthonormal frames*, modulo

[1] Some authors find it convenient to call "spinor field" sections of any bundle associated to the spin group. This by itself *does not* guarantee the existence of a representation of the Clifford algebra; *cf.*, *e.g.,* [44]. However, in all applications known to us the Clifford algebra plays a key role, therefore we will assume that a representation thereof exists on the bundle under consideration. In this terminology, Penrose's two component spinors are *not* spinors from a space-time point of view, as they do not carry a representation of the Clifford algebra associated to the space-time metric. On the other hand they are "space spinors", since carry a representation of the Clifford algebra associated to the positive-definite three-dimensional scalar product.

a sign ambiguity. This sign ambiguity arises as follows: by definition, the spinor fields transform in a specific way (which will be explained below) under changes of frames. Because the orthogonal groups are not simply connected, there exist closed paths in the set of orthonormal frames such that, if you apply the transformation rule for spinors along the path, you will end up with the negative of the initial spinor after having gone around the loop. One way out of this problem would be to work with equivalence classes of objects, defined up to a sign. However, this would prevent one to be able to add spinors in a consistent way, so this is not a good solution.

Now, the problem with such closed loops does not arise if one works on subsets of the bundle of orthonormal frames which are of the form $\pi^{-1}\mathscr{U}$, where $\mathscr{U}$ is a coordinate ball on the manifold $\mathscr{M}$, and π is the canonical projection on the bundle of frames. Hence, there exist many subsets of $\mathscr{M}$ where the ambiguities can be resolved.

So, consider a subset $\mathscr{U}$ of the manifold $\mathscr{M}$ with a globally defined field of orthonormal frames e_a; $\mathscr{U}$ can be taken as $\mathscr{M}$ if the bundle of orthonormal frames is trivial. Having chosen a field of orthonormal frames e_a over $\mathscr{U}$, a spinor field is then a function on $\mathscr{U}$ valued in a finite-dimensional, complex or real, vector space V. As long as one does not need to make changes of frames, one can ignore the frame dependence. Further, no ambiguities arise for frames which are "not too far from e_a", in the following sense: if all the frames e'_a over $\mathscr{U}$ can be obtained from e_a by group elements in a fixed sufficiently small neighborhood of the origin in $\mathrm{SO}_0(p,q)$, where $SO_0(p,q)$ is the connected component of the identity in $SO(p,q)$, then one can determine the value of the spinor field at this frame using the transformation rule (3.1.11) below. In practice, one simply chooses a frame adapted to the problem at hand, and the determination of the values of a spin or field in an other frame is rarely needed.

To get rid of the ambiguity in sign which arises *if* one needs to consider all possible frames, one introduces a double cover of the bundle of orthonormal frames, with spinor fields being V-valued functions on this new bundle. The double cover allows to take care of the sign ambiguity, at least locally: a closed path in the set of orthonormal frames along which the spinor field changes sign is not closed in the double cover anymore, ending on a different branch of the double cover instead. Now, to carry this out we need to assume that we are able to synchronize the sign ambiguity over the whole manifold, which is not always possible. (However, this is always possible in dimension three, or in globally hyperbolic four-dimensional Lorentzian manifolds [97, 98]). We say that the manifold is *spin* if such a global synchronisation exists. Whenever the manifold is spin, some choices might have to be made when synchronising between distinct sets $\mathscr{U}$ and $\mathscr{V}$ as in the previous paragraph. This choice is called the choice of a spin structure. Sometimes only one choice is possible (*e.g.,* for simply connected spin manifolds), sometimes many, sometimes none; this depends upon the topology of the manifold.

Given the above general remarks, let us give some more details. So, let $\mathscr{U}$ be as above and let e_a be a field of orthonormal frames defined over $\mathscr{U}$. Let V be a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, carrying a representation of the Clifford algebra: this can be understood as a set $\{\gamma_a\}$ of endomorphisms of

V satisfying

$$\boxed{\gamma_a\gamma_b + \gamma_b\gamma_a = -2g_{ab}}\ . \tag{3.1.1}$$

Given a vector field $X = X^a e_a$ tangent to $\mathscr{M}$ and defined over $\mathscr{U}$, we assign to X a function $X^a\gamma_a$ on $\mathscr{U}$ with values in the space of endomorphisms of V. This gives a precise sense in which the γ_a's are Clifford-representing the e_a's.

(Strictly speaking, we should write

$$\gamma_a\gamma_b + \gamma_b\gamma_a = -2g_{ab}\mathrm{Id}_V\ , \tag{3.1.2}$$

where Id_V is the identity matrix of V, but the identity matrix will always be taken for granted.)

We can now think of spinor fields on $\mathscr{U}$ as "functions on $\mathscr{U}$ with values in V once an orthonormal frame e_a has been chosen". The case of main interest for the positive-energy theorem is when the metric g is Riemannian. It is shown in Appendix A.18, in the Riemannian case, that V can be equipped with a hermitian product (in the complex case) or scalar product (in the real case) so that the γ_a's are anti-hermitian or anti-symmetric,

$$\gamma_a^\dagger = -\gamma_a\ , \tag{3.1.3}$$

and we will always assume, again in the Riemannian case, that some such scalar product has been chosen. The reader is warned that (3.1.3) will not hold for spinors associated to metrics with other signatures. However, in the Lorentzian case we can always choose a representation so that (3.1.3) holds for spacelike elements of the basis, while $\gamma_0^\dagger = \gamma_0$ (compare (3.3.4a) below).

Some more information about Clifford algebras can be found in Appendix A.18.

3.1.2 The spinorial connection

Note that so far the γ_a's have been assumed to be fixed maps from V to V, in particular they do not depend upon $p \in \mathscr{U}$. Now, a proper way of understanding spinors is that they are sections of a vector bundle, the fibers of which are modeled on $V = \mathbb{K}^N$, for some N, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. This means that a representation of a spinor field as N functions with values in $\mathbb{K}$ requires a choice of trivialisation of the bundle; equivalently, a choice of a local basis for the bundle. Under changes of trivialisations the basis-transformation-matrices will typically depend upon the points in $\mathscr{U}$, so that the γ_a's, when represented as matrices in the new basis, will not have constant entries, but will depend upon $p \in \mathscr{U}$. This will not affect (3.1.1), but this means that there is no good reason to assume that the γ_a's, when represented as matrices, have constant entries. Nevertheless, it is a fundamental consequence of the construction of spinor bundles as *associated bundles* (outlined in Section 3.1.5 below) that, for such bundles, *near every point we can find a (perhaps point dependent) basis of V so that the matrices satisfying (3.1.1) will have constant entries*, assuming of course that the g_{ab}'s are constants. In <u>such a basis</u>, if we think of φ as a function on the bundle of frames with values in $\mathbb{K}^N$, we have the *fundamental formula for the spinor connection*:

$$\boxed{D_X\varphi = X(\varphi) - \frac{1}{4}\omega_{ab}(X)\gamma^a\gamma^b\varphi}\ , \tag{3.1.4}$$

where the ω_{ab}'s are the connection coefficients associated with the frame e_a, *cf.* (A.17.3), Appendix A.17, while

$$\gamma^a = g^{ab}\gamma_b \ .$$

Further, $X(\varphi)$ should be understood as the element of the vector space V obtained by acting with X on the components of φ with respect to the basis of V in which the γ^a's have constant entries.

In line with our conventions elsewhere in this work, in a Riemannian context we will write D for the covariant derivative of spinors, and use ∇ when a Lorentzian metric is considered.

For many purposes, including the proof of the positive energy theorem that we are about to present, formulae (3.1.1) and (3.1.4) are all that is needed to carry-out calculations involving spinors.

3.1.3 Transformation law

When manipulating spinors the following two healthy rules apply:

1. never change tetrads unless you absolutely have too, and
2. always use bases in which the γ_a's have constant entries, so that (3.1.4) holds.

Should the reader ever have to change e_a after all, let us present the rule how to do that, regardless of the signature of the metric. The transformation rule arises from the fact, that spinors transform according to a certain representation of the Spin(p,q) group, which is a double-covering group of the group SO(p,q), the group of linear transformations preserving a pseudo-Riemannian metric g of signature (p,q).

REMARK 3.1.1 To dispel confusion with terminology and notation,[2] following [18, 178] we define the group Pin(p,q) to be the double cover of O(p,q) realized as the group of elements in the Clifford algebra of the form $u = x_1 \cdots x_k$, $k \in \mathbb{N}$, where the x_i's are vectors of length 1 or -1. The subgroup Spin$(p,q) \subset$ Pin(p,q) is then the corresponding cover of SO$(p,q) \subset O(p,q)$.

The covering map $\lambda : \mathrm{Pin}(p,q) \to O(p,q)$ is given by

$$\lambda(x_1 \cdots x_k) := S_{x_1} \circ \cdots \circ S_{x_k} ,$$

where $S_x : \mathbb{R}^{p+q} \to \mathbb{R}^{p+q}$, for x a vector of length one or minus one, denotes the reflection on the hyperplane $x^\perp$.

For special signatures (p,q) this double cover is in fact universal. This is exactly the case if the fundamental group of SO$_0(p,q)$ is Z_2, then Pin$_0(p,q)$ is simply connected. For $p \le q$ this happens exactly when (here p denotes the number of minus signs and q the number of plus signs in the signature)

1. $p = 0$, $q \ge 3$, which corresponds to Riemannian manifolds of dimension greater than or equal to three, and
2. $p = 1$, $q \ge 3$, which are Lorentzian manifolds of dimension greater than or equal to four.

[2] I am grateful to Helga Baum for clarifying this point.

For all other signatures with $p \le q$ the double cover $\lambda : \mathrm{Pin}(p,q) \to \mathrm{O}(p,q)$ fails to be universal.

For example, SO(2) is connected, smoothly homomorphic to $U(1) \approx S^1$, its double cover is again homomorphic to S^1 but its universal cover is homomorphic to $\mathbb{R}$. □

In order to continue we need, first, to make a detour through the Lie algebra $\mathrm{so}(p,q)$ of the group $\mathrm{SO}(p,q)$.

It is a fundamental fact from the theory of Lie groups that any map $\Lambda \in SO_0(p,q)$ can be written as $\exp(\lambda)$, where the matrix $(\lambda^a{}_b)$ representing λ in some ON basis is anti-symmetric after raising or lowering its indices:

$$\lambda^{ab} := g^{bc}\lambda^a{}_c = -\lambda^{ca}\ , \qquad \lambda_{ab} := g_{bc}\lambda^c{}_a = -\lambda_{ba}\ .$$

The map $\mathrm{so}(p,q) \supset \mathscr{V} \ni \lambda \mapsto \exp(\lambda) \in \mathrm{SO}(p,q)$ is a diffeomorphism between a neighborhood $\mathscr{V}$ of zero in $\mathrm{so}(p,q)$ and a neighborhood of the identity in $\mathrm{SO}(p,q)$. However, it is not a bijection from $\mathrm{so}(p,q)$ to $\mathrm{SO}(p,q)$, which is at the origin of the discussion that follows.

For any two vectors X and Y let $X \wedge Y$ be the linear map defined as

$$(X \wedge Y)(Z) = g(Y,Z)X - g(X,Z)Y\ . \tag{3.1.5}$$

(The symbol $\wedge$ is of course closely related to, but should not be confused with, the exterior product, which maps forms to forms; here we have mapped two vectors in W to a linear map from W to itself.)

Note that adding to Y any multiple of X does not change $X \wedge Y$, so to understand $X \wedge Y$ one can without loss of generality assume that X is orthogonal to Y. Then

$$(X \wedge Y)(Z) = \begin{cases} g(Y,Y)X, & Z = Y; \\ -g(X,X)Y, & Z = X; \\ 0, & Z \in \{X,Y\}^{\perp}. \end{cases} \tag{3.1.6}$$

In particular if $X = e_1$ and $Y = e_2$ are the first two vectors of an ON basis, then $X \wedge Y$ is represented by the matrix

$$\Big((X \wedge Y)^a{}_b\Big) = \begin{pmatrix} 0 & g(Y,Y) & 0 & \cdots & 0 \\ -g(X,X) & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & & \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & \cdots & 0 \end{pmatrix}.$$

(By hypothesis, both $g(X,X)$ $g(Y,Y)$ are in $\{-1,1\}$). After lowering an index, this becomes

$$\Big(g_{ac}(X \wedge Y)^c{}_b\Big) = g(X,X)\,g(Y,Y) \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & & \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & \cdots & 0 \end{pmatrix}, \tag{3.1.7}$$

which is antisymmetric. Hence $X \wedge Y$ is an element of the Lie algebra $\mathrm{so}(p,q)$ for all X and Y.

Set

$$\epsilon := -g(X,X)\,g(Y,Y) \in \{-1,1\}\ .$$

Iterating (3.1.6), again for X orthogonal to Y, we find

$$(X\wedge Y)^{2n}(Z) = \begin{cases} \epsilon^n Z, & Z = X \text{ or } Z = Y; \\ 0, & Z \in \{X,Y\}^\perp, \end{cases} \tag{3.1.8}$$

which gives

$$\begin{aligned} \exp(\theta X\wedge Y)(Z) &= \sum_{m=0}^{\infty} \frac{1}{(2m)!}(\theta X\wedge Y)^{2m}(Z) + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!}(\theta X\wedge Y)^{2m+1}(Z) \\ &= \begin{cases} \sum_{m=0}^{\infty} \frac{\epsilon^m\theta^{2m}}{(2m)!} Z + \sum_{m=0}^{\infty} \frac{\epsilon^m\theta^{2m+1}}{(2m+1)!}(X\wedge Y)(Z), & Z = X \text{ or } Z = Y; \\ Z, & Z \in \{X,Y\}^\perp, \end{cases} \end{aligned}$$

When X has length-squared equal to minus one and Y has length one this equals

$$\exp(\theta X\wedge Y)(Z) = \begin{cases} \cosh(\theta)Y + \sinh(\theta)X, & Z = Y; \\ \cosh(\theta)X + \sinh(\theta)Y, & Z = X; \\ Z, & Z \in \{X,Y\}^\perp. \end{cases} \tag{3.1.9}$$

So $X\wedge Y$ generates a boost with velocity parameter θ in the (X,Y) plane.

On the other hand, if X and Y both have length one, we obtain

$$\exp(\theta X\wedge Y)(Z) = \begin{cases} \cos(\theta)Y + \sin(\theta)X, & Z = Y; \\ \cos(\theta)X - \sin(\theta)Y, & Z = X; \\ Z, & Z \in \{X,Y\}^\perp. \end{cases} \tag{3.1.10}$$

Thus the map $X\wedge Y$ generates a rotation by angle $-\theta$ in the (X,Y) plane.

Given an ON basis e_a, it follows from (3.1.7) that $e_a\wedge e_b$ is just a fancy notation for the anti-symmetric matrix which, up to an overall sign, equals plus one in the a'th row and b'th column, minus one in the b'th row and a'th column, and has zeros elsewhere. So, in such a basis, any $\lambda \in so(p,q)$ is represented uniquely by the matrix

$$\lambda = \frac{1}{2}\lambda^{ab} e_a \wedge e_b = \sum_{a<b} \lambda^{ab} e_a\wedge e_b\ ,$$

for some anti-symmetric matrix λ^{ab}. Matrices λ where only one summand above is non-zero generate rotations or boosts, see (3.1.9)-(3.1.10).

Consider an orthonormal basis e'_a given by

$$e'_a = \Lambda^b{}_a e_b\ , \quad \text{with} \quad \Lambda^a{}_b = (\exp(\theta\lambda))^a{}_b\ .$$

Then, *for* $|\theta| < \pi$, the rule for calculating the spinor φ' associated to the frame e'_a from the spinor φ, associated to the frame e_a, is:

$$\boxed{\varphi' = \exp\Big(-\tfrac{1}{4}\theta\lambda^{ab}\gamma_a\gamma_b\Big)\varphi\ .} \tag{3.1.11}$$

There is a problem when one attempts to extend this formula beyond $\theta = \pi$: equation (3.1.12), that we are about to derive, shows that

$$\exp(-\frac{\pi}{2}X\cdot Y\cdot) = \cos\Big(\frac{\pi}{2}\Big) - \sin\Big(\frac{\pi}{2}\Big)X\cdot Y\cdot = -X\cdot Y\cdot\ ,$$

while

$$\exp(-\frac{\pi}{2}X\cdot Y\cdot) \quad = \quad \cos\Big(-\frac{\pi}{2}\Big) - \sin\Big(-\frac{\pi}{2}\Big)X\cdot Y\cdot = X\cdot Y\cdot\,,$$

This is, however, incompatible with the fact that a rotation by π along an axis equals a rotation by $-\pi$: $\exp(\pi X\wedge y) = \exp(-\pi X\wedge Y)$.

However, for all $\theta\in[-\pi,\pi]$ the above formula can be used *up to a sign.* This sign ambiguity can only be resolved when invoking the group $\mathrm{Spin}(p,q)$, which is a double cover of $\mathrm{SO}(p,q)$, see Remark 3.1.1 and Section 3.1.5.

REMARK 3.1.2 Should a reader already familiar with the subject be perplexed by the minus sign in (3.1.11), we note that, in the Riemannian case, with our definitions $\exp(\theta e_a\wedge e_b)$ is a rotation around the axis orthogonal to the plane spanned by $\{e_a,e_b\}$ with angle $-\theta$, adopting the usual convention that a rotation by $+\pi/2$ rotates e_a to e_b. □

The factor 1/2 in (3.1.11) is at the origin of the sign problem seen above, the calculation proceeds as follows: Suppose that $\lambda=\theta X\wedge Y$, where X and Y are any two unit orthogonal vectors. Then $\Lambda := \exp(\theta X\wedge Y)$ is a rotation of angle $-\theta$ in the plane $\mathrm{Vect}\{X,Y\}$, in particular Λ is the identity if $\theta=2\pi$. Let $X=X^ie_i$, and

$$\boxed{\text{let us write } X\cdot \text{ for } X^i\gamma_i.}$$

Thus, $X\cdot$ denotes the linear map from V to V associated with the vector X within the given representation of the Clifford algebra. We have

$$X\cdot Y\cdot = X^iY^j\gamma_i\gamma_j = X^iY^j(-\gamma_j\gamma_i-2g_{ij}) = -Y\cdot X\cdot -2\underbrace{g(X,Y)}_{=0} = -Y\cdot X\cdot\,, \tag{3.1.12}$$

$$X\cdot X\cdot = \underbrace{X^iX^j}_{\text{symmetric in } i \text{ and } j}\gamma_i\gamma_j = X^iX^j\tfrac{1}{2}(\gamma_i\gamma_j+\gamma_j\gamma_i) = -g(X,X) = -1\,,$$

similarly $Y\cdot Y\cdot = -1$. (Alternatively, we could have chosen a basis in which $X=e_1$ and $Y=e_2$, and we would obtained the last two equations directly from (3.1.1); but this argument assumes that if (3.1.1) holds in one ON basis, then it will hold in any other one, which might not be clear at this stage.)

Now, since $\lambda = X^{[i}Y^{j]}e_i\wedge e_j$ in our case, in view of (3.1.12) equation (3.1.11) becomes

$$\varphi' = \exp\Big(-\frac{\theta}{2}X^{[i}Y^{j]}\gamma_i\gamma_j\Big)\varphi = \exp\Big(-\frac{\theta}{2}X\cdot Y\cdot\Big)\varphi\,. \tag{3.1.13}$$

We have

$$(X\cdot Y\cdot)^2 = X\cdot\underbrace{Y\cdot X\cdot}_{=-X\cdot Y\cdot}Y\cdot = X\cdot X\cdot Y\cdot Y = -1\,,$$

so that

$$(X\cdot Y\cdot)^{2n} = \Big((X\cdot Y\cdot)^2\Big)^n = (-1)^n\,,\quad (X\cdot Y\cdot)^{2n+1} = (-1)^nX\cdot Y\cdot\,,$$

which gives

$$\exp(-\frac{\theta}{2}X\cdot Y\cdot) \quad = \quad \sum_{n=0}^{\infty}\frac{1}{n!}\Big(-\frac{\theta}{2}X\cdot Y\cdot\Big)^n$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty}\frac{1}{(2n)!}\Big(-\frac{\theta}{2}X\cdot Y\cdot\Big)^{2n}+\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\Big(-\frac{\theta}{2}X\cdot Y\cdot\Big)^{2n+1}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\Big(\frac{\theta}{2}\Big)^{2n}-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\Big(\frac{\theta}{2}\Big)^{2n+1}X\cdot Y\cdot\\
&= \cos\Big(\frac{\theta}{2}\Big)-\sin\Big(\frac{\theta}{2}\Big)X\cdot Y\cdot\ . \qquad (3.1.14)
\end{aligned}$$

We see in particular that $\varphi'=-\varphi$ when $\theta=2\pi$, even though $\Lambda=\exp(2\pi\lambda)$ is the identity map.

The transformation rule (3.1.11) stems from the fact, that every representation of the Clifford algebra naturally induces a representation of the Lie algebra associated with the corresponding orthonormal group. This proceeds via the map

$$X\wedge Y\mapsto -\frac{1}{2}(X\cdot Y\cdot -Y\cdot X\cdot)\ . \qquad (3.1.15)$$

To simplify calculations, we can and will assume that vectors appearing in a wedge product are mutually orthogonal, in which case the above simplifies to

$$X\wedge Y\mapsto -\frac{1}{2}X\cdot Y\cdot\ \ . \qquad (3.1.16)$$

We claim that this map, defined on pairs of orthogonal vectors, extends by linearity to an isomorphism of the corresponding Lie algebras. To prove this, it is convenient to start by checking that $\mathrm{Span}\{X\wedge Y\}$ forms a Lie algebra; equivalently, we need to check that $\mathrm{Span}\{X\wedge Y\}$ is closed under commutation:

$$\begin{aligned}
[X\wedge Y,V\wedge W](Z) &= (X\wedge Y)\,(V\wedge W)(Z))-((V,W)\longleftrightarrow(X,Y))\\
&= (X\wedge Y)(g(W,Z)V-g(V,Z)W)-((V,W)\longleftrightarrow(X,Y))\\
&= g(W,Z)g(Y,V)X-g(W,Z)g(X,V)Y-g(V,Z)g(Y,W)X\\
&\quad +g(V,Z)g(X,W)Y-((V,W)\longleftrightarrow(X,Y))\\
&= g(Y,V)\underbrace{g(W,Z)X}_{(X\wedge W)(Z)+g(X,Z)W}-g(X,V)\underbrace{g(W,Z)Y}_{(Y\wedge W)(Z)+g(Y,Z)W}\\
&\quad -g(Y,W)\underbrace{g(V,Z)X}_{(X\wedge V)(Z)+g(X,Z)V}+g(X,W)\underbrace{g(V,Z)Y}_{(Y\wedge V)(Z)+g(Y,Z)V}\\
&\quad -\Big(g(Y,Z)g(W,X)V-g(Y,Z)g(X,V)W-g(X,Z)g(Y,W)V\\
&\quad +g(X,Z)g(V,Y)W\Big)\\
&= \Big(g(Y,V)\,X\wedge W-g(X,V)\,Y\wedge W-g(Y,W)\,X\wedge V\\
&\quad +g(X,W)\,Y\wedge V\Big)(Z)\ .
\end{aligned}$$

Thus

$$\begin{aligned}
[X\wedge Y,V\wedge W] &= g(Y,V)\,X\wedge W-g(X,V)\,Y\wedge W-g(Y,W)\,X\wedge V\\
&\quad +g(X,W)\,Y\wedge V\ . \qquad (3.1.17)
\end{aligned}$$

Similarly, the collection $\mathrm{Span}\{X\cdot Y\cdot\}$ of linear combinations of endomorphisms of V of the form $X\cdot Y\cdot$ forms a Lie algebra:

$$\begin{aligned}
[X\cdot Y\cdot, V\cdot W\cdot] &= X\cdot \underbrace{Y\cdot V\cdot}_{-2g(Y,V)-V\cdot Y\cdot} W\cdot - V\cdot W\cdot X\cdot Y\cdot \\
&= -2g(Y,V)X\cdot W\cdot - X\cdot V\cdot \underbrace{Y\cdot W\cdot}_{-2g(Y,W)-W\cdot Y\cdot} - V\cdot W\cdot X\cdot Y\cdot \\
&= -2g(Y,V)X\cdot W\cdot + 2g(Y,W)X\cdot V\cdot \\
&\quad + \underbrace{X\cdot V\cdot}_{-2g(V,X)-V\cdot X\cdot} W\cdot Y\cdot - V\cdot W\cdot X\cdot Y\cdot \\
&= -2g(Y,V)X\cdot W\cdot + 2g(Y,W)X\cdot V\cdot - 2g(V,X)W\cdot Y\cdot \\
&\quad - V\cdot \underbrace{X\cdot W\cdot}_{-2g(X,W)-W\cdot X\cdot} Y\cdot - V\cdot W\cdot X\cdot Y\cdot\,,
\end{aligned}$$

hence

$$\begin{aligned}
[\frac{1}{2}X\cdot Y\cdot, \frac{1}{2}V\cdot W\cdot] &= -g(Y,V)\frac{1}{2}X\cdot W\cdot + g(Y,W)\frac{1}{2}X\cdot V\cdot - g(V,X)\frac{1}{2}W\cdot Y\cdot \\
&\quad + g(X,W)\frac{1}{2}V\cdot Y\cdot\ . \qquad (3.1.18)
\end{aligned}$$

Comparing with (3.1.17), we see that the factor $-1/2$ is precisely what is needed in (3.1.16) to obtain the desired isomorphism of Lie algebras.

For further reference, if we set $\gamma_a := e_a\cdot$, then (3.1.18) can be rewritten as

$$[\gamma_{[a}\gamma_{b]}, \gamma_{[c}\gamma_{d]}] = -2g_{bc}\gamma_{[a}\gamma_{d]} + 2g_{bd}\gamma_{[a}\gamma_{c]} - 2g_{ac}\gamma_{[d}\gamma_{b]} + 2g_{ad}\gamma_{[c}\gamma_{b]}\ . \qquad (3.1.19)$$

This can be seen by first noting that both sides of (3.1.19) with $a=b$ or $c=d$ are zero so there is nothing to check. Hence to obtain (3.1.19) it suffices to apply (3.1.18) with $a\neq b$ and $c\neq d$ using an orthonormal basis, in which case $\gamma_{[a}\gamma_{b]} = \gamma_a\gamma_b = e_a\cdot e_b\cdot$, etc..)

An alternative justification of (3.1.19) proceeds by first writing down a general formula compatible with the symmetries of the left-hand side, and then checking the coefficients by applying the formula to sufficiently general sets of indices. In this case it suffices to check that both sides of (3.1.19) are equal in an ON frame with $abcd = 1213$ and $abcd = 1234$: indeed, any other nontrivial possibilities can be reduced to those by a renaming of basis vectors. But then, for example,

$$\gamma^1 \underbrace{\gamma^2\gamma^1}_{-\gamma^1\gamma^2} \gamma^3 = \underbrace{-\gamma^1\gamma^1}_{1} \gamma^2\gamma^3 = \gamma^2\gamma^3\ ,$$

by symmetry

$$\gamma^1\gamma^3\gamma^1\gamma^2 = \gamma^3\gamma^2\ ,$$

and the difference coincides with the right-hand side of (3.1.19) in this case. The checking of the second possibility proceeds by a similar calculation.

EXAMPLE 3.1.3 A truthful representation of the Lie algebra of SO(3) is provided by linear combination of the three vector fields defined as[3]

$$J_i = -\epsilon_{ijk} x^j \partial_k \ . \tag{3.1.20}$$

Here we use the summation convention even though the indices are in the same position. The Lie bracket is

$$[J_i, J_j] = \epsilon_{ijk} J_k \ . \tag{3.1.21}$$

For example,

$$[J_1, J_2] = [-y\partial_z + z\partial_y, -z\partial_x + x\partial_z] = y\partial_x - x\partial_y = J_3 \ ,$$

from which one can infer (3.1.21) by invariance of both sides of that equation under orthogonal maps. Alternatively, a brute-force calculation proceeds as follows:

$$\begin{aligned}
[J_i, J_j] &= \epsilon_{i\ell m}\epsilon_{jrs} \underbrace{[x^\ell \partial_m, x^r \partial_s]}_{x^\ell \delta^r_m \partial_s - x^r \delta^\ell_s \partial_m} = \underbrace{\epsilon_{i\ell r}\epsilon_{jrs}}_{-\delta^j_i \delta^s_\ell + \delta^s_i \delta^j_\ell} x^\ell \partial_s - \underbrace{\epsilon_{i\ell m}\epsilon_{jr\ell}}_{-\delta^j_i \delta^r_m + \delta^r_i \delta^j_m} x^r \partial_m \\
&= -\delta^j_i x^s \partial_s + x^j \partial_i + \delta^j_i x^m \partial_m - x^i \partial_j = x^j \partial_i - x^i \partial_j \\
&= (\delta^m_i \delta^\ell_j - \delta^\ell_i \delta^m_j) x^\ell \partial_m = -\epsilon_{ijk}\epsilon_{k\ell m} x^\ell \partial_m \\
&= \epsilon_{ijk} J_k \ .
\end{aligned}$$

The Lie algebra isomorphism (3.1.16) is then provided by the formula

$$J_i \mapsto \frac{1}{4} \epsilon_{ijk} \gamma^j \gamma^k \ .$$

More precisely, we claim that the linear map defined on the basis elements by the last equation is compatible with the Lie algebra structures of $\mathrm{Span}\{J_1, J_2, J_3\}$ and $\mathrm{Span}\{\frac{1}{2}\gamma^2\gamma^3, \frac{1}{2}\gamma^3\gamma^1, \frac{1}{2}\gamma^1\gamma^2\}$. Indeed, the above reads

$$J_1 \mapsto \frac{1}{4}(\gamma^2\gamma^3 - \gamma^3\gamma^2) = \frac{1}{2}\gamma^2\gamma^3 \ , \quad J_2 \mapsto \frac{1}{2}\gamma^3\gamma^1 \ , \quad J_3 \mapsto \frac{1}{2}\gamma^1\gamma^2 \ ,$$

which is compatible with the commutation relations (3.1.21): for example,

$$\begin{aligned}
[J_1, J_2] \quad \longleftrightarrow \quad & \left[\frac{1}{2}\gamma^2\gamma^3, \frac{1}{2}\gamma^3\gamma^1\right] = \frac{1}{4}(\gamma^2 \underbrace{\gamma^3\gamma^3}_{-1} \gamma^1 - \gamma^3 \underbrace{\gamma^1\gamma^2\gamma^3}_{\gamma^3\gamma^1\gamma^2}) \\
& = \frac{1}{4}\left(-\gamma^2\gamma^1 + \gamma^1\gamma^2\right) = \frac{1}{2}\gamma^1\gamma^2 \\
\longleftrightarrow \quad & J_3 \ ,
\end{aligned}$$

as desired. The general formula follows by renaming coordinates.

Let us show that the connection defined above satisfies *the Leibniz rule with respect to Clifford multiplication*:

$$D_X(Y \cdot \varphi) = (D_X Y) \cdot \varphi + Y \cdot D_X \varphi \ . \tag{3.1.22}$$

[3]Some readers might find the minus sign in (3.1.20) perplexing, since a positively oriented rotation by angle θ around the i-th axis defines a family of curves with tangent vector $\epsilon_{ijk} x^j \partial_k$. One way of understanding the sign is that elements g of the rotation group define naturally vector fields through the representation $g \mapsto (gf)(x) := f(g^{-1}x)$. The g^{-1} in the argument of f leads then to the minus sign. Another thing to keep in mind that the quantum mechanical commutation relation $[J_i, J_j] = \hbar\epsilon_{ijk} J_k$ involves the momentum operators $p_i = -\sqrt{-1}\hbar\partial_i$, while the differential-geometric definitions of J_i do not involve a square root of minus one.

Indeed, keeping in mind that the γ_a's are assumed to be matrices with constant entries,

$$\begin{aligned} D_X(Y\cdot\varphi) &= D_X(Y^a\gamma_a\varphi) = X(Y^a\gamma_a\varphi) - \frac{1}{4}\omega_{abc}X^c\gamma^a\gamma^bY^d\gamma_d\varphi \\ &= Y^a\gamma_a \underbrace{X(\varphi)}_{D_X\varphi+\frac{1}{4}\omega_{abc}X^c\gamma^a\gamma^b\varphi} \\ &\quad +(\underbrace{X(Y^a)}_{D_XY^a-\omega^a{}_{cd}X^dY^c}\gamma_a - \frac{1}{4}\omega_{abc}X^cY^d\gamma^a\gamma^b\gamma_d)\varphi \\ &= Y\cdot D_X\varphi + D_XY\cdot\varphi \\ &\quad +\frac{1}{4}Y^dX^c\omega_{abc}(\gamma_d\gamma^a\gamma^b - 4\delta^b_d\gamma^a - \gamma^a\gamma^b\gamma_d)\varphi\ . \end{aligned}$$

We claim that the second line above vanishes, which establishes (3.1.22):

$$\underbrace{\gamma_d\gamma^a}_{-2\delta^a_d-\gamma^a\gamma_d}\gamma^b = 2\delta^a_d\gamma^b - \gamma^a\underbrace{\gamma_d\gamma^b}_{-2\delta^b_d-\gamma^b\gamma_d} = -2\delta^a_d\gamma^b + 2\delta^b_d\gamma^a + \gamma^a\gamma^b\gamma_d\ ;$$

antisymmetrising in a and b gives the desired result.

For further reference we summarise the last calculation as

$$[\gamma_d,\gamma^a\gamma^b] = -2\delta^a_d\gamma^b + 2\delta^b_d\gamma^a\ . \tag{3.1.23}$$

If we set

$$\omega(X) = -\frac{1}{4}\omega_{ab}(X)\gamma^a\gamma^b\ , \tag{3.1.24}$$

then (3.1.23) gives

$$[\omega(X),\gamma_d] = \frac{1}{4}\omega_{ab}(X)(-2\delta^a_d\gamma^b + 2\delta^b_d\gamma^a) = \omega_{ad}(X)\gamma^a\ . \tag{3.1.25}$$

In the γ matrices notation, using the Leibniz rule for triple products one has

$$D_X(Y^a\gamma_a\varphi) = (D_XY^a)\gamma_a\varphi + Y^a(D_X\gamma_a)\varphi + Y^a\gamma_aD_X\varphi\ .$$

Equation (3.1.22) has thus the interpretation that *the γ matrices are covariantly constant:*

$$D_a(\gamma^b\varphi) = \gamma^bD_a\varphi\ , \qquad D_a(\gamma_b\varphi) = \gamma_bD_a\varphi\ . \tag{3.1.26}$$

3.1.4 Spinor curvature

We continue by calculating the curvature of the connection (3.1.4), defined through the usual formula:

$$R(X,Y)\varphi := D_XD_Y\varphi - D_YD_X\varphi - D_{[X,Y]}\varphi\ .$$

For this we write

$$\begin{aligned} D_X\varphi &= X(\varphi) + \omega(X)\varphi\ , \qquad \omega(X) := -\frac{1}{4}\omega_{ab}\gamma^a\gamma^b \equiv -\frac{1}{4}\omega_{ab}\gamma^{[a}\gamma^{b]}\ , \\ D_XD_Y\varphi &= X(D_Y\varphi) + \omega(X)D_Y\varphi \\ &= X(Y(\varphi) + \omega(Y)\varphi) + \omega(X)(Y(\varphi) + \omega(Y)\varphi) \\ &= X(Y(\varphi)) + X(\omega(Y))\varphi + \omega(Y)X(\varphi) + \omega(X)Y(\varphi) + \omega(X)\omega(Y)\varphi \\ D_{[X,Y]}\varphi &= X(Y(\varphi)) - Y(X(\varphi)) + \omega([X,Y])\varphi\ . \end{aligned}$$

After obvious simplifications we find

$$\begin{aligned} R(X,Y)\varphi &= (X(\omega(Y)) - Y(\omega(X)) - \omega((X,Y]) + [\omega(X),\omega(Y)])\varphi \\ &= (d\omega(X,Y) + [\omega(X),\omega(Y)])\varphi \ . \end{aligned}$$

In a frame in which the γ^a's are point-independent it holds that

$$d\omega(X,Y) = -\frac{1}{4} d\omega_{ab}(X,Y)\gamma^a\gamma^b \ .$$

Next, in view of (3.1.19) the commutator term equals

$$\begin{aligned} [\omega(X),\omega(Y)] &= \frac{1}{16}\omega_{ab}(X)\omega_{cd}(Y)[\gamma^{[a}\gamma^{b]},\gamma^{[c}\gamma^{d]}] \\ &= \frac{1}{16}\omega_{ab}(X)\omega_{cd}(Y)\left(-2g^{bc}\gamma^{[a}\gamma^{d]} + 2g^{bd}\gamma^{[a}\gamma^{c]} - 2g^{ac}\gamma^{[d}\gamma^{b]} + 2g^{ad}\gamma^{[c}\gamma^{b]}\right) \\ &= -\frac{1}{2}\omega_{ab}(X)\omega^b{}_d(Y)\gamma^{[a}\gamma^{d]} \\ &= -\frac{1}{4}(\omega_{ab}(X)\omega^b{}_d(Y) - \omega_{ab}(Y)\omega^b{}_d(X))\gamma^a\gamma^d \\ &= -\frac{1}{4}(\omega_{ab}\wedge\omega^b{}_d)(X,Y)\gamma^a\gamma^d \ . \end{aligned}$$

We thus have

$$D_X D_Y\varphi - D_Y D_X\varphi - D_{[X,Y]}\varphi = -\frac{1}{4}\Omega_{ab}(X,Y)\gamma^a\gamma^b\varphi \ , \qquad (3.1.27)$$

where

$$\Omega_{ab} = d\omega_{ab} + \omega_{ac}\wedge\omega^c{}_b \ ,$$

and we recognize the curvature two-form (A.17.20) of Appendix A.17.

3.1.5 The origin of the spinorial connection

In this section we will justify that the operation defined by (3.1.4) maps spinors to spinors.[4]

Recall that we defined a spinor field φ as a function valued in a vector space V defined over patches $\mathscr{U}$ of a manifold on which an orthonormal frame e_a has been given. We further required a specific transformation law of φ under changes of frames. A fancy way of saying this is that *spinor fields are sections of a bundle associated to a covering bundle of the bundle of orthonormal frames of* $(\mathscr{M},g)$. Here the qualification "covering bundle" is related to the change of sign of spinors after 2π-rotations.

Lie groups and their representations

To understand (3.1.4), some facts from the theory of Lie groups will be needed. Recall that a *Lie group* is a group which is also a manifold. The *Lie algebra*

[4] I am very grateful to Helmuth Urbantke for enlightning discussions concerning this question.

$\mathfrak{G}$ of a Lie group G can be defined as the tangent space of G at the the unity element e of G, equipped with a *Lie bracket* structure which we will define shortly. Thus, each $a \in \mathfrak{G}$ can be represented by the tangent at zero to a curve $g(t) \in G$ such that $g(0) = e$:

$$a = \frac{dg}{dt}(0) \ .$$

We will often use the following: Let $g(t)$ be a curve in G such that $g(0) = 0$, with corresponding tangent a. Let b denote the tangent to the curve $g(t)^{-1}$. Differentiating the identity $g(t)^{-1}g(t) = e$ at $t = 0$ one finds

$$b := \frac{d(g(t)^{-1})}{dt}|_{t=0} = -\frac{dg(t)}{dt}|_{t=0} =: -a \ .$$

Given any $a \in \mathfrak{G}$ represented by the tangent to a curve $t \mapsto g(t)$, and given any $h \in G$, the curve $t \mapsto h^{-1}g(t)h$ also passes through e at $t = 0$, and therefore defines an element of $\mathfrak{G}$, called $\mathrm{Ad}_h a$. In this way we obtain an action of G on its Lie algebra,

$$G \ni h \mapsto \mathrm{Ad}_h \in \mathrm{End}(\mathfrak{G}) \ ,$$

called the *adjoint action*.

In this section we are essentially interested in groups of matrices, where the composition is the matrix multiplication. Then the elements of the Lie algebra are also matrices, with

$$\mathrm{Ad}_h a = h^{-1}ah \ ,$$

all products being again matrix multiplications.

EXAMPLE 3.1.4 Let g be a quadratic form of signature (p, q) over a vector space W. The group $\mathrm{SO}(p, q)$ is defined as the group of matrices Λ of determinant one which preserve g: in a basis:

$$g_{ab}\Lambda^a{}_c\Lambda^b{}_d = g_{cd} \ . \tag{3.1.28}$$

If $t \mapsto \Lambda(t)$ is a curve of such matrices with $\Lambda(t) = \mathrm{Id}$, the identity matrix, set

$$\lambda^a{}_b := \frac{d}{dt}\Lambda^a{}_b\big|_{t=0} \ .$$

Differentiating (3.1.28) with respect to t, with Λ there replaced by $\Lambda(t)$, one finds

$$0 = g_{ab}(\lambda^a{}_c\delta^b{}_d + \delta^a_c\lambda^b{}_d) = \lambda_{cd} + \lambda_{dc} \ ,$$

so the Lie algebra $\mathrm{so}(p, q)$ of $\mathrm{SO}(p, q)$ is contained in the collection of matrices which are anti-symmetric after lowering the first index.

To show that any such matrix $\lambda^a{}_b$ belongs to $\mathrm{so}(p, q)$, let e_a be an ON basis of W, and define $\Lambda^a{}_b(t)$ to be the matrix whose rows, say $f(t)_a$, are obtained by applying a Gram-Schmidt orthonormalisation to the basis $e_a + t\lambda^b{}_a e_b$, for t small. We leave it as an exercise to the reader to check that $\frac{d}{dt}\Lambda^a{}_b(t)|_{t=0} = \lambda^a{}_b$.

Consider, now, an element b of $\mathfrak{G}$ represented by the tangent at zero to a curve $t \mapsto h(t)$. The Lie-bracket $[a, b]$ is then defined as

$$[a, b] := \frac{d}{dt}\left(h(t)^{-1}ah(t)\right)|_{t=0} \ .$$

For a matrix group, this is clearly the usual commutator of matrices,

$$[a, b] = ab - ba \ ,$$

and therefore is anti-symmetric, and satisfies the *Jacobi identity*

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0 \ .$$

A *representation* of G is a pair (V, ρ), where V is a vector space and ρ is a map from G to the group $\mathrm{End}(V)$ of endomorphisms of V such that

$$\rho(gh) = \rho(g)\rho(h) \ .$$

This easily implies

$$\rho(e) = \mathrm{Id}_V \ , \qquad \rho(g^{-1}) = \rho(g)^{-1} \ ,$$

where Id_V is the identity map of V.

If V has dimension $N < \infty$, $\mathrm{End}(V)$ is a Lie group, which can be thought of as the group of all invertible $N \times N$ matrices. Its Lie algebra is then the collection of all $N \times N$ matrices.

A representation ρ of G is a smooth map between manifolds, and so the push-forward map ρ_* is a map between their tangent spaces. Since ρ maps the identity element e of G to the identity element Id_V of $\mathrm{End}(V)$, $\rho_*|_e$ maps the Lie algebra $\mathfrak{G}$ of G to the Lie algebra of $\mathrm{End}(V)$. We will simply write ρ_* for $\rho_*|_e$ when ambiguities are unlikely to occur.

The tangent map ρ_* preserves the Lie bracket: indeed, let $a = \dot{g}(0)$, and $b = \dot{h}(0)$. From the representation property we have

$$\rho(h(s)^{-1} g(t) h(s)) = \rho(h(s))^{-1} \rho(g(t)) \rho(h(s)) \ .$$

Differentiating with respect to t at $t = 0$ gives

$$\rho_*(h(s)^{-1} a \, h(s)) = \rho(h(s))^{-1} \rho_*(a) \, \rho(h(s)) \ ,$$

and differentiating with respect to s at $s = 0$ one obtains

$$\rho_*([a, b]) = [\rho_*(a), \rho_*(b)] \ .$$

Associated bundles

To continue, it is convenient to recall various descriptions of the collection of *vectors tangent to an n-dimensional manifold.* The first approach is to use local coordinate patches, with vector fields being collections of n functions which, under coordinate transformations $x^i \mapsto y^j(x^i)$ transform as

$$X^i \mapsto \bar{X}^i = X^j \frac{\partial y^i}{\partial x^j} \ .$$

The second point of view, routinely used in this work, is to identify vectors with partial differential operators

$$X = X^i \partial_i \ .$$

Invoking orthonormal *frames* $e_a = e_c{}^i \partial_i$, partial differential operators can also be written as

$$X = X^a e_a \ , \tag{3.1.29}$$

with the collection of numbers X^a transforming via $\mathrm{SO}(p,q)$ matrices under changes of frame:

$$e_a \mapsto \Lambda^b{}_a e_b \quad \Longrightarrow \quad X^a \mapsto \Lambda^a{}_b X^b \ . \tag{3.1.30}$$

The *associated bundle* point of view forgets about the identification of vectors with partial differential operators as in (3.1.29), but adopts (3.1.30) as the defining property of a vector: vectors are elements of an N-dimensional vector space W associated to a local frame e_a. Under a change of frame as in (3.1.30), vectors transform under the fundamental defining representation of $\mathrm{SO}(p,q)$:

$$X^a \mapsto \bar{X}^a = \Lambda^a{}_b X^b \ . \tag{3.1.31}$$

This last point of view conveniently generalises to encompass spinors: the only thing which needs changing above is the use of a spinorial representation of $\mathrm{Spin}(p,q)$, where $\mathrm{Spin}(p,q)$ denotes a *double covering group* of $\mathrm{SO}(p,q)$, instead of the defining one (universal covering in Lorentzian or Riemannian signature).

The need to consider the group $\mathrm{Spin}(p,q)$ instead of $\mathrm{SO}(p,q)$ leads to a subtlety, which is irrelevant for almost all purposes, but needs to be mentioned: when defining spinor fields, instead of considering ON frames one needs to introduce the notion of *spin frames*: by definition, a spin frame is an ON frame "which knows about the ambiguity of sign related to the definition of spinors". As the Lie algebras of $\mathrm{Spin}(p,q)$ and $\mathrm{SO}(p,q)$ are identical, the collection of spin frames is locally identical to that of ON frames. In Riemannian or Lorentzian signature the bundle of spin frames forms a double cover of the bundle of frames: a 2π rotation around a fixed axis will not take us back to the original *spin frame*, we need to rotate by 4π to return where we started. For parallelisable manifolds, or locally, this whole discussion is essentially irrelevant, since then we can choose a global frame once and for all and treat spinor fields on our manifold as fields with values in $\mathbb{R}^N$ for some convenient N.

Now, let ρ be any representation of $\mathrm{Spin}(p,q)$ on a vector space V, and let $h \in \mathrm{Spin}(p,q)$. We shall say that a field φ with values in V is *of type* ρ if, under a change of frame spin frame $e_a \mapsto \Lambda^b{}_a(h) e_b$, where $\Lambda(h) = (\Lambda^b{}_a(h)) \in SO(p,q)$, the field φ transforms as

$$\varphi \mapsto \rho(h)\varphi \ .$$

So, for example, vectors are such objects: ρ in this case is the composition of the defining representation of the group $\mathrm{SO}(p,q)$ on vectors with the projection map from $\mathrm{Spin}(p,q)$ to $\mathrm{SO}(p,q)$.

Another example is provided by a space V carrying a representation of the Clifford algebra as discussed in Section 3.1.3. There we have defined a representation of the algebra $\mathrm{spin}(p,q) = \mathrm{so}(p,q)$. In the Riemannian or Lorentzian case, where $\mathrm{Spin}(p,q)$ is simply connected, a fundamental theorem of the theory of representations of Lie groups and algebras asserts that each such a representation defines a unique representation on V of the group $\mathrm{Spin}(p,q)$, such that the map (3.1.16) is the corresponding tangent map ρ_*.

In what follows we will need the following two identities:

$$\rho_*(\Lambda^{-1}d\Lambda) = \rho(\Lambda)^{-1}d\left(\rho(\Lambda)\right) \ , \tag{3.1.32}$$

and

$$\forall\ a \in \mathfrak{g} \qquad \rho_*(\Lambda^{-1}a\Lambda) = \rho(\Lambda^{-1})\rho_*(a)\rho(\Lambda)\ . \tag{3.1.33}$$

To prove (3.1.33), let $h(t)$ be a curve such that $\dot h(0) = a$, since ρ is a representation we have $\rho(\Lambda^{-1}h(t)\Lambda) = \rho(\Lambda)^{-1}\rho(h(t))\rho(\Lambda)$; differentiation and the definition of ρ_* provide the result.

For (3.1.32), let $\Lambda(t)$ be any curve such that $\Lambda(0) = \Lambda$, then $\Lambda^{-1}\Lambda(t)$ is a curve that defines the Lie algebra element $\Lambda^{-1}\dot\Lambda(0)$. Its image by ρ is

$$\rho(\Lambda^{-1}\Lambda(t)) = \rho(\Lambda)^{-1}\rho(\Lambda(t))\ ,$$

which defines in $\mathrm{End}(V)$ the Lie algebra element

$$\rho_*(\Lambda^{-1}\dot\Lambda(0)) = \rho(\Lambda)^{-1}\frac{d(\rho(\Lambda(t)))}{dt}|_{t=0}\ ,$$

as desired.

After those preliminaries, we are ready now to address the question of construction of a connection on spinor fields. The starting point is the Levi-Civita connection on the tangent bundle. We adopt the associated-bundle point of view, with $X = (X^a)$ denoting a field with values in a given vector space W. Given a differential operator Y, we will write $Y(X)$ for the collection of derivatives $(Y(X^a))$. We will also write

$$D_Y X = Y(X) + \omega(Y)X\ ,$$

for the covariant derivative. For example, if X is a vector field on a manifold $(\mathscr{M}, g)$, then $\omega(Y)X$ stands for the vector with components $\omega^a{}_b(Y)X^b$, the $\omega^a{}_b(X)$'s being the connection coefficients defined in Appendix A.17. Because this case plays a distinguished role, and by visual analogy with the Christoffel symbols, we shall write Γ instead of ω:

$$D_Y X = Y(X) + \Gamma(Y)X \text{ when } X \text{ is a vector field}\ .$$

Suppose, now, that X is a field of type ρ, then under a change of frame we will have

$$X \mapsto \bar X = \rho(\Lambda)X\ , \tag{3.1.34}$$

where $\rho(\Lambda)X$ denotes matrix multiplication of the vector X by the matrix $\rho(\Lambda)$. For example, $\rho(\Lambda) = \Lambda$ if one uses the defining representation of $\mathrm{SO}(p,q)$. By an abuse of notation, we will also write $\rho(\Lambda) = \Lambda$ when the group is $\mathrm{Spin}(p,q)$ but the covering ambiguities are irrelevant. By definition of covariant derivative, $D_Y X$ should also be of type ρ, so it holds that

$$D_Y X \mapsto \overline{D_Y X} \equiv \overline{D}_Y \bar{X} = \rho(\Lambda) D_Y X \ .$$

Writing

$$\overline{D}_Y \bar{X} = Y(\bar{X}) + \bar{\omega}(Y)\bar{X} \ ,$$

we have

$$\begin{aligned}
Y(\bar{X}) + \bar{\omega}(Y)\bar{X} &= Y(\rho(\Lambda)X) + \bar{\omega}(Y)\rho(\Lambda)X \\
&= Y\left(\rho(\Lambda)\right)X + \rho(\Lambda)Y(X) + \bar{\omega}(Y)\rho(\Lambda)X \\
&= \rho(\Lambda)D_Y X \\
&= \rho(\Lambda)Y(X) + \rho(\Lambda)\omega(Y)X \ .
\end{aligned}$$

Comparing the second and last lines, the string of equalities will hold for all X if

$$\omega(Y) = \rho(\Lambda)^{-1} Y\left(\rho(\Lambda)\right) + \rho(\Lambda)^{-1}\bar{\omega}(Y)\rho(\Lambda)$$

which is often written as, in hopefully obvious notation:

$$\bar{\omega} = \rho(\Lambda)^{-1} d\left(\rho(\Lambda)\right) + \rho(\Lambda)^{-1}\omega\rho(\Lambda) \ . \tag{3.1.35}$$

Now, the point is that we know that this equation holds when $\rho(\Lambda) = \Lambda$ and ω equals the connection-form Γ arising from the Levi-Civita connection:

$$\bar{\Gamma} = \Lambda^{-1} d\Lambda + \Lambda^{-1}\Gamma\Lambda \ . \tag{3.1.36}$$

So if we set

$$\boxed{\omega = \rho_* \Gamma} \ , \tag{3.1.37}$$

then (3.1.35) will hold by (3.1.36), (3.1.32) and (3.1.33).

To justify our formula for the spinorial connection it remains to show that (3.1.37) coincides with (3.1.4),

$$\boxed{D_X \varphi = X(\varphi) - \frac{1}{4}\omega_{ab}(X)\gamma^a\gamma^b\varphi} \ , \tag{3.1.38}$$

Since the tangent map ρ_* has been defined in formula (3.1.16) on the basis vectors $e_a \wedge e_b$

$$\rho_*(e_a \wedge e_b) = -\frac{1}{2}\gamma_a\gamma_b \ , \tag{3.1.39}$$

we need to decompose the connection form $\omega^a{}_b$ in terms of this basis. From the definition (3.1.5) we have

$$(e_c \wedge e_d(X))^a = \theta^a(X_d e_c - X_c e_d) = X_d\delta^a_c - X_c\delta^a_d = 2\delta^a_{[c}X_{d]} \ ,$$

which gives

$$\omega^a{}_b X^b = \omega^{cd}\delta^a_c X_d = \frac{1}{2}\omega^{cd}\delta^a_{[c}X_{d]} = \frac{1}{2}\theta^a(\omega^{cd}e_c \wedge e_d(X)) \ ,$$

so

$$\Gamma(Y)X = \frac{1}{2}\omega^{cd}(Y)e_c \wedge e_d(X) \ ,$$

and

$$\rho_*\Gamma(Y) = -\frac{1}{4}\omega^{cd}(Y)\gamma_c\gamma_d \ ,$$

as desired.

3.2 Witten's positivity proof

The *positive mass theorem* asserts that the mass of an asymptotically Euclidean Riemannian manifold with non-negative scalar curvature is non-negative. There exist by now at least four different proofs of this result, the first one due to Schoen and Yau [163], shortly followed by a spinor-based proof by Witten [187]. Another argument has been given by Lohkamp [135], while positivity of mass can also be obtained from the proof of the Penrose inequality of Huisken and Ilmanen [120]. From all those proofs the simplest one by far is that of Witten, and this is the one which we will present here. An advantage thereof is that it can be adapted to provide further global inequalities; some such inequalities will be presented in Section 3.3.

Witten's proof of positivity of mass can be broken into three steps, as follows:

Step 1: Write the ADM mass m in terms of spinors.

Step 2: Use the Schrödinger-Lichnerowicz identity to write m as a volume integral, which is manifestly positive if the spinor field satisfies a Dirac-type equation.

Step 3: Prove existence of spinors satisfying the Dirac equation of Step 2.

We start with the simplest version of the theorem; various extensions will be presented in Section 3.3.

Let D be the standard spin connection for spinor fields which, locally, are represented by fields with values in a real vector space V. (A hermitian scalar product over a complex vector space is also a scalar product over the same space viewed as a real vector space, so this involves no loss of generality, and avoids the nuisance of taking the real part of the scalar product in several calculations. It also saves us the trouble of discussing complex bundles over a real manifold.) We shall also use the symbol D for the usual Levi-Civita derivative associated to the metric g acting on tensors, *etc.* The matrices γ_i stand for $c_g(e_i)$, with c_g — the canonical injection of TM into the representation under consideration on V of the Clifford algebra associated with the metric Riemannian metric g, and are D-covariantly constant,

$$D_i\gamma_j = 0 \, .$$

Step 1: Let $\not{D}$ be the Dirac operator associated with g,

$$\not{D} := \gamma^i D_i \, ,$$

we will show that under natural asymptotic conditions the ADM mass m equals

$$m = \alpha_n \int \mathscr{U}^i dS_i \, , \tag{3.2.1}$$

$$\mathscr{U}^i = \langle \phi, D^i\phi + \gamma^i \not{D}\phi \rangle \, , \tag{3.2.2}$$

where α_n is a dimension-dependent constant, and ϕ is a spinor field which asymptotes to a constant spinor at an appropriate rate.

We will have to defer a complete justification of (3.2.1) until the notion of "appropriate rate" has been clarified, but some preliminary analysis is in order. We thus start by considering a non-zero spinor field $\mathring{\varphi}$ with constant entries, $e_i(\mathring{\varphi}) = 0$, in a given spin frame (which will be specified later). From the definition of the covariant derivative of a spinor we then have

$$D_{e_i}\mathring{\varphi} = -\frac{1}{4}\omega_{jki}\gamma^j\gamma^k\mathring{\varphi} = -\frac{1}{4}\omega_{jki}\gamma^{[j}\gamma^{k]}\mathring{\varphi} \ ,$$

so that

$$\mathscr{U}_i = -\frac{1}{4}\langle\mathring{\varphi}, (\omega_{jki} + \omega_{jk\ell}\gamma_i\gamma^\ell)\gamma^j\gamma^k\mathring{\varphi}\rangle \ . \tag{3.2.3}$$

Since for any A we have

$$\langle\mathring{\varphi}, A\mathring{\varphi}\rangle = \langle A\mathring{\varphi}, \mathring{\varphi}\rangle = \langle\mathring{\varphi}, A^t\mathring{\varphi}\rangle = \frac{1}{2}\langle\mathring{\varphi}, (A + A^t)\mathring{\varphi}\rangle \ ,$$

we calculate the transpose of $\gamma^j\gamma^k$. We assume that $(V, \langle\cdot,\cdot\rangle)$ and the γ_i have been chosen so that

$$(\gamma^i)^t = -\gamma^i \ ;$$

this is always possible provided that the manifold is spin. This gives

$$\gamma^j\gamma^k + (\gamma^j\gamma^k)^t = \gamma^j\gamma^k + \gamma^k\gamma^j = -2g^{jk} \ . \tag{3.2.4}$$

Anti-symmetry of ω_{jki} implies that the term $\omega_{jki}\gamma^j\gamma^k$ gives no contribution in (3.2.3):

$$\omega_{jki}\gamma^j\gamma^k + (\omega_{jki}\gamma^j\gamma^k)^t = -2\omega_{jki}g^{jk} = 0 \ .$$

Consider, next, the terms $\gamma^i\gamma^\ell\gamma^j\gamma^k$; again by antisymmetry only $j \neq k$ matters. If $i = \ell$ one has $\gamma^i\gamma^\ell\gamma^j\gamma^k = -\gamma^j\gamma^k$, giving no contribution as before, thus it remains to consider the cases $i \neq \ell$. If $i = j$ one obtains $\gamma^i\gamma^\ell\gamma^j\gamma^k = -\gamma^\ell\gamma^i\gamma^j\gamma^k = \gamma^\ell\gamma^k$, and (3.2.4) implies a contribution $\langle\mathring{\varphi}, \gamma^i\gamma^\ell\gamma^j\gamma^k\mathring{\varphi}\rangle = -g^{ij}g^{\ell k}|\mathring{\varphi}|^2$ from such terms. The case $i = k$ differs from the last one by a sign and an interchange of j and k. The only terms which have not been accounted for so far are those in which all indices are distinct, which does not occur in dimension three, so in this dimension we obtain

$$\mathscr{U}^i = \frac{1}{2}g^{ij}g^{k\ell}\omega_{jk\ell}|\mathring{\varphi}|^2 \ . \tag{3.2.5}$$

For further reference, it follows from what has been said so far that

$$\langle\phi, \gamma^i\gamma^j\gamma^{[k}\gamma^{\ell]}\phi\rangle = (g^{i\ell}g^{jk} - g^{ik}g^{j\ell})|\phi|^2 + \langle\phi, \gamma^{[i}\gamma^j\gamma^k\gamma^{\ell]}\phi\rangle \ . \tag{3.2.6}$$

Indeed, (3.2.6) has been established when two indices coincide, since the last term at the right-hand side is then zero. On the other hand, the first two-terms in (3.2.6) vanish in an ON frame when all indices are distinct, but then $\gamma^{[i}\gamma^j\gamma^k\gamma^{\ell]} = \gamma^i\gamma^j\gamma^k\gamma^\ell$, whence the result.

In the coordinate system of (1.1.57)-(1.1.58), p. 19, consider the collection of vector fields $\mathring{e}_i$ defined as

$$\mathring{e}_i = \partial_i - \frac{1}{2}\sum_j h_{ij}\partial_j \ . \tag{3.2.7}$$

The $\mathring{e}_i$'s have been chosen to be "orthonormal to leading order":

$$g(\mathring{e}_i, \mathring{e}_j) = \delta_{ij} + O(r^{-2\alpha}) \; .$$

If we perform a Gram-Schmidt orthonormalization starting from the $\mathring{e}_i$'s we will therefore obtain an ON basis e_i such that

$$e_i = \mathring{e}_i + O(r^{-2\alpha}) \; , \quad \partial_k e_i = \partial_k \mathring{e}_i + O(r^{-2\alpha-1}) \; . \tag{3.2.8}$$

We can use (A.17.9), Appendix A.17, to calculate the connection coefficients with respect to this frame:

$$\omega_{jk\ell} = \frac{1}{2}\Big(g(e_\ell, [e_j, e_k]) - g(e_k, [e_\ell, e_j]) - g(e_j, [e_k, e_\ell])\Big) \; . \tag{3.2.9}$$

Inserting this into (3.2.5) and using symmetry in k and ℓ, in space-dimension three one obtains

$$\mathscr{U}^i = \frac{1}{2} g^{ij} g^{\ell k} g(e_\ell, [e_j, e_k]) |\mathring{\varphi}|^2 \; . \tag{3.2.10}$$

From (3.2.7)-(3.2.8) the commutators are, whatever the dimension,

$$[e_j, e_k] = \frac{1}{2}\sum_s (\partial_k h_{sj} - \partial_j h_{sk}) e_s + O(r^{-2\alpha-1}) \; ,$$

leading to

$$\begin{aligned} g(e_\ell, [e_j, e_k]) &= \frac{1}{2}\sum_s g(e_\ell, (\partial_k h_{sj} - \partial_j h_{sk}) e_s) + O(r^{-2\alpha-1}) \\ &= \frac{1}{2}(\partial_k h_{\ell j} - \partial_j h_{\ell k}) + O(r^{-2\alpha-1}) \; . \end{aligned} \tag{3.2.11}$$

It follows that

$$\mathscr{U}^i = \frac{1}{4}\sum_{i,\ell} (\partial_\ell h_{\ell i} - \partial_i h_{\ell\ell}) |\mathring{\varphi}|^2 + O(r^{-2\alpha-1}) \; .$$

We choose $\mathring{\varphi}$ so that $|\mathring{\varphi}|^2 = 1$, though any non-zero asymptotic value will work. Equation (1.1.65) shows that $\int_{S_\infty} \mathscr{U}^i dS_i$ is indeed proportional to the ADM mass in dimension three, provided that $\alpha > 1/2$.

In higher dimensions it remains to show that the terms

$$\langle \mathring{\varphi}, \omega_{jk\ell} \gamma_i \gamma^\ell \gamma^j \gamma^k \mathring{\varphi} \rangle$$

in (3.2.3) with all indices distinct give a vanishing contribution. The commutation properties of the γ^i's give

$$\sum_{\ell, j, k \text{ distinct}} \langle \mathring{\varphi}, \omega_{jk\ell} \gamma_i \gamma^\ell \gamma^j \gamma^k \mathring{\varphi} \rangle = \langle \mathring{\varphi}, \omega_{jk\ell} \gamma_i \gamma^{[\ell} \gamma^j \gamma^{k]} \mathring{\varphi} \rangle = \langle \mathring{\varphi}, \omega_{[jk\ell]} \gamma_i \gamma^\ell \gamma^j \gamma^k \mathring{\varphi} \rangle \; .$$

From (3.2.9) we obtain

$$\begin{aligned} \omega_{[jk\ell]} &= \frac{1}{2}\Big(g(e_{[\ell}, [e_j, e_{k]}]) - g(e_{[k}, [e_\ell, e_{j]}]) - \underbrace{g(e_{[j}, [e_k, e_{\ell]}])}_{= g(e_{[\ell}, [e_j, e_{k]}])} \Big) \\ &= -\frac{1}{2} g(e_{[k}, [e_\ell, e_{j]}]) \; . \end{aligned} \tag{3.2.12}$$

Equation (3.2.11) shows that $\omega_{[jk\ell]}$ vanishes, finishing the proof of this step in higher dimensions.

