Viscosity Solutions of Fully Nonlinear Second Order Parabolic Equations with L^1 Dependence in Time and Neumann Boundary Conditions

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Abstract : In this article, we are interested in viscosity solutions for second-order fully nonlinear parabolic equations having a L^1 dependence in time and associated with nonlinear Neumann boundary conditions. The main contributions of our study are, not only to treat the case of nonlinear Neumann boundary conditions, but also to revisit the theory of viscosity solutions for such equations and to extend it in order to take in account singular geometrical equations. In particular, we provide comparison results, both for the cases of standard and geometrical equations, which extend the known results for Neumann boundary conditions even in the framework of continuous equations.

Résumé : Dans cet article, nous nous intéressons aux solutions de viscosité d'équations paraboliques fortement non linéaires avec une dépendance L^1 en temps, associées à des conditions de Neumann éventuellement non linéaires. Les contributions de notre étude sont non seulement de traiter le cas de conditions au bord de Neumann non linéaires, mais aussi de revisiter la théorie des solutions de viscosité dans le cadre d'équations avec une dépendance L^1 en temps et de l'étendre afin de prendre en compte les équations singulières de type géométriques. En particulier, nous obtenons des résultats de comparaison qui s'appliquent à la fois à des équations standard et de type géométrique et qui généralisent les résultats connus pour les conditions aux limites de Neumann, même dans le cas où l'équation a une dépendance continue en temps.

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1 Introduction

In this article, we consider fully nonlinear parabolic equations, having a L^1 dependence in time, associated with nonlinear Neumann boundary conditions. The interest of this paper is twofold : first we provide new results, in particular new comparison results, for nonlinear Neumann boundary conditions which extends those of G. Barles [3] *even in the framework of equations with a continuous dependence in time.* Then we revisit viscosity solutions' theory for equations, having a L^1 dependence in time and we extend it to the case of singular equations, typically geometrical equations.

In order to be more specific, we consider the following boundary value problem

$$\frac{\partial u}{\partial t} + F(t, x, u, Du, D^2u) = 0 \quad \text{in} \quad (0, T) \times \Omega, \tag{1.1}$$

$$L(t, x, u, Du) = 0 \quad \text{on} \quad (0, T) \times \partial\Omega, \tag{1.2}$$

where $T > 0$ and Ω is a bounded domain of $I\!R^N$. The solution u is scalar, $\frac{\partial u}{\partial t}$, Du, D^2u denote respectively the derivate of u with respect to t, the gradient and the Hessian matrix of u with respect to the space variable x. The function F is real-valued and defined for almost every $t \in (0, T)$ and, at least, for every $\xi = (x, r, p, X) \in \overline{\Omega} \times I\!R \times I\!R^N \setminus \{0\} \times \mathcal{S}(N)$, where $\mathcal{S}(N)$ denotes the space of $N \times N$ symmetric matrices. For almost every t, it is continuous with respect to ξ and satisfies $t \mapsto F(t, \xi) \in L^1(0, T)$ for every ξ.

We always assume that the function F is *degenerate elliptic*, i.e. for almost every $t \in (0, T)$, for every $(x, r, p) \in \overline{\Omega} \times I\!R \times I\!R^N \setminus \{0\}$, and $X, Y \in \mathcal{S}(N)$

$$F(t, x, r, p, X) \le F(t, x, r, p, Y) \quad \text{if } X \ge Y, \tag{1.3}$$

where " $\le$ " stands for the usual partial ordering on symmetric matrices.

The function $L : [0, T] \times \partial\Omega \times I\!R \times I\!R^N \to I\!R$ is continuous and satisfies the characteristic property of a Neumann type condition, namely: for every $R > 0$, there exists $\nu_R > 0$, such that, for every $(t, x, p) \in [0, T] \times \partial\Omega \times I\!R^N$, $|r| \le R$ and $\lambda \ge 0$, one has,

$$L(t, x, r, p + \lambda n(x)) - L(t, x, r, p) \ge \nu_R \lambda, \tag{1.4}$$

where $n(x)$ is the unit outward normal vector to $\partial\Omega$ at x. More precise conditions on the functions F and L will be given later on.

Our study is motivated by several types of applications: on one hand, by stochastic optimal control problems of (reflected) diffusion processes with L^1 dependence in the data. The associated Hamilton-Jacobi-Bellman Equations takes the classical form

$$\frac{\partial u}{\partial t} - \sup_{\alpha \in \mathcal{A}} \Big\{ \frac{1}{2} \mathrm{Tr}(A_\alpha(t, x) D^2u) + b_\alpha(t, x) \cdot Du + c_\alpha(t, x) u + f_\alpha(t, x) \Big\} = 0, \tag{1.5}$$

where Tr is the trace operator and $p_1 \cdot p_2$ stands for the usual inner product of p_1, p_2 in $I\!R^N$. As in the continuous case, $A_\alpha = \sigma_\alpha \sigma_\alpha^T$, σ_α, b_α, c_α, f_α are functions defined on $(0, T) \times \overline{\Omega}$ which take values respectively in the sets of $N \times m$ matrices for some $m \ge 1$, $I\!R^N$ and $I\!R$ and which are continuous (and even Lipschitz continuous) with respect to $x \in \overline{\Omega}$ but here we assume them to have a L^1 dependence in time. More precise conditions are given in the third section.

On the other hand, we consider geometrical equations, arising in the so-called "level-set approach" for defining the weak notion of evolution of hypersurfaces with curvature dependent normal velocities (and angle boundary conditions), i.e.

$$\frac{\partial u}{\partial t} - a(t, x) \left(\Delta u - \frac{D^2u Du \cdot Du}{|Du|^2} \right) + b(t, x) \cdot Du = 0 \quad \text{in} \quad (0, T) \times \Omega, \tag{1.6}$$

where $a : (0,T) \times \overline{\Omega} \to I\!R^+$ and $b : (0,T) \times \overline{\Omega} \to I\!R^N$ are continuous with respect to x and have a L^1 dependence in time. Again precise conditions on these functions are given in the fourth section.

