Tutorials for the course “Allgemeine Relativitätstheorie und Kosmologie” (2021S and later)

1. **[Summation convention]** For each of the following, either write out the equation with the summation signs included explicitly or explain why the equation is ambiguous or does not make sense. Provide a possible correct version, or versions, of the wrong or incoherent equations. (Recall that $\delta^a_b = 1$ if $a = b$ and is zero otherwise.)

   (a) $X^a = L^a{}_b M^{bc} \hat{X}_c$
   (b) $X^a = L^b{}_c M^c{}_d \hat{X}^d$
   (c) $\delta^a_b = \delta^a_c \delta^c_d$
   (d) $\delta^a_b = \delta^a_b \delta^c_c$
   (e) $X^a = L^a{}_b \hat{X}^b + M^{ab} \hat{X}^b$
   (f) $X^a = L^a{}_b \hat{X}^b + M^a{}_c \hat{X}^c$
   (g) $X^a = L^a{}_c \hat{X}^c + M^b{}_c \hat{X}^c$
   (h) $X^a = L^a{}_c \hat{X}^c + \sum_c M^{ac} \hat{X}^c$
   (i) $R^a{}_{bac} = g^{ad} R_{bdca}$
   (j) $R_{\alpha\beta} = \partial_\rho \Gamma^\rho_{\beta\alpha} - \partial_\beta \Gamma^\rho_{\rho\alpha} + \Gamma^\rho_{\rho\lambda} \Gamma^\lambda_{\beta\alpha} - \Gamma^\xi_{\beta\eta} \Gamma^\eta_{\rho\alpha}$

2. Let a bracket over indices denote complete antisymmetrisation, and a parenthesis over indices denote complete symmetrisation: for example,

$$A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu}) \,, \quad A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu}) \,, \quad \delta^{[\alpha}_\mu \delta^{\gamma]}_\rho = \frac{1}{2}(\delta^\alpha_\mu \delta^\gamma_\rho - \delta^\gamma_\mu \delta^\alpha_\rho) \,,$$

$$A_{[\mu\nu\rho]} = \frac{1}{6}(A_{\mu\nu\rho} + A_{\nu\rho\mu} + A_{\rho\mu\nu} - A_{\nu\mu\rho} - A_{\mu\rho\nu} - A_{\rho\nu\mu}) \,,$$

   and similarly for four or more indices (with combinatorial prefactors $1/n!$). Show that

   (a) $A_{(\mu\nu)} = A_{(\nu\mu)}$, $A_{[\mu\nu]} = -A_{[\nu\mu]}$,
   (b) $A_{\mu\nu} = A_{[\mu\nu]} + A_{(\mu\nu)}$,
   (c) $A^{[\mu\nu]} B_{\mu\nu} = A^{\mu\nu} B_{[\mu\nu]} = A^{[\mu\nu]} B_{[\mu\nu]}$,
   (d) $A^{(\mu\nu)} B_{\mu\nu} = A^{\mu\nu} B_{(\mu\nu)} = A^{(\mu\nu)} B_{(\mu\nu)}$,
   (e) if $C_{\alpha\beta}$ is symmetric, i.e. $C_{\alpha\beta} = C_{\beta\alpha}$, and if $D^{\alpha\beta}$ is antisymmetric, i.e. $D^{\alpha\beta} = -D^{\beta\alpha}$, then $C_{\alpha\beta} D^{\alpha\beta} = 0$,
   (f) $A^{[\mu\nu\rho]} B_{\mu\nu\rho} = A^{\mu\nu\rho} B_{[\mu\nu\rho]}$,
   (g) $\delta^{[\alpha}_\mu \delta^{\gamma]}_\rho = \delta^\alpha_{[\mu} \delta^\gamma_{\rho]} = \delta^{[\alpha}_{[\mu} \delta^{\gamma]}_{\rho]}$,
   (h) $\delta^{[\alpha}_\mu \delta^\beta_\nu \delta^{\gamma]}_\rho = \delta^\alpha_{[\mu} \delta^\beta_\nu \delta^\gamma_{\rho]} = \delta^{[\alpha}_{[\mu} \delta^\beta_\nu \delta^{\gamma]}_{\rho]}$.

[*Reminder:* Let $x^i \mapsto \overline{x}^i$ be a coordinate transformation of $\mathbb{R}^n$, which we write for short as $x \mapsto \overline{x}$. A *scalar field* f on $\mathbb{R}^n$ satisfies the transformation law

$$\overline{f}(\overline{x}) = f(x(\overline{x})) \,.$$

A *vector field* (X^i) on $\mathbb{R}^n$ satisfies the transformation law

$$\overline{X}^i(\overline{x}) = \frac{\partial \overline{x}^i}{\partial x^j} X^j(x(\overline{x})) \,,$$

where the summation convention is used. A *covector field* (X_i) on $\mathbb{R}^n$ satisfies the transformation law

$$\overline{X}_i(\overline{x}) = \frac{\partial x^j}{\partial \overline{x}^i} X_j(x(\overline{x}))\,. \tag{1}$$

We write ∂_i for $\partial/\partial x^i$, and $g(X,Y) = g_{ij}X^iY^j$.]

3. **[Changes of coordinates]** Let $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$ be inertial coordinates on flat space-time, so the Minkowski metric has components

$$(g_{ab}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let X be the vector field which in the above coordinate system equals $(1,1,0,0)^\mathsf{T}$, and let α be a one-form which in the above coordinate system equals $(1,1,0,0)$.

Find the metric coefficients $\tilde{g}_{ab}$, and the components of X and α, in each of the following coordinate systems.

(a) $\tilde{x}^0 = t - z$, $\tilde{x}^1 = r$, $\tilde{x}^2 = \theta$, $\tilde{x}^3 = z$,

(b) $\bar{x}^0 = \tau$, $\bar{x}^1 = \phi$, $\bar{x}^2 = y$, $\bar{x}^3 = z$,

(c) $\hat{x}^0 = t$, $\hat{x}^1 = r$, $\hat{x}^2 = \theta$, $\hat{x}^3 = \phi$,

where, in the first case, (r,θ) are plane polar coordinates in the (x,y) plane: $x = r\cos\theta$, $y = r\sin\theta$; in the second, (τ,ϕ) are 'Rindler coordinates', defined by $t = \tau\cosh\phi$, $x = \tau\sinh\phi$; and, in the third, (r,θ,ϕ) are spherical polar coordinates. In each case, state which region of Minkowski space the coordinate system covers. [*Hint: A quick method for the metric is to write it as* $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ *and substitute, for example,*

$$\mathrm{d}x = \mathrm{d}(r\cos\theta) = \cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta\,,$$

and so on. Here we have used the definition $\mathrm{d}f = \partial_\mu f\,\mathrm{d}x^\mu$ *of the differential* $\mathrm{d}f$ *of a function* f. *Of course you should convince yourself that this is legitimate.*]

4. **[Stereographic projection]** Let $(x,y) \in \mathbb{R}^2$ be local coordinates on a two-dimensional sphere obtained from a stereographic projection from the north pole onto the plane tangent to the south pole of the sphere. Express the standard metric on the sphere, namely $g = d\theta^2 + \sin^2(\theta)d\varphi^2$, in these coordinates.

5. **[Coordinates on a manifold]**

(a) Find an atlas of coordinate charts and give the change of coordinates (transition functions) between overlapping charts for the circle S^1.

(b) An infinite cylinder can be covered with just one chart, e.g. $\varphi : \mathbb{R} \times S^1 \to \mathbb{R}^2 \setminus (0,0)$,

$$\varphi(x, y, z) = (X, Y) = (xe^z, ye^z) , \tag{2}$$

where x, y, z are restricted to a two-dimensional cylinder $x^2 + y^2 = 1$, $z \in (-\infty, \infty)$ and X, Y span a flat plane with the origin excluded. Given this mapping, find the inverse map and express the following trajectories on the cylinder in the (X, Y) plane:

- a straight line $(x, y, z) = (1, 0, z)$
- circles $z = 1, z = -1, z = 0$
- parametrically defined spiral $(x, y, z) = (\cos(a), \sin(a), a)$ with $a \in (-\infty, \infty)$

6. **[Vector as differential operator]** Show that (X^i) is a vector field in the sense of (1) if and only if for every scalar field f the expression

$$X^i \partial_i f$$

is a scalar field.