Step 2: We calculate the divergence of $\mathscr{U}$ as defined by (3.2.2):

$$
\begin{aligned}
D_i\langle\phi, D^i\phi\rangle &= |D\phi|^2 + \langle\phi, D_iD^i\phi\rangle \;, \\
D_i\langle\phi, \gamma^i \not{D}\phi\rangle &= D_i\langle\phi, \gamma^i\gamma^j D_j\phi\rangle \\
&= -|\not{D}\phi|^2 + \langle\phi, \not{D}^2\phi\rangle \;,
\end{aligned}
$$

so that, adding, we obtain

$$
\begin{aligned}
D_i\mathscr{U}^i &= |D\phi|^2 - |\not{D}\phi|^2 && \text{(3.2.13a)}\\
&\quad +\langle\phi, (D_iD^i + \not{D}^2)\phi\rangle \;. && \text{(3.2.13b)}
\end{aligned}
$$

The last term can be rewritten using the Schrödinger-Lichnerowicz identity:

$$\langle\phi, (D_iD^i + \not{D}^2)\phi\rangle = \frac{1}{4}R|\phi|^2 \;, \tag{3.2.14}$$

which is justified as follows:

$$
\begin{aligned}
(D_iD^i + \not{D}^2)\phi &= (g^{ij}D_iD_j + \gamma^iD_i\gamma^jD_j)\phi \\
&= (g^{ij} + \gamma^i\gamma^j)D_iD_j\phi \\
&= (g^{ij} + \gamma^i\gamma^j)(D_{(i}D_{j)} + D_{[i}D_{j]})\phi \\
&= \gamma^i\gamma^jD_{[i}D_{j]}\phi \;. && \text{(3.2.15)}
\end{aligned}
$$

From the definitions

$$
\begin{aligned}
D_k\phi &= e_k(\phi) - \frac{1}{4}\omega_{ijk}\gamma^i\gamma^j\phi \;, \\
\omega_{ijk} &= g(e_i, D_{e_k}e_j) \;, \\
D_{e_k}X^i &= e_k(X^i) + \omega^i{}_{jk}X^k \;, \\
R^i{}_{\ell jk}X^\ell &= D_{e_j}D_{e_k}X^i - D_{e_k}D_{e_j}X^i - D_{[e_j,e_k]}X^i \;,
\end{aligned}
$$

one readily finds (see Section 3.1.4)

$$D_{[i}D_{j]}\phi = -\frac{1}{8}R_{ijk\ell}\gamma^k\gamma^\ell\phi \;. \tag{3.2.16}$$

From (3.2.6) we have

$$
\begin{aligned}
-\frac{1}{8}R_{ijk\ell}\langle\phi, \gamma^i\gamma^j\gamma^k\gamma^\ell\phi\rangle &= -\frac{1}{8}R_{ijk\ell}((g^{i\ell}g^{jk} - g^{ik}g^{j\ell})|\phi|^2 + \langle\phi, \gamma^{[i}\gamma^j\gamma^k\gamma^{\ell]}\phi\rangle) \\
&= \frac{1}{4}R|\phi|^2 \;, && \text{(3.2.17)}
\end{aligned}
$$

since the last term vanishes identically in dimension three, and is zero by the Bianchi identity $R_{i[jk\ell]} = 0$ in the remaining dimensions. This establishes (3.2.14).

In fact, it holds directly that

$$R_{ijk\ell}\gamma^i\gamma^j\gamma^k\gamma^\ell = -2R\,, \tag{3.2.18}$$

which can be seen as follows: First,

$$\begin{aligned}
R_{ijk\ell}\gamma^i\gamma^j\gamma^k\gamma^\ell &= (R_{i(jk)\ell}\gamma^i\gamma^j\gamma^k\gamma^\ell + \underbrace{R_{i[jk]\ell}}_{=\frac{1}{2}(R_{ijk\ell}-R_{ikj\ell})=\frac{1}{2}(R_{ijk\ell}+R_{ik\ell j})=-\frac{1}{2}R_{i\ell jk}}\gamma^i\gamma^j\gamma^k\gamma^\ell)\\
&= R_{i(jk)\ell}\gamma^i\gamma^{(j}\gamma^{k)}\gamma^\ell - \frac{1}{2}R_{i\ell jk}\gamma^i\gamma^j\gamma^k\gamma^\ell\\
&= -R_{i(jk)\ell}\gamma^i g^{jk}\gamma^\ell - \frac{1}{2}R_{i\ell jk}\gamma^i\gamma^j\gamma^k\gamma^\ell\\
&= R_{i\ell}\gamma^i\gamma^\ell - \frac{1}{2}R_{i\ell jk}\gamma^i\gamma^j\gamma^k\gamma^\ell\\
&= -R - \frac{1}{2}R_{i\ell jk}\gamma^i\gamma^j\gamma^k\gamma^\ell\,.
\end{aligned} \tag{3.2.19}$$

Next,

$$\begin{aligned}
R_{i\ell jk}\gamma^i\gamma^j\gamma^k\gamma^\ell &= R_{i\ell jk}\gamma^i\gamma^j(-2g^{k\ell}-\gamma^\ell\gamma^k)\\
&= -2R_{ij}\gamma^i\gamma^j - R_{i\ell jk}\gamma^i\gamma^j\gamma^\ell\gamma^k\\
&= 2R - R_{i\ell jk}\gamma^i(-2g^{j\ell}-\gamma^\ell\gamma^j)\gamma^k\\
&= 2R - 2R_{ik}\gamma^i\gamma^k + R_{i\ell jk}\gamma^i\gamma^\ell\gamma^j\gamma^k\\
&= 4R + R_{ijk\ell}\gamma^i\gamma^j\gamma^k\gamma^\ell\,.
\end{aligned}$$

Inserting this into (3.2.19) leads to (3.2.18).

Suppose, thus, that $R \geq 0$; (3.3.7b)-(3.2.14) show that $D_i\mathscr{U}^i \geq 0$ if φ satisfies the Dirac equation,

$$\not{D}\varphi = 0\,. \tag{3.2.20}$$

This gives

$$\lim_{r\to\infty}\int_{S_r}\mathscr{U}^i dS_i = \lim_{r\to\infty}\int_{B_r} D_i\mathscr{U}^i\, d\mu_g \geq 0\,,$$

where S_r is a coordinate sphere in $M_{\rm ext}$, which establishes Step 2.

Step 3: Let $\mathring{\varphi}$ be a differentiable spinor field which has constant entries in the frame e_i constructed in Step 1 for $r \geq 2R$, and which vanishes away from the region $r \geq R$; we want to show existence of solutions φ of (3.2.20) of the form

$$\varphi = \mathring{\varphi} + \psi\,,$$

with ψ in an appropriate functional space so that (3.2.1) holds. Clearly (3.2.20) is equivalent to

$$\not{D}\psi = \chi\,, \qquad \chi := -\not{D}\mathring{\varphi}\,. \tag{3.2.21}$$

Note that if the connection coefficients are in $L^2(\mathbb{R}^n \setminus B(0,R))$ (which will be the case if $\alpha > (n-2)/2$ in (1.1.57)-(1.1.58); compare Exercice 1.1.6), then $\chi \in L^2(M)$. We will show that the existence of solutions of (3.2.21) with any

$$\chi \in L^2(M)$$

is a simple consequence of elementary Hilbert space theory together with the Schrödinger-Lichnerowicz identity (3.2.14). We start by defining

$$\|\psi\|^2_{\mathcal{H}} := \int_M |D\psi|^2 + \frac{R}{4}|\psi|^2 \qquad (3.2.22)$$

for $\psi \in C^1_c$, where C^k_c denotes the set of C^k compactly supported spinor fields, compactly supported spinor fields. This provides a norm if $R \geq 0$: indeed, in this last case, $\|\psi\|_{\mathcal{H}} = 0$ implies $D\psi = 0$, hence ψ is covariantly constant. It follows that

$$d|\psi|^2 = 2\langle\psi, D\psi\rangle = 0$$

so that ψ has constant norm. If ψ is compactly supported and the manifold is not compact (which is the case here) we obtain $|\psi| = 0$, so $\psi = 0$ as desired.

We set

$$\mathcal{H} = \|\cdot\|_{\mathcal{H}} - \text{completion of } C^1_c \,. \qquad (3.2.23)$$

THEOREM 3.2.1 *Let (M, g) be a complete Riemannian manifold with a $W^{1,\infty}_{\rm loc}$ metric and positive scalar curvature, and suppose that M contains an asymptotic region $M_{\rm ext} := \mathbb{R}^n \setminus B(0, R)$ in which the metric satisfies the asymptotic flatness conditions (1.1.57)-(1.1.58), for some $\alpha > (n-2)/2$. Then the Dirac operator is an isomorphism from $\mathcal{H}$ to L^2.*

PROOF: The starting element of the proof is a *weighted Poincaré inequality*:

DEFINITION 3.2.2 *We say that the covariant derivative D on E over M admits a* weighted Poincaré inequality *if there is a weight function $w \in L^1_{\rm loc}(M)$ with* $\operatorname{ess\,inf}_\Omega w > 0$ *for all relatively compact $\Omega \Subset M$, such that for all $u \in C^1_c(M)$ we have*

$$\int_M |u|^2\, w\, dv_M \leq \int_M |Du|^2\, dv_M \,. \qquad (3.2.24)$$

We show in Appendix B that (3.2.24) holds for manifolds with an asymptotically flat end, see Proposition B.0.5.

We note that the weighted Poincaré inequality (3.2.24), together with the positivity of R, leads to the implication

$$\|\psi\|_{\mathcal{H}} = 0 \implies \psi = 0 \,,$$

giving an alternative proof of the fact that $\|\cdot\|_{\mathcal{H}}$ is a norm on C^1_c.

By construction, $\mathcal{H}$ is a Hilbert space with the obvious scalar product inherited from C^1_c,

$$\langle\psi, \chi\rangle_{\mathcal{H}} = \int_M \langle D\psi, D\chi\rangle + \frac{R}{4}\langle\psi, \chi\rangle \,. \qquad (3.2.25)$$

Now, $\mathcal{H}$ is obtained by the standard process of completing a metric space, yielding a somewhat abstract set of objects. Anticipating, to prove the isomorphism property we will need to invoke elliptic regularity; for that, the objects involved have to be spinor fields, rather than some abstract equivalence classes of Cauchy sequences. So, our next step is to prove that elements of $\mathcal{H}$ can be identified with spinor fields on M:

LEMMA 3.2.3 *Suppose that the weighted Poincaré inequality holds. Then any $\psi \in \mathcal{H}$ can be represented by a spinor field in $H^1_{\rm loc}$, with (3.2.24), as well as (3.2.25), holding for ψ. Moreover, convergence of a sequence ψ_n to ψ in $\mathcal{H}$ implies convergence of $D\psi_n$ to $D\psi$ in L^2, as well as convergence of ψ_n to ψ in $H^1(\Omega)$ for any conditionally compact domain $\Omega \subset M$.*

PROOF: Let $\psi_n \in C^1_c$ be Cauchy in $\mathcal{H}$; since $R \geq 0$, by (3.2.25) we have

$$\int_M |D\psi_n - D\psi_m|^2 < \epsilon \ .$$

By hypothesis there exists a measurable strictly positive function w such that the last inequality implies

$$\int_M |\psi_n - \psi_m|^2 w \, dv_M < \epsilon \ .$$

Completeness of $L^2(w\, dv_M)$ implies that ψ_n converges in $L^2(w\, dv_M)$ to some $\psi \in L^2(w\, dv_M)$.

Similarly completeness of $L^2(dv_M)$ shows that $D\psi_n$ converges in $L^2(dv_M)$ to some $\theta \in L^2(dv_M)$. For any $\chi \in C^1_c$, we have the identity

$$\int_K \langle D^*\chi, \psi_n \rangle = \int_K \langle \chi, D\psi_n \rangle \ ,$$

for any compact set K containing the support of χ. Since ψ_n converges to ψ in $L^2(K, dv_M)$ we can pass to the limit in this equation, obtaining

$$\int_K \langle D^*\chi, \psi \rangle = \int_K \langle \chi, \theta \rangle \ .$$

This shows that $\theta = D\psi$ in the sense of distributions, hence

$$\psi \in H^1_{\rm loc} \ .$$

weighted Poincaré inequality for $\psi \in \mathcal{H}$ follows by passing to the limit:

$$\int_M |\psi|^2 \, w \, dv_M \longleftarrow \int_M |\psi_n|^2 \, w \, dv_M \leq \int_M |D\psi_n|^2 \, dv_M \longrightarrow \int_M |D\psi|^2 \, dv_M \ .$$

Finally, both the left-hand-side and (from what has been said) the first term of the right-hand-side (3.2.25) are continuous on $\mathcal{H}$, therefore the second term of the right-hand-side (3.2.25) also is; passing to the limit one thus finds that (3.2.25) holds on $\mathcal{H}$. □

Returning to the proof of Theorem 3.2.1, note that if $\psi \in C^1_c$, and if we integrate (3.2.13) over M and use (3.2.14) we obtain (since the boundary term vanishes)

$$\|\psi\|^2_{\mathcal{H}} = \int_M |\not{D}\psi|^2 \ . \tag{3.2.26}$$

It follows that the map

$$\mathcal{H} \supset C^1_c \ni \psi \to \not{D}\psi \in L^2$$

is continuous, and so extends by continuity to a continuous map from $\mathcal{H}$ to L^2 (in fact, an isometry), with (3.2.26) holding on $\mathcal{H}$. In particular $\not{D}$ maps $\mathcal{H}$ to L^2, and is injective on $\mathcal{H}$. By polarisation, or by continuity,

$$\langle \psi, \chi \rangle_{\mathcal{H}} = \langle \not{D}\psi, \not{D}\chi \rangle_{L^2} \ . \tag{3.2.27}$$

For $\theta \in L^2$ and $\psi \in \mathcal{H}$ define

$$F_\theta(\psi) = \int_M \langle \not{D}\psi, \theta \rangle \ . \tag{3.2.28}$$

By (3.2.26) we have

$$|F_\theta(\psi)| \le \|\not{D}\psi\|_{L^2} \|\theta\|_{L^2} = \|\psi\|_{\mathcal{H}} \|\theta\|_{L^2} \ ,$$

showing continuity of F_θ. By the Riesz representation theorem, and by (3.2.27), there exists $\varphi \in \mathcal{H}$ such that

$$F_\theta(\psi) = \langle \psi, \varphi \rangle_{\mathcal{H}} = \langle \not{D}\psi, \not{D}\varphi \rangle_{L^2} \ .$$

In particular, for all $\psi \in C^1_c$ we have

$$\int_M \langle \not{D}\psi, \not{D}\varphi - \theta \rangle = 0 \ .$$

Since $\not{D}$ is formally self-adjoint, we obtain that

$$\chi := \not{D}\varphi - \theta \in L^2 \tag{3.2.29}$$

is a weak solution of the equation

$$\not{D}\chi = 0 \ . \tag{3.2.30}$$

We claim:

LEMMA 3.2.4 *Let (M, g) be complete with positive scalar curvature, then the L^2–kernel of $\not{D}$ is trivial.*

REMARK 3.2.5 The hypothesis of completeness of M guarantees that there exists on M an increasing sequence of compactly supported functions $\varphi_n \in W^{1,\infty}$ such that $0 \le \varphi_n \le 1$, $|d\varphi_n| \le C$ and $M = \cup_{n\in\mathbb{N}}\{p : \varphi_n(p) = 1\}$. The Lemma remains true whenever this last property is satisfied.

PROOF: Let $\chi \in L^2$ be in the kernel of $\not{D}$, thus for any $\psi \in C^1_c$ we have

$$\int_M \langle \not{D}\psi, \chi \rangle = 0 \ . \tag{3.2.31}$$

Let $\varphi \in C^\infty(\mathbb{R})$ be a smooth function such that $0 \le \varphi \le 1$, $\varphi = 1$ on $[0, 1]$, and $\varphi = 0$ on $[2, \infty)$. Choose any $q \in M$ and let $\sigma_q(p)$ be the distance function from q. By the triangle inequality σ_q is Lipschitz with Lipschitz constant one, hence differentiable almost everywhere by Rademacher's theorem, with $|D\sigma_q| = 1$ wherever defined. Set

$$\varphi_n(p) = \varphi(\sigma_q(p)/n) .$$

By elliptic regularity we have $\chi \in H^1_{\mathrm{loc}}$. As (M,g) is complete, the Hopf-Rinow theorem implies that metrics balls are compact, hence $\varphi_n\chi \in \mathcal{H}$. By the dominated convergence theorem we have $\varphi_n\chi \to \chi$ in L^2. Equation (3.2.31) with ψ replaced by $\varphi_n\psi$ gives

$$\int_M \langle \not{D}\psi, \varphi_n\chi\rangle + e_i(\varphi_n)\langle\gamma^i\psi,\chi\rangle = 0 ,$$

so that

$$\not{D}(\varphi_n\chi) = e_i(\varphi_n)\gamma^i\chi$$

in the sense of distributions. As $\chi \in H^1_{\mathrm{loc}}$ this equality remains true in L^2. Since $d\varphi_n$ is supported in $B(2n)\setminus B(n)$ we have

$$\int_M |e_i(\varphi_n)\gamma^i\chi|^2 = \int_{B(2n)\setminus B(n)} |e_i(\varphi_n)\gamma^i\chi|^2 \le |d\varphi|_{L^\infty}\int_{B(2n)\setminus B(n)} |\chi|^2 \to_{n\to\infty} 0$$

(again by dominated convergence), so $\not{D}(\varphi_n\chi)$ tends to zero in L^2. Equation (3.2.26) shows that $\varphi_n\chi$ tends to zero in $\mathcal{H}$, hence $\chi = 0$. □

Lemma 3.2.4 together with (3.2.29)-(3.2.30) shows that $\not{D}\varphi = \theta$, establishing that $\not{D}$ is surjective, which finishes the proof of Theorem 3.2.1. □

Return to Step 1: We are ready now to finish the analysis of (3.2.1)-(3.2.2). Recall that to prove positivity we need to use a spinor field φ of the form

$$\varphi = \mathring{\varphi} + \psi ,$$

where $\mathring{\varphi}$ be a differentiable spinor field which has constant entries in the frame e_i, and $\psi \in \mathcal{H}$. We start be rewriting the Schrödinger-Lichnerowicz identity in an integral form,

$$\int_\Omega |D\varphi|^2 - |\not{D}\varphi|^2 + \frac{R}{4}|\varphi|^2 = \oint_{\partial\Omega} \mathscr{U}^i dS_i . \qquad (3.2.32)$$

with a spinor field $\varphi = \mathring{\varphi} + \psi$, with ψ differentiable and compactly supported, while $\partial\Omega = S_R$, a coordinate sphere of radius R in the exterior region, with R large enough so that ψ vanishes there. Passing to the limit $R \to \infty$, our previous calculations of Step 1 give

$$\int_M |D\varphi|^2 - |\not{D}\varphi|^2 + \frac{R}{4}|\varphi|^2 = \alpha_n m , \qquad (3.2.33)$$

still for C^1 compactly supported ψ's. Let $F(\psi)$ denote the left-hand-side of Equation (3.2.33) with $\varphi = \mathring{\varphi} + \psi$ there

$$F(\psi) = \int_M |D\mathring{\varphi} + D\psi|^2 - |\not{D}\mathring{\varphi} + \not{D}\psi|^2 + \frac{R}{4}|\mathring{\varphi} + \psi|^2 . \qquad (3.2.34)$$

We want to show that F is continuous on $\mathcal{H}$. In order to do that, suppose that $\psi_i \in \mathcal{H}$ converges in $\mathcal{H}$ to $\psi \in \mathcal{H}$. We have

$$\begin{aligned} F(\psi) - F(\psi_i) &= \|\psi\|^2_{\mathcal{H}} - \|\psi_i\|^2_{\mathcal{H}} + 2\int_M \langle D^k \mathring{\varphi} \,, D_k(\psi - \psi_i)\rangle \\ &\quad -2\int_M \langle \not{D}\mathring{\varphi} \,, \not{D}(\psi-\psi_i)\rangle + \frac{1}{2}\int_M R\langle \mathring{\varphi} \,, (\psi-\psi_i)\rangle \,. \end{aligned}$$

The first two terms at the right-hand-side of this equation converge to zerom as $i \to \infty$ by continuity of the norm. The third term converges to zero, using Cauchy-Schwarz in $L^2(M)$, because $D\mathring{\varphi} \in L^2(M)$ while $D\psi_i$ converges to $D\psi$ in L^2 by Lemma 3.2.3. The convergence of the last term can be justified by applying the Cauchy-Schwarz inequality as follows:

$$\begin{aligned} \left|\int_M R\langle \mathring{\varphi} \,, \psi - \psi_i\rangle\right| &= \left|\int_M \langle \sqrt{R}\mathring{\varphi} \,, \sqrt{R}(\psi-\psi_i)\rangle\right| \\ &\le \left(\int_M R\langle \mathring{\varphi} \,, \mathring{\varphi}\rangle\right)^{1/2} \left(\int_M R\langle \psi-\psi_i, \psi-\psi_i\rangle\right)^{1/2} \\ &\le C(\|R\|_{L^1(M_{\mathrm{ext}})})\|\psi-\psi_i\|_{\mathcal{H}} \,, \end{aligned}$$

which tends to zero if $\|R\|_{L^1(M_{\mathrm{ext}}} < \infty$ Now, since the ψ_i's are compactly supported we have $F(\psi_i) = F(0) = \alpha_n m$, and density (which holds by construction of $\mathcal{H}$) implies that (3.2.33) remains true for any φ of the form $\mathring{\varphi}+\psi$, with $\psi \in \mathcal{H}$.

If $m = 0$ then clearly $R = D\varphi = 0$ by (3.2.33) when $\not{D}\varphi = 0$ and $R \geq 0$, showing that vanishing mass implies existence of a covariantly constant spinor. Summarising, we have proved:

THEOREM 3.2.6 *Let (M, g) be a complete Riemannian manifold with a $W^{1,\infty}_{\mathrm{loc}}$ metric, $\dim M \geq 3$. Suppose that M contains an asymptotic region $M_{\mathrm{ext}} := \mathbb{R}^n \setminus B(0, R)$ in which the metric satisfies the asymptotic flatness conditions (1.1.57)-(1.1.58), for some $\alpha > 1/2$. If*

$$0 \leq R \in L^1(M_{\mathrm{ext}}) \,,$$

then

$$m \geq 0 \,,$$

with equality if and only if $R = 0$ and there exists a non-trivial covariantly constant spinor field on M.

3.3 Generalised Schrödinger-Lichnerowicz identities

Consider, now, a connection ∇_i of the form

$$\nabla_i = D_i + A_i \,, \tag{3.3.1}$$

where D_i is the standard spin connection for spinor fields which, locally, are represented by fields with values in V. As elsewhere, space-dimension n is assumed unless explicitly indicated otherwise. In this section we also use the symbol D_i for the usual Levi-Civita derivative associated to the metric g acting on tensors, *etc.* The matrices γ_i stand for $c_g(e_i) \equiv e_i\cdot$, with c_g — the canonical injection of TM into the representation under consideration on V of the Clifford algebra associated with a Riemannian metric g, and are D-covariantly constant,

$$D_i\gamma_\mu = 0 \; .$$

We will shortly need a matrix γ_0 with constant entries satisfying

$$\gamma_0^t = \gamma_0 \; , \quad \gamma_0^2 = 1 \; , \quad \gamma_i\gamma_0 = -\gamma_0\gamma_i \; , \quad D_i\gamma_0 = 0 \; . \tag{3.3.2}$$

When the manifold M is a spacelike hypersurface in a spin Lorentzian manifold $(\mathscr{M}, \gamma)$, γ_0 can be obtained by setting

$$\gamma_0 := c_\gamma(N) \; ,$$

where N is the field of unit normals to M, and c_γ is the canonical injection of $T\mathscr{M}$ into the representation under consideration on V of the Clifford algebra associated with the Lorentzian metric γ.

If, however, the desired $\gamma_0 \in \mathrm{End}(V)$ does not exist, we proceed as follows: Let $\mathfrak{S}$ denote the bundle, over M, of spinors under consideration (thus the fibers of $\mathfrak{S}$ are isomorphic to V). Let $\mathfrak{S}' = \mathfrak{S} \oplus \mathfrak{S}$ be the direct sum of two copies of $\mathfrak{S}$, equipped with the direct sum metric $\langle \cdot, \cdot \rangle_\oplus$:

$$\langle (\psi_1, \psi_2), (\varphi_1, \varphi_2) \rangle_\oplus := \langle \psi_1, \varphi_1 \rangle + \langle \psi_2, \varphi_2 \rangle \; . \tag{3.3.3}$$

We set, for $X \in TM$,

$$\gamma_0(\psi_1, \psi_2) := (\psi_2, \psi_1) \; , \tag{3.3.4a}$$
$$X \cdot (\psi_1, \psi_2) := (X \cdot \psi_1, -X \cdot \psi_2) \; , \tag{3.3.4b}$$
$$D_X(\psi_1, \psi_2) := (D_X\psi_1, D_X\psi_2) \; . \tag{3.3.4c}$$

One readily verifies that (3.3.4b) defines a representation of the Clifford algebra on $\mathfrak{S}'$, and that (3.3.2) holds.

We set

$$\gamma^0 := -\gamma_0 \; .$$

We want to derive divergence identities involving the Dirac operator associated with ∇_i. For this we define,

$$\nabla\!\!\!\!/ := \gamma^i\nabla_i \; , \qquad D\!\!\!\!/ := \gamma^i D_i \; ,$$

and we calculate

$$\begin{aligned} D_i\langle \phi, \nabla^i\phi \rangle &= |\nabla\phi|^2 + \langle \phi, D_iD^i\phi \rangle \\ &\quad + \langle \phi, D_i(A^i\phi) \rangle - \langle A^i\phi, (D_i + A_i)\phi \rangle \; , \end{aligned} \tag{3.3.5}$$

$$\begin{aligned} D_i\langle \phi, \gamma^i\nabla\!\!\!\!/\phi \rangle &= D_i\langle \phi, \gamma^i\gamma^j\nabla_j\phi \rangle \\ &= -|\nabla\!\!\!\!/\phi|^2 + \langle \phi, D\!\!\!\!/^2\phi \rangle \\ &\quad + \langle \phi, \gamma^i\gamma^j D_i(A_j\phi) \rangle + \langle \gamma^i A_i\phi, \gamma^j(D_j + A_j)\phi \rangle \; . \end{aligned} \tag{3.3.6}$$

Adding we obtain

$$
\begin{aligned}
D_i\mathscr{U}^i &:= D_i\langle\phi, \left(\nabla^i + \gamma^i \nabla\!\!\!\!/\right)\phi\rangle && (3.3.7a)\\
&= |\nabla\phi|^2 - |\nabla\!\!\!\!/\phi|^2 && (3.3.7b)\\
&\quad +\langle\phi, (D_iD^i + D\!\!\!\!/^2)\phi\rangle && (3.3.7c)\\
&\quad +\langle\phi, \left[A^i - (A^i)^t + \gamma^i\gamma^j A_j + (\gamma^j A_j)^t\gamma^i\right] D_i\phi\rangle && (3.3.7d)\\
&\quad +\langle\phi, \left[D_iA^i + \gamma^i\gamma^j D_iA_j\right]\phi\rangle && (3.3.7e)\\
&\quad +\langle\phi, \left[(\gamma^i A_i)^t\gamma^j A_j - (A^i)^t A_i\right]\phi\rangle\ . && (3.3.7f)
\end{aligned}
$$

The term (3.3.7c) is independent of the A_i's, and is the one that arises in the original Schrödinger-Lichnerowicz identity (3.2.14):

$$\langle\phi, (D_iD^i + D\!\!\!\!/^2)\phi\rangle = \frac{1}{4}R|\phi|^2\ . \tag{3.3.8}$$

In order to work out the remaining terms in (3.3.7), an explicit form of the A_i's is needed.

3.3.1 A connection involving extrinsic curvature

Suppose, first, that ∇ is the "space-time spin connection":

$$A_i = \frac{1}{2}K_i{}^j\gamma_j\gamma_0\ , \tag{3.3.9}$$

where K_{ij} is a symmetric tensor field on M. A_i is then symmetric and we have, by symmetry of K_{jk},

$$
\begin{aligned}
A^i - (A^i)^t + \gamma^i\gamma^j A_j + (\gamma^j A_j)^t\gamma^i &= \frac{1}{2}K_{jk}\left(\gamma^i\gamma^j\gamma^k\gamma_0 + \gamma_0\gamma^k\gamma^j\gamma^i\right)\\
&= -\frac{1}{2}\mathrm{tr}_g K\left(\gamma^i\gamma_0 + \gamma_0\gamma^i\right)\\
&= 0\ , && (3.3.10)
\end{aligned}
$$

so that there is no contribution from (3.3.7d). We set

$$
\begin{aligned}
\mu &:= R - |K|^2 + (\mathrm{tr}K)^2\ , && (3.3.11a)\\
\nu^j &:= 2D_i(K^{ij} - \mathrm{tr}K g^{ij}) && (3.3.11b)
\end{aligned}
$$

(the reader will recognize the right-hand-sides as occurring in the general relativistic constraint equations.) Next,

$$
\begin{aligned}
D^iA_i + \gamma^j\gamma^i D_jA_i &= \frac{1}{2}\left(D^iK_{ij}\gamma^j + D_jK_{ik}\gamma^j\gamma^i\gamma^k\right)\gamma_0\\
&= \frac{1}{2}\left(D^iK_{ij}\gamma^j - D_j\mathrm{tr}_gK\gamma^j\right)\gamma_0\\
&= \frac{1}{4}\nu_j\gamma^j\gamma_0\ , && (3.3.12)
\end{aligned}
$$

with ν as in Equation (3.3.11); this gives the contribution from (3.3.7e). Using symmetry of $K_{ij}K^i{}_k$ we further have

$$\begin{aligned}(A^i)^t A_i &= \frac{1}{4}K_{ij}K^i{}_k\gamma^j\gamma_0\gamma^k\gamma_0 \\ &= -\frac{1}{4}K_{ij}K^i{}_k\gamma^j\gamma^k \\ &= -\frac{1}{8}K_{ij}K^i{}_k(\gamma^j\gamma^k+\gamma^k\gamma^j) \\ &= \frac{1}{4}|K|_g^2 \,.\end{aligned}$$

From

$$\gamma^i A_i = -\mathrm{tr}_g K\gamma_0/2 = (\gamma^i A_i)^t \tag{3.3.13}$$

one obtains

$$(\gamma^i A_i)^t\gamma^j A_j - (A^i)^t A_i = \frac{1}{4}\left(-|K|_g^2 + (\mathrm{tr}_g K)^2\right) \,.$$

Collecting all this we are led to

$$D_i\langle\phi,\left(\nabla^i+\gamma^i\nabla\!\!\!\!/\right)\phi\rangle = |\nabla\phi|^2 - |\nabla\!\!\!\!/\phi|^2 + \frac{1}{4}\langle\phi,(\mu+\nu_j\gamma^j\gamma_0)\phi\rangle \,, \tag{3.3.14}$$

where μ is given by Equation (3.3.11a).

Consider, now, the vector field $\mathscr{U}^i$ defined by (3.3.7a):

$$\mathscr{U}^i = \langle\phi,\left(\nabla^i+\gamma^i\nabla\!\!\!\!/\right)\phi\rangle = \langle\phi,\left(D^i+\gamma^i D\!\!\!\!/\right)\phi\rangle + \langle\phi,\left(A^i+\gamma^i\gamma^j A_j\right)\phi\rangle \,.$$

We have, using (3.3.13),

$$\begin{aligned}A^i+\gamma^i\gamma^j A_j &= \frac{1}{2}(K_{ij}\gamma^j - \mathrm{tr}_g K\gamma^i)\gamma_0 \\ &= \frac{1}{2}(\mathrm{tr}_g K g^{ij} - K^{ij})\gamma_0\gamma_j \,,\end{aligned}$$

which integrated upon a coordinate sphere S_R in M_{ext} gives

$$\begin{aligned}&\lim_{R\to\infty}\oint_{S(R)}\langle\phi_\infty,\left(A^i+\gamma^i\gamma^j A_j\right)\phi_\infty\rangle dS_i \\ &= \left(\lim_{R\to\infty}\frac{1}{2}\oint_{S(R)}(\mathrm{tr}_g K g^{ij} - K^{ij})dS_i\right)\langle\phi_\infty,\gamma_0\gamma_j\phi_\infty\rangle \\ &= \omega_n p^i\langle\phi_\infty,\gamma_0\gamma_i\phi_\infty\rangle = \omega_n p_i\langle\phi_\infty,\gamma^i\gamma^0\phi_\infty\rangle \,,\end{aligned} \tag{3.3.15}$$

with p^i — the (suitable normalised) ADM momentum of M_{ext} (recall that $\gamma^0 = -\gamma_0$). Here ϕ_∞ is a covariantly constant spinor field of the Euclidean metric associated to the natural coordinates on M_{ext}. Combining this with the calculations in Section 3.2 we obtain

$$\lim_{R\to\infty}\oint_{S(R)}\mathscr{U}^i dS_i = 4\pi p_\mu\langle\phi_\infty,\gamma^\mu\gamma^0\phi_\infty\rangle \,. \tag{3.3.16}$$

Modulo the question of weak differentiability conditions on the metric, which has been addressed in detail in [14, 15], the algebra above together with the analytic arguments of the previous section give:

THEOREM 3.3.1 *Let (M, g, K) be initial data for the Einstein equations with $g \in W^{2,2}_{\mathrm{loc}}$, $K \in W^{1,2}_{\mathrm{loc}}$, with (M, g) complete (without boundary). Suppose that M contains an asymptotically flat end and let $p_\alpha = (m, \vec{p})$ be the associated ADM four-momentum.*[5] *If*

$$\mu \geq |\nu|_g \;, \tag{3.3.17}$$

then

$$m \geq |\vec{p}|_\delta \;, \tag{3.3.18}$$

with equality if and only if m vanishes. Further, in that last case there exists a non-trivial covariantly constant (with respect to the space-time spin connection) spinor field on M.

REMARK 3.3.2 Under the supplementary assumption of smoothness of g and K, it has been shown in [21] that the existence of a covariantly constant spinor implies that the initial data can be isometrically embedded into Minkowski space-time, *cf.* also [188]. We expect this result to remain true under the current hypotheses, but we have not attempted to prove this.

As pointed out by Yvonne Choquet-Bruhat [49], the choice

$$A_k = \frac{1}{2}\sqrt{-1}K_k{}^j\gamma_j \tag{3.3.19}$$

leads to a similar identity

$$D_i\langle\phi, \left(\nabla^i + \gamma^i \not{\nabla}\right)\phi\rangle = |\nabla\phi|^2 - |\not{\nabla}\phi|^2 + \frac{1}{4}\langle\phi, (\mu + \sqrt{-1}\nu_j\gamma^j)\phi\rangle \;. \tag{3.3.20}$$

(The calculations are essentially identical, except that transpositions have to be replaced by hermitian conjugations.) The drawback of (3.3.19) is that the resulting Dirac-type operator $\not{\nabla}$ is not formally self-adjoint any more. However, the formal adjoint of $\not{\nabla}$ differs from $\not{\nabla}$ by a change of sign of the K_{ij} term, and the existence theory for such an operator can be repeated without essential changes. Note that (3.3.19) does not require introducing the matrix γ_0, leading to a certain economy in the argument.

3.3.2 Maxwell fields

We consider an initial data set (M, g, K) on which a Maxwell field $F = (E, B)$ is given. Thus, $E = E_i dx^i$ is the electric field on M related to the space-time Maxwell tensor field $F = F_{\mu\nu}dx^\mu dx^\nu$ through the equation $E_i = F_{i\mu}n^\mu$, where n^μ is the future-pointing unit normal to M considered as a hypersurface in space-time. Similarly the magnetic field $B = B_i dx^i$ on M is defined as $B_i = *_4 F_{i\mu}n^\mu$, where $*_4$ is the space-time Hodge dual. (Here, we use a coordinate system so that $M = \{t = 0\}$.) Note that those definitions require space-time

[5] There is a signature-dependent ambiguity in the relationship between p^0, p_0 and the mass m: in the space-time signature $(-,+,+,+)$ used in this work this sign is determined by the fact that p_0, obtained by Hamiltonian methods, is usually positive in Lagrangean theories on Minkowski space-time such as the Maxwell theory, while the mass m is a quantity which is expected to be positive.

dimension four. However, one can work in any space dimension using E_i, and *either* B_i *or* F_{ij} as the fundamental objects. (In higher dimensions this leads of course to different theories.) In space-dimension three we note the relations

$$B_i = \tfrac{1}{2}\epsilon_i{}^{jk}F_{jk} \quad \Longleftrightarrow \quad F_{ij} = \epsilon_{ijk}B^k \ ,$$
$$*_4 F_{ij} = \epsilon_{ijk0}F^{k0} = \epsilon_{ijk}E^k \ ,$$

where ϵ_{ijk} is completely antisymmetric and equals $\sqrt{\det g}$ for $ijk = 123$.

Following Gibbons and Hull [105], when $n = 3$ we set

$$A_i = \underbrace{\frac{1}{2}K_{ij}\gamma^j\gamma_0}_{A_i(K)} \underbrace{-\frac{1}{2}E^k\gamma_k\gamma_i\gamma_0}_{A_i(E)} \underbrace{-\frac{1}{4}\epsilon_{jk\ell}B^j\gamma^k\gamma^\ell\gamma_i}_{A_i(B)} \ . \tag{3.3.21}$$

For $n > 4$ we could drop the term involving B, and hope for the best. But we will shortly see that the electric field will give undesirable terms in the divergence identity in space-dimension different from $n = 3$.

We return to (3.3.7). The contribution of K to the linear terms (3.3.7d) and (3.3.7e) has already been worked out, so it remains to evaluate that of E and B. We have:

$$\begin{aligned}
A_i(E)^t &= -\frac{1}{2}E^k\gamma_0\gamma_i\gamma_k = -\frac{1}{2}E^k\gamma_i\gamma_k\gamma_0 = -\frac{1}{2}E^k\left(-\gamma_k\gamma_i - 2g_{ki}\right)\gamma_0 \\
&= -A_i(E) + E_i\gamma_0 \ , && (3.3.22a)\\
\gamma^j A_j(E) &= -\frac{1}{2}E^k\gamma^j\gamma_k\gamma_j\gamma_0 = -\frac{1}{2}E^k\left(-\gamma_k\gamma^j - 2\delta^j_k\right)\gamma_j\gamma_0 = -\frac{1}{2}E^k\left(n\gamma_k - 2\gamma_k\right)\gamma_0 \\
&= -\frac{(n-2)}{2}E^k\gamma_k\gamma_0 \ , && (3.3.22b)\\
(\gamma^j A_j(E))^t &= \gamma^j A_j(E) \ . && (3.3.22c)
\end{aligned}$$

This gives

$$\begin{aligned}
&A^i(E) - (A^i(E))^t + \gamma^i\gamma^j A_j(E) + (\gamma^j A_j(E))^t\gamma^i \\
&\qquad = -E_k\Big(\underbrace{\gamma^k\gamma^i}_{-g^{ki}+\gamma^{[k}\gamma^{i]}} + \frac{(n-2)}{2}(\gamma^i\gamma^k - \gamma^k\gamma^i)\Big)\gamma_0 - E^i\gamma_0 \\
&\quad = \frac{(n-3)}{2}E_k\gamma^{[k}\gamma^{i]}\gamma_0 \ .
\end{aligned}$$

So we see that this unwanted term will drop out only if $n = 3$. From now on we assume that this last property holds.

We continue with the observation that

$$\begin{aligned}
A_1(B) &= -\frac{1}{2}\left[B^1\gamma^2\gamma^3 + B^2\gamma^3\gamma^1 + B^3\gamma^1\gamma^2\right]\gamma_1 \\
&= -\frac{1}{2}\left[B_1\gamma^1\gamma^2\gamma^3 - B^2\gamma^3 + B^3\gamma^2\right] \\
&= -\frac{1}{2}\left[B_1\gamma^1\gamma^2\gamma^3 - \epsilon_{1jk}B^j\gamma^k\right] \ , && (3.3.23)
\end{aligned}$$

so, since the e_1 direction can be chosen at will,

$$A_i(B) \;=\; \frac{1}{2}\left[\epsilon_{ijk}B^j\gamma^k - B_i\gamma^1\gamma^2\gamma^3\right] . \tag{3.3.24}$$

Now

$$(\gamma^1\gamma^2\gamma^3)^t = (\gamma^3)^t(\gamma^2)^t(\gamma^1)^t = -\gamma^3\gamma^2\gamma^1 = -\gamma^2\gamma^1\gamma^3 = \gamma^1\gamma^2\gamma^3 , \tag{3.3.25}$$

$$\gamma^i\gamma^1\gamma^2\gamma^3 = -\tfrac{1}{2}\epsilon^{ijk}\gamma_j\gamma_k , \tag{3.3.26}$$

$$\gamma^i\tfrac{1}{2}\epsilon_{jk\ell}B^j\gamma^k\gamma^\ell = B^i\gamma^1\gamma^2\gamma^3 + \epsilon^{ijk}B_j\gamma_k , \tag{3.3.27}$$

where the last two equations have been obtained by a calculation similar to that of Equation (3.3.23). This leads to

$$A_i(B)^t \;=\; -\frac{1}{2}\left[\epsilon_{ijk}B^j\gamma^k + B_i\gamma^1\gamma^2\gamma^3\right] , \tag{3.3.28a}$$

$$\begin{aligned}\gamma^iA_i(B) \;&=\; \frac{1}{2}\left[\epsilon_{ijk}B^j\gamma^i\gamma^k + \frac{1}{2}\epsilon^{ijk}B_i\gamma_j\gamma_k\right]\\ &=\; -\frac{1}{4}\epsilon_{ijk}B^i\gamma^j\gamma^k ,\end{aligned} \tag{3.3.28b}$$

$$(\gamma^jA_j(B))^t \;=\; -\gamma^jA_j(B) , \tag{3.3.28c}$$

$$\begin{aligned}\gamma^i\gamma^jA_j(B) \;&=\; -\frac{1}{2}\left[B^i\gamma^1\gamma^2\gamma^3 + \epsilon^{ijk}B_j\gamma_k\right]\\ &=\; (A^i(B))^t ,\end{aligned} \tag{3.3.28d}$$

$$(\gamma^jA_j(B))^t\gamma^i \;=\; -A^i(B) . \tag{3.3.28e}$$

Here (3.3.28d) follows from (3.3.27) and (3.3.28b), while (3.3.28e) is obtained by comparing minus (3.3.28b) multiplied from the right by γ^i, as justified by (3.3.28c), with the definition (3.3.21) of $A_i(B)$. Using (3.3.28d) and (3.3.28e) we conclude that

$$\begin{aligned}&A^i(B) - (A^i(B))^t + \gamma^i\gamma^jA_j(B) + (\gamma^jA_j(B))^t\gamma^i\\ &\quad= A^i(B) - (A^i(B))^t + (A^i(B))^t - A^i(B)\\ &\quad= 0 ,\end{aligned} \tag{3.3.29}$$

which shows that the contribution (3.3.7d) to (3.3.7) vanishes. We consider next (3.3.7e):

$$\begin{aligned}D_jA^j(E) + \gamma^i\gamma^jD_iA_j(E) \;&=\; -\frac{1}{2}D_iE^k\{\gamma_k\gamma^i + \gamma^i\underbrace{\gamma^j\gamma_k}_{-\gamma_k\gamma^j-2\delta^j_k}\gamma_j\}\gamma_0\\ &=\; -\frac{1}{2}D_iE_k\{\gamma^k\gamma^i + 3\gamma^i\gamma^k - 2\gamma^i\gamma^k\}\gamma_0\\ &=\; D_iE^i\gamma_0 =: \mathrm{div}(E)\gamma_0 .\end{aligned} \tag{3.3.30}$$

To analyse the contribution of $A_i(B)$ to (3.3.7e) it is convenient to use (3.3.24), which gives

$$D_iA^i(B) + \gamma^i\gamma^jD_iA_j(B) \;=\; \frac{1}{2}D_iB_k[\epsilon^{jk\ell}(\delta^i_j\gamma_\ell + \gamma^i\gamma_j\gamma_\ell) - (g^{ik} + \gamma^i\gamma^k)\gamma^1\gamma^2\gamma^3] . \tag{3.3.31}$$

Fortunately it is not necessary to evaluate this expression in detail, because any antisymmetric matrix appearing above gives a zero contribution after insertion into (3.3.7e). This follows immediately from the fact that for any linear map F we have

$$\langle \phi, F\phi\rangle = \langle \phi, F^t\phi\rangle = \langle \phi, \frac{1}{2}(F+F^t)\phi\rangle \,. \qquad (3.3.32)$$

Now, by Clifford algebra rules, the right-hand-side of (3.3.31) will be a linear combination of γ^i's and of $\gamma^1\gamma^2\gamma^3$; the γ^i's are antisymmetric and can thus be ignored, and it remains to work out the coefficient in front of the symmetric matrix $\gamma^1\gamma^2\gamma^3$. The first term gives no such contribution, the second one will contribute when i equals k, producing then a contribution $-2g^{ik}\gamma^1\gamma^2\gamma^3$,. The only possible contribution from the last two terms could occur when $i=k$, but then they cancel out each other. We are thus led to

$$\begin{aligned} D_iA^i(B) + \gamma^i\gamma^j D_iA_j(B) &= -D_iB^i\gamma^1\gamma^2\gamma^3 + \text{antisymmetric} \\ &=: -\mathrm{div}(B)\gamma^1\gamma^2\gamma^3 + \text{antisymmetric}\,. \qquad (3.3.33)\end{aligned}$$

Let us, finally, consider the quadratic term (3.3.7f); from Equations (3.3.13), (3.3.22b) and (3.3.28b) together with (3.3.26) we have

$$\begin{aligned} \gamma_iA^i &= -\frac{1}{2}(\mathrm{tr}_gK + E^i\gamma_i)\gamma_0 - \frac{1}{4}\epsilon_{ijk}B^i\gamma^j\gamma^k \\ &= -\frac{1}{2}\left[(\mathrm{tr}_gK + E^i\gamma_i)\gamma_0 - B^i\gamma_i\gamma^1\gamma^2\gamma^3\right]\,. \qquad (3.3.34)\end{aligned}$$

It follows that

$$(\gamma_iA^i)^t = -\frac{1}{2}\left[(\mathrm{tr}_gK + E^i\gamma_i)\gamma_0 + B^i\gamma_i\gamma^1\gamma^2\gamma^3\right]\,, \qquad (3.3.35)$$

and

$$\begin{aligned} (\gamma_iA^i)^t\gamma_jA^j &= \frac{1}{4}\left[(\mathrm{tr}_gK + E^i\gamma_i)\gamma_0 + B^i\gamma_i\gamma^1\gamma^2\gamma^3\right]\left[(\mathrm{tr}_gK + E^j\gamma_j)\gamma_0 - B^j\gamma_j\gamma^1\gamma^2\gamma^3\right] \\ &= \frac{1}{4}\Big\{(\mathrm{tr}_gK + E^i\gamma_i)\gamma_0(\mathrm{tr}_gK + E^j\gamma_j)\gamma_0 - B^iB^j\gamma_i\gamma^1\gamma^2\gamma^3\gamma_j\gamma^1\gamma^2\gamma^3 \\ &\qquad -E^iB^j\left[\gamma_i\gamma_0\gamma_j\gamma^1\gamma^2\gamma^3 - \gamma_j\gamma^1\gamma^2\gamma^3\gamma_i\gamma_0\right]\Big\} \\ &= \frac{1}{4}\Big\{(\mathrm{tr}_gK + E^i\gamma_i)(\mathrm{tr}_gK - E^j\gamma_j) - B^iB^j\gamma_i\gamma_j(\gamma^1\gamma^2\gamma^3)^2 \\ &\qquad -E^iB^j\left[\gamma_i\gamma_j - \gamma_j\gamma_i\right]\gamma^1\gamma^2\gamma^3\gamma_0\Big\} \\ &= \frac{1}{4}\Big\{(\mathrm{tr}_gK)^2 + |E|^2_g + |B|^2_g + 2\epsilon_{ijk}E^iB^j\gamma^k\gamma_0\Big\}\,. \qquad (3.3.36)\end{aligned}$$

Next, from the definition (3.3.21) together with (3.3.22a) and (3.3.28a) we obtain

$$(A_i)^t = \frac{1}{2}\left[(K_{ij}\gamma^j + 2E_i + E^k\gamma_k\gamma_i)\gamma_0 - \epsilon_{ijk}B^j\gamma^k - B_i\gamma^1\gamma^2\gamma^3\right]\,. \qquad (3.3.37)$$

Using the form (3.3.24) of $A_i(B)$ one has

$$\begin{aligned} (A_i)^tA^i &= \frac{1}{4}\left[(K_{ij}\gamma^j + 2E_i + E^k\gamma_k\gamma_i)\gamma_0 - \epsilon_{ijk}B^j\gamma^k - B_i\gamma^1\gamma^2\gamma^3\right] \\ &\quad\times\left[(K^i{}_j\gamma^j - E^k\gamma_k\gamma^i)\gamma_0 + \epsilon^{ijk}B_j\gamma_k - B^i\gamma^1\gamma^2\gamma^3\right]\,. \qquad (3.3.38)\end{aligned}$$

Again, we do not need to calculate all the terms above, only the symmetric part matters. It is straightforward to check that the following symmetry properties hold:

$$\text{symmetric}: \quad \gamma_0 \,, \quad \gamma_i\gamma_0 \,, \quad \gamma^1\gamma^2\gamma^3 \,, \tag{3.3.39a}$$

$$\text{antisymmetric}: \quad \gamma_i \,,\ \gamma_i\gamma_j \ \text{ and } \ \gamma_i\gamma_j\gamma_0 \ \text{ both with } \ i\neq j \,,\ \gamma^1\gamma^2\gamma^3\gamma_0 \,. \tag{3.3.39b}$$

For example,

$$(\gamma^1\gamma^2\gamma^3\gamma_0)^t = \gamma_0^t(\gamma^1\gamma^2\gamma^3)^t = \gamma_0\gamma^1\gamma^2\gamma^3 = -\gamma^1\gamma^2\gamma^3\gamma_0 \,, \tag{3.3.40}$$

the remaining claims in (3.3.39) being proved similarly, *cf.* also (3.3.25). Using (3.3.39), Equation (3.3.38) can be manipulated[6] as follows

$$\begin{aligned}
(A_i)^tA^i &= \frac{1}{4}\Big\{(K_{i\ell}\gamma^\ell + 2E_i + E^\ell\gamma_\ell\gamma_i)\gamma_0 \\
&\qquad \times\Big[(K^i{}_j\gamma^j - E^k\gamma_k\gamma^i)\gamma_0 + \epsilon^{ijk}B_j\gamma_k - B^i\gamma^1\gamma^2\gamma^3\Big] \\
&\quad -\epsilon_{i\ell m}B^\ell\gamma^m\Big[(K^i{}_j\gamma^j - E^k\gamma_k\gamma^i)\gamma_0 + \epsilon^{ijk}B_j\gamma_k - B^i\gamma^1\gamma^2\gamma^3\Big] \\
&\quad -B_i\gamma^1\gamma^2\gamma^3\Big[(K^i{}_j\gamma^j - E^k\gamma_k\gamma^i)\gamma_0 + \epsilon^{ijk}B_j\gamma_k - B^i\gamma^1\gamma^2\gamma^3\Big]\Big\} \\
&= \frac{1}{4}\Big\{(K_{i\ell}\gamma^\ell + 2E_i + E^\ell\gamma_\ell\gamma_i) \\
&\qquad \times\Big[-K^i{}_j\gamma^j - E^k\gamma_k\gamma^i - \epsilon^{ijk}B_j\gamma_k\gamma_0 + B^i\gamma^1\gamma^2\gamma^3\gamma_0\Big] \\
&\quad -\epsilon_{i\ell m}B^\ell\gamma^m(K^i{}_j\gamma^j - E^k\gamma_k\gamma^i)\gamma_0 + \epsilon_{i\ell m}B^\ell\epsilon^{ijm}B_j \\
&\quad -B_i\gamma^1\gamma^2\gamma^3\Big[-E^k\gamma_k\gamma^i\gamma_0 - B^i\gamma^1\gamma^2\gamma^3\Big]\Big\} + \text{antisymmetric} \\
&= \frac{1}{4}\Big\{(K_{i\ell}\gamma^\ell\Big[-K^i{}_j\gamma^j - E^k\gamma_k\gamma^i - \epsilon^{ijk}B_j\gamma_k\gamma_0\Big] \\
&\quad +2E_i\Big[-E^k\gamma_k\gamma^i - \epsilon^{ijk}B_j\gamma_k\gamma_0\Big] \\
&\quad +E^\ell\gamma_\ell\gamma_i\Big[-K^i{}_j\gamma^j - E^k\gamma_k\gamma^i - \epsilon^{ijk}B_j\gamma_k\gamma_0 + B^i\gamma^1\gamma^2\gamma^3\gamma_0\Big] \\
&\quad +\underbrace{\epsilon_{i\ell m}B^\ell K^{im}}_{0} + \underbrace{\underbrace{B^\ell E^k\epsilon_{i\ell m}}_{\text{antisym. in } i,m}\ \underbrace{(-\gamma^m\delta^i_k - \delta^m_k\gamma^i)}_{\text{sym. in } i,m}}_{0}\gamma_0 + 2|B|^2_g \\
&\quad +\epsilon_{ik\ell}B^iE^k\gamma^\ell\gamma_0 + |B|^2_g\Big\} + \text{antisymmetric} \\
&= \frac{1}{4}\Big\{|K|^2_g - \underbrace{K_{i\ell}E_k\epsilon^{\ell ki}}_{0}\gamma^1\gamma^2\gamma^3 + \underbrace{\epsilon^{ijk}K_{ik}B_j}_{0}\gamma_0 \\
&\quad +2|E|^2_g - 2\epsilon^{ijk}E_iB_j\gamma_k\gamma_0 \\
&\quad -E^\ell E^k\gamma_\ell\underbrace{(-2g_{ik} - \gamma_k\gamma_i)\gamma^i}_{\gamma_k} - \underbrace{\epsilon^{ijk}B_jE^\ell(-g_{\ell i}\gamma_k + g_{k\ell}\gamma_i)\gamma_0}_{-2\epsilon_{ijk}E^iB^j\gamma^k\gamma_0} + \epsilon_{ijk}B^iE^j\gamma^k\gamma_0
\end{aligned}$$

---

[6] The calculation here can be somewhat simplified by noting at the outset that there is no symmetric nonzero matrix which can be built by contraction with K, E, and the γ-matrices with no indices left, similarly for K and B, hence the contributions from $A_i(K)$ and that of $A_i(E) + A_i(B)$ can be computed separately.

$$
\begin{aligned}
& +2|B|_g^2 \\
& +\epsilon_{ijk}B^iE^j\gamma^k\gamma_0 + |B|_g^2\Big\} + \text{antisymmetric} \\
= \ & \frac{1}{4}\Big\{|K|_g^2 + 3|E|_g^2 + 3|B|_g^2 - 2\epsilon^{ijk}E_iB_j\gamma_k\gamma_0\Big\}\ , \qquad (3.3.41)
\end{aligned}
$$

where the last equality is justified by the fact that all the antisymmetric matrices have to cancel out, since the matrix at the left-hand-side of the first line of (3.3.41) is symmetric. Subtracting (3.3.41) from (3.3.36) we thus find the following formula for the term (3.3.7f):

$$
\begin{aligned}
& \langle\phi, \{(\gamma^iA_i)^t\gamma^jA_j - (A^i)^tA_i\}\,\phi\rangle = \\
& \quad \frac{1}{4}\langle\phi, \Big\{|K|_g^2 - (\mathrm{tr}_gK)^2 + 2|E|_g^2 + 2|B|_g^2 - 4\epsilon^{ijk}E_iB_j\gamma_k\gamma_0\Big\}\phi\rangle\ . \qquad (3.3.42)
\end{aligned}
$$

Summarising, Equations (3.3.7), (3.3.8), (3.3.10), (3.3.12), (3.3.23), (3.3.29), (3.3.30), (3.3.33) and (3.3.42) lead to

$$
\begin{aligned}
D_i\langle\phi, (\nabla^i + \gamma^i\gamma^j\nabla_j)\phi\rangle &= |\nabla\phi|^2 - |\nabla\!\!\!/\phi|^2 \\
&+ \frac{1}{4}\langle\phi, \Big\{\mu + (\nu_i\gamma^i + 4\mathrm{div}(E))\gamma_0 - 4\mathrm{div}(B)\gamma^1\gamma^2\gamma^3\Big\}\phi\rangle\ , \qquad (3.3.43)
\end{aligned}
$$

where

$$\mu := R - |K|_g^2 + (\mathrm{tr}_gK)^2 - 2|E|_g^2 - 2|B|_g^2\ , \qquad (3.3.44a)$$

$$\nu_j = 2D_i(K^{ij} - \mathrm{tr}Kg^{ij}) - 4\epsilon_{jk\ell}E^kB^\ell\ . \qquad (3.3.44b)$$

We turn now our attention to the electromagnetic field contribution to the boundary integrand $\langle\phi, (\nabla^i + \gamma^i\gamma^j\nabla_j)\phi\rangle$:

$$
\begin{aligned}
A^i(E) + \gamma^i\gamma^jA_j(E) &= E^i\gamma_0 + \text{antisymmetric}\ , \qquad (3.3.45a) \\
A^i(B) + \gamma^i\gamma^jA_j(B) &= -\frac{1}{2}B^i\gamma^1\gamma^2\gamma^3 - \frac{1}{4}\epsilon_{jk\ell}B^j\gamma^i\gamma^k\gamma^\ell + \text{antisymmetric} \\
&= -B^i\gamma^1\gamma^2\gamma^3 + \text{antisymmetric}\ , \qquad (3.3.45b)
\end{aligned}
$$

and Equation (3.3.16) gives

$$\lim_{R\to\infty}\oint_{S(R)}\mathscr{U}^i dS_i = 4\pi\langle\phi_\infty, \left[p_\mu\gamma^\mu\gamma^0 + Q\gamma_0 - P\gamma^1\gamma^2\gamma^3\right]\phi_\infty\rangle\ . \qquad (3.3.46)$$

Our algebraic and analytic considerations so far lead to (see [72] for the analysis of the equality case):

Theorem 3.3.3 *Let (M, g, K) be a smooth three-dimensional initial data set, with (M, g) complete, and with an asymptotically flat end $M_{\rm ext}$, and with ∂M weakly outer trapped, if not empty. Suppose, further, that we are given on $\mathscr{S}$ two smooth vector fields $\hat{E}$ and B satisfying*

$$4\pi\rho_B := D_iB^i \in L^1(\mathscr{S})\ , \quad 4\pi\rho_E := D_iE^i \in L^1(\mathscr{S})\ .$$

Set

$$4\pi Q^E := \lim_{R\to\infty}\int_{r=R} E^i dS_i\ , \quad 4\pi Q^B := \lim_{R\to\infty}\int_{r=R} B^i dS_i\ .$$

Let R be the Ricci scalar of g and assume

$$0 \le R - |K|^2 + (trK)^2 - 2g(\hat E,\hat E) - 2g(B,B) =: 16\pi\rho_m \in L^1(M_{\rm ext})\ . \quad (3.3.47)$$

If

$$\rho_E^2 + \rho_B^2 + |J|_g^2 \le \rho_m^2\ , \quad (3.3.48)$$

where

$$16\pi J^i = 2D_j(K^{ij} - {\rm tr}K g^{ij}) - 4\epsilon^i{}_{k\ell}E^k B^\ell\ , \quad (3.3.49)$$

then the ADM mass m of $M_{\rm ext}$ satisfies

$$m \ge \sqrt{|\vec p|^2 + (Q^E)^2 + (Q^B)^2}\ . \quad (3.3.50)$$

*If the equality is attained in (3.3.50) then (3.3.48) is also an equality, and there exists on M a ∇-parallel spinor field. Furthermore, the associated space-time metric is, locally, an Israel-Wilson-Perjes (*not *necessarily electro-vacuum) metric.*

3.3.3 Cosmological constant

Let us turn our attention now to the hyperbolic case: let $\alpha \in \mathbb{R}$, we set

$$\mathscr{A}_i = \underbrace{A_i + \frac{\alpha\sqrt{-1}}{2}\gamma_i}_{A_i(\alpha)}\ , \qquad \nabla_i = D_i + \mathscr{A}_i\ , \quad (3.3.51)$$

with A_i as in Equation (3.3.21). (This is the standard way of taking into account a cosmological constant when no Maxwell field is present [102].) Here $\sqrt{-1} : V \to V$ is any map satisfying

$$(\sqrt{-1})^2 = -{\rm id}_V\ , \quad \sqrt{-1}\gamma_i = \gamma_i\sqrt{-1}\ , \quad (\sqrt{-1})^t = -\sqrt{-1}\ . \quad (3.3.52)$$

(If V is a complex vector space understood as a vector space over $\mathbb{R}$, with the real scalar product $\langle\cdot,\cdot\rangle$ arising from a sesquilinear form $\langle\cdot,\cdot\rangle_{\mathbb{C}}$, then $\sqrt{-1}$ can be taken as multiplication by i. On the other hand, if no such map $\sqrt{-1}$ exists on V, one can always replace V by its complexification $V_{\mathbb{C}} := V \otimes \mathbb{C}$, with new matrices $\gamma_\mu \otimes {\rm id}_{\mathbb{C}}$, and use multiplication by ${\rm id}_V \otimes (i\ {\rm id}_{\mathbb{C}})$ on $V_{\mathbb{C}}$ as the desired map.) We consider again Equation (3.3.7), with A there replaced by $\mathscr{A}$. Now, the terms (3.3.7d) and (3.3.7e) are linear in $\mathscr{A}$, and they have already been shown to vanish when $\alpha = 0$; thus, to show that they vanish for $\alpha \neq 0$ it suffices to show that they do so when $A_i = A_i(\alpha)$. This is obvious for (3.3.7e), while for (3.3.7d) we have

$$(A_i(\alpha))^t = A_i(\alpha)\ ,$$

$$\gamma^i A_i(\alpha) = -3\frac{\alpha\sqrt{-1}}{2}\ , \qquad (\gamma^i A_i(\alpha))^t = 3\frac{\alpha\sqrt{-1}}{2}\ ,$$

$$A^i(\alpha)-(A^i(\alpha))^t+\gamma^i\gamma^jA_j(\alpha)+(\gamma^jA_j(\alpha))^t\gamma^i = \frac{\alpha\sqrt{-1}}{2}\left(\gamma^i+(\gamma^i)^t-\frac{3}{2}\gamma^i+\frac{3}{2}\gamma^i\right) = 0 \ .$$

It follows that the only new terms that can perhaps occur in Equations (3.3.7c)-(3.3.7f) arise from (3.3.7f). We calculate

$$\begin{aligned}
\left(\gamma^iA_i+\gamma^iA_i(\alpha)\right)^t\left(\gamma^jA_j+\gamma^jA_j(\alpha)\right) &= \left((\gamma^iA_i)^t+3\frac{\alpha\sqrt{-1}}{2}\right)\left(\gamma^jA_j-3\frac{\alpha\sqrt{-1}}{2}\right)\\
&= (\gamma^iA_i)^t\gamma^jA_j+\frac{9\alpha^2}{4}\\
&\quad +3\frac{\alpha\sqrt{-1}}{2}\left(\gamma^jA_j-(\gamma^iA_i)^t\right) \ , \qquad (3.3.53)\\
\left(A^i+A^i(\alpha)\right)^t\left(A_i+A_i(\alpha)\right) &= \left((A^i)^t+\frac{\alpha\sqrt{-1}}{2}\gamma^i\right)\left(A_i+\frac{\alpha\sqrt{-1}}{2}\gamma_i\right)\\
&= (A^i)^tA_i+\frac{3\alpha^2}{4}+\frac{\alpha\sqrt{-1}}{2}\times\\
&\quad (\underbrace{(A_i)^t\gamma^i}_{-(A_i)^t(\gamma^i)^t=-(\gamma^iA_i)^t}+\gamma^jA_j) \ , \qquad (3.3.54)
\end{aligned}$$

so that

$$\begin{aligned}
(\gamma^i\mathscr{A}_i)^t\gamma^j\mathscr{A}_j-(\mathscr{A}^i)^t\mathscr{A}_i &= (\gamma^iA_i)^t\gamma^jA_j-(A^i)^tA_i+\frac{3\alpha^2}{2}\\
&\quad +\alpha\sqrt{-1}\left(\gamma^jA_j-(\gamma^iA_i)^t\right) \ . \qquad (3.3.55)
\end{aligned}$$

Equations (3.3.34)-(3.3.35) show that

$$\alpha\sqrt{-1}\left(\gamma^jA_j-(\gamma^iA_i)^t\right) = \alpha\sqrt{-1}B^i\gamma_i\gamma^1\gamma^2\gamma^3 \ , \qquad (3.3.56)$$

so that we obtain

$$\begin{aligned}
&D_i\langle\phi,(\nabla^i+\gamma^i\gamma^j\nabla_j)\phi\rangle = |\nabla\phi|^2-|\nabla\!\!\!/\phi|^2\\
&\quad +\frac{1}{4}\langle\phi,\left\{\mu+6\alpha^2+(\nu_i\gamma^i-\operatorname{div}(E))\gamma_0+\left(4\alpha\sqrt{-1}B^i\gamma_i-\operatorname{div}(B)\right)\gamma^1\gamma^2\gamma^3\right\}\phi\rangle \ ,
\end{aligned} \qquad (3.3.57)$$

where μ,ν is as in (3.3.44), $\operatorname{div}(E)$ is the divergence of E and $\operatorname{div}(B)$ that of B. Somewhat surprisingly, the term $B^i\gamma_i\gamma^1\gamma^2\gamma^3$ occurring above does not seem to combine in any obvious way with the remaining ones to yield a useful identity except when B vanishes. In any case the identity (3.3.57) can then be used to prove a mass-charge inequality in an asymptotically hyperboloidal setting, in the spirit of Theorem 3.3.3 with $B\equiv 0$ there, or if the positive terms there dominate the undesirable B term.