A well-known particularity of geometrical equations is that they present a singularity for $p = 0$ while, in the case of Hamilton-Jacobi-Bellman Equations, F is continuous in $\xi \in \overline{\Omega} \times I\!R \times I\!R^N \times \mathcal{S}(N)$. This implies a different treatment of them and we refer below as "standard case", the case when F is defined and continuous for $\xi \in \overline{\Omega} \times I\!R \times I\!R^N \times \mathcal{S}(N)$, for almost every $t \in (0,T)$, while we refer to the "singular case" when F presents a singularity for $p = 0$. Comparison results for each case are described separately in the third and fourth sections respectively. However, it is worth mentioning that we provide only one definition of viscosity solutions, which is valid both for the standard and the singular case.

Concerning the Neumann boundary conditions, in the standard case, the more natural conditions are either the classical homogeneous Neumann boundary condition, namely

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on} \quad (0,T] \times \partial\Omega, \tag{1.7}$$

or, more generally, the oblique derivative boundary condition

$$Du(t,x) \cdot \gamma(t,x) = \Theta(t,x) \qquad \text{on} \quad (0,T] \times \partial\Omega, \tag{1.8}$$

where $\gamma : [0,T] \times \partial\Omega \to I\!R^N$ and $\Theta : [0,T] \times \partial\Omega \to I\!R$ are continuous functions. The condition (1.4) holds when the function γ satisfies: for every $(t,x) \in [0,T] \times \partial\Omega$, $\gamma(t,x) \cdot n(x) \geq \alpha > 0$, for some $\alpha > 0$.

In the singular case, because of the geometrical aspect, capillarity boundary conditions are more natural,

$$Du(t,x) \cdot n(x) = \Theta(t,x)|Du(t,x)|, \quad \text{on} \quad (0,T] \times \overline{\Omega}, \tag{1.9}$$

where $\Theta : [0,T] \times \partial\Omega \to I\!R$ is continuous and satisfies, in order to have (1.4), $|\Theta(t,x)| \leq \delta < 1$, on $[0,T] \times \partial\Omega$. More generally, in the singular case, the function L is supposed to be independent of r and homogeneous of degree one in p.

The case where F is continuous in time (referred in the sequel as the "classical case"), was extensively studied, both for equations set in $I\!R^N$ and also with various type of boundary conditions. We refer the reader to the "Users' guide" of Crandall, Ishii and Lions [9] and the books of Bardi and Capuzzo- Dolcetta [1], Barles [7] and Fleming and Soner [12], for a complete presentation of the theory of viscosity solutions.

The case of geometrical equations was first studied by L.C. Evans et J. Spruck in [10] for the Mean Curvature Equation and then by Chen, Giga and Goto [8] for more general equations.

Viscosity solutions for equations with L^1-dependences in time were first studied by H. Ishii [13] for first-order Hamilton-Jacobi equations. In this article, H. Ishii introduces a definition of viscosity solutions, taking into account the weak regularity of the Hamiltonians in time; he proves stability and comparison results. Then P.L. Lions and B. Perthame [16] simplified Ishii's proofs by using simpler equivalent definitions. The first paper (to the best of our knowledge) dealing with the second-order case is the one of D. Nunziante

[17]: she adapts Ishii's definition to the second-order case and provide a maximum principle type result in bounded domains for fully nonlinear second-order parabolic equations. Then, in [18], she studies the existence and uniqueness of unbounded viscosity solutions for such equations.

Up to now, again to the best of our knowledge, the singular case and the Neumann boundary conditions have not been studied, when F has a L^1 dependence in time.

The study of continuous equations associated with Neumann boundary conditions started with the work of P.L. Lions [15] on homogeneous Neumann boundary conditions for first-order Hamilton-Jacobi Equations. In particular, this work contains the first definition of boundary condition in the viscosity sense and a comparison result in this framework. These results, still in the first order case, were extended to the case of nonlinear boundary conditions by G. Barles and P.L Lions [5]. The case of second-order equations was first considered by Ishii [13] and then by G. Barles [2]: while [13] provides results in the case of C^1-boundaries but rather restrictive assumptions on the boundary conditions, [2] imposes stronger assumptions on the boundary ($W^{3,\infty}$) but weaker ones on the boundary conditions. These results were extended to the case of geometrical equations respectively by H. Ishii and M.H. Sato [14] and G. Barles [3].

As we point out above, one of the aims of this paper is to revisit the L^1 theory for viscosity solutions of fully nonlinear second-order parabolic equations and to extend it to the case of geometrical equations and nonlinear Neumann boundary conditions. This explains the length of this paper and the fact that it is impossible to give an idea of all the results of this work in this introduction. Thus, we are going to point out only the main contributions of this article.

First we provide a definition of viscosity solutions of (1.1)-(1.2), consistent with the classical one, adapting the one given by D. Nunziante in [17]. Then we give several equivalent definitions, whose interest is to simplify the proofs of different results; a striking example is the change of variable (a key result in the "level-sets approach") $u \to \Psi(u)$ where Ψ is a smooth function such that $\Psi' > 0$ on $I\!R$: whereas this result can be proved very easily in the classical case, it is much harder in the L^1-case because of the specific form of the test-functions used in the definition. Therefore, it is convenient to provide a new definition where the space of the test-functions is stable by composition with a smooth function Ψ.

Of course, a second contribution are the comparison results which are provided both for the standard and singular case. For the treatment of the Neumann boundary condition, we use the assumptions of G. Barles [3] and adapt the methods of this paper to our framework. We point out that these results are obtained using natural hypothesis on F and L.

These results, even when F is continuous, improved the ones of G. Barles [3] because we only assume $\dfrac{\partial L}{\partial t}$ to be in L^1 while, in [3], this derivative has to be in L^∞.

The paper is organized as follows: in the second section, we introduce a definition of viscosity solutions for the Neumann problem in the L^1 framework, we provide several equivalent ones and give some properties of these solutions. The third and fourth sections are devoted to the study of the standard and singular case respectively. In each section, after describing the precise assumptions used on the functions F and L, we give and prove the comparison results.

