QUASILINEAR EQUATIONS AND SINGULAR PROBLEMS

QUESP

Nonlinear partial differential equations (PDE's) play a central role in the modelling of a great number of phenomena, ranging from Theoretical Physics, Astrophysics and Chemistry to Economy, Medicine and Population Dynamics. Among the phenomena encountered, the diffusion processes play a fundamental role.

In the last 25 years a great deal of work has been devoted to semi-linear equations. In this type of equations, the interaction between a linear partial differential operator and the superlinear reaction term (source or absorption) can be understood, at least in part, thanks to the linear theory. One of the main observations, valid in the most interesting cases is the existence of critical exponents, for example the Fujita exponents for nonlinear heat equation, the Pohozaev exponent, the Sobolev exponent. In general, the first results (blow-up, global estimates, decay estimate), were proved up to a critical exponent by more or less easy applications of linear energy estimates, linked to ODE techniques. Then the study of what happens if the exponent is critical or even supercritical involves a very delicate analysis, often based upon very sharp linear estimates or even, in some cases a completely new approach.

The main areas of research in the current proposal include the description of singular phenomenon: blow-up, singularities, problems with singular measure data in a large class of reaction diffusion equations, with quasi-linear or fully nonlinear diffusion and strong reaction.

1- Singularities of solutions of equations with absorption

The description of isolated singularities for solutions of nonlinear semi-linear equations of the type

(1.1) $-\Delta u + u^q = 0$

or

(1.2) $-\Delta u = u^q$

has been performed in the heighties by works of Véron, Gidas and Spruck, Matano and Véron. The tools are combination of sharp a priori estimates and dynamical systems methods. Later, on the study was extended to boundary singularities through the works of Gmira and Véron, Bidaut-Véron and Yarur, Bidaut-Véron, Ponce and Véron. Recent progesses are based upon new estimates due to Polácik-Quittner-Souplet and connected to Liouville type theorems. The role of isolated singularities is fundamental since they are the miscroscope which allows the desription of larger singularities. For equations of Hamilton-Jacobi type

(1.3) $-\Delta u + |Du|^q = 0$,

internal singularities have been considered by Lions, but the description of boundary singularities is much more recent (Nguyen and Véron). Concerning equations with source reaction

(1.4) $-\Delta u = |Du|^q$

it appears that nothing important is known.

A natural extension of the above problems have been to replace the Laplace operator by the p-Laplace operator : $u \rightarrow \Delta_p u := \mathrm{div}\,(|Du|^{p-2} Du)$ and replace (1.1)-(1.4)

(1.5) $-\Delta_p u + u^q = 0$

(1.6) $-\Delta_p u = u^q$

(1.7) $-\Delta_p u + |Du|^q = 0$

(1.8) $-\Delta_p u = |Du|^q$

Equations (1.5) and (1.6) have been treated by several mathematicians in the eighties, including Serrin and Ni, Friedman and Véron, Serrin and Zou and more recently by Porretta and Véron and Borghol and Véron. The existence of anisotropic singularities was in particular shown. Concerning equation (1.7), works in progress are performed by Bidaut-Véron, Garcia-Huidobro and Véron. Up to now they only concern isolated singularities, but the study of anisotropic singularities as well as equations with reaction terms such as (1.8) will be started.

2- Nonlinear trace theory

The boundary trace theory for nonlinear equations is the natural extension of what have been done by Fatou, Riesz, Doob and Herglotz for positive harmonic or super-harmonic functions. For semi-linear elliptic equations of type (1.1); it has been initiated by Le Gall and Dynkin using probabilistic tools, but with strong limitation on q. Later on Marcus and Véron have developed the theory using purely analytic tools. The idea is the represent a solution by mean of a generalized Borel measure on the boundary, the boundary trace. As in many problems of interaction of nonlinear absorption or reaction and diffusion it occurs a critical exponent. Usually a critical exponent corresponds to the removability of a certain type isolated singularities. Below this critical exponent the things are well understood while the matter become much more difficult in the supercritical regime. A new approach is based upon the introduction of fine topologies (naturally associated to a suitable Bessel capacity) as it has been pointed out by Dynkin and Kuznetsov, Mselati, Marcus and Véron. One of the tools is the quasi representation of solutions, that is the two-side estimate of solutions by capacitary potentials. If the study of (1.1) is essentially understood, (although many deep questions remain unsolved), the parabolic problem