Chapter 4

The Trautman-Bondi mass

4.1 Introduction

In 1958 Trautman [175] (see also [177]) has introduced a notion of energy suitable for asymptotically Minkowskian radiating gravitational fields, and proved that it is monotonically decreasing; this mass has been further studied by Bondi *et al.* [108] and Sachs [160]. Several other definitions of mass have been given in the radiation setting, and it is convenient to start with a general overview of the subject. Our presentation follows closely [65].

First, there are at least seven methods for defining energy-momentum ("mass" for short) in the current context:

1. The already-mentioned definition of Trautman [175], based on the Freud integral [87], that involves asymptotically Minkowskian coordinates in space-time. The definition stems from a Hamiltonian analysis in a fixed global coordinate system.

2. A definition of Bondi *et al.* [108], also alluded-to above, which uses space-time "Bondi coordinates".

3. A definition of Abbott-Deser [1], originally introduced in the context of space-times with negative cosmological constant, which (as we will see) is closely related to the problem at hand. The Abbott-Deser integrand turns out to coincide with the linearisation of the Freud integrand, up to a total divergence [70].

4. The space-time "charge integrals", derived in a geometric Hamiltonian framework [53, 64, 69, 112]. A conceptually distinct, but closely related, variational approach, has been presented in [40].

5. The "initial data charge integrals", presented below, expressed in terms of data (g, K) on an initial data manifold.

6. The Hawking mass and its variations such as the Brown-York mass, using two-dimensional spheres.

7. A purely Riemannian definition, that provides a notion of mass for asymptotically hyperbolic Riemannian metrics [63, 185].

Each of the above typically comes with several distinct variations.

Those definitions have the following properties:

1. The Bondi mass, say m_B, requires in principle a space-time on which Bondi coordinates can be introduced. However, a null hypersurface extending to future null infinity suffices. Neither analysis is directly adapted to an analysis in terms of usual spacelike initial data sets. The mass m_B is an invariant under Bondi-van der Burg-Metzner-Sachs coordinate transformations.

   The Bondi mass m_{B} has been shown to be the unique functional, within an appropriate class, which is non-increasing with respect to deformations of the section to the future [66].

   A formulation of that mass in terms of "quasi-spherical" foliations of null cones has been recently given in [13] under rather weak differentiability conditions.

   The question, whether m_B is uniquely defined by the asymptotic structure of the space-time is not clear, because there could exist Bondi coordinates which are not related to each other by a Bondi-van der Burg-Metzner-Sachs coordinate transformation.

2. Trautman's definition, say m_T, requires existence of a certain class of asymptotically Minkowskian coordinates, with m_T being invariant under a class of coordinate transformations that arise naturally in this context [175]. The definition is obtained by evaluating the Freud integral in Trautman's coordinates. Asymptotically Minkowskian coordinates associated with the Bondi coordinates belong to the Trautman class, and the definition is invariant under a natural class of coordinate transformations. Trautman's conditions for existence of mass are less stringent, at least in principle[1], than the Bondi ones. Bondi's mass m_B equals Trautman's mass m_T of the associated quasi-Minkowskian coordinate system, whenever both m_B and m_T can be simultaneously defined.

   Uniqueness is not clear, because there could exist Trautman coordinates which are not related to each other by the coordinate transformations analyzed by Trautman.

3. The Hawking mass, and its variations, are *a priori* highly sensitive to the way that a family of spheres approaches a cut of conformal null infinity $\mathscr{I}$. (This is one of the major problems which one faces when trying to generalise the proof of the Penrose inequality to a hyperboloidal setting [146].) It is known that those masses converge to the Trautman-Bondi mass when

[1] It is difficult to make a clear cut statement here because existence theorems that lead to space-times with Trautman coordinates seem to provide Bondi coordinates as well (though perhaps in a form that is weaker than required in the original definition of mass, but still compatible with an extension of Bondi's definition).

evaluated on Bondi spheres, but this result is useless in a Cauchy data context, as general initial data sets will not be collections of Bondi spheres.

4. From a Cauchy problem point of view, a suitable framework for discussing the mass in the radiation regime is provided by "hyperboloidal initial data sets". Such data sets are in general relativity in two different contexts: as hyperboloidal hypersurfaces in asymptotically Minkowskian space-times on which K approaches a multiple of g as one recedes to infinity, or as spacelike hypersurfaces in space-times with a negative cosmological constant on which $|K|_g$ approaches zero as one recedes to infinity. This indicates that the Abbott-Deser integrals, which arose in the context of space-times with non-zero cosmological constant, could be related to the Trautman-Bondi mass. It follows from our analysis below that these masses do, in fact, coincide with the initial data charge integrals under certain strong decay conditions.

   In space-times with a cosmological constant, the strong decay conditions are satisfied on hypersurfaces which are, roughly speaking, orthogonal to high order to the conformal boundary, but will not be satisfied on more general hypersurfaces.

5. The Abbott-Deser definition of mass is based on the analysis of the linearized field equations. It proceeds through a linearized version of boundary integrals in the spirit of that of Freud, introduced in Section 1.2. Hence the need to understand the relation between such integrals and their linearisations. We will show that the linearisation of the Freud integral coincides with the linearisation of the initial data "charge integrals".

6. However, we will see that the linearisation of the Freud integral does *not* coincide in general with the Freud integral for radiating metrics. Nevertheless, the resulting numbers which are assigned to hypersurfaces coincide when decay conditions, referred to as *strong decay conditions*, are imposed. The strong decay conditions turn out to be incompatible with existence of gravitational radiation.

7. We will show that a version of the Brown-York mass, as well as the Hawking mass, evaluated on a specific foliation within a hyperboloidal initial data set, converges to the Trautman-Bondi mass.

   This will be used to show that the Freud integrals coincide with the initial data charge integrals, for asymptotically CMC initial data sets on which a space-equivalent of Bondi coordinates can be constructed.

8. It has been shown in [63] that the strong decay conditions on g, mentioned above, are *necessary* for a well defined Riemannian definition of mass and of momentum. This appears to be paradoxical at first sight, since the strong decay conditions are incompatible with gravitational radiation; on the other hand one expects the Trautman-Bondi mass to be well defined even if there is gravitational radiation. It turns out that the initial data charge integrals involve delicate cancelations between the metric g and

the extrinsic curvature tensor K, leading to a well defined notion of mass of initial data sets without the stringent restrictions of the Riemannian definition (which does not involve K).

The definition of the Trautman-Bondi mass presented here requires a conformal completion at null infinity of our space-time. We assign a mass, which we denote by m_{TB}, to sections of the conformal boundary $\mathscr{I}$ using Bondi coordinates. Now, there is a potential ambiguity arising from the possibility of existence of non-equivalent conformal completions of a Lorentzian space-time (see [52] for an explicit example). From a physical point of view, sections of Scri represent the asymptotic properties of a radiating system at a given moment of retarded time. Thus, one faces the curious possibility that two different masses could be assigned to the same state of the system, at the same retarded time, depending upon which of the conformally inequivalent completions one chooses. We show that the Trautman-Bondi mass of a section S of $\mathscr{I}^+$ is a geometric invariant, in a sense which is made precise in Section 4.6.2 below. (See [60, Section 5.1] for an alternative proof within the frameworks of [6, 101], under the supplementary hypothesis of existence of a strongly causal conformal completion at $\mathscr{I}$.)

The proof of positivity of m_{TB} can be found in Theorems 4.6.4 and 4.6.7 below; see [118, 136, 159, 165] for alternative arguments. The reader is warned that some of the proofs available in the literature prove positivity of something which might be different from the TB mass, or are not detailed enough to be able to assert correctedness, or are incomplete, or wrong.

An interesting property of the Trautman-Bondi mass is that it can be given a Hamiltonian interpretation [64]. Now, from a Hamiltonian point of view, it is natural to assign a Hamiltonian to an initial data set $(\mathscr{S}, g, K)$, where $\mathscr{S}$ is a three-dimensional manifold, without the need of invoking a four-dimensional space-time. If there is an associated conformally completed space-time in which the completion $\widetilde{\mathscr{S}}$ of $\mathscr{S}$ meets $\mathscr{I}^+$ in a sufficiently regular, say differentiable, section S, then $m_{\mathrm{TB}}(\mathscr{S})$ can be defined as the Trautman-Bondi mass of the section S. For the purposes of this chapter, hypersurfaces satisfying the above will be called *hyperboloidal*.

Our main approach to the mass does not involve any space-time constructions. This has several advantages. First, for non-vacuum initial data sets an existence theorem for an associated space-time might be lacking. Further, the initial data might not be sufficiently differentiable to guarantee existence of an associated space-time using known evolution theorems. Next, there might be a loss of differentiability arising from evolution theorems which will not allow one to perform the space-time constructions needed for the space-time definition of mass. Finally, a proof of uniqueness of the definition of mass of an initial data set could perhaps be easier to achieve than the space-time one. Last but not least, most proofs of positivity use three-dimensional hypersurfaces anyway. For all those reasons it is of interest to obtain a definition of mass, momentum, *etc.*, in an initial data setting.

The final $(3 + 1)$-dimensional formulae for the Hamiltonian charges turn out to be rather complicated. We close this chapter by deriving a considerably simpler expression for the charges in terms of the geometry of "approximate Bondi spheres" near $\mathscr{I}^+$, Equations (4.7.1)-(4.7.2) below. The expression is similar in spirit to that of Hawking and of Brown, Lau and York [40]. It applies to mass as well as momentum, angular momentum and centre of mass.

It should be said that our three-dimensional and two-dimensional versions of the definitions do not cover all possible hyperboloidal initial hypersurfaces, because of a restrictive assumption on the asymptotic behavior of $g^{ij}K_{ij}$, see (4.10.1). This condition arises from the need to reduce the calculational complexity of our problem; without (4.10.1) the calculations needed seem to exceed the limit of what one can calculate by hand with a reasonable degree of confidence in the final formulae. However, the results obtained are sufficient to prove positivity of $m_{\mathrm{TB}}(S)$ for all smooth sections of $\mathscr{I}^+$ which bound some smooth complete hypersurface $\mathscr{S}$, because then $\mathscr{S}$ can be deformed in space-time to a hypersurface which satisfies (4.10.1), while retaining the same conformal boundary S; compare Theorem 4.6.7 below.

Let us expand on our comments above concerning the Riemannian definition of mass: consider a CMC initial data set with $\mathrm{tr}_g K = -3$ and $\Lambda = 0$, corresponding to a hyperboloidal hypersurface in an asymptotically Minkowskian space-time as constructed in [3], so that the g-norm of $K_{ij} + g_{ij}$ tends to zero as one approaches $\mathscr{I}^+$. It then follows from the vacuum constraint equations that $R(g)$ approaches -6 as one recedes to infinity, and one can enquire whether the metric satisfies the conditions needed for the Riemannian definition of mass for such metrics [63]. Now, one of the requirements in [63] is that the background derivatives $\mathring{\nabla} g$ of g be in $L^2(M)$. A simple calculation (see Section 4.5 below) shows that for smoothly compactifiable $(\mathscr{S}, g)$ this will only be the case if the extrinsic curvature χ of the conformally rescaled metric vanishes at the conformal boundary $\partial\mathscr{S}$. It has been shown in [64, Appendix C.3] that the u-derivative of χ coincides with the Bondi "news function", and therefore the Riemannian definition of mass can *not* be used for families of hypersurfaces in space-times with non-zero flux of Trautman-Bondi energy, yielding an unacceptable restriction. Clearly one needs a definition which would allow less stringent conditions than the ones in [63, 69], but this seems incompatible with the examples in [69] which show sharpness of the conditions assumed. The answer to this apparent paradox turns out to be the following: in contradistinction with the asymptotically flat case, in the hyperboloidal one the definition of mass does involve the extrinsic curvature tensor K in a non-trivial way. The leading behavior of the latter combines with the leading behavior of the metric to give a well defined, convergent, geometric invariant. It is only when K (in the $\Lambda < 0$ case) or $K + g$ (in the $\Lambda = 0$ case) vanishes to sufficiently high order that one recovers the purely Riemannian definition; however, because the leading order of K, or $K + g$, is coupled to that of g via the constraint equations, one obtains – in the purely Riemannian case – more stringent conditions on the difference between the metric g and its asymptotic value.

We will analyse invariance and finiteness properties of the charge integrals in any dimension under asymptotic conditions analogous to those in [63, 69]. However, the analysis under boundary conditions appropriate for existence of gravitational radiation will only be done in space-time dimension four.

4.2 Bondi mass

The starting point of the Bondi-Sachs definition of mass is the existence of a suitable coordinate system: We shall suppose that there exists an open subset of the space-time M on which *Bondi–Sachs coordinates* can be introduced; by definition, these are coordinates in which the metric takes the form

$$g = -\frac{V}{r}e^{2\beta}du^2 - 2e^{2\beta}dudr + r^2h_{AB}(dx^A - U^Adu)(dx^B - U^Bdu) , \quad (4.2.1)$$

$$\frac{\partial(\det h_{AB})}{\partial r} = 0 , \quad (4.2.2)$$

$$r \in (r_0, \infty) , \quad u \in (u_-, u_+) , \quad (x^A) \text{ — local coordinates on } S^2 .$$

One is then interested in the behaviour of the gravitational field as r tends to infinity, at constant u.

It follows from (4.5.2) that the area of spheres of constant r is proportional to r^2. The coordinate r is therefore called *area coordinate*.

For example, in Minkowski space-time we replace t by $u = t-r$, which leads to

$$\begin{aligned} \eta &= -dt^2 + dr^2 + r^2d\Omega^2 = -(du + dr)^2 + dr^2 + r^2d\Omega^2 \\ &= -du^2 - 2\,du\,dr + r^2d\Omega^2 . \end{aligned}$$

As another example, consider the Schwarzschild metric, and replace t by $u = t - r - 2m\ln(r - 2m)$:

$$\begin{aligned} g &= -(1 - \frac{2m}{r})dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2d\Omega^2 \\ &= -(1 - \frac{2m}{r})(du + \frac{dr}{1 - \frac{2m}{r}})^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2d\Omega^2 \\ &= -(1 - \frac{2m}{r})du^2 - 2\,du\,dr + r^2d\Omega^2 . \end{aligned}$$

We will ignore the question, how general are the metrics which can be brought to the form (4.2.1)-(4.2.2). The reader is referred to [26, 68, 174] for constructions under suitable conditions.

It is convenient to replace the area coordinate r by its inverse,

$$x = 1/r \in (0, 1/r_0) ,$$

and use the coordinate system (u, x^A, x) for small x to describe the asymptotic behaviour of the gravitational field. Then

$$g = \frac{1}{x^2}\Big(-Vx^3e^{2\beta}du^2 + 2e^{2\beta}du\,dx + h_{AB}(dx^A - U^Adu)(dx^B - U^Bdu)\Big) . \quad (4.2.3)$$

Bondi *et al.* require that in the coordinate system (u, x^A, x) the functions V, β, the S^2-vector fields U^A, and the S^2-Riemannian metrics h_{AB} extend smoothly by continuity to the boundary

$$\mathscr{I}^+ := \{x = 0\} .$$

It is also required that

$$\lim_{x\to 0} h_{AB} = \breve{h}_{AB} \ ,$$

where $\breve{h}_{AB}$ is the standard round metric on the sphere,

$$\breve{h}_{AB} dx^A dx^B = d\theta^2 + \sin^2\theta \, d\varphi^2 \ .$$

The *conformal completion* $\overline{M}$ of M is defined as

$$\overline{M} \equiv M \cup \mathscr{I}^+ \ ,$$

with the obvious differential structure defined by the coordinate system (u, x^A, x). Equation (4.2.3) shows that the conformally rescaled metric

$$\tilde{g} := \Omega^2 g \ ,$$

where

$$\Omega \equiv x \ ,$$

extends smoothly as a Lorentzian metric across the *conformal boundary* $\mathscr{I}^+$.

The above construction of $\mathscr{I}^+$ and $\overline{M}$ is equivalent [174] to the geometric approach of Penrose [153] as far as local considerations near $\mathscr{I}^+$ are concerned.

The vacuum Einstein equations lead to the following expansions for the coefficients of the metric (the formulae below are taken from [66]; they are essentially due to [179], *cf.* also [26]):

$$\begin{aligned}
h_{AB} &= \breve{h}_{AB}\left(1 + \frac{1}{4r^2}\chi^{CD}\chi_{CD}\right) + \frac{\chi_{AB}(v)}{r} + O(r^{-3}) \ , && (4.2.4)\\
\beta &= -\frac{\breve{h}^{AB}\breve{h}^{CD}\chi_{AC}\chi_{BD}}{32r^2} + O(r^{-3}) \ , \\
U^A &= -\frac{\breve{\mathcal{D}}_B\chi^{AB}}{2r^2} + \frac{2N^A(v)}{r^3} + \frac{\breve{\mathcal{D}}^A\left(\chi^{CD}\chi_{CD}\right)}{16r^3} \\
&\quad + \frac{\chi^A{}_B\breve{\mathcal{D}}_C\chi^{BC}}{2r^3} + O(r^{-4}) \, , && (4.2.5)\\
V &= r - 2M(v) + \frac{\breve{\mathcal{D}}_B\chi^{AB}\breve{\mathcal{D}}^C\chi_{AC} - 4\breve{\mathcal{D}}_A N^A}{4r} \\
&\quad + \frac{\chi^{CD}\chi_{CD}}{16r} + O(r^{-2}) \, . && (4.2.6)
\end{aligned}$$

Here $v \equiv (u, x^A)$, and $\breve{\mathcal{D}}_A$ is the covariant derivative operator associated with the metric $\breve{h}_{AB}$ on S^2. Indices A, B, *etc.*, take values 1 and 2, and are raised and lowered with $\breve{h}^{AB}$. The tensor field χ_{AB} is trace-free with respect to the metric $\breve{h}$:

$$\breve{h}^{AB}\chi_{AB} = 0 \ . \qquad (4.2.7)$$

Further, the functions M and N^A satisfy the following evolution equations

$$\frac{\partial M}{\partial u} = -\frac{1}{8}\breve{h}^{AC}\breve{h}^{BD}\partial_u\chi_{AB}\partial_u\chi_{CD} + \frac{1}{4}\breve{\mathcal{D}}_A\breve{\mathcal{D}}_B\partial_u\chi^{AB} \ , \qquad (4.2.8)$$

$$\begin{aligned}
3\frac{\partial N^A}{\partial u} &= -\breve{\mathcal{D}}^A M + \frac{1}{4}\epsilon^{AB}\breve{\mathcal{D}}_B\tilde{\lambda} - K^A \,, \qquad (4.2.9)\\
K^A &\equiv \frac{3}{4}\chi^A{}_B\breve{\mathcal{D}}_C\partial_u\chi^{BC} + \frac{1}{4}\partial_u\chi^{CD}\breve{\mathcal{D}}_D\chi^A{}_C \,,\\
\tilde{\lambda} &\equiv \breve{h}^{BD}\epsilon^{AC}\breve{\mathcal{D}}_C\breve{\mathcal{D}}_B\chi_{DA} \,.
\end{aligned}$$

Here ϵ_{AB} is an anti-symmetric tensor field on S^2 defined by the formula

$$\sin\theta\, d\theta\wedge d\varphi = \frac{1}{2}\epsilon_{AB}\, dx^A\wedge dx^B \,.$$

If we fix some $u_0 \in I$, then the Einstein equations do not impose any local restrictions on the function $M(u_0,\,\theta,\,\varphi)$, and on the vector field $N^A(u_0,\,\theta,\,\varphi)$ on S^2, or on the u-dependent family of tensor fields $\chi_{CD}(u,\,\theta,\,\varphi)$ on S^2.

The function M is called *mass aspect*, while the tensor field χ_{AB} is called *Bondi news*. The *Bondi mass* $m_{\rm B}$ is defined as

$$m_{\rm B}(u) = \frac{1}{4\pi}\int_{S^2} M(u,\theta,\varphi)\sin\theta\, d\theta\, d\varphi \,. \qquad (4.2.10)$$

It follows from (4.2.8) that

$$\frac{\partial m_{\rm B}}{\partial u} = -\frac{1}{32\pi}\int_{S^2}\breve{h}^{AC}\breve{h}^{BD}\partial_u\chi_{AB}\partial_u\chi_{CD}\sin\theta\, d\theta\, d\varphi \,. \qquad (4.2.11)$$

Hence $m_{\rm B}$ is monotonically decreasing. Equation (4.2.11) is known as the *Bondi mass-loss formula.*

4.3 Global charges of initial data sets

4.3.1 The charge integrals

Let g and b be two Riemannian metrics on an n-dimensional manifold M, $n \geq 2$, and let V be any function there. We set

$$e_{ij} := g_{ij} - b_{ij} \,. \qquad (4.3.1)$$

We denote by $\mathring{D}$ the Levi-Civita connection of b and by R_f the scalar curvature of a metric f. In [63] the following identity has been proved:

$$\sqrt{\det g}\, V(R_g - R_b) = \partial_i\left(\mathbb{U}^i(V)\right) + \sqrt{\det g}\,(s+Q) \,, \qquad (4.3.2)$$

where

$$\begin{aligned}
\mathbb{U}^i(V) &:= 2\sqrt{\det g}\,\left(Vg^{i[k}g^{j]l}\mathring{D}_j g_{kl} + D^{[i}Vg^{j]k}e_{jk}\right) \,, \qquad (4.3.3)\\
s &:= (-V\mathrm{Ric}(b)_{ij} + \mathring{D}_i\mathring{D}_j V - \Delta_b V b_{ij})g^{ik}g^{j\ell}e_{k\ell} \,, \qquad (4.3.4)\\
Q &:= V(g^{ij} - b^{ij} + g^{ik}g^{j\ell}e_{k\ell})\mathrm{Ric}(b)_{ij} + Q' \,. \qquad (4.3.5)
\end{aligned}$$

Brackets over a symbol denote anti-symmetrisation, with an appropriate numerical factor (1/2 in the case of two indices).[2] The symbol Δ_f denotes the Laplace operator of a metric f. The result is valid in any dimension $n \geq 2$. Here Q' denotes an expression which is bilinear in e_{ij} and $\mathring{D}_k e_{ij}$, linear in V, dV and Hess$V$, with coefficients which are uniformly bounded in a b-ON frame, as long as g is uniformly equivalent to b. The idea behind the calculation leading to (4.3.2) is to collect all terms in R_g that contain second derivatives of the metric in $\partial_i \mathbb{U}^i$; in what remains one collects in s the terms which are linear in e_{ij}, while the remaining terms are collected in Q; one should note that the first term at the right-hand-side of (4.3.5) does indeed not contain any terms linear in e_{ij} when Taylor expanded at $g_{ij} = b_{ij}$.

We wish to present a generalisation of this formula — Equation (4.3.11) below — which takes into account the physical extrinsic curvature tensor K and its background equivalent $\mathring{K}$; this requires introducing some notation. For any scalar field V and vector field Y we define

$$\mathring{A}_{kl} \equiv A_{kl}(b, \mathring{K}) := \mathcal{L}_Y b_{kl} - 2V \mathring{K}_{kl} \,, \tag{4.3.6}$$

$$A \equiv A(g,K) := V \underbrace{\left(R_g - K^{kl}K_{kl} + (\mathrm{tr}_g K)^2\right)}_{=\rho(g,K)} - 2Y^k \underbrace{D_l \left(K^l{}_k - \delta^l{}_k \mathrm{tr}_g K\right)}_{=-J_k(g,K)/2}$$

$$= -\frac{2\mathcal{G}^0{}_\mu X^\mu}{\sqrt{\det g}} = 2G_{\mu\nu} n^\mu X^\nu \,. \tag{4.3.7}$$

The symbol $\mathscr{L}$ denotes a Lie derivative. If we were in a space-time context, then $\mathcal{G}^\lambda{}_\mu$ would be the Einstein tensor density, $G_{\mu\nu}$ would be the Einstein tensor, while n^μ would be the future directed normal to the initial data hypersurface. Finally, an associated space-time vector field X would then be defined as

$$X = V n^\mu \partial_\mu + Y^k \partial_k = \frac{V}{N}\partial_0 + (Y^k - \frac{V}{N}N^k)\partial_k \,, \tag{4.3.8}$$

where N and N^k are the lapse and shift functions. However, as far as possible we will forget about any space-time structures. It should be pointed out that our ρ here can be interpreted as the energy-density of the matter fields when the cosmological constant Λ vanishes; it is, however, shifted by a constant in the general case.

We set

$$P^{kl} := g^{kl}\mathrm{tr}_g K - K^{kl} \,, \qquad \mathrm{tr}_g K := g^{kl} K_{kl} \,,$$

with a similar definition relating the background quantities $\mathring{K}$ and $\mathring{P}$; indices on K and P are always moved with g while those on $\mathring{K}$ and $\mathring{P}$ are always moved with b.

We shall say that (V, Y) satisfy the (background) *vacuum Killing Initial Data* (KID) equations if

$$A_{ij}(b, \mathring{K}) = 0 = S^{kl}(b, \mathring{K}) \,, \tag{4.3.9}$$

[2] In general relativity a normalising factor $1/16\pi$, arising from physical considerations, is usually thrown in into the definition of $\mathbb{U}^i$. From a geometric point view this seems purposeful when the boundary at infinity is a round two dimensional sphere; however, for other topologies and dimensions, this choice of factor does not seem very useful, and for this reason we do not include it in $\mathbb{U}^i$.

where

$$\mathring{S}^{kl} \equiv S^{kl}(b,\mathring{K}) := V\left(2\mathring{P}^{ml}\mathring{P}_m{}^k - \frac{3}{n-1}\mathrm{tr}_b\mathring{P}\mathring{P}^{kl} + \mathrm{Ric}(b)^{kl} - R_b b^{kl}\right) - \pounds_Y\mathring{P}^{kl} + \Delta_b V b^{kl} - \mathring{D}^k\mathring{D}^l V\ . \qquad (4.3.10)$$

Vacuum initial data with this property lead to space-times with Killing vectors, see [141] (compare [23]). We will, however, not assume at this stage that we are dealing with vacuum initial data sets, and we will do the calculations in the general case.

We have the following counterpart of Equation (4.3.2):

$$\partial_i\left(\mathbb{U}^i(V) + \mathbb{V}^i(Y)\right) = \sqrt{\det g}\left[V\left(\rho(g,K) - \rho(b,\mathring{K})\right) + s' + Q''\right] + Y^k\sqrt{\det g}\left(J_k(g,K) - J_k(b,\mathring{K})\right)\ , \qquad (4.3.11)$$

where

$$\mathbb{V}^l(Y) := 2\sqrt{\det g}\left[(P^l{}_k - \mathring{P}^l{}_k)Y^k - \frac{1}{2}Y^l\mathring{P}^{mn}e_{mn} + \frac{1}{2}Y^k\mathring{P}^l{}_k b^{mn}e_{mn}\right]\ . \qquad (4.3.12)$$

Further, Q'' contains terms which are quadratic in the deviation of g from b and its derivatives, and in the deviations of K from $\mathring{K}$, while s', obtained by collecting all terms linear or linearised in e_{ij}, except for those involving ρ and J, reads

$$s' = (\mathring{S}^{kl} + \mathring{B}^{kl})e_{kl} + (P^{kl} - \mathring{P}^{kl})\mathring{A}_{kl}\ ,$$
$$\mathring{B}^{kl} := \frac{1}{2}\left[b^{kl}\mathring{P}^{mn}\mathring{A}_{mn} - b^{mn}\mathring{A}_{mn}\mathring{P}^{kl}\right]\ . \qquad (4.3.13)$$

We postpone the derivation of (4.3.11) to Section 4.8.

4.4 Initial data sets with rapid decay

4.4.1 The reference metrics

Consider a manifold M which contains a region $M_{\mathrm{ext}} \subset M$ together with a diffeomorphism

$$\Phi^{-1} : M_{\mathrm{ext}} \to [R,\infty)\times N\ , \qquad (4.4.1)$$

where N is a compact boundaryless manifold. Suppose that on $[R,\infty)\times N$ we are given a Riemannian metric b_0 of a product form

$$b_0 := \frac{dr^2}{r^2+k} + r^2\breve{h}\ , \qquad (4.4.2)$$

as well as a symmetric tensor field K_0; conditions on K_0 will be imposed later on. We assume that $\breve{h}$ is a Riemannian metric on N with constant scalar curvature $R_{\breve{h}}$ equal to

$$R_{\breve{h}} = \begin{cases}(n-1)(n-2)k\ , \quad k\in\{0,\pm1\}\ , & \text{if } n>2,\\ 0\ , \quad k=1\ , & \text{if } n=2.\end{cases} \qquad (4.4.3)$$

In (4.4.2) the coordinate r runs along the $[R,\infty)$ factor of $[R,\infty)\times N$. As such, the dimension of N equals to $n-1$; we will later on specialise to the case $n+1=4$ but we allow a general n in this section. There is some freedom in the choice of k when $n=2$, associated with the range of the angular variable φ on $N=S^1$, and we make the choice $k=1$ which corresponds to the usual form of the two-dimensional hyperbolic space. When $(N,\breve h)$ is the unit round $(n-1)$–dimensional sphere $(\mathbb{S}^{n-1},g_{\mathbb{S}^{n-1}})$, then b_0 is the hyperbolic metric.

Pulling-back b_0 using Φ^{-1} we define on $M_{\rm ext}$ a reference metric b,

$$\Phi^*b=b_0\ . \tag{4.4.4}$$

Equations (4.4.2)–(4.4.3) imply that the scalar curvature R_b of the metric b is constant:

$$R_b=R_{b_0}=n(n-1)k\ .$$

Moreover, the metric b will be Einstein if and only if $\breve h$ is. We emphasise that for all our purposes we only need b on $M_{\rm ext}$, and we continue b in an arbitrary way to $M\setminus M_{\rm ext}$ whenever required.

Anticipating, the "charge integrals" will be defined as the integrals of $\mathbb{U}+\mathbb{V}$ over "the boundary at infinity", *cf.* Proposition 4.4.2 below. The convergence of the integrals there requires appropriate boundary conditions, which are defined using the following b-orthonormal frame $\{f_i\}_{i=1,n}$ on $M_{\rm ext}$:

$$\Phi^{-1}_*f_i=r\epsilon_i\ ,\quad i=1,\dots,n-1\ ,\quad \Phi^{-1}_*f_n=\sqrt{r^2+k}\,\partial_r\ , \tag{4.4.5}$$

where the ϵ_i's form an orthonormal frame for the metric $\breve h$. We moreover set

$$g_{ij}:=g(f_i,f_j)\ ,\quad K_{ij}:=K(f_i,f_j)\ , \tag{4.4.6}$$

etc., and throughout this section only tetrad components will be used.

4.4.2 The charges

We start by introducing a class of boundary conditions for which convergence and invariance proofs are particularly simple. We emphasise that the asymptotic conditions of Definition 4.4.1 are too restrictive for general hypersurfaces meeting $\mathscr{I}$ in anti-de Sitter space-time, or – perhaps more annoyingly – for general radiating asymptotically flat metrics. We will return to that last case in Section 4.6; this requires considerably more work.

DEFINITION 4.4.1 (Strong asymptotic decay conditions) *We shall say that the initial data* (g,K) *are* strongly asymptotically hyperboloidal *if:*

$$\begin{aligned}\int_{M_{\rm ext}}\Bigg(&\sum_{i,j}\Big(|g_{ij}-\delta_{ij}|^2+|K_{ij}-\mathring{K}_{ij}|^2\Big)+\sum_{i,j,k}|f_k(g_{ij})|^2\\ &+\sum_{i,j}\Big(|\mathring{S}^{ij}+\mathring{B}^{ij}|^2+|\mathring{A}_{ij}|^2\Big)+\sum_k|J_k(g,K)-J_k(b,\mathring{K})|\\ &+|\rho(g,K)-\rho(b,\mathring{K})|\Bigg)\,r\circ\Phi\,d\mu_g<\infty\ ,\end{aligned} \tag{4.4.7}$$

$$\exists\, C > 0 \quad \textit{such that} \quad C^{-1} b(X,X) \le g(X,X) \le C b(X,X)\ . \tag{4.4.8}$$

Of course, for vacuum metrics g and b (with or without cosmological constant) and for background KIDs (V,Y) (which will be mostly of interest to us) all the quantities appearing in the second and third lines of (4.4.7) vanish.

For hyperboloids in Minkowski space-time, or for static hypersurfaces in anti de Sitter space-time, the V's and Y's associated to the translational Killing vectors satisfy

$$V = O(r)\ , \quad \sqrt{b^{\#}(dV,dV)} = O(r)\ , \qquad |Y|_b = O(r)\ , \tag{4.4.9}$$

where $b^{\#}$ is the metric on T^*M associated to b, and this behavior will be assumed in what follows.

Let $\mathcal{N}_{b_0,\mathring{K}_0}$ denote[3] the *space of background KIDs*:

$$\mathcal{N}_{b_0,\mathring{K}_0} := \{(V_0,Y_0) \mid A_{ij}(b_0,\mathring{K}_0) = 0 = S^{kl}(b_0,\mathring{K}_0)\}\ , \tag{4.4.10}$$

compare Equations (4.3.6) and (4.3.10), where it is understood that V_0 and Y_0 have to be used instead of V and Y there. The geometric character of (4.3.6) and (4.3.10) shows that if (V_0,Y_0) is a background KID for $(b_0,\mathring{K}_0)$, then

$$(V := V_0 \circ \Phi^{-1}, Y := \Phi_* Y_0)$$

will be a background KID for $(b = (\Phi^{-1})^* b_0, \mathring{K} = (\Phi^{-1})^*\mathring{K}_0)$. The introduction of the (V_0,Y_0)'s provides a natural identification of KIDs for different backgrounds $((\Phi_1^{-1})^* b_0, (\Phi_1^{-1})^*\mathring{K}_0)$ and $((\Phi_2^{-1})^* b_0, (\Phi_2^{-1})^*\mathring{K}_0)$. We have

PROPOSITION 4.4.2 *Let the reference metric b on $M_{\rm ext}$ be of the form (4.4.4), suppose that V and Y satisfy (4.4.9), and assume that Φ is such that Equations (4.4.7)-(4.4.8) hold. Then for all $(V_0,Y_0) \in \mathcal{N}_{b_0,\mathring{K}_0}$ the limits*

$$H_\Phi(V_0,Y_0) := \lim_{R\to\infty} \int_{r\circ\Phi^{-1}=R} \big(\mathbb{U}^i(V_0\circ\Phi^{-1}) + \mathbb{V}^i(\Phi_* Y_0)\big)\, dS_i \tag{4.4.11}$$

exist, and are finite.

The integrals (4.4.11) will be referred to as *the Riemannian charge integrals*, or simply *charge integrals*.

PROOF: We work in coordinates on $M_{\rm ext}$ such that Φ is the identity. For any R_1, R_2 we have

$$\int_{r=R_2} (\mathbb{U}^i + \mathbb{V}^i) dS_i = \int_{r=R_1} (\mathbb{U}^i + \mathbb{V}^i) dS_i + \int_{[R_1,R_2]\times N} \partial_i(\mathbb{U}^i + \mathbb{V}^i)\, d^n x\ , \tag{4.4.12}$$

and the result follows from (4.3.11)-(4.3.13), together with (4.4.7)-(4.4.8) and the Cauchy-Schwarz inequality, by passing to the limit $R_2 \to \infty$. □

[3] We denote by $\mathring{K}$ the background extrinsic curvature on the physical initial data manifold M, and by $\mathring{K}_0$ its equivalent in the model manifold $[R,\infty)\times N$, $\mathring{K}_0 := \Phi^*\mathring{K}$.

In order to continue we need some more restrictions on the extrinsic curvature tensor $\mathring{K}$. In the physical applications we have in mind in this section the tensor field $\mathring{K}$ will be pure trace, which is certainly compatible with the following hypothesis:

$$|\mathring{K}^i{}_j - \frac{\mathrm{tr}_{b_0}\mathring{K}}{n}\delta^i_j|_{b_0} = o(r^{-n/2})\ . \tag{4.4.13}$$

Under (4.4.13) and (4.4.15) one easily finds from (4.3.12) that

$$\lim_{R\to\infty}\int_{r\circ\Phi^{-1}=R}\mathbb{V}^i(Y)dS_i = 2\lim_{R\to\infty}\int_{r\circ\Phi^{-1}=R}\sqrt{\det b}\left[(P^l{}_k-\mathring{P}^l{}_k)Y^k\right]dS_i\ , \tag{4.4.14}$$

which gives a slightly simpler expression for the contribution of P to H_Φ.

Under the conditions of Proposition 4.4.2, the integrals (4.4.11) define a linear map from $\mathcal{N}_{b_0,\mathring{K}_0}$ to $\mathbb{R}$. Now, each map Φ used in (4.4.4) defines in general a different background metric b on $M_{\rm ext}$, so that the maps H_Φ are potentially dependent[4] upon Φ. (It should be clear that, given a fixed $\breve{h}$, (4.4.11) does not depend upon the choice of the frame ϵ_i in (4.4.5).) It turns out that this dependence can be controlled:

Theorem 4.4.3 *Under (4.4.13), consider two maps Φ_a, $a=1,2$, satisfying (4.4.7) together with*

$$\sum_{i,j}\Big(|g_{ij}-\delta_{ij}|+|P^i{}_j-\mathring{P}^i{}_j|\Big)+\sum_{i,j,k}|f_k(g_{ij})| =$$
$$= \begin{cases} o(r^{-n/2})\ , & \text{if } n¿2,\\ O(r^{-1-\epsilon})\ , & \text{if } n{=}2, \text{ for some } \epsilon>0. \end{cases} \tag{4.4.15}$$

Then there exists an isometry A of b_0, defined perhaps only for r large enough, such that

$$H_{\Phi_2}(V_0,Y_0) = H_{\Phi_1}\left(V_0\circ A^{-1}, A_*Y_0\right)\ . \tag{4.4.16}$$

Remark 4.4.4 The examples in [70] show that the decay rate (4.4.15) is sharp when $\mathring{P}_{ij}=0$, or when $Y^i=0$, compare [63].

Proof: When $\mathring{K}=0$ the result is proved, using a space-time formalism, at the beginning of Section 4 in [70]. When $Y=0$ this is Theorem 2.3 of [63]. It turns out that under (4.4.13) the calculation reduces to the one in that last theorem, and that under the current conditions the integrals of $\mathbb{U}$ and $\mathbb{V}$ are separately covariant, which can be seen as follows: On $M_{\rm ext}$ we have three pairs of fields:

$$(g,K)\ ,\quad \Big((\Phi_1^{-1})^*b_0,(\Phi_1^{-1})^*\mathring{K}_0\Big)\ ,\quad\text{and}\quad\Big((\Phi_2^{-1})^*b_0,(\Phi_2^{-1})^*\mathring{K}_0\Big)\ .$$

Pulling back everything by Φ_2 to $[R,\infty)\times N$ we obtain there

$$\Big((\Phi_2)^*g,(\Phi_2)^*K\Big)\ ,\quad\Big((\Phi_1^{-1}\circ\Phi_2)^*b_0,(\Phi_1^{-1}\circ\Phi_2)^*\mathring{K}_0\Big)\ ,\quad\text{and}\quad(b_0,\mathring{K}_0)\ .$$

[4]Note that the space of KIDs is fixed, as $\mathcal{N}_{b_0,\mathring{K}_0}$ is tied to $(b_0,\mathring{K}_0)$ which are fixed once and for all.

Now, $(\Phi_2)^*g$ is simply "the metric g as expressed in the coordinate system Φ_2", similarly for $(\Phi_2)^*K$, and following the usual physicist's convention we will instead write

$$(g,K)\ ,\quad (b_1,\mathring{K}_1):=\Big((\Phi_1^{-1}\circ\Phi_2)^*b_0,(\Phi_1^{-1}\circ\Phi_2)^*\mathring{K}_0\Big)\ ,\ \text{ and }\ (b_2,\mathring{K}_2)=(b_0,\mathring{K}_0)\ ,$$

which should be understood in the sense just explained.

As discussed in more detail in [63, Theorem 2.3], there exists an isometry A of the background metric b_0, defined perhaps only for r large enough, such that $\Phi_1^{-1}\circ\Phi_2$ is a composition of A with a map which approaches the identity as one approaches the conformal boundary, see (4.4.21)-(4.4.22) below. It can be checked that the calculation of the proof of [63, Theorem 2.3] remains valid, and yields

$$H_{\Phi_2}(V_0,0)=H_{\Phi_1}(V_0\circ A^{-1},0)\ . \tag{4.4.17}$$

(In [63, Theorem 2.3] Equation (4.3.10) with $Y=0$ has been used. However, under the hypothesis (4.4.13) the supplementary terms involving Y in (4.4.13) cancel out in that calculation.) It follows directly from the definition of H_Φ that

$$H_{\Phi_1\circ A}(0,Y_0)=H_{\Phi_1}\left(0,A_*Y_0\right)\ . \tag{4.4.18}$$

Since

$$H_\Phi(V_0,Y_0)=H_\Phi(V_0,0)+H_\Phi(0,Y_0)\ ,$$

we need to show that

$$H_{\Phi_2}(0,Y_0)=H_{\Phi_1}\left(0,A_*Y_0\right)\ . \tag{4.4.19}$$

In order to establish (4.4.19) it remains to show that for all Y_0 we have

$$H_{\Phi_1\circ A}(0,Y_0)=H_{\Phi_2}(0,Y_0)\ . \tag{4.4.20}$$

Now, Corollary 3.5 of [70] shows that the pull-back of the metrics by $\Phi_1\circ A$ has the same decay properties as that by Φ_1, so that — replacing Φ_1 by $\Phi_1\circ A$ — to prove (4.4.20) it remains to consider two maps $\Phi_1^{-1}=(r_1,v_1^A)$ and $\Phi_2^{-1}=(r_2,v_2^A)$ (where v^A denote abstract local coordinates on N) satisfying

$$r_2 \quad = r_1+o(r_1^{1-\frac{n}{2}}), \tag{4.4.21}$$
$$v_2^A \quad = v_1^A+o(r_1^{-(1+\frac{n}{2})}), \tag{4.4.22}$$

together with similar derivative bounds. In that case one has, in tetrad components, by elementary calculations,

$$P^i{}_j-\mathring{P}_1{}^i{}_j=P^i{}_j-\mathring{P}_2{}^i{}_j+o(r^{-n})\ , \tag{4.4.23}$$

leading immediately to (4.4.20). We point out that it is essential that $P^i{}_j$ appears in (4.4.23) with one index up and one index down. For example, the difference

$$P_{ij}-\mathring{P}_{ij}=o(r^{-n/2})\ ,$$

transforms as

$$P_{ij} - \mathring{P}_{1ij} = P_{ij} - \mathring{P}_{2ij} - \mathrm{tr}_{b_1}\mathring{P}_1\mathcal{L}_\zeta b_1 + o(r^{-n}) ,$$

where

$$\zeta = (r_2 - r_1)\frac{\partial}{\partial r_1} + \sum_A (v_2^A - v_1^A)\frac{\partial}{\partial v_1^A} .$$

□

4.5 Problems with the extrinsic curvature of the conformal boundary

Consider a vacuum space-time $(\mathscr{M}, \mathfrak{g})$, with cosmological constant $\Lambda = 0$, which possesses a smooth conformal completion $(\overline{\mathscr{M}}, {}^4\overline{g})$ with conformal boundary $\mathscr{I}^+$. Consider a hypersurface $\mathscr{S}$ such that its completion $\widetilde{\mathscr{S}}$ in $\overline{\mathscr{M}}$ is a smooth spacelike hypersurface intersecting $\mathscr{I}^+$ transversally, with $\widetilde{\mathscr{S}} \cap \mathscr{I}^+$ being smooth two-dimensional sphere; no other completeness conditions upon $\mathscr{M}$, $\overline{\mathscr{M}}$, or upon $\mathscr{I}^+$ are imposed. In a neighborhood of $\widetilde{\mathscr{S}} \cap \mathscr{I}^+$ one can introduce Bondi coordinates [174], in terms of which $\mathfrak{g}$ takes the form

$$\mathfrak{g} = -xVe^{2\beta}\, du^2 + 2e^{2\beta}x^{-2}\, du\, dx + x^{-2}h_{AB}\left(dx^A - U^A\, du\right)\left(dx^B - U^B\, du\right) . \tag{4.5.1}$$

(The usual radial Bondi coordinate r equals $1/x$, compare Section 4.2.) One has

$$h_{AB} = \breve{h}_{AB} + x\chi_{AB} + O(x^2) , \tag{4.5.2}$$

where $\breve{h}$ is the round unit metric on S^2, and the whole information about gravitational radiation is encoded in the tensor field χ_{AB}. It has been shown in [64, Appendix C.3] that the trace-free part of the extrinsic curvature of $\widetilde{\mathscr{S}} \cap \mathscr{I}$ within $\mathscr{S}$ is proportional to χ. In coordinate systems on $\mathscr{S}$ of the kind used in (4.4.2) this leads to a $1/r$ decay of the tensor field e of (4.3.1), so that the decay condition (4.4.7) is not satisfied. In fact, (4.3.1) "doubly fails" as the $K - \mathring{K}$ contribution also falls off too slowly for convergence of the integral. Thus the decay conditions of Definition 4.4.1 are not suitable for the problem at hand.

Similarly, let $\mathscr{S}$ be a space-like hypersurface in a vacuum space-time $(\mathscr{M}, \mathfrak{g})$ with strictly negative cosmological constant Λ, with a smooth conformal completion $(\widetilde{\mathcal{M}}, {}^4\overline{g})$ and conformal boundary $\mathscr{I}$, as considered *e.g.* in [73, Section 5]. Then a generic smooth deformation of $\mathscr{S}$ at fixed conformal boundary $\widetilde{\mathscr{S}} \cap \mathscr{I}$ will lead to induced initial data which will not satisfy (4.4.7).

As already mentioned, in some of the calculations we will not consider the most general hypersurfaces compatible with the set-ups just described, because the calculations required seem to be too formidable to be performed by hand. We will instead impose the following restriction:

$$d(\mathrm{tr}_g K) \text{ vanishes on the conformal boundary.} \tag{4.5.3}$$

In other words, $\mathrm{tr}_g K$ is constant on $\widetilde{\mathscr{S}} \cap \mathscr{I}$, with the transverse derivatives of $\mathrm{tr}_g K$ vanishing there as well.

Equation (4.5.3) is certainly a restrictive assumption. We note, however, the following:

1. It holds for all initial data constructed by the conformal method, both if $\Lambda = 0$ [3] and if $\Lambda < 0$ [122], for then $d(\mathrm{tr}_g K)$ is zero throughout $\mathscr{S}$.
2. One immediately sees from the equations in [64, Appendix C.3] that in the case $\Lambda = 0$ Equation (4.5.3) can be achieved by deforming $\mathscr{S}$ in $\mathscr{M}$, while keeping $\widetilde{\mathscr{S}} \cap \mathscr{I}^+$ fixed, whenever an associated space-time exists. It follows that for the proof of positivity of m in space-times with a conformal completion it suffices to consider hypersurfaces satisfying (4.5.3).

4.6 Convergence, uniqueness, and positivity of the Trautman-Bondi mass

From now on we assume that the space dimension is three, and that

$$\Lambda = 0 .$$

An extension to higher dimension would require studying Bondi expansions in $n+1 > 4$, which appears to be quite a tedious undertaking. On the other hand the adaptation of our results here to the case $\Lambda < 0$ should be straightforward, but we have not attempted such a calculation.

The metric g of a Riemannian manifold (M, g) will be said to be C^k *compactifiable* if there exists a compact Riemannian manifold with boundary $(\overline{M} \approx M \cup \partial_\infty M \cup \partial M, \widetilde{g})$, where $\partial\overline{M} = \partial M \cup \partial_\infty M$ is the metric boundary of $(\overline{M}, \widetilde{g})$, with ∂M — the metric boundary of (M, g), together with a diffeomorphism

$$\psi : \mathrm{int}\overline{M} \to M$$

such that

$$\psi^* g = x^{-2}\widetilde{g} \,, \tag{4.6.1}$$

where x is a defining function for $\partial_\infty M$ (*i.e.*, $x \geq 0$, $\{x = 0\} = \partial_\infty M$, and dx is nowhere vanishing on $\partial_\infty M$), with $\widetilde{g}$ — a metric which is C^k up–to–boundary on $\overline{M}$. The triple $(\overline{M}, \widetilde{g}, x)$ will then be called a C^k *conformal completion* of (M, g). Clearly the definition allows M to have a usual compact boundary. (M, g) will be said to have a *conformally compactifiable end* M_{ext} if M contains an open submanifold M_{ext} (of the same dimension that M) such that $(M_{\mathrm{ext}}, g|_{M_{\mathrm{ext}}})$ is conformally compactifiable, with a connected conformal boundary $\partial_\infty M_{\mathrm{ext}}$.

In the remainder of this chapter we shall assume for simplicity that the conformally rescaled metric $\widetilde{g}$ is *polyhomogeneous* and C^1 near the conformal boundary; this means that $\widetilde{g}$ is C^1 up-to-boundary and has an asymptotic expansion with smooth expansion coefficients to any desired order in terms of powers of x and of $\ln x$. (In particular, smoothly compactifiable metrics belong to the polyhomogeneous class; the reader unfamiliar with polyhomogeneous expansions might wish to assume smoothness throughout.) It should be clear that the conditions here can be adapted to metrics which are polyhomogeneous plus a weighted Hölder or Sobolev lower order term decaying sufficiently fast. In fact, a very conservative estimate, obtained by inspection of the calculations below, shows that relative $O_{\ln^* x}(x^4)$ error terms introduced in the metric because of matter fields or because of sub-leading non-polyhomogeneous behavior do not affect the validity of the calculations below, provided the derivatives of those error term behave under differentiation in the obvious way (an x derivative lowers the powers of x by one, other derivatives preserve the powers). (We use the symbol $f = O_{\ln^* x}(x^p)$ to denote the fact that there exists $N \in \mathbb{N}$ and a constant C such that $|f| \le Cx^p(1 + |\ln x|^N)$.)

An initial data set (M, g, K) will be said to be $C^k(\overline{M}) \times C^\ell(\overline{M})$ *conformally compactifiable* if (M, g) is $C^k(\overline{M})$ conformally compactifiable and if K is of the form

$$K^{ij} = x^3 L^{ij} + \frac{\mathrm{tr}_g K}{3} g^{ij} , \tag{4.6.2}$$

with the trace-free tensor L^{ij} in $C^\ell(\overline{M})$, and with $\mathrm{tr}_g K$ in $C^\ell(\overline{M})$, strictly bounded away from zero on $\overline{M}$. We note that (4.4.15) would have required $|L|_{\tilde{g}} = o(x^{1/2})$, while we allow $|L|_{\tilde{g}} = O(1)$. The slower decay rate is necessary in general for compatibility with the constraint equations if the trace-free part of the extrinsic curvature tensor of the conformal boundary does not vanish (equivalently, if the tensor field χ_{AB} in (4.5.2) does not vanish); this follows from the calculations in Section 4.10 below.

4.6.1 The Trautman-Bondi four-momentum of asymptotically hyperboloidal initial data sets – the four-dimensional definition

The definition (4.4.11) of global charges requires a background metric b, a background extrinsic curvature tensor $\mathring{K}$, and a map Φ. For initial data which are vacuum near $\mathscr{I}^+$ all these objects will now be defined using Bondi coordinates, as follows: Let $(\mathscr{S}, g, K)$ be a hyperboloidal initial data set, by [67, 132] the associated vacuum space-time $(\mathscr{M}, \mathfrak{g})$ has a conformal completion $\mathscr{I}^+$, with perhaps a rather low degree of differentiability. One expects that $(\mathscr{M}, \mathfrak{g})$ will indeed be polyhomogeneous, but such a result has not been established so far. However, the analysis of [68] shows that one can formally determine all the expansion coefficients of a polyhomogeneous space-time metric *on* $\mathscr{S}$, as well as all their time-derivatives on $\mathscr{S}$. This is sufficient to carry out all the calculations here *as if the resulting completion were polyhomogeneous.* In all our calculations from now on we shall therefore assume that $(\mathscr{M}, \mathfrak{g})$ has a polyhomogeneous conformal completion, this assumption being understood in the sense just explained.

In $(\mathscr{M}, \mathfrak{g})$ we can always [68] introduce a Bondi coordinate system (u, x, x^A) such that $\mathscr{S}$ is given by an equation

$$u = \alpha(x, x^A) , \quad \text{with} \quad \alpha(0, x^A) = 0 , \ \alpha_{,x}(0, x^A) > 0 , \tag{4.6.3}$$

where α is polyhomogeneous. There is *exactly* a six-parameter family of such coordinate systems, parameterised by the Lorentz group (the supertranslation freedom is gotten rid of by requiring that α vanishes on $\widetilde{\mathscr{S}} \cap \mathscr{I}$). We use the Bondi coordinates to define the background ${}^4b$:

$$ {}^4b := -\,du^2 + 2x^{-2}\,du\,dx + x^{-2}\check{h}_{AB}\,dx^A\,dx^B \ . \tag{4.6.4} $$

This Lorentzian background metric ${}^4b$ is independent of the choice of Bondi coordinates as above. One then defines the Trautman-Bondi four-momentum p_μ of the asymptotically hyperboloidal initial data set we started with as the Trautman-Bondi four-momentum of the cut $u = 0$ of the resulting $\mathscr{I}^+$; the latter is defined as follows: Let X be a translational Killing vector of ${}^4b$, it is shown *e.g.* in [64, Section 6.1, 6.2 and 6.10] that the integrals

$$ H(\mathscr{S}, X, \mathfrak{g}, {}^4b) := \lim_{\epsilon\to 0} \int_{\{x=\epsilon\}\cap\mathscr{S}} \mathbb{W}^{\nu\lambda}(X, \mathfrak{g}, {}^4b) dS_{\nu\lambda} $$

converge. Here $\mathbb{W}^{\nu\lambda}(X, \mathfrak{g}, {}^4b)$ is given by (4.9.2). Choosing an ON basis X_μ for the X's one then sets

$$ p_\mu(\mathscr{S}) := H(\mathscr{S}, X_\mu, \mathfrak{g}, {}^4b) \ . $$

(The resulting numbers coincide with the Trautman-Bondi four-momentum; we emphasise that the whole construction depends upon the use of Bondi coordinates.)

4.6.2 Geometric invariance

The definition just given involves two arbitrary elements: the first is the choice of a conformal completion, the second is that of a Bondi coordinate system. While the latter is easily taken care of, the first requires attention. Suppose, for example, that a prescribed region of a space-time $(\mathscr{M}, g)$ admits two completely unrelated conformal completions, as is the case for the Taub-NUT space-time. In such a case the resulting p_μ's might have nothing to do with each other. Alternatively, suppose that there exist two conformal completions which are homeomorphic but not diffeomorphic. Because the objects occurring in the definition above require derivatives of various tensor fields, one could a priori again obtain different answers. In fact, the construction of the approximate Bondi coordinates above requires expansions to rather high order of the metric at $\mathscr{I}^+$, which is closely related to high differentiability of the metric at $\mathscr{I}^+$, so even if we have two diffeomorphic completions such that the diffeomorphism is not smooth enough, we might still end up with unrelated values of p_μ.

It turns out that none of the above can happen. The key element of the proof is the following result, which is essentially Theorem 6.1 of [63]; the proof there was given for C^∞ completions, but an identical argument applies under the hypotheses here:

THEOREM 4.6.1 *Let (M, g) be a Riemannian manifold endowed with two C^k, $k \geq 1$ and polyhomogeneous conformal compactifications $(\overline{M}_1, \widetilde{g}_1, x_1)$ and $(\overline{M}_2, \widetilde{g}_2, x_2)$ with compactifying maps ψ_1 and ψ_2. Then*

$$ \psi_1^{-1} \circ \psi_2 : \mathrm{int} M_2 \to \mathrm{int} M_1 $$

extends by continuity to a C^k *and polyhomogeneous conformal up-to-boundary diffeomorphism from* $(\overline{M}_2, \widetilde{g}_2)$ *to* $(\overline{M}_1, \widetilde{g}_1)$, *in particular* $\overline{M}_1$ *and* $\overline{M}_2$ *are diffeomorphic as manifolds with boundary.*

We are ready now to prove definitional uniqueness of four-momentum. Some remarks are in order:

1. It should be clear that the proof below generalises to matter fields near $\mathscr{I}^+$ which admit a well posed conformal Cauchy problem *à la* Friedrich, *e.g.* to Einstein-Yang-Mills fields [88].

2. The differentiability conditions below have been chosen to ensure that the vacuum conformal Cauchy problem of Friedrich [89] is well posed; we have taken a very conservative estimate for the differentiability thresholds, and for simplicity we have chosen to present the results in terms of classical rather than Sobolev differentiability. One expects that $C^1(\overline{M}) \times C^0(\overline{M})$ and polyhomogeneous CMC initial data $(\widetilde{g}, L)$ will lead to existence of a polyhomogeneous $\mathscr{I}^+$; such a theorem would immediately imply a corresponding equivalent of Theorem 4.6.2.

3. It is clear that there exists a purely three-dimensional version of the proof below, but we have not attempted to find one; the argument given seems to minimise the amount of calculations needed. Such a three-dimensional proof would certainly provide a result under much weaker asymptotic conditions concerning both the matter fields and the requirements of differentiability at $\mathscr{I}^+$.

THEOREM 4.6.2 *Let* (M, g, K) *be a* $C^7(\overline{M}) \times C^6(\overline{M})$ *and polyhomogeneous conformally compactifiable initial data set which is vacuum near the conformal boundary, and consider two* $C^7(\overline{M})$ *and polyhomogeneous compactifications thereof as in Theorem 4.6.1, with associated four-momenta* p^a_μ, $a = 1, 2$. *Then there exists a Lorentz matrix* $\Lambda_\mu{}^\nu$ *such that*

$$p^1_\mu = \Lambda_\mu{}^\nu p^2_\nu \ .$$

PROOF: By the results of Friedrich (see [89] and references therein) the maximal globally hyperbolic development $(\mathscr{M}, \mathfrak{g})$ of (M, g, K) admits C^4 conformal completions $(\overline{\mathscr{M}}_a, \widetilde{\mathfrak{g}}_a)$, $a = 1, 2$ with conformal factors Ω_a and diffeomorphisms $\Psi_a : \mathrm{int}\, \overline{\mathscr{M}}_a \to \mathscr{M}$ such that

$$\Psi_a^*(\mathfrak{g}) = \Omega_a^{-2}\, \widetilde{\mathfrak{g}}_a \ , \quad \Psi_a|_M = \psi_a \ .$$

The uniqueness-up-to-conformal-diffeomorphism property of the conformal equations of Friedrich together with Theorem 4.6.1 show that $\Psi_2^{-1} \circ \Psi_1$ extends by continuity to a C^4-up-to-boundary map from a neighborhood of $\Psi_a^{-1}(\overline{M}) \subset \overline{\mathscr{M}}_1$ to $\overline{\mathscr{M}}_2$. Let b_a be the Minkowski background metrics constructed near the respective conformal boundaries $\mathscr{I}^+_a$ as in Section 4.6.1, we have

$$\left(\Psi_1 \circ \Psi_2^{-1}\right)^* b_2 = b_1 \ ,$$

so that $\left(\Psi_1 \circ \Psi_2^{-1}\right)^*$ defines a Lorentz transformation between the translational Killing vector fields of b_1 and b_2, and the result follows *e.g.* from [64, Section 6.9]. $\square$

4.6.3 The Trautman-Bondi four-momentum of asymptotically CMC hyperboloidal initial data sets – a three-dimensional definition

Consider a conformally compactifiable initial data set (M, g, K) as defined in Section 4.6, see (4.6.2). We shall say that (M, g, K) is *asymptotically CMC* if $\mathrm{tr}_g K$ is in $C^1(\overline{M})$ and if

$$\text{the differential of } \mathrm{tr}_g K \text{ vanishes on } \partial_\infty M_{\mathrm{ext}} . \tag{4.6.5}$$

The vacuum scalar constraint equation ($\rho = 0$ in (4.3.7)) shows that, for $C^1(\overline{M}) \times C^1(\overline{M})$ (or for $C^1(\overline{M}) \times C^0(\overline{M})$ and polyhomogeneous) conformally compactifiable initial data sets, Equation (4.6.5) is equivalent to

$$\text{the differential of the Ricci scalar } R(g) \text{ vanishes on } \partial_\infty M_{\mathrm{ext}} . \tag{4.6.6}$$

We wish, now, to show that for asymptotically CMC initial data sets one can define a mass in terms of limits (4.4.11). The construction is closely related to that presented in Section 4.6.1, except that everything will be directly read off from the initial data: *If* a space-time as in Section 4.6.1 exists, then we define the Riemannian background metric b on $\mathscr{S}$ as the metric induced by the metric ${}^4b$ of Section 4.6.1 on the hypersurface $u = \alpha$, and $\mathring{K}$ is defined as the extrinsic curvature tensor, with respect to ${}^4b$, of that hypersurface. The map Φ needed in (4.4.11) is defined to be the identity in the Bondi coordinate system above, and the metric b_0 is defined to coincide with b in the coordinate system above. The four translational Killing vectors X_μ of ${}^4b$ induce on $\mathscr{S}$ four KIDs $(V, Y)_\mu$, and one can plug those into (4.4.11) to obtain a definition of four-momentum. *However*, the question of existence and/or of construction of the space-time there is completely circumvented by the fact that *the asymptotic development of the function α, and that of b, can be read off directly from g and K*, using the equations of [64, Appendix C.3]. The method is then to read-off the restrictions $x|_{\mathscr{S}}$ and $x^A|_{\mathscr{S}}$ of the space-time Bondi functions x and x^A to $\mathscr{S}$ from the initial data, and henceforth the asymptotic expansions of all relevant Bondi quantities in the metric, up to error terms $O_{\ln^* x}(x^4)$ (order $O(x^4)$ in the smoothly compactifiable case); equivalently, one needs an approximation of the Bondi coordinate x on $\mathscr{S}$ up to error terms $O_{\ln^* x}(x^5)$. The relevant coefficients can thus be recursively read from the initial data by solving a finite number of recursive equations. The resulting approximate Bondi function x induces a foliation of a neighborhood of the conformal boundary, which will be called the *approximate Bondi foliation*. The asymptotic expansion of α provides an identification of $\mathscr{S}$ with a hypersurface $u = \alpha$ in Minkowski space-time with the flat metric (4.6.4). The Riemannian background metric b is defined to be the metric induced by ${}^4b$ on this surface, and Φ is defined to be the identity in the approximate Bondi coordinates. As already indicated, the KIDs are obtained on $\mathscr{S}$ from the translational Killing vector fields of ${}^4b$. The charge integrals (4.4.11) have to be calculated *on the approximate Bondi spheres $x = \epsilon$* before passing to the limit $\epsilon \to 0$.