Our proofs (even without considering the singular case and the Neumann boundary condition) differ from the ones of D. Nunziante in [17]. In particular, in Section 5, we show a Maximum Principle type result for viscosity solutions adapted to the L^1 theory and largely inspired of Theorem 3.2 in [9], which can be used independently. The sixth section is dedicated to the construction of a suitable test-function. In the seventh section, we provide the proofs of the properties of viscosity solutions given in the first section, of the auxiliary lemmas given in the second and third sections and deal in details with the quasilinear case.

2 Definition and properties of viscosity sub and supersolution for the Neumann problem

We first introduce some notations. In the sequel, we set $\Omega_T^- = (0,T) \times \overline{\Omega}$, $\Omega_T = (0,T] \times \overline{\Omega}$ and $\overline{\Omega}_T = [0,T] \times \overline{\Omega}$. We also use the notations $\Gamma = \overline{\Omega} \times I\!R \times I\!R^N \times \mathcal{S}(N)$, $\Gamma^* = \overline{\Omega} \times I\!R \times I\!R^N \setminus \{0\} \times \mathcal{S}(N)$, $\Sigma = \overline{\Omega} \times I\!R \times I\!R^N$, $\Sigma^* = \overline{\Omega} \times I\!R \times I\!R^N \setminus \{0\}$ and $\Sigma_T = [0,T] \times \Sigma$, $\Gamma_T = [0,T] \times \Gamma$, $\Gamma_T^* = [0,T] \times \Gamma^*$.

For $m \geq 1$, if A is a subset of $I\!R^m$, $USC(A)$ (resp. $LSC(A)$) is the set of real-valued upper (resp. lower) semicontinuous functions on A. If $A \subset \Gamma_T$ (resp. $\subset \Gamma$), the notation A^* will be used for the set $\{(t,x,r,p,X)$ (resp. $(x,r,p,X)) \in A$ such that $p \neq 0\}$.

If A is a subset of $I\!R \times I\!R^N$, we say that $f \in C^{1,2}_{t,x}(A)$ if there exists a function $\tilde{f} : I\!R \times I\!R^N \to I\!R$ with $f(t,x) = \tilde{f}(t,x)$ if $(t,x) \in A$ and $\tilde{f}$ is a C^1 function in t and C^2 in x, all derivatives being continuous on A.

Before providing the definition, we recall that F is assumed to be a locally bounded degenerate elliptic function, defined for almost every $t \in (0,T)$ and (at least) for every $\xi := (x,u,p,M) \in \Gamma^*$ while L is a real-valued function continuous on $[0,T] \times \partial\Omega \times I\!R \times I\!R^N$.

Definition 2.1 *A function $u \in USC(\Omega_T)$ (resp. $\in LSC(\Omega_T)$) is a viscosity subsolution (resp. supersolution) of (1.1)-(1.2) in Ω_T, if and only if for every $(t_0,x_0) \in \Omega_T$, $b \in L^1(0,T)$, $\varphi \in C^{1,2}_{t,x}(\Omega_T)$ and $G \in C(\Gamma_T)$, such that the function $(t,x) \mapsto u(t,x) + \int_0^t b(s)ds - \varphi(t,x)$, has a local maximum point (resp. minimum point) at (t_0,x_0) and such that*

$$b(t) + G(t,\xi) \leq F(t,\xi), \quad (\textit{resp.} \geq) \tag{2.1}$$

for almost every $t \in (0,T)$ in some neighborhood of t_0 and for every $\xi \in \Gamma^*$ in some neighborhood of $\xi_0 = (x_0, u(t_0,x_0), p_0, X_0)$, $p_0 = D\varphi(t_0,x_0)$, $X_0 = D^2\varphi(t_0,x_0)$, then

$$\alpha_0 := \frac{\partial \varphi}{\partial t}(t_0,x_0) + G(t_0,\xi_0) \leq 0\,(\textit{resp.} \geq 0) \quad \textit{if} \quad x_0 \in \Omega, \tag{2.2}$$

$$\min\,(\textit{resp.} \max)\Big(\alpha_0, L(t_0,x_0,u(t_0,x_0),p_0)\Big) \leq 0\,(\textit{resp.} \geq 0) \quad \textit{if} \quad x_0 \in \partial\Omega. \tag{2.3}$$

A locally bounded function u in Ω_T is said to be a solution of (1.1)-(1.2) in Ω_T, if u^* and u_* are respectively subsolution and supersolution of (1.1)-(1.2) in Ω_T.

It is worth pointing out that the above definition does not use the upper and lower semicontinuous envelope of F as the classical definition does but this fact is hidden in the roles of G and b. In particular, the possible singularity for $p = 0$ is taken in account by the inequalities (2.1) for which we remark that they have to hold only for $\xi \in \Gamma^*$.

Despite of this rather complicated definition, the classical basic reductions or extensions in the definition hold as in the classical case : we may assume that $u(t_0, x_0) + \int_0^{t_0} b(s)ds = \varphi(t_0, x_0)$, the space $C^{1,2}_{t,x}(\Omega_T)$ may be replaced by $C^{\infty}_{t,x}(\Omega_T)$ and we can also assume that we have a strict maximum (resp. minimum) point.

2.1 A characterization in terms of sub and superdifferential

In order to give an equivalent definition in terms of sub and superdifferential, we recall the following notions. If $I \subset I\!R$ and $U \subset I\!R^m$ $(m \geq 1)$ and if v is a real-valued function defined in $I \times U$, for $(t_0, x_0) \in I \times U$, we denote by $\mathcal{P}^{2,+}_{I\times U} v(t_0, x_0)$ (resp. $\mathcal{P}^{2,-}_{I\times U}$), the subset of $I\!R \times I\!R^m \times \mathcal{S}(m)$, defined as follows : $(a, p, X) \in \mathcal{P}^{2,+}_{I\times U} v(t_0, x_0)$ (resp. $\mathcal{P}^{2,-}_{I\times U} v(t_0, x_0)$) if, for every $(t, x) \in I \times U$, we have

$$v(t, x) \leq v(t_0, x_0) + a(t - t_0) + p \cdot (x - x_0) + \frac{1}{2}\, X(x - x_0) \cdot (x - x_0) + o(|t - t_0| + |x - x_0|^2),$$

(resp. $\geq$).