(2.1) $u_t - \Delta u + u^q = 0$

has not yet been treated in the supercritical case. Concerning the evolution equations

(2.1) $u_t - \Delta u + |Du|^q = 0$

and

(2.2) $u_t - \Delta u = |Du|^q$,

important results are due to Souplet and Zhang, Ben Artzi, Souplet and Weissler, Laurençot and Benachour. More recently Bidaut-Véron and Dao have extended the analysis of isolated singularities of Laurençot and Benachour. When the Laplacian is replaced by a p-laplacian, existence of singular solutions (the VSS) are obtained by Peletier, Vazquez, Kamin in the case of

(2.3) $u_t - \Delta_p u + u^q = 0$.

Concerning

(2.5) $u_t - \Delta_p u+|Du|^q=0$

(2.6) $u_t - \Delta_p u =|Du|^q$

the situation is still to be explored. Very deep breakthrough can be expected.

3- Measure data

Concerning semi-linear problems

(3.1) $- \Delta u +u^q=\mu$

(3.2) $- \Delta u =u^q+\mu$

(3.3) $u_t - \Delta u +u^q=\mu$

or

(3.4) $u_t - \Delta u =u^q+\mu$

where μ is a Radon measure, much is known since the works of Baras and Pierre or Adams and Pierre in the heighties. Essentially, in the supercritical regime, admissible measures have not to be too concentrated with respect to some Bessel capacity. The corresponding Dirichlet problems with measure have been intensively studied by Dynkin and Kuznetsov or Marcus and Véron in connexion with the key role they play in the description of boundary trace. The results still holds, under a slightly different formulation if the Radon measure is replaced by a generalized Borel measure. Concerning equation

(3.5) $- \Delta u +|Du|^q=\mu$,

existence results are more or less standard in the sub-critical regime. Using singular integrals and Riesz operator Jayes, Mazy'a and Verbitsky have pointed out a necessary and sufficient condition for existence of solutions to

(3.6) $- \Delta u =|Du|^q + \mu$

in the supercritical case. We intend to develop and combined this method with others to treat the case of (3.5). In the same type of problems we shall try to adapt the methods of Wolff potentials developed by the Finish School (Kilpelainen, Martio, Heinonen, etc) for representing super-p-harmonic functions or more generally solutions of

(3.7) $- \Delta_p u =\mu$,

in order to solve the supercritical problems

(3.8) $- \Delta_p u +u^q= \mu$

and

(3.9) $- \Delta_p u +|Du|^q= \mu$.

Notice that the reaction case of (3.8),

(3.10) $-\Delta_p u = u^q + \mu$

has been considered in a celebrated article by Phuc and Verbitsky. Since our group possesses many excellent specialists of fully nonlinear equations, we hope to obtain significant results for more general equations such as

(3.11) $F[u] + |Du|^q = \mu$,

where F is a fully nonlinear second order operator (notice again that many results have been obtained for Monge Ampere type operators by Trudinger and Wang).

4- Blowup of solutions of nonlinear reaction-diffusion equations and systems

Another area of research is the development of criteria on initial values which imply that the resulting solution of a semi-linear equation blows up in finite time. For the nonlinear heat equation, the first results were proved by Kaplan, Fujita and Levine. Since these earlier papers, more effort have been devoted to studying the nature of blow-up rather than developing more sophisticated criteria which imply that the solution blows up. Among the relatively few papers in this area we cite for the nonlinear heat equation Mizoguchi and Yanagida, Dickstein, Gazzola and Weth. More recently Cazenave, Dickstein and Weissler have shown that for the nonlinear heat equation the set of (not necessarily positive) initial values for which the resulting solution is global is not necessarily star-shaped around 0.