The simplest question one can ask is whether the linearisation of the integrals (4.4.11) reproduces the linearisation of the Freud integrals under the procedure above. We will show in Section 4.9 that this is indeed the case. We also show in that section that the linearisation of the Freud integrals *does not* reproduce the Trautman-Bondi four-momentum in general, see Equation (4.9.21). On the other hand that linearisation provides the right expression for p_μ when the extrinsic curvature χ of the conformal boundary vanishes. Note that under the boundary conditions of Section 4.4 the extrinsic curvature χ vanishes, and the values of the charge integrals coincide with the values of their linearised counterparts for translations, so that the calculations in Section 4.9 prove the equality of the $3+1$ charge integrals and the Freud ones for translational background KIDs under the conditions of Section 4.4.

The main result of this section is the following:

THEOREM 4.6.3 *Consider an asymptotically CMC initial data set which is C^1 and polyhomogeneously (or smoothly) conformally compactifiable. Let Φ be defined as above and let $(V,Y)_\mu$ be the background KIDs associated to space-time translations ∂_μ. Then the limits (4.4.11) $H_\Phi\left((V,Y)_\mu\right)$ taken along* approximate Bondi spheres $\{x=\epsilon\}\subset\mathscr{S}$ *exist and are finite. Further, the numbers*

$$p_\mu := H_\Phi\left((V,Y)_\mu\right) \tag{4.6.7}$$

coincide with the Trautman-Bondi four-momentum *of the associated cut in the Lorentzian space-time, whenever such a space-time exists.*

PROOF: We will show that $\mathbb{U}^x+\mathbb{V}^x$ coincides with (4.7.2) below up to a complete divergence and up to lower order terms not contributing in the limit, the result follows then from (4.7.3)-(4.7.4). It is convenient to rewrite the last two terms in (4.3.12) as

$$-\frac{1}{2}Y^x\mathring{P}^m{}_n e^m{}_n + \frac{1}{2}Y^k\mathring{P}^x{}_k b^{mn}e_{mn}\;, \tag{4.6.8}$$

so that we can use (4.12.3)-(4.12.4) with $M=\chi_{AB}=\beta=N^A=0$ there. We further need the following expansions (all indices are coordinate ones)

$$\begin{aligned}
e^k{}_l &:= b^{km}(g_{ml}-b_{ml})\;,\\
e^x{}_x &= 2\beta + x^3\alpha_{,x}M + O_{\ln^* x}(x^4)\;,\\
e^x{}_A &= \frac{1}{4}x^2\chi_{AC}{}^{||C} + \frac{\alpha_{,A}}{2\alpha_{,x}} - x^3N_A - \frac{1}{32}x^3(\chi^{CD}\chi_{CD})_{||A} + O_{\ln^* x}(x^4)\;,\\
e^A{}_x &= \frac{1}{2}x^2\alpha_{,x}\chi^{AC}{}_{||C} + \alpha^{,A} - 2x^3\alpha_{,x}\left(N^A + \frac{1}{32}(\chi^{CD}\chi_{CD})^{||A}\right) + O_{\ln^* x}(x^4)\;,\\
e^A{}_B &= x\chi^A{}_B + \frac{1}{4}x^2\chi^{CD}\chi_{CD}\delta^A{}_B + x^3\xi^A{}_B + O_{\ln^* x}(x^4)\;,
\end{aligned}$$

with $||$ denoting a covariant derivative with respect to $\breve{h}$. One then finds

$$\begin{aligned}
-\frac{1}{2}Y^x\mathring{P}^{mn}e_{mn} + \frac{1}{2}Y^k\mathring{P}^x{}_k b^{mn}e_{mn} &= Y^x\cdot O_{\ln^* x}(x^4) + Y^B\cdot O_{\ln^* x}(x^5)\;,\\
-\frac{1}{2}Y^A\mathring{P}^{mn}e_{mn} + \frac{1}{2}Y^k\mathring{P}^A{}_k b^{mn}e_{mn} &= Y^x\cdot O_{\ln^* x}(x^5) + Y^B\cdot O_{\ln^* x}(x^4)\;.
\end{aligned}$$

This shows that for Y^i which are $O(1)$ in the (x, x^A) coordinates, as is the case here (see Section 4.13), the terms above multiplied by $\sqrt{\det g} = O(x^{-3})$ will give zero contribution in the limit, so that in (4.3.12) only the first two terms will survive. Those are clearly equal to the first two terms in (4.7.2) when a minus sign coming from the change of the orientation of the boundary is taken into account.

On the other hand $\mathbb{U}^x$ does not coincide with the remaining terms in (4.7.2), instead with some work one finds

$$\begin{aligned}
2(\lambda k - \mathring{\lambda}\mathring{k})V - \mathbb{U}^x &= V\Big[\sqrt{\det g}\cdot {}^2g^{AB}g^{xm}\mathring{D}_A g_{mB} \\
&\quad +2\Big(\sqrt{\frac{b^{xx}}{g^{xx}}}-1\Big)\mathring{\lambda}\mathring{k} + 2\sqrt{\det g}({}^2g^{AB} - {}^2b^{AB})\mathring{\Gamma}^x_{AB}\Big] \\
&\quad -2\sqrt{\det g}D^{[x}Vg^{j]k}e_{jk} \\
&= V\sqrt{\breve{h}}\left(\frac{x}{4}\chi^{AB}{}_{||AB} + O(x^2)\right) , \qquad (4.6.9)
\end{aligned}$$

where ${}^2g^{AB}g_{BC} = \delta^A{}_C$ and similarly for ${}^2b^{AB}$. However, integration over S^2 yields equality in the limit; here one has to use the fact that V is a linear combination of $\ell = 0$ and 1 spherical harmonics in the relevant order in x (see Section 4.13), so that the trace-free part of $V_{||AB}$ is $O(x)$. □

4.6.4 Positivity

We pass now to the proof of positivity of the Trautman-Bondi mass:

THEOREM 4.6.4 *Suppose that (M, g) is geodesically complete without boundary. Assume that (M, g, K) contains an end which is $C^4 \times C^3$, or C^1 and polyhomogeneously, compactifiable and asymptotically CMC. If*

$$\sqrt{g_{ij}J^iJ^j} \le \rho \in L^1(M) , \qquad (4.6.10)$$

then p_μ is timelike future directed or vanishes, *in the following sense:*

$$p_0 \ge \sqrt{\sum p_i^2} . \qquad (4.6.11)$$

Further, equality holds if and only if there exists a ∇-covariantly constant spinor field on M.

REMARK 4.6.5 We emphasise that no assumptions about the geometry or the behavior of the matter fields except geodesic completeness of (M, g) are made on $M \setminus M_{\text{ext}}$.

REMARK 4.6.6 In vacuum one expects that equality in (4.6.11) is possible only if the future maximal globally hyperbolic development of (M, g, K) is isometrically diffeomorphic to a subset of the Minkowski space-time; compare [21] for the corresponding statement for initial data which are asymptotically flat in space-like directions. When K is pure trace one can use a result of Baum [17] to conclude that (M, g) is the three-dimensional hyperbolic space, which implies the rigidity result. A corresponding result with a general K is still lacking.

Before passing to the proof of Theorem 4.6.4, we note the following variation thereof, where no restrictions on $\mathrm{tr}_g K$ are made:

THEOREM 4.6.7 *Suppose that (M, g) is geodesically complete without boundary. If (M, g, K) contains an end which is $C^4 \times C^3$, or C^1 and polyhomogeneously, compactifiable and which is vacuum near the conformal boundary, then the conclusions of Theorem 4.6.4 hold.*

PROOF: It follows from [64, Eq. (C.84)] and from the results of Friedrich that one can deform M near $\mathscr{I}^+$ in the maximal globally hyperbolic vacuum development of the initial data there so that the hypotheses of Theorem 4.6.4 hold. The Trautman-Bondi four-momentum of the original hypersurface coincides with that of the deformed one by [64, Section 6.1]. □

PROOF OF THEOREM 4.6.4: The proof follows the usual argument as proposed by Witten. While the method of proof is well known, there are tedious technicalities which need to be taken care of to make sure that the argument applies.

Let D be the covariant-derivative operator of the metric g, let ∇ be the covariant-derivative operator of the space-time metric $\mathfrak{g}$, and let $\nabla\!\!\!\!/$ be the Dirac operator associated with ∇ along M,

$$\nabla\!\!\!\!/\psi = \gamma^i \nabla_i \psi$$

(summation over space indices only). Recall the Sen-Witten identity of Section 3.3.1, Equation (3.3.14), p. 124, as rewritten in the current notation:

$$\int_{M\setminus\{r\geq R\}} \|\nabla\psi\|_g^2 + \langle\psi, (\rho + J^i\gamma_i\gamma_0)\psi\rangle ngle - \|\nabla\!\!\!\!/\psi\|_g^2 = \int_{S_R} B^i(\psi) dS_i \ , \quad (4.6.12)$$

where the boundary integrand is

$$B^i(\psi) = \langle\nabla^i\psi + \gamma^i\nabla\!\!\!\!/\psi, \psi\rangle ngle_g \ . \quad (4.6.13)$$

We have the following:

LEMMA 4.6.8 *Let ψ be a Killing spinor for the space-time background metric b, and let $A^\mu\partial_\mu$ be the associated translational Killing vector field $A^\mu = \psi^\dagger\gamma^\mu\psi$. We have*

$$\lim_{R\to\infty} \int_{S(R)} < \psi, \nabla^i\psi + \gamma^i\gamma^j\nabla_j\psi > dS_i = \frac{1}{4\pi} p_\mu A^\mu \ . \quad (4.6.14)$$

PROOF: For $A^\mu\partial_\mu = \partial_0$ the calculations are carried out in detail in Section 4.10. For general A^μ the result follows then by the well known Lorentz-covariance of p_μ under changes of Bondi frames (see *e.g.* [64, Section 6.8] for a proof under the current asymptotic conditions). □

LEMMA 4.6.9 *Let ψ be a spinor field on M which vanishes outside of M_{ext}, and coincides with a Killing spinor for the background metric b for R large enough. Then*

$$\nabla\psi \in L^2(M) \ .$$

PROOF: In Section 4.11 we show that

$$\nabla\!\!\!\!/\,\psi \in L^2 \,. \tag{4.6.15}$$

We then rewrite (4.6.12) as

$$\int_{M\setminus\{r\geq R\}} \|\nabla\psi\|_g^2 + \langle\psi,(\rho+J^i\gamma_i\gamma_0)\psi\rangle = \int_{M\setminus\{r\geq R\}} \|\nabla\!\!\!\!/\,\psi\|_g^2 + \int_{S_R} B^i(\psi)dS_i \,. \tag{4.6.16}$$

By the dominant energy condition (4.6.10) the function $\langle\psi,(\rho+J^i\gamma_i)\psi\rangle$ is non-negative. Passing with R to infinity the right-hand-side is bounded by (4.6.15) and by Lemma 4.6.8. The result follows from the monotone convergence theorem. □

Lemmata 4.6.8 and 4.6.9 are the two elements needed to establish positivity of the right-hand-side of (4.6.14) whatever A^μ, see *e.g.* [4, 63] for a detailed exposition in a related setting, or Section 3.2 in the asymptotically flat context. For instance, Lemma 4.6.9 justifies the right-hand-side of the implication [63, Equation (4.17)], while Lemma 4.6.8 replaces all the calculations that follow [63, Equation (4.18)]. The remaining arguments in [63] require only trivial modifications. □

One also has a version of Theorem 4.6.4 with trapped boundary, using solutions of the Dirac equation with the boundary conditions of [103] (compare [114]):

THEOREM 4.6.10 *Let (M,g) be a geodesically complete manifold with compact boundary ∂M, and assume that the remaining hypotheses of Theorem 4.6.4 or of Theorem 4.6.7 hold. If ∂M is either outwards-past trapped, or outwards-future trapped, then p^μ is* timelike future directed*:*

$$p^0 > \sqrt{\sum_i p_i^2} \,.$$

□

4.7 The mass of approximate Bondi foliations

The main tool in our analysis so far was the foliation of the asymptotic region by spheres arising from space-times Bondi coordinates adapted to the initial data surface. Such foliations will be called *Bondi foliations*. The aim of this section is to reformulate our definition of mass as an object directly associated to this foliation. The definition below is somewhat similar to that of Brown, Lau and York [40], but the normalisation (before passing to the limit) used here seems to be different from the one used by those authors.

Let us start by introducing some notation (for ease of reference we collect here all notations, including some which have already been introduced elsewhere): Let $\mathfrak{g}_{\mu\nu}$ be a metric of a spacetime which is asymptotically flat in null directions. Let g_{ij} be the metric induced on a three-dimensional surface $\mathscr{S}$ with extrinsic curvature K_{ij}. We denote by ${}^3\Gamma^i{}_{jk}$ the Christoffel symbols of g_{ij}.

We suppose that we have a function $x > 0$ the level sets $x =$ const. of which provide a foliation of $\mathscr{S}$ by two-dimensional submanifolds, denoted by $\mathscr{S}_x$, each of them homeomorphic to a sphere. The level set "$\{x = 0\}$" (which does not exist in $\mathscr{S}$) should be thought of as corresponding to $\mathscr{I}^+$. The metric g_{ij} induces a metric ${}^2g_{AB}$ on each of these spheres, with area element $\lambda = \sqrt{\det {}^2g_{AB}}$. By k_{AB} we denote the extrinsic curvature of the leaves of the x-foliation, $k_{AB} = {}^3\Gamma^x{}_{AB}/\sqrt{g^{xx}}$, with mean curvature $k = {}^2g^{AB}k_{AB}$. (The reader is warned that in this convention the outwards extrinsic curvature of a sphere in a flat Riemannian metric is negative.) There are also the corresponding objects for the background (Minkowski) metric ${}^4b_{\mu\nu}$, denoted by letters with a circle.

Let X be a field of space-time vectors defined along $\mathscr{S}$. Such a field can be decomposed as $X = Y + Vn$, where Y is a field tangent to $\mathscr{S}$, n — the unit ($n^2 = -1$), future-directed vector normal to $\mathscr{S}$. Motivated by (4.6.9), we define the following functional depending upon various objects defined on the hypersurface $\mathscr{S}$ and on a vector field X:

$$\frac{1}{16\pi}\lim_{x\to 0}\int_{\mathscr{S}_x}\mathcal{F}(X)\,d^2x\ , \tag{4.7.1}$$

where

$$\mathcal{F}(X) = 2\sqrt{\det g}(\mathring{P}^x{}_i - P^x{}_i)Y^i + 2(\mathring{\lambda}\mathring{k} - \lambda k)V. \tag{4.7.2}$$

So far the considerations were rather general, from now on we assume that the initial data set contains an *asymptotically CMC* conformally compactifiable end and that x provides an approximate Bondi foliation, as defined in Section 4.6.3. (To avoid ambiguities, we emphasise that we do impose the restrictive condition (4.6.5), which in terms of the function α of (4.10.2) translates into Equation (4.10.3).) We then have $\lambda = \mathring{\lambda}$ up to terms which do not affect the limit $x \to 0$ (see Section 4.12.1) so that the terms containing λ above can also be written as $\mathring{\lambda}(\mathring{k} - k)$ or $\lambda(\mathring{k} - k)$. We will show that in the limit $x \to 0$ the functional (4.7.1) gives the Trautman-Bondi mass and momentum, as well as angular momentum and centre of mass, for appropriately chosen fields X corresponding to the relevant generators of Poincaré group.

To study the convergence of the functional when $x \to 0$ we need to calculate several objects on $\mathscr{S}$. We write the spacetime metric in Bondi-Sachs coordinates, as in (4.10.8) and use the standard expansions for the coefficients of the metric (see *eg.* [64, Equations (5.98)-(5.101)]). The covariant derivative operator associated with the metric $\breve{h}_{AB}$ is denoted by $_{||A}$. Some intermediate results needed in those calculations are presented in Section 4.12.1, full details of the calculations can be found[5] in [133]. Using formulae (4.12.2)-(4.12.5) and the decomposition of Minkowski spacetime Killing vectors given in Section 4.13 we get (both here and in Section 4.13 all indices are coordinate ones):

$$\mathcal{F}(X_{time}) = \sqrt{\breve{h}}[4M - \chi^{CD}{}_{||CD} + O(x)]\ ,$$

$$\mathcal{F}(X_{trans}) = -v\sqrt{\breve{h}}[4M - \chi^{CD}{}_{||CD} + O(x)]\ ,$$

[5] The reader is warned that $\mathcal{F}$ there is $-\mathcal{F}$ here.

$$\begin{aligned}
\mathcal{F}(X_{rot}) &= -\sqrt{\breve{h}}\left[\varepsilon^{AB}\left(-\underline{\frac{\chi^C{}_{A||C}}{x}}+\frac{3(\chi^{CD}\chi_{CD})_{||A}}{16}\right.\right.\\
&\qquad\left.\left.+6N_A+\frac{1}{2}\chi_{AC}\chi^{CD}{}_{||D}\right)v_{,B}+O(x)\right]\,,
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}(X_{boost}) &= -\sqrt{\breve{h}}\left[-\underline{\frac{\chi^C{}_{A||C}v^{,A}}{x}}+\frac{1}{8}\chi^{CD}\chi_{CD}v\right.\\
&\quad\left.+\left(\frac{3}{16}(\chi^{CD}\chi_{CD})_{||A}+6N_A+\frac{1}{2}\chi_{AC}\chi^{CD}{}_{||D}\right)v^{,A}+O(x)\right]\,.
\end{aligned}$$

The underlined terms in integrands corresponding to boosts and rotations diverge when x tends to zero, but they yield zero when integrated over a sphere. In the formulae above we set $u-\alpha$ equal to any constant, except for the last one where $u-\alpha=0$. Moreover, v is a function on the sphere which is a linear combination of $\ell=1$ spherical harmonics and ε^{AB} is an antisymmetric tensor (more precise definitions are given in Section 4.13).

In particular we obtain:

$$E_{TB}=\frac{1}{16\pi}\lim_{x\to 0}\int_{\mathscr{S}_x}\mathcal{F}(X_{time})\,d^2x\,, \tag{4.7.3}$$

$$P_{TB}=\frac{1}{16\pi}\lim_{x\to 0}\int_{\partial\mathscr{S}_R}\mathcal{F}(X_{trans})\,d^2x\,, \tag{4.7.4}$$

where the momentum is computed for a space-translation generator corresponding to the function v (see Section 4.13). It follows that the integrals of $\mathcal{F}(X)$ are convergent for all four families of fields X.

4.7.1 Polyhomogeneous metrics

In this section we consider polyhomogeneous metrics. More precisely, we will consider metrics of the form (4.5.1) for which the V, β, U^A and h_{AB} have asymptotic expansions of the form

$$f\simeq\sum_i\sum_{j=1}^{N_i}f_{ij}(x^A)x^i\log^j x\,,$$

where the coefficients f_{ij} are smooth functions. This means that f can be approximated up to terms $O(x^N)$ (for any N) by a finite sum of terms of the form $f_{ij}(x^A)x^i\log^j x$, and that this property is preserved under differentiation in the obvious way.

As mentioned in [64, Section 6.10], allowing a polyhomogeneous expansion of h_{AB} of the form

$$h_{AB}=\breve{h}_{AB}+x\chi_{AB}+x\log xD_{AB}+o(x)$$

is not compatible with the Hamiltonian approach presented there because the integral defining the symplectic structure diverges. (For such metrics it is still possible to define the Trautman-Bondi mass and the momentum [66].) It is also noticed in [64, Section 6.10], that when logarithmic terms in h_{AB} are allowed only at the x^3 level and higher, then all integrals of interest converge. It is therefore natural to study metrics for which h_{AB} has the intermediate behavior

$$h_{AB} = \breve{h}_{AB} + x\chi_{AB} + x^2\zeta_{AB}(\log x) + O(x^2) \ , \tag{4.7.5}$$

where $\zeta_{AB}(\log x)$ is a polynomial of order N in $\log x$ with coefficients being smooth, symmetric tensor fields on the sphere. Under (4.7.5), for the $\mathcal{F}$ functional we find:

$$\mathcal{F}(X_{time}) = \sqrt{\breve{h}}[4M - \chi^{CD}{}_{||CD} + O(x\log^{N+1}x)] \ ,$$

$$\mathcal{F}(X_{trans}) = -v\sqrt{\breve{h}}[4M - \chi^{CD}{}_{||CD} + O(x\log^{N+1}x)] \ .$$

The only difference with the power-series case is that terms with $O(x\log^{N+1}x)$ asymptotic behavior appear in the error term, while previously we had $O(x)$. Thus $\mathcal{F}$ reproduces the Trautman-Bondi four-momentum for such metrics as well.

In [64] no detailed analysis of the corresponding expressions for the remaining generators (*i.e.*, X_{rot} and X_{boost}) was carried out, except for a remark in Section 6.10, that the asymptotic behavior (4.7.5) leads to potentially divergent terms in the Freud potential, which might lead to a logarithmic divergence for boosts and rotations in the relevant Hamiltonians, which then could cease to be well defined. Here we check that the values $\mathcal{F}(X_{rot})$ and $\mathcal{F}(X_{boost})$ remain well defined for certain metrics of the form (4.7.5). Supposing that h_{AB} is of such form one finds (see [64, Appendix C.1.2])

$$\begin{aligned}
\partial_u\zeta_{AB} &= 0 \ , \\
U^A &= -\frac{1}{2}x^2\chi^{AC}{}_{||C} + x^3W^A(\log x) + O(x) \ , \\
\beta &= -\frac{1}{32}x^2\chi^{CD}\chi_{CD} + O(x^3\log^{2N}x) \ , \\
V &= \frac{1}{x} - 2M + O(x\log^{N+1}x) \ ,
\end{aligned}$$

where $W^A(\log x)$ is a polynomial of order $N+1$ in $\log x$ with coefficients being smooth vector fields on the sphere. The W^A's can be calculated by solving Einstein equations (which are presented in convenient form in [68]).

The calculation in case of polyhomogeneous metrics is very similar to the one in case of metrics allowing power-series expansion (see Section 4.12.2), leading to

$$\mathcal{F}(X_{rot}) = -\sqrt{\breve{h}}\left[\varepsilon^{AB}\left(\underline{-\frac{\chi^C{}_{A||C}}{x}} - 3W_A - xW_{A\wr x}\right)v_{,B} + O(1)\right] \ ,$$

$$\mathcal{F}(X_{boost}) = -\sqrt{\breve{h}}\left[\left(\underline{-\frac{\chi^C{}_{A||C}}{x}} - 3W_A - xW_{A\wr x}\right)v^{,A} + O(1)\right] \ ,$$

where $\wr x$ denotes derivation at constant u. As in the previous section, the underlined terms tend to infinity for $x \to 0$, but give zero when integrated over the sphere. However some potentially divergent terms remain, which can be rewritten as

$$\frac{1}{x^2}\varepsilon^{AB}(x^3 W_A)_{\wr x} v_{,B}\ , \qquad \frac{1}{x^2}(x^3 W_A)_{\wr x} v^{,A}\ .$$

To obtain convergence one must therefore have:

$$\lim_{x\to 0}\int_{\mathscr{S}_x} \frac{1}{x^2}\varepsilon^{AB}(x^3 W_A)_{\wr x} v_{,B}\sqrt{\breve{h}}\, d^2x = 0$$

and

$$\lim_{x\to 0}\int_{\mathscr{S}_x} \frac{1}{x^2}(x^3 W_A)_{\wr x} v^{,A}\sqrt{\breve{h}}\, d^2x = 0$$

for any function v which is a linear combination of the $\ell = 1$ spherical harmonics.[6] It turns out that in the simplest case $N = 0$ those integrals are identically zero when vacuum Einstein equations are imposed [68].

4.7.2 The Hawking mass of approximate Bondi spheres

One of the objects of interest associated to the two dimensional surfaces $\mathscr{S}_x$ is their Hawking mass,

$$\begin{aligned} m_H(\mathscr{S}_x) &= \sqrt{\frac{A}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{\mathscr{S}_x}\theta^-\theta^+ d^2\mu\right) \\ &= \sqrt{\frac{A}{16\pi}}\left(1 + \frac{1}{16\pi}\int_{\mathscr{S}_x}\lambda\left(\frac{P^{xx}}{g^{xx}} - k\right)\left(\frac{P^{xx}}{g^{xx}} + k\right)d^2x\right) \end{aligned} \tag{4.7.6}$$

We wish to show that for asymptotically CMC initial data the Hawking mass of approximate Bondi spheres converges to the Trautman Bondi mass. In fact, for $a = (a_\mu)$ let us set

$$v(a)(\theta,\varphi) = a_0 + a_i\frac{x^i}{r}\ ,$$

where x^i has to be expressed in terms of the spherical coordinates in the usual way, and

$$\begin{aligned} p_H(a,\mathscr{S}_x) &= p_H^\mu a_\mu \\ &= -\frac{1}{16\pi}\sqrt{\frac{A}{16\pi}}\int_{\mathscr{S}_x} v(a)\left(\frac{16\pi}{A} - \theta^-\theta^+\right)d^2\mu \\ &= -\frac{1}{16\pi}\sqrt{\frac{A}{16\pi}}\int_{\mathscr{S}_x} v(a)\left(\frac{16\pi}{A} + \left(\frac{P^{xx}}{g^{xx}}\right)^2 - k^2\right)d^2\mu\ , \end{aligned}$$

with $d^2\mu = \lambda d^2x$. Up to the order needed to calculate the limit of the integral, on approximate Bondi spheres satisfying (4.10.3) we have (see Sections 4.12.1 and 4.12.2)

$$\frac{P^{xx}}{g^{xx}} = P^x{}_x + O(x^4)\ , \quad \lambda = \frac{\sqrt{\breve{h}}}{x^2} + O(x^3)\ ,$$

[6] See Section 4.13.

and

$$\begin{aligned}\left(\frac{P^{xx}}{g^{xx}}\right)^2 &= \frac{2}{\alpha_{,x}} - \frac{4\beta}{\alpha_{,x}} - 3x^2(1-2Mx) - x^3\chi^{CD}{}_{||CD} + O_{\ln^* x}(x^4)\ ,\\ k^2 &= \frac{2}{\alpha_{,x}} - \frac{4\beta}{\alpha_{,x}} + x^2(1-2Mx) + x^3\chi^{CD}{}_{||CD} + O_{\ln^* x}(x^4)\ .\end{aligned}$$

Therefore

$$\lambda\left(\left(\frac{P^{xx}}{g^{xx}}\right)^2 - k^2\right) = -4\sqrt{\breve h} + \sqrt{\breve h}[2x(4M - \chi^{CD}{}_{||CD}) + O_{\ln^* x}(x^2)]\ ,$$

which, together with

$$A = \frac{4\pi}{x^2} + O(x^3)\ , \quad \sqrt{\frac{A}{16\pi}} = \frac{1}{2x} + O(x^4)\ ,$$

yields

$$p_H(a, \mathscr{S}_x) = -\frac{1}{16\pi}\int_{\mathscr{S}_x} v(a)[\sqrt{\breve h}(4M - \chi^{CD}{}_{||CD}) + O_{\ln^* x}(x)]\, d^2x\ .$$

Passing to the limit $x \to 0$ the $\chi^{CD}{}_{||CD}$ term integrates out to zero, leading to equality of the Trautman-Bondi four-momentum $p_{\mathrm{TB}}(a, \mathscr{S})$ with the limit of $p_H(a, \mathscr{S}_x)$ defined above.

4.8 Proof of (4.3.11)

From

$$2D_l(Y^k P^l{}_k) = P^{kl}\mathring{A}_{kl} + Y^k J_k(g,K) + P^{kl}\left(2V\mathring{K}_{kl} + \pounds_Y e_{kl}\right) \tag{4.8.1}$$

we get

$$\begin{aligned}VR_g + 2D_l(Y^k P^l{}_k) &= V\rho(g,K) + Y^k J_k(g,K) + P^{kl}\mathring{A}_{kl}\\ &\quad -V\left(\mathring{P}^{mn}\mathring{P}_{mn} - \frac{1}{n-1}(\mathrm{tr}_b\mathring{P})^2\right) + \mathring{P}^{kl}\pounds_Y e_{kl}\\ &\quad +Ve_{kl}\left(2\mathring{P}^{ml}\mathring{P}_{mk} - \frac{2}{n-1}\mathring{P}^{kl}\mathrm{tr}_b\mathring{P}\right) + Q_2\ ,\end{aligned} \tag{4.8.2}$$

where, here and below, we use the symbol Q_i to denote terms which are quadratic or higher order in e and $P - \mathring{P}$. Moreover, we have

$$\mathring{P}^{kl}\pounds_Y e_{kl} = -e_{kl}\pounds_Y \mathring{P}^{kl} + D_m(Y^m\mathring{P}^{kl}e_{kl}) - \mathring{P}^{kl}e_{kl}D_mY^m\ , \tag{4.8.3}$$

$$\begin{aligned}D_mY^m &= \frac{1}{2}g^{kl}(\mathring{A}_{kl} + \pounds_Y e_{kl}) + V(g^{kl} - b^{kl})\mathring{K}_{kl} + V\mathrm{tr}_g\mathring{K}\\ &=: \frac{1}{2}g^{kl}\mathring{A}_{kl} + \frac{V}{n-1}\mathrm{tr}_b\mathring{P} + L_1\ ,\end{aligned} \tag{4.8.4}$$

$$\begin{aligned}2D_l(Y^k\mathring{P}^l{}_k) &= \mathring{P}^{kl}\mathring{A}_{kl} + Y^kJ_k(b,\mathring{K}) - 2V\left(\mathring{P}^{mn}\mathring{P}_{mn} - \frac{1}{n-1}(\mathrm{tr}_b\mathring{P})^2\right)\\ &\quad + D_l\left(Y^k\mathring{P}^l{}_k b^{mn}e_{mn}\right) - b^{mn}e_{mn}\mathring{D}_l(Y^k\mathring{P}^l{}_k) + Q_1\ .\end{aligned} \tag{4.8.5}$$

(The term L_1 in (4.8.4) is a linear remainder term which, however, will give a quadratic contribution in equations such as (4.8.7) below.) Using

$$\begin{aligned} -\mathring{D}_l(Y^k\mathring{P}^l{}_k) &= V\left(\mathring{P}^{mn}\mathring{P}_{mn} - \frac{1}{n-1}(\mathrm{tr}_b\mathring{P})^2\right) - \frac{1}{2}\mathring{P}^{kl}\mathring{A}_{kl} - \frac{1}{2}Y^kJ_k(b,\mathring{K}) \\ &= VR_b - V\rho(b,\mathring{K}) - \frac{1}{2}\mathring{P}^{kl}\mathring{A}_{kl} - \frac{1}{2}Y^kJ_k(b,\mathring{K}) \end{aligned} \quad (4.8.6)$$

and (4.8.2)-(4.8.5) we are led to

$$\begin{aligned} V(R_g - R_b) &= V\left(\rho(g,K) - \rho(b,\mathring{K})\right) + Y^k\left(J_k(g,K) - J_k(b,\mathring{K})\right) - \frac{\partial_l\mathbb{V}^l}{\sqrt{\det g}} \\ &\quad + V\left[2\mathring{P}^{ml}\mathring{P}_m{}^k - \frac{3}{n-1}\mathrm{tr}_b\mathring{P}\mathring{P}^{kl} - \left(\mathring{P}^{mn}\mathring{P}_{mn} - \frac{1}{n-1}(\mathrm{tr}_b\mathring{P})^2\right)b^{kl}\right]e_{kl} \\ &\quad - e_{kl}\pounds_Y\mathring{P}^{kl} + \frac{1}{2}e_{kl}\left[b^{kl}\left(\mathring{P}^{mn}\mathring{A}_{mn} + Y^mJ_m(b,\mathring{K})\right) - b^{mn}\mathring{A}_{mn}\mathring{P}^{kl}\right] \\ &\quad + (P^{kl} - \mathring{P}^{kl})\mathring{A}_{kl} + Q_3\ , \end{aligned} \quad (4.8.7)$$

where

$$\mathbb{V}^l(Y) := 2\sqrt{\det g}\left[(P^l{}_k - \mathring{P}^l{}_k)Y^k - \frac{1}{2}Y^l\mathring{P}^{mn}e_{mn} + \frac{1}{2}Y^k\mathring{P}^l{}_kb^{mn}e_{mn}\right]\ . \quad (4.8.8)$$

Inserting Equation (4.3.2) into (4.8.7) one obtains the following counterpart of Equation (4.3.2)

$$\begin{aligned} \partial_i\left(\mathbb{U}^i(V) + \mathbb{V}^i(Y)\right) &= \sqrt{\det g}\ \Big[V\left(\rho(g,K) - \rho(b,\mathring{K})\right) \\ &\quad + Y^k\sqrt{\det g}\left(J_k(g,K) - J_k(b,\mathring{K})\right) + s' + Q''\Big]\ , \end{aligned} \quad (4.8.9)$$

where Q'' contains terms which are quadratic in the deviation of g from b and its derivatives, and in the deviations of K from $\mathring{K}$, while s', obtained by collecting all terms linear or linearised in e_{ij}, except for those involving ρ and J, reads

$$\begin{aligned} s' &= (\mathring{S}^{kl} + \mathring{B}^{kl})e_{kl} + (P^{kl} - \mathring{P}^{kl})\mathring{A}_{kl}\ , \\ \mathring{B}^{kl} &:= \frac{1}{2}\left[b^{kl}\mathring{P}^{mn}\mathring{A}_{mn} - b^{mn}\mathring{A}_{mn}\mathring{P}^{kl}\right]\ . \end{aligned} \quad (4.8.10)$$

4.9 $3+1$ charge integrals *vs* the Freud integrals

In this section we wish to show that under the boundary conditions of Section 4.4 the numerical value of the Riemannian charge integrals coincides with that of the Hamiltonians derived in a space-time setting in [53, 69]:

$$H^\mu \equiv p^\mu_{\alpha\beta}\pounds_X\mathfrak{g}^{\alpha\beta} - X^\mu L = \partial_\alpha\mathbb{W}^{\mu\alpha} + \frac{1}{8\pi}\mathscr{G}^\mu{}_\beta(\Lambda)X^\beta\ , \quad (4.9.1)$$

$$\mathbb{W}^{\nu\lambda} = \mathbb{W}^{\nu\lambda}{}_\beta X^\beta - \frac{1}{8\pi}\sqrt{|\det g_{\rho\sigma}|}g^{\alpha[\nu}\delta^{\lambda]}_\beta X^\beta{}_{;\alpha}\ , \quad (4.9.2)$$

$$
\begin{aligned}
\mathbb{W}^{\nu\lambda}{}_{\beta} &= \frac{2|\det b_{\mu\nu}|}{16\pi\sqrt{|\det g_{\rho\sigma}|}} g_{\beta\gamma}\left(\frac{|\det g_{\rho\sigma}|}{|\det b_{\mu\nu}|} g^{\gamma[\lambda}g^{\nu]\kappa}\right)_{;\kappa} \\
&= 2\mathfrak{g}_{\beta\gamma}\left(\mathfrak{g}^{\gamma[\lambda}\mathfrak{g}^{\nu]\kappa}\right)_{;\kappa} \\
&= 2\mathfrak{g}^{\mu[\nu}p^{\lambda]}_{\mu\beta} - 2\delta^{[\nu}_{\beta}p^{\lambda]}_{\mu\sigma}\mathfrak{g}^{\mu\sigma} - \frac{2}{3}\mathfrak{g}^{\mu[\nu}\delta^{\lambda]}_{\beta}p^{\sigma}_{\mu\sigma}\ ,
\end{aligned} \tag{4.9.3}
$$

where a semi-colon denotes the covariant derivative of the metric b, square brackets denote anti-symmetrisation (with a factor of $1/2$ when two indices are involved), as before $\mathfrak{g}_{\beta\gamma} \equiv (\mathfrak{g}^{\alpha\sigma})^{-1} = 16\pi g_{\beta\gamma}/\sqrt{|\det g_{\rho\sigma}|}$. Further, $\mathscr{G}^{\alpha}{}_{\beta}(\Lambda)$ is the Einstein tensor density eventually shifted by a cosmological constant,

$$
\mathscr{G}^{\alpha}{}_{\beta}(\Lambda) := \sqrt{|\det g_{\rho\sigma}|}\left(R^{\alpha}{}_{\beta} - \frac{1}{2}g^{\mu\nu}R_{\mu\nu}\delta^{\alpha}_{\beta} + \Lambda\delta^{\alpha}_{\beta}\right) \tag{4.9.4}
$$

(equal, of course, to the energy-momentum tensor density of the matter fields in models with matter, and vanishing in vacuum), while

$$
\begin{aligned}
p^{\lambda}_{\mu\nu} &= \frac{1}{2}\mathfrak{g}_{\mu\alpha}\mathfrak{g}^{\lambda\alpha}{}_{;\nu} + \frac{1}{2}\mathfrak{g}_{\nu\alpha}\mathfrak{g}^{\lambda\alpha}{}_{;\mu} - \frac{1}{2}\mathfrak{g}^{\lambda\alpha}\mathfrak{g}_{\sigma\mu}\mathfrak{g}_{\rho\nu}\mathfrak{g}^{\sigma\rho}{}_{;\alpha} \\
&\quad + \frac{1}{4}\mathfrak{g}^{\lambda\alpha}\mathfrak{g}_{\mu\nu}\mathfrak{g}_{\sigma\rho}\mathfrak{g}^{\sigma\rho}{}_{;\alpha}\ ,
\end{aligned} \tag{4.9.5}
$$

where by $\mathfrak{g}_{\mu\nu}$ we denote the matrix inverse to $\mathfrak{g}^{\mu\nu}$, and

$$
L := \mathfrak{g}^{\mu\nu}\left[\left(\Gamma^{\alpha}_{\sigma\mu} - B^{\alpha}_{\sigma\mu}\right)\left(\Gamma^{\sigma}_{\alpha\nu} - B^{\sigma}_{\alpha\nu}\right) - \left(\Gamma^{\alpha}_{\mu\nu} - B^{\alpha}_{\mu\nu}\right)\left(\Gamma^{\sigma}_{\alpha\sigma} - B^{\sigma}_{\alpha\sigma}\right) + r_{\mu\nu}\right]\ . \tag{4.9.6}
$$

Finally, $B^{\alpha}_{\mu\nu}$ is the connection of the background metric, and $r_{\mu\nu}$ its Ricci tensor – zero in our case. For typesetting reasons in the remainder of this section we write η for ${}^4b$, while the symbol b will be reserved for the space-part of ${}^4b$ and its inverse. While η is flat, the reader should not assume that it takes the usual diagonal form. Let us denote by $\approx$ the linearisation at ${}^4b \equiv \eta$; we find

$$
16\pi\mathbb{W}^{0l}{}_{0} \approx \sqrt{-\eta}\left(e^{l}{}_{m}{}^{;m} - e^{m}{}_{m}{}^{;l}\right)\ , \tag{4.9.7}
$$

$$
16\pi\mathbb{W}^{0l}{}_{k} \approx \sqrt{-\eta}\left[\delta^{l}{}_{k}\left(e^{m}{}_{m}{}^{;0} - e^{0}{}_{m}{}^{;m}\right) + e^{0}{}_{k}{}^{;l} - e^{l}{}_{k}{}^{;0}\right]\ , \tag{4.9.8}
$$

leading to

$$
16\pi\mathbb{W}^{0l}{}_{\beta}n^{\beta} \approx \sqrt{b}\left(b^{kn}b^{lm} - b^{km}b^{ln}\right)e_{mk;n}\ , \tag{4.9.9}
$$

$$
16\pi\mathbb{W}^{0l}{}_{k}Y^{k} \approx \sqrt{-\eta}\left(Y^{l}b^{km} - Y^{k}b^{lm}\right)\left(e_{mk}{}^{;0} - e^{0}{}_{k;m}\right)\ . \tag{4.9.10}
$$

Here

$$
e_{\mu\nu} := g_{\mu\nu} - \eta_{\mu\nu}\ ,
$$

and in the linearised expressions of this section all space-time indices are raised and lowered with η. Similarly,

$$
\begin{aligned}
&\left[\sqrt{-g}(g^{l\alpha}\eta^{0\mu} - g^{0\alpha}\eta^{l\mu}) - \sqrt{-\eta}(\eta^{l\alpha}\eta^{0\mu} - \eta^{0\alpha}\eta^{l\mu})\right]X_{\mu;\alpha} \approx \\
&\sqrt{-\eta}\left[(\eta^{l\alpha}\eta^{0\mu} - \eta^{0\alpha}\eta^{l\mu})\frac{1}{2}e^{\sigma}{}_{\sigma} - e^{l\alpha}\eta^{0\mu} + e^{0\alpha}\eta^{l\mu}\right]X_{\mu;\alpha} = \\
&\sqrt{-\eta}\left(b^{mn}e_{mn}b^{lk}X^{0}{}_{;k} - b^{ln}b^{mk}e_{mn}X^{0}{}_{;k} + e^{0}{}_{m}b^{mk}b^{ln}X_{n;k}\right)\ .
\end{aligned} \tag{4.9.11}
$$

We also have

$$X^0{}_{;k} = \frac{1}{\mathring{N}}\left(V_{,k} - Y^l\mathring{K}_{lk}\right) , \tag{4.9.12}$$

$$X_{l;k} = b_{lm}\mathring{D}_kY^m - V\mathring{K}_{lk} , \tag{4.9.13}$$

and we have of course assumed that

$$X = Vn + Y$$

is a background Killing vector field, with Y tangent to the hypersurface of interest, so that

$$b_{lm}\mathring{D}_kY^m + b_{km}\mathring{D}_lY^m = 2V\mathring{K}_{lk} . \tag{4.9.14}$$

We recall that

$$n^\mu = -\frac{\eta^{0\mu}}{\sqrt{-\eta^{00}}} ,$$

which gives $X^0 = \frac{V}{\mathring{N}}$, $X^k = Y^k - \frac{V}{\mathring{N}}\mathring{N}^k$, with the lapse and shift given by the formulae $\mathring{N} = \frac{1}{\sqrt{-\eta^{00}}}$, $\mathring{N}^k = -\frac{\eta^{0k}}{\eta^{00}}$. We will also need the $3+1$ decomposition of the Christoffel symbols $B^\alpha_{\beta\gamma}$ of the four-dimensional background metric in terms of those, denoted by $\mathring{\Gamma}^m{}_{kl}(b)$, associated with the three-dimensional one:

$$B^m{}_{kl} = \mathring{\Gamma}^m{}_{kl}(b) + \frac{\mathring{N}^m}{\mathring{N}}\mathring{K}_{lk} , \quad B^0{}_{0k} = \partial_k\log\mathring{N} - \frac{\mathring{N}^l}{\mathring{N}}\mathring{K}_{lk} ,$$

$$B^0{}_{kl} = -\frac{1}{\mathring{N}}\mathring{K}_{lk} , \quad B^l{}_{k0} = \mathring{D}_k\mathring{N}^l - \frac{\mathring{N}^l}{\mathring{N}}\mathring{D}_k\mathring{N} - \mathring{N}\mathring{K}^l{}_k + \frac{\mathring{N}^l}{\mathring{N}}\mathring{N}^m\mathring{K}_{mk} .$$

Linearising the equation for K_{ij} one finds

$$\delta K_{kl} := -\frac{1}{2}\mathring{N}\left(e^0{}_{k;l} + e^0{}_{l;k} - e_{kl}{}^{;0}\right) - \frac{1}{2}(\mathring{N})^2e^{00}\mathring{K}_{kl} , \tag{4.9.15}$$

where the relevant indices have been raised with the space-time background metric. This leads to the following formula for the linearised ADM momentum:

$$\delta P^l{}_k := \delta^l{}_kb^{mn}\delta K_{mn} - b^{ml}\delta K_{mk} + \left(b^{li}\mathring{K}^j{}_k - \delta^l{}_k\mathring{K}^{ij}\right)e_{ij} . \tag{4.9.16}$$

A rather lengthy calculation leads then to

$$\begin{aligned}
&\left[\sqrt{-g}(g^{l\alpha}\eta^{0\mu} - g^{0\alpha}\eta^{l\mu}) - \sqrt{-\eta}(\eta^{l\alpha}\eta^{0\mu} - \eta^{0\alpha}\eta^{l\mu})\right]X_{\mu;\alpha}\\
&+16\pi V\mathbb{W}^{0l}{}_\beta n^\beta + 16\pi\mathbb{W}^{0l}{}_kY^k \approx\\
&\sqrt{b}Vb^{ij}b^{lm}(\mathring{D}_ie_{jm} - \mathring{D}_me_{ji}) + \sqrt{b}e_{mn}(b^{mn}b^{lk} - b^{ln}b^{mk})\mathring{D}_kV\\
&+\sqrt{b}\mathring{D}_k\left[\mathring{N}e^0{}_m(Y^lb^{km} - Y^kb^{lm})\right] + 2\sqrt{b}(Y^lb^{km} - Y^kb^{lm})\delta K_{km}\\
&+\sqrt{b}e_{mn}\left[2Y^k\mathring{K}_k{}^mb^{ln} - Y^k\mathring{K}_k{}^lb^{mn} - Y^l\mathring{K}^{mn}\right] =\\
&\sqrt{b}Vb^{ij}b^{lm}(\mathring{D}_ie_{jm} - \mathring{D}_me_{ji}) + \sqrt{b}e_{mn}(b^{mn}b^{lk} - b^{ln}b^{mk})\mathring{D}_kV\\
&+\partial_k\left[\sqrt{b}\mathring{N}e^0{}_m(Y^lb^{km} - Y^kb^{lm})\right] + 2\sqrt{b}Y^k\delta P^l{}_k\\
&+\sqrt{b}e_{mn}\left(\mathring{P}^l{}_kY^kb^{mn} - Y^l\mathring{P}^{mn}\right) .
\end{aligned} \tag{4.9.17}$$

The integral over a sphere of (4.9.17) coincides with the linearisation of the integral over a sphere of $\mathbb{U}^i + \mathbb{V}^i$, as the difference between those expressions is a complete divergence.

Let us finally show that the linearised expression above does *not* reproduce the Trautman-Bondi mass. It is most convenient to work directly in space-time Bondi coordinates (u, x, x^A) rather than in coordinates adapted to $\mathscr{S}$. We consider a metric of the Bondi form (4.5.1)–(4.5.2) with the asymptotic behavior (4.10.11)–(4.10.13); we have

$$
\begin{aligned}
e_{00} &= 1 - xV + x^{-2} h_{AB} U^A U^B = 2Mx + O_{\ln^* x}(x^2) \;, \\
e_{03} &= x^{-2}(e^{2\beta} - 1) = O(1) \;, \\
e_{0A} &= -x^{-2} h_{AB} U^B = \frac{1}{2} \chi_A{}^B{}_{||B} + O(x) \;, \\
e_{AB} &= x^{-2}(h_{AB} - \breve{h}_{AB}) = x^{-1}\chi_{AB} + O(1) \;, \\
e_{33} &= e_{3A} = 0 \;.
\end{aligned}
$$

(Recall that $\breve{h}$ denotes the standard round metric on S^2.) For translational ${}^4b$-Killing vector fields X^μ we have $X_{\mu;\nu} = 0$, hence

$$
\mathbb{W}^{0l} = \mathbb{W}^{0l}{}_\mu X^\mu \;.
$$

Now, taking into account (4.9.7) and (4.9.8), the linearised Freud superpotential takes the form:

$$
\begin{aligned}
16\pi \mathbb{W}^{03} \;=\;& \sqrt{-\eta}\,\big[(e^A{}_A{}^{;3} - e^3{}_A{}^{;A})X^0 + (e^0{}_A{}^{;A} - e^A{}_A{}^{;0})X^3 \\
& + (e^3{}_A{}^{;0} - e^0{}_A{}^{;3})X^A\big] \;,
\end{aligned}
\tag{4.9.18}
$$

The following formulae for the non-vanishing ${}^4B^\sigma{}_{\beta\gamma}$'s for the flat metric (4.10.5) are useful when working out (4.9.18):

$$
{}^4B^u{}_{AB} = x^{-1}\breve{h}_{AB}, \quad {}^4B^x{}_{AB} = x\breve{h}_{AB}, \quad {}^4B^A{}_{xB} = -x^{-1}\delta^A_B, \tag{4.9.19}
$$

$$
{}^4B^x{}_{xx} = -2x^{-1}, \quad {}^4B^A{}_{BC} = \breve{\Gamma}^A{}_{BC}(\breve{h}) \;. \tag{4.9.20}
$$

The Killing field corresponding to translations in spacetime can be characterised by a function κ which is a linear combination of the $\ell = 0$ and $\ell = 1$ spherical harmonics:

$$
X^0 = \kappa \;, \quad X^3 = -\frac{1}{2}x^2 \triangle \kappa \;, \quad X^A = -x\breve{h}^{AB}\kappa_{,B} \;.
$$

With some work one finds the following formula for the linearised superpotential for time translations ($\kappa = 1$)

$$
\begin{aligned}
16\pi \mathbb{W}^{03} &= \sqrt{-\eta}(e^A{}_A{}^{;3} - e^3{}_A{}^{;A}) \\
&= \sin\theta \left[4M - \frac{1}{2}\chi^{AB}{}_{||AB} + \chi^{CD}\partial_u \chi_{CD} + O_{\ln^* x}(x)\right] \;.
\end{aligned}
$$

The resulting integral reproduces the Trautman-Bondi mass if and only if the $\chi^{CD}\partial_u\chi_{CD}$ term above gives a zero contribution after being integrated upon; in general this will not be the case.

4.10 Proof of Lemma 4.6.8

The object of this section is to calculate, for large R, the boundary integrand that appears in the integral identity (4.6.12), for a class of hyperboloidal initial data sets made precise in Theorem 4.6.4. We consider a conformally compactifiable polyhomogeneous initial data set $(\mathscr{S}, g, K)$, such that

$$\mathrm{tr}_g K \text{ is constant to second order at } \partial\mathscr{S}. \tag{4.10.1}$$

In $(\mathscr{M}, \mathfrak{g})$ we can always [68] introduce a Bondi coordinate system (u, x, x^A) such that $\mathscr{S}$ is given by an equation

$$u = \alpha(x, x^A)\ , \quad \text{with} \ \ \alpha(0, x^A) = 0\ ,\ \alpha_{,x}(0, x^A) > 0\ , \tag{4.10.2}$$

where α is polyhomogeneous. It follows from [64, Equation (C.83)] that (4.10.1) is equivalent to

$$\alpha_{,xx}|_{x=0} = 0\ . \tag{4.10.3}$$

(Throughout this section we will make heavy use of the formulae of [64, Appendix C] without necessarily indicating this fact.) Polyhomogeneity implies then

$$\alpha_{,A} = O_{\ln^* x}(x^3)\ , \tag{4.10.4}$$

where we use the symbol $f = O_{\ln^* x}(x^p)$ to denote the fact that there exists $N \in \mathbb{N}$ and a constant C such that

$$|f| \le Cx^p(1 + |\ln x|^N)\ .$$

We also assume that this behaviour is preserved under differentiation in the obvious way:

$$|\partial_x f| \le Cx^{p-1}(1 + |\ln x|^N)\ , \quad |\partial_A f| \le Cx^p(1 + |\ln x|^N)\ ,$$

similarly for higher derivatives. The reason for imposing (4.10.1) is precisely Equation (4.10.3). For more general α's the expansions below acquire many further terms, and we have not attempted to carry through the (already scary) calculations below if (4.10.3) does not hold.

Alternatively, one can start from a space-time with a polyhomogeneous $\mathscr{I}^+$ and choose any space-like hypersurface $\mathscr{S}$ so that (4.10.2)–(4.10.3) hold. Such an approach can be used to study the positivity properties of the Trautman-Bondi mass, viewed as a function on the set of space-times rather than a function on the set of initial data sets.

We use the Bondi coordinates to define the background ${}^4b$:

$${}^4b = -\,du^2 + 2x^{-2}\,du\,dx + x^{-2}\breve{h}_{AB}\,dx^A\,dx^B\ , \tag{4.10.5}$$

so that the components of the inverse metric read

$${}^4b^{ux} = x^2 \qquad {}^4b^{xx} = x^4 \qquad {}^4b^{AB} = x^2\breve{h}^{AB}\ .$$

The metric b induced on $\mathscr{S}$ takes thus the form

$$\begin{aligned} b &= x^{-2}\Bigg[\Big(\breve{h}_{AB}+O_{\ln^* x}(x^8)\Big)\, dx^A\, dx^B + 2(1-x^2\alpha_{,x})\alpha_{,A}\, dx\, dx^A \\ &\quad +2\alpha_{,x}\left(1-\frac{1}{2}x^2\alpha_{,x}\right)(\,dx)^2\Bigg]\,. \end{aligned} \tag{4.10.6}$$

Let $\tilde{e}^a$ be a local orthonormal co-frame for the unit round metric $\breve{h}$ on S^2 (outside of the south and north pole one can, *e.g.*, use $\tilde{e}^1 = d\theta$, $\tilde{e}^2 = \sin\theta\, d\varphi$), we set

$$e^a := x^{-1}\tilde{e}^a\ , \quad e^3 := x^{-1}\tilde{e}^3\ ,$$

where

$$\tilde{e}^3 := \sqrt{\alpha_{,x}\,(2-x^2\alpha_{,x})}\, dx + \frac{1-x^2\alpha_{,x}}{\sqrt{\alpha_{,x}\,(2-x^2\alpha_{,x})}}\alpha_{,A}\, dx^A\ .$$

Assuming (4.10.3)-(4.10.4), it follows that the co-frame $\tilde{e}^i$ is close to being orthonormal for b:

$$b = x^{-2}\left[\tilde{e}^1\tilde{e}^1+\tilde{e}^2\tilde{e}^2+\tilde{e}^3\tilde{e}^3+O_{\ln^* x}(x^6)\right] = e^1e^1+e^2e^2+e^3e^3+O_{\ln^* x}(x^4)\ . \tag{4.10.7}$$

Here and elsewhere, an equality $f = O_{\ln^* x}(x^p)$ *for a tensor field* f *means that the components of* f *in the coordinates* (x, x^A) *are* $O_{\ln^* x}(x^p)$.

Recall that in Bondi-Sachs coordinates (u, x, x^A) the space-time metric takes the form:

$$\mathfrak{g} = -xV\mathrm{e}^{2\beta}\, du^2 + 2\mathrm{e}^{2\beta}x^{-2}\, du\, dx + x^{-2}h_{AB}\left(dx^A - U^A\, du\right)\left(dx^B - U^B\, du\right)\ . \tag{4.10.8}$$

This leads to the following form of the metric g induced on $\mathscr{S}$

$$\begin{aligned} x^2 g &= 2\Big[(h_{AB}U^AU^B - x^3V\mathrm{e}^{2\beta})\alpha_{,x}\alpha_{,C} + \mathrm{e}^{2\beta}\alpha_{,C} - h_{CB}U^B\alpha_{,x}\Big]\, dx^C\, dx \\ &\quad + \Big[(h_{AB}U^AU^B - x^3V\mathrm{e}^{2\beta})\alpha_{,D}\alpha_{,C} - 2h_{CB}U^B\alpha_{,D} + h_{CD}\Big]\, dx^C\, dx^D \\ &\quad + \Big[\Big(h_{AB}U^AU^B - x^3V\mathrm{e}^{2\beta}\Big)(\alpha_{,x})^2 + 2\mathrm{e}^{2\beta}\alpha_{,x}\Big](\,dx)^2 \end{aligned} \tag{4.10.9}$$

Let γ_{ij} be defined as

$$x^2 g = \gamma_{AB}\, dx^A\, dx^B + 2\gamma_{xA}\, dx^A\, dx + \gamma_{xx}(\,dx)^2\ , \tag{4.10.10}$$

so that

$$\begin{aligned} \gamma_{CD} &:= h_{CD} - 2\alpha_{,(D}h_{C)B}U^B + (h_{AB}U^AU^B - x^3V\mathrm{e}^{2\beta})\alpha_{,D}\alpha_{,C}\ , \\ \gamma_{xC} &:= (h_{AB}U^AU^B - x^3V\mathrm{e}^{2\beta})\alpha_{,x}\alpha_{,C} + \mathrm{e}^{2\beta}\alpha_{,C} - h_{CB}U^B\alpha_{,x}\ , \\ \gamma_{xx} &:= \Big[\Big(h_{AB}U^AU^B\mathrm{e}^{-2\beta} - x^3V\Big)\alpha_{,x} + 2\Big]\mathrm{e}^{2\beta}\alpha_{,x}\ . \end{aligned}$$

If we assume that $(\mathscr{S}, g, K)$ is polyhomogeneous and conformally $C^1 \times C^0$-compactifiable, it follows that

$$h_{AB} = \breve{h}_{AB}(1+\frac{x^2}{4}\chi^{CD}\chi_{CD}) + x\chi_{AB} + x^2\zeta_{AB} + x^3\xi_{AB} + O_{\ln^* x}(x^4)\ \ ,$$

where ζ_{AB} and ξ_{AB} are polynomials in $\ln x$ with coefficients which smoothly depend upon the x^A's. By definition of the Bondi coordinates we have $\det h = \det \breve{h}$, which implies

$$\breve{h}^{AB}\chi_{AB} = \breve{h}^{AB}\zeta_{AB} = 0\ .$$

Further,

$$\beta = -\frac{1}{32}\chi^{CD}\chi_{CD}x^2 + Bx^3 + O_{\ln^* x}(x^4)\ , \tag{4.10.11}$$

$$xV = 1 - 2Mx + O_{\ln^* x}(x^2)\ , \tag{4.10.12}$$

$$h_{AB}U^B = -\frac{1}{2}\chi_A{}^B{}_{||B}x^2 + u_Ax^3 + O_{\ln^* x}(x^4)\ , \tag{4.10.13}$$

where B and u_A are again polynomials in $\ln x$ with smooth coefficients depending upon the x^A's, while $||$ denotes covariant differentiation with respect to the metric $\breve{h}$. This leads to the following approximate formulae

$$(h_{AB}U^AU^B - x^3V\mathrm{e}^{2\beta})\alpha_{,D} = O_{\ln^* x}(x^5)\ , \quad h_{CB}U^B\alpha_{,D} = O_{\ln^* x}(x^5)\ ,$$

$$\gamma_{xA} = \alpha_{,A} + \alpha_{,x}\left[\frac{1}{2}\chi_A{}^B{}_{||B}x^2 - u_Ax^3\right] + O_{\ln^* x}(x^4)\ , \tag{4.10.14}$$

$$\sqrt{\gamma_{xx}} = \sqrt{2\alpha_{,x}}\left[1 - \frac{1}{4}(\alpha_{,x} + \frac{1}{8}\chi^{CD}\chi_{CD})x^2 + \left(\frac{1}{2}\alpha_{,x}M + B\right)x^3\right] + O_{\ln^* x}(x^4)\ . \tag{4.10.15}$$

Let

$$h_{ab} = h(\tilde{e}_a, \tilde{e}_b)\ ,$$

where $\tilde{e}_a$ is a basis of vectors tangent to S^2 dual to $\tilde{e}^a$, and let $\mu^a{}_b$ be the symmetric root of h_{ab},

$$\mu^a{}_c h_{ab}\mu^b{}_d = \delta_{cd}\ ,$$

or, in matrix notation,

$${}^t\mu h\mu = id\ , \tag{4.10.16}$$

where ${}^t\mu^c{}_a = \mu^a{}_c$ stands for the transpose of μ. Let $\tilde{f}^a$, $a = 1, 2$, be the field of local orthonormal co-frames for the metric h_{AB} defined by the formula

$$\tilde{e}^a = \mu^a{}_b\tilde{f}^b\ . \tag{4.10.17}$$

We set

$$\tilde{f}^3 := \sqrt{\gamma_{xx}}\,dx + \frac{\gamma_{xA}}{\sqrt{\gamma_{xx}}}\,dx^A\ , \tag{4.10.18}$$

so that

$$x^2g = \tilde{f}^1\tilde{f}^1 + \tilde{f}^2\tilde{f}^2 + \tilde{f}^3\tilde{f}^3 + O_{\ln^* x}(x^4)\ . \tag{4.10.19}$$

There exists a matrix $M^k{}_l$ such that

$$\tilde{e}^k = M^k{}_l\tilde{f}^l\ .$$

Now,

$$\tilde{e}^3 = (1 - x^2a)\tilde{f}^3 - x^2b_a\mu^a{}_b\tilde{f}^b + O_{\ln^* x}(x^4)\ , \tag{4.10.20}$$

with

$$a := -\frac{1}{32}\chi^{CD}\chi_{CD} + \left(\frac{1}{2}\alpha_{,x}M + B\right) x \ ,$$

$$b_a\tilde{e}^a = b_A dx^A := \sqrt{\frac{\alpha_{,x}}{2}}\left[\frac{1}{2}\chi_A{}^B{}_{||B} - u_A x\right] dx^A \ .$$

The matrix M is easily calculated to be

$$M^3{}_3 = 1 - x^2 a + O_{\ln^* x}(x^4) \ , \quad M^3{}_b = -x^2 b_a \mu^a{}_b + O_{\ln^* x}(x^4) \ , \qquad (4.10.21)$$

$$M^a{}_3 = 0 \ , \quad M^a{}_b = \mu^a{}_b \ . \qquad (4.10.22)$$

Since

$$e^k = x^{-1}\tilde{e}^k \ , \quad f^k = x^{-1}\tilde{f}^k \ ,$$

it follows that

$$e^k = M^k{}_l f^l \ , \quad f_l = M^k{}_l e_k \ ,$$

where f_l and e_k stand for frames dual to f^i and e^i. If we choose

$$\tilde{e}^1 = d\theta \ , \quad \tilde{e}^2 = \sin\theta \, d\varphi \ ,$$

then

$$\tilde{e}^3 := \sqrt{\alpha_{,x}\,(2 - x^2\alpha_{,x})}\, dx + \frac{1 - x^2\alpha_{,x}}{\sqrt{\alpha_{,x}\,(2 - x^2\alpha_{,x})}}\alpha_{,A}\, dx^A \ ,$$

$$\tilde{e}_1 = \frac{\partial}{\partial\theta} - \alpha_{,\theta}\frac{1 - x^2\alpha_{,x}}{\alpha_{,x}\,(2 - x^2\alpha_{,x})}\frac{\partial}{\partial x} \ ,$$

$$\tilde{e}_2 = \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\varphi} - \alpha_{,\varphi}\frac{1 - x^2\alpha_{,x}}{\alpha_{,x}\,(2 - x^2\alpha_{,x})}\frac{\partial}{\partial x}\right] \ ,$$

$$\tilde{e}_3 = \frac{1}{\sqrt{\alpha_{,x}\,(2 - x^2\alpha_{,x})}}\frac{\partial}{\partial x} \ ,$$

$$e_k = x\tilde{e}_k \ , \quad f_k = x\tilde{f}_k \ .$$

We also have the relation

$$e_i = (M^{-1})^k{}_i f_k \ ,$$

with

$$(M^{-1})^3{}_3 = \frac{1}{M^3{}_3} = 1 + x^2 a + O_{\ln^* x}(x^4) \ , \quad (M^{-1})^a{}_b = (\mu^{-1})^a{}_b \ , \qquad (4.10.23)$$

$$(M^{-1})^a{}_3 = 0 \ , \quad (M^{-1})^3{}_b = -\frac{M^3{}_a(\mu^{-1})^a{}_b}{M^3{}_3} = x^2 b_b + O_{\ln^* x}(x^4) \ . \qquad (4.10.24)$$

Consider, now, the integrand in (4.6.12):

$$B^3 = - < \psi, \nabla^3\psi + \gamma^3\gamma^i\nabla_i\psi > = - < \psi, \gamma^3\gamma^a\nabla_a\psi > \ ,$$

where the minus sign arises from the fact that we will use a g–orthonormal frame $\hat{f}_i$ in which $\hat{f}_3$ is *minus* the outer-directed normal to the boundary. Here

ψ is assumed to be the restriction to $\mathscr{S}$ of a space-time covariantly constant spinor with respect to the background metric ${}^4b$:

$$\mathring{\nabla}\psi = 0\ .$$

It follows that

$$\begin{aligned}
\nabla_a\psi &= \hat{f}_a(\psi) + \left[-\frac{1}{2}K(\hat{f}_a,\hat{f}_j)\gamma^j\gamma_0 - \frac{1}{4}\omega_{ij}(\hat{f}_a)\gamma^i\gamma^j\right]\psi \qquad (4.10.25)\\
&= \left[-\frac{1}{2}\Big(K(\hat{f}_a,\hat{f}_j)-\mathring{K}(\hat{f}_a,\hat{e}_j)\Big)\gamma^j\gamma_0 + \frac{1}{4}\Big(\mathring{\omega}_{ij}(\hat{f}_a)-\omega_{ij}(\hat{f}_a)\Big)\gamma^i\gamma^j\right]\psi\ .
\end{aligned}$$

This allows us to rewrite B^3 as

$$\begin{aligned}
B^3 &= \frac{1}{2}\Big(K(\hat{f}_a,\hat{f}_j)-\mathring{K}(\hat{f}_a,\hat{e}_j)\Big) < \psi,\gamma^3\gamma^a\gamma^j\gamma_0\psi >\\
&\quad -\frac{1}{4}\Big(\mathring{\omega}_{ij}(\hat{f}_a)-\omega_{ij}(\hat{f}_a)\Big) < \psi,\gamma^3\gamma^a\gamma^i\gamma^j\psi >\\
&= \frac{1}{2}\Big(K(\hat{f}_a,\hat{f}_j)-\mathring{K}(\hat{f}_a,\hat{e}_j)\Big)\left(g^{j3}Y^a-g^{ja}Y^3\right)\\
&\quad +\frac{1}{2}\Big(\mathring{\omega}_{3a}(\hat{f}_a)-\omega_{3a}(\hat{f}_a)\Big)W\ , \qquad (4.10.26)
\end{aligned}$$

where (W,Y^i) denotes the KID[7] associated to the spinor field ψ

$$W := < \psi,\psi >\ ,\quad Y^k = < \psi,\gamma^k\gamma_0\psi >\ .$$

We use the convention in which

$$\{\gamma^i,\gamma^j\} = -2\delta^{ij}\ ,$$

with γ^0 — anti-hermitian, and γ^i — hermitian. Because ψ is covariantly constant, W and Y^i satisfy the following equation

$$\partial_i W = K_{ij}Y^j\ .$$

Let $\hat{e}_i$ be an orthonormal frame for b; we will shortly see that we have the following asymptotic behaviors,

$$\begin{aligned}
\mathring{\omega}_{3a}(\hat{f}_a)-\omega_{3a}(\hat{f}_a) &= O(x^2)\ ,\\
K(\hat{f}_a,\hat{f}_j)-\mathring{K}(\hat{f}_a,\hat{e}_j) &= O(x^2)\ ,\\
Y^k = O(x^{-1})\ ,\quad W &= O(x^{-1})\ ,
\end{aligned}$$

which determines the order to which various objects above have to be expanded when calculating B^3. In particular, these equations show that some non-obvious cancelations have to occur for the integral of B^3 to converge.