Moreover, we say that $(a, p, X) \in \overline{\mathcal{P}}^{2,+}_{I\times U} v(t_0, x_0)$ (resp. $\overline{\mathcal{P}}^{2,-}_{I\times U}$), if there exists a sequence $((t_k, x_k))_k$ converging to (t_0, x_0) and $(a_k, p_k, X_k) \in \mathcal{P}^{2,+}_{I\times U} v(t_k, x_k)$ (resp. $\overline{\mathcal{P}}^{2,-}_{I\times U} v(t_k, x_k)$) such that

$$(a_k, p_k, X_k) \to (a, p, X)\ .$$

Finally, if $I = I\!R$ and $U = I\!R^m$, we set to simplify $\mathcal{P}^{2,+}$, $\mathcal{P}^{2,-}$,$\overline{\mathcal{P}}^{2,+}$ and $\overline{\mathcal{P}}^{2,-}$.

Proposition 2.1 *A function $u \in USC(\Omega_T)$ (resp. $\in LSC(\Omega_T)$) is a subsolution (resp. supersolution) of (1.1)-(1.2) in Ω_T, if and only if*
for every $(t_0, x_0) \in \Omega_T$, $b \in L^1(0, T)$, $(a_0, p_0, X_0) \in \overline{\mathcal{P}}^{2,+}_{\Omega_T}\Big(u(t_0, x_0) + \int_0^{t_0} b(s)ds\Big)$ (resp. $\overline{\mathcal{P}}^{2,-}_{\Omega_T}$), $G \in C(\Gamma_T)$, such that (2.1) holds then (2.2) and (2.3) hold.

2.2 A new space of test-functions

As we mentioned it in the introduction, Definition 2.1 which essentially relies on test-functions with separated variables, may not be convenient to prove some results : for example, if u is a solution of some equation, it is not so clear that $\chi(u)$ is a solution of the transformed equation, even if χ is a smooth, strictly increasing function.

To produce a more convenient definition, we first introduce some notations. If $g : I\!R \times I\!R^+ \to I\!R^+$, we say that $g \in \mathcal{M}$ if it satisfies

$$t \mapsto g(t, r) \in L^1_{loc}(I\!R) \text{ for all } r \geq 0, \quad g(\cdot, r) \to 0 \quad \text{when } r \to 0, \text{ in } \quad L^1_{loc}(I\!R),$$
$$r \to g(t, r) \quad \text{is non-decreasing in } I\!R^+, \text{ for almost every } t \in I\!R. \qquad (2.4)$$

In the same way, a function $f : I\!R \times I\!R^N \to I\!R$ is in the space C^2_s if it satisfies

$$\forall t \in I\!R, \quad x \mapsto f(t,x) \in C^2(I\!R^N), \quad \text{with} \quad f,\, Df,\, D^2 f \in C(I\!R \times I\!R^N).$$

Finally, the space $\mathcal{H}$ is defined by : $\varphi \in \mathcal{H}$ if $\varphi \in C^2_s$ and if, in the sense of distributions,

$$\frac{\partial \varphi}{\partial t}(.,x) \in L^1_{loc}(I\!R), \quad |\frac{\partial \varphi}{\partial t}(.,x) - \frac{\partial \varphi}{\partial t}(.,y)| \le g_\varphi(.,|x-y|), \quad \forall x,y \in I\!R^N \tag{2.5}$$

for some $g_\varphi \in \mathcal{M}$.

One checks easily that the functions of this space $\mathcal{H}$ have the following (clear) property.

Lemma 2.1 *For every $\varphi \in \mathcal{H}$ and $\psi \in C^2(I\!R)$, $\psi \circ \varphi \in \mathcal{H}$.*

Then, we have the following proposition, providing an equivalent definition using $\mathcal{H}$ as space of test-functions.

Proposition 2.2 *A function $u \in USC(\Omega_T)$ (resp. $\in LSC(\Omega_T)$) is a subsolution (resp. supersolution) of (1.1)-(1.2) in Ω_T, if and only if*
for every $(t_0,x_0) \in \Omega_T$, $\varphi \in \mathcal{H}$ and $G \in C(\Gamma_T)$ such that the function $u - \varphi$ has a local maximum point (resp. minimum point) at (t_0,x_0) and such that

$$G(t,\xi) \le \frac{\partial \varphi}{\partial t}(t,x) + F(t,\xi), \quad (\textit{resp. } \ge) \tag{2.6}$$

for almost every $t \in (0,T)$ in some neighborhood of t_0 and for every $\xi \in \Gamma^*$ in some neighborhood of ξ_0, then

$$G(t_0,\xi_0) \le 0 \quad (\textit{resp. } \ge 0) \quad \textit{if } x_0 \in \Omega, \tag{2.7}$$

$$\min(\textit{resp. } \max)\Big(G(t_0,\xi_0), L(t_0,u(t_0,x_0),x_0,p_0)\Big) \le 0 \quad (\textit{resp. } \ge 0) \quad \textit{if } x_0 \in \partial\Omega. \tag{2.8}$$

2.3 Further Remarks and Results

We first remark that all the above equivalent definitions are consistent with the classical definition when F is continuous in time. We consider two cases, corresponding respectively to the standard and singular one.
(i) $F \in C(\Gamma_T)$.
(ii) $F \in C(\Gamma^*_T)$ is locally bounded in Γ^*_T and

$$F^*(t,x,r,0,0) = F_*(t,x,r,0,0), \quad \forall\,(t,x,r) \in [0,T] \times \overline{\Omega} \times I\!R.$$

Proposition 2.3 *Assume that F satisfies (i) or (ii), with, in this latter case, L being homogeneous of degree 1 in p. Then the classical and the L^1- notions of viscosity sub and supersolutions are equivalent.*

The proof of this result is given in Section 7.