[*Conclusion: vector fields can be identified with homogeneous linear first order partial differential operators* $X = X^i \partial_i$ *acting on scalar fields as* $X(f) = X^i \partial_i f$.]

7. **[Lie bracket]**

(a) The Lie-bracket $[X, Y]$ of two vector fields X and Y is defined as

$$[X, Y](f) = X(Y(f)) - Y(X(f)) .$$

Show that, in local coordinates,

$$[X, Y]^i = X^j \partial_j Y^i - Y^j \partial_j X^i .$$

(b) Show that $[X, Y]$ is a vector field if X and Y are.

(c) Prove the *Jacobi identity*:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 .$$

8. **[Raising and lowering of indices]** Let $g_{\mu\nu}$ be a symmetric non-degenerate tensor field. Define

$$B_\alpha := g_{\alpha\beta} A^\beta , \quad C^\gamma := g^{\gamma\sigma} B_\sigma . \tag{3}$$

Show that

$$C^\gamma = A^\gamma . \tag{4}$$

The first operation in (3) is called "lowering an index with the metric"; the second "raising an index with the metric". What does (4) say about these operations?

From now on we shall simply write

$$A_\alpha := g_{\alpha\beta}A^\beta\ , \quad B^\gamma := g^{\gamma\sigma}B_\sigma\ .$$

Show that

$$A_\alpha B^\alpha = A^\alpha B_\alpha\ .$$

9. **[Killing vectors]**

(a) Let X be a vector field which satisfies the following:

for every affinely parameterised geodesic γ, $g(X, \dot\gamma)$ is constant along γ.

Show that this implies that X satisfies the following equation, known as *the Killing equation*:

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0\ . \tag{5}$$

Solutions of the Killing equation are called *Killing vectors*.

(b) Show that a linear combination of Killing vectors with constant coefficients is a Killing vector. Thus, the set of Killing vectors forms a vector space.

(c) Let X be a Killing vector field, and let $s \mapsto x^\mu(s)$ be an integral curve of X: by definition, this means that

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}s}(s) = X^\mu(x(s))\ . \tag{6}$$

In other words, the vector field X is tangent to the curves $s \mapsto x^\mu(s)$. Show that the integral curves of X are geodesics if and only if we have

$$\nabla_\alpha(g(X, X)) = 0 \ \text{ along the curve } x(s). \tag{7}$$

[*Reminder:* The Christoffel symbols of the Levi-Civita connection are defined by the formula

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc})\ . \tag{8}$$

For functions we have $\nabla_i f = \partial_i f$, while for vector fields we have

$$\nabla_i X^j = \partial_i X^j + \Gamma^j_{ki}X^k\ ,$$

and it holds that

$$\partial_i(g(X, Y)) = \nabla_i(g(X, Y)) = g(\nabla_i X, Y) + g(X, \nabla_i Y)\,.$$

A curve $s \mapsto \gamma(s)$ is called an *affinely parameterised geodesic* if and only if

$$\frac{d^2\gamma^i}{ds^2} + \Gamma^i_{\ jk}\frac{d\gamma^j}{ds}\frac{d\gamma^k}{ds} = 0\,.$$

Unless explicitly indicated otherwise, "geodesic" is usually meant as "affinely parameterised geodesic".]

10. **[Christoffel symbols]**

(a) Using the definition calculate the Christoffel symbols for the Euclidean metric on $\mathbb{R}^2$ in polar coordinates: $g = \mathrm{d}\rho^2 + \rho^2\, \mathrm{d}\varphi^2$, and the unit round metric on S^2: $h = \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2$. [*Hint: Calculate first the Christoffel symbols for a metric of the form* $\mathrm{d}x^2 + e^{2f(x)}\mathrm{d}y^2$.]

(b) Show that the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = \frac{\partial L}{\partial x^a} \tag{9}$$

associated with the Lagrange function

$$L(x^c, \dot{x}^c) = \frac{1}{2} g_{ab}\dot{x}^a\dot{x}^b \tag{10}$$

(here, a dot denotes $\mathrm{d}/\mathrm{d}s$) can be written as

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0\ . \tag{11}$$

(c) Show that if the metric does not explicitly depend upon a coordinate, say x^1, then $g(\dot{x}, \partial_1)$ is constant along every geodesic.

(d) Using the above variational principle for geodesics, write down the geodesic equations, and give the obvious constants of motion, for a metric of the form

$$-e^{2f(r)}\mathrm{d}t^2 + e^{-2f(r)}\mathrm{d}r^2 + r^2\mathrm{d}\varphi^2\ . \tag{12}$$

(e) Use (9) to calculate the Christoffel symbols for the metric (12).

(f) Establish the transformation law

$$\Gamma^a_{bc} = \tilde{\Gamma}^d_{ef}\frac{\partial x^a}{\partial \tilde{x}^d}\frac{\partial \tilde{x}^e}{\partial x^b}\frac{\partial \tilde{x}^f}{\partial x^c} + \frac{\partial x^a}{\partial \tilde{x}^d}\frac{\partial^2 \tilde{x}^d}{\partial x^b \partial x^c}$$

by direct calculation. Explain why this implies that the Christoffel symbols do *not* define a tensor.

11. **[Some tensor manipulations]**

(a) Show that for all functions f

$$\nabla_a\nabla_b f = \nabla_b\nabla_a f$$

if and only if ∇ is torsion-free.

(b) Recall that ∇f is defined as the vector field $g^{ab}\partial_a f\partial_b$. Explain what $df(\nabla f)$ and $\nabla f(f)$ mean, and prove the equalities

$$g(\nabla f, \nabla f) = g^{ab}\partial_a f\partial_b f = df(\nabla f) = \nabla f(f)\ .$$

Conclude that if $f = x^\alpha$ is a coordinate, you can determine whether ∇f is timelike, spacelike, or null, by looking at a component of the inverse metric tensor (which?).

12. **[Symmetries of the curvature tensor]**

    (a) What does it mean for a connection ∇ on a space-time with metric g_{ab} to be: a *metric connection*, *torsion free*?

    Assume henceforth that ∇ is torsion free.

    In what follows, it is often useful to use the preceding results to do the next ones.

    (b) Given an arbitrary smooth covector field A_a, and a smooth antisymmetric tensor field F_{ab}, show that

    $$H_{ab} := \nabla_a A_b - \nabla_b A_a \quad \text{and} \quad \nabla_a F_{bc} + \nabla_b F_{ca} + \nabla_c F_{ab}$$

    are both independent of the choice of the connection.

    (c) Hence or otherwise show that

    $$\nabla_a H_{bc} + \nabla_b H_{ca} + \nabla_c H_{ab} = 0\ .$$

    Recall that the curvature tensor is defined as

    $$\nabla_a \nabla_b V^c - \nabla_b \nabla_a V^c = R^c{}_{dab} V^d\ .$$

    (d) Show that

    $$\nabla_a \nabla_b A_c - \nabla_b \nabla_a A_c = -R^d{}_{cab} A_d\ .$$

    (e) Hence show that $R^d{}_{abc} + R^d{}_{bca} + R^d{}_{cab} = 0$ for a torsion-free connection.

    (f) Show further that, for a tensor T_{ab},

    $$\nabla_a \nabla_b T_{cd} - \nabla_b \nabla_a T_{cd} = -R^e{}_{cab} T_{ed} - R^e{}_{dab} T_{ce}\ .$$

    (g) Hence show that $R_{abcd} = -R_{bacd}$ if ∇ is metric and torsion-free.

    (h) Show that the symmetry $R_{abcd} = R_{cdab}$ follows from $R_{abcd} = R_{[ab]cd} = R_{ab[cd]}$ and $R_{a[bcd]} = 0$.