Since $\hat{e}_i$ is b–orthonormal, it holds that

$$2\mathring{\omega}_{klj} = b([\hat{e}_k,\hat{e}_l],\hat{e}_j) + b([\hat{e}_k,\hat{e}_j],\hat{e}_l) - b([\hat{e}_l,\hat{e}_j],\hat{e}_k)\ . \qquad (4.10.27)$$

[7]To avoid a clash of notation with the Bondi function V we are using the symbol W for the normal component of the KID here.

It follows from (4.10.7) that we can choose $\hat{e}_k$ so that

$$\hat{e}_k = (\delta_k^\ell + O_{\ln^* x}(x^6))e_\ell \; . \qquad (4.10.28)$$

This choice leads to the following commutators:

$$[\hat{e}_1, \hat{e}_2] = -x\cot\theta \hat{e}_2 + O_{\ln^* x}(x^3)\hat{e}_1 + O_{\ln^* x}(x^3)\hat{e}_2 + O_{\ln^* x}(x^4)\hat{e}_3 \; , \qquad (4.10.29)$$

$$\begin{aligned} [\hat{e}_a, \hat{e}_3] &= -\left[\alpha_{,x}(2 - x^2\alpha_{,x})\right]^{-1/2} \hat{e}_a - (\frac{\alpha_{,a}}{2\alpha_{,x}} + x\frac{\alpha_{,xa}}{4\alpha_x} + O_{\ln^* x}(x^4))\hat{e}_3 \\ &\quad + O_{\ln^* x}(x^4)\hat{e}_1 + O_{\ln^* x}(x^4)\hat{e}_2 \; , \qquad (4.10.30) \end{aligned}$$

$$[\hat{e}_i, \hat{e}_j] = c_{ij}{}^k \hat{e}_k \; , \qquad (4.10.31)$$

$$\begin{aligned} c_{21}{}^1 = -c_{12}{}^1 &= O_{\ln^* x}(x^3) \; , & (4.10.32) \\ c_{21}{}^2 = -c_{12}{}^2 &= x\cot\theta + O_{\ln^* x}(x^3) \; , & (4.10.33) \\ c_{12}{}^3 = -c_{21}{}^3 &= O_{\ln^* x}(x^4) \; , & (4.10.34) \\ c_{3a}{}^3 = -c_{a3}{}^3 &= \tfrac{\alpha_{,a}}{2\alpha_{,x}} + x\tfrac{\alpha_{,xa}}{4\alpha_x} + O_{\ln^* x}(x^4) \; , & (4.10.35) \\ c_{3a}{}^b = -c_{a3}{}^b &= \left[\alpha_{,x}(2 - x^2\alpha_{,x})\right]^{-1/2} \delta^b{}_a + O_{\ln^* x}(x^4) \; . & (4.10.36) \end{aligned}$$

It follows from (4.10.19) that we can choose $\hat{f}_j$ so that

$$\hat{f}_j = (\delta_j^k + O_{\ln^* x}(x^4))f_k \; . \qquad (4.10.37)$$

Let $\hat{M}$ be the transition matrix from the frame $\hat{e}_i$ to the frame $\hat{f}_j$,

$$\hat{f}_k = \hat{M}^\ell{}_k \hat{e}_\ell \; ,$$

if we define the functions $d_{ij}{}^k$ as

$$[\hat{f}_i, \hat{f}_j] = d_{ij}{}^k \hat{f}_k \; , \qquad (4.10.38)$$

then we have

$$d_{ml}{}^k = \hat{M}^j{}_l \hat{M}^i{}_m c_{ij}{}^n (\hat{M}^{-1})^k{}_n + (\hat{M}^{-1})^k{}_j \hat{f}_m(\hat{M}^j{}_l) - (\hat{M}^{-1})^k{}_j \hat{f}_l(\hat{M}^j{}_m) \; . \qquad (4.10.39)$$

Chasing through the definitions one also finds that

$$\hat{M}^i{}_j = M^i{}_j + O_{\ln^* x}(x^4) \; . \qquad (4.10.40)$$

This leads to

$$d_{3aa} = M^3{}_3 c_{3aa} + O_{\ln^* x}(x^4) = 2\left[\alpha_{,x}(2 - x^2\alpha_{,x})\right]^{-1/2} M^3{}_3 + O_{\ln^* x}(x^4) \; .$$

Now,

$$2\mathring{\omega}_{klj} = c_{klj} + c_{kjl} - c_{ljk} \; , \qquad (4.10.41)$$

$$2\omega_{klj} = d_{klj} + d_{kjl} - d_{ljk} \; , \qquad (4.10.42)$$

$$\omega_{3ab} = d_{3(ab)} - \frac{1}{2} d_{ab3} \ , \tag{4.10.43}$$

$$\omega_{3aa} = d_{3aa} \ , \tag{4.10.44}$$

$$\mathring{\omega}_{3ab} = \left[\alpha_{,x}(2 - x^2\alpha_{,x})\right]^{-1/2} \delta_{ab} + O_{\ln^* x}(x^4) = O(1) \ , \tag{4.10.45}$$

$$\mathring{\omega}_{3a3} = O_{\ln^* x}(x^3) \ . \tag{4.10.46}$$

Using obvious matrix notation, from (4.10.16) we obtain

$$\mu = \mathrm{id} - \frac{x}{2}\chi + x^2 d + x^3 w + O_{\ln^* x}(x^4) \ , \tag{4.10.47}$$

where d and w are polynomials in $\ln x$ with coefficients which are x^A–dependent symmetric matrices. The condition that $\det \mu = 1$ leads to the relations

$$\mathrm{tr}\chi = 0 \ , \quad \mathrm{tr}d = \frac{1}{8}\mathrm{tr}\chi^2 \ , \quad \mathrm{tr}w = -\frac{1}{2}\mathrm{tr}(\chi d) \ .$$

Using $(\mu^{-1})^a{}_b \hat{f}_3(\mu^b{}_a) = \hat{f}_3(\det \mu) = 0$, the asymptotic expansions (4.10.33)-(4.10.36) together with (4.10.23)-(4.10.24), (4.10.28) and (4.10.37) one finds the following contribution to the first term $\frac{1}{2}\left(\mathring{\omega}_{3a}(\hat{f}_a) - \omega_{3a}(\hat{f}_a)\right) W$ in B^3:

$$\begin{aligned} \mathring{\omega}_{3a}(\hat{f}_a) - \omega_{3a}(\hat{f}_a) &= \hat{M}^l{}_a \mathring{\omega}_{3al} - \omega_{3aa} \\ &= M^3{}_a \mathring{\omega}_{3a3} + (\mu^b{}_a - \delta^b{}_a)\mathring{\omega}_{3ab} + \mathring{\omega}_{3aa} - \omega_{3aa} + O_{\ln^* x}(x^4) \\ &= \frac{x^2}{\sqrt{2\alpha_{,x}}} \left[\underline{\frac{1}{16}\chi_{AB}\chi^{AB}} + x\left(M\alpha_{,x} + 2B - \frac{1}{2}\mathrm{tr}(\chi d)\right)\right] \\ &\quad + O_{\ln^* x}(x^4) \ . \end{aligned} \tag{4.10.48}$$

The underlined term would give a diverging contribution to the integral of B^3 over the conformal boundary if it did not cancel out with an identical term from the K contribution, except when $\chi = 0$.

We choose now the Killing spinor ψ so that

$$W\mathring{\nabla} + Y = \frac{\partial}{\partial t}$$

in the usual Minkowskian coordinates, where $\mathring{\nabla}$ is the unit normal to $\mathscr{S}$. This leads to

$$W = \frac{x^{-1}}{\sqrt{2\alpha_{,x}}}\left[1 + O(x^2)\right] \ , \quad Y^x = \frac{1}{2\alpha_{,x}}\left[1 + O(x^2)\right] \ , \quad Y^A = O(x^2) \ ,$$

$$Y^3 = \frac{x^{-1}}{\sqrt{2\alpha_{,x}}} + O(x) \ , \quad Y^a = O(x) \ .$$

In order to calculate the remaining terms in (4.10.26) we begin with

$$K(\hat{f}_a, \hat{f}_a) - \mathring{K}(\hat{f}_a, \hat{e}_a) = x^2 \mu^c{}_a \mu^d{}_a K(\tilde{e}_c, \tilde{e}_d) - x^2 \mu^c{}_a \mathring{K}(\tilde{e}_c, \tilde{e}_a) + O_{\ln^* x}(x^4) \ , \tag{4.10.49}$$

Using the formulae of [64, Appendix C.3] one finds

$$\mu^c{}_a\mu^d{}_a K(\tilde e_c,\tilde e_d) = h^{cd}K(\tilde e_c,\tilde e_d) = h^{AB}K_{AB} + O_{\ln^* x}(x^2)\ ,$$

$$\Gamma^\omega{}_{AB} = \Gamma^u{}_{AB} - \alpha_{,x}\Gamma^x{}_{AB} + O_{\ln^* x}(x^3)\ , \tag{4.10.50}$$

$$\Gamma^u{}_{AB} = x^{-1}\mathrm{e}^{-2\beta}\left(h_{AB} - \frac{x}{2}h_{AB,x}\right)\ , \tag{4.10.51}$$

$$\Gamma^x{}_{AB} = -\frac{1}{2}\mathrm{e}^{-2\beta}\left(2\mathscr{D}_{(A}U_{B)} + \partial_u h_{AB} - 2Vx^2h_{AB} + Vx^3h_{AB,x}\right)\ , \tag{4.10.52}$$

where $\mathscr{D}_A$ denotes the covariant derivative with respect to the metric h. It follows that

$$\Gamma^\omega{}_{AB}h^{AB} = x^{-1}\mathrm{e}^{-2\beta}\left[2 + x\alpha_{,x}\left(U^A{}_{||A} - 2Vx^2\right)\right] + O_{\ln^* x}(x^3)\ . \tag{4.10.53}$$

Further

$$N = \frac{x^{-1}}{\sqrt{2\alpha_{,x}}}\left[1 + \beta + \frac{1}{4}x^3\alpha_{,x}V + O(x^4)\right]\ .$$

Equation (4.10.50) specialised to Minkowski metric reads

$$\mathring{\Gamma}^\omega{}_{AB} = x^{-1}\breve h_{AB} - x\alpha_{,x}\breve h_{AB} + O_{\ln^* x}(x^3)\ , \tag{4.10.54}$$

yielding

$$\mathring{K}_{AB} \;=\; -\frac{x^{-2}}{\sqrt{2\alpha_{,x}}}\left[\breve h_{AB}\left(1 - \frac{3}{4}x^2\alpha_{,x}\right) + O_{\ln^* x}(x^4)\right]\ . \tag{4.10.55}$$

We therefore have

$$\begin{aligned} x^2h^{AB}K_{AB} - x^2\mu^b{}_a\mathring{K}_{ab} \;=\; & \frac{1}{\sqrt{2\alpha_{,x}}}\Big[2\beta + \frac{1}{8}x^2\chi_{CD}\chi^{CD} - \frac{3}{2}x^2\alpha_{,x}(1 - xV) \\ & + x\alpha_{,x}U^A{}_{||A} - \frac{x^3}{2}\mathrm{tr}(\chi d)\Big] + O_{\ln^* x}(x^4)\ , \end{aligned} \tag{4.10.56}$$

which implies

$$\begin{aligned} \left(K(\hat f_a,\hat f_a) - \mathring{K}(\hat f_a,\hat e_a)\right)Y^3 \;=\; & \frac{x}{2\alpha_{,x}}\Big[\underline{\frac{1}{16}\chi_{AB}\chi^{AB}} + \\ & + x\left(2B - 3M\alpha_{,x} + \frac{1}{2}\alpha_{,x}\chi^{AB}{}_{||AB} - \frac{1}{2}\mathrm{tr}(\chi d)\right)\Big] + O_{\ln^* x}(x^3)\ . \end{aligned} \tag{4.10.57}$$

Here we have again underlined a potentially divergent term. Using the formulae of [64, Appendix C] one further finds

$$\left(K(\hat f_a,\hat f_3) - \mathring{K}(\hat f_a,\hat e_3)\right)Y^a = O_{\ln^* x}(x^3)\ ,$$

which will give a vanishing contribution to B^3 in the limit. Collecting this together with (4.10.57) and (4.10.48) we finally obtain

$$B^3 = \frac{1}{4}x^2\left(4M - \frac{1}{2}\chi^{AB}{}_{||AB}\right) + O_{\ln^* x}(x^3)\ , \tag{4.10.58}$$

We have thus shown that the integral of B^3 over the conformal boundary is proportional to the Trautman-Bondi mass, as desired.

4.11 Proof of Lemma 4.6.9

From
$$\mathring{\nabla}\psi = 0$$
we have
$$\begin{aligned}\gamma^\ell \nabla_\ell \psi &= \gamma^\ell \hat{f}_\ell(\psi) + \gamma^\ell \left[-\frac{1}{2}K(\hat{f}_\ell,\hat{f}_j)\gamma^j\gamma_0 - \frac{1}{4}\omega_{ij}(\hat{f}_\ell)\gamma^i\gamma^j\right]\psi \\ &= \left[-\frac{1}{2}\left(K(\hat{f}_\ell,\hat{f}_j) - \mathring{K}(\hat{f}_\ell,\hat{e}_j)\right)\gamma^\ell\gamma^j\gamma_0 \right. \\ &\quad \left. +\frac{1}{4}\left(\mathring{\omega}_{ij}(\hat{f}_\ell) - \omega_{ij}(\hat{f}_\ell)\right)\gamma^\ell\gamma^i\gamma^j\right]\psi \ . \end{aligned} \tag{4.11.1}$$

We start by showing that
$$\left[\mathring{\omega}_{ij}(\hat{f}_l) - \omega_{ij}(\hat{f}_l)\right]\gamma^l\gamma^i\gamma^j = O(x^2) \ .$$

In order to do that, note first
$$\gamma^i\gamma^j\gamma^k\Delta_{jki} = \varepsilon^{ijk}\Delta_{jki}\gamma^1\gamma^2\gamma^3 - 2g^{ij}\gamma^k\Delta_{jki} \ ,$$
where
$$\Delta_{jki} := \mathring{\omega}_{jk}(\hat{f}_i) - \omega_{jk}(\hat{f}_i) = \hat{M}^l{}_i\mathring{\omega}_{jkl} - \omega_{jki} \ .$$
We claim that
$$\varepsilon^{ijk}\Delta_{jki} = O(x^2) \ , \quad \Delta_{jkk} = O(x^2) \ . \tag{4.11.2}$$
The intermediate calculations needed for this are as follows:
$$[\hat{e}_1,\hat{e}_2] = -x\cot\theta\hat{e}_2 + O_{\ln^* x}(x^4) \tag{4.11.3}$$
(which follows from (4.10.29)),
$$[\hat{e}_a,\hat{e}_3] = -\left[2\alpha_{,x}\right]^{-1/2}\hat{e}_a + O_{\ln^* x}(x^4) \ , \tag{4.11.4}$$
$$[\hat{f}_1,\hat{f}_2] = -x\cot\theta\hat{f}_2 + O(x^2) \ , \tag{4.11.5}$$
$$[\hat{f}_3,\hat{f}_a] = \left[2\alpha_{,x}\right]^{-1/2}\left(\hat{f}_a - \frac{1}{2}x\chi^b{}_a\hat{f}_b\right) + O(x^2) \ . \tag{4.11.6}$$
In order to calculate $\Delta_{123} + \Delta_{312} + \Delta_{231}$, we note that
$$\begin{aligned}\mathring{\omega}_{12}(\hat{f}_3) + \mathring{\omega}_{31}(\hat{f}_2) + \mathring{\omega}_{23}(\hat{f}_1) &= \mathring{\omega}_{123} + \mu_2{}^a\mathring{\omega}_{31a} + \mu_1{}^a\mathring{\omega}_{23a} + O(x^2) \\ &= (\mathring{\omega}_{123} + \mathring{\omega}_{312} + \mathring{\omega}_{231}) - \frac{1}{2}x\left(\chi_2{}^a\mathring{\omega}_{31a} + \chi_1{}^a\mathring{\omega}_{23a}\right) + O(x^2) \ .\end{aligned}$$

From $\mathring{\omega}_{3ab} = \left[2\alpha_{,x}\right]^{-1/2}\delta_{ab} + O_{\ln^* x}(x^3)$ the χ terms drops out. Equations (4.11.3)-(4.11.4) show that
$$2\mathring{\omega}_{123} + 2\mathring{\omega}_{312} + 2\mathring{\omega}_{231} = b([\hat{e}_2,\hat{e}_1],\hat{e}_3) + b([\hat{e}_3,\hat{e}_2],\hat{e}_1) + b([\hat{e}_1,\hat{e}_3],\hat{e}_2) = O(x^2) \ .$$

Similarly it follows from (4.11.5)-(4.11.6) that

$$2\omega_{123}+2\omega_{312}+2\omega_{231}=g([\hat f_2,\hat f_1],\hat f_3)+g([\hat f_3,\hat f_2],\hat f_1)+g([\hat f_1,\hat f_3],\hat f_2)=O(x^2)\ ,$$

yielding finally

$$\Delta_{123}+\Delta_{312}+\Delta_{231}=O(x^2)\ .$$

Next, $\Delta_{3kk}=\Delta_{3aa}$ is given by (4.10.48), and is thus $O(x^2)$. We continue with $\Delta_{akk}=\Delta_{abb}+\Delta_{a33}$. For $a=1$ one finds

$$\omega_{122}+\omega_{133}=g([\hat f_1,\hat f_2],\hat f_2)+g([\hat f_1,\hat f_3],\hat f_3)=-x\cot\theta+O(x^2)\ .$$

Further,

$$\mathring\omega_{11}(\hat f_1)+\mathring\omega_{12}(\hat f_2)+\mathring\omega_{13}(\hat f_3)=\mathring\omega_{133}+\mu_a{}^b\mathring\omega_{1ab}+O(x^2)=\mathring\omega_{1kk}-\frac12 x\chi_a{}^b\mathring\omega_{1ab}+O(x^2)\ .$$

We have $\mathring\omega_{122}=-x\cot\theta+O_{\ln^* x}(x^3)=-x\cot\theta+O(x^2)$, while $\mathring\omega_{1ab}=O(x^2)$ as well, so that the χ's can be absorbed in the error terms, leading to

$$\Delta_{1kk}=\mathring\omega_{11}(\hat f_1)+\mathring\omega_{12}(\hat f_2)+\mathring\omega_{13}(\hat f_3)-(\omega_{122}+\omega_{133})=O(x^2)\ .$$

For $a=2$ we calculate

$$\omega_{211}+\omega_{233}=g([\hat f_2,\hat f_1],\hat f_1)+g([\hat f_2,\hat f_3],\hat f_3)=O(x^2)\ ,$$

and it is easy to check now that $\Delta_{2kk}=O(x^2)$. This establishes (4.11.2).

To estimate the contribution to (4.11.1) of the terms involving K we will need the following expansions

$$d_{3a}{}^b=\frac{1}{\sqrt{2\alpha_{,x}}}\left[\delta^b{}_a-\frac12 x\chi^b{}_a\right]+O_{\ln^* x}(x^2)\ , \tag{4.11.7}$$

$$d_{3a}{}^3=-\frac{x^2b_a}{\sqrt{2\alpha_{,x}}}+O_{\ln^* x}(x^3)=O_{\ln^* x}(x^2)\ , \tag{4.11.8}$$

$$d_{ab}{}^3=x^2c_{ab}{}^cb_c+O_{\ln^* x}(x^3)=O(x^2)\ . \tag{4.11.9}$$

Now, it follows from [64, Appendix C.3] that

$$tr_gK-\mathring K(\hat f_i,\hat e_i)=O_{\ln^* x}(x^2)\ .$$

Next, we claim that

$$\mathring K(\hat f_j,\hat e_k)-\mathring K(\hat f_k,\hat e_j)=O_{\ln^* x}(x^2)\ .$$

We have the following asymptotic formulae for the connection coefficients:

$$\Gamma^\omega{}_{xx}=2x^{-1}\alpha_{,x}+O(x)\ ,\quad \Gamma^\omega{}_{xA}=O(x)\ ,$$

$$\Gamma^\omega{}_{AB}=x^{-1}\breve h_{AB}+\frac12\chi_{AB}+O(x)\ ,$$

and for the extrinsic curvature tensor:

$$K_{xx} = -N\Gamma^{\omega}{}_{xx} = -x^{-2}\left[\sqrt{2\alpha_{,x}} + O(x^2)\right] = \mathring{K}_{xx} ,$$

$$K_{xA} = -N\Gamma^{\omega}{}_{xA} = O(1) , \quad \mathring{K}_{xA} = O(x) ,$$

$$\begin{aligned} K_{AB} = -N\Gamma^{\omega}{}_{AB} &= -\frac{x^{-2}}{\sqrt{2\alpha_{,x}}}\left[\breve{h}_{AB} + \frac{1}{2}x\chi_{AB} + O(x^2)\right] , \\ \mathring{K}_{AB} &= -\frac{x^{-2}}{\sqrt{2\alpha_{,x}}}\left[\breve{h}_{AB} + O(x^2)\right] . \end{aligned} \tag{4.11.10}$$

From the above and [64, Appendix C.3] one obtains the following formulae

$$\mathring{K}(\hat{e}_k, \hat{e}_l) = -\frac{1}{\sqrt{2\alpha_{,x}}}\delta_{kl} + O_{\ln^* x}(x^2) , \tag{4.11.11}$$

$$K(\hat{e}_k, \hat{e}_l) = -\frac{1}{\sqrt{2\alpha_{,x}}}\left[\delta_{kl} + \frac{1}{2}x\chi_{kl}\right] + O_{\ln^* x}(x^2) , \tag{4.11.12}$$

where we have set $\chi_{3k} = 0$. Further

$$\hat{M}^l{}_k = \delta^l{}_k - \frac{1}{2}x\chi^l{}_k + O_{\ln^* x}(x^2) ,$$

$$\begin{aligned} &\mathring{K}(\hat{f}_j, \hat{e}_k) - \mathring{K}(\hat{f}_k, \hat{e}_j) = \hat{M}^l{}_j\mathring{K}(\hat{e}_l, \hat{e}_k) - \hat{M}^l{}_k\mathring{K}(\hat{e}_l, \hat{e}_j) \\ &\quad = -\frac{1}{\sqrt{2\alpha_{,x}}}\left(\delta^l{}_j - \frac{1}{2}x\chi^l{}_j\right)\delta_{lk} + \frac{1}{\sqrt{2\alpha_{,x}}}\left(\delta^l{}_k - \frac{1}{2}x\chi^l{}_k\right)\delta_{lj} + O_{\ln^* x}(x^2) \\ &\quad = O_{\ln^* x}(x^2) . \end{aligned}$$

This, together with [64, Equation (C.83)] yields

$$tr_g K - \mathring{K}(\hat{f}_i, \hat{e}_i) = -\frac{3}{\sqrt{2\alpha_{,x}}} + \frac{1}{\sqrt{2\alpha_{,x}}}\left(\delta^l{}_i - \frac{1}{2}x\chi^l{}_i\right)\delta_{li} + O_{\ln^* x}(x^2) = O_{\ln^* x}(x^2) .$$

We also have

$$\begin{aligned} \left[K(\hat{f}_l, \hat{f}_j) - \mathring{K}(\hat{f}_l, \hat{e}_j)\right]\gamma^l\gamma^j &= -K(\hat{f}_k, \hat{f}_k) - \mathring{K}(\hat{f}_l, \hat{e}_j)\gamma^l\gamma^j \\ &= -tr_g K + \mathring{K}(\hat{f}_k, \hat{e}_k) \\ &\quad + \frac{1}{2}\left(\mathring{K}(\hat{f}_j, \hat{e}_k) - \mathring{K}(\hat{f}_k, \hat{e}_j)\right)\gamma^k\gamma^j , \end{aligned}$$

so that

$$\left[K(\hat{f}_l, \hat{f}_j) - \mathring{K}(\hat{f}_l, \hat{e}_j)\right]\gamma^l\gamma^j = O_{\ln^* x}(x^2) .$$

Since

$$\sqrt{\det g} = O(x^3) , \quad < \psi, \psi >= O(x^{-1}) ,$$

we obtain

$$\sqrt{\det g}\left|\gamma^k\nabla_k\psi\right|^2 = O_{\ln^* x}(1) \in L^1 .$$

4.12 Asymptotic expansions of objects on $\mathscr{S}$

Throughout this section coordinate indices are used.

4.12.1 Smooth case

Induced metric g. We write the spacetime metric in Bondi-Sachs coordinates, as in (4.10.8), and use the standard expansions for the coefficients of the metric (see *e.g.* [64, Equations (5.98)-(5.101)]). Let $\mathscr{S}$ be given by $\omega = \text{const.}$, where

$$\omega = u - \alpha(x, x^A) .$$

Two different coordinate systems will be used: (u, x, x^A) and (ω, x, x^A) — coordinates adapted to $\mathscr{S}$. To avoid ambiguity two different symbols for partial derivatives will be used: the comma stands for the derivative with $\omega = \text{const.}$ and $\wr$ stands for the derivative with $u = \text{const.}$ These two derivatives can be transformed into each other:

$$A_{,x} = A_{\wr x} + \alpha_{\wr x}\partial_u A ,$$

$$A_{,A} = A_{\wr A} + \alpha_{\wr A}\partial_u A .$$

For functions not depending on u (*e.g.*, α) the symbols mean the same. Derivations in covariant derivatives ${}_{||A}$ and ${}_{|i}$ are with $\omega = \text{const.}$

Three-dimensional reciprocal metric. We have the following implicit formulae for the three dimensional inverse metric g^{ij}:

$$-\frac{g^{xA}}{g^{xx}} = {}^2g^{AB}g_{xB} ,$$

$$\frac{1}{g^{xx}} = g_{xx} + \frac{g^{xA}}{g^{xx}}g_{xA} ,$$

$$g^{AB} = {}^2g^{AB} + \frac{g^{xA}g^{xB}}{g^{xx}} ,$$

where ${}^2g^{AB}$ denotes the matrix inverse to (g_{AB}). The calculations get very complicated in general. To simplify them we will assume a particular form of α, *i.e.*,

$$\alpha = \text{const.} \cdot x + O(x^3) . \tag{4.12.1}$$

This choice is motivated by the form of α for standard hyperboloid $t^2 - r^2 = 1$ in Minkowski spacetime. In that case $\alpha = \frac{1}{x}\sqrt{1+x^2} - \frac{1}{x} = \frac{1}{2}x + O(x^3)$. It is further equivalent to the asymptotically CMC condition (4.5.3). With the above assumptions we have the following asymptotic expansions:

$${}^2g^{AB} = x^2 h^{AB} + O(x^7) ,$$

$$-\frac{g^{xA}}{g^{xx}} = \alpha^{,A} + x^2\alpha_{,x}\left(\frac{\chi^{AC}{}_{||C}}{2} - 2N^A x - \frac{(\chi^{CD}\chi_{CD})^{||A}}{16}x - \frac{\chi^{AB}\chi_B{}^C{}_{||C}}{2}x\right)$$
$$+O(x^4) ,$$

$$\frac{1}{g^{xx}} = \frac{2\alpha_{,x}}{x^2}\left[1+2\beta-\frac{x^2\alpha_{,x}}{2}+Mx^3\alpha_{,x}+O(x^4)\right] = g_{xx}\ ,$$

$$g^{xx} = \frac{x^2}{2\alpha_{,x}}\left[1-2\beta+\frac{x^2\alpha_{,x}}{2}-Mx^3\alpha_{,x}+O(x^4)\right]\ ,$$

$$\frac{1}{\sqrt{g^{xx}}} = \frac{\sqrt{2\alpha_{,x}}}{x}\left[1+\beta-\frac{1}{4}x^2\alpha_{,x}(1-2Mx)+O(x^4)\right]\ ,$$

$$-g^{Ax} = \frac{x^4}{2}\left[\frac{\chi^{AC}{}_{||C}}{4}-2xN^A-x\frac{(\chi^{CD}\chi_{CD})^{||A}}{16}-x\frac{\chi_C{}^A\chi^{CD}{}_{||D}}{2}+\frac{\alpha_{,A}}{x^2\alpha_{,x}}+O(x^2)\right]\ ,$$

$$g^{AB} = x^2h^{AB}+O(x^6)\ .$$

Determinant of the induced metric ^2g. The determinant of the 2-dimensional metric induced on the Bondi spheres $u=$ const., $x=$ const., equals

$$\det\left(\frac{1}{x^2}h\right) = \det\left(\frac{1}{x^2}\breve{h}\right)\ .$$

In our case the $\omega=$ const., $x=$ const. spheres are not exactly the Bondi ones, and one finds

$$\lambda := \sqrt{\det{}^2g} = \frac{\sqrt{\breve{h}}}{x^2}+O(x^3)\ ,\quad \mathring{\lambda} := \sqrt{\det{}^2b} = \frac{\sqrt{\breve{h}}}{x^2}+O(x^6)\ .$$

We also note the following purely algebraic identities:

$$\lambda = \sqrt{g^{xx}}\cdot\sqrt{\det g}\ ,\quad \mathring{\lambda} = \sqrt{b^{xx}}\cdot\sqrt{\det b}\ , \tag{4.12.2}$$

Lapse and shift. The function N (the *lapse*) and the vector S_i (the *shift*) can be calculated as follows:

$$N = \frac{1}{\sqrt{-\mathfrak{g}^{\omega\omega}}}\ ,$$

$$S_i = \mathfrak{g}_{\omega i}\ .$$

Hence

$$N = \frac{1}{x\sqrt{2\alpha_{,x}}}\left[1+\beta+\frac{1}{4}x^2\alpha_{,x}-\frac{1}{2}x^3\alpha_{,x}M+O(x^4)\right]\ ,$$

$$S_x = \frac{1}{x^2}\left[1+2\beta-x^2\alpha_{,x}+2x^3\alpha_{,x}M+O(x^4)\right]\ ,$$

$$S_A = \frac{1}{2}\chi^C{}_{A||C}-2xN_A-\frac{1}{16}x(\chi^{CD}\chi_{CD})_{||A}+O(x^2)\ .$$

Christoffel coeficients. Using the induced metric (and the reciprocal metric) we can compute the coefficients ${}^3\Gamma^i{}_{jk}$. The results are:

$$ {}^3\Gamma^x{}_{xx} = -\frac{1}{x} + \frac{\alpha_{,xx}}{2\alpha_{,x}} + \beta_{\wr x} + \alpha_{,x}\left(\frac{3}{2}x^2 M - \frac{1}{2}x + \partial_u\beta\right) + O(x^3) \, , $$

$$ \begin{aligned} {}^3\Gamma^x{}_{xA} \;&=\; \frac{1}{4}x\left[\chi^C{}_{A||C} - 4xN_A - \frac{1}{8}x(\chi^{CD}\chi_{CD})_{||A} - \frac{1}{2}x\chi_{AC}\chi^{CD}{}_{||D}\right. \\ &\left. + \frac{2\alpha_{,A}}{x^2\alpha_{,x}} + \frac{4\beta_{\wr A}}{x} + \frac{2\alpha_{,xA}}{x\alpha_{,x}} + O(x^2)\right] \, , \end{aligned} $$

$$ {}^3\Gamma^x{}_{AB} = \frac{\breve{h}_{AB}}{2x\alpha_{,x}} + \frac{\chi_{AB}}{4\alpha_{,x}} + \frac{1}{4}x\breve{h}_{AB} - \frac{1}{4}x\partial_u\chi_{AB} - \frac{\beta\breve{h}_{AB}}{x\alpha_{,x}} + O(x^2) \, , $$

$$ {}^3\Gamma^A{}_{xx} = \frac{1}{2}x\alpha_{,x}\chi^{AC}{}_{||C} + O(x^2) \, , $$

$$ \begin{aligned} {}^3\Gamma^A{}_{xB} \;&=\; -\frac{1}{x}\delta^A{}_B + \frac{1}{2}\chi^A{}_B + \frac{1}{2}x\alpha_{,x}\partial_u\chi^A{}_B + \frac{1}{2}x^2\left(3H^A{}_B - \frac{\chi^{CD}\chi_{CD}}{4}\chi^A{}_B\right) \\ &+ \frac{1}{4}x^2\alpha_{,x}\left(\chi_{BC}\partial_u\chi^{AC} - \chi^{AC}\partial_u\chi_{BC} + \chi^{AD}{}_{||DB} - \chi_{BD}{}^{||DA}\right) + O(x^3) \, , \end{aligned} $$

$$ {}^3\Gamma^A{}_{BC} = \Gamma(\breve{h})^A{}_{BC} + O(x) \, . $$

Extrinsic curvature. The tensor field K_{ij} can be computed as

$$ K_{ij} = \frac{1}{2N}(S_{i|j} + S_{j|i} - \partial_u g_{ij}) \, , $$

where by $A_{|i}$ we denote the covariant derivative of a quantity A with respect to g_{ij}. These covariant derivatives read:

$$ \begin{aligned} S_{x|x} \;&=\; \frac{1}{x^3}\left[-1 - \frac{x\alpha_{,xx}}{2\alpha_{,x}} + x\beta_{\wr x} - 2\beta - \frac{1}{2}x^2\alpha_{,x}\right. \\ &\left. + \frac{5}{2}x^3\alpha_{,x}M + x\alpha_{,x}\partial_u\beta + O(x^4)\right] \, , \end{aligned} $$

$$ \begin{aligned} S_{x|A} \;&=\; \frac{\beta_{\wr A}}{x^2} - \frac{\alpha_{,A}}{2x^3\alpha_{,x}} - \frac{\alpha_{,xA}}{2x^2\alpha_{,x}} + \frac{1}{4x}\chi^C{}_{A||C} - N_A \\ &- \frac{1}{32}(\chi^{CD}\chi_{CD})_{||A} - \frac{1}{8}\chi_{AC}\chi^{CD}{}_{||D} + O(x) \, , \end{aligned} $$

$$ \begin{aligned} S_{A|x} \;&=\; -\frac{\beta_{\wr A}}{x^2} - \frac{\alpha_{,A}}{2x^3\alpha_{,x}} - \frac{\alpha_{,xA}}{2x^2\alpha_{,x}} + \frac{1}{4x}\chi^C{}_{A||C} - 3N_A \\ &- \frac{3}{32}(\chi^{CD}\chi_{CD})_{||A} - \frac{1}{8}\chi_{AC}\chi^{CD}{}_{||D} \\ &+ \frac{1}{2}\alpha_{,x}\partial_u(\chi^C{}_{A||C}) + O(x) \, , \end{aligned} $$

$$S_{A|B} = -\frac{1}{x^2}\left[\frac{\breve{h}_{AB}}{2x\alpha_{,x}} + \frac{\chi_{AB}}{4\alpha_{,x}} - \frac{1}{4}x\breve{h}_{AB} - \frac{1}{4}x\partial_u\chi_{AB} + O(x^2)\right] .$$

Derivatives of g_{ij} with respect to u :

$$\partial_u g_{xx} = 4x^{-2}\alpha_{,x}\partial_u\beta + O(x) ,$$

$$\partial_u g_{xA} = \frac{1}{2}\alpha_{,x}\partial_u(\chi^C{}_{A||C}) + O(x) ,$$

$$\partial_u g_{AB} = x^{-1}\partial_u\chi_{AB} + O(1) .$$

Hence the extrinsic curvature:

$$\begin{aligned} K_{xx} &= \frac{\sqrt{2\alpha_{,x}}}{x^2}\Big[-1 - \frac{x\alpha_{,xx}}{2\alpha_{,x}} + x\beta_{\wr x} - \beta - \frac{1}{4}x^2\alpha_{,x} \\ &\quad + 2x^3\alpha_{,x}M - x\alpha_{,x}\partial_u\beta + O(x^4)\Big] , \end{aligned}$$

$$\begin{aligned} K_{xA} &= \frac{\sqrt{2\alpha_{,x}}}{2}\Big[\frac{1}{2}\chi^C{}_{A||C} + \frac{\alpha_{,A}}{x^2\alpha_{,x}} - \frac{\alpha_{,xA}}{x\alpha_{,x}} - 4xN_A \\ &\quad - \frac{1}{8}x(\chi^{CD}\chi_{CD})_{||A} - \frac{1}{4}x\chi_{AC}\chi^{CD}{}_{||D} + O(x^2)\Big] , \end{aligned}$$

$$K_{AB} = -\frac{\sqrt{2\alpha_{,x}}}{2x}\left[\frac{\breve{h}_{AB}}{x\alpha_{,x}} - \frac{\beta\breve{h}_{AB}}{x\alpha_{,x}} + \frac{\chi_{AB}}{2\alpha_{,x}} - \frac{3}{4}x\breve{h}_{AB} + \frac{1}{2}x\partial_u\chi_{AB} + O(x^2)\right] .$$

The trace $\mathrm{tr}_g K = K^i{}_i$ can be easily calculated more precisely as

$$\mathrm{tr}_g K = \frac{1}{2N}[2S^i{}_{|i} - g^{ij}\partial_u g_{ij}] ,$$

or after multiplying by $\sqrt{\det g}$ and changing the covariant divergence to the ordinary one:

$$\sqrt{\det g}\,\mathrm{tr}_g K = \frac{1}{2N}\left[2(\sqrt{\det g}S^i)_{,i} - \frac{\partial_u(\det g)}{\sqrt{\det g}}\right] .$$

After some calculations we get

$$\begin{aligned} \mathrm{tr}_g K &= -\tfrac{1}{\sqrt{2\alpha_{,x}}}\Big[3 - 3\beta - x\beta_{\wr x} + \frac{x\alpha_{,xx}}{2\alpha_{,x}} - \frac{3}{4}x^2\alpha_{,x} \\ &\quad - \frac{1}{2}x^3\alpha_{,x}\chi^{CD}{}_{||CD} + x\alpha_{,x}\beta_{,u} + O(x^4)\Big] . \end{aligned}$$

ADM momenta. The ADM momenta can be expressed in terms of the extrinsic curvature:

$$P^k{}_l = g^{ki}(g_{il}\mathrm{tr}_g K - K_{il}) .$$

Substituting the previously calculated $\mathrm{tr}_g K$ and K_{ij} we get

$$P^x{}_x = -\frac{1}{\sqrt{2\alpha_{,x}}}\left[2 - 2\beta - \frac{3}{2}x^2\alpha_{,x}(1 - 2Mx) - \frac{1}{2}x^3\alpha_{,x}\chi^{CD}{}_{||CD} + O(x^4)\right] ,$$

$$P^x{}_x - \mathring{P}^x{}_x = \frac{-1}{\sqrt{2\alpha_{,x}}}\left[-2\beta + 3x^3\alpha_{,x}M - \frac{1}{2}x^3\alpha_{,x}\chi^{CD}{}_{||CD} + O(x^4)\right] , \quad (4.12.3)$$

$$\begin{aligned} P^A{}_B &= -\frac{1}{\sqrt{2\alpha_{,x}}}\left[(2 - 2\beta - x\beta_{\wr x} + \frac{x\alpha_{,xx}}{2\alpha_{,x}})\delta^A{}_B + \frac{1}{2}x\chi^A{}_B - \frac{1}{2}x^2\alpha_{,x}\partial_u\chi^A{}_B\right] \\ &\quad +O(x^3) , \end{aligned}$$

$$P^A{}_x = -\frac{1}{2}x^2\sqrt{2\alpha_{,x}}\chi^{AC}{}_{||C} + O(x^3) ,$$

$$\begin{aligned} P^x{}_A &= \frac{-x^3}{2\sqrt{2\alpha_{,x}}}\left[\frac{\chi^C{}_{A||C}}{x} - \frac{\alpha_{,xA}}{x^2\alpha_{,x}} - 6N_A - \frac{3(\chi^{CD}\chi_{CD})_{||A}}{16} - \frac{1}{2}\chi_{AC}\chi^{CD}{}_{||D}\right] \\ &\quad +O(x^4) , \end{aligned}$$

$$\begin{aligned} P^x{}_A - \mathring{P}^x{}_A &= \frac{-x^3}{2\sqrt{2\alpha_{,x}}}\left[\frac{\chi^C{}_{A||C}}{x} - 6N_A - \frac{3(\chi^{CD}\chi_{CD})_{||A}}{16} - \frac{1}{2}\chi_{AC}\chi^{CD}{}_{||D}\right] \\ &\quad +O(x^4) . \end{aligned} \quad (4.12.4)$$

The second fundamental form k_{AB}. The extrinsic curvature of the leaves of the "2+1 foliation" can be computed from the formula

$$k_{AB} = \frac{{}^3\Gamma^x{}_{AB}}{\sqrt{g^{xx}}} .$$

Hence

$$\begin{aligned} k_{AB} &= \frac{\sqrt{2\alpha_{,x}}}{2x}\left[\frac{1}{x\alpha_{,x}}\breve{h}_{AB} + \frac{1}{2\alpha_{,x}}\chi_{AB}\right. \\ &\quad \left. + \frac{1}{4}x\breve{h}_{AB} - \frac{1}{2}x\partial_u\chi_{AB} - \frac{\beta}{x\alpha_{,x}}\breve{h}_{AB} + O(x^2)\right] . \end{aligned}$$

We need a more accurate expansion of the trace $k = {}^2g^{AB}k_{AB}$. The formula

$$k_{AB} = \frac{\Gamma^x{}_{AB}}{\sqrt{g^{xx}}} - \frac{\mathfrak{g}^{x\omega}\Gamma^\omega{}_{AB}}{\mathfrak{g}^{\omega\omega}\sqrt{g^{xx}}} .$$

can be used. It is convenient to calculate ${}^2g^{AB}\Gamma^x{}_{AB}$ and ${}^2g^{AB}\Gamma^\omega{}_{AB}$ using the expressions for the Christoffel symbols given in [64, Appendix C]:

$$\Gamma^\omega{}_{AB} = x^{-1}\mathrm{e}^{-2\beta}(h_{AB} - \frac{1}{2}xh_{AB\wr x}) - \alpha_{,x}\Gamma^x{}_{AB} + O(x^3) ,$$

$$\Gamma^x{}_{AB} = -\frac{1}{2}\mathrm{e}^{-2\beta}(2\mathcal{D}(h)_{(A}U_{B)} + \partial_u h_{AB} - 2Vx^2h_{AB} + Vx^3h_{AB\wr x}) ,$$

where $\mathcal{D}(h)_A$ is a covariant derivative with respect to h_{AB} (we use the (u, x, x^A) coordinate system and all the differentiations with respect to x, x^A are at constant u). Hence:

$${}^2g^{AB}\Gamma^\omega{}_{AB} = 2x - 4\beta x + x^2\alpha_{,x}(U^A{}_{||A} - 2Vx^2) + O(x^5) ,$$

$$ {}^2g^{AB}\Gamma^x{}_{AB} = -x^2[U^A{}_{||A} - 2Vx^2 + O(x^3)] \ . $$

After substitution of relevant asymptotic expansions:

$$ k = \frac{1}{\sqrt{g^{xx}}}\left[\frac{1}{4}x^4\chi^{CD}{}_{||CD} + \frac{1}{2}x^3(1-2Mx) + \frac{x}{\alpha_{,x}} - \frac{2\beta x}{\alpha_{,x}} + O(x^5)\right] \ , $$

$$ k = \sqrt{2\alpha_{,x}}\left[\frac{1}{4}x^3\chi^{CD}{}_{||CD} + \frac{1}{4}x^2(1-2Mx) + \frac{1}{\alpha_{,x}} - \frac{\beta}{\alpha_{,x}} + O(x^4)\right] \ , $$

$$ k - \mathring{k} = \sqrt{2\alpha_{,x}}\left[\frac{1}{4}x^3\chi^{CD}{}_{||CD} - \frac{1}{2}x^3 M - \frac{\beta}{\alpha_{,x}} + O(x^4)\right] \ . \qquad (4.12.5) $$

4.12.2 The polyhomogenous case

We give only the most important intermediate results which differ from the power-series case:

$$ g^{xB} = -\frac{1}{4}x^4\chi^{AC}{}_{||C} + \frac{1}{2}x^5W^A + O(x^5) \ , $$

$$ S_A = \frac{1}{2}\chi^C{}_{A||C} - xW_A + O(x) \ , $$

$$ K_{xA} = \sqrt{2\alpha_{,x}}[\frac{1}{4}\chi^C{}_{A||C} - xW_A - \frac{1}{2}x^2W_{A\wr x} + O(x)] \ , $$

$$ P^x{}_A = -\frac{x^2}{2\sqrt{2\alpha_{,x}}}[\chi^C{}_{A||C} - 3xW_A - x^2W_{A\wr x} + O(x)] \ . $$

4.13 Decomposition of Poincaré group vectors into tangential and normal parts

The generators of Poincaré group are given after [64]:

$$ X_{time} = \partial_\omega = \partial_u \ , $$

$$ X_{rot} = -\varepsilon^{AB}\alpha_{,A}v_{,B}\partial_\omega + \varepsilon^{AB}v_{,B}\partial_A \ , $$

$$ X_{trans} = (-v - x\alpha^{,A}v_{,A} + x^2v\alpha_{,x})\partial_\omega - x^2v\partial_x + xv^{,A}\partial_A \ , $$

$$ \begin{aligned} X_{boost} &= [xv((\alpha+\omega)x+1)\alpha_{,x} - (\alpha+\omega)v \\ &- \alpha^{,A}v_{,A}((\alpha+\omega)x+1)]\partial_\omega \\ &- xv[(\alpha+\omega)x+1]\partial_x + v^{,A}[(\alpha+\omega)x+1]\partial_A \ . \end{aligned} $$

The tensor ε^{AB} is defined as

$$ \varepsilon^{AB} = \frac{1}{\sqrt{\breve{h}}}\{A,B\} \ , $$

where $\{1,2\} = -\{2,1\} = 1$ and $\{1,1\} = \{2,2\} = 0$. By v we denote a function on the sphere which is a combination of $\ell = 1$ spherical harmonics. If we consider embedding of the sphere into $\mathbb{R}^3$, then

$$v(x^A) = v^i \frac{x_i}{r}$$

and we get a bijection between functions v and vectors $(v^i) \in \mathbb{R}^3$.

Let us now decompose a vector field X into parts tangent and normal to $\mathscr{S}$:

$$X = Y + Vn \, .$$

Y is a vector tangent to $\mathscr{S}$ and n is a unit ($n^2 = -1$), future-directed normal vector. Setting

$$\tau = 2\alpha_{,x} - \breve{h}^{AB}\alpha_{,A}\alpha_{,B} \, ,$$

we have the following decomposition of respective vectors :

$$V_{time} = \frac{1}{x\sqrt{\tau}}\left(1 - \frac{x\chi^{CD}\alpha_{,C}\alpha_{,D}}{2\tau} + O(x^2)\right) \, ,$$

$$Y^x_{time} = \frac{1}{\tau}\left(1 - \frac{x\chi^{CD}\alpha_{,C}\alpha_{,D}}{\tau} + O(x^2)\right) \, ,$$

$$Y^A_{time} = -\frac{1}{\tau}\left(a^{,A} - x\chi^{AC}\alpha_{,C} - \frac{x\chi^{CD}\alpha_{,C}\alpha_{,D}\alpha^{,A}}{\tau} + O(x^2)\right) \, ,$$

$$V_{rot} = -\varepsilon^{AB}\alpha_{,A}v_{,B}\cdot\frac{1}{x\sqrt{\tau}}\left(1 - \frac{x\chi^{CD}\alpha_{,C}\alpha_{,D}}{2\tau} + O(x^2)\right) \, ,$$

$$Y^x_{rot} = -\varepsilon^{AB}\alpha_{,A}v_{,B}\cdot\frac{1}{\tau}\left(1 - \frac{x\chi^{CD}\alpha_{,C}\alpha_{,D}}{\tau} + O(x^2)\right) \, ,$$

$$\begin{aligned} Y^A_{rot} &= \varepsilon^{AB}v_{,B} \\ &\quad + \varepsilon^{CB}\alpha_{,C}v_{,B}\cdot\frac{1}{\tau}\left(\alpha^{,A} - x\chi^{AC}\alpha_{,C} - \frac{x\chi^{CD}\alpha_{,C}\alpha_{,D}\alpha^{,A}}{\tau} + O(x^2)\right) \, , \end{aligned}$$

$$V_{trans} = \frac{1}{x\sqrt{\tau}}\left(-v - x\alpha^{,A}v_{,A} + v\frac{x\chi^{CD}\alpha_{,C}\alpha_{,D}}{2\tau} + O(x^2)\right) \, ,$$

$$Y^x_{trans} = \frac{1}{\tau}\left(-v - x\alpha^{,A}v_{,A} + v\frac{x\chi^{CD}\alpha_{,C}\alpha_{,D}}{\tau} + O(x^2)\right) \, ,$$

$$\begin{aligned} Y^A_{trans} &= xv^{,A} \\ &\quad - \frac{1}{\tau}\left(-v\alpha^{,A} + vx\chi^{AC}\alpha_{,C} + v\frac{x\chi^{CD}\alpha_{,C}\alpha_{,D}}{\tau}\alpha^{,A} - x\alpha^{,C}v_{,C}\alpha^{,A} + O(x^2)\right) \, , \end{aligned}$$

$$\begin{aligned} V_{boost} &= \frac{1}{x\sqrt{\tau}}\Big[- (\omega+\alpha)v + xv\alpha_{,x} - v^{,C}\alpha_{,C}(\omega x + \alpha x + 1) \\ &\quad + \frac{x\chi^{CD}\alpha_{,C}\alpha_{,D}}{2\tau}((\omega+\alpha)v + v^{,C}\alpha_{,C}) + O(x^2)\Big] \, , \end{aligned}$$

$$\begin{aligned} Y^x_{boost} &= -xv + \frac{1}{\tau}\Big[-(\omega+\alpha)v + xv\alpha_{,x} - v^{,C}\alpha_{,C}(\omega x + \alpha x + 1) \\ &\quad + \frac{x\chi^{CD}\alpha_{,C}\alpha_{,D}}{\tau}((\omega+\alpha)v + v^{,C}\alpha_{,C}) + O(x^2)\Big] \,, \end{aligned}$$

$$\begin{aligned} Y^A_{boost} &= v^{,A}(\omega x + \alpha x + 1) \\ &\quad + \frac{1}{\tau}\Big[(\omega+\alpha)v\alpha^{,A} - xv\alpha_{,x}\alpha^{,A} + v^{,C}\alpha_{,C}\alpha^{,A}(\omega x + \alpha x + 1) \\ &\quad - x(\frac{\chi^{CD}\alpha_{,C}\alpha_{,D}\alpha^{,A}}{\tau} + \chi^{AC}\alpha_{,C})((\omega+\alpha)v + v^{,C}\alpha_{,C}) + O(x^2)\Big] \,. \end{aligned}$$

Part II

Background Material

Appendix A

Introduction to pseudo-Riemannian geometry

A.1 Manifolds

DEFINITION A.1.1 *An n–dimensional manifold is a set M equipped with the following:*

1. *topology: a "connected Hausdorff paracompact topological space" (think of nicely looking subsets of $\mathbb{R}^{1+n}$, like spheres, hyperboloids, and such), together with*
2. *local charts: a collection of coordinate patches $(\mathscr{U}, x^i)$ covering M, where $\mathscr{U}$ is an open subset of M, with the functions $x^i : \mathscr{U} \to \mathbb{R}^n$ being continuous. One further requires that the maps*

   $$M \supset \mathscr{U} \ni p \mapsto (x^1(p), \ldots, x^n(p)) \in \mathscr{V} \subset \mathbb{R}^n$$

   are homeomorphisms.
3. *compatibility: given two overlapping coordinate patches, $(\mathscr{U}, x^i)$ and $(\widetilde{\mathscr{U}}, \tilde{x}^i)$, with corresponding sets $\mathscr{V}, \widetilde{\mathscr{V}} \subset \mathbb{R}^n$, the maps $\tilde{x}^j \mapsto x^i(\tilde{x}^j)$ are smooth diffeomorphisms wherever defined: this means that they are bijections differentiable as many times as one wishes, with*

   $$\det\left[\frac{\partial x^i}{\partial \tilde{x}^j}\right] \text{ nowhere vanishing}\,.$$

Definition of differentiability: A function on M is smooth if it is smooth when expressed in terms of local coordinates. Similarly for tensors.

EXAMPLES:
1. $\mathbb{R}^n$ with the usual topology, one single global coordinate patch.
2. A sphere: use stereographic projection to obtain two overlapping coordinate systems (or use spherical angles, but then one must avoid borderline angles, so they don't cover the whole manifold!).

3. We will use several coordinate patches (in fact, five), to describe the Schwarzschild black hole, though one spherical coordinate system would suffice.

4. Let $f : \mathbb{R}^n \to \mathbb{R}$, and define $N := f^{-1}(0)$. If ∇f has no zeros on N, then every connected component of N is a smooth $(n-1)$–dimensional manifold. This construction leads to a plethora of examples. For example, if $f = \sqrt{(x^1)^2 + \ldots + (x^n)^2} - R$, with $R > 0$, then N is a sphere of radius R.

In this context a useful example is provided by the function $f = t^2 - x^2$ on $\mathbb{R}^2$: its zero-level-set is the light-cone $t = \pm x$, which is a manifold except at the origin; note that $\nabla f = 0$ there, which shows that the criterion is sharp.

A.2 Scalar functions

Let M be an n-dimensional manifold. Since manifolds are defined using coordinate charts, we need to understand how things behave under coordinate changes. For instance, under a change of coordinates $x^i \to y^j(x^i)$, to a function $f(x)$ we can associate a new function $\bar{f}(y)$, using the rule

$$\bar{f}(y) = f(x(y)) \quad \Longleftrightarrow \quad f(x) = \bar{f}(y(x)) \;.$$

In general relativity it is a common abuse of notation to write the same symbol f for what we wrote $\bar{f}$, when we think that this is the same function but expressed in a different coordinate system. We then say that a real- or complex-valued f is a *scalar function* when, under a change of coordinates $x \to y(x)$, the function f transforms as $f \to f(x(y))$.

In this section, to make things clearer, we will write $\bar{f}$ for $f(x(y))$ even when f is a scalar, but this will almost never be done in the remainder of these notes. For example we will systematically use the same symbol $g_{\mu\nu}$ for the metric components, whatever the coordinate system used.

A.3 Vector fields

Physicists often think of vector fields in terms of coordinate systems: a vector field X is an object which in a coordinate system $\{x^i\}$ is represented by a collection of functions X^i. In a new coordinate system $\{y^j\}$ the field X is represented by a new set of functions:

$$X^i(x) \to X^j(y) := X^j(x(y))\frac{\partial y^i}{\partial x^j}(x(y)) \;. \tag{A.3.1}$$

(The summation convention is used throughout, so that the index j has to be summed over.)

The notion of a vector field finds its roots in the notion of the tangent to a curve, say $s \to \gamma(s)$. If we use local coordinates to write $\gamma(s)$ as $(\gamma^1(s), \gamma^2(s), \ldots, \gamma^n(s))$, the tangent to that curve at the point $\gamma(s)$ is defined as the set of numbers

$$(\dot\gamma^1(s), \dot\gamma^2(s), \ldots, \dot\gamma^n(s)) \;.$$

Consider, then, a curve $\gamma(s)$ given in a coordinate system x^i and let us perform a change of coordinates $x^i \to y^j(x^i)$. In the new coordinates y^j the curve γ is represented by the functions $y^j(\gamma^i(s))$, with new tangent

$$\frac{dy^j}{ds}(y(\gamma(s))) = \frac{\partial y^j}{\partial x^i}(\gamma(s))\dot\gamma^i(s) \; .$$

This motivates (A.3.1).

In modern differential geometry a different approach is taken: one identifies vector fields with homogeneous first order differential operators acting on real valued functions $f : M \to \mathbb{R}$. In local coordinates $\{x^i\}$ a vector field X will be written as $X^i\partial_i$, where the X^i's are the "physicists's functions" just mentioned. This means that the action of X on functions is given by the formula

$$\boxed{X(f) := X^i\partial_i f} \tag{A.3.2}$$

(recall that ∂_i is the partial derivative with respect to the coordinate x^i). Conversely, given some abstract first order homogeneous derivative operator X, the (perhaps locally defined) functions X^i in (A.3.2) can be found by acting on the coordinate functions:

$$X(x^i) = X^i \; . \tag{A.3.3}$$

One justification for the differential operator approach is the fact that the tangent $\dot\gamma$ to a curve γ can be calculated — in a way independent of the coordinate system $\{x^i\}$ chosen to represent γ — using the equation

$$\dot\gamma(f) := \frac{d(f\circ\gamma)}{dt} \; .$$

Indeed, if γ is represented as $\gamma(t) = \{x^i = \gamma^i(t)\}$ within a coordinate patch, then we have

$$\frac{d(f\circ\gamma)(t)}{dt} = \frac{d(f(\gamma(t)))}{dt} = \frac{d\gamma^i(t)}{dt}(\partial_i f)(\gamma(t)) \; ,$$

recovering the previous coordinate formula $\dot\gamma = (d\gamma^i/dt)$.

An even better justification is that *the transformation rule (A.3.1) becomes implicit in the formalism.* Indeed, consider a (scalar) function f, so that the differential operator X acts on f by differentiation:

$$X(f)(x) := \sum_i X^i \frac{\partial f(x)}{\partial x^i} \; . \tag{A.3.4}$$

If we make a coordinate change so that

$$x^j = x^j(y^k) \quad \Longleftrightarrow \quad y^k = y^k(x^j) \; ,$$

keeping in mind that

$$\bar f(y) = f(x(y)) \quad \Longleftrightarrow \quad f(x) = \bar f(y(x)) \; ,$$

then

$$
\begin{aligned}
X(f)(x) &:= \sum_i X^i(x)\frac{\partial f(x)}{\partial x^i} \\
&= \sum_i X^i(x)\frac{\partial \bar f(y(x))}{\partial x^i} \\
&= \sum_{i,k} X^i(x)\frac{\partial \bar f(y(x))}{\partial y^k}\frac{\partial y^k}{\partial x^i}(x) \\
&= \sum_k \bar X^k(y(x))\frac{\partial \bar f(y(x))}{\partial y^k} \\
&= \left(\sum_k \bar X^k \frac{\partial \bar f}{\partial y^k}\right)(y(x)) \ ,
\end{aligned}
$$

with $\bar X^k$ given by the right-hand-side of (A.3.1). So

$\boxed{X(f) \text{ is a scalar iff the coefficients } X^i \text{ satisfy the transformation law of a vector.}}$

EXERCICE A.3.1 Check that this is a necessary and sufficient condition.

One often uses the middle formula in the above calculation in the form

$$
\frac{\partial}{\partial x^i} = \frac{\partial y^k}{\partial x^i}\frac{\partial}{\partial y^k} \ . \tag{A.3.5}
$$

Note that the tangent to the curve $s \to (s, x^2, x^3, \ldots x^n)$, where $(x^2, x^3, \ldots x^n)$ are constants, is identified with the differential operator

$$
\partial_1 \equiv \frac{\partial}{\partial x^1} \ .
$$

Similarly the tangent to the curve $s \to (x^1, s, x^3, \ldots x^n)$, where $(x^1, x^3, \ldots x^n)$ are constants, is identified with

$$
\partial_2 \equiv \frac{\partial}{\partial x^2} \ ,
$$

etc. Thus, $\dot\gamma$ is identified with

$$
\dot\gamma(s) = \dot\gamma^i \partial_i
$$

At any given point $p \in M$ the set of vectors forms a vector space, denoted by T_pM. The collection of all the tangent spaces is called the tangent bundle to M, denoted by TM.