The next remark consists in comparing viscosity sub and supersolutions of (1.1)-(1.2) in Ω^-_T and in Ω_T : it is a standard result in the classical framework that locally bounded subsolution and supersolution of (1.1)-(1.2) in Ω^-_T are in fact sub and supersolutions in Ω_T.

Proposition 2.4 *Assume that, for every compact subset K of Γ, one has*

$$\sup_{\xi \in K^*} |F(.,\xi)| \in L^1(0,T), \tag{2.9}$$

and that L is continuous on Σ_T, then a subsolution (resp. supersolution) u of (1.1)-(1.2) in Ω_T^- which is locally bounded up to time T, is a subsolution (resp. supersolution) of (1.1)-(1.2) in Ω_T.

The proof of this result is also given in Section 7.

Finally we present a results on a change of variables which leads to simplifications in the proof of the comparison results below.

Proposition 2.5 *Let $u \in USC(\Omega_T)$ (resp. $v \in LSC(\Omega_T)$) a subsolution (resp. supersolution) of (1.1)-(1.2) in Ω_T. For every $\gamma \in L^1(0,T)$, $(x,r,p,X) \in \Gamma^*$ and almost every $t \in (0,T)$, we set*

$$\begin{aligned} u_\gamma(t,x) &= e^{\int_0^t \gamma(s)ds} u(t,x), \qquad v_\gamma(t,x) = e^{\int_0^t \gamma(s)ds} v(t,x), \\ F_\gamma(t,x,r,p,X) &= -\gamma(t)r + e^{\int_0^t \gamma(s)ds} F(t,x,re^{-\int_0^t \gamma(s)ds}, pe^{-\int_0^t \gamma(s)ds}, Xe^{-\int_0^t \gamma(s)ds}), \\ L_\gamma(t,x,r,p) &= e^{\int_0^t \gamma(s)ds} L(t,x,re^{-\int_0^t \gamma(s)ds}, pe^{-\int_0^t \gamma(s)ds}), \end{aligned} \tag{2.10}$$

then u_γ (resp. v_γ) is a subsolution of (1.1)-(1.2), with (F_γ, L_γ), in Ω_T.

We leave the proof of this result to the reader : it is rather straightforward by using Proposition 2.2.

3 Comparison Result I : the Standard Case

In this section, F is defined for almost every $t \in (0,T)$ and for every $\xi \in \Gamma$; F is also continuous in ξ for almost every t. The following conditions on the function F are the natural adaptations to the L^1-case from the conditions given by G. Barles [3] for the classical one. We use below the notations : $a \vee b = \max(a,b)$ and $a \wedge b = \min(a,b)$.

(H0) $F(.,\chi) \in L^1(0,T)$, for every $\chi \in \Gamma$.

(H1) For any $R > 0$, there exists $\gamma_R \in L^1(0,T)$ such that

$$F(t,x,r,p,X) - F(t,x,z,p,X) \geq \gamma_R(t)(r-z),$$

for almost every $t \in (0,T)$, for every $r,z \in I\!R$, such that $-R \leq z \leq r \leq R$, and for every $(x,p,X) \in \overline{\Omega} \times I\!R^N \times \mathcal{S}(N)$.

(H2) For any $R > 0$, there exists $m_R \in \mathcal{M}$, such that, for every $0 < \nu, \varepsilon \leq 1$

$$F(t,y,r,q,Y) - F(t,x,r,p,X) \leq m_R\Big(t, \nu + |x-y|(1+|p| \vee |q|)\Big), \tag{3.1}$$

for almost all $t \in (0,T)$, for any $x,y \in \overline{\Omega}$, $p,q \in I\!R^N$, $r \in I\!R$ with $|r| \leq R$ and for any matrices $X,Y \in \mathcal{S}(N)$, satisfying the following properties

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{1}{\varepsilon^2} \begin{pmatrix} Id & -Id \\ -Id & Id \end{pmatrix} + \nu Id, \tag{3.2}$$

$$|p-q| \leq \min\Big(\nu\varepsilon(1+|p| \wedge |q|), \nu\Big), \tag{3.3}$$

$$|x-y| \leq \nu\varepsilon. \tag{3.4}$$

(H3) For every $R > 0$, there exists $g_R \in \mathcal{M}$, such that

$$|F(t,x,r,p,X) - F(t,x,z,p,X)| \leq g_R(t,|r-z|), \tag{3.5}$$

for almost every $t \in (0,T)$, for every $r, z \in I\!R$, $x \in \overline{\Omega}$, $p \in I\!R^N$, $X \in \mathcal{S}(N)$ with $|r| \vee |z| \vee |p| \vee ||X|| \leq R$.

For the function L, the following assumptions are used.

(H4) For any $R > 0$, there exists $\nu_R > 0$, such that, for any $\lambda \geq 0$, $t \in [0,T], x \in \partial\Omega$, $-R \leq z \leq r \leq R$, $p \in I\!R^N$, one has

$$L\big(t,x,r,p+\lambda n(x)\big) - L(t,x,z,p) \geq \nu_R\, \lambda,$$

where $n(x)$ denotes the unit outward normal vector to $\partial\Omega$ at x.

(H5) For any $R > 0$, there exists $C_R > 0$ such that, for every $t \in [0,T], x, y \in \partial\Omega$, $p, q \in I\!R^N$, $r, z \in I\!R$, with $|r| \vee |z| \leq R$, one has

$$|L(t,x,r,p) - L(t,y,z,q)| \leq C_R\Bigg(\big(1+|p|+|q|\big)|x-y| + |r-z| + |p-q|\Bigg).$$

(H6) For any $R > 0$, there exists a nonnegative function $h_R \in L^1(0,T)$, such that, for every $0 \leq t \leq s \leq T$, $|r| \leq R$, $x \in \partial\Omega$ and $p \in I\!R^N$, one has

$$|L(t,x,r,p) - L(s,x,r,p)| \leq (1+|p|)\int_t^s h_R(w)dw.$$

Finally, the assumption on the domain Ω is the following.