13. **[Covariant derivative]** Let ∇ be a covariant derivative operator defined on vector fields

    (a) Given three vector fields X, Y, Z define

    $$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

    Given a function f, show that

    $$R(fX,Y)Z = R(X,fY)Z = R(X,Y)fZ = fR(X,Y)Z.$$

(b) Define $\nabla_X(Y\otimes Z) := (\nabla_X Y)\otimes Z + Y\otimes\nabla_X Z$. Using the fact that every two-contravariant tensor field T can be written as $T^{ij}\partial_i\otimes\partial_j$, this can be extended to any tensor T by using the defining properties of a connection. Show that this extension defines a connection on two-contravariant tensors. Using the notation

$$\nabla_X T \equiv (\nabla_X T)^{ij}\partial_i\otimes\partial_j = X^k\nabla_k T^{ij}\partial_i\otimes\partial_j\,,$$

show that

$$\nabla_k T^{ij} = \partial_k T^{ij} + \Gamma^i_{lk}T^{lj} + \Gamma^j_{lk}T^{il}\,.$$

(c) Given a covector field $\alpha = \alpha_i\,\mathrm{d}x^i$ define

$$(\nabla_X\alpha)(Y) := X(\alpha(Y)) - \alpha(\nabla_X Y)$$

Show that this defines a connection on covector fields. Using the notation

$$\nabla_X\alpha \equiv (\nabla_X\alpha)_i\,\mathrm{d}x^i = X^k\,\nabla_k\alpha_i\,\mathrm{d}x^i\,,$$

show that

$$\nabla_k\alpha_i = \partial_k\alpha_i - \Gamma^l_{ik}\alpha_l\,.$$

14. **[Counting components]**

(a) In four dimensions, a tensor satisfies $T_{abcde} = T_{[abcde]}$. Show that $T_{abcde} = 0$.

(b) A tensor T_{ab} is *symmetric* if $T_{ab} = T_{(ab)}$. In n-dimensional space, it has n^2 components, but only $n(n+1)/2$ of these can be specified independently—for example the components T_{ab} for $a \le b$. How many independent components do the following tensors have (in n dimensions)?

i. F_{ab} with $F_{ab} = F_{[ab]}$.

ii. A tensor of type $(0,k)$ such that $T_{ab\ldots c} = T_{[ab\ldots c]}$ (distinguish the cases $k \le n$ and $k > n$, bearing in mind the result of question (a)).

iii. R_{abcd} with $R_{abcd} = R_{[ab]cd} = R_{ab[cd]}$.

iv. R_{abcd} with $R_{abcd} = R_{[ab]cd} = R_{ab[cd]} = R_{cdab}$.

(c) Show that, in four dimensions, a tensor with the symmetries of the Riemann tensor has 20 independent components.

15. **[Divergence]**

(a) Show that for any connection we have

$$\nabla_\alpha\delta^\beta_\gamma = 0.$$

Hence, or otherwise, show that for the Levi-Civita connection it holds that

$$\nabla_\alpha g^{\beta\gamma} = 0.$$

(b) Let $\Delta^{\alpha\beta}$ denote the adjoint matrix of g (recall that $\Delta^{\alpha\beta} = \det g \, g^{\alpha\beta}$). Show that

$$\frac{\partial \det g}{\partial g_{\alpha\beta}} = \Delta^{\alpha\beta} \, .$$

Conclude that

$$g^{\alpha\beta}\partial_\mu g_{\alpha\beta} = \partial_\mu \ln(|\det g_{\alpha\beta}|) \, ,$$

as well as

$$\Gamma^\alpha_{\mu\alpha} = \frac{1}{\sqrt{|\det g_{\alpha\beta}|}}\partial_\mu(\sqrt{|\det g_{\alpha\beta}|}) \, .$$

Find likewise a simple expression for $g^{\mu\nu}\Gamma^\alpha_{\mu\nu}$, and deduce that

$$\Box_g f := g^{\mu\nu}\nabla_\mu\nabla_\nu f = \frac{1}{\sqrt{|\det g_{\alpha\beta}|}}\partial_\mu(\sqrt{|\det g_{\alpha\beta}|}g^{\mu\nu}\partial_\nu f)$$

(c) Show that for a vector field U^α, we have

$$\nabla_\mu U^\mu = \frac{1}{\sqrt{|\det g_{\alpha\beta}|}}\partial_\mu(\sqrt{|\det g_{\alpha\beta}|}U^\mu) \, .$$

Similarly, for an anti-symmetric tensor $F^{\alpha\beta}$, show that

$$\nabla_\mu F^{\mu\nu} = \frac{1}{\sqrt{|\det g_{\alpha\beta}|}}\partial_\mu(\sqrt{|\det g_{\alpha\beta}|}F^{\mu\nu}) \, .$$

Does this remain true for totally-antisymmetric tensors with more indices?

16. **[Parallel transport I]** Consider a unit two-sphere S^2 with the round metric $h = \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2$ and a vector $V = \partial_\theta$ at the point $(\theta, \varphi) = (\theta_0, 0)$. Compute the change of V after it is parallel transported around the circle $\theta = \theta_0$. What is the magnitude of V?

17. **[Parallel transport II]** Consider an infinitesimal loop specified by two infinitesimal vectors X^μ and Y^μ. Show that the change δV^α of a vector V^μ parallel transported around the loop has the following form

$$\delta V^\alpha = R^\alpha{}_{\beta\mu\nu} X^\mu Y^\nu V^\beta \, .$$

18. **[Killing vectors II]** Show that a Killing vector K^μ satisfies

(a) $\nabla_\alpha \nabla^\alpha K^\beta + R^\beta{}_\sigma K^\sigma = 0 \, ,$

(b) $\nabla_\alpha \nabla_\beta K_\mu = R^\gamma{}_{\alpha\beta\mu} K_\gamma \, ,$

(c) $K^\sigma \nabla_\sigma R = 0 \, .$

Can you find a Lagrangian-type variational principle from which (a) can be derived?

19. **[Killing vectors III]** Show that a Killing vector solves the Maxwell's equations, when considered as a vector potential, for a test field in vacuum. What electromagnetic field corresponds to the Killing vector ∂_ϕ in Minkowski space?

20. **[Conformal transformation I]** Show that a conformal transformation of a metric, i.e. $\tilde{g} = f(x^\mu)g$ (for an arbitrary function f of coordinates), preserves angles. Show that under such a transformation null curves remain null.

21. **[Conformal transformation II]** Compute the Riemann tensor, Ricci tensor, and scalar curvature of the conformally-flat metric $g_{\alpha\beta} = e^{2\phi}\eta_{\alpha\beta}$ where $\phi = \phi(x^\mu)$ is an arbitrary function.

22. **[Geodesics I]** Let $\lambda \mapsto \gamma(\lambda)$ be a curve such that

$$\frac{D\dot{\gamma}}{d\lambda} = \alpha\dot{\gamma} \,, \tag{13}$$

for some function $\alpha = \alpha(\lambda)$, where $\dot{\gamma} = d\gamma/d\lambda$ (cf. (15) below). Show that there exists a change of parameter $\tau \mapsto \lambda(\tau)$ so that

$$\frac{D}{d\tau}\frac{d\gamma}{d\tau} = 0 \,. \tag{14}$$

[*Remark: It follows that solutions of (13) are geodesics, but* not *affinely parameterised.*]

23. **[Geodesics II]** Given a curve $\lambda \mapsto x^\mu(\lambda)$, the acceleration vector a^μ is defined as

$$a^\mu := \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} \,.$$

Show that a^μ transforms as a vector under changes of coordinates. (Use the the transformation law of Christoffel symbols under coordinate transformations established in Problem 10.)

More generally, given a vector field $\lambda \mapsto W^\mu(\lambda) \in T_{\gamma(\lambda)}M$ defined along the curve $R \ni \lambda \mapsto \gamma(\lambda) \in M$, check that the formula

$$\frac{DW^\mu}{d\lambda} := \frac{dW^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta}W^\alpha\frac{d\gamma^\beta}{d\lambda} \tag{15}$$

defines a field along γ which transforms as a vector under changes of coordinates.