It has been conjectured that for $1<p<(n+2)/(n-2)_+$, the non-linear heat equation

(4.1) $u_t - \Delta u = u^q$

has no positive classical solution on $\mathbf{R}\times\mathbf{R}^n$. It is to be noted that important partial results on this conjecture have been obtained by Bidaut-Véron, Matos and Souplet, Polácik, Quittner and Souplet. This result would have many applications concerning decay rates of global solutions and asymptotic behavior of blow-up solutions. In fact, the Liouville property is strongly connected with the question of universal bounds (for local in time solutions), first studied by Fila-Souplet-Weissler by different methods. For instance, the combination of parabolic Liouville-type theorems and recent techniques from enable one to obtain universal bounds for local positive solution of (4.1) in $(0,T)\times\mathbf{R}^n$, of the form

(4.1) $u(x,t) \leq C(t^{-a}+(T-t)^{-b})$,

with $b=1/(p-1)$, $a=a(n,p)>0$ and $C>0$ independent of u.

To prove the conjecture, a natural direction would be to use moving plane techniques and reduce the problem to the (already solved) radial case. However some new idea is still missing to be able to do so. The expertise of P. Felmer, Sirakov and Esteban among others should be a great help.

Alternatively, a different approach based on suitable pseudo-differential operators could shed some new light on the problem.

Among the deep blow-up problems we would like to investigate the single-point blow-up for system

(4.2) $u_t - \Delta u = v^p$

(4.3) $v_t - \Delta v = u^q$

Partial results have been obtained by Friedman and Giga, Andreucci, Herrero and Velazquez, Zaag and Souplet. In particular we will further study the space profiles, which are known only for $|p-q|<<1$. Concerning the stationary Hamiltonian system which is associated to (4.2)-(4.3) one important result which will be a key stone for proving general estimates is the so-called Lane-Emden conjecture :
It has been conjectured that the elliptic system

(1.1) $-\Delta u = v^p$

(1.2) $-\Delta v = u^q$

with $p,q>1$, has no positive solutions in $\mathbf{R}^n$ if (p,q) lies below the so-called Sobolev hyperbola (which is a natural analogue of the Sobolev exponent from the scalar case). Many mathematicians have contributed to this problem (Mitidieri, de Figueiredo-Felmer, Busca-Manasevich, Serrin-Zou). The conjecture for n=3 has been settled by Serrin and Zou for polynomialy bounded solutions and by Polácik-Quittner-Souplet in the general case. Very recently, the conjecture for n=4 has been solved by Souplet. The case of higher dimensions is a challenging open problem and will be studied intensively.

5- nonlocal operators

Nonlocal diffusion operators appear to play a more and more important role, in connexion with probability theory (Levy processes) and geometry or mechanics (Steklov problem).
For example the following problem appears in Mechanics and Differential Geometry:

(5.1) $-\Delta u+u=0$ in Ω, $\partial u/\partial n=\pm u^q$ on $\partial\Omega$.

The + sign arises in the study of conformal transformations of a metric with prescribed constant Gaussian curvature on the boundary. This is a reaction problem of pseudo-differential type. Partial results have already been obtained by Escobar in the critical case $q=2(N-1)/N$ (critical with respect to the imbedding of $H^{1/2}(\partial\Omega)$ into $L^q(\partial\Omega)$) using geometric methods. Results of Gidas-Spruck type are also expected. In the case of – sign, the problem is of absorption type. We intend to study the following equation with boundary measure:

(5.2) $-\Delta u+u=0$ in Ω, $\partial u/\partial n+u^q=\mu$ on $\partial\Omega$.

This problem appears to have several critical exponents. More generally we shall study problems of the type

$$(-\Delta)^{\alpha/2}(u) + g((-\Delta)^{\beta/2}u) = \mu$$

For example the questions of large solutions have already been considered by Felmer and Quass and we intend to develop this question in connexion with the regularity of the boundary.