(H7) Ω is a bounded domain with a $W^{3,\infty}$-boundary.

The comparison result is the following.

Theorem 3.1 (comparison result in the standard case)
Assume that u, $v : \overline{\Omega}_T \to I\!R$ are respectively a bounded upper semicontinuous subsolution and a bounded lower semicontinuous supersolution of (1.1)-(1.2) in Ω_T. If F, L and Ω satisfy conditions **(H0)-(H7)** *and if $u(0,.) \leq v(0,.)$ in $\overline{\Omega}$, then*

$$u \leq v \qquad in \quad \overline{\Omega}_T. \tag{3.6}$$

Before providing the proof of Theorem 3.1, we point out easy consequences of it. First, if $F \in C(\Gamma_T)$ satisfies the conditions given in [3] and conditions **(H4)-(H6)** hold for L, then Theorem 3.1 applies; it is worth point out that **(H6)** is a weaker requirement on L than in [3] where it was supposed to be locally Lipschitz in time.

Next we come back on the Hamilton-Jacobi-Bellman Equation (1.5) and the conditions on $A_\alpha = \sigma_\alpha \sigma_\alpha^T$, σ_α, b_α, c_α, f_α in order to have the assumptions of Theorem 3.1 being satisfied for this equation. A natural condition is the existence of $g \in L^1(0,T)$ such that, for every α, $\psi = b_\alpha, c_\alpha, f_\alpha$ satisfies, for almost every $t \in (0,T)$ and for every $x, y \in \overline{\Omega}$

$$|\psi(t,x)| \ \leq g(t), \quad |\psi(t,x) - \psi(t,y)| \ \leq g(t)|x-y| \ , \tag{3.7}$$

while, for σ_α, this inequality has to be satisfied with some $g \in L^2(0,T)$.

In the proof of Theorem 3.1 below, we are going to regularize F; this is why we require the unusual additional condition **(H3)** and this is also partly the role of **(H2)**. A precise result in this direction is the following lemma proved in Section 7.

Lemma 3.1 *Assume that F satisfies* **(H0)**, **(H2)** *and* **(H3)** *then F is degenerate elliptic. Moreover, if K is a compact subset of Γ, we have*

$$\sup_{\chi \in K} |F(.,\chi)| \in L^1(0,T), \tag{3.8}$$

and

$$h_K^r(.) := \sup_{\substack{(\xi_1,\xi_2)\in K^2 \\ |\xi_1-\xi_2|\le r}} |F(.,\xi_1) - F(.,\xi_2)| \in L^1(0,T), \tag{3.9}$$

and satisfies

$$h_K^r(.) \to 0 \quad \text{in } L^1(0,T) \text{ as } r \to 0. \tag{3.10}$$

Proof of Theorem 3.1 : As in the standard case, the proof relies on the construction of a suitable test-function (cf Lemma 6.1) and a suitable extension of the Maximum Principle for viscosity solutions (cf. Lemma 5.1). As in the classical case, we argue by contradiction by assuming that

$$\max_{\overline{\Omega}_T} (u - v) = M > 0. \tag{3.11}$$

0. First, since u and v are bounded, we can set $R := \max(||u||_\infty, ||v||_\infty)$ and drop any dependence in R in all the modulus and functions appearing in assumptions **(H0)-(H6)**. Moreover, by Proposition 2.5, we can assume w.l.o.g. that $\gamma_R \equiv 0$ in $(0,T)$ in condition **(H1)**. Finally we denote by $\tilde{K}$, $\tilde{h}$ and δ the constants and function appearing in Lemma 6.1
1. For $\varepsilon > 0$, we consider the following compact subset of Γ

$$I_\varepsilon = \left\{ (x,r,p,X) \in \Gamma, \quad \text{such that} \quad |r| \le R+1,\ |p| \le \frac{1}{\varepsilon},\ ||X|| \le \frac{1}{\varepsilon^5} \right\}. \tag{3.12}$$

Next we introduce a sequence $(\rho_n)_{n\ge1}$ of smooth mollifiers, satisfying, for every $n \ge 1$,

$$\rho_n \in \mathcal{D}(I\!R), \quad \operatorname{supp}(\rho_n) \subset (-\frac{1}{n}, \frac{1}{n}), \quad \int_{I\!R} \rho_n(s)ds = 1 \quad \text{and} \quad \rho_n \ge 0 \text{ on } I\!R.$$

If $(\rho_n)_{n\ge1}$ is such a sequence and $h \in L^1(0,T)$, we introduce the sequence $(h_n)_n$ as follows : we first extend h to a $L^1(I\!R)$ function, extending it by 0 for $t \notin (0,T)$ and still denoting by h this extension. Then we set $h_n = h * \rho_n$. For $n \ge 1$, we define the functions F_n, m_n, g_n, by $F_n(.,\xi) = (F(.,\xi))_n$, $m_n(.,s) = (m(.,s))_n$ and $g_n(.,s) = (g(.,s))_n$, for every $\xi \in \Gamma$ and $s \in I\!R^+$.

By standard arguments, it is easy to show that, if F satisfies conditions **(H0)**, **(H2)** and **(H3)**, then the following holds : for every $n \in I\!N$, $F_n \in C(\Gamma_T)$ satisfies **(H2)** and **(H3)** with m_n, g_n and for every compact subset K of Γ and for any $n \in I\!N$, we have

$$\sup_{\chi\in K} |F_n(.,\chi) - F(.,\chi)| \to 0 \quad \text{in} \quad L^1(0,T). \tag{3.13}$$

2. For $\varepsilon > 0$, for almost every $t \in (0,T)$, we set $b_n^\varepsilon(t) = -\sup_{\xi \in I_\varepsilon} |F_n(t,\xi) - F(t,\xi)|$. By the above recalled properties, for every $n \in I\!N$ and $\varepsilon > 0$, $F_n \in C(\Gamma_T)$ and $b_n^\varepsilon \in L^1(0,T)$.