[*Interpretation: The equation for an affinely parameterised geodesic is $a^\mu = 0$. The exercise shows that this is a coordinate-independent notion.*]

[*Remark: As already mentioned in Problem 22., the curve γ is said to be a (non-affinely-parameterised) geodesic if a^μ is proportional to the velocity vector $\frac{dx^\mu}{d\lambda}$.*]

24. **[Geodesics III]** If a geodesic is timelike/spacelike/null at a given point $p \in M$, show that it is timelike/spacelike/null.

25. Let $u^\mu = K^\mu / \sqrt{-K^\nu K_\nu}$ where K is a time-like Killing vector. Show that $a^\mu \equiv u^\nu \nabla_\nu u^\mu = \frac{1}{2} \nabla^\mu \log(-K^\alpha K_\alpha)$.

26. If u is the four-velocity of a fluid show that ∇u can be decomposed as follows

$$\nabla_\mu u_\nu = \omega_{\nu\mu} + \sigma_{\nu\mu} + \frac{\theta}{3} P_{\nu\mu} - a_\nu u_\mu \tag{16}$$

where $a_\mu \equiv u^\nu \nabla_\nu u_\mu$ is the four-acceleration of the fluid, θ is the expansion of the fluid world lines $\theta \equiv \nabla_\mu u^\mu$, $\omega_{\mu\nu} \equiv \frac{1}{2}\left(\nabla_\alpha u_\mu P^\alpha{}_\nu - \nabla_\alpha u_\nu P^\alpha{}_\mu\right)$ is the rotation 2-form of the fluid, and $\sigma_{\mu\nu} \equiv \frac{1}{2}\left(\nabla_\alpha u_\mu P^\alpha{}_\nu + \nabla_\alpha u_\nu P^\alpha{}_\mu\right) - \frac{\theta}{3} P_{\mu\nu}$ is the shear tensor. In the above, the projection tensor $P_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu$ projects onto the three-surface orthogonal to u.

27. Using result of Problem 26. derive the Raychaudhuri equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \nabla_\mu a^\mu + 2(\omega^2 - \sigma^2) - \frac{\theta^2}{3} - R_{\mu\nu} u^\mu u^\nu \,, \tag{17}$$

where $\omega^2 := \frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu}$, and $\sigma^2 := \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}$.

28. **[Units I]** In geometrized units $G = c = k = 1$(where G is the Newton constant, c is the speed of light and k is the Boltzmann constant) give the values of the following, expressed in terms of centimeters: (a) $\hbar$; (b) e the charge of an electron; (c) e/m for an electron; (d) $M_\odot$ the mass of the sun; (e) $L_\odot$ the luminosity of the sun; (f) the temperature of 300 K; (g) one year; (h) one volt.

29. **[Units II]** Find natural units of mass, length, and time out of the physical constants $\hbar$, G, and c.

30. Show that for a diagonal metric the Christoffel symbols, defined in Eq. (8) are

$$\begin{aligned}
\Gamma^a_{aa} &= \partial_a \log \sqrt{|g_{aa}|}, \\
\Gamma^a_{ab} &= \partial_b \log \sqrt{|g_{aa}|}, \\
\Gamma^a_{bb} &= -\frac{1}{2g_{aa}} \partial_a g_{bb}, \\
\Gamma^a_{bc} &= 0,
\end{aligned}$$

where $a \neq b \neq c$ and no summation over repeated indices.

31. **[A naked singularity]** Let g be the Schwarzschild metric with $\boxed{m < 0}$ in the usual coordinate system

$$g = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\Omega^2 \,. \tag{18}$$

Given constants $(t_0, \theta_0, \varphi_0)$, consider the curve

$$(0, \infty) \ni r \mapsto \gamma(r) = (t(r), r, \theta_0, \varphi_0)\,, \tag{19}$$

where the function $t(r)$ is obtained by solving the equation

$$\frac{dt}{dr} = \frac{1}{1 - \frac{2m}{r}}\,, \tag{20}$$

with the condition $t(r) \to_{r \to 0} t_0$.

(a) Solve the equation to make sure that the function $t(r)$ is defined for all $r \in (0, \infty)$. Will this remain true if $m > 0$?

(b) Check that the curve γ defined by (19) is null, i.e., its tangent is lightlike everywhere.

(c) Calculate the $\Gamma^\mu_{\alpha\beta}$'s, for $\alpha, \beta \in \{0, 1\}$.

[*Hint: Group the indices as* $(x^A) = (\theta, \varphi)$, $(x^a) = (t, r)$ *Check by hand that* $\Gamma^A_{ab} = 0$*. Use the Lagrangean* $\mathscr{L} = \frac{1}{2}(g_{tt}(\frac{dt}{ds})^2 + g_{rr}(\frac{dr}{ds})^2)$ *to find* Γ^a_{bc}.]

(d) Using (c), or otherwise, show that γ is a geodesic.

[*Remark: The knowledge of the* Γ*'s from iii) should suffice for this.*]

[*Interpretation: Keeping in mind that photons travel on null geodesics, we see that, when* $m < 0$*, photons starting at the singularity* $r = 0$ *can reach any event in spacetime. Equivalently, the singularity can be seen from any point in spacetime.*]

32. **[Isotropic coordinates]**

(a) Given a spherically symmetric metric in Schwarzschild coordinates (t, r, θ, φ)

$$g = -e^{2A(r)}dt^2 + e^{2B(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\,, \tag{21}$$

with arbitrary functions $A(r)$ and $B(r)$ find a transformation to isotropic coordinate system $(t, \bar{r}, \theta, \varphi)$ where the metric takes the form:

$$g = -e^{2A(\bar{r})}dt^2 + e^{2C(\bar{r})}\left(d\bar{r}^2 + \bar{r}^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right)\,. \tag{22}$$

(b) Find an explicit form of the transformation and the metric in isotropic coordinates for the Schwarzschild solution (18).

(c) Is the area $\mathcal{A}$ of the surface $t = \text{const}$, $\bar{r} = \text{const}$ given by $\mathcal{A} = 4\pi\bar{r}^2$?

33. **[Integral curves]**

(a) Given a vector field X^μ, recall that its integral curves are defined as the solutions of the equations

$$\frac{dx^\mu}{d\lambda} = X^\mu(x(\lambda))\,.$$

Find the integral curves of the following vector fields on $\mathbb{R}^2$: ∂_x, $x\partial_y + y\partial_x$, $x\partial_y - y\partial_x$, $x\partial_x + y\partial_y$.

(b) Let f be a function satisfying

$$g(\nabla f, \nabla f) = \psi(f) \;,$$

for some function $\psi : \mathbb{R} \mapsto \mathbb{R}$. Show that

$$\nabla^\mu f \, \nabla_\mu \nabla^\alpha f = \frac{1}{2}\psi'(f)\nabla^\alpha f \;. \tag{23}$$

Let $\lambda \mapsto \gamma(\lambda)$ be any integral curve of ∇f; thus $d\gamma^\mu/d\lambda = \nabla^\mu f$. Rewrite (23) as an equation for $D\dot{\gamma}/d\lambda$. Find the differential equation for a reparameterization $\tau \mapsto \gamma(\lambda(\tau))$ of γ so that

$$\frac{D}{d\tau}\frac{d\gamma^\mu}{d\tau} = 0 \;.$$

[*Hint: Use problem 22.*]

(c) Recall that, for the Schwarzschild metric, we may define the Eddington–Finkelstein coordinate v by

$$dv = dt + \frac{r}{r-2m}dr.$$

Show that, in the coordinates (v, r, θ, φ), the integral curves of the vector field ∇r meeting $\{r = 2m\}$ are null geodesics.

(d) Let f be one of the coordinates, say $f = x^1$, in a coordinate system $\{x^i\}$. Verify that

$$g(\nabla f, \nabla f) = g^{11} \;.$$

Using this observation find a family of spacelike geodesics in the (t, r, θ, φ) coordinate system, as well as two distinct families of geodesics in the (v, r, θ, φ) coordinate system. Do any members of the second family coincide with members of the first?