A.3.1 Lie bracket

Vector fields can be added and multiplied by functions in the obvious way. Another useful operation is the *Lie bracket*, or *commutator*, defined as

$$\boxed{[X,Y](f) := X(Y(f)) - Y(X(f))}\,. \tag{A.3.6}$$

One needs to check that this does indeed define a new vector field: the simplest way is to use local coordinates,

$$\begin{aligned}
[X,Y](f) &= X^j\partial_j(Y^i\partial_i f) - Y^j\partial_j(X^i\partial_i f)\\
&= X^j(\partial_j(Y^i)\partial_i f + Y^i\partial_j\partial_i f) - Y^j(\partial_j(X^i)\partial_i f + X^i\partial_j\partial_i f)\\
&= (X^j\partial_j Y^i - Y^j\partial_j X^i)\partial_i f + \underbrace{X^jY^i\partial_j\partial_i f - Y^jX^i\partial_j\partial_i f}_{=X^jY^i\underbrace{(\partial_j\partial_i f - \partial_i\partial_j f)}_{0}}\\
&= (X^j\partial_j Y^i - Y^j\partial_j X^i)\partial_i f\,,
\end{aligned} \tag{A.3.7}$$

which is indeed a homogeneous first order differential operator. Here we have used the symmetry of the matrix of second derivatives of twice differentiable functions. We note that the last line of (A.3.7) also gives an explicit coordinate expression for the commutator of two differentiable vector fields.

The Lie bracket satisfies the *Jacobi identity*:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0\,.$$

Indeed, if we write $S_{X,Y,Z}$ for a cyclic sum, then

$$\begin{aligned}
&([X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]])(f) = S_{X,Y,Z}[X,[Y,Z]](f)\\
&\quad = S_{X,Y,Z}\left\{X([Y,Z](f)) - [Y,Z](X(f))\right\}\\
&\quad = S_{X,Y,Z}\left\{X(Y(Z(f))) - X(Z(Y(f))) - Y(Z(X(f))) + Z(Y(X(f)))\right\}\,.
\end{aligned}$$

The third term is a cyclic permutation of the first, and the fourth a cyclic permutation of the second, so the sum gives zero.

A.4 Covectors

Covectors are *maps from the space of vectors to functions which are linear under addition and multiplication by functions.*

The basic object is the *coordinate differential* dx^i, defined by its action on vectors as follows:

$$dx^i(X^j\partial_j) := X^i\,. \tag{A.4.1}$$

Equivalently,

$$dx^i(\partial_j) := \delta^i_j := \begin{cases} 1, & i=j;\\ 0, & \text{otherwise.}\end{cases}$$

The dx^i's form a basis for the space of covectors: indeed, let φ be a linear map on the space of vectors, then

$$\varphi(\underbrace{X}_{X^i\partial_i}) = \varphi(X^i\partial_i) \underbrace{=}_{\text{linearity}} X^i \underbrace{\varphi(\partial_i)}_{\text{call this }\varphi_i} = \varphi_i dx^i(X) \underbrace{=}_{\text{def. of sum of functions}} (\varphi_i dx^i)(X)\,,$$

hence

$$\varphi = \varphi_i dx^i \ ,$$

and every φ can indeed be written as a linear combination of the dx^i's. Under a change of coordinates we have

$$\bar\varphi_i \bar X^i = \bar\varphi_i \frac{\partial y^i}{\partial x^k} X^k = \varphi_k X^k \ ,$$

leading to the following transformation law for components of covectors:

$$\varphi_k = \bar\varphi_i \frac{\partial y^i}{\partial x^k} \ , \tag{A.4.2}$$

Given a scalar f, we define its *differential* df as

$$df = \frac{\partial f}{\partial x^1} dx^1 + \ldots + \frac{\partial f}{\partial x^n} dx^n \ .$$

With this definition, dx^i *is* the differential of the coordinate function x^i.

As presented above, the differential of a function is a covector by definition. As an exercice, you should check directly that the collection of functions $\varphi_i := \partial_i f$ satisfies the transformation rule (A.4.2).

We have a formula which is often used in calculations

$$dy^j = \frac{\partial y^j}{\partial x^k} dx^k \ .$$

An elegant approach to the definition of differentials proceeds as follows: Given any function f, we define:

$$df(X) := X(f) \ . \tag{A.4.3}$$

(Recall that here we are viewing a vector field X as a differential operator on functions, defined by (A.3.4).) The map $X \mapsto df(X)$ is linear under addition of vectors, and multiplication of vectors by numbers: if λ, μ are real numbers, and X and Y are vector fields, then

$$\begin{aligned} df(\lambda X + \mu Y) & \underbrace{=}_{\text{by definition (A.4.3)}} (\lambda X + \mu Y)(f) \\ & \underbrace{=}_{\text{by definition (A.3.4)}} \lambda X^i \partial_i f + \mu Y^i \partial_i f \\ & \underbrace{=}_{\text{by definition (A.4.3)}} \lambda df(X) + \mu df(Y) \ . \end{aligned}$$

Applying (A.4.3) to the function $f = x^i$ we obtain

$$dx^i(\partial_j) = \frac{\partial x^i}{\partial x^j} = \delta^i_j \ ,$$

recovering (A.4.1).

EXAMPLE A.4.1 Let (ρ, φ) be polar coordinates on $\mathbb{R}^2$, thus $x = \rho \cos \varphi$, $y = \rho \sin \varphi$, and then

$$dx = d(\rho \cos \varphi) = \cos \varphi d\rho - \rho \sin \varphi d\varphi \ ,$$
$$dy = d(\rho \sin \varphi) = \sin \varphi d\rho + \rho \cos \varphi d\varphi \ .$$

At any given point $p \in M$, the set of covectors forms a vector space, denoted by $T^*_p M$. The collection of all the tangent spaces is called the cotangent bundle to M, denoted by T^*M.

Summarising, covectors are dual to vectors. It is convenient to define

$$\boxed{dx^i(X) := X^i} \ ,$$

where X^i is as in (A.3.2). With this definition the (locally defined) bases $\{\partial_i\}_{i=1,\dots,\dim M}$ of TM and $\{dx^j\}_{i=1,\dots,\dim M}$ of T^*M are dual to each other:

$$\langle dx^i, \partial_j \rangle := dx^i(\partial_j) = \delta^i_j \ ,$$

where δ^i_j is the Kronecker delta, equal to one when $i = j$ and zero otherwise.

A.5 Bilinear maps, two-covariant tensors

A map is said to be multi-linear if it is linear in every entry; e.g. g is bilinear if

$$g(aX + bY, Z) = ag(X, Z) + bg(Y, Z) \ ,$$

and

$$g(X, aZ + bW) = ag(X, Z) + bg(X, W) \ .$$

Here, as elsewhere when talking about *tensors*, bilinearity is meant with respect to addition and to multiplication by functions.

A map g which is bilinear on the space of vectors can be represented by a matrix with two indices down:

$$g(X, Y) = g(X^i\partial_i, Y^j\partial_j) = X^iY^j \underbrace{g(\partial_i, \partial_j)}_{=:g_{ij}} = g_{ij}X^iY^j = g_{ij}dx^i(X)dx^j(Y) \ .$$

We say that g is a *covariant tensor of valence two*.

We say that g is *symmetric* if $g(X, Y) = g(Y, X)$ for all X, Y; equivalently, $g_{ij} = g_{ji}$.

A symmetric bilinear tensor field is said to be *non-degenerate* if $\det g_{ij}$ has no zeros.

By Sylvester's inertia theorem, there exists a basis θ^i of the space of covectors so that a symmetric bilinear map g can be written as

$$g(X,Y) = \theta^1(X)\theta^1(Y)+\ldots+\theta^s(X)\theta^s(Y)-\theta^{s+1}(X)\theta^{s+1}(Y)-\ldots-\theta^{s+r}(X)\theta^{s+r}(Y)$$

(s, r) is called the signature of g; in geometry, unless specifically said otherwise, one always assumed that the signature does not change from point to point.

If $s = n$, in dimension n, then g is said to be a Riemannian metric tensor.

A canonical example is provided by the flat Riemannian metric on $\mathbb{R}^n$ is

$$g = (dx^1)^2 + \ldots + (dx^n)^2 \ .$$

By definition, a *Riemannian metric* is a field of symmetric two-covariant tensors with signature $(+, \ldots, +)$ and with $\det g_{ij}$ without zeros.

A Riemannian metric can be used to define the length of curves: if $\gamma : [a, b] \ni s \to \gamma(s)$, then

$$\ell_g(\gamma) = \int_a^b \sqrt{g(\dot\gamma, \dot\gamma)} ds \ .$$

One can then define the distance between points by minimizing the length of the curves connecting them.

If $s = 1$ and $r = N - 1$, in dimension N, then g is said to be a *Lorentzian metric tensor.*

For example, the *Minkowski metric* on $\mathbb{R}^{1+n}$ is

$$\eta = (dx^0)^2 - (dx^1)^2 - \ldots - (dx^n)^2 \ .$$

A.6 Tensor products

If φ and θ are covectors we can define a bilinear map using the formula

$$(\varphi \otimes \theta)(X, Y) = \varphi(X)\theta(Y) \ . \tag{A.6.1}$$

For example

$$(dx^1 \otimes dx^2)(X, Y) = X^1 Y^2 \ .$$

Using this notation we have

$$g(X, Y) = g(X^i \partial_i, Y^j \partial_j) = \underbrace{g(\partial_j, \partial_j)}_{=:g_{ij}} \underbrace{\underbrace{X^i}_{dx^i(X)} \ \underbrace{Y^j}_{dx^j(Y)}}_{(dx^i \otimes dx^j(X,Y)} = (g_{ij} dx^i \otimes dx^j)(X, Y)$$

We will write $dx^i dx^j$ for the symmetric product,

$$dx^i dx^j := \frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i) \ ,$$

and $dx^i \wedge dx^j$ for the anti-symmetric one,

$$dx^i \wedge dx^j := \frac{1}{2}(dx^i \otimes dx^j - dx^j \otimes dx^i) \ .$$

It should be clear how this generalises: the tensors $dx^i \otimes dx^j \otimes dx^k$, defined as

$$(dx^i \otimes dx^j \otimes dx^k)(X, Y, Z) = X^i Y^j Z^k \ ,$$

form a basis of three-linear maps on the space of vectors, which are objects of the form

$$X = X_{ijk} dx^i \otimes dx^j \otimes dx^k \ .$$

Here X is a called *tensor of valence* $(0,3)$. Each index transforms as for a covector:

$$X = X_{ijk} dx^i \otimes dx^j \otimes dx^k = X_{ijk} \frac{\partial x^i}{\partial y^m} \frac{\partial x^j}{\partial y^\ell} \frac{\partial x^k}{\partial y^n} dy^m \otimes dy^\ell \otimes dy^n \ .$$

It is sometimes useful to think of vectors as linear maps on co-vectors, using a formula which looks funny when first met: if θ is a covector, and X is a vector, then

$$X(\theta) := \theta(X) \ .$$

So if $\theta = \theta_i dx^i$ and $X = X^i \partial_i$ then

$$\theta(X) = \theta_i X^i = X^i \theta_i = X(\theta) \ .$$

It then makes sense to define e.g. $\partial_i \otimes \partial_j$ as a bilinear map on covectors:

$$(\partial_i \otimes \partial_j)(\theta, \psi) := \theta_i \psi_j \ .$$

And one can define a map $\partial_i \otimes dx^j$ which is linear on forms in the first slot, and linear in vectors in the second slot as

$$(\partial_i \otimes dx^j)(\theta, X) := \partial_i(\theta) dx^j(X) = \theta_i X^j \ . \tag{A.6.2}$$

The $\partial_i \otimes dx^j$'s form the basis of the space of *tensors of rank* $(1,1)$:

$$T = T^i{}_j \partial_i \otimes dx^j \ .$$

Generally, a *tensor of valence, or rank,* (r,s) can be defined as an object which has r vector indices and s covector indices, so that it transforms as

$$S^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} \to S^{m_1 \ldots m_r}{}_{\ell_1 \ldots \ell_s} \frac{\partial y^{i_1}}{\partial x^{m_1}} \cdots \frac{\partial y^{i_s}}{\partial x^{m_r}} \frac{\partial x^{\ell_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{\ell_s}}{\partial y^{j_s}}$$

For example, if $X = X^i \partial_i$ and $Y = Y^j \partial_j$ are vectors, then $X \otimes Y = X^i Y^j \partial_i \otimes \partial_j$ forms a contravariant tensor of valence two.

Tensors of same valence can be added in the obvious way: *e.g.*

$$(A+B)(X,Y) := A(X,Y) + B(X,Y) \quad \Longleftrightarrow \quad (A+B)_{ij} = A_{ij} + B_{ij} \ .$$

Tensors can be multiplied by scalars: *e.g.*

$$(fA)(X,Y,Z) := fA(X,Y,Z) \quad \Longleftrightarrow \quad f(A_{ijk}) := (fA_{ijk}) \ .$$

Finally, we have seen in (A.6.1) how to take tensor products for one forms, and in (A.6.2) how to take a tensor product of a vector and a one form, but this can also be done for higher order tensor; e.g., if S is of valence (a,b) and T is a multilinear map of valence (c,d), then $S \otimes T$ is a multilinear map of valence $(a+c, b+d)$, defined as

$$(S \otimes T)(\underbrace{\theta, \ldots}_{a \text{ covectors and } b \text{ vectors}}, \underbrace{\psi, \ldots}_{c \text{ covectors and } d \text{ vectors}}) := S(\theta, \ldots) T(\psi, \ldots) \ .$$

A.6.1 Contractions

Given a tensor field $S^i{}_j$ with one index down and one index up one can perform the sum

$$S^i{}_i \; .$$

This defines a scalar, i.e., a function on the manifold. Indeed, using the transformation rule

$$S^i{}_j \to \bar{S}^\ell{}_k = S^i{}_j \frac{\partial x^j}{\partial y^k} \frac{\partial y^\ell}{\partial x^i} \; ,$$

one finds

$$\bar{S}^\ell{}_\ell = S^i{}_j \underbrace{\frac{\partial x^j}{\partial y^\ell} \frac{\partial y^\ell}{\partial x^i}}_{\delta^j_i} = S^i{}_i \; ,$$

as desired.

One can similarly do contractions on higher valence tensors, e.g.

$$S^{i_1 i_2 \ldots i_r}{}_{j_1 j_2 j_3 \ldots j_s} \to S^{\ell i_2 \ldots i_r}{}_{j_1 \ell j_3 \ldots j_s} \; .$$

After contraction, a tensor of rank $(r+1, s+1)$ becomes of rank (r, s).

A.7 Raising and lowering of indices

Let g be a symmetric two-covariant tensor field on M, by definition such an object is the assignment to each point $p \in M$ of a bilinear map $g(p)$ from $T_pM \times T_pM$ to $\mathbb{R}$, with the additional property

$$g(X, Y) = g(Y, X) \; .$$

In this work the symbol g will be reserved to *non-degenerate* symmetric two-covariant tensor fields. It is usual to simply write g for $g(p)$, the point p being implicitly understood. We will sometimes write g_p for $g(p)$ when referencing p will be useful.

The usual Sylvester's inertia theorem tells us that at each p the map g will have a well defined signature; clearly this signature will be point-independent on a connected manifold when g is non-degenerate. A pair (M, g) is said to be a *Riemannian manifold* when the signature of g is $(\dim M, 0)$; equivalently, when g is a positive definite bilinear form on every product $T_pM \times T_pM$. A pair (M, g) is said to be a *Lorentzian manifold* when the signature of g is $(\dim M - 1, 1)$. One talks about *pseudo-Riemannian* manifolds whatever the signature of g, as long as g is non-degenerate, but we will only encounter Riemannian and Lorentzian metrics in this work.

Since g is non-degenerate it induces an isomorphism

$$\flat : T_pM \to T_p^*M$$

by the formula

$$\boxed{X_\flat(Y) = g(X, Y)} \; .$$

In local coordinates this gives

$$X_\flat = g_{ij}X^i dx^j =: X_j dx^j \ . \tag{A.7.1}$$

This last equality defines X_j — "the vector X^j with the index j lowered":

$$\boxed{X_i := g_{ij}X^j}\ . \tag{A.7.2}$$

The operation (A.7.2) is called the *lowering of indices* in the physics literature and, again in the physics literature, one does not make a distinction between the one-form $X_\flat$ and the vector X.

The inverse map will be denoted by $\sharp$ and is called the *raising of indices*; from (A.7.1) we obviously have

$$\alpha^\sharp = g^{ij}\alpha_i\partial_j =: \alpha^i\partial_i \quad \Longleftrightarrow \quad dx^i(\alpha^\sharp) = \boxed{\alpha^i = g^{ij}\alpha_j}\ ,$$

where g^{ij} is the matrix inverse to g_{ij}. For example,

$$(dx^i)^\sharp = g^{ik}\partial_k \ .$$

Clearly g^{ij}, understood as the matrix of a bilinear form on T_p^*M, has the same signature as g, and can be used to define a scalar product $g^\sharp$ on $T_p^*(M)$:

$$g^\sharp(\alpha,\beta) := g(\alpha^\sharp,\beta^\sharp) \quad \Longleftrightarrow \quad g^\sharp(dx^i,dx^j) = g^{ij} \ .$$

This last equality is justified as follows:

$$g^\sharp(dx^i,dx^j) = g((dx^i)^\sharp,(dx^j)^\sharp) = g(g^{ik}\partial_k, g^{j\ell}\partial_\ell) = \underbrace{g^{ik}g_{k\ell}}_{=\delta^i_\ell} g^{j\ell} = g^{ji} = g^{ij} \ .$$

It is convenient to use the same letter g for $g^\sharp$ — physicists do it all the time — or for scalar products induced by g on all the remaining tensor bundles, and we will sometimes do so.

One might wish to check by direct calculations that $g_{\mu\nu}X^\nu$ transforms as a one-form if X^μ transforms as a vector. The simplest way is to notice that $g_{\mu\nu}X^\nu$ is a contraction, over the last two indices, of the three-index tensor $g_{\mu\nu}X^\alpha$. Hence it is a one-form by the analysis at the end of the previous section. Alternatively, if we write $\bar g_{\mu\nu}$ for the transformed $g_{\mu\nu}$'s, and $\bar X^\alpha$ for the transformed X^α's, then

$$\underbrace{\bar g_{\alpha\beta}}_{g_{\mu\nu}\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}} \bar X^\beta = g_{\mu\nu}\frac{\partial x^\mu}{\partial y^\alpha}\underbrace{\frac{\partial x^\nu}{\partial y^\beta}\bar X^\beta}_{X^\nu} = g_{\mu\nu}X^\nu\frac{\partial x^\mu}{\partial y^\alpha} \ ,$$

which is indeed the transformation law of a covector.

A.8 The Lie derivative

A.8.1 A pedestrian approach

We start with a pedestrian approach to the definition of Lie derivative; the elegant geometric definition will be given in the next section.

Given a vector field X, the *Lie derivative* $\mathscr{L}_X$ is an operation on tensor fields, defined as follows:

For a function f, one sets

$$\mathscr{L}_X f := X(f) . \tag{A.8.1}$$

For a vector field Y, the Lie derivative coincides with the Lie bracket:

$$\mathscr{L}_X Y := [X,Y] . \tag{A.8.2}$$

For a one form α, $\mathscr{L}_X\alpha$ is defined by imposing the Leibniz rule written backwards:

$$(\mathscr{L}_X\alpha)(Y) := \mathscr{L}_X(\alpha(Y)) - \alpha(\mathscr{L}_X Y) . \tag{A.8.3}$$

(Indeed, the Leibniz rule applied to the contraction $\alpha_i X^i$ would read

$$\mathscr{L}_X(\alpha_i Y^i) = (\mathscr{L}_X\alpha)_i Y^i + \alpha_i(\mathscr{L}_X Y)^i ,$$

which can be rewritten as (A.8.3).)

Let us check that (A.8.3) defines a one form. Clearly, the right-hand side transforms in the desired way when Y is replaced by Y_1+Y_2. Now, if we replace Y by fY, where f is a function, then

$$\begin{aligned}
(\mathscr{L}_X\alpha)(fY) &= \mathscr{L}_X(\alpha(fY)) - \alpha(\underbrace{\mathscr{L}_X(fY)}_{X(f)Y+f\mathscr{L}_X Y})\\
&= X(f\alpha(Y)) - \alpha(X(f)Y + f\mathscr{L}_X Y))\\
&= X(f)\alpha(Y) + fX(\alpha(Y)) - \alpha(X(f)Y) - \alpha(f\mathscr{L}_X Y))\\
&= fX(\alpha(Y)) - f\alpha(\mathscr{L}_X Y))\\
&= f((\mathscr{L}_X\alpha)(Y)) .
\end{aligned}$$

So $\mathscr{L}_X\alpha$ is a C^∞-linear map on vector fields, hence a covector field.

In coordinate-components notation we have

$$(\mathscr{L}_X\alpha)_a = X^b\partial_b\alpha_a + \alpha_b\partial_a X^b . \tag{A.8.4}$$

Indeed,

$$\begin{aligned}
(\mathscr{L}_X\alpha)_i Y^i &:= \mathscr{L}_X(\alpha_i Y^i) - \alpha_i(\mathscr{L}_X Y)^i\\
&= X^k\partial_k(\alpha_i Y^i) - \alpha_i(X^k\partial_k Y^i - Y^k\partial_k X^i)\\
&= X^k(\partial_k\alpha_i)Y^i + \alpha_i Y^k\partial_k X^i\\
&= \left(X^k\partial_k\alpha_i + \alpha_k\partial_i X^k\right) Y^i ,
\end{aligned}$$

as desired

For tensor products, the Lie derivative is defined by imposing linearity under addition together with the Leibniz rule:

$$\mathscr{L}_X(\alpha\otimes\beta) = (\mathscr{L}_X\alpha)\otimes\beta + \alpha\otimes\mathscr{L}_X\beta\ .$$

Since a general tensor A is a sum of tensor products,

$$A = A^{a_1\ldots a_p}{}_{b_1\ldots b_q}\partial_{a_1}\otimes\ldots\partial_{a_p}\otimes dx^{b_1}\otimes\ldots\otimes dx^{a_p}\ ,$$

requiring linearity with respect to addition of tensors gives thus a definition of Lie derivative for any tensor.

For example, we claim that

$$\mathscr{L}_X T^a{}_b = X^c\partial_c T^a{}_b - T^c{}_b\partial_c X^a + T^a{}_c\partial_b X^c\ , \tag{A.8.5}$$

To see this, call a tensor $T^a{}_b$ *simple* if it is of the form $Y\otimes\alpha$, where Y is a vector and α is a covector. Using indices, this corresponds to $Y^a\alpha_b$ and so, by the Leibniz rule,

$$\begin{aligned}
\mathscr{L}_X(Y\otimes\alpha)^a{}_b &= \mathscr{L}_X(Y^a\alpha_b)\\
&= (\mathscr{L}_X Y)^a\alpha_b + Y^a(\mathscr{L}_X\alpha)_b\\
&= (X^c\partial_c Y^a - Y^c\partial_c X^a)\alpha_b + Y^a(X^c\partial_c\alpha_b + \alpha_c\partial_b X^c)\\
&= X^c\partial_c(Y^a\alpha_b) - Y^c\alpha_b\partial_c X^a + Y^a\alpha_c\partial_b X^c\ ,
\end{aligned}$$

which coincides with (A.8.5) if $T^a{}_b = Y^b\alpha_b$. But a general $T^a{}_b$ can be written as a linear combination with constant coefficients of simple tensors,

$$T = \sum_{a,b}\underbrace{T^a{}_b\partial_a\otimes dx^b}_{\text{no summation, so simple}}\ ,$$

and the result follows.

Similarly, one has, e.g.,

$$\mathscr{L}_X R^{ab} = X^c\partial_c R^{ab} - R^{ac}\partial_c X^b - R^{bc}\partial_c X^a\ ,$$

$$\mathscr{L}_X S_{ab} = X^c\partial_c S_{ab} + S_{ac}\partial_b X^c + S_{bc}\partial_a X^c\ , \tag{A.8.6}$$

etc. Those are all special cases of the general formula for the Lie derivative $\mathscr{L}_X A^{a_1\ldots a_p}{}_{b_1\ldots b_q}$:

$$\begin{aligned}
\mathscr{L}_X A^{a_1\ldots a_p}{}_{b_1\ldots b_q} &= X^c\partial_c A^{a_1\ldots a_p}{}_{b_1\ldots b_q} - A^{ca_2\ldots a_p}{}_{b_1\ldots b_q}\partial_c X^{a_1} - \ldots\\
&\quad + A^{a_1\ldots a_p}{}_{cb_1\ldots b_q}\partial_{b_1}X^c + \ldots\ .
\end{aligned}$$

A useful property of Lie derivatives is

$$\mathscr{L}_{[X,Y]} = [\mathscr{L}_X,\mathscr{L}_Y]\ , \tag{A.8.7}$$

where, for a tensor T, the commutator $[\mathscr{L}_X,\mathscr{L}_Y]T$ is defined in the usual way:

$$[\mathscr{L}_X,\mathscr{L}_Y]T := \mathscr{L}_X(\mathscr{L}_Y T) - \mathscr{L}_Y(\mathscr{L}_X T)\ . \tag{A.8.8}$$

To see this, we first note that if $T = f$ is a function, then the right-hand-side of (A.8.8) is the definition of $[X,Y](f)$, which in turn coincides with the definition of $\mathscr{L}_{[X,Y]}(f)$.

Next, for a vector field $T = Z$, (A.8.7) reads

$$\mathscr{L}_{[X,Y]}Z = \mathscr{L}_X(\mathscr{L}_Y Z) - \mathscr{L}_Y(\mathscr{L}_X Z) \,, \tag{A.8.9}$$

which is the same as

$$[[X,Y],Z] = [X,[Y,Z]] - [Y,[X,Z]] \,, \tag{A.8.10}$$

which is the same as

$$[Z,[Y,X]] + [X,[Z,Y]] + [Y,[X,Z]] = 0 \,, \tag{A.8.11}$$

which is the Jacobi identity. Hence (A.8.7) holds for vector fields.

We continue with a one form α. We use the definitions, with Z any vector field:

$$\begin{aligned}
(\mathscr{L}_X\mathscr{L}_Y\alpha)(Z) &= X(\underbrace{(\mathscr{L}_Y\alpha)(Z)}_{Y(\alpha(Z))-\alpha(\mathscr{L}_Y Z))}) - \underbrace{(\mathscr{L}_Y\alpha)(\mathscr{L}_X Z)}_{Y(\alpha(\mathscr{L}_X Z))-\alpha(\mathscr{L}_Y\mathscr{L}_X Z)} \\
&= X(Y(\alpha(Z))) - X(\alpha(\mathscr{L}_Y Z))) - Y(\alpha(\mathscr{L}_X Z)) + \alpha(\mathscr{L}_Y\mathscr{L}_X Z) \,.
\end{aligned}$$

Antisymmetrizing over X and Y, the second and third term above cancel out, so that

$$\begin{aligned}
((\mathscr{L}_X\mathscr{L}_Y\alpha - \mathscr{L}_Y\mathscr{L}_X)\alpha)(Z) &= X(Y(\alpha(Z))) + \alpha(\mathscr{L}_Y\mathscr{L}_X Z) - (X \longleftrightarrow Y) \\
&= [X,Y](\alpha(Z)) - \alpha(\mathscr{L}_X\mathscr{L}_Y Z - \mathscr{L}_Y\mathscr{L}_X Z) \\
&= \mathscr{L}_{[X,Y]}(\alpha(Z)) - \alpha(\mathscr{L}_{[X,Y]}Z) \\
&= (\mathscr{L}_{[X,Y]}\alpha)(Z) \,.
\end{aligned}$$

Since Z is arbitrary, (A.8.7) for covectors follows.

To conclude that (A.8.7) holds for arbitrary tensor fields, we note that by construction we have

$$\mathscr{L}_{[X,Y]}(A\otimes B) = \mathscr{L}_{[X,Y]}A\otimes B + A\otimes\mathscr{L}_{[X,Y]}B \,. \tag{A.8.12}$$

Similarly

$$\begin{aligned}
\mathscr{L}_X\mathscr{L}_Y(A\otimes B) &= \mathscr{L}_X(\mathscr{L}_Y A\otimes B + A\otimes\mathscr{L}_Y B) \\
&= \mathscr{L}_X\mathscr{L}_Y A\otimes B + \mathscr{L}_X A\otimes\mathscr{L}_Y B + \mathscr{L}_Y A\otimes\mathscr{L}_X B \\
&\quad + A\otimes\mathscr{L}_X\mathscr{L}_Y B \,.
\end{aligned} \tag{A.8.13}$$

Exchanging X with Y and subtracting, the middle terms drop out:

$$[\mathscr{L}_X,\mathscr{L}_Y](A\otimes B) = [\mathscr{L}_X,\mathscr{L}_Y]A\otimes B + A\otimes[\mathscr{L}_X,\mathscr{L}_Y]B \,. \tag{A.8.14}$$

Basing on what has been said, the reader should have no difficulties finishing the proof of (A.8.7).

EXAMPLE A.8.1 As an example of application of the formalism, suppose that there exists a coordinate system in which $(X^a) = (1,0,0,0)$ and $\partial_0 g_{bc} = 0$. Then

$$\mathscr{L}_X g_{ab} = \partial_0 g_{ab} = 0 \; .$$

But the Lie derivative of a tensor field is a tensor field, and we conclude that $\mathscr{L}_X g_{ab} = 0$ holds in every coordinate system.

Vector fields for which $\mathscr{L}_X g_{ab} = 0$ are called *Killing vectors*: they arise from symmetries of space-time. We have the useful formula

$$\mathscr{L}_X g_{ab} = \nabla_a X_b + \nabla_b X_a \; . \tag{A.8.15}$$

An effortless proof of this proceeds as follows: in adapted coordinates in which the derivatives of the metric vanish at a point p, one immediately checks that equality holds at p. But both sides are tensor fields, therefore the result holds at p for all coordinate systems, and hence also everywhere.

The brute-force proof of (A.8.15) proceeds as follows:

$$\begin{aligned}
\mathscr{L}_X g_{ab} &= X^c\partial_c g_{ab} + \partial_a X^c g_{cb} + \partial_b X^c g_{ca} \\
&= X^c\partial_c g_{ab} + \partial_a (X^c g_{cb}) - X^c\partial_a g_{cb} + \partial_b (X^c g_{ca}) - X^c\partial_b g_{ca} \\
&= \partial_a X_b + \partial_b X_a + X^c \underbrace{(\partial_c g_{ab} - \partial_a g_{cb} - \partial_b g_{ca})}_{-2 g_{cd}\Gamma^d_{ab}} \\
&= \nabla_a X_b + \nabla_b X_a \; .
\end{aligned}$$

A.8.2 The geometric approach

We pass now to a geometric definition of Lie derivative. This requires, first, an excursion through the land of push-forwards and pull-backs.

Transporting tensor fields

We start by noting that, given a point p_0 in a manifold M, every vector $X \in T_{p_0}M$ is tangent to some curve. To see this, let $\{x^i\}$ be any local coordinates near p_0, with $x^i(p_0) = x^i_0$, then X can be written as $X^i(p_0)\partial_i$. If we set $\gamma^i(s) = x^i_0 + sX^i(p_0)$, then $\dot\gamma^i(0) = X^i(p_0)$, which establishes the claim. This observation shows that studies of vectors can be reduced to studies of curves.

Let, now, M and N be two manifolds, and let $\phi : M \to N$ be a differentiable map between them. Given a vector $X \in T_pM$, the *push-forward* $\phi_* X$ of X is a vector in $T_{\phi(p)}N$ defined as follows: let γ be any curve for which $X = \dot\gamma(0)$, then

$$\phi_* X := \left.\frac{d(\phi\circ\gamma)}{ds}\right|_{s=0} . \tag{A.8.16}$$

In local coordinates y^A on N and x^i on M, so that $\phi(x) = (\phi^A(x^i))$, we find

$$\begin{aligned}
(\phi_* X)^A &= \left.\frac{d\phi^A(\gamma^i(s))}{ds}\right|_{s=0} = \frac{\partial\phi^A(\gamma^i(s))}{\partial x^i}\dot\gamma^i(s)|_{s=0} \\
&= \frac{\partial \phi^A(x^i)}{\partial x^i} X^i \; .
\end{aligned} \tag{A.8.17}$$

This makes it clear that the definition is independent of the choice of the curve γ satisfying $X = \dot\gamma(0)$.

If we apply this formula to a vector field X defined on M we obtain

$$(\phi_* X)^A(\phi(x)) = \frac{\partial \phi^A}{\partial x^i}(x)\, X^i(x)\ . \tag{A.8.18}$$

The equation shows that if a point $y \in N$ has more than one pre-image, say $y = \phi(x_1) = \phi(x_2)$ with $x_1 \neq x_2$, then (A.8.18) might will define more than one tangent vector at y in general. This leads to an important caveat: we will be certain that the push-forward of a vector field on M defines a vector field on N only when ϕ is a diffeomorphism. More generally, $\phi_* X$ defines locally a vector field on $\phi(M)$ if and only if ϕ is a local diffeomorphism. In such cases we can invert ϕ (perhaps locally) and write (A.8.18) as

$$(\phi_* X)^j(x) = \left(\frac{\partial \phi^j}{\partial x^i} X^i\right)(\phi^{-1}(x))\ . \tag{A.8.19}$$

When ϕ is understood as a coordinate change rather than a diffeomorphism between two manifolds, this is simply the standard transformation law of a vector field under coordinate transformations.

The push-forward operation can be extended to *contravariant* tensors by defining it on tensor products in the obvious way, and extending by linearity: for example, if X, Y and Z are vectors, then

$$\phi_*(X \otimes Y \otimes Z) := \phi_* X \otimes \phi_* Y \otimes \phi_* Z\ .$$

Consider, next, a k-multilinear map α from $T_{\phi(p_0)}M$ to $\mathbb{R}$. The *pull-back* $\phi^*\alpha$ of α is a multilinear map on $T_{p_0}M$ defined as

$$T_pM \ni (X_1, \dots X_k) \mapsto \phi^*(\alpha)(X_1, \dots, X_k) := \alpha(\phi^* X_1, \dots, \phi_* X_k)\ .$$

As an example, let $\alpha = \alpha_A dy^A$ be a one-form, if $X = X^i\partial_i$ then

$$(\phi^*\alpha)(X) = \alpha(\phi_* X) = \alpha(\frac{\partial \phi^A}{\partial x^i} X^i \partial_A) = \alpha_A \frac{\partial \phi^A}{\partial x^i} X^i = \alpha_A \frac{\partial \phi^A}{\partial x^i} dx^i(X)\ .$$

Equivalently,

$$(\phi^*\alpha)_i = \alpha_A \frac{\partial \phi^A}{\partial x^i}\ . \tag{A.8.20}$$

If α is a one-form field on N, this reads

$$(\phi^*\alpha)_i(x) = \alpha_A(\phi(x)) \frac{\partial \phi^A(x)}{\partial x^i}\ . \tag{A.8.21}$$

It follows that $\phi^*\alpha$ is a field of one-forms on M, irrespective of injectivity or surjectivity properties of ϕ. Similarly, pull-backs of covariant tensor fields of higher rank are smooth tensor fields.

For a function f equation (A.8.21) reads

$$(\phi^* df)_i(x) = \frac{\partial f}{\partial y^A}(\phi(x)) \frac{\partial \phi^A(x)}{\partial x^i} = \frac{\partial (f \circ \phi)}{\partial x^i}(x)\ , \tag{A.8.22}$$

which can be succinctly written as

$$\phi^* df = d(f \circ \phi) \ . \tag{A.8.23}$$

Using the notation

$$\phi^* f := f \circ \phi \ , \tag{A.8.24}$$

we can write (A.8.23) as

$$\phi^* d = d\phi^* \ \text{ for functions.} \tag{A.8.25}$$

Summarising:

1. Pull-backs of covariant tensor fields define covariant tensor fields. In particular the metric can *always* be pulled back.

2. Push-forwards of contravariant tensor fields can be used to define contravariant tensor fields when ϕ is a diffeomorphism.

In this context it is thus clearly of interest to consider diffeomorphisms ϕ, as then tensor products can now be transported in the following way; we will denote by $\hat\phi$ the associated map: We define $\hat\phi f = f \circ \phi$ for functions, $\hat\phi = \phi_*$ for covariant fields, $\hat\phi = (\phi^{-1})_*$ for contravariant tensor fields, we use the rule

$$\hat\phi(A \otimes B) = \hat\phi A \otimes \hat\phi B$$

for tensor products, and the definition is extended by linearity under multiplication by functions to any tensor fields.

So, for example, if X is a vector field and α is a field of one-forms, one has

$$\hat\phi(X \otimes \alpha) := (\phi^{-1})_* X \otimes \phi^* \alpha \ . \tag{A.8.26}$$

Flows of vector fields

Let X be a vector field on M. For every $p_0 \in M$ consider the (maximally extended, as a function of t) solution to the problem

$$\frac{dx^i}{dt} = X^i(x(t)) \ , \quad x^i(0) = x_0^i \ . \tag{A.8.27}$$

The map

$$(t, x_0) \mapsto \phi_t[X](x_0) := x(t)$$

where $x^i(t)$ is the solution of (A.8.27), is called *the local flow of X*. We say that X generates $\phi_t[X]$. We will write ϕ_t for $\phi_t[X]$ when X is unambiguous in the context.

The time of existence of solutions of (A.8.27) depends upon x_0 in general.

EXAMPLE A.8.2 As an example, let $M = \mathbb{R}$ and $X = x^2\partial_x$. We then have to solve

$$\frac{dx}{dt} = x^2 \ , \ x(0) = x_0 \quad \Longrightarrow \quad x = \begin{cases} 0, & x_0 = 0; \\ \frac{x_0}{1 - x_0 t}, & x_0 \neq 0 \ , \ 1 - x_0 t > 0. \end{cases}$$

Hence

$$\phi_t(x) = \frac{x}{1 - xt} \ ,$$

with $t \in \mathbb{R}$ when $x = 0$, with $t \in (-\infty, 1/x)$ when $x > 0$ and with $t \in (1/x, \infty)$ when $x < 0$.

We say that X is *complete* if $\phi_t[X](p)$ is defined for all $(t,p) \in \mathbb{R} \times M$.

The following standard facts are left as exercices to the reader:

1. ϕ_0 is the identity map.
2. $\phi_t \circ \phi_s = \phi_{t+s}$.
   In particular, $\phi_t^{-1} = \phi_{-t}$, and thus:
3. The maps $x \mapsto \phi_t(x)$ are local diffeomorphisms; global if for all $x \in M$ the maps ϕ_t are defined for all $t \in \mathbb{R}$.
4. $\phi_{-t}[X]$ is generated by $-X$:
$$\phi_{-t}[X] = \phi_t[-X] \ .$$

A family of diffeomorphisms satisfying property 2. above is called a *one parameter group of diffeomorphisms*. Thus, *complete* vector fields generate one-parameter families of diffeomorphisms via (A.8.27).

Reciprocally, suppose that a local one-parameter group ϕ_t is given, then the formula
$$X = \frac{d\phi}{dt}\Big|_{t=0}$$
defines a vector field, said to be *generated by* ϕ_t.

The Lie derivative revisited

The idea of the *Lie transport*, and hence of the *Lie derivative*, is to be able to compare objects along integral curves of a vector field X. This is pretty obvious for scalars: we just compare the values of $f(x)$ with $f(\phi_t(x))$, leading to a derivative
$$\mathscr{L}_X f := \lim_{t\to 0} \frac{f\circ\phi_t - f}{t} \equiv \lim_{t\to 0}\frac{\phi_t^* f - f}{t} \equiv \lim_{t\to 0}\frac{\hat\phi_t f - f}{t} \equiv \frac{d(\hat\phi_t T)}{dt}\Big|_{t=0} \ . \tag{A.8.28}$$

We wish, next, to compare the value of a vector field Y at $\phi_t(x)$ with the value at x. For this, we move from x to $\phi_t(x)$ following the integral curve of X, and produce a new vector at x by applying $(\phi_t^{-1})_*$ to $Y|_{\phi_t(x)}$. This makes it perhaps clearer why we introduced the transport map $\hat\phi$, since $(\hat\phi Y)(x)$ is precisely the value at x of $(\phi_t^{-1})_* Y$. We can then calculate
$$\mathscr{L}_X Y(x) := \lim_{t\to 0}\frac{(\phi_t^{-1}Y)(\phi_t(x)) - Y(x)}{t} \equiv \lim_{t\to 0}\frac{(\hat\phi_t Y)(x) - Y(x)}{t} \equiv \frac{d(\hat\phi_t Y(x))}{dt}\Big|_{t=0} \ . \tag{A.8.29}$$

In general, let X be a vector field and let ϕ_t be the associated local one-parameter family of diffeomorphisms. Let $\hat\phi_t$ be the associated family of transport maps for tensor fields. For any tensor field T one sets
$$\mathscr{L}_X T := \lim_{t\to 0}\frac{\hat\phi_t T - T}{t} \equiv \frac{d(\hat\phi_t T)}{dt}\Big|_{t=0} \ . \tag{A.8.30}$$

We want to show that this operation coincides with that defined in Section A.8.1.

The result for functions should be clear:

$$\mathscr{L}_X f = X(f) \ .$$

Consider, next, a vector field Y. From (A.8.19), setting $\psi_t := \phi_{-t} \equiv (\phi_t)^{-1}$ we have

$$\hat{\phi}_t Y^j(x) := ((\phi_t^{-1})_* Y)^j(x) = (\frac{\partial \psi_t^j}{\partial x^i} Y^i)(\phi_t(x)) \ . \tag{A.8.31}$$

Hence

$$\begin{aligned}
\frac{d\hat{\phi}_t Y^j}{dt}(x)|_{t=0} &= \frac{\partial \dot{\psi}_0^j}{\partial x^i}(x)\, Y^i(x) + \partial_j(\underbrace{\frac{\partial \psi_0^j}{\partial x^i}\, Y^i}_{Y^j})(x)\dot{\phi}^j(x) \\
&= -\partial_i X^j(x) Y^i(x) + \partial_j Y^i(x) X^j(x) \\
&= [X,Y]^j(x) \ ,
\end{aligned}$$

and we have obtained (A.8.2), p. 198.

For a covector field α, it seems simplest to calculate directly from (A.8.21):

$$(\hat{\phi}_t \alpha)_i(x) = (\phi_t^* \alpha)_i(x) = \alpha_k(\phi_t(x)) \frac{\partial \phi_t^k(x)}{\partial x^i} \ .$$

Hence

$$\mathscr{L}_X \alpha_i = \left.\frac{d(\phi_t^* \alpha)_i(x)}{dt}\right|_{t=0} = \partial_j \alpha_k(x) X^j(x) + \alpha_k(x) \frac{\partial X^k(x)}{\partial x^i}(x) \ , \tag{A.8.32}$$

as in (A.8.4).

It now follows from (A.8.3) that the Leibniz rule under duality holds.

Alternatively, one can start by showing that the Leibniz rule under duality holds, and then use the calculations in Section A.8.1 to derive (A.8.32): For this, by definition we have

$$\phi_t^* \alpha(Y) = \alpha((\phi_t)_* Y) \ ,$$

hence

$$\alpha(Y)|_{\phi_t(x)} = \alpha((\phi_t)_* (\phi_t^{-1})_* Y)|_{\phi_t(x)} = \phi_t^* \alpha|_x((\phi_t^{-1})_* Y|_{\phi_t(x)}) = \hat{\phi}_t \alpha(\hat{\phi}_t Y)|_x \ .$$

Equivalently,

$$\hat{\phi}_t(\alpha(Y)) = (\hat{\phi}_t \alpha)(\hat{\phi}_t Y) \ ,$$

from which the Leibniz rule under duality immediately follows.

A similar calculation leads to the Leibniz rule under tensor products.

The reader should have no difficulties checking that the remaining requirements set forth in Section A.8.1 are satisfied.

A.9 Covariant derivatives

When dealing with $\mathbb{R}^n$, or subsets thereof, there exists an obvious prescription for how to differentiate tensor fields: in this case we have at our disposal the canonical "trivialization $\{\partial_i\}_{i=1,\ldots,n}$ of $T\mathbb{R}^n$" (this means: a globally defined set of vectors which, at every point, form a basis of the tangent space), together with its dual trivialization $\{dx^j\}_{i=1,\ldots,n}$ of $T^*\mathbb{R}^n$. We can expand a tensor field T of valence (k,ℓ) in terms of those bases,

$$\begin{aligned} T = T^{i_1\ldots i_k}{}_{j_1\ldots j_\ell}\partial_{i_1}\otimes\ldots\otimes\partial_{i_k}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_\ell} \\ \Longleftrightarrow \quad T^{i_1\ldots i_k}{}_{j_1\ldots j_\ell} = T(dx^{i_1},\ldots,dx^{i_k},\partial_{j_1},\ldots,\partial_{j_\ell})\ , \end{aligned} \tag{A.9.1}$$

and differentiate each component $T^{i_1\ldots i_k}{}_{j_1\ldots j_\ell}$ of T separately:

$$X(T)_{\text{in the coordinate system } x^i} := X^i\frac{\partial T^{i_1\ldots i_k}{}_{j_1\ldots j_\ell}}{\partial x^i}\partial_{x^{i_1}}\otimes\ldots\otimes\partial_{x^{i_k}}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_\ell}\ . \tag{A.9.2}$$

The resulting object does, however, *not* behave as a tensor under coordinate transformations, in the sense that the above form of the right-hand-side will *not* be preserved under coordinate transformations: as an example, consider the one-form $T = dx$ on $\mathbb{R}^n$, which has vanishing derivative as defined by (A.9.2). When expressed in spherical coordinates we have

$$T = d(\rho\cos\varphi) = -\rho\sin\varphi d\varphi + \cos\varphi d\rho\ ,$$

the partial derivatives of which are non-zero (both with respect to the original cartesian coordinates (x,y) and to the new spherical ones (ρ,φ)).

The Lie derivative $\mathscr{L}_X$ of Section A.8 maps tensors to tensors but does not resolve this question, because it is *not* linear under multiplication of X by a function.

The notion of *covariant derivative*, sometimes also referred to as *connection*, is introduced precisely to obtain a notion of derivative which has tensorial properties. By definition, a covariant derivative is a map which to a vector field X and a tensor field T assigns a tensor field of the same type as T, denoted by $\nabla_X T$, with the following properties:

1. $\nabla_X T$ is linear with respect to addition both with respect to X and T:

$$\nabla_{X+Y}T = \nabla_X T + \nabla_Y T\ , \qquad \nabla_X(T+Y) = \nabla_X T + \nabla_X Y\ ; \tag{A.9.3}$$

2. $\nabla_X T$ is linear with respect to multiplication of X by functions f,

$$\nabla_{fX}T = f\nabla_X T\ ; \tag{A.9.4}$$

3. and, finally, $\nabla_X T$ satisfies the *Leibniz rule* under multiplication of T by a differentiable function f:

$$\nabla_X(fT) = f\nabla_X T + X(f)T\ . \tag{A.9.5}$$

By definition, if T is a tensor field of rank (p,q), then for any vector field X the field $\nabla_X T$ is again a tensor of type (p,q). Since $\nabla_X T$ is linear in X, the field ∇T can naturally be viewed as a tensor field of rank $(p,q+1)$.

It is natural to ask whether covariant derivatives do exist at all in general and, if so, how many of them can there be. First, it immediately follows from the axioms above that if D and ∇ are two covariant derivatives, then

$$\Delta(X,T) := D_X T - \nabla_X T$$

is multi-linear both with respect to addition and multiplication by functions — the non-homogeneous terms $X(f)T$ in (A.9.5) cancel — and is thus a tensor field. Reciprocally, if ∇ is a covariant derivative and $\Delta(X,T)$ is bilinear with respect to addition and multiplication by functions, then

$$D_X T := \nabla_X T + \Delta(X,T) \qquad \text{(A.9.6)}$$

is a new covariant derivative. So, at least locally, on tensors of valence (r,s) there are as many covariant derivatives as tensors of valence $(r+s, r+s+1)$.

We note that the sum of two covariant derivatives is *not* a covariant derivative. However, *convex* combinations of covariant derivatives, with coefficients which may vary from point to point, are again covariant derivatives. This remark allows one to construct covariant derivatives using partitions of unity: Let, indeed, $\{\mathscr{O}_i\}_{i\in\mathbb{N}}$ be an open covering of M by coordinate patches and let φ_i be an associated partition of unity. In each of those coordinate patches we can decompose a tensor field T as in (A.9.1), and define

$$D_X T := \sum_i \varphi_i X^j \partial_j (T^{i_1\ldots i_k}{}_{j_1\ldots j_\ell})\partial_{i_1}\otimes\ldots\otimes\partial_{i_k}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_\ell}\ . \qquad \text{(A.9.7)}$$

This procedure, which depends upon the choice of the coordinate patches and the choice of the partition of unity, defines *one* covariant derivative; all other covariant derivatives are then obtained from D using (A.9.6). Note that (A.9.2) is a special case of (A.9.7) when there exists a global coordinate system on M. Thus (A.9.2) *does* define a covariant derivative. However, the associated operation on tensor fields will *not* take the simple form (A.9.2) when we go to a different coordinate system $\{y^i\}$ in general.

A.9.1 Functions

The *canonical covariant derivative on functions* is defined as

$$\nabla_X(f) = X(f)\ ,$$

and we will always use the above. This has all the right properties, so obviously covariant derivatives of functions exist. From what has been said, any covariant derivative on functions is of the form

$$\nabla_X f = X(f) + \alpha(X)f\ , \qquad \text{(A.9.8)}$$

where α is a one-form. Conversely, given any one form α, (A.9.8) defines a covariant derivative on functions. The addition of the lower-order term $\alpha(X)f$

(A.9.8) does not appear to be very useful here, but it turns out to be useful in geometric formulation of electrodynamics, or in *geometric quantization.* In any case such lower-order terms play an essential role when defining covariant derivatives of tensor fields.

A.9.2 Vectors

The simplest next possibility is that of a covariant derivative of vector fields. Let us not worry about existence at this stage, but assume that a covariant derivative exists, and work from there. (Anticipating, we will show shortly that a metric defines a covariant derivative, called the *Levi-Civita* covariant derivative, which is the unique covariant derivative operator satisfying a natural set of conditions, to be discussed below.)

We will first assume that we are working on a set $\Omega \subset M$ over which we have a *global trivialization* of the tangent bundle TM; by definition, this means that there exist vector fields e_a, $a = 1, \ldots, \dim M$, such that at every point $p \in \Omega$ the fields $e_a(p) \in T_pM$ form a basis of T_pM.[1]

Let θ^a denote the dual trivialization of T^*M — by definition the θ^a's satisfy

$$\boxed{\theta^a(e_b) = \delta^a_b} \, .$$

Given a covariant derivative ∇ on vector fields we set

$$\Gamma^a{}_b(X) := \theta^a(\nabla_X e_b) \quad \Longleftrightarrow \quad \nabla_X e_b = \Gamma^a{}_b(X) e_a \, , \quad (A.9.9\text{a})$$

$$\boxed{\Gamma^a{}_{bc} := \Gamma^a{}_b(e_c) = \theta^a(\nabla_{e_c} e_b)} \quad \Longleftrightarrow \quad \nabla_X e_b = \Gamma^a{}_{bc} X^c e_a \, . \quad (A.9.9\text{b})$$

The (locally defined) functions $\Gamma^a{}_{bc}$ are called *connection coefficients.* If $\{e_a\}$ is the coordinate basis $\{\partial_\mu\}$ we shall write

$$\Gamma^\mu{}_{\alpha\beta} := dx^\mu(\nabla_{\partial_\beta}\partial_\alpha) \quad \Big(\Longleftrightarrow \quad \nabla_{\partial_\mu}\partial_\nu = \Gamma^\sigma{}_{\nu\mu}\partial_\sigma \Big) \, , \qquad (A.9.10)$$

etc. In this particular case the connection coefficients are usually called *Christoffel symbols.* We will sometimes write $\Gamma^\sigma_{\nu\mu}$ instead of $\Gamma^\sigma{}_{\nu\mu}$; note that the former convention is more common. By using the Leibniz rule (A.9.5) we find

$$\begin{aligned}
\nabla_X Y &= \nabla_X (Y^a e_a) \\
&= X(Y^a) e_a + Y^a \nabla_X e_a \\
&= X(Y^a) e_a + Y^a \Gamma^b{}_a(X) e_b \\
&= (X(Y^a) + \Gamma^a{}_b(X) Y^b) e_a \\
&= (X(Y^a) + \Gamma^a{}_{bc} Y^b X^c) e_a \, , \qquad (A.9.11)
\end{aligned}$$

which gives various equivalent ways of writing $\nabla_X Y$. The (perhaps only locally defined) $\Gamma^a{}_b$'s are linear in X, and the collection $(\Gamma^a{}_b)_{a,b=1,\ldots,\dim M}$ is sometimes

[1] This is the case when Ω is a coordinate patch with coordinates (x^i), then the $\{e_a\}_{a=1,\ldots,\dim M}$ can be chosen to be equal to $\{\partial_i\}_{a=1,\ldots,\dim M}$. Recall that a manifold is said to be parallelizable if a basis of TM can be chosen globally over M — in such a case Ω can be taken equal to M. We emphasize that we are *not* assuming that M is parallelizable, so that equations such as (A.9.9) have only a local character in general.

referred to as the *connection one-form*. The one-covariant, one-contravariant tensor field ∇Y is defined as

$$\nabla Y := \nabla_a Y^b \theta^a \otimes e_b \Longleftrightarrow \nabla_a Y^b := \theta^b(\nabla_{e_a} Y) \Longleftrightarrow \boxed{\nabla_a Y^b = e_a(Y^b) + \Gamma^b{}_{ca} Y^c} . \tag{A.9.12}$$

We will often write ∇_a for ∇_{e_a}. Further, $\nabla_a Y^b$ will sometimes be written as $Y^b{}_{;a}$.

A.9.3 Transformation law

Consider a coordinate basis ∂_{x^i}, it is natural to enquire about the transformation law of the connection coefficients $\Gamma^i{}_{jk}$ under a change of coordinates $x^i \to y^k(x^i)$. To make things clear, let us write $\Gamma^i{}_{jk}$ for the connection coefficients in the x–coordinates, and $\hat\Gamma^i{}_{jk}$ for the ones in the y–cordinates. We calculate:

$$\begin{aligned}
\Gamma^i{}_{jk} &:= dx^i\Big(\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j}\Big) \\
&= dx^i\Big(\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial y^\ell}{\partial x^j}\frac{\partial}{\partial y^\ell}\Big) \\
&= dx^i\Big(\frac{\partial^2 y^\ell}{\partial x^k \partial x^j}\frac{\partial}{\partial y^\ell} + \frac{\partial y^\ell}{\partial x^j}\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial y^\ell}\Big) \\
&= \frac{\partial x^i}{\partial y^s} dy^s\Big(\frac{\partial^2 y^\ell}{\partial x^k \partial x^j}\frac{\partial}{\partial y^\ell} + \frac{\partial y^\ell}{\partial x^j}\nabla_{\frac{\partial y^r}{\partial x^k}\frac{\partial}{\partial y^r}}\frac{\partial}{\partial y^\ell}\Big) \\
&= \frac{\partial x^i}{\partial y^s} dy^s\Big(\frac{\partial^2 y^\ell}{\partial x^k \partial x^j}\frac{\partial}{\partial y^\ell} + \frac{\partial y^\ell}{\partial x^j}\frac{\partial y^r}{\partial x^k}\nabla_{\frac{\partial}{\partial y^r}}\frac{\partial}{\partial y^\ell}\Big) \\
&= \frac{\partial x^i}{\partial y^s}\frac{\partial^2 y^s}{\partial x^k \partial x^j} + \frac{\partial x^i}{\partial y^s}\frac{\partial y^\ell}{\partial x^j}\frac{\partial y^r}{\partial x^k}\hat\Gamma^s{}_{\ell r} .
\end{aligned} \tag{A.9.13}$$

Summarising,

$$\boxed{\Gamma^i{}_{jk} = \hat\Gamma^s{}_{\ell r}\frac{\partial x^i}{\partial y^s}\frac{\partial y^\ell}{\partial x^j}\frac{\partial y^r}{\partial x^k} + \frac{\partial x^i}{\partial y^s}\frac{\partial^2 y^s}{\partial x^k \partial x^j}} . \tag{A.9.14}$$

Thus, the $\Gamma^i{}_{jk}$'s do *not* form a tensor; instead they transform as a tensor *plus* a non-homogeneous term containing second derivatives, as seen above.

EXERCICE A.9.1 Let $\Gamma^i{}_{jk}$ transform as in (A.9.14) under coordinate transformations. If X and Y are vector fields, define in local coordinates

$$\nabla_X Y := \Big(X(Y^i) + \Gamma^i{}_{jk} X^k Y^k\Big)\,\partial_i . \tag{A.9.15}$$

Show that $\nabla_X Y$ transforms as a vector field under coordinate transformations (and thus is a vector field). Hence, a collection of fields $\{\Gamma^i{}_{jk}\}$ satisfying the transformation law (A.9.14) can be used to define a covariant derivative using (A.9.15).

A.9.4 Torsion

Because the inhomogeneous term in (A.9.14) is symmetric under the interchange of i and j, it follows from (A.9.14) that

$$T^i_{jk} := \Gamma^i{}_{kj} - \Gamma^i{}_{jk}$$

does transform as a tensor, called *the torsion tensor* of ∇.

An index-free definition of torsion proceeds as follows: Let ∇ be a covariant derivative defined for vector fields, the *torsion tensor* T is defined by the formula

$$\boxed{T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]}\,, \tag{A.9.16}$$

where $[X,Y]$ is the Lie bracket. We obviously have

$$T(X,Y) = -T(Y,X)\,. \tag{A.9.17}$$

Let us check that T is actually a tensor field: multi-linearity with respect to addition is obvious. To check what happens under multiplication by functions, in view of (A.9.17) it is sufficient to do the calculation for the first slot of T. We then have

$$\begin{aligned} T(fX,Y) &= \nabla_{fX}Y - \nabla_Y(fX) - [fX,Y] \\ &= f\Big(\nabla_X Y - \nabla_Y X\Big) - Y(f)X - [fX,Y]\,. \end{aligned} \tag{A.9.18}$$

To work out the last commutator term we compute, for any function g,

$$[fX,Y](g) = fX(Y(g)) - \underbrace{Y(fX(g))}_{=Y(f)X(g)+fY(X(g))} = f[X,Y](g) - Y(f)X(g)\,,$$

hence

$$[fX,Y] = f[X,Y] - Y(f)X\,, \tag{A.9.19}$$

and the last term here cancels the undesirable second-to-last term in (A.9.18), as required.

In a coordinate basis ∂_μ we have $[\partial_\mu,\partial_\nu]=0$ and one finds from (A.9.10)

$$\boxed{T(\partial_\mu,\partial_\nu) = (\Gamma^\sigma{}_{\nu\mu} - \Gamma^\sigma{}_{\mu\nu})\partial_\sigma}\,, \tag{A.9.20}$$

which shows that T is determined by twice the antisymmetrization of the $\Gamma^\sigma{}_{\mu\nu}$'s over the lower indices. In particular that last antisymmetrization produces a tensor field.

A.9.5 Covectors

Suppose that we are given a covariant derivative on vector fields, there is a natural way of inducing a covariant derivative on one-forms by imposing the condition that *the duality operation be compatible with the Leibniz rule*: given two vector fields X and Y together with a field of one-forms α, one sets

$$\boxed{(\nabla_X\alpha)(Y) := X(\alpha(Y)) - \alpha(\nabla_X Y)}\,. \tag{A.9.21}$$

Let us, first, check that (A.9.21) indeed defines a field of one-forms. The linearity, in the Y variable, with respect to addition is obvious. Next, for any function f we have

$$\begin{aligned}(\nabla_X\alpha)(fY) &= X(\alpha(fY)) - \alpha(\nabla_X(fY))\\ &= X(f)\alpha(Y) + fX(\alpha(Y)) - \alpha(X(f)Y + f\nabla_X Y)\\ &= f(\nabla_X\alpha)(Y)\ ,\end{aligned}$$

as should be the case for one-forms. Next, we need to check that ∇ defined by (A.9.21) does satisfy the remaining axioms imposed on covariant derivatives. Again multi-linearity with respect to addition is obvious, as well as linearity with respect to multiplication of X by a function. Finally,

$$\begin{aligned}\nabla_X(f\alpha)(Y) &= X(f\alpha(Y)) - f\alpha(\nabla_X Y)\\ &= X(f)\alpha(Y) + f(\nabla_X\alpha)(Y)\ ,\end{aligned}$$

as desired.

The duality pairing

$$T^*_pM \times T_pM \ni (\alpha, X) \to \alpha(X) \in \mathbb{R}$$

is sometimes called *contraction*. As already pointed out, the operation ∇ on one forms has been defined in (A.9.21) so as to satisfy the *Leibniz rule under duality pairing*:

$$X(\alpha(Y)) = (\nabla_X\alpha)(Y) + \alpha(\nabla_X Y)\ ; \tag{A.9.22}$$

this follows directly from (A.9.21). This should not be confused with the Leibniz rule under multiplication by functions, which is part of the definition of a covariant derivative, and therefore always holds. It should be kept in mind that (A.9.22) does not necessarily hold for all covariant derivatives: if ${}^v\nabla$ is some covariant derivative on vectors, and ${}^f\nabla$ is some covariant derivative on one-forms, in general one will have

$$X(\alpha(Y)) \neq ({}^f\nabla_X)\alpha(Y) + \alpha({}^v\nabla_X Y)\ .$$

Using the basis-expression (A.9.11) of $\nabla_X Y$ and the definition (A.9.21) we have

$$\nabla_X\alpha = X^a\nabla_a\alpha_b\,\theta^b\ ,$$

with

$$\begin{aligned}\boxed{\nabla_a\alpha_b} &:= (\nabla_{e_a}\alpha)(e_b)\\ &= e_a(\alpha(e_b)) - \alpha(\nabla_{e_a}e_b)\\ &= \boxed{e_a(\alpha_b) - \Gamma^c{}_{ba}\alpha_c}\ .\end{aligned}$$

A.9.6 Higher order tensors

It should now be clear how to extend ∇ to tensors of arbitrary valence: if T is r covariant and s contravariant one sets

$$\begin{aligned}(\nabla_X T)(X_1,\ldots,X_r,\alpha_1,\ldots\alpha_s) &:= X\Big(T(X_1,\ldots,X_r,\alpha_1,\ldots\alpha_s)\Big)\\ &-T(\nabla_X X_1,\ldots,X_r,\alpha_1,\ldots\alpha_s)-\ldots-T(X_1,\ldots,\nabla_X X_r,\alpha_1,\ldots\alpha_s)\\ &-T(X_1,\ldots,X_r,\nabla_X\alpha_1,\ldots\alpha_s)-\ldots-T(X_1,\ldots,X_r,\alpha_1,\ldots\nabla_X\alpha_s)\,.\end{aligned} \tag{A.9.23}$$

The verification that this defines a covariant derivative proceeds in a way identical to that for one-forms. In a basis we have

$$\nabla_X T = X^a\nabla_a T_{a_1\ldots a_r}{}^{b_1\ldots b_s}\theta^{a_1}\otimes\ldots\otimes\theta^{a_r}\otimes e_{b_1}\otimes\ldots\otimes e_{b_s}\ ,$$

and (A.9.23) gives

$$\begin{aligned}\nabla_a T_{a_1\ldots a_r}{}^{b_1\ldots b_s} &:= (\nabla_{e_a}T)(e_{a_1},\ldots,e_{a_r},\theta^{b_1},\ldots,\theta^{b_s})\\ &= e_a(T_{a_1\ldots a_r}{}^{b_1\ldots b_s})-\Gamma^c{}_{a_1 a}T_{c\ldots a_r}{}^{b_1\ldots b_s}-\ldots-\Gamma^c{}_{a_r a}T_{a_1\ldots c}{}^{b_1\ldots b_s}\\ &\quad+\Gamma^{b_1}{}_{ca}T_{a_1\ldots a_r}{}^{c\ldots b_s}+\ldots+\Gamma^{b_s}{}_{ca}T_{a_1\ldots a_r}{}^{b_1\ldots c}\,.\end{aligned} \tag{A.9.24}$$

Carrying over the last two lines of (A.9.23) to the left-hand-side of that equation one obtains the Leibniz rule for ∇ under pairings of tensors with vectors or forms. It should be clear from (A.9.23) that ∇ defined by that equation is the *only covariant derivative which agrees with the original one on vectors, and which satisfies the Leibniz rule under the pairing operation.* We will only consider such covariant derivatives in this work.