Moreover, for every $\varepsilon > 0$ fixed, $||b_n^\varepsilon||_{L^1(0,T)} \to 0$. Therefore, for every $\varepsilon > 0$, we can choose $n_\varepsilon \in I\!N$, such that

$$||b_{n_\varepsilon}^\varepsilon||_{L^1(0,T)} \leq \varepsilon. \tag{3.14}$$

For the sake of simplicity of notations, from now on, we write b_ε instead of $b_{n_\varepsilon}^\varepsilon$ and F_ε instead of F_{n_ε}. By the definition of b_ε, we clearly have, for almost every $t \in (0,T)$ and for every $\xi \in I_\varepsilon$

$$b_\varepsilon(t) + F_\varepsilon(t,\xi) \leq F(t,\xi), \quad -b_\varepsilon(t) + F_\varepsilon(t,\xi) \geq F(t,\xi). \tag{3.15}$$

3. For $0 < \varepsilon \leq \nu \leq 1$, for almost every $t \in (0,T)$, we set $m_\nu(t) = m\Big(t, (2\tilde{K}^{\frac{3}{2}} + 1)\nu\Big)$ and $m_{\nu,\varepsilon} = m_{n_\varepsilon,\nu}$, where we recall that m appears in condition **(H2)** on F. As $m_R \in \mathcal{H}$, it is clear that $m_{\nu,\varepsilon} \in C(I\!R)$. Let $\psi_{\nu,\varepsilon} \in C_s^2$ the function given by Lemma 6.1, we define the function $\Phi_{\nu,\varepsilon} \in C_s^2$, for $(t,x,y) \in I\!R \times I\!R^N \times I\!R^N$, by

$$\Phi_{\nu,\varepsilon}(t,x,y) = \psi_{\nu,\varepsilon}(t,x,y) + \nu \int_0^t \tilde{h}(s)ds + \int_0^t m_{\nu,\varepsilon}(s)ds + \nu t.$$

Then, we consider the function defined, for $t \in [0,T]$, $(x,y) \in \overline{\Omega} \times \overline{\Omega}$, by

$$\Psi_{\nu,\varepsilon}(t,x,y) = u(t,x) + \int_0^t b_\varepsilon(s)ds - \left(v(t,y) - \int_0^t b_\varepsilon(s)ds\right) - \Phi_{\nu,\varepsilon}(t,x,y). \tag{3.16}$$

As $\Psi_{\nu,\varepsilon} \in USC([0,T] \times \overline{\Omega} \times \overline{\Omega})$, it achieves its maximum over $[0,T] \times \overline{\Omega} \times \overline{\Omega}$ at some point $(t_{\nu,\varepsilon}, x_{\nu,\varepsilon}, y_{\nu,\varepsilon})$. And to simplify, we set $(\bar{t},\bar{x},\bar{y}) = (t_{\nu,\varepsilon}, x_{\nu,\varepsilon}, y_{\nu,\varepsilon})$. In the sequel, we set $p_x = D_x\Phi_{\nu,\varepsilon}(\bar{t},\bar{x},\bar{y})$, $p_y = D_y\Phi_{\nu,\varepsilon}(\bar{t},\bar{x},\bar{y})$ and $A = D^2\Phi_{\nu,\varepsilon}(\bar{t},\bar{x},\bar{y})$.
4. As $m_R \in \mathcal{M}$, we can choose $\nu > 0$ small enough, such that

$$||m_{\varepsilon,\nu}||_{L^1(0,T)} + \nu||\tilde{h}||_{L^1(0,T)} + \nu T \leq \frac{\delta}{2}. \tag{3.17}$$

With this choice of ν, classical arguments allow to prove that we have

$$\frac{|\bar{x}-\bar{y}|}{\varepsilon} \to 0 \quad \text{as} \quad \varepsilon \to 0, \tag{3.18}$$

$$u(\bar{t},\bar{x}) - v(\bar{t},\bar{y}) \geq M - \delta > 0, \quad \bar{t} > 0, \quad \text{for } \varepsilon \text{ small enough.} \tag{3.19}$$

Noticing that $p_x = D_x\psi_{\nu,\varepsilon}(\bar{t},\bar{x},\bar{y})$, $p_y = D_y\psi_{\nu,\varepsilon}(\bar{t},\bar{x},\bar{y})$, $A = D^2\psi_{\nu,\varepsilon}(\bar{t},\bar{x},\bar{y})$, Lemma 6.1 show that, for $0 < \varepsilon \leq \nu \leq 1$ small enough, we have

$$|p_x| \vee |p_y| < \varepsilon^{-1}, \quad ||A|| < 3\tilde{K}\varepsilon^{-2}, \tag{3.20}$$

$$L(\bar{t},\bar{x},u(\bar{t},\bar{x}),p_x) > 0 \quad \text{if} \quad \bar{x} \in \partial\Omega, \tag{3.21}$$

$$L(\bar{t},\bar{y},v(\bar{t},\bar{y}),-p_y) < 0 \quad \text{if} \quad \bar{y} \in \partial\Omega. \tag{3.22}$$

By (3.18), for ε small enough, $|\bar{x}-\bar{y}| \leq \nu\varepsilon$ and therefore, inequality (6.6) in Lemma 6.1, shows that $\dfrac{\partial \psi_{\nu,\varepsilon}}{\partial t}(.,x,y) \geq -\nu\tilde{h}$ in $\mathcal{D}'(0,2T)$, for every $(x,y) \in B_{\frac{\nu\varepsilon}{2},\overline{\Omega}}(\bar{x},\bar{y})$. This implies the following inequality

$$\frac{\partial \Phi_{\nu,\varepsilon}}{\partial t}(.,x,y) \geq m_{\nu,\varepsilon}(.) + \nu \quad \text{in} \quad \mathcal{D}'(0,2T), \quad \text{for all } (x,y) \in B_{\frac{\nu\varepsilon}{2},\overline{\Omega}}(\bar{x},\bar{y}). \tag{3.23}$$