34. **[White hole]** Let $m > 0$. Show that the Eddington-Finkelstein extension using the coordinate u defined as

$$u = t - r - 2m \ln\left(\frac{r}{2m} - 1\right) ,$$

extends the region $\{r > 2m\}$ by a *white hole region*, namely a region that you can exit but you cannot enter on future directed causal curves.

35. Show that an observer which crosses the Schwarzschild radius $r = 2M$ of a black hole reaches the central singularity $r = 0$ in a proper time $\tau \leq \pi M$, independently of initial velocity.

36. **[Penrose diagrams]**

(a) Consider the four-dimensional Minkowski metric η with the usual spherical coordinates (t, r, θ, φ). Introduce new coordinates $\hat{u} = t - r$, $\hat{v} = t + r$. Set

$$\hat{u} = \tan \overline{u}\,, \quad \hat{v} = \tan \overline{v}\,, \quad \bar{t} = \frac{\overline{v} + \overline{u}}{2}\,, \quad \overline{r} = \frac{\overline{v} - \overline{u}}{2}\,. \tag{24}$$

Rewrite η in the coordinates $(\bar{t}, \overline{r}, \theta, \varphi)$. Find the set, say Ω, on which the map

$$(t, r) \mapsto (\bar{t}, \overline{r})$$

is a diffeomorphism. [*Hint: do not forget that* $r > 0$.]

(b) Consider the Schwarzschild metric g in Kruskal-Szekeres coordinates $(\hat{u}, \hat{v}, \theta, \varphi)$,

$$g = -\frac{32m^3 \exp(-\frac{r}{2m})}{r} d\hat{u}\, d\hat{v} + r^2 d\Omega^2\,. \tag{25}$$

Rewrite g in the coordinates $(\bar{t}, \overline{r}, \theta, \varphi)$ of (24). Find the set, say Ω_S, on which the map

$$(\hat{u}, \hat{v}) \mapsto (\bar{t}, \overline{r})$$

is a diffeomorphism.

[*Hint: do not forget that* $r > 0$*, where* r *is a function of the product* $\hat{u}\,\hat{v}$ *as explained in the lecture.*]

(c) Show that if a vector X is causal for the original Minkowski or Schwarzschild metric, then its two-dimensional projection $X^{\bar{t}}\partial_{\bar{t}} + X^{\overline{r}}\partial_{\overline{r}}$ is causal for the two-dimensional Minkowski metric $-d\bar{t}^2 + d\overline{r}^2$.

(d) Using (a) and (b) draw a conformal diagrams of Minkowski and Schwarzschild solutions.

37. **[Metric connection with non-zero torsion]** Repeat part (a-g) of Problem 12. with a connection ∇ that is metric (i.e. $\nabla_\alpha g_{\beta\gamma} = 0$) but has torsion (i.e. $T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$ non-zero). Show that, for such a connection, the first Bianchi identity takes the form

$$R^\delta{}_{[\alpha\beta\gamma]} = \nabla_{[\alpha} T^\delta{}_{\beta\gamma]} - T^\epsilon{}_{[\alpha\beta} T^\delta{}_{\gamma]\epsilon}.$$

[*Hint: One way is to let* ϕ *be an arbitrary function and start from the identity* $-R^\delta{}_{\gamma\alpha\beta}\nabla_\delta\phi = \nabla_\alpha\nabla_\beta\nabla_\gamma\phi - \nabla_\beta\nabla_\alpha\nabla_\gamma\phi + T^\delta{}_{\alpha\beta}\nabla_\delta\nabla_\gamma\phi$*. Rewrite the second term using* $\nabla_\alpha\nabla_\gamma\phi = \nabla_\gamma\nabla_\alpha\phi - T^\delta{}_{\alpha\gamma}\nabla_\delta\phi$*. Then skew-symmetrise over* α, β, γ*. Manipulating what you get and stripping off the* $\nabla\phi$ *terms (using the fact that* ϕ *is arbitrary) should give you the desired result.*]

38. **[Wormhole]** An observer is moving in the spacetime of a wormhole connecting wro asymptotically flat regions and described by the metric

$$g = -\mathrm{d}t^2 + \mathrm{d}r^2 + \left(b^2 + r^2\right)\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) \tag{26}$$

Assume that the observer starts at $r = R$ with $u^r = -U$ ($U > 0$, initial infall). Compute the proper time for the observer it takes him to travel from $r = R$ to $r = -R$ through the wormhole.

39. **[Some timelike geodesics in Schwarzschild]** Let $V^2 = 1 - 2m/r$, and consider the Schwarzschild metric:

$$g = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\Omega^2\ , \quad t \in \mathbb{R}\ ,\ r \neq 2m, 0\ , \tag{27}$$

Recall that for geodesics in the Schwarzschild metric lying in the equatorial plane $\theta = \pi/2$ we have the following constants of motion (you are strongly encouraged to check that you can reproduce the result, but this will not be done again in class):

$$\frac{d}{ds}\left(V^2 \frac{dt}{ds}\right) = 0 \quad \Longrightarrow \quad \frac{dt}{ds} = \frac{E}{1 - \frac{2m}{r}}\ . \tag{28}$$

$$\frac{d}{ds}\left(r^2 \frac{d\varphi}{ds}\right) = 0 \quad \Longrightarrow \quad \frac{d\varphi}{ds} = \frac{J}{r^2}\ . \tag{29}$$

$$\underbrace{V^2\left(\frac{dt}{ds}\right)^2}_{E^2V^{-2}} - V^{-2}\left(\frac{dr}{ds}\right)^2 - \underbrace{r^2\left(\frac{d\varphi}{ds}\right)^2}_{J^2/r^2} = \lambda \in \{0, \pm 1\}\ . \tag{30}$$

Consider a timelike geodesic in the Schwarzschild metric parameterized by proper time, thus $\lambda = 1$, and lying in the equatorial plane $\theta = \pi/2$.

(a) Verify that

$$\frac{E^2 - \dot{r}^2}{1 - \frac{2m}{r}} - \frac{J^2}{r^2} = 1\ .$$

(b) Deduce that if $E = 1$ and $J = 4m$ then

$$\frac{\sqrt{r} - 2\sqrt{m}}{\sqrt{r} + 2\sqrt{m}} = Ae^{\epsilon\varphi/\sqrt{2}}\ ,$$

where $\epsilon = \pm 1$ and A is a constant. Describe the orbit that starts at $\varphi = 0$ in each of the cases (i) $A = 0$, (ii) $A = 1$, $\epsilon = -1$, (iii) $r(0) = 3m$, $\epsilon = -1$.

(c) Consider the case $E = 1$, $J = 0$ and deduce a relation for $r = r(s)$.

40. **[Circular geodesics]** Consider the equations of motion in the Schwarzschild geometry (see question 39.). Show that for any given $r > 3m$ there exists an equatorial future-directed time-like geodesic with a constant r and

$$t - t_0 = \frac{|J|}{\sqrt{mr}}(s - s_0), \tag{31}$$

$$\varphi - \varphi_0 = \frac{J}{r^2}(s - s_0) = \pm\sqrt{\frac{m}{r^3}}(t - t_0), \tag{32}$$

where s is the proper time along the geodesic and J is given by

$$J^2 = \frac{mr^2}{r - 3m}. \tag{33}$$

41. **[Null Geodesics in Schwarzschild]** Let γ be an affinely parameterized null geodesic, thus $\lambda = 0$, in the Schwarzschild metric lying in the equatorial plane $\theta = \pi/2$. Assume that $J \neq 0$, hence we can make a change of parameter $\varphi \mapsto s(\varphi)$ using the implicit equation

$$\frac{d\varphi}{ds} = \frac{J}{r^2} \quad \Longrightarrow \quad \frac{ds}{d\varphi} = \frac{r^2}{J} \ .$$

Set

$$u(\varphi) = \frac{m}{r(s(\varphi))} \ , \quad p(\varphi) = \frac{du(\varphi)}{d\varphi} \ .$$

(a) Show that along γ, for $u \neq \frac{1}{2}$, we have

$$p^2 = 2u^3 - u^2 + \alpha^2 \ . \tag{34}$$

(b) Show that (29)-(30) imply the two-dimensional autonomous first order system

$$\frac{du}{d\varphi} = p \ , \quad \frac{dp}{d\varphi} = 3u^2 - u \ . \tag{35}$$

Recall that critical points of a dynamical system $d\vec{x}/d\varphi = \vec{Y}$ (here $\vec{x} = (u, p)$) are defined as points where $\vec{Y}$ vanishes. Find the critical points of (35). Can you sketch the trajectories in the (u, p) phase-plane?