A.9.7 Geodesics and Christoffel symbols

A geodesic can be defined as the stationary point of the action

$$I(\gamma) = \int_a^b \underbrace{\frac{1}{2}g(\dot\gamma,\dot\gamma)(s)}_{=:\mathscr{L}(\gamma,\dot\gamma)}ds\ , \tag{A.9.25}$$

where $\gamma:[a,b]\to M$ is a differentiable curve. Thus,

$$\mathscr{L}(x^\mu,\dot x^\nu)=\frac{1}{2}g_{\alpha\beta}(x^\mu)\dot x^\alpha\dot x^\beta\ .$$

One readily finds the Euler-Lagrange equations for this Lagrange function:

$$\frac{d}{ds}\left(\frac{\partial\mathscr{L}}{\partial\dot x^\mu}\right)=\frac{\partial\mathscr{L}}{\partial x^\mu}\quad\Longleftrightarrow\quad\frac{d^2x^\mu}{ds^2}+\Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=0\,. \tag{A.9.26}$$

This provides a very convenient way of calculating the Christoffel symbols: given a metric g, write down $\mathscr{L}$, work out the Euler-Lagrange equations, and identify the Christoffels as the coefficients of the first derivative terms in those equations.

EXERCICE A.9.2 Prove (A.9.26).

(The Euler-Lagrange equations for (A.9.25) are identical with those of

$$\tilde{I}(\gamma) = \int_a^b \sqrt{|g(\dot\gamma,\dot\gamma)(s)|}ds \;, \tag{A.9.27}$$

but (A.9.25) is more convenient to work with. For example, $\mathscr{L}$ is differentiable at points where $\dot\gamma$ vanishes, while $\sqrt{|g(\dot\gamma,\dot\gamma)(s)|}$ is not. The aesthetic advantage of (A.9.27), of being reparameterization-invariant, is more than compensated by the calculational convenience of $\mathscr{L}$.)

EXAMPLE A.9.3 As an example, consider a metric of the form

$$g = dr^2 + f(r)d\varphi^2 \;.$$

Special cases of this metric include the Euclidean metric on $\mathbb{R}^2$ (then $f(r) = r^2$), and the canonical metric on a sphere (then $f(r) = \sin^2 r$, with r actually being the polar angle θ). The Lagrangian (A.9.27) is thus

$$L = \frac{1}{2}\left(\dot r^2 + f(r)\dot\varphi^2\right) \;.$$

The Euler-Lagrange equations read

$$\underbrace{\frac{\partial L}{\partial \varphi}}_{0} = \frac{d}{ds}\left(\frac{\partial L}{\partial \dot\varphi}\right) = \frac{d}{ds}(f(r)\dot\varphi) \;,$$

so that

$$0 = f\ddot\varphi + f'\dot r\dot\varphi = f\left(\ddot\varphi + \Gamma^\varphi_{\varphi\varphi}\dot\varphi^2 + 2\Gamma^\varphi_{r\varphi}\dot r\dot\varphi + \Gamma^\varphi_{rr}\dot r^2\right) \quad\Longrightarrow\quad \Gamma^\varphi_{\varphi\varphi} = \Gamma^\varphi_{rr} = 0\;, \quad \Gamma^\varphi_{r\varphi} = \frac{f'}{2f}\;.$$

Similarly

$$\underbrace{\frac{\partial L}{\partial r}}_{f'\dot\varphi^2/2} = \frac{d}{ds}\left(\frac{\partial L}{\partial \dot r}\right) = \ddot r \;,$$

so that

$$\Gamma^r_{r\varphi} = \Gamma^r_{rr} = 0\;, \quad \Gamma^r_{\varphi\varphi} = -\frac{f'}{2}\;.$$

A.10 The Levi-Civita connection

One of the fundamental results in pseudo-Riemannian geometry is that of the existence of a torsion-free connection which preserves the metric:

THEOREM A.10.1 *Let g be a two-covariant symmetric non-degenerate tensor field on a manifold M. Then there exists a unique connection ∇ such that*

1. *$\nabla g = 0$,*

2. *the torsion tensor T of ∇ vanishes.*

PROOF: Using the definition of $\nabla_i g_{jk}$ we have

$$0 = \nabla_i g_{jk} \equiv \partial_i g_{jk} - \Gamma^\ell{}_{ji} g_{\ell k} - \Gamma^\ell{}_{ki} g_{\ell j} \; ; \tag{A.10.1}$$

here we have written $\Gamma^i{}_{jk}$ instead of $\Gamma^i{}_{jk}$, as is standard in the literature. We rewrite this equation making cyclic permutations of indices, and changing the overall sign:

$$0 = -\nabla_j g_{ki} \equiv -\partial_j g_{ki} + \Gamma^\ell{}_{kj} g_{\ell i} + \Gamma^\ell{}_{ij} g_{\ell k} \; .$$
$$0 = -\nabla_k g_{ij} \equiv -\partial_k g_{ij} + \Gamma^\ell{}_{ik} g_{\ell j} + \Gamma^\ell{}_{jk} g_{\ell i} \; .$$

Adding the three equations and using symmetry of Γ^k_{ji} in ij one obtains

$$0 = \partial_i g_{jk} - \partial_j g_{ki} - \partial_k g_{ij} + 2\Gamma^\ell{}_{jk} g_{\ell i} \; ,$$

Multiplying by g^{im} we obtain

$$\Gamma^m{}_{jk} = g^{mi} \Gamma^\ell_{jk} g_{\ell i} = \frac{1}{2} g^{mi} (\partial_i g_{jk} - \partial_j g_{ki} - \partial_k g_{ij}) \; . \tag{A.10.2}$$

This proves uniqueness.

A straightforward, though somewhat lengthy, calculation shows that the $\Gamma^m{}_{jk}$'s defined by (A.10.2) satisfy the transformation law (A.9.14). Exercice A.9.1 shows that the formula (A.9.15) defines a torsion-free connection. It then remains to check that the insertion of the $\Gamma^m{}_{jk}$'s, as given by (A.10.2), into the right-hand-side of (A.10.1), indeed gives zero, proving existence. □

Let us give a coordinate-free version of the above, which turns out to require considerably more work: Suppose, first, that a connection satisfying the above is given. By the Leibniz rule we then have for any vector fields X, Y and Z,

$$0 = (\nabla_X g)(Y, Z) = X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) \; . \tag{A.10.3}$$

One then rewrites the same equation applying cyclic permutations to X, Y, and Z, with a minus sign for the last equation:

$$\begin{aligned} g(\nabla_X Y, Z) + g(Y, \nabla_X Z) &= X(g(Y, Z)) \; , \\ g(\nabla_Y Z, X) + g(Z, \nabla_Y X) &= Y(g(Z, X)) \; , \\ -g(\nabla_Z X, Y) - g(X, \nabla_Z Y) &= -Z(g(X, Y)) \; . \end{aligned} \tag{A.10.4}$$

As the torsion tensor vanishes, the sum of the left-hand-sides of these equations can be manipulated as follows:

$$\begin{aligned} & g(\nabla_X Y, Z) + g(Y, \nabla_X Z) + g(\nabla_Y Z, X) + g(Z, \nabla_Y X) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) \\ &= g(\nabla_X Y + \nabla_Y X, Z) + g(Y, \nabla_X Z - \nabla_Z X) + g(X, \nabla_Y Z - \nabla_Z Y) \\ &= g(2\nabla_X Y - [X, Y], Z) + g(Y, [X, Z]) + g(X, [Y, Z]) \\ &= 2g(\nabla_X Y, Z) - g([X, Y], Z) + g(Y, [X, Z]) + g(X, [Y, Z]) \; . \end{aligned}$$

This shows that the sum of the three equations (A.10.4) can be rewritten as

$$\begin{aligned} 2g(\nabla_X Y, Z) &= g([X, Y], Z) - g(Y, [X, Z]) - g(X, [Y, Z]) \\ &\quad + X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \; . \end{aligned} \tag{A.10.5}$$

Since Z is arbitrary and g is non-degenerate, the left-hand-side of this equation determines the vector field $\nabla_X Y$ uniquely, and uniqueness of ∇ follows.

To prove existence, let $S(X,Y)(Z)$ be defined as one half of the right-hand-side of (A.10.5),

$$\begin{aligned} S(X,Y)(Z) &= \frac{1}{2}\Big(X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &\quad + g(Z,[X,Y]) - g(Y,[X,Z]) - g(X,[Y,Z])\Big) . \end{aligned} \tag{A.10.6}$$

Clearly S is linear with respect to addition in all fields involved. It is straightforward to check that it is linear with respect to multiplication of Z by a function, and since g is non-degenerate there exists a unique vector field $W(X,Y)$ such that

$$S(X,Y)(Z) = g(W(X,Y),Z) .$$

One readily checks that the assignment

$$(X,Y) \to W(X,Y)$$

satisfies all the requirements imposed on a covariant derivative $\nabla_X Y$. With some more work one checks that ∇_X so defined is torsion free, and metric compatible. □

Let us check that (A.10.5) reproduces (A.10.2): Consider (A.10.5) with $X = \partial_\gamma$, $Y = \partial_\beta$ and $Z = \partial_\sigma$,

$$\begin{aligned} 2g(\nabla_\gamma \partial_\beta, \partial_\sigma) &= 2g(\Gamma^\rho{}_{\beta\gamma}\partial_\rho, \partial_\sigma) \\ &= 2g_{\rho\sigma}\Gamma^\rho{}_{\beta\gamma} \\ &= \partial_\gamma g_{\beta\sigma} + \partial_\beta g_{\gamma\sigma} - \partial_\sigma g_{\beta\gamma} \end{aligned} \tag{A.10.7}$$

Multiplying this equation by $g^{\alpha\sigma}/2$ we then obtain

$$\boxed{\Gamma^\alpha{}_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\sigma}\{\partial_\beta g_{\sigma\gamma} + \partial_\gamma g_{\sigma\beta} - \partial_\sigma g_{\beta\gamma}\}} . \tag{A.10.8}$$

A.11 "Local inertial coordinates"

PROPOSITION A.11.1 *1. Let g be a Lorentzian metric, for every $p \in M$ there exists a neighborhood thereof with a coordinate system such that* $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(1, -1, \cdots, -1)$ *at p.*

2. If g is differentiable, then the coordinates can be further chosen so that

$$\partial_\sigma g_{\alpha\beta} = 0 \tag{A.11.1}$$

at p.

The coordinates above will be referred to as *local inertial coordinates near p.*

REMARK A.11.2 An analogous result holds for any pseudo-Riemannian metric. Note that the "normal coordinates" satisfy the above. However, for metrics of finite differentiability, the introduction of normal coordinates leads to a loss of differentiability of the metric components, while the construction below preserves the order of differentiability.

PROOF: 1. Let y^μ be any coordinate system around p, shifting by a constant vector we can assume that p corresponds to $y^\mu = 0$. Let $e_a = e_a{}^\mu \partial/\partial y^\mu$ be any frame at p such that $g(e_a, e_b) = \eta_{ab}$ — such frames can be found by, *e.g.*, a Gram-Schmidt orthogonalisation. Calculating the determinant of both sides of the equation

$$g_{\mu\nu} e_a{}^\mu e_b{}^\nu = \eta_{ab}$$

we obtain, at p,

$$\det(g_{\mu\nu}) \det(e_a{}^\mu)^2 = -1 ,$$

which shows that $\det(e_a{}^\mu)$ is non-vanishing. It follows that the formula

$$y^\mu = e^\mu{}_a x^a$$

defines a (linear) diffeomorphism. In the new coordinates we have, again at p,

$$g\Big(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\Big) = e^\mu{}_a e^\nu{}_b g\Big(\frac{\partial}{\partial y^\mu}, \frac{\partial}{\partial y^\nu}\Big) = \eta_{ab} . \tag{A.11.2}$$

2. We will use (A.9.14), which uses latin indices, so let us switch to that notation. Let x^i be the coordinates described in point 1., recall that p lies at the origin of those coordinates. The new coordinates $\hat x^j$ will be implicitly defined by the equations

$$x^i = \hat x^i + \frac{1}{2} A^i{}_{jk} \hat x^j \hat x^k ,$$

where $A^i{}_{jk}$ is a set of constants, symmetric with respect to the interchange of j and k. Recall (A.9.14),

$$\hat\Gamma^i{}_{jk} = \Gamma^s{}_{\ell r} \frac{\partial \hat x^i}{\partial x^s} \frac{\partial x^\ell}{\partial \hat x^j} \frac{\partial x^r}{\partial \hat x^k} + \frac{\partial \hat x^i}{\partial x^s} \frac{\partial^2 x^s}{\partial \hat x^k \partial \hat x^j} ; \tag{A.11.3}$$

here we use $\hat\Gamma^s_{\ell r}$ to denote the Christoffel symbols of the metric in the hatted coordinates. Then, at $x^i = 0$, this equation reads

$$\begin{aligned} \hat\Gamma^i{}_{jk} &= \Gamma^s{}_{\ell r} \underbrace{\frac{\partial \hat x^i}{\partial x^s}}_{\delta^i_s} \underbrace{\frac{\partial x^\ell}{\partial \hat x^j}}_{\delta^\ell_j} \underbrace{\frac{\partial x^r}{\partial \hat x^k}}_{\delta^r_k} + \underbrace{\frac{\partial x^i}{\partial x^s}}_{\delta^i_s} \underbrace{\frac{\partial^2 x^s}{\partial \hat x^k \partial \hat x^j}}_{A^s_{kj}} \\ &= \Gamma^i{}_{jk} + A^i{}_{kj} . \end{aligned}$$

Choosing A^i_{jk} as $-\Gamma^i{}_{jk}(0)$, the result follows.

If you do not like to remember formulae such as (A.9.14), proceed as follows: Let x^μ be the coordinates described in point 1. The new coordinates $\hat x^\alpha$ will be implicitly defined by the equations

$$x^\mu = \hat x^\mu + \frac{1}{2} A^\mu{}_{\alpha\beta} \hat x^\alpha \hat x^\beta ,$$

where $A^\mu{}_{\alpha\beta}$ is a set of constants, symmetric with respect to the interchange of α and β. Set

$$\hat g_{\alpha\beta} := g\Big(\frac{\partial}{\partial \hat x^\alpha}, \frac{\partial}{\partial \hat x^\beta}\Big) , \qquad g_{\alpha\beta} := g\Big(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\Big) .$$

Recall the transformation law

$$\hat g_{\mu\nu}(\hat x^\sigma) = g_{\alpha\beta}(x^\rho(\hat x^\sigma))\frac{\partial x^\alpha}{\partial \hat x^\mu}\frac{\partial x^\beta}{\partial \hat x^\nu} \ .$$

By differentiation one obtains at $x^\mu = \hat x^\mu = 0$,

$$\begin{aligned}\frac{\partial \hat g_{\mu\nu}}{\partial \hat x^\rho}(0) &= \frac{\partial g_{\mu\nu}}{\partial x^\rho}(0) + g_{\alpha\beta}(0)\Big(A^\alpha{}_{\mu\rho}\delta^\beta_\nu + \delta^\alpha_\mu A^\beta{}_{\nu\rho}\Big) \\ &= \frac{\partial g_{\mu\nu}}{\partial x^\rho}(0) + A_{\nu\mu\rho} + A_{\mu\nu\rho} \ , \end{aligned} \qquad (A.11.4)$$

where

$$A_{\alpha\beta\gamma} := g_{\alpha\sigma}(0)A^\sigma{}_{\beta\gamma} \ .$$

It remains to show that we can choose $A^\sigma{}_{\beta\gamma}$ so that the left-hand-side can be made to vanish at p. An explicit formula for $A_{\sigma\beta\gamma}$ can be obtained from (A.11.4) by a cyclic permutation calculation similar to that in (A.10.4). After raising the first index, the final result is

$$A^\alpha{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\rho}\left\{\frac{\partial g_{\beta\gamma}}{\partial x^\rho} - \frac{\partial g_{\beta\rho}}{\partial x^\gamma} - \frac{\partial g_{\rho\gamma}}{\partial x^\beta}\right\}(0) \ ;$$

the reader may wish to check directly that this does indeed lead to a vanishing right-hand-side of (A.11.4).

□

A.12 Curvature

Let ∇ be a covariant derivative defined for vector fields, the curvature tensor is defined by the formula

$$\boxed{R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z} \ , \qquad (A.12.1)$$

where, as elsewhere, $[X,Y]$ is the Lie bracket defined in (A.3.6). We note the anti-symmetry

$$R(X,Y)Z = -R(Y,X)Z \ . \qquad (A.12.2)$$

It turns out the this defines a tensor. Multi-linearity with respect to addition is obvious, but multiplication by functions require more work.

First, we have (see (A.9.19))

$$\begin{aligned} R(fX,Y)Z &= \nabla_{fX}\nabla_Y Z - \nabla_Y\nabla_{fX}Z - \nabla_{[fX,Y]}Z \\ &= f\nabla_X\nabla_Y Z - \nabla_Y(f\nabla_X Z) - \underbrace{\nabla_{f[X,Y]-Y(f)X}Z}_{=f\nabla_{[X,Y]}Z - Y(f)\nabla_X Z} \\ &= fR(X,Y)Z \ . \end{aligned}$$

The simplest proof of linearity in the last slot proceeds via an index calculation in adapted coordinates; so while we will do the "elegant", index-free version shortly, let us do the ugly one first. We use the coordinate system of Proposition A.11.1 below, in which the first derivatives of the metric vanish at the prescribed point p:

$$\begin{aligned}\nabla_i\nabla_j Z^k &= \partial_i(\partial_j Z^k - \Gamma^k{}_{\ell j}Z^\ell) + \underbrace{0\times\nabla Z}_{\text{at } p} \\ &= \partial_i\partial_j Z^k - \partial_i\Gamma^k{}_{\ell j}Z^\ell \quad \text{at } p\ . \qquad \text{(A.12.3)}\end{aligned}$$

Antisymmetrising in i and j, the terms involving the second derivatives of Z drop out, so the result is indeed linear in Z. So $\nabla_i\nabla_j Z^k - \nabla_j\nabla_i Z^k$ is a tensor field linear in Z, and therefore can be written as $R^k{}_{\ell ij}Z^\ell$.

Note that $\nabla_i\nabla_j Z^k$ is, by definition, the tensor field of first covariant derivatives of the tensor field $\nabla_j Z^k$, while (A.12.1) involves covariant derivatives of vector fields only, so the equivalence of both approaches requires a further argument. This is provided in the calculation below leading to (A.12.6).

Next,

$$\begin{aligned}R(X,Y)(fZ) &= \nabla_X\nabla_Y(fZ) - \nabla_Y\nabla_X(fZ) - \nabla_{[X,Y]}(fZ) \\ &= \Big\{\nabla_X\Big(Y(f)Z + f\nabla_Y Z\Big)\Big\} - \Big\{\cdots\Big\}_{X\leftrightarrow Y} \\ &\quad -[X,Y](f)Z - f\nabla_{[X,Y]}Z \\ &= \Big\{\underbrace{X(Y(f))Z}_{a} + \underbrace{Y(f)\nabla_X Z + X(f)\nabla_Y Z}_{b} + f\nabla_X\nabla_Y Z\Big\} - \Big\{\cdots\Big\}_{X\leftrightarrow Y} \\ &\quad - \underbrace{[X,Y](f)Z}_{c} - f\nabla_{[X,Y]}Z\ .\end{aligned}$$

Now, a together with its counterpart with X and Y interchanged cancel out with c, while b is symmetric with respect to X and Y and therefore cancels out with its counterpart with X and Y interchanged, leading to the desired equality

$$R(X,Y)(fZ) = fR(X,Y)Z\ .$$

In a coordinate basis $\{e_a\} = \{\partial_\mu\}$ we find[2] (recall that $[\partial_\mu,\partial_\nu]=0$)

$$\begin{aligned}R^\alpha{}_{\beta\gamma\delta} &:= \langle dx^\alpha, R(\partial_\gamma,\partial_\delta)\partial_\beta\rangle \\ &= \langle dx^\alpha, \nabla_\gamma\nabla_\delta\partial_\beta\rangle - \langle\cdots\rangle_{\delta\leftrightarrow\gamma} \\ &= \langle dx^\alpha, \nabla_\gamma(\Gamma^\sigma{}_{\beta\delta}\partial_\sigma)\rangle - \langle\cdots\rangle_{\delta\leftrightarrow\gamma} \\ &= \langle dx^\alpha, \partial_\gamma(\Gamma^\sigma{}_{\beta\delta})\partial_\sigma + \Gamma^\rho{}_{\sigma\gamma}\Gamma^\sigma{}_{\beta\delta}\partial_\rho\rangle - \langle\cdots\rangle_{\delta\leftrightarrow\gamma} \\ &= \{\partial_\gamma\Gamma^\alpha{}_{\beta\delta} + \Gamma^\alpha{}_{\sigma\gamma}\Gamma^\sigma{}_{\beta\delta}\} - \{\cdots\}_{\delta\leftrightarrow\gamma}\ ,\end{aligned}$$

leading finally to

$$\boxed{\boxed{R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma\Gamma^\alpha{}_{\beta\delta} - \partial_\delta\Gamma^\alpha{}_{\beta\gamma} + \Gamma^\alpha{}_{\sigma\gamma}\Gamma^\sigma{}_{\beta\delta} - \Gamma^\alpha{}_{\sigma\delta}\Gamma^\sigma{}_{\beta\gamma}}}\ . \qquad \text{(A.12.4)}$$

In a general frame some supplementary commutator terms will appear in the formula for $R^a{}_{bcd}$.

We note the following:

[2]The reader is warned that certain authors use a different sign convention either for $R(X,Y)Z$, or for $R^\alpha{}_{\beta\gamma\delta}$, or both. A useful table that lists the sign conventions for a series of standard GR references can be found on the backside of the front cover of [140].

THEOREM A.12.1 *There exists a coordinate system in which the metric tensor field has vanishing second derivatives at p if and only if its Riemann tensor vanishes at p. Furthermore, there exists a coordinate system in which the metric tensor field has constant entries near p if and only if the Riemann tensor vanishes near p.*

PROOF: The condition is necessary, since Riem is a tensor. The sufficiency will be admitted. □

The calculation of the curvature tensor is often a very traumatic experience. There is one obvious case where things are painless, when all $g_{\mu\nu}$'s are constants: in this case the Christoffels vanish, and so does the curvature tensor.

For more general metrics one way out is to use symbolic computer algebra, this can, e.g., be done online on `http://grtensor.phy.queensu.ca/NewDemo`. The MATHEMATICA package XACT [139] provides a very powerful tool for all kinds of calculations involving curvature.

EXAMPLE A.12.2 As a less trivial example, consider the round two sphere, which we write in the form

$$g = d\theta^2 + e^{2f} d\varphi^2 \ , \qquad e^{2f} = \sin^2\theta \ .$$

As seen in Example A.9.3, the Christoffel symbols are easily founds from the Lagrangean for geodesics:

$$\mathscr{L} = \frac{1}{2}(\dot\theta^2 + e^{2f}\dot\varphi^2) \ .$$

The Euler-Lagrange equations give

$$\Gamma^\theta{}_{\varphi\varphi} = -f' e^{2f} \ , \quad \Gamma^\varphi{}_{\theta\varphi} = \Gamma^\varphi{}_{\varphi\theta} = f' \ ,$$

with the remaining Christoffel symbols vanishing. Using the definition of the Riemann tensor we then immediately find

$$R^\varphi{}_{\theta\varphi\theta} = -f'' - (f')^2 = -e^{-f}(e^f)'' = 1 \ . \tag{A.12.5}$$

All remaining components of the Riemann tensor can be obtained from this one by raising and lowering of indices, together with the symmetry operations which we are about to describe. This leads to

$$R_{ab} = g_{ab} \ , \qquad R = 2 \ .$$

Equation (A.12.1) is most frequently used "upside-down", not as a definition of the Riemann tensor, but as a tool for calculating what happens when one changes the order of covariant derivatives. Recall that for partial derivatives we have

$$\partial_\mu\partial_\nu Z^\sigma = \partial_\nu\partial_\mu Z^\sigma \ ,$$

but this is not true in general if partial derivatives are replaced by covariant ones:

$$\nabla_\mu\nabla_\nu Z^\sigma \neq \nabla_\nu\nabla_\mu Z^\sigma \ .$$

To find the correct formula let us consider the tensor field S defined as

$$Y \longrightarrow S(Y) := \nabla_Y Z \ .$$

In local coordinates, S takes the form

$$S = \nabla_\mu Z^\nu \, dx^\mu \otimes \partial_\nu \ .$$

It follows from the Leibniz rule — or, equivalently, from the definitions in Section A.9 — that we have

$$\begin{aligned}(\nabla_X S)(Y) &= \nabla_X(S(Y)) - S(\nabla_X Y)\\ &= \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z \ .\end{aligned}$$

The commutator of the derivatives can then be calculated as

$$\begin{aligned}(\nabla_X S)(Y) - (\nabla_Y S)(X) &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\nabla_X Y} Z + \nabla_{\nabla_Y X} Z\\ &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\\ &\quad +\nabla_{[X,Y]} Z - \nabla_{\nabla_X Y} Z + \nabla_{\nabla_Y X} Z\\ &= R(X,Y)Z - \nabla_{T(X,Y)} Z \ . \end{aligned} \tag{A.12.6}$$

Writing ∇S in the usual form

$$\nabla S = \nabla_\sigma S_\mu{}^\nu \, dx^\sigma \otimes dx^\mu \otimes \partial_\nu = \nabla_\sigma \nabla_\mu Z^\nu \, dx^\sigma \otimes dx^\mu \otimes \partial_\nu \ ,$$

we are thus led to

$$\nabla_\mu \nabla_\nu Z^\alpha - \nabla_\nu \nabla_\mu Z^\alpha = R^\alpha{}_{\sigma\mu\nu} Z^\sigma - T^\sigma{}_{\mu\nu} \nabla_\sigma Z^\alpha \ . \tag{A.12.7}$$

In the important case of vanishing torsion, the coordinate-component equivalent of (A.12.1) is thus

$$\boxed{\nabla_\mu \nabla_\nu X^\alpha - \nabla_\nu \nabla_\mu X^\alpha = R^\alpha{}_{\sigma\mu\nu} X^\sigma} \ . \tag{A.12.8}$$

An identical calculation gives, still for torsionless connections,

$$\nabla_\mu \nabla_\nu a_\alpha - \nabla_\nu \nabla_\mu a_\alpha = -R^\sigma{}_{\alpha\mu\nu} a_\sigma \ . \tag{A.12.9}$$

For a general tensor t and torsion-free connection each tensor index comes with a corresponding Riemann tensor term:

$$\begin{aligned}&\nabla_\mu \nabla_\nu t_{\alpha_1 \ldots \alpha_r}{}^{\beta_1 \ldots \beta_s} - \nabla_\nu \nabla_\mu t_{\alpha_1 \ldots \alpha_r}{}^{\beta_1 \ldots \beta_s} =\\ &\quad -R^\sigma{}_{\alpha_1 \mu\nu} t_{\sigma \ldots \alpha_r}{}^{\beta_1 \ldots \beta_s} - \ldots - R^\sigma{}_{\alpha_r \mu\nu} t_{\alpha_1 \ldots \sigma}{}^{\beta_1 \ldots \beta_s}\\ &\quad +R^{\beta_1}{}_{\sigma\mu\nu} t_{\alpha_1 \ldots \alpha_r}{}^{\sigma \ldots \beta_s} + \ldots + R^{\beta_s}{}_{\sigma\mu\nu} t_{\alpha_1 \ldots \alpha_r}{}^{\beta_1 \ldots \sigma} \ . \end{aligned} \tag{A.12.10}$$

A.12.1 Bianchi identities

We have already seen the anti-symmetry property of the Riemann tensor, which in the index notation corresponds to the equation

$$R^\alpha{}_{\beta\gamma\delta} = -R^\alpha{}_{\beta\delta\gamma} \ . \tag{A.12.11}$$

There are a few other identities satisfied by the Riemann tensor, we start with the *first Bianchi identity*. Let $A(X,Y,Z)$ be any expression depending upon three vector fields X, Y, Z which is antisymmetric in X and Y, we set

$$\sum_{[XYZ]} A(X,Y,Z) := A(X,Y,Z) + A(Y,Z,X) + A(Z,X,Y) \; , \tag{A.12.12}$$

thus $\sum_{[XYZ]}$ is a sum over cyclic permutations of the vectors X, Y, Z. Clearly,

$$\sum_{[XYZ]} A(X,Y,Z) = \sum_{[XYZ]} A(Y,Z,X) = \sum_{[XYZ]} A(Z,X,Y) \; . \tag{A.12.13}$$

Suppose, first, that X, Y and Z commute. Using (A.12.13) together with the definition (A.9.16) of the torsion tensor T we calculate

$$\begin{aligned}
\sum_{[XYZ]} R(X,Y)Z &= \sum_{[XYZ]} \Big(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z\Big) \\
&= \sum_{[XYZ]} \Big(\nabla_X\nabla_Y Z - \nabla_Y \underbrace{(\nabla_Z X + T(X,Z))}_{\text{we have used } [X,Z]=0, \text{ see (A.9.16)}}\Big) \\
&= \underbrace{\sum_{[XYZ]} \nabla_X\nabla_Y Z - \sum_{[XYZ]} \nabla_Y\nabla_Z X}_{=0 \text{ (see (A.12.13))}} - \sum_{[XYZ]} \nabla_Y(\underbrace{T(X,Z)}_{=-T(Z,X)}) \\
&= \sum_{[XYZ]} \nabla_X(T(Y,Z)) \; ,
\end{aligned}$$

and in the last step we have again used (A.12.13). This can be somewhat rearranged by using the definition of the covariant derivative of a higher order tensor (compare (A.9.23)) — equivalently, using the Leibniz rule rewritten upside-down:

$$(\nabla_X T)(Y,Z) = \nabla_X(T(Y,Z)) - T(\nabla_X Y, Z) - T(Y, \nabla_X Z) \; .$$

This leads to

$$\begin{aligned}
\sum_{[XYZ]} \nabla_X(T(Y,Z)) &= \sum_{[XYZ]} \Big((\nabla_X T)(Y,Z) + T(\nabla_X Y, Z) + T(Y, \underbrace{\nabla_X Z}_{=T(X,Z)+\nabla_Z X})\Big) \\
&= \sum_{[XYZ]} \Big((\nabla_X T)(Y,Z) - T(\underbrace{T(X,Z)}_{=-T(Z,X)}, Y)\Big) \\
&\quad \underbrace{+ \sum_{[XYZ]} T(\nabla_X Y, Z) + \sum_{[XYZ]} \underbrace{T(Y, \nabla_Z X)}_{=-T(\nabla_Z X, Y)}}_{=0 \text{ (see (A.12.13))}} \\
&= \sum_{[XYZ]} \Big((\nabla_X T)(Y,Z) + T(T(X,Y),Z)\Big) \; .
\end{aligned}$$

Summarizing, we have obtained the first Bianchi identity:

$$\sum_{[XYZ]} R(X,Y)Z = \sum_{[XYZ]} \Big((\nabla_X T)(Y,Z) + T(T(X,Y),Z) \Big) , \qquad \text{(A.12.14)}$$

under the hypothesis that X, Y and Z commute. However, both sides of this equation are tensorial with respect to X, Y and Z, so that they remain correct without the commutation hypothesis.

We are mostly interested in connections with vanishing torsion, in which case (A.12.14) can be rewritten as

$$\boxed{R^\alpha{}_{\beta\gamma\delta} + R^\alpha{}_{\gamma\delta\beta} + R^\alpha{}_{\delta\beta\gamma} = 0} \, . \qquad \text{(A.12.15)}$$

Our next goal is the *second Bianchi identity.* We consider four vector fields X, Y, Z and W and we assume again that everybody commutes with everybody else. We calculate

$$\begin{aligned} \sum_{[XYZ]} \nabla_X (R(Y,Z)W) &= \sum_{[XYZ]} \Big(\underbrace{\nabla_X \nabla_Y \nabla_Z W}_{=R(X,Y)\nabla_Z W + \nabla_Y \nabla_X \nabla_Z W} - \nabla_X \nabla_Z \nabla_Y W \Big) \\ &= \sum_{[XYZ]} R(X,Y)\nabla_Z W \\ &\quad + \underbrace{\sum_{[XYZ]} \nabla_Y \nabla_X \nabla_Z W - \sum_{[XYZ]} \nabla_X \nabla_Z \nabla_Y W}_{=0} \\ &\quad . \end{aligned} \qquad \text{(A.12.16)}$$

Next,

$$\begin{aligned} \sum_{[XYZ]} (\nabla_X R)(Y,Z)W &= \sum_{[XYZ]} \Big(\nabla_X (R(Y,Z)W) - R(\nabla_X Y, Z)W \\ &\quad - R(Y, \underbrace{\nabla_X Z}_{=\nabla_Z X + T(X,Z)})W - R(Y,Z)\nabla_X W \Big) \\ &= \sum_{[XYZ]} \nabla_X (R(Y,Z)W) \\ &\quad \underbrace{- \sum_{[XYZ]} R(\nabla_X Y, Z)W - \sum_{[XYZ]} \underbrace{R(Y,\nabla_Z X)W}_{=-R(\nabla_Z X,Y)W}}_{=0} \\ &\quad - \sum_{[XYZ]} \Big(R(Y, T(X,Z))W + R(Y,Z)\nabla_X W \Big) \\ &= \sum_{[XYZ]} \Big(\nabla_X (R(Y,Z)W) - R(T(X,Y),Z)W - R(Y,Z)\nabla_X W \Big) . \end{aligned}$$

It follows now from (A.12.16) that the first term cancels out the third one, leading to

$$\sum_{[XYZ]} (\nabla_X R)(Y,Z)W = - \sum_{[XYZ]} R(T(X,Y),Z)W \, , \qquad \text{(A.12.17)}$$

which is the desired second Bianchi identity for commuting vector fields. As before, because both sides are multi-linear with respect to addition and multiplication by functions, the result remains valid for arbitrary vector fields.

For torsionless connections the components equivalent of (A.12.17) reads

$$\boxed{R^\alpha{}_{\mu\beta\gamma;\delta} + R^\alpha{}_{\mu\gamma\delta;\beta} + R^\alpha{}_{\mu\delta\beta;\gamma} = 0}\ . \tag{A.12.18}$$

A.12.2 Pair interchange symmetry

There is one more identity satisfied by the curvature tensor which is specific to the curvature tensor associated with the Levi-Civita connection, namely

$$g(X, R(Y,Z)W) = g(Y, R(X,W)Z)\ . \tag{A.12.19}$$

If one sets

$$\boxed{R_{abcd} := g_{ae}R^e{}_{bcd}}\ , \tag{A.12.20}$$

then (A.12.19) is equivalent to

$$\boxed{R_{abcd} = R_{cdab}}\ . \tag{A.12.21}$$

We will present two proofs of (A.12.19). The first is direct, but not very elegant. The second is prettier, but less insightful.

For the ugly proof, we suppose that the metric is twice-differentiable. By point 2. of Proposition A.11.1, in a neighborhood of any point $p \in M$ there exists a coordinate system in which the connection coefficients $\Gamma^\alpha{}_{\beta\gamma}$ vanish at p. Equation (A.12.4) evaluated at p therefore reads

$$\begin{aligned}
R^\alpha{}_{\beta\gamma\delta} &= \partial_\gamma\Gamma^\alpha{}_{\beta\delta} - \partial_\delta\Gamma^\alpha{}_{\beta\gamma} \\
&= \frac{1}{2}\Big\{g^{\alpha\sigma}\partial_\gamma(\partial_\delta g_{\sigma\beta} + \partial_\beta g_{\sigma\delta} - \partial_\sigma g_{\beta\delta}) \\
&\quad - g^{\alpha\sigma}\partial_\delta(\partial_\gamma g_{\sigma\beta} + \partial_\beta g_{\sigma\gamma} - \partial_\sigma g_{\beta\gamma})\Big\} \\
&= \frac{1}{2}g^{\alpha\sigma}\Big\{\partial_\gamma\partial_\beta g_{\sigma\delta} - \partial_\gamma\partial_\sigma g_{\beta\delta} - \partial_\delta\partial_\beta g_{\sigma\gamma} + \partial_\delta\partial_\sigma g_{\beta\gamma}\Big\}\ .
\end{aligned}$$

Equivalently,

$$R_{\sigma\beta\gamma\delta}(0) = \frac{1}{2}\Big\{\partial_\gamma\partial_\beta g_{\sigma\delta} - \partial_\gamma\partial_\sigma g_{\beta\delta} - \partial_\delta\partial_\beta g_{\sigma\gamma} + \partial_\delta\partial_\sigma g_{\beta\gamma}\Big\}(0)\ . \tag{A.12.22}$$

This last expression is obviously symmetric under the exchange of $\sigma\beta$ with $\gamma\delta$, leading to (A.12.21).

The above calculation traces back the pair-interchange symmetry to the definition of the Levi-Civita connection in terms of the metric tensor. As already mentioned, there exists a more elegant proof, where the origin of the symmetry is perhaps somewhat less apparent, which proceeds as follows: We start by noting that

$$0 = \nabla_a\nabla_b g_{cd} - \nabla_b\nabla_a g_{cd} = -R^e{}_{cab}g_{ed} - R^e{}_{dab}g_{ce}\ , \tag{A.12.23}$$

leading to anti-symmetry in the first two indices:

$$R_{abcd} = -R_{bacd} \; .$$

Next, using the cyclic symmetry for a torsion-free connection, we have

$$\begin{aligned} R_{abcd} + R_{cabd} + R_{bcad} &= 0 \; , \\ R_{bcda} + R_{dbca} + R_{cdba} &= 0 \; , \\ R_{cdab} + R_{acdb} + R_{dacb} &= 0 \; , \\ R_{dabc} + R_{bdac} + R_{abdc} &= 0 \; . \end{aligned}$$

The desired equation (A.12.21) follows now by adding the first two and subtracting the last two equations, using (A.12.23).

It is natural to enquire about the number of independent components of a tensor with the symmetries of a metric Riemann tensor in dimension n, the calculation proceeds as follows: as R_{abcd} is symmetric under the exchange of ab with cd, and anti-symmetric in each of these pairs, we can view it as a symmetric map from the space of anti-symmetric tensor with two indices. Now, the space of anti-symmetric tensors is $N = n(n-1)/2$ dimensional, while the space of symmetric maps in dimension N is $N(N+1)/2$ dimensional, so we obtain at most $n(n-1)(n^2-n+2)/8$ free parameters. However, we need to take into account the cyclic identity:

$$R_{abcd} + R_{bcad} + R_{cabd} = 0 \; . \tag{A.12.24}$$

If $a = b$ this reads

$$R_{aacd} + R_{acad} + R_{caad} = 0 \; ,$$

which has already been accounted for. Similarly if $a = d$ we obtain

$$R_{abca} + R_{bcaa} + R_{caba} = 0 \; ,$$

which holds in view of the previous identities. We conclude that the only new identities which could possibly arise are those where $abcd$ are all distinct. Clearly no expression involving three such components of the Riemann tensor can be obtained using the previous identities, so this is an independent constraint. In dimension four (A.12.24) provides thus four candidate equations for another constraint, labeled by d, but it is easily checked that they all coincide; this leads to 20 free parameters at each space point. The reader is encouraged to finish the counting in higher dimensions.

A.13 Geodesic deviation (Jacobi equation)

Suppose that we have a one parameter family of geodesics

$$\gamma(s,\lambda) \text{ (in local coordinates, } (\gamma^\alpha(s,\lambda))),$$

where s is the parameter along the geodesic, and λ is a parameter which distinguishes the geodesics. Set

$$Z(s,\lambda) := \frac{\partial\gamma(s,\lambda)}{\partial\lambda} \equiv \frac{\partial\gamma^\alpha(s,\lambda)}{\partial\lambda}\partial_\alpha \; ;$$

for each λ this defines a vector field Z along $\gamma(s,\lambda)$, which measures how nearby geodesics deviate from each other, since, to first order, using a Taylor expansion,

$$\gamma^\alpha(s,\lambda) = \gamma^\alpha(s,\lambda_0) + Z^\alpha(\lambda-\lambda_0) + O((\lambda-\lambda_0)^2)\ .$$

To measure how a vector field W changes along $s\mapsto\gamma(s,\lambda)$, one introduces the differential operator D/ds, defined as

$$\frac{DW^\mu}{ds} := \frac{\partial(W^\mu\circ\gamma)}{\partial s} + \Gamma^\mu{}_{\alpha\beta}\dot\gamma^\beta W^\alpha \tag{A.13.1}$$
$$= \dot\gamma^\beta\frac{\partial W^\mu}{\partial x^\beta} + \Gamma^\mu{}_{\alpha\beta}\dot\gamma^\beta W^\alpha \tag{A.13.2}$$
$$= \dot\gamma^\beta\nabla_\beta W^\mu\ . \tag{A.13.3}$$

(It would perhaps be more logical to write $\frac{DW^\mu}{\partial s}$ in the current context, but people never do that.) The last two lines only make sense if W is defined in a whole neighbourhood of γ, but for the first it suffices that $W(s)$ be defined along $s\mapsto\gamma(s,\lambda)$. (One possible way of making sense of the last two lines is to extend W^μ to any smooth vector field defined in a neighorhood of $\gamma^\mu(s,\lambda)$, and note that the result is independent of the particular choice of extension because the equation involves only derivatives tangential to $s\mapsto\gamma^\mu(s,\lambda)$.)

Analogously one sets

$$\frac{DW^\mu}{d\lambda} := \frac{\partial(W^\mu\circ\gamma)}{\partial\lambda} + \Gamma^\mu{}_{\alpha\beta}\partial_\lambda\gamma^\beta W^\alpha \tag{A.13.4}$$
$$= \partial_\lambda\gamma^\beta\frac{\partial W^\mu}{\partial x^\beta} + \Gamma^\mu{}_{\alpha\beta}\partial_\lambda\gamma^\beta W^\alpha \tag{A.13.5}$$
$$= Z^\beta\nabla_\beta W^\mu\ . \tag{A.13.6}$$

Note that since $s\to\gamma(s,\lambda)$ is a geodesic we have from (A.13.1) and (A.13.3)

$$\frac{D^2\gamma^\mu}{ds^2} := \frac{D\dot\gamma^\mu}{ds} = \frac{\partial^2\gamma^\mu}{\partial s^2} + \Gamma^\mu{}_{\alpha\beta}\dot\gamma^\beta\dot\gamma^\alpha = 0\ . \tag{A.13.7}$$

(This is sometimes written as $\dot\gamma^\alpha\nabla_\alpha\dot\gamma^\mu = 0$, which is again an abuse of notation since typically we will only know $\dot\gamma^\mu$ as a function of s, and so there is no such thing as $\nabla_\alpha\dot\gamma^\mu$.) Furthermore,

$$\frac{DZ^\mu}{ds} \underbrace{=}_{\text{(A.13.1)}} \frac{\partial^2\gamma^\mu}{\partial s\partial\lambda} + \Gamma^\mu{}_{\alpha\beta}\dot\gamma^\beta\partial_\lambda\gamma^\alpha \underbrace{=}_{\text{(A.13.4)}} \frac{D\dot\gamma^\mu}{d\lambda}\ , \tag{A.13.8}$$

(The abuse-of-notation derivation of the same formula proceeds as:

$$\nabla_{\dot\gamma}Z^\mu = \dot\gamma^\nu\nabla_\nu Z^\mu = \dot\gamma^\nu\nabla_\nu\partial_\lambda\gamma^\mu \underbrace{=}_{\text{(A.13.3)}} \frac{\partial^2\gamma^\mu}{\partial s\partial\lambda} + \Gamma^\mu{}_{\alpha\beta}\dot\gamma^\beta\partial_\lambda\gamma^\alpha \underbrace{=}_{\text{(A.13.6)}} Z^\beta\nabla_\beta\dot\gamma^\mu = \nabla_Z\dot\gamma^\mu\ , \tag{A.13.9}$$

which can then be written as

$$\nabla_{\dot\gamma}Z = \nabla_Z\dot\gamma\ .) \tag{A.13.10}$$

One can now repeat the calculation leading to (A.12.8) to obtain, for any vector field W defined along $\gamma^\mu(s,\lambda)$,

$$\frac{D}{ds}\frac{D}{d\lambda}W^\mu - \frac{D}{d\lambda}\frac{D}{ds}W^\mu = R_{\alpha\beta\delta}{}^{\mu}\dot\gamma^\alpha Z^\beta W^\delta \,. \tag{A.13.11}$$

If $W^\mu = \dot\gamma^\mu$ the second term at the left-hand-side is zero, and from $\frac{D}{d\lambda}\dot\gamma = \frac{D}{ds}Z$ we obtain

$$\frac{D^2 Z^\mu}{ds^2}(s) = R_{\alpha\beta\sigma}{}^{\mu}\dot\gamma^\alpha Z^\beta \dot\gamma^\sigma \,. \tag{A.13.12}$$

We have obtained an equation known as the *Jacobi equation*, or as the *geodesic deviation equation*; in index-free notation:

$$\boxed{\frac{D^2 Z}{ds^2} = R(\dot\gamma, Z)\dot\gamma} \,. \tag{A.13.13}$$

Solutions of (A.13.13) are called *Jacobi fields* along γ.

A.14 Null hyperplanes and hypersurfaces

One of the objects that occur in Lorentzian geometry and which posses rather disturbing properties are *null hyperplanes* and *null hypersurfaces*, and it appears useful to include a short discussion of those. Perhaps the most unusual feature of such objects is that the direction normal is actually tangential as well. Furthermore, because the normal has no natural normalization, there is no natural measure induced on a null hypersurface by the ambient metric.

We start with some algebraic preliminaries. Let W be a real vector space, and recall that its dual W^* is defined as the set of all linear maps from W to $\mathbb{R}$ in the applications (in this work only vector spaces over the reals are relevant, but the field makes no difference for the discussion below). To avoid unnecessary complications we assume that W is finite dimensional. It is then standard that W^* has the same dimension as W.

We suppose that W is equipped with a *a) bilinear, b) symmetric, and c) non-degenerate form* q. Thus

$$q : W \to W$$

satisfies

$$a) \quad q(\lambda X + \mu Y, Z) = \lambda q(X,Z) + \mu q(Y,Z) \,, \qquad b) \quad q(X,Y) = q(Y,X) \,,$$

and we also have the implication

$$c) \quad \forall Y \in W \;\; q(X,Y) = 0 \implies X = 0 \,. \tag{A.14.1}$$

(Strictly speaking, we should have indicated linearity with respect to the second variable in a) as well, but this property follows from a) and b) as above). By an abuse of terminology, we will call q a *scalar product*; note that standard algebra textbooks often add the condition of positive-definiteness to the definition of scalar product, which we do not include here.

Let $V \subset W$ be a vector subspace of W. The *annihilator* V^0 of W is defined as the set of linear forms on W which vanish on V:

$$V^0 := \{\alpha \in W^* : \ \forall Y \in V \quad \alpha(Y) = 0\} \subset W^* \ .$$

V^0 is obviously a linear subspace of W^*.

Because q non-degenerate, it defines a linear isomorphism, denoted by $\flat$, between W and W^* by the formula:

$$X^\flat(Y) = g(X, Y) \ .$$

Indeed, the map $X \mapsto X^\flat$ is clearly linear. Next, it has no kernel by (A.14.1). Since the dimensions of W and W^* are the same, it must be an isomorphism. The inverse map is denoted by $\sharp$. Thus, by definition we have

$$g(\alpha^\sharp, Y) = \alpha(Y) \ .$$

The map $\flat$ is nothing but "the lowering of the index on a vector using the metric q", while $\sharp$ is the "raising of the index on a one-form using the inverse metric".

For further purposes it is useful to recall the standard fact:

PROPOSITION A.14.1

$$\dim V + \dim V^0 = \dim W \ .$$

PROOF: Let $\{e_i\}_{i=1,\dots,\dim V}$ be any basis of V, we can complete $\{e_i\}$ to a basis $\{e_i, f_a\}$, with $a = 1, \dots, \dim W - \dim V$, of W. Let $\{e_i^*, f_a^*\}$ be the dual basis of W^*. It is straightforward to check that V^0 is spanned by $\{f_a^*\}$, which gives the result. □

The quadratic form q defines the notion of orthogonality:

$$V^\perp := \{Y \in W : \forall X \in V \ g(X, Y) = 0\} \ .$$

A chase through the definitions above shows that

$$V^\perp = (V^0)^\sharp \ .$$

Proposition A.14.1 implies:

PROPOSITION A.14.2

$$\dim V + \dim V^\perp = \dim W \ .$$

This implies, again regardless of signature:

PROPOSITION A.14.3

$$(\dim V^\perp)^\perp = V \ .$$

PROOF: The inclusion $(\dim V^{\perp})^{\perp} \supset V$ is obvious from the definitions. The equality follows now because both spaces have the same dimension, as a consequence of Proposition (A.14.2). $\Box$

Now,

$$X \in V \cap V^{\perp} \implies q(X,X) = 0 \;, \tag{A.14.2}$$

so that X vanishes if q is positive- or negative-definite, leading to $\dim V \cap \dim V^{\perp} = \{0\}$ in those cases. However, this does not have to be the case anymore for non-definite scalar products q.

A vector subspace V of W is called a *hyperplane* if

$$\dim V = \dim W - 1 \;.$$

Proposition A.14.2 implies then

$$\dim V^{\perp} = 1 \;,$$

regardless of the signature of q. Thus, given a hyperplane V there exists a vector w such that

$$V^{\perp} = \mathbb{R}w \;.$$

If q is Lorentzian, we say that

$$V \text{ is } \begin{cases} \text{spacelike} & \text{if } w \text{ is timelike;} \\ \text{timelike} & \text{if } w \text{ is spacelike;} \\ \text{null} & \text{if } w \text{ is null.} \end{cases}$$

An argument based e.g. on Gram-Schmidt orthonormalization shows that if V is spacelike, then the scalar product defined on V by restriction is positive-definite; similarly if V is timelike, then the resulting scalar product is Lorentzian. The last case, of a null V, leads to a degenerate induced scalar product. In fact, we claim that

$$V \text{ is null if and only if } V \text{ contains its normal. .} \tag{A.14.3}$$

To see (A.14.3), suppose that $V^{\perp} = \mathbb{R}w$, with w null. Since $g(w,w) = 0$ we have $w \in (\mathbb{R}w)^{\perp}$, and from Proposition A.14.3

$$w \in (\mathbb{R}w)^{\perp} = (V^{\perp})^{\perp} = V \;.$$

Since V does not contain its normal in the remaining cases, the equivalence is established.

A hypersurface is $\mathscr{N} \subset \mathscr{M}$ called *null* if at every $p \in \mathscr{N}$ the tangent space $T_p\mathscr{N}$ is a null subspace of $T_p\mathscr{M}$. So (A.14.2) shows that a normal to a null hypersurface $\mathscr{N}$ is also tangent to $\mathscr{N}$.

A.15 Isometries

Let (M, g) be a pseudo-Riemannian manifold. A map ψ is called an *isometry* if

$$\psi^* g = g \ , \qquad \text{(A.15.1)}$$

where ψ^* is the pull-back map defined in Section A.8.2, p. 201.

A standard fact is that the group $\mathrm{Iso}(M, g)$ of isometries of (M, g) carries a natural manifold structure; such groups are called *Lie groups*. If (M, g) is Riemannian and compact, then $\mathrm{Iso}(M, g)$ is compact.

It is also a standard fact that any element of the connected component of the identity of a Lie group G belongs to a one-parameter subgroup $\{\phi_t\}_{t\in\mathbb{R}}$ of G. This allows one to study actions of isometry groups by studying the generators of one-parameter subgroups, defined as

$$X(f)(x) = \left.\frac{d(f(\phi_t(x)))}{dt}\right|_{t=0} \qquad \Longleftrightarrow \qquad X = \left.\frac{d\phi_t}{dt}\right|_{t=0} .$$

The vector fields X obtained in this way are called *Killing vectors*. The knowledge of Killing vectors provides considerable amount of information on the isometry group, and we thus continue with an analysis of their properties. We will see shortly that the collection of Killing vectors forms a Lie algebra: by definition, this is a vector space equipped with a bracket operation such that

$$[X, Y] = -[Y, X] \ ,$$

and

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y] = 0 \ .$$

In the case of Killing vectors, the bracket operation will be the usual bracket of vector fields.

Of key importance to us will be the fact, that the dimension of the isometry group of $(\mathscr{M}, g)$ equals the dimension of the space of the Killing vectors.

A.16 Killing vectors

Let ϕ_t be a one-parameter group of isometries of $(\mathscr{M}, g)$, thus

$$\phi_t^* g = g \quad \Longrightarrow \quad \mathscr{L}_X g = 0 \ . \qquad \text{(A.16.1)}$$

Recall that

$$\mathscr{L}_X g_{\mu\nu} = X^\alpha \partial_\alpha g_{\mu\nu} + \partial_\mu X^\alpha g_{\alpha\nu} + \partial_\nu X^\alpha g_{\mu\alpha} \ .$$

In a coordinate system where the partial derivatives of the metric vanish at a point p, the right-hand-side equals $\nabla_\mu X_\nu + \nabla_\nu X_\mu$. But the left-hand-side is a tensor field, and two tensor fields equal in one coordinate system coincide in all coordinate systems. We have thus proved that generators of isometries satisfy the equation

$$\nabla_\alpha X_\beta + \nabla_\beta X_\alpha = 0 \ . \qquad \text{(A.16.2)}$$

Conversely, consider a solution of (A.16.2); any such solution is called a *Killing vector*. From the calculation just carried out, the Lie derivative of the metric with respect to X vanishes. This means that the local flow of X preserves the metric. In other words, X generates local isometries of g. To make sure that X generates a one-parameter group of isometries one needs moreover to make sure that X is complete; this requires separate considerations.

Recall the identity (A.8.7), p. 199:

$$\mathscr{L}_{[X,Y]} = [\mathscr{L}_X, \mathscr{L}_Y] \; . \tag{A.16.3}$$

This implies that the commutator of two Killing vector fields is a Killing vector field:

$$\mathscr{L}_{[X,Y]}g = \mathscr{L}_X(\underbrace{\mathscr{L}_Y g}_{0}) - \mathscr{L}_Y(\underbrace{\mathscr{L}_X g}_{0}) = 0 \; .$$

We say that the collection of all Killing vector fields, equipped with the Lie bracket, forms a *Lie algebra*.

Remark A.16.1 Let (M,g) be a complete Riemannian manifold, than all Killing vector fields are complete. To see this, let ϕ_t be generated by a Killing vector X, let $p \in M$ and let $\gamma(t) = \phi_t(p)$ be the integral curve of X through p, thus $\dot\gamma = X$. We claim, first, that the length of X is preserved along the orbits of X. Indeed:

$$\frac{d(X^i X_i)}{dt} = 2X^k X^i \nabla_k X_i = 0 \; ,$$

as $\nabla_k X_i$ is antisymmetric. Next, the length of any segment of $\gamma(t)$ is

$$\int_{t_1}^{t_2} |\dot\gamma| dt = \int_{t_1}^{t_2} |X| dt = |X(p)|(t_2 - t_1) \; .$$

The fact that γ is defined for all t follows now immediately from completeness of (M,g).

Remark A.16.2 Let p be a fixed point of an isometry ϕ. For $W \in T_pM$ let $s \mapsto \gamma_W(s)$ be an affinely parameterised geodesic with $\gamma_W(0) = p$ and $\dot\gamma(0) = W$. Since isometries map geodesics to geodesics, the curve $s \mapsto \phi(\gamma_W(s))$ is a geodesic that passes through p and has tangent vector $\phi_* W$ there. As affine parameterisation is also preserved by isometries, we conclude that

$$\phi(\gamma_W(s)) = \gamma_{\phi_* W}(s) \; . \tag{A.16.4}$$

In particular, in the Riemannian case ϕ maps the metric spheres

$$S(p,r) := \{q \in M : \; d(p,q) = r\}$$

to themselves. (This remains true in the Lorentzian case, except that the $S(p,r)$'s are not spheres anymore.)

If (M,g) is isotropic at p, then the action of $\mathrm{Iso}(M,g)$ on T_pM is transitive which, in the complete Riemannian case, implies that the action of $\mathrm{Iso}(M,g)$ on $S(p,r)$ is transitive as well.

In Riemannian geometry, the *sectional curvature* κ of a plane spanned by two vectors $X, Y \in T_pM$ is defined as

$$\kappa(X,Y) := \frac{g(R(X,Y)X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2} \ .$$

Note that κ depends only upon the plane, and not the choice of the vectors X and Y spanning the plane. The definition extends to pseudo-Riemannian manifolds as long as the denominator does not vanish; equivalently, the plane spanned by X and Y should *not* be null.

For maximally symmetric Riemannian manifolds the action of the isometry group on the ollection of two-dimensional subspaces of the tangent bundle is transitive, which implies that κ is independent of p. *Complete* Riemannian manifolds, not necessarily simply connected, with *constant* κ are called *space forms.*

REMARK A.16.3 A *complete Riemannian* manifold (M, g) which is isotropic around every point is necessarily homogeneous. To see this, let $p, p' \in M$, and let q be any point such that the distance from q to p equals that from q to p', say r. Then both p and p' lie on the distance sphere $S(q, r)$, and since (M, g) is isotropic at q, it follows from Remark A.16.2 that there exists an isometry which leaves q fixed and which maps p into p'.

Equation (A.16.2) leads to a second order system of equations, as follows: Taking cyclic permutations of the equation obtained by differentiating (A.16.2) one has

$$\begin{aligned}
-\nabla_\gamma\nabla_\alpha X_\beta - \nabla_\gamma\nabla_\beta X_\alpha &= 0\ ,\\
\nabla_\alpha\nabla_\beta X_\gamma + \nabla_\alpha\nabla_\gamma X_\beta &= 0\ ,\\
\nabla_\beta\nabla_\gamma X_\alpha + \nabla_\beta\nabla_\alpha X_\gamma &= 0\ .
\end{aligned}$$

Adding, and expressing commutators of derivatives in terms of the Riemann tensor, one obtains

$$\begin{aligned}
2\nabla_\alpha\nabla_\beta X_\gamma &= (R_{\sigma\gamma\beta\alpha} + R_{\sigma\alpha\beta\gamma} + \underbrace{R_{\sigma\beta\alpha\gamma}}_{=-R_{\sigma\alpha\gamma\beta}-R_{\sigma\gamma\beta\alpha}})X^\sigma\\
&= 2R_{\sigma\alpha\beta\gamma}X^\sigma\ .
\end{aligned}$$

Thus

$$\nabla_\alpha\nabla_\beta X_\gamma = R_{\sigma\alpha\beta\gamma}X^\sigma\ . \qquad \text{(A.16.5)}$$

EXAMPLE A.16.4 As an example of application of (A.16.5), let (M, g) be flat. In a coordinate system $\{x^\mu\}$ in which the metric has constant entries (A.16.5) reads

$$\partial_\alpha\partial_\beta X_\gamma = 0\ .$$

The solutions are therefore linear,

$$X^\alpha = A^\alpha + B^\alpha{}_\beta x^\gamma\ .$$

Plugging this into (A.16.2), one finds that $B_{\alpha\beta}$ must be anti-symmetric. Hence, the dimension of the set of all Killing vectors of $\mathbb{R}^{n,m}$, and thus of $\mathrm{Iso}(\mathbb{R}^{n,m})$, is $n(n+1)/2$, independently of signature.

Consider, next, a torus $\mathbb{T}^n := S^1 \times \ldots \times S^1$, equipped again with a flat metric. In this case none of the locally defined Killing vectors of the form $B^i{}_j x^j$ survive the periodic identifications, hence the dimension of $\mathrm{Iso}(\mathbb{T}^n, \delta)$ is n: indeed, from (A.16.5) and integrating by parts we have

$$\int X^i \underbrace{D_j D_i X^j}_{=0} = -\int D^j X^i \underbrace{D_i X_j}_{=-D_j X_i} = \int |DX|^2 , \tag{A.16.6}$$

and so $B_{ij} \equiv D_i X_j = 0$: all Killing vectors on a flat Riemannian $\mathbb{T}^n$ are covariantly constant.

In fact, an obvious modification of the last calculation shows that *the isometry group of compact Riemannian manifolds with* <u>*strictly negative*</u> *Ricci tensor is finite*, and that *non-trivial Killing vectors of compact Riemannian manifolds with* <u>*non-positive*</u> *Ricci tensor are covariantly constant*. Indeed, for such manifolds the left-hand side does not vanish anymore in general, but instead we have

$$\int |DX|^2 = \int X^i D_j D_i X^j = \int X^i R_{kji}{}^j X^j = \int X^i R_{ki} X^j , \tag{A.16.7}$$

The left-hand side is always positive. If the Ricci tensor is non-positive, then the right-hand-side is non-positive, which is only possible if both vanish, hence $DX = 0$ and $R_{ij} X^i X^j = 0$. If the Ricci tensor is strictly negative, then $X = 0$. Hence there are no non-trivial Kiling vectors, and the dimension of the group of isometries is zero. Since the group is compact when (M, g) is Riemannian and compact, it must be finite.

An important consequence of (A.16.5) is:

PROPOSITION A.16.5 *Let M be connected and let $p \in M$. A Killing vector is uniquely defined by its value $X(p)$ and the value at p of the anti-symmetric tensor $\nabla X(p)$.*

PROOF: Consider two Killing vectors X and Y such that $X(p) = Y(p)$ and $\nabla X(p) = \nabla Y(p)$. Let $q \in M$ and let γ be any curve from p to q. Set

$$Z^\beta := X^\beta - Y^\beta , \qquad A_{\alpha\beta} = \nabla_\alpha (X_\beta - Y_\beta) .$$

Along the curve γ we have

$$\begin{aligned} \frac{DZ_\alpha}{ds} &= \dot\gamma^\mu \nabla_\mu Z_\alpha = \dot\gamma^\mu A_{\mu\alpha} \\ \frac{DA_{\alpha\beta}}{ds} &= \dot\gamma^\mu \nabla_\mu \nabla_\alpha Z_\beta = R_{\gamma\mu\alpha\beta} \dot\gamma^\mu Z^\gamma . \end{aligned}$$

This is a linear first order system of ODEs along γ with vanishing Cauchy data at p. Hence the solution vanishes along γ, and thus $X^\mu(q) = Y^\mu(q)$. □

Note that there are at most n values of X at p and, of view of anti-symmetry, at most $n(n-1)/2$ values of ∇X at p. Since the dimension of the space of Killing vectors equals the dimension of the group of isometries, as a Corollary we obtain:

PROPOSITION A.16.6 *The dimension of the group of isometries of an n-dimensional pseudo-Riemannian manifold (M, g) is less than or equal to $n(n+1)/2$.*

A manifold (M,g) is called *maximally symmetric* if the dimension of $\mathrm{Iso}(M,g)$ equals the maximum allowed value, $n(n+1)/2$ in dimension n.

A manifold (M,g) is called *homogeneous* if the action of $\mathrm{Iso}(M,g)$ is transitive. Thus the dimension of the orbit of the isometry group through each point is n. This implies that the dimension of $\mathrm{Iso}(M,g)$ is at least n.

A manifold (M,g) is called *isotropic* at $p\in M$ if for every antisymmetric tensor $B_{\mu\nu}\in T^*_pM\otimes T^*_pM$ there exists a Killing vector on M such that $\nabla_\mu X_\nu(p)=B_{\mu\nu}$. In particular the dimension of $\mathrm{Iso}(M,g)$ is at least $n(n-1)/2$.

Examples of flat maximally symmetric manifolds are provided by $\mathbb{R}^{n,m}$. It follows e.g. from the (pseudo-Riemannian version of the) *Hadamard-Cartan* theorem that these are the only *simply connected geodesically complete* such manifolds.