(c) For the case $\alpha = 0$ and $u > 1/2$ show that the geodesic has an equation of the form

$$r = 2m\cos^2\left(\frac{\varphi - \varphi_0}{2}\right) \ , \tag{36}$$

with $t = t(\varphi)$ that you should determine.

42. **[Weak field light bending]** Let $u = m/r$, $\alpha = m/d$, where d is the minimal value of the coordinate r along a null geodesic. In the lecture we derived the approximate equation

$$u = (\alpha - \alpha^2)\cos\varphi + \alpha^2(1 + \sin^2\varphi) + O(\alpha^3)\,. \tag{37}$$

After forgetting error term there, (37) can be rewritten as

$$\frac{m}{r} = (\alpha - \alpha^2)\frac{x}{r} + \alpha^2(1 + \frac{y^2}{r^2})\,. \tag{38}$$

Assuming that α is small, find the approximate values of $a, b \in \mathbb{R}$ so that the curve defined by (38) asymptotes, as $y \to \infty$, to the straight line $x = ay + b$.

43. **[Weak field light bending]** Does the light-bending formula,

$$\delta\varphi = \frac{4mG}{dc^2}\,, \tag{39}$$

have a coordinate-independent meaning and, if so, in which sense? Note that the symbol d there denotes the value of the Schwarzschild radial *coordinate* r at the point of closest approach, and $\pi + \delta\varphi$ is the change of the Schwarzschild angular *coordinate* along the orbit.

44. **[Radial geodesics in Schwarzschild]** Consider the geodesic equations for a massive particle initially at rest in the Schwarzschild metric. The tangent vector $U := \frac{dx}{ds}$ therefore satisfies $U^a\nabla_a U^b = 0$ and $g_{ab}U^aU^b = -1$ with the initial conditions that $U^r(s=0) = U^\theta(s=0) = U^\phi(s=0) = 0$ and $U^t(s=0) > 0$.

   (a) Show that we have

$$U^t(0) \equiv \frac{dt}{ds}(s=0) = \frac{1}{\sqrt{1 - \frac{2m}{r(0)}}}.$$

   (b) Derive the part of the geodesic equations satisfied by $\frac{d}{ds}U^\theta$, $\frac{d}{ds}U^\phi$. [*Hint: Repeat the argument presented in the lecture.*] Show that the initial conditions $U^\theta(0) = U^\phi(0) = 0$ imply that $U^\theta(s) = U^\phi(s) = 0$ for all s. Therefore, the geodesics are radial (i.e. the only spatial motion is in the radial direction).

   (c) Using a formula derived in the lecture (the derivation of which you should reproduce), show that

$$\frac{dt}{ds} = \frac{\sqrt{1 - \frac{2m}{r(0)}}}{1 - \frac{2m}{r(s)}}$$

   (d) From the result of part (c), and the fact that $g_{ab}U^aU^b = -1$, deduce that $r(s)$ satisfies the differential equation

$$\frac{dr}{ds} = -\sqrt{\frac{2m}{r(s)} - \frac{2m}{r(0)}}.$$

[*Note: At one point, you will need to take a square root, which involves a choice of sign. In order that $\frac{2m}{r(s)} - \frac{2m}{r(0)}$ is positive, you need $r(s) \le r(0)$, which tells you that you should choose the negative root.*] Integrating this equation, show that $r(s)$ is implicitly given by the equation

$$s = +\frac{r(0)^{3/2}}{\sqrt{2m}}\left[\cos^{-1}\sqrt{\frac{r}{r_0}} + \sqrt{\frac{r}{r_0}}\left(1-\left(\frac{r}{r^0}\right)\right)^{1/2}\right].$$

[*Hint: When doing the integration, you may find it useful to use the substitution $r = r_0 \cos^2 x$.*]

45. **[A non-physical black hole metric]** Let a be a strictly positive constant, and for $r > a$ let g be a Lorentzian metric of the form

$$g = -\left(1-\frac{a}{r}\right)^3 dt^2 + \frac{dr^2}{\left(1-\frac{a}{r}\right)^3} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 \ . \tag{40}$$

(a) Proceeding in a way analogous to the analysis of the Schwarzschild metric, replace t by a new coordinate v so that the metric, in the new coordinates, can be smoothly extended from the original manifold $\{t \in \mathbb{R}\} \times \{r > a\} \times S^2$ to a Lorentzian metric on a new manifold $\{v \in \mathbb{R}\} \times \{r > 0\} \times S^2$.

(b) Show that, with an appropriate choice of the coordinate v, the set $\{r < a\}$ in the extended manifold is a black hole region.

(c) Find the four-acceleration of stationary observers for this metric.

(d) Find the gravitational red-shift formula for this metric.

(e) One can calculate (for the ambitious: use an algebraic manipulation program to carry this out):

$$R = 2a^3 r^{-5}\,, \quad R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 4a^2 r^{-10}(46a^4 - 150a^3 r + 186a^2 r^2 - 108ar^3 + 27r^4)\,.$$

How does this imply that $\{r = 0\}$ is a singular set for the metric (40)?

46. Alice circles planet X freely for a long time on a circular orbit of radius R, while her twin Bob remains motionless on the surface of the planet, at radius r_0. For $r \ge r_0$ the geometry of the gravitational field of the planet X is described by the Schwarzschild metric with mass $0 < m < r_0/2$. Derive a necessary and sufficient condition on R which guarantees that, on meeting Bob again, Alice will have the same age as Bob. You should assume that the time of travel back and forth from radius R to radius r_0 can be neglected compared to the time that Alice spent on the circular orbit.

47. Assuming a circular orbit, and using units so that the result is dimensionless, calculate m_s/r, $J_N^2/(m_e r)^2$, and $m_s^2 m_e^2/J_N^2$ for a) the orbit of the earth around the sun, and for b) the orbit of Mercury around the sun (the orbit of Mercury is actually far from

circular, use a radius calculated from the mean of the aphelion and perihelion). Here m_s is the mass of the sun, and the remaining parameters concern the moving object.

Using the formula from the lecture (you might wish, though, to verify that you can reproduce its derivation), calculate the shift of the perihelion for Mercury and the earth in orbit around the Sun. Repeat the calculation for an exoplanet with the mass of Jupiter on a circular orbit around a star of 10 solar masses with a period of 1 day.

48. Consider a gyroscope with spin vector s circling on a circular timelike geodesic of radius r in the Schwarzschild metric. Let τ denote the proper time along the geodesic and let u be the unit-normalised tangent to the geodesic.Thus

$$\frac{Ds}{d\tau} = 0\,, \quad g(u,s) = 0\,.$$

Show that

$$\frac{ds^\theta}{dt} = 0\,, \quad s^t = \frac{r^2 u^\varphi}{\left(1-\frac{2m}{r}\right)u^t} s^\varphi\,, \quad \frac{ds^r}{dt} = (r-3m)\Omega s^\varphi\,, \quad \frac{ds^\varphi}{dt} = -\frac{\Omega}{r} s^r\,,$$

where $\Omega = \sqrt{\frac{m}{r^3}}$, and find $s(t)$, where t is the Schwarzschild coordinate time. [*Hint: check first which Christoffel symbols you need for this calculation, and calculate them directly from the formula for the Christoffels in terms of the metric and its derivatives.*]

49. Consider timelike geodesics in the $\theta = \pi/2$ plane in the Schwarzschild geometry,

(a) Show that for every $r > 3m$ there exist timelike geodesics for which $dr/ds \equiv 0$.

(b) Show that on such geodesics we have

$$\varphi = \varphi_0 + \Omega t\,, \quad \Omega = \frac{m^{1/2}}{r^{3/2}}\,, \qquad t = t_0 \pm \frac{Js}{\sqrt{mr}}\,,$$

where $J = r^2 d\varphi/ds$, and s is the proper time along the geodesic.

50. Consider the metric

$$g = g_{\text{Schw}} - \frac{4J\sin^2\theta}{r} dt\, d\varphi\,,$$

where g_{Schw} denotes the Schwarzschild metric (18).

(a) Show that are geodesics for the metric g of the form $\gamma(\tau) = (t(\tau), r(\tau), \theta = 0)$.

(b) Show that if $s^r = 0$ at some point then it is zero along the whole such geodesic γ.

(c) Write-down explicitly the gyroscope equation on γ assuming that $s^r = 0$. (Can you solve it?)

51. Calculate the total gravitational potential energy of a spherically symmetric Newtonian star with mass density ρ which is constant within a ball o radius R. [*Hint: Start by explaining why a shell at radius r of thickness dr contributes*

$$dU = -4\pi\rho r^2 \frac{m(r)}{r} dr$$

to the total potential energy U of the star.]]

52. **[The Lane-Emden equation]** Recall from lectures that, for a Newtonian static fluid, p and ρ satisfy the equation

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi\rho.$$

Assume that $\rho = \rho_c\theta(r)^n$ and $p = p_c\theta(r)^{n+1}$, where n, ρ_c, p_c are constants, and θ is a function to be determined. Check that

$$p = K\rho^{\frac{n+1}{n}},$$

for a constant K that you will determine. Show that the function θ satisfies the differential equation

$$\frac{K(n+1)}{4\pi}\rho_c^{\frac{1}{n}-1}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n.$$

Let $r_n > 0$ be defined by

$$r_n^2 = \frac{K(n+1)}{4\pi}\rho_c^{\frac{1}{n}-1},$$

and define a new variable ξ by $r = r_n\xi$. Deduce that

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n.$$

In the case $n = 0$ solve this equation with the boundary conditions

$$\theta(0) = 1\,, \quad \theta'(0) = 0\,. \tag{41}$$

In the case $n = 1$ solve again with the boundary conditions (41). [*Hint: When $n = 1$ use the substitution $\theta(\xi) = \psi(\xi)/\xi$.*]

Finally, verify that the function

$$\theta(\xi) = \frac{1}{\left(1 + \frac{1}{3}\xi^2\right)^{1/2}}$$

is a solution in the case $n = 5$.

53. **[A gravitational wave]** Let η_{ab} and η^{ab} be the covariant and contravariant metric tensors on Minkowski space $\mathcal{M}$, with standard Lorentzian coordinates x^a so that

$$(\eta_{ab}) = -\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{42}$$

Let n_a be a constant covector on $\mathcal{M}$ satisfying $\eta^{ab}n_a n_b = 0$. Define a new metric on $\mathcal{M}$ by

$$g_{ab} = \eta_{ab} + f n_a n_b \ ,$$

where f is a function on $\mathcal{M}$ such that $\eta^{ab}n_a\partial_b f = 0$, where $\partial_a = \partial/\partial x^a$.

(a) Show that the connection derived from g_{ab} is given by

$$\Gamma^a{}_{bc} = \tfrac{1}{2}\eta^{ad}\left(n_d n_b \partial_c f + n_d n_c \partial_b f - n_b n_c \partial_d f\right) \ .$$

[*Hint: look for g^{ab} of the form $\eta^{ab} + hn^a n^b$, where $n^a = \eta^{ab}n_b$.*]

(b) The Ricci tensor is defined as

$$-R_{ac} = \partial_d \Gamma^d{}_{ac} - \partial_a \Gamma^d{}_{dc} + \Gamma^d{}_{de}\Gamma^e{}_{ac} - \Gamma^d{}_{ae}\Gamma^e{}_{dc} \ . \tag{43}$$

Show that

$$R_{ab} \text{ is proportional to } n_a n_b \eta^{cd}\partial_c\partial_d f \ , \tag{44}$$

[*Hint: Check that $\Gamma^d{}_{de}$ vanishes. Show, next, that any contraction of n^a with the Christoffel symbols vanishes, and conclude that the last term in* (43) *gives no contribution either.*]

(c) Show that Einstein's vacuum field equations,

$$R_{ab} = 0 \ ,$$

have solutions as above with $f = \alpha \sin(k_a x^a)$, with $\alpha \in \mathbb{R}$, and with k_a having constant components in the coordinate system where η_{ab} takes the form (42), provided that k_a satisfies $\eta^{ab}k_a k_b = \eta^{ab}k_a n_b = 0$. Deduce that such a k_a is proportional to n_a.

54. **[Small perturbations of Minkowski spacetime]** Consider $\mathbb{R}^{n+1}$ with a metric which in the natural coordinates on $\mathbb{R}^{n+1}$, takes the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \tag{45}$$

and suppose that there exists a small constant ϵ such that we have

$$|h_{\mu\nu}| \ , \ |\partial_\sigma h_{\mu\nu}| \ , \ |\partial_\sigma\partial_\rho h_{\mu\nu}| \le \epsilon \ . \tag{46}$$

(a) Check that
$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O(\epsilon^2)\,. \tag{47}$$

(b) Using the metric η to raise and lower indices, *e.g.*,
$$h^\alpha{}_\beta := \eta^{\alpha\mu} h_{\mu\beta}\,, \qquad h^{\alpha\beta} := \eta^{\alpha\mu}\eta^{\beta\nu} h_{\mu\nu} = \eta^{\beta\nu} h^\alpha{}_\nu\,.$$
show that
$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}\{\partial_\beta h^\alpha{}_\gamma + \partial_\gamma h^\alpha{}_\beta - \partial^\alpha h_{\beta\gamma}\} + O(\epsilon^2) = O(\epsilon) \tag{48}$$

(c) Show further that
$$\begin{aligned} R_{\beta\delta} &= \partial_\alpha \Gamma^\alpha{}_{\beta\delta} - \partial_\delta \Gamma^\alpha{}_{\beta\alpha} + O(\epsilon^2) \\ &= \frac{1}{2}\Big[\partial_\alpha\{\partial_\beta h^\alpha{}_\delta + \partial_\delta h^\alpha{}_\beta - \partial^\alpha h_{\beta\delta}\} - \partial_\delta\{\partial_\beta h^\alpha{}_\alpha + \partial_\alpha h^\alpha{}_\beta - \partial^\alpha h_{\beta\alpha}\}\Big] + O(\epsilon^2) \\ &= \frac{1}{2}\Big[\partial_\alpha\{\partial_\beta h^\alpha{}_\delta + \partial_\delta h^\alpha{}_\beta - \partial^\alpha h_{\beta\delta}\} - \partial_\delta\partial_\beta h^\alpha{}_\alpha\Big] + O(\epsilon^2)\,. \end{aligned} \tag{49}$$

55. Let $g_{\mu\nu} = \eta_{\mu\nu} + A_{\mu\nu}\cos(k_\alpha x^\alpha)$ be the metric of a linearized plane wave in the gauge $A_{\mu\nu}k^\mu = 0 = A_{\mu\nu}u^\mu = A_\mu{}^\mu$, where $k^\mu\partial_\mu = \omega(\partial_t + \partial_z)$, $u^\mu\partial_\mu = \partial_t$. Introducing the variables $u = t - z$, $v = t + z$, and using the variational principle for geodesics, write down the geodesic equations.

    Show that there are two classes of geodesics:

    The first class have u = constant, in which case you should be able to integrate the geodesic equations completely.