Curved examples of maximally symmetric manifolds can be constructed as follows: Let $\mathbb{R}^{n,m}:=(\mathbb{R}^{n+m},\eta)$, where η is a quadratic form of signature (n,m). For $a\in\mathbb{R}^*$ consider the submanifold

$$\mathscr{S}_a:=\{\eta_{\alpha\beta}x^\alpha x^\beta=a\}\ .$$

Note that the covector field $N_\alpha:=\eta_{\alpha\beta}x^\alpha/\sqrt{|a|}$ is conormal to $\mathscr{S}_a$. Since $\eta(N,N)=a/|a|\neq 0$, the tensor field η induces on $\mathscr{S}_a$ a pseudo-Riemannian metric, which will be denoted by h.

As both η and $\mathscr{S}_a$ are invariant under the defining action of $SO(n,m)$ on $\mathbb{R}^{n+m}$, the metric h is also invariant under this action.

Further, the action is *effective*; this means that the only element $g\in SO(n,m)$ leaving all points invariant is the identity map. Indeed, suppose that $\phi\in SO(n,m)$ leaves invariant every point $p\in\mathscr{S}_a$. Now, every point $x=(x^\mu)$ for which $\eta_{\alpha\beta}x^\alpha x^\beta$ has the same sign as a can be written in the form $x^\mu=\beta x^\mu_0$, where $x_0\in\mathscr{S}_a$ and $\beta\in\mathbb{R}$. By linearity of the action, ϕ leaves x invariant. Hence ϕ is a linear map which is the identity on an open set, and thus the identity everywhere.

We conclude that the dimension of the isometry group of $(\mathscr{S}_a,h)$ equals that of $SO(n,m)$, namely $(n+m)(n+m-1)/2$. Since the dimension of $\mathscr{S}_a$ is $n+m-1$, this implies that $(\mathscr{S}_a,h)$ is maximally symmetric.

As a first explicit example, let η be either the Minkowski or the flat metric in dimension $n+1$, which we write in the form

$$\eta=\epsilon dw^2+\delta=\epsilon dw^2+dr^2+r^2d\Omega^2\ ,\qquad \epsilon\in\{\pm1\}\ ,$$

hence $\mathscr{S}_a$ is given by the equation

$$\epsilon w^2=a-r^2\ .$$

Away from the set $\{a=r^2\}$, differentiation gives $2\epsilon w\,dw=-2r\,dr$, hence

$$dw^2=\frac{r^2}{w^2}dr^2=\epsilon\frac{r^2}{a-r^2}dr^2\ .$$

Thus the induced metric equals

$$h=\frac{r^2}{a-r^2}dr^2+dr^2+r^2d\Omega^2=\frac{a}{a-r^2}dr^2+r^2d\Omega^2\ . \tag{A.16.8}$$

When $\epsilon = 1$ and $a = R^2$, $\mathscr{S}_a$ is a sphere in Euclidean space, $w^2 = R^2 - r^2 \le R^2$ and we find the following representation of the upper-hemisphere metric:

$$h = \frac{1}{1 - \frac{r^2}{R^2}} dr^2 + r^2 d\Omega^2 \ , \quad 0 < r < R \ . \tag{A.16.9}$$

Note that the same formula defines a Lorentzian metric for $r > R$:

$$h = -\frac{1}{\frac{r^2}{R^2} - 1} dr^2 + r^2 d\Omega^2 \ , \quad r > R \ . \tag{A.16.10}$$

This corresponds to the case $a = R^2$ and $\epsilon = -1$ of our construction, in which case $\mathscr{S}_a$ is the timelike hyperboloid $r^2 = R^2 + t^2 \ge R^2$ in Minkowski space-time. This is *de Sitter space-time* (in a non-standard coordinate system), solution of the vacuum Einstein equations with positive cosmological constant.

When $\epsilon = -1$ and $a = -R^2$, $\mathscr{S}_a$ consists of two copies of a spacelike hyperboloid in Minkowski space, $t^2 = R^2 + r^2$, with induced metric of constant negative curvature:

$$h = \frac{1}{1 + \frac{r^2}{R^2}} dr^2 + r^2 d\Omega^2 \ . \tag{A.16.11}$$

This is the metric on *hyperbolic space*.

Introducing $\hat{r} = Rr$, the above Riemannian metrics can be written in a more standard form

$$h = R^2 \left(\frac{d\hat{r}^2}{1 + k\hat{r}^2} + \hat{r}^2 d\Omega^2 \right) \ , \quad k \in \{0, \pm 1\} \ , \tag{A.16.12}$$

where k determines whether the metric is flat, or positively curved, or negatively curved.

To obtain negatively curved Lorentzian metrics we take η of signature $(2, n)$:

$$\eta = -dw^2 - dz^2 + \delta = -dw^2 - dz^2 + dr^2 + r^2 d\Omega^2 \ .$$

The hypersurface $\mathscr{S}_{-R^2}$ is then given by the equation

$$w^2 + z^2 = R^2 + r^2 \ ,$$

and can be thought of as a surface of revolution obtained by rotating a hyperboloid $w = 0$, $z^2 = R^2 + r^2$, around the $w = z = 0$ axis. Setting

$$w = \rho \cos t \ , \quad z = \rho \sin t \ ,$$

one has

$$dw^2 + dz^2 = d\rho^2 + \rho^2 dt^2 \ ,$$

and the equation for $\mathscr{S}_{-R^2}$ becomes

$$\rho^2 = r^2 + R^2 \ .$$

Hence

$$d\rho^2 = \frac{r^2}{\rho^2} dr^2 = \frac{r^2 dr^2}{r^2 + R^2} \ ,$$

and since

$$\eta = -dw^2 - dz^2 + dr^2 + r^2 d\Omega^2 = -(d\rho^2 + \rho^2 dt^2) + dr^2 + r^2 d\Omega^2 \; .$$

we obtain

$$\begin{aligned} h &= -\left(\frac{r^2 dr^2}{r^2+R^2} + (r^2+R^2)dt^2\right) + dr^2 + r^2 d\Omega^2 \\ &= -(r^2+R^2)dt^2 + \frac{dr^2}{1+\frac{r^2}{R^2}} + r^2 d\Omega^2 \; . \end{aligned}$$

Note that the slices $\{t = \mathrm{const}\}$ are maximally symmetric hyperbolic. Somewhat amusingly, in the embedded model the time coordinate t is periodic. The universal cover of $\mathscr{S}_{-R^2}$, where t is assumed to run over $\mathbb{R}$ instead of S^1, is called the *anti-de Sitter* space-time.

It is a standard fact that all maximally symmetric metrics are *locally isometric*. So the above formulae give the local form for all maximally symmetric Lorentzian or Riemannian metrics.

A.17 Moving frames

A formalism which is very convenient for practical calculations is that of *moving frames*; it also plays a key role when considering spinors, compare Section 3.1. By definition, a moving frame is a (locally defined) field of bases $\{e_a\}$ of TM such that the scalar products

$$g_{ab} := g(e_a, e_b) \tag{A.17.1}$$

are point independent. In most standard applications one assumes that the e_a's form an orthonormal basis, so that g_{ab} is a diagonal matrix with plus and minus ones on the diagonal. However, it is sometimes convenient to allow other such frames, *e.g.* with isotropic vectors being members of the frame.

It is customary to denote by $\omega^a{}_{bc}$ the associated connection coefficients:

$$\omega^a{}_{bc} := \theta^a(\nabla_{e_c} e_b) \quad \Longleftrightarrow \quad \nabla_X e_b = \omega^a{}_{bc} X^c e_a \; , \tag{A.17.2}$$

where, as elsewhere, $\{\theta^a(p)\}$ is a basis of T^*_pM dual to $\{e_a(p)\} \subset T_pM$; we will refer to θ^a as a *coframe*. The *connection one forms* $\omega^a{}_b$ are defined as

$$\omega^a{}_b(X) := \theta^a(\nabla_X e_b) \quad \Longleftrightarrow \quad \nabla_X e_b = \omega^a{}_b(X) e_a \; ; . \tag{A.17.3}$$

As always we use the metric to raise and lower indices, even though the $\omega^a{}_{bc}$'s do not form a tensor, so that

$$\omega_{abc} := g_{ad}\omega^e{}_{bc} \; , \quad \omega_{ab} := g_{ae}\omega^e{}_b \; . \tag{A.17.4}$$

When ∇ is metric compatible, the ω_{ab}'s are anti-antisymmetric: indeed, as the g_{ab}'s are point independent, for any vector field X we have

$$\begin{aligned} 0 = X(g_{ab}) = X(g(e_a, e_b)) &= g(\nabla_X e_a, e_b) + g(e_a, \nabla_X e_b) \\ &= g(\omega^c{}_a(X) e_c, e_b) + g(e_a, \omega^d{}_b(X) e_d) \\ &= g_{cb}\omega^c{}_a(X) + g_{ad}\omega^d{}_b(X) \\ &= \omega_{ba}(X) + \omega_{ab}(X) \; . \end{aligned}$$

Hence

$$\boxed{\omega_{ab} = -\omega_{ba} \quad \Longleftrightarrow \quad \omega_{abc} = -\omega_{bac}} \,. \tag{A.17.5}$$

One can obtain a formula for the ω_{ab}'s in terms of Christoffels, the frame vectors and their derivatives: In order to see this, we note that

$$g(e_a, \nabla_{e_c} e_b) = g(e_a, \omega^d{}_{bc} e_d) = g_{ad}\omega^d{}_{bc} = \omega_{abc} \,. \tag{A.17.6}$$

Rewritten the other way round this gives an alternative equation for the ω's with all indices down:

$$\omega_{abc} = g(e_a, \nabla_{e_c} e_b) \quad \Longleftrightarrow \quad \omega_{ab}(X) = g(e_a, \nabla_X e_b) \,. \tag{A.17.7}$$

Then, writing

$$e_a = e_a{}^\mu \partial_\mu \,,$$

we find

$$\begin{aligned} \omega_{abc} &= g(e_a{}^\mu \partial_\mu, e_c{}^\lambda \nabla_\lambda e_b) \\ &= g_{\mu\sigma} e_a{}^\mu e_c{}^\lambda (\partial_\lambda e_b{}^\sigma + \Gamma^\sigma_{\lambda\nu} e_b{}^\nu) \,. \end{aligned} \tag{A.17.8}$$

Next, it turns out that we can calculate the ω_{ab}'s in terms of the Lie brackets of the vector fields e_a, without having to calculate the Christoffel symbols. This shouldn't be too surprising, since an ON frame defines the metric uniquely. If ∇ has no torsion, from (A.17.7) we find

$$\omega_{abc} - \omega_{acb} = g(e_a, \nabla_{e_c} e_b - \nabla_{e_b} e_c) = g(e_a, [e_c, e_b]) \,.$$

We can now carry-out the usual cyclic-permutations calculation to obtain

$$\begin{aligned} \omega_{abc} - \omega_{acb} &= g(e_a, [e_c, e_b]) \,, \\ -(\omega_{bca} - \omega_{bac}) &= -g(e_b, [e_a, e_c]) \,, \\ -(\omega_{cab} - \omega_{cba}) &= -g(e_c, [e_b, e_a]) \,. \end{aligned}$$

So, if the connection is the Levi-Civita connection, summing the three equations and using (A.17.5) leads to

$$\boxed{\omega_{cba} = \frac{1}{2}\Big(g(e_a, [e_c, e_b]) - g(e_b, [e_a, e_c]) - g(e_c, [e_b, e_a])\Big)} \,. \tag{A.17.9}$$

Equations (A.17.8)-(A.17.9) provide explicit expressions for the ω's; yet another formula can be found in (A.17.10) below. While it is useful to know that there are such expressions, and while those expressions are useful to estimate things for PDE purposes, they are rarely used for practical calculations; see Example A.17.3 for more comments about that last issue.

Exercice A.17.1 Use (A.17.9) to show that

$$\omega_{cba} = \frac{1}{2}\Big(\eta_{ad}\, d\theta^d(e_b, e_c) + \eta_{bd}\, d\theta^d(e_a, e_c) + \eta_{cd}\, d\theta^d(e_b, e_a)\Big) \,. \tag{A.17.10}$$

It turns out that one can obtain a simple expression for the torsion of ω using exterior differentiation. Recall that if α is a one-form, then its exterior derivative $d\alpha$ can be calculated using the formula

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]) \ . \qquad \text{(A.17.11)}$$

We set

$$T^a(X,Y) := \theta^a(T(X,Y)) \ ,$$

and using (A.17.11) together with the definition (A.9.16) of the torsion tensor T we calculate as follows:

$$\begin{aligned} T^a(X,Y) &= \theta^a(\nabla_X Y - \nabla_Y X - [X,Y]) \\ &= X(Y^a) + \omega^a{}_b(X)Y^b - Y(X^a) - \omega^a{}_b(Y)X^b - \theta^a([X,Y]) \\ &= X(\theta^a(Y)) - Y(\theta^a(X)) - \theta^a([X,Y]) + \omega^a{}_b(X)\theta^b(Y) - \omega^a{}_b(Y)\theta^b(X) \\ &= d\theta^a(X,Y) + (\omega^a{}_b \wedge \theta^b)(X,Y) \ . \end{aligned}$$

It follows that

$$T^a = d\theta^a + \omega^a{}_b \wedge \theta^b \ . \qquad \text{(A.17.12)}$$

In particular when the torsion vanishes we obtain the so-called *Cartan's first structure equation*

$$\boxed{d\theta^a + \omega^a{}_b \wedge \theta^b = 0} \ . \qquad \text{(A.17.13)}$$

EXAMPLE A.17.2 As a simple example, we consider a two-dimensional metric of the form

$$g = dx^2 + e^{2f}dy^2 \ , \qquad \text{(A.17.14)}$$

where f could possibly depend upon x and y. A natural frame is given by

$$\theta^1 = dx \ , \quad \theta^2 = e^f dy \ .$$

The first Cartan structure equations read

$$0 = \underbrace{d\theta^1}_{0} + \omega^1{}_b \wedge \theta^b = \omega^1{}_2 \wedge \theta^2 \ ,$$

since $\omega^1{}_1 = \omega_{11} = 0$ by antisymmetry, and

$$0 = \underbrace{d\theta^2}_{e^f \partial_x f dx \wedge dy} + \omega^2{}_b \wedge \theta^b = \partial_x f \theta^1 \wedge \theta^2 + \omega^2{}_1 \wedge \theta^1 \ .$$

It should then be clear that both equations can be solved by choosing ω_{12} proportional to θ^2, and such an ansatz leads to

$$\omega_{12} = -\omega_{21} = -\partial_x f\, \theta^2 = -\partial_x(e^f)\, dy \ . \qquad \text{(A.17.15)}$$

EXAMPLE A.17.3 As another example of the moving frame technique we consider (the most general) three-dimensional spherically symmetric metric

$$g = e^{2\beta(r)}dr^2 + e^{2\gamma(r)}d\theta^2 + e^{2\gamma(r)}\sin^2\theta d\varphi^2 \ . \qquad \text{(A.17.16)}$$

There is an obvious choice of ON coframe for g given by

$$\theta^1 = e^{\beta(r)}dr \ , \ \theta^2 = e^{\gamma(r)}d\theta \ , \ \theta^3 = e^{\gamma(r)}\sin\theta d\varphi \ , \tag{A.17.17}$$

leading to

$$g = \theta^1 \otimes \ \theta^1 + \theta^2 \otimes \ \theta^2 + \theta^3 \otimes \ \theta^3 \ ,$$

so that the frame e_a dual to the θ^a's will be ON, as desired:

$$g_{ab} = g(e_a, e_b) = \mathrm{diag}(1,1,1) \ .$$

The idea of the calculation which we are about to do is the following: there is only one connection which is compatible with the metric, and which is torsion free. If we find a set of one forms ω_{ab} which exhibit the properties just mentioned, then they have to be the connection forms of the Levi-Civita connection. As shown in the calculation leading to (A.17.5), the compatibility with the metric will be ensured if we require

$$\omega_{11} = \omega_{22} = \omega_{33} = 0 \ ,$$

$$\omega_{12} = -\omega_{21} \ , \quad \omega_{13} = -\omega_{31} \ , \quad \omega_{23} = -\omega_{32} \ .$$

Next, we have the equations for the vanishing of torsion:

$$\begin{aligned}
0 = d\theta^1 &= -\underbrace{\omega^1{}_1}_{=0}\theta^1 - \omega^1{}_2\theta^2 - \omega^1{}_3\theta^3 \\
&= -\omega^1{}_2\theta^2 - \omega^1{}_3\theta^3 \ , \\
d\theta^2 &= \gamma' e^{\gamma} dr\wedge d\theta = \gamma' e^{-\beta}\theta^1\wedge\theta^2 \\
&= -\underbrace{\omega^2{}_1}_{=-\omega^1{}_2}\theta^1 - \underbrace{\omega^2{}_2}_{=0}\theta^2 - \omega^2{}_3\theta^3 \\
&= \omega^1{}_2\theta^1 - \omega^2{}_3\theta^3 \ , \\
d\theta^3 &= \gamma' e^{\gamma}\sin\theta\, dr\wedge d\varphi + e^{\gamma}\cos\theta\, d\theta\wedge d\varphi = \gamma' e^{-\beta}\theta^1\wedge\theta^3 + e^{-\gamma}\cot\theta\,\theta^2\wedge\theta^3 \\
&= -\underbrace{\omega^3{}_1}_{=-\omega^1{}_3}\theta^1 - \underbrace{\omega^3{}_2}_{=-\omega^2{}_3}\theta^2 - \underbrace{\omega^3{}_3}_{=0}\theta^3 \\
&= \omega^1{}_3\theta^1 + \omega^2{}_3\theta^2 \ .
\end{aligned}$$

Summarising,

$$\begin{aligned}
-\omega^1{}_2\theta^2 - \omega^1{}_3\theta^3 \qquad\qquad &= 0 \ , \\
\omega^1{}_2\theta^1 \qquad - \omega^2{}_3\theta^3 &= \gamma' e^{-\beta}\theta^1\wedge\theta^2 \ , \\
\omega^1{}_3\theta^1 + \omega^2{}_3\theta^2 &= \gamma' e^{-\beta}\theta^1\wedge\theta^3 + e^{-\gamma}\cot\theta\,\theta^2\wedge\theta^3 \ .
\end{aligned}$$

It should be clear from the first and second line that an $\omega^1{}_2$ proportional to θ^2 should do the job; similarly from the first and third line one sees that an $\omega^1{}_3$ proportional to θ^3 should work. It is then easy to find the relevant coefficient, as well as to find $\omega^2{}_3$:

$$\begin{aligned}
\omega^1{}_2 &= -\gamma' e^{-\beta}\theta^2 = -\gamma' e^{-\beta+\gamma}d\theta \ , &(A.17.18a)\\
\omega^1{}_3 &= -\gamma' e^{-\beta}\theta^3 = -\gamma' e^{-\beta+\gamma}\sin\theta\, d\varphi \ , &(A.17.18b)\\
\omega^2{}_3 &= -e^{-\gamma}\cot\theta\,\theta^3 = -\cos\theta\, d\varphi \ . &(A.17.18c)
\end{aligned}$$

It is convenient to define *curvature two-forms*:

$$\Omega^a{}_b = R^a{}_{bcd}\theta^c\otimes\theta^d = \frac{1}{2}R^a{}_{bcd}\theta^c\wedge\theta^d \ . \tag{A.17.19}$$

The *second Cartan structure equation* then reads

$$\boxed{\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b} \,. \tag{A.17.20}$$

This identity is easily verified using (A.17.11):

$$\begin{aligned}
\Omega^a{}_b(X,Y) &= \frac{1}{2} R^a{}_{bcd} \underbrace{\theta^c \wedge \theta^d(X,Y)}_{=X^cY^d - X^dY^c} \\
&= R^a{}_{bcd} X^c Y^d \\
&= \theta^a(\nabla_X \nabla_Y e_b - \nabla_Y \nabla_X e_b - \nabla_{[X,Y]} e_b) \\
&= \theta^a(\nabla_X(\omega^c{}_b(Y)e_c) - \nabla_Y(\omega^c{}_b(X)e_c) - \omega^c{}_b([X,Y])e_c) \\
&= \theta^a\Big(X(\omega^c{}_b(Y))e_c + \omega^c{}_b(Y)\nabla_X e_c \\
&\quad -Y(\omega^c{}_b(X))e_c - \omega^c{}_b(X)\nabla_Y e_c - \omega^c{}_b([X,Y])e_c\Big) \\
&= X(\omega^a{}_b(Y)) + \omega^c{}_b(Y)\omega^a{}_c(X) \\
&\quad -Y(\omega^a{}_b(X)) - \omega^c{}_b(X)\omega^a{}_c(Y) - \omega^a{}_b([X,Y]) \\
&= \underbrace{X(\omega^a{}_b(Y)) - Y(\omega^a{}_b(X)) - \omega^a{}_b([X,Y])}_{=d\omega^a{}_b(X,Y)} \\
&\quad +\omega^a{}_c(X)\omega^c{}_b(Y) - \omega^a{}_c(Y)\omega^c{}_b(X) \\
&= (d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b)(X,Y) \,.
\end{aligned}$$

Equation (A.17.20) provides an efficient way of calculating the curvature tensor of any metric.

Example A.17.4 In dimension two the only non-vanishing components of $\omega^a{}_b$ are $\omega^1{}_2 = -\omega^2{}_1$, and it follows from (A.17.20) that

$$\Omega^1{}_2 = d\omega^1{}_2 + \omega^1{}_a \wedge \omega^a{}_2 = d\omega^1{}_2 \,. \tag{A.17.21}$$

In particular (assuming that θ^2 is dual to a spacelike vector, whatever the signature of the metric)

$$\begin{aligned}
R\, d\mu_g &= R\,\theta^1 \wedge \theta^2 = 2R^{12}{}_{12}\theta^1 \wedge \theta^2 = R^1{}_{2ab}\theta^a \wedge \theta^b = 2\Omega^1{}_2 \\
&= 2d\omega^1{}_2 \,,
\end{aligned} \tag{A.17.22}$$

where $d\mu_g$ is the volume two-form.

Example A.17.2 continued We have seen that the connection one-forms for the metric

$$g = dx^2 + e^{2f} dy^2 \tag{A.17.23}$$

read

$$\omega_{12} = -\omega_{21} = -\partial_x f\, \theta^2 = -\partial_x(e^f)\, dy \,.$$

By symmetry the only non-vanishing curvature two-forms are $\Omega_{12} = -\Omega_{21}$. From (A.17.20) we find

$$\Omega_{12} = d\omega_{12} + \underbrace{\omega_{1b} \wedge \omega^b{}_2}_{=\omega_{12}\wedge\omega^2{}_2=0} = -\partial_x^2(e^f)\, dx \wedge dy = -e^{-f}\partial_x^2(e^f)\, \theta^1 \wedge \theta^2 \,.$$

We conclude that

$$R_{1212} = -e^{-f}\partial_x^2(e^f) \; . \tag{A.17.24}$$

(Compare Example A.12.2, p. 219.) For instance, if g is the unit round metric on the two-sphere, then $e^f = \sin x$, and $R_{1212} = 1$. If $e^f = \sinh x$, then g is the canonical metric on hyperbolic space, and $R_{1212} = -1$. Finally, the function $e^f = \cosh x$ defines a *hyperbolic wormhole*, with again $R_{1212} = -1$.

Example A.17.3 continued: From (A.17.18) we find:

$$\begin{aligned}
\Omega^1{}_2 &= d\omega^1{}_2 + \underbrace{\omega^1{}_1}_{=0} \wedge\omega^1{}_2 + \omega^1{}_2 \wedge \underbrace{\omega^2{}_2}_{=0} + \underbrace{\omega^1{}_3 \wedge \omega^3{}_2}_{\sim\theta^3\wedge\theta^3=0} \\
&= -d(\gamma' e^{-\beta+\gamma} d\theta) \\
&= -(\gamma' e^{-\beta+\gamma})' dr \wedge d\theta \\
&= -(\gamma' e^{-\beta+\gamma})' e^{-\beta-\gamma}\theta^1 \wedge \theta^2 \\
&= \sum_{a<b} R^1{}_{2ab}\theta^a \wedge \theta^b \; ,
\end{aligned}$$

which shows that the only non-trivial coefficient (up to permutations) with the pair 12 in the first two slots is

$$R^1{}_{212} = -(\gamma' e^{-\beta+\gamma})' e^{-\beta-\gamma} \; . \tag{A.17.25}$$

A similar calculation, or arguing by symmetry, leads to

$$R^1{}_{313} = -(\gamma' e^{-\beta+\gamma})' e^{-\beta-\gamma} \; . \tag{A.17.26}$$

Finally,

$$\begin{aligned}
\Omega^2{}_3 &= d\omega^2{}_3 + \omega^2{}_1 \wedge \omega^1{}_3 + \underbrace{\omega^2{}_2}_{=0} \wedge\omega^2{}_3 + \omega^2{}_3 \wedge \underbrace{\omega^3{}_3}_{=0} \\
&= -d(\cos\theta\, d\varphi) + (\gamma' e^{-\beta}\theta^2) \wedge (-\gamma' e^{-\beta}\theta^3) \\
&= (e^{-2\gamma} - (\gamma')^2 e^{-2\beta})\theta^2 \wedge \theta^3 \; ,
\end{aligned}$$

yielding

$$R^2{}_{323} = e^{-2\gamma} - (\gamma')^2 e^{-2\beta} \; . \tag{A.17.27}$$

The curvature scalar can easily be calculated now to be

$$\begin{aligned}
R = R^{ij}{}_{ij} &= 2(R^{12}{}_{12} + R^{13}{}_{13} + R^{23}{}_{23}) \\
&= -4(\gamma' e^{-\beta+\gamma})' e^{-\beta-\gamma} + 2(e^{-2\gamma} - (\gamma')^2 e^{-2\beta}) \; .
\end{aligned} \tag{A.17.28}$$

The Bianchi identities have a particularly simple proof in the moving frame formalism. For this, let ψ^a be any vector-valued differential form, and define

$$D\psi^a = d\psi^a + \omega^a{}_b \wedge \psi^b \; . \tag{A.17.29}$$

Thus, using this notation the vanishing of torsion can be written as

$$D\theta^a = 0 \; . \tag{A.17.30}$$

In situations where the torsion does not vanish, we calculate

$$\begin{aligned} D\tau^a &= d\tau^a + \omega^a{}_b \wedge \tau^b = d(d\theta^a + \omega^a{}_b \wedge \theta^b) + \omega^a{}_c \wedge (d\theta^c + \omega^c{}_b \wedge \theta^b) \\ &= d\omega^a{}_b \wedge \theta^b - \omega^a{}_b \wedge d\theta^b + \omega^a{}_c \wedge (d\theta^c + \omega^c{}_b \wedge \theta^b) \\ &= \Omega^a{}_b \wedge \theta^b \ . \end{aligned}$$

If the torsion vanishes the left-hand side is zero. We leave it as an exercice for the reader to check that the equation

$$\Omega^a{}_b \wedge \theta^b = 0 \tag{A.17.31}$$

is equivalent to the first Bianchi identity.

Next, for any differential form with two-frame indices, such as the curvature two-form, we define

$$D\Omega^a{}_b := d\Omega^a{}_b + \omega^a{}_c \wedge \Omega^c{}_b - \omega^c{}_b \wedge \Omega^a{}_c \ . \tag{A.17.32}$$

Then

$$\begin{aligned} D\Omega^a{}_b &= d(d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b) + \omega^a{}_c \wedge \Omega^c{}_b - \omega^c{}_b \wedge \Omega^a{}_c \\ &= d\omega^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge d\omega^c{}_b + \omega^a{}_c \wedge \Omega^c{}_b - \omega^c{}_b \wedge \Omega^a{}_c \\ &= (\Omega^a{}_c - \omega^a{}_e \wedge \omega^e{}_c) \wedge \omega^c{}_b - \omega^a{}_c \wedge (\Omega^c{}_b - \omega^c{}_e \wedge \omega^e{}_b) \\ &\quad + \omega^a{}_c \wedge \Omega^c{}_b - \omega^c{}_b \wedge \Omega^a{}_c = 0 \ . \end{aligned}$$

Thus

$$D\Omega^a{}_b \quad = \quad 0 \ , \tag{A.17.33}$$

which can be checked to coincide with the second Bianchi identity.

A.18 Clifford algebras

Our approach is a variation upon [42]. Let q be a *non-degenerate* quadratic form on a vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathscr{A}$ be an algebra over $\mathbb{K}$. A *Clifford map* of (W, q) into $\mathscr{A}$ is a linear map $f : W \to \mathscr{A}$ with the property that, for any $X \in W$,

$$f(X)^2 = -q(X, X) \ . \tag{A.18.1}$$

By polarisation, this is equivalent to

$$f(X)f(Y) + f(Y)f(X) = -2q(X, Y) \ . \tag{A.18.2}$$

for any $X, Y \in W$.

Note that a Clifford map is necessarily injective: if $f(X) = 0$ then $q(X, Y) = 0$ for all Y by (A.18.2), hence $X = 0$. Thus $\dim \mathscr{A} \geq \dim W$ whenever a Clifford map exists.

The *Clifford algebra* $C\ell(W, q)$ is the unique (up to homomorphism) associative algebra with unity defined by the following two properties:

1. there exists a Clifford map $\kappa : W \to C\ell(W,q)$, and
2. for any Clifford map $f : W \to \mathscr{A}$ there exists exactly one homomorphism $\tilde{f}$ of algebras with unity $\tilde{f} : C\ell(W,q) \to \mathscr{A}$ such that

$$f = \tilde{f} \circ \kappa \ .$$

The definition is somewhat roundabout, and takes a while to absorb. The key property is the Clifford anti-commutation rule (A.18.2). The second point is a way of saying that $C\ell(W,q)$ is the smallest algebra for which (A.18.2) holds.

Now, uniqueness of $C\ell(W,q)$, up to algebra homomorphism, follows immediately from its definition. Existence is a consequence of the following construction: let $\mathscr{T}(W)$ be the tensor algebra of W,

$$\mathscr{T}(W) := \mathbb{K} \oplus W \oplus_{\ell=2}^{\infty} \underbrace{W \otimes \ldots \otimes W}_{\ell \text{ factors}} ,$$

it being understood that only elements with a finite number of non-zero components in the infinite sum are allowed. Then $\mathscr{T}(W)$ is an associative algebra with unity, the product of two elements $a, b \in \mathscr{T}(W)$ being the tensor product $a \otimes b$. Let I_q be the two-sided ideal generated by all tensors of the form $X \otimes X + q(X,X)$, $X \in W$. Then the quotient algebra

$$\mathscr{T}(W)/I_q$$

has the required property. Indeed, let κ be the map which to $X \in W \subset \mathscr{T}(W)$ assigns the equivalence class $[X] \in \mathscr{T}(W)/I_q$. Then κ is a Clifford map by definition. Further, if $f : V \to \mathscr{A}$ is a Clifford map, let $\hat{f}$ be the unique linear map $\hat{f} : \mathscr{T}(W) \to \mathscr{A}$ satisfying

$$\hat{f}(X_1 \otimes \ldots \otimes X_k) = f(X_1) \cdots f(X_k) \ .$$

Then $\hat{f}$ vanishes on I_q, hence provides the desired map $\tilde{f}$ defined on the quotient, $\tilde{f}([X]) := \hat{f}(X)$.

Example A.18.1 Let $W = \mathbb{R}$, with $q(x) = x^2$. Then $\mathbb{C}$ with $\kappa(x) = xi$, satisfies the Clifford product rule. Clearly (A.18.1) cannot be satisfied in any smaller algebra, so (up to homomorphism) $C\ell(W,q) = \mathbb{C}$.

Example A.18.2 Let $W = \mathbb{R}$, with $q'(x) = -x^2$. Then $C\ell(W,q') = \mathbb{R}$, $\kappa(x) = x$. Comparing with Example A.18.1, one sees that passing to the opposite signature matters.

Example A.18.3 Consider the hermitian, traceless, *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad \text{(A.18.3)}$$

and set $\sigma^i = \sigma_i$. One readily checks that

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma^k \qquad \Longrightarrow \qquad \{\sigma_i, \sigma_j\} := \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij} \ , \qquad \text{(A.18.4)}$$

where for any two matrices a, b the *anti-commutator* $\{a, b\}$ is defined as

$$\{a, b\} = ab + ba \ .$$

Hence, for any $X \in \mathbb{R}^3$ it holds that

$$(X^k i\sigma_k)(X^\ell i\sigma_\ell) = -\delta(X, X) \ ,$$

where δ is the standard scalar product on $W = \mathbb{R}^3$. Thus, the map

$$X = (X^k) \to X^k i\sigma_k$$

is a Clifford map on $(\mathbb{R}^3, \delta)$, and in fact the algebra generated by the matrices $\gamma_k := i\sigma_k$ is homomorphic to $C\ell(\mathbb{R}^3, \delta)$. This follows again from the fact that no smaller dimension is possible.

EXAMPLE A.18.4 Let σ_i be the Pauli matrices (A.18.3) and let the 4×4 complex valued matrices be defined as

$$\gamma^0 = \begin{pmatrix} 0 & \mathrm{id}_{\mathbb{C}^2} \\ \mathrm{id}_{\mathbb{C}^2} & 0 \end{pmatrix} = -\gamma_0 \ , \qquad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = \gamma^i \ . \tag{A.18.5}$$

We note that γ_0 is hermitian, while the γ_i's are anti-hermitian with respect to the canonical hermitian scalar product $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ on $\mathbb{C}^4$. From Equation (A.18.5) one immediately finds

$$\{\gamma_i, \gamma_j\} = \begin{pmatrix} -\{\sigma_i, \sigma_j\} & 0 \\ 0 & -\{\sigma_i, \sigma_j\} \end{pmatrix} \ , \quad \{\gamma_i, \gamma_0\} = 0 \ , \quad (\gamma_0)^2 = 1 \ ,$$

and (A.18.4) leads to the Clifford product relation

$$\gamma_a\gamma_b + \gamma_b\gamma_a = -2g_{ab} \tag{A.18.6}$$

for the Minkowski metric $g_{ab} = \mathrm{diag}(-1, 1, 1, 1)$.

A real representation of the commutation relations (A.18.6) on $\mathbb{R}^8$ can be obtained by viewing $\mathbb{C}^4$ as a vector space over $\mathbb{R}$, so that 1) each 1 above is replaced by $\mathrm{id}_{\mathbb{R}^2}$, and 2) each i is replaced by the antisymmetric 2×2 matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

More precisely, let us define the 4×4 matrices $\hat\sigma_i$ by

$$\hat\sigma_1 = \begin{pmatrix} 0_{\mathrm{End}(\mathbb{R}^2)} & \mathrm{id}_{\mathbb{R}^2} \\ \mathrm{id}_{\mathbb{R}^2} & 0_{\mathrm{End}(\mathbb{R}^2)} \end{pmatrix} \ , \quad \hat\sigma_3 = \begin{pmatrix} \mathrm{id}_{\mathbb{R}^2} & 0_{\mathrm{End}(\mathbb{R}^2)} \\ 0_{\mathrm{End}(\mathbb{R}^2)} & -\mathrm{id}_{\mathbb{R}^2} \end{pmatrix} \ , \tag{A.18.7}$$

$$\hat\sigma_2 = \begin{pmatrix} 0_{\mathrm{End}(\mathbb{R}^2)} & -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & 0_{\mathrm{End}(\mathbb{R}^2)} \end{pmatrix} \ , \tag{A.18.8}$$

which are clearly symmetric, and the new γ's by

$$\gamma^0 = \begin{pmatrix} 0_{\mathrm{End}(\mathbb{R}^4)} & \mathrm{id}_{\mathbb{R}^4} \\ \mathrm{id}_{\mathbb{R}^4} & 0_{\mathrm{End}(\mathbb{R}^4)} \end{pmatrix} = -\gamma_0 \ , \qquad \gamma_i = \begin{pmatrix} 0_{\mathrm{End}(\mathbb{R}^4)} & \hat\sigma_i \\ -\hat\sigma_i & 0_{\mathrm{End}(\mathbb{R}^4)} \end{pmatrix} = \gamma^i \ . \tag{A.18.9}$$

It should be clear that the γ's satisfy (A.18.6), with γ_0 symmetric, and γ_i's - antisymmetric.

Let us return to general considerations. Choose a basis e_i of W, and consider any element $a \in \mathscr{T}(W)$. Then a can be written as

$$a = \alpha + \sum_{k=1}^{N} \sum_{i_1,\dots,i_k} a^{i_1\dots i_k} e_{i_1} \otimes \dots \otimes e_{i_k} \ ,$$

for some N depending upon a. When passing to the quotient, every tensor product $e_{i_j} \otimes e_{i_r}$ with $i_j > i_r$ can be replaced by $-2g_{i_j i_r} - e_{i_r} \otimes e_{i_j}$, leaving eventually only those indices which are increasingly ordered. Thus, a is equivalent to

$$\beta + b^i e_i + \sum_{k=N}^{k} \sum_{i_1<\dots<i_k} b^{i_1\dots i_k} e_{i_1} \otimes \dots \otimes e_{i_k} \ ,$$

for some new coefficients. For example

$$\begin{aligned}
\alpha + a^i e_i + a^{ij} e_i \otimes e_j &= \alpha + a^i e_i + a^{ij} (\underbrace{e_{(i} \otimes e_{j)}}_{\sim -q(e_i,e_j)} + e_{[i} \otimes e_{j]}) \\
&\sim \alpha - a^{ij} q(e_i,e_j) + a^i e_i + a^{ij} e_{[i} \otimes e_{j]} \ .
\end{aligned}$$

This implies that elements of the form

$$\gamma_{i_1\dots i_k} := [e_{i_1} \otimes \dots \otimes e_{i_k}] \ , \quad i_1 < \dots < i_k$$

span $C\ell(W,q)$. (Here the outermost bracket is the equivalence relation in $\mathscr{T}(W)$.) Equivalently,

$$C\ell(W,q) = \mathbb{K} \oplus \mathrm{Vect}\{\gamma_{i_1\dots i_k}\} \ , \qquad \text{where } \gamma_i := \kappa(e_i) \ , \ \gamma_{i_1\dots i_k} := \gamma_{[i_1} \cdots \gamma_{i_k]} \ .$$

We conclude that the dimension of $C\ell(W,q)$ is less than or equal to that of the exterior algebra of W, in particular $C\ell(W,q)$ is finite dimensional (recall that it was part of our definition that $\dim W < \infty$). The reader is warned that the above elements of the algebra are not necessarily linearly independent, as can be seen in Examples A.18.2 and A.18.3.

It should be clear to the reader that the linear map, which is deduced by the considerations above, from the exterior algebra to the Clifford algebra, does *not* preserve the product structures in those algebras.

A *representation* (V,ρ) of a Clifford algebra $C\ell(W,q)$ on a vector space V over $\mathbb{K}$ is a map $\rho : C\ell(W,q) \to \mathrm{End}(V)$ such that $\rho \circ \kappa$ is a Clifford map. It immediately follows from the definition of the Clifford algebra that ρ is uniquely defined by its restriction to $\kappa(W)$.

A fundamental fact is the following:

PROPOSITION A.18.5 *Let q be positive definite and let (V,ρ) be a representation of $C\ell(W,q)$. If $\mathbb{K} = \mathbb{R}$, then there exists a scalar product $\langle \cdot,\cdot \rangle$ on V so that $\rho \circ \kappa(X)$ is antisymmetric for all $X \in W$. Similarly if $\mathbb{K} = \mathbb{C}$, then there exists a hermitian product $\langle \cdot,\cdot \rangle$ on V so that $\rho \circ \kappa(X)$ is antihermitian for all $X \in W$.*

PROOF: Let e_i be any basis of W, set $\gamma_i := \rho(\kappa(e_i))$, since ρ is a representation the γ_i's satisfy the relation (A.18.6). Let γ_I run over the set

$$\Omega := \{\pm 1 \ , \pm\gamma_i \ , \pm\gamma_{i_1\ldots i_k}\}_{1\le i_1<\cdots<i_k\le n} \ .$$

It is easy to check that $\Omega\gamma_i \subset \Omega$, but since

$$(\Omega\gamma_i)\gamma_i = \Omega \underbrace{\gamma_i\gamma_i}_{=-1} = -\Omega = \Omega \ ,$$

we conclude that $\Omega\gamma_i = \Omega$. Let $(\cdot,\cdot)$ denote any scalar product on V, and for $\psi, \varphi \in V$ set

$$\langle\psi,\varphi\rangle := \sum_{\gamma_I\in\Omega} (\gamma_I\psi, \gamma_I\varphi) \ .$$

Then for any $1 \le \ell \le n$ we have (no summation over ℓ)

$$\begin{aligned}
\langle(\gamma_\ell)^t\gamma_\ell\psi,\varphi\rangle &= \langle\gamma_\ell\psi,\gamma_\ell\varphi\rangle = \sum_{\gamma_I\in\Omega}(\gamma_I\gamma_\ell\psi,\gamma_I\gamma_\ell\varphi) \\
&= \sum_{\gamma_I\in\Omega\gamma_\ell}(\gamma_I\psi,\gamma_I\varphi) = \sum_{\gamma_I\in\Omega}(\gamma_I\psi,\gamma_I\varphi) \\
&= \langle\psi,\varphi\rangle \ .
\end{aligned}$$

Since this holds for all ψ and φ we conclude that $(\gamma_\ell)^t\gamma_\ell = \mathrm{Id}$. Multiplying from the right with γ_ℓ, and recalling that $(\gamma_\ell)^2 = -\mathrm{Id}$ we obtain $(\gamma_\ell)^t = -\gamma_\ell$. Now, by definition,

$$(\rho\circ\kappa(X))^t = (X^a\gamma_a)^t = -X^a\gamma_a = -\rho\circ\kappa(X) \ ,$$

as desired.

An identical calculation applies for hermitian scalar products. □

The scalar product constructed above is likely to depend upon the initial choice of basis e_a, but this is irrelevant for the problem at hand, since the statement that $\rho\circ\kappa(X)$ is anti-symmetric, or anti-hermitian, is basis-independent.

Throughout most of this work, when q is positive definite we will only use scalar products on V for which the representation of $C\ell(W,q)$ is anti-symmetric or anti-hermitian.

Let $\dim V > 0$. A representation (V,ρ) of $C\ell(W,q)$ is said to be *reducible* if V can be decomposed as a direct sum $V_1 \oplus V_2$ of nontrivial subspaces, each of them being invariant under all maps in $\rho(C\ell(W,q))$. The representation (V,ρ) is said to be *irreducible* if it is not reducible. Note that the existence of an invariant space does not *a priori* imply the existence of a complementing space which is invariant as well. However, we have the following:

PROPOSITION A.18.6 *Every finite dimensional representation*

$$\rho : C\ell(W,q) \to End(V)$$

of $C\ell(W,q)$ such that V contains a non-trivial invariant subspace is reducible. Hence, $V = \oplus_{i=1}^k V_i$, $\rho = \oplus_{i=1}^k \rho_i$, with $(V_i, \rho_i := \rho|_{V_i})$ irreducible.

PROOF: Suppose that there exists a subspace $V_1 \subset V$ invariant under ρ. We can assume that V_1 has no invariant subspaces, otherwise we pass to this subspace and call it V_1; in a finite number of steps we obtain a subspace V_1 such that $\rho|_{V_1}$ is irreducible. The proof of Proposition A.18.5 provides a scalar or hermitian product $\langle \cdot, \cdot \rangle$ on V which is invariant under the action of all maps in Ω. Then $V = V_1 \oplus (V_1)^\perp$, and it is easily checked that $(V_1)^\perp$ is also invariant under maps in Ω, hence under all maps in the image of ρ. One can repeat now the whole argument with V replaced by $(V_1)^\perp$, and the claimed decomposition is obtained after a finite number of steps. $\Box$

It is sometimes convenient to use irreducible representations, which involves no loss of generality in view of Proposition A.18.6. However, we will not assume irreducibility unless explicitly specified otherwise.

A.18.1 Eigenvalues of γ-matrices

In the proofs of the energy-momentum inequalities the positivity properties of several matrices acting on the space of spinors have to be analyzed. It is sufficient to make a pointwise analysis, so we consider a real vector space V equipped with a scalar product $\langle \cdot, \cdot \rangle$ together with matrices γ_μ, $\mu = 0, 1, \cdots, n$ satisfying

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = -2\eta_{\mu\nu} \ , \tag{A.18.10}$$

where $\eta = \mathrm{diag}(-1, 1, \cdots, 1)$. We further suppose that the matrices γ_μ^t, transposed with respect to $\langle \cdot, \cdot \rangle$, satisfy

$$\gamma_0^t = \gamma_0 \ , \quad \gamma_i^t = -\gamma_i \ ,$$

where the index i runs from one to n. Let us start with

$$a^\mu \gamma_0 \gamma_\mu = a^0 + a^i \gamma_0 \gamma_i \ , \quad (a^\mu) = (a^0, \vec{a}) = (a^0, (a^i)) \ .$$

The matrices $a^i \gamma_0 \gamma_i$ are symmetric and satisfy

$$(a^i \gamma_0 \gamma_i)^2 = a^i a^j \gamma_0 \gamma_i \gamma_0 \gamma_j = -a^i a^j \gamma_0 \gamma_0 \gamma_i \gamma_j = |\vec{a}|_\delta^2 \ ,$$

so that the eigenvalues belong to the set $\{\pm |\vec{a}|_\delta\}$. Since γ_0 anticommutes with $a^i \gamma_0 \gamma_i$, it interchanges the eigenspaces with positive and negative eigenvalues. Let ψ_i, $i = 1, \ldots, N$, be an ON basis of the $|\vec{a}|_\delta$ eigenspace of $a^i \gamma_0 \gamma_i$, set

$$\phi_{2i-1} = \psi_i \ , \quad \phi_{2i} = \gamma_0 \psi_i \ .$$

It follows that $\{\phi_i\}_{i=1}^{2N}$ forms an ON basis of V (in particular $\dim V = 2N$), and in that basis $a^\mu \gamma_0 \gamma_\mu$ is diagonal with entries $a^0 \pm |\vec{a}|_\delta$. We have thus proved

PROPOSITION A.18.7 *The quadratic form $\langle \psi, a^\mu \gamma_0 \gamma_\mu \psi \rangle$ is non-negative if and only if $a^0 \geq |\vec{a}|_\delta$.*

Let us consider, next, the symmetric matrix

$$A := a^\mu \gamma_0 \gamma_\mu + b\gamma_0 + c\gamma_1 \gamma_2 \gamma_3 \ . \tag{A.18.11}$$

Let ψ_1 be an eigenvector of $a^i\gamma_0\gamma_i$ with eigenvalue $|\vec{a}|_\delta$, set

$$\phi_1 = \psi_1 \;, \quad \phi_2 = \gamma_0\psi_1 \;, \quad \phi_3 = \gamma_1\gamma_2\gamma_3\psi_1 \;, \quad \phi_4 = \gamma_1\gamma_2\gamma_3\gamma_0\psi_1 \;.$$

From the commutation relations (A.18.10) one easily finds

$$a^i\gamma_0\gamma_i\phi_1 = |\vec{a}|_\delta\phi_1 \;, \quad a^i\gamma_0\gamma_i\phi_2 = -|\vec{a}|_\delta\phi_2 \;, \quad a^i\gamma_0\gamma_i\phi_3 = -|\vec{a}|_\delta\phi_3 \;, \quad a^i\gamma_0\gamma_i\phi_4 = |\vec{a}|_\delta\phi_4 \;,$$

$$\gamma_0\phi_1 = \phi_2 \;, \quad \gamma_0\phi_2 = \phi_1 \;, \quad \gamma_0\phi_3 = -\phi_4 \;, \quad \gamma_0\phi_4 = -\phi_3 \;,$$

$$\gamma_1\gamma_2\gamma_3\phi_1 = \phi_3 \;, \quad \gamma_1\gamma_2\gamma_3\phi_2 = \phi_4 \;, \quad \gamma_1\gamma_2\gamma_3\phi_3 = \phi_1 \;, \quad \gamma_1\gamma_2\gamma_3\phi_4 = \phi_2 \;.$$

It is simple to check that the ϕ_i's so defined are ON; proceeding by induction one constructs an ON-basis $\{\phi_i\}_{i=1}^{2N}$ of V (in particular $\dim V$ is a multiple of 4) in which A is block-diagonal, built-out of blocks of the form

$$\begin{pmatrix} a^0 + |\vec{a}|_\delta & b & c & 0 \\ b & a^0 - |\vec{a}|_\delta & 0 & c \\ c & 0 & a^0 - |\vec{a}|_\delta & -b \\ 0 & c & -b & a^0 + |\vec{a}|_\delta \end{pmatrix} .$$

The eigenvalues of this matrix are easily found to be $a^0 \pm \sqrt{|\vec{a}|^2_\delta + b^2 + c^2}$. We thus have:

PROPOSITION A.18.8 *We have the sharp inequality*

$$\langle\psi, (a^\mu\gamma_0\gamma_\mu + b\gamma_0 - c\gamma_1\gamma_2\gamma_3)\,\psi\rangle \geq \left(a^0 - \sqrt{|\vec{a}|^2_\delta + b^2 + c^2}\right)|\psi|^2 \;,$$

in particular the quadratic form $\langle\psi, A\psi\rangle$, with A defined in (A.18.11), is non-negative if and only if

$$a^0 \geq \sqrt{|\vec{a}|^2_\delta + b^2 + c^2} \;.$$

A.19 Elements of causality theory

It is convenient to recall some definitions from causality theory. Given a manifold $\mathscr{M}$ equipped with a Lorentzian metric g, at each point $p \in \mathscr{M}$ the set of timelike vectors in T_pM has precisely two components. A *time-orientation of* $T_p\mathscr{M}$ is the assignment of the name "future pointing vectors" to one of those components; vectors in the remaining component are then called "past pointing". A Lorentzian manifold is said to be *time-orientable* if such locally defined time-orientations can be defined globally in a consistent way. A *space-time* is a time-orientable Lorentzian manifold on which a time-orientation has been chosen.

A differentiable path γ will be said to be *timelike* if at each point the tangent vector $\dot\gamma$ is timelike; it will be said *future directed* if $\dot\gamma$ is future directed. There is an obvious extension of this definition to *null, causal* or *spacelike* curves. We define an *observer* to be an *inextendible, future directed timelike path.* In these notes the names "path" and "curve" will be used interchangedly.

Let $\mathscr{U} \subset \mathscr{O} \subset \mathscr{M}$. One sets

$$\begin{aligned} I^+(\mathscr{U};\mathscr{O}) \quad &:= \quad \{q \in \mathscr{O} : \text{there exists a timelike future directed path} \\ & \qquad \text{from } \mathscr{U} \text{ to } q \text{ contained in } \mathscr{O}\}\ , \\ J^+(\mathscr{U};\mathscr{O}) \quad &:= \quad \{q \in \mathscr{O} : \text{there exists a causal future directed path} \\ & \qquad \text{from } \mathscr{U} \text{ to } q \text{ contained in } \mathscr{O}\} \cup \mathscr{U}\ . \end{aligned}$$

$I^-(\mathscr{U};\mathscr{O})$ and $J^-(\mathscr{U};\mathscr{O})$ are defined by replacing "future" by "past" in the definitions above. The set $I^+(\mathscr{U};\mathscr{O})$ is called the *timelike future of $\mathscr{U}$ in $\mathscr{O}$*, while $J^+(\mathscr{U};\mathscr{O})$ is called the *causal future of $\mathscr{U}$ in $\mathscr{O}$*, with similar terminology for the timelike past and the causal past. We will write $I^\pm(\mathscr{U})$ for $I^\pm(\mathscr{U};\mathscr{M})$, similarly for $J^\pm(\mathscr{U})$, and one then omits the qualification "in $\mathscr{M}$" when talking about the causal or timelike futures and pasts of $\mathscr{U}$. We will write $I^\pm(p;\mathscr{O})$ for $I^\pm(\{p\};\mathscr{O})$, $I^\pm(p)$ for $I^\pm(\{p\};\mathscr{M})$, *etc.*

A function f will be called a *time function* if its gradient is *timelike, past pointing*. Similarly a function f will be said to be a *causal function* if its gradient is *causal, past pointing*. The choice "past-pointing" here has to do with our choice $(-,+,\ldots,+)$ of the signature of the metric. This is easily understood on the example of Minkowski space-time $(\mathbb{R}^{n+1},\eta)$, where the gradient of the usual time coordinate t is $-\partial_t$, since $\eta^{00}=-1$. Had we chosen to work with the signature $(+,-,\ldots,-)$, time functions would have been defined to have future pointing gradients.

A.19.1 Geodesics

An *affinely parameterised geodesic* γ is a maximally extended solution of the equation

$$\nabla_{\dot\gamma}\dot\gamma = 0\ .$$

It is a fundamental postulate of general relativity that physical observers move on timelike geodesics. This motivates the following useful definition: an *observer* is a maximally extended future directed timelike geodesics.

It is sometimes convenient to consider geodesics which are not necessarily affinely parameterised. Those are solutions of

$$\nabla_{\frac{d\gamma}{d\lambda}}\frac{d\gamma}{d\lambda} = \chi\frac{d\gamma}{d\lambda}\ . \tag{A.19.1}$$

Indeed, let us show that a change of parameter obtained by solving the equation

$$\frac{d^2\lambda}{ds^2} + \chi\left(\frac{d\lambda}{ds}\right)^2 = 0 \tag{A.19.2}$$

brings (A.19.2) to the form (A.19.2): under a change of parameter $\lambda=\lambda(s)$ we have

$$\frac{d\gamma^\mu}{ds} = \frac{d\lambda}{ds}\frac{d\gamma^\mu}{d\lambda}\ ,$$

and

$$\frac{D}{ds}\frac{d\gamma^\nu}{ds} \quad = \quad \frac{D}{ds}(\frac{d\lambda}{ds}\frac{d\gamma^\nu}{d\lambda})$$

$$
\begin{aligned}
&= \frac{d^2\lambda}{ds^2}\frac{d\gamma^\nu}{d\lambda} + \frac{d\lambda}{ds}\frac{D}{ds}\frac{d\gamma^\nu}{d\lambda} \\
&= \frac{d^2\lambda}{ds^2}\frac{d\gamma^\nu}{d\lambda} + \left(\frac{d\lambda}{ds}\right)^2\frac{D}{d\lambda}\frac{d\gamma^\nu}{d\lambda} \\
&= \frac{d^2\lambda}{ds^2}\frac{d\gamma^\nu}{d\lambda} + \left(\frac{d\lambda}{ds}\right)^2\chi\frac{d\gamma^\nu}{d\lambda} ,
\end{aligned}
$$

and the choice indicated above gives zero, as desired.

Let f be a smooth function and let $\lambda \mapsto \gamma(\lambda)$ be any integral curve of ∇f; by definition, this means that $d\gamma^\mu/d\lambda = \nabla^\mu f$. The following provides a convenient tool for finding geodesics:

PROPOSITION A.19.1 (Integral curves of gradients) *Let f be a function satisfying*

$$g(\nabla f, \nabla f) = \psi(f) ,$$

for some function ψ. Then the integral curves of ∇f are geodesics, affinely parameterised if $\psi' = 0$.

PROOF: We have

$$\dot\gamma^\alpha\nabla_\alpha\dot\gamma^\beta = \nabla^\alpha f\nabla_\alpha\nabla^\beta f = \nabla^\alpha f\nabla^\beta\nabla_\alpha f = \frac{1}{2}\nabla^\beta(\nabla^\alpha f\nabla_\alpha f) = \frac{1}{2}\nabla^\beta\psi(f) = \frac{1}{2}\psi'\nabla^\beta f . \tag{A.19.3}$$

Let us denote by λ the parameter such that

$$\frac{d\gamma^\mu}{d\lambda} = \nabla^\mu f ,$$

then (A.19.3) can be rewritten as

$$\frac{D}{d\lambda}\frac{d\gamma^\mu}{d\lambda} = \frac{1}{2}\psi'\frac{d\gamma^\mu}{d\lambda} .$$

□

A significant special case is that of a coordinate function $f = x^i$. Then

$$g(\nabla f, \nabla f) = g(\nabla x^i, \nabla x^i) = g^{ii} \text{ (no summation)} .$$

For example, in Minkowski space-time, all $g^{\mu\nu}$'s are constant, which shows that the integral curves of the gradient of any coordinate, and hence also of any linear combination of coordinates, are affinely parameterized geodesics. An other example is provided by the coordinate r in Schwarzschild space-time, where $g^{rr} = 1 - 2m/r$; this is indeed a function of r, so the integral curves of $\nabla r = (1 - 2m/r)\partial_r$ are (non-affinely parameterized) geodesics.

Appendix B

Weighted Poincaré inequalities

The following is a simplified version of [14] (most intricacies in [14] are due to low differentiability hypotheses on the metric). We start with a Lemma, essentially due to Geroch and Perng [100]:

LEMMA B.0.2 *Let $\Omega, \tilde\Omega$ be any two relatively compact domains in M, there is a constant $\epsilon > 0$ such that for all sections $u \in H^1_{\mathrm{loc}}(M)$ of a vector bundle E we have*

$$\epsilon \int_{\tilde\Omega} |u|^2 \, dv_M \le \int_\Omega |u|^2 \, dv_M + \int_M |Du|^2 \, dv_M \; . \tag{B.0.1}$$

PROOF: Let $\mathring{g}$ be any smooth auxiliary Riemannian metric on M, let q be any point of $\tilde\Omega$, fix $p \in \Omega$ and let r_p be small enough that the $\mathring{g}$-geodesic ball $B(p, r_p)$ of radius r_p and centred at p, lies within Ω. Let X be a C^∞ compactly supported vector field, such that the associated flow ϕ_t satisfies $\phi_1(B(p, r_p)) \supset B(q, r_q)$ for some $r_q > 0$. (Since M is C^∞ and connected, it is always possible to construct such an X.) Let $\Omega_t = \phi_t(B(p, r_p))$.

By direct calculation and Hölder's inequality we have, for any $u \in H^1_{\mathrm{loc}}(M)$,

$$\begin{aligned} \frac{d}{dt} \int_{\Omega_t} |u|^2 dv_M &= \int_{\Omega_t} \left(2\langle u, D_X u\rangle + |u|^2 \mathrm{div}_{\mathring{g}} X \right) \, dv_M \\ &\le C \int_{\Omega_t} \left(|u|^2 + |Du|^2 \right) \, dv_M \; , \end{aligned}$$

where C depends on $\|X\|_{L^\infty}$, $\|\mathrm{div}_{\mathring{g}} X\|_{L^\infty}$. Defining $F(t) = \int_{\Omega_t} |u|^2 dv_M$, we have

$$\frac{d}{dt} F(t) \le C F(t) + C \int_M |Du|^2 dv_M \; ,$$

and Gronwall's lemma gives $F(1) \le e^C (F(0) + \int_M |Du|^2 dv_M)$. Thus there is $\epsilon > 0$ such that

$$\epsilon \int_{B(q, r_q)} |u|^2 \, dv_M \le \int_\Omega |u|^2 \, dv_M + \int_M |Du|^2 \, dv_M \; .$$

Since $\tilde\Omega$ has compact closure, it is covered by finitely many such balls $B(q, r_q)$ and (B.0.1) follows. □

As a corollary of Lemma B.0.2 we obtain:

Corollary B.0.3 *If there is a domain $\Omega \subset M$ and a constant $\epsilon > 0$ such that*

$$\epsilon \int_\Omega |u|^2 \, dv_M \leq \int_M |Du|^2 \, dv_M \tag{B.0.2}$$

for all $u \in C^1_c(M)$, then (M, D) admits a weighted Poincaré inequality (3.2.24).

Proof: By paracompactness and Lemma B.0.2, there is a countable locally finite covering of M by domains Ω_k and constants $1 \geq \epsilon_k > 0$, $k \in \mathbb{Z}^+$, such that for each k,

$$\epsilon_k \int_{\Omega_k} |u|^2 \, dv_M \leq \int_\Omega |u|^2 \, dv_M + \int_M |Du|^2 \, dv_M \ .$$

This can be further estimated using (B.0.2), so the function

$$w(x) = \sum_{k : x \in \Omega_k} \frac{2^{-k} \epsilon \epsilon_k}{1 + \epsilon} \tag{B.0.3}$$

is bounded, strictly positive, and satisfies

$$\int_M |u|^2 \, w \, dv_M \leq \int_M |Du|^2 \, dv_M \ ,$$

which is the required weighted Poincaré inequality. □

We will show that the inequality needed in Corollary B.0.3 holds under rather weak asymptotic conditions on the metric:

Definition B.0.4 *A* uniform end *$\tilde{M} \subset M$ is a set such that $\tilde{M} \simeq \mathbb{R}^n \backslash B(0,1)$ and there is a constant $\eta > 0$ such that*

$$\eta \, \delta_{ij} \xi^i \xi^j \leq g_{ij}(x) \xi^i \xi^j \leq \eta^{-1} \delta_{ij} \xi^i \xi^j ;$$

for all $x \in \mathbb{R}^n \backslash B(0,1)$ and all vectors $\xi \in \mathbb{R}^n$.

We have:

Proposition B.0.5 *Suppose (M, g) is a (connected) manifold of dimension $n \geq 3$, with $g \in C^0(M)$, containing a uniform end $\tilde{M}$. Then (M, D) admits a weighted Poincaré inequality.*

Proof: Let $r = (\sum (x^i)^2)^{1/2} \in C^\infty(\tilde{M})$ and $\chi = \chi(r) \in C^1_c(\tilde{M})$ satisfy, for some $R_0 > 1$ and $k \geq 10$,

$$\chi(r) = \frac{\log(r/R_0)}{\log k}, \quad 2R_0 \leq r \leq (k-1) R_0$$

and $\chi(r) = 1$ for $r > kR_0$, $\chi(r) = 0$ for $r \leq R_0$. Then $|\chi'(r)| \leq 2/(r \log k)$, so for any $u \in C^1_c(M)$

$$\int_M |D(\chi u)|^2 \, dv_M \leq 2 \int_M |Du|^2 \, dv_M + \frac{4}{(\log k)^2} \int_{R_0 \leq r \leq kR_0} \frac{1}{r^2} |u|^2 \, dv_M . \tag{B.0.4}$$

Now $\Delta_0(r^{2-n}) = 0$ for $r \geq 1$ in $\mathbb{R}^n$, $n \geq 3$, so for any $v \in C^1_c(\mathbb{R}^n \backslash B(0,1))$ we have

$$\begin{aligned} 0 &= -\int_{\mathbb{R}^n} \partial_i(\partial_i(r^{2-n})\,|v|^2 r^{n-2})\,dx \\ &= (n-2)^2 \int_{\mathbb{R}^n} r^{-2}|v|^2\,dx + (n-2)\int_{\mathbb{R}^n} r^{-1} 2\langle v, D_r v\rangle\,dx. \end{aligned}$$

Using Hölder's inequality we obtain

$$\frac{(n-2)^2}{4}\int_{\mathbb{R}^n} r^{-2}|v|^2\,dx \leq \int_{\mathbb{R}^n} |Dv|^2\,dx$$

and thus there is $\epsilon > 0$ such that for all $v \in C^1_c(\tilde{M})$,

$$\epsilon \int_{\tilde{M}} r^{-2}|v|^2\,dv_M \leq \int_{\tilde{M}} |Dv|^2\,dv_M. \tag{B.0.5}$$

Thus combining (B.0.5) with $v = \chi u$ and (B.0.4) gives

$$\begin{aligned} \int_{r>kR_0} r^{-2}|u|^2\,dv_M &\leq \int_{\tilde{M}} r^{-2}|\chi u|^2\,dv_M \\ &\leq C\int_{\tilde{M}} |D(\chi u)|^2\,dv_M \\ &\leq C\int_{\tilde{M}} |Du|^2\,dv_M + \frac{C}{(\log k)^2}\int_{\tilde{M}} r^{-2}|u|^2\,dv_M. \end{aligned}$$

If k is chosen so that $C/(\log k)^2 \leq \frac{1}{2}$ then the last term may be absorbed into the left hand side, giving

$$\int_{r\geq kR_0} r^{-2}|u|^2\,dv_M \leq C\int_M |Du|^2\,dv_M. \tag{B.0.6}$$

Lemma B.0.2 now applies and gives the required weighted Poincaré inequality. $\square$

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