    For the geodesics with u non-constant, use u as a parameter along the geodesics, and solve the equations neglecting terms which are quadratic in ϵ, assuming that $|A_{\mu\nu}| \le \epsilon$ and that the space-velocities are smaller than ϵ.

    For the ambitious: Find all geodesics, without assuming the smallness conditions above.

56. **[Laser Interferometric Space Observatory (LISA), or: if LIGO mirrors were freely falling]** Consider two freely falling mirrors separated by a coordinate distance L lying in a plane perpendicular to the direction of propagation of a linearised plane wave with $h_\times = 0$. Assuming that the time of flight of light between the mirrors is very small compared to the period of the wave, and neglecting all terms quadratic in $g_{\mu\nu} - \eta_{\mu\nu}$, calculate the time needed for a photon to travel back and forth from one mirror to another. Compare the result with the calculation in the lecture (which you might wish to repeat), of the space-distance between both mirrors.

    [*Hint: Light follows null geodesics. For this, you can use the results of Problem 55.. Alternatively, use directly the Killing vectors, there are at least three of them.*]

57. Calculate the field of unit normals and the induced metric for

(a) S^2 included in a flat $\mathbb{R}^3$ (use polar coordinates on S^2)

(b) S^2 viewed as a sphere of constant radius in Schwarzschild (use polar coordinates on S^2)

(c) $\mathcal{S} = \{x^0 = \sqrt{1+r^2}\}$ in four dimensional Minkowski; use the cartesian space coordinates x^i as coordinates on $\mathcal{S}$

(d) $\mathcal{S}' = \{r = \sqrt{1+t^2}\}$ in four dimensional Minkowski; use t and polar coordinates as coordinates on $\mathcal{S}'$

58. Find a function f so that the metric induced on the hypersurface $\{t = f(r)\}$ in Schwarzschild space-time is flat. [*Hint: The metric induced on $t = f(r)$ is obtained by replacing every occurrence of dt in the metric by df.*]

59. Let φ satisfy the wave equation in a general space-time with Lorentzian metric g, $\Box_g\varphi := \nabla^\mu\nabla_\mu\varphi = 0$. Set

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}\nabla^\alpha\varphi\partial_\alpha\varphi g_{\mu\nu} \,.$$

Check that $\nabla_\mu T^\mu{}_\nu = 0$.

Recall that a vector field X^μ is called *Killing* if

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0 \,.$$

(a) Check that ∂_t and ∂_φ are Killing vectors both in the Minkowski metric and in the Schwarzschild metric.

(b) Check that you remember the proof of the following: if γ is a geodesic and X is Killing, then $g(\gamma, X)$ is constant along γ.

(c) Show that if $\nabla_\mu T^\mu{}_\nu = 0$ and X is Killing then the vector field $J^\mu := T^\mu{}_\nu X^\nu$ has vanishing divergence, $\nabla_\mu J^\mu = 0$.

Let $X = \partial_t$ be a Killing vector. Given a spacelike hypersurface $\mathscr{S}$ with unit future-directed timelike normal n^μ, set

$$E := \int_{\mathscr{S}} T_{\mu\nu}X^\nu n^\mu d^3x \,.$$

E is called the total energy E of the field contained in $\mathscr{S}$. Give an explicit expression for E for the surface $\{t = 0\}$ in the Minkowski space-time, and in the Schwarzschild space-time.

What is the conservation law associated with

$$J := \int_{\mathscr{S}} T_{\mu\nu}X^\nu n^\mu d^3x \,,$$

where $X = \partial_\varphi$? Likewise write an explicit formula for J in Minkowski and in Schwarzschild.

60. Consider a pointlike body of mass m_0 and angular momentum J_N moving in an elliptic, Kepler orbit of eccentricity e in the field of a central mass of mass m. The density distribution of the central mass is spherically symmetric and time-independent. Assume that, in Cartesian coordinates, this motion takes place in the $z = 0$ plane. Show that the *time-dependent part* of the quadrupole moment of the system takes the form

$$q_{ij} = m_0 r(\varphi(t))^2 \begin{pmatrix} \cos^2\varphi(t) & \sin\varphi(t)\cos\varphi(t) & 0 \\ \sin\varphi(t)\cos\varphi(t) & \sin^2\varphi(t) & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

Using the explicit form of $r(\varphi(t))$ for Newtonian orbits seen in the lectures, derive an expression for the linearised metric perturbation $\overline{h}_{xx}$, to first order in the eccentricity e.

61. **[Cosmological redshift]** A distant galaxy has a redshift $z = (\lambda_{\text{observed}} - \lambda_{\text{emitted}})/\lambda_{\text{emitted}}$ of 0.2. According to Hubble's law, how far away was the galaxy when the light was emitted if the Hubble constant is 72 (km/s)/Mpc?

62. **[Cepheid]** A Cepheid variable star is observed with an apparent magnitude of 22 (see `http://outreach.atnf.csiro.au/education/senior/astrophysics/photometry_magnitude.html#magnapparent` for the notion of the magnitute of a star) and a period of 28 days. Using data from `http://hyperphysics.phy-astr.gsu.edu/hbase/astro/cepheid.html`, determine the distance to this star.

[*Hint: The apparent magnitude m of the star is related to that of the Sun by the formula*

$$m - m_\odot = -2.5 \log(\ell/\ell_\odot) ,$$

where ℓ is the apparent luminosity, $\ell_\odot$ is the apparent luminosity of the Sun, and log *is the logarithm at base ten. Use the fact that distance is inversely proportional to the square root of luminosity. Use $m_\odot = -26.5$, $d_\odot$ = 1 astronomical unit* $= 149.60 \times 10^6$ *km* $= 149.60 \times 10^9$ *m* $= 4.8481 \times 10^{-6}$ *parsecs,1 parsec= 3.26 light years* $= 3 \times 10^{13}$ *km.*]

63. **[Null rays in FLRW]** Check, by a direct calculation of the Christoffel symbols or otherwise, that the radial null rays in a FLRW metric are geodesics (perhaps, but not necessarily, affinely parameterised).

64. Verify that for FLRW models with $\rho + 3p \geq 0$ and with non-positive cosmological constant the scale factor R is a concave function of t (i.e. $\frac{d^2R}{dt^2} \leq 0$). Assuming that $R(t) \approx ct^\alpha$ as $t \to 0$, deduce from this that $1/H(t) \geq t$ for $t > 0$. [*Hint: Compute time derivative of* $(1/H(t) - t)$.]

65. Check, or derive the $k \neq 0$ solutions of the Friedman equation in a non-empty matter-dominated universe with $\Lambda = 0$:

$$k = 1 : \qquad R = C(1 - \cos\eta) , \quad t = C(\eta - \sin\eta) , \tag{50}$$

$$k = -1 : \qquad R = C(\cosh\eta - 1) , \quad t = C(-\eta + \sinh\eta) . \tag{51}$$

66. Suppose that the spatial volume of a closed, matter dominated, FLRW universe with spherical space sections and vanishing cosmological constant is $10^{12}\mathrm{Mpc}^3$ at the moment of maximum expansion. What is the duration of this universe from big bang to big crunch in years?

67. Show that all solutions of the linearisation of the Friedman equation of the radiation and dust universe at $\dot{R} = 0$ are unstable.

68. Using Friedmann equations and the continuity equation, derive how the density of the universe changes as a function of the scale factor for a universe dominated by a) radiation, b) matter, or c) cosmological constant, assuming that the curvature is zero. Sketch these dependencies on a log-log graph of $\rho(R)$. Using that the densities of the matter and radiation today are $\rho_{m,0} = 1.88 \times 10^{-32}\Omega_m h^2 \mathrm{kg\ cm}^{-3}$ and $\rho_{r,0} = 7.8 \times 10^{-37} \mathrm{kg\ cm}^{-3}$ respectively, derive the redshift of matter-radiation equality. (Ω_m is the total matter density today (baryons + cold dark matter) divided by the critical density, and h is the dimensionless Hubble parameter defined by the equation $H = 100\ h$ km/s/Mpc.)