5. Next we use Lemma 5.1. We choose $\Delta = \varepsilon^{-5}$. By (3.20), it is clear that for ε small enough, we have $\Delta > 3||A||$. Then, let $(X,Y) \in (\mathcal{S}(N))^2$, with $||X|| \vee ||Y|| \leq \Delta$. By (3.20), for every (x,r,p), (y,v,q) in Σ^* closed enough to $(\bar{x}, u(\bar{t},\bar{x}), p_x)$ and $(\bar{y}, v(\bar{t},\bar{y}), -p_y)$ respectively, we have $(\bar{x}, u(\bar{t},\bar{x}), p_x, X), (\bar{y}, v(\bar{t},\bar{y}), -p_y, Y) \in I_\varepsilon$. And therefore, (3.15) shows (5.2) and (5.3), with $b_1 = -b_2 = b_\varepsilon$, $G_1 = G_2 = F_\varepsilon$. Thus, using also (3.18), (3.21), (3.22) and (3.23), we can use Lemma 5.1 with $\varphi = \psi_{\nu,\varepsilon}$, $\Delta, b_1, b_2, G_1, G_2$ and with $\vartheta = m_{\nu,\varepsilon} + \nu$. Therefore, there exists $(a,b) \in I\!R^2$, $(X,Y) \in (\mathcal{S}(N))^2$, such that

$$
\begin{aligned}
(a, p_x, X) &\in \overline{\mathcal{P}}^{2,+}_{\Omega_T}\left(u(\bar{t},\bar{x}) + \int_0^{\bar{t}} b^\varepsilon(s)ds\right), && (3.24)\\
(b, -p_y, Y) &\in \overline{\mathcal{P}}^{2,-}_{\Omega_T}\left(v(\bar{t},\bar{y}) - \int_0^{\bar{t}} b^\varepsilon(s)ds\right), && (3.25)\\
a - b &\geq m_{\nu,\varepsilon}(\bar{t}) + \nu, && (3.26)\\
-\left(\frac{1}{3\varepsilon^5} + ||A||\right) Id &\leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + 3\varepsilon^5 A^2. && (3.27)
\end{aligned}
$$

From inequalities (3.20) and (3.27), it is easy to show that, for ε small enough,

$$\max(||X||, ||Y||) \leq \frac{1}{3\varepsilon^5} + ||A|| < \frac{2}{3\varepsilon^5}. \tag{3.28}$$

Therefore, by (3.12), (3.20) and (3.28), if $\xi, \xi' \in \Gamma$ are closed enough to $(\bar{x}, u(\bar{t},\bar{x}), p_x, X)$ and $(\bar{y}, v(\bar{t},\bar{y}), -p_y, Y)$ respectively, then $\xi, \xi' \in I_\varepsilon$. This shows that the inequalities (3.15) hold true for almost every t in some neighborhood of $\bar{t}$ in $(0,T)$, for every $\xi, \xi' \in \Gamma$ in some neighborhood of $(\bar{x}, u(\bar{t},\bar{x}), p_x, X)$ and $(\bar{t}, v(\bar{t},\bar{y}), -p_y, Y)$, respectively.

6. Since u and v are respectively subsolution and supersolution of (1.1)-(1.2) in Ω_T, we get using (3.21) and (3.22) and then (3.15), (3.24), (3.25)

$$a + F_\varepsilon(\bar{t}, \bar{x}, u(\bar{t},\bar{x}), p_x, X) \leq 0, \qquad b + F_\varepsilon(\bar{t}, \bar{y}, v(\bar{t},\bar{y}), -p_y, Y) \geq 0.$$

We have used in the above inequalities that $F_\varepsilon \in C(\Gamma_T)$. Moreover since **(H1)** holds with $\gamma_R \equiv 0$, this implies by (3.19) and (3.26)

$$m_{\nu,\varepsilon}(\bar{t}) + \nu \leq \mathcal{A}_{\varepsilon,\nu} := F_\varepsilon(\bar{t}, \bar{y}, v(\bar{t},\bar{y}), -p_y, Y) - F_\varepsilon(\bar{t}, \bar{x}, v(\bar{t},\bar{y}), p_x, X). \tag{3.29}$$

7. To estimate $\mathcal{A}_{\varepsilon,\nu}$, we are going to use condition **(H2)** on F_ε. To this end, we need the following technical lemma, the proof of which is postponed.

Lemma 3.2 *For ε small enough, (X,Y), $(\bar{x},\bar{y})$, $(p_x, -p_y)$ satisfy conditions (3.2), (3.3) and (3.4) of **(H2)**, with (ε,ν) replaced by $(\tilde{\varepsilon},\tilde{\nu})$, where*

$$\tilde{\varepsilon} = \varepsilon \tilde{K}^{-1/2}, \quad \tilde{\nu} = 2\tilde{K}^{\frac{3}{2}}\nu.$$

Thanks to Lemma 3.2, we can use **(H2)** for F_ε to get, for ε small enough,

$$
\begin{aligned}
\mathcal{A}_{\varepsilon,\nu} &\leq m_{n_\varepsilon,R}\left(\bar{t}, 2\tilde{K}^{\frac{3}{2}}\nu + |\bar{x}-\bar{y}|(1 + |p_x| \vee |p_y|)\right), && (3.30)\\
&\leq m_{n_\varepsilon,R}\left(\bar{t}, 2\tilde{K}^{\frac{3}{2}}\nu + 2\frac{|\bar{x}-\bar{y}|}{\varepsilon}\right) \quad \text{by (3.20)}, && (3.31)\\
&\leq m_{n_\varepsilon,R}\left(\bar{t}, (2\tilde{K}^{\frac{3}{2}} + 1)\nu\right) = m_{\nu,\varepsilon}(\bar{t}) \quad \text{by (3.18)}. && (3.32)
\end{aligned}
$$

This gives a contradiction with (3.29) and the proof of Theorem 3.1 is then complete.

Now, it remains to provide the **proof of Lemma 3.2**: inequality (3.20) shows that, for ε small enough,

$$3\varepsilon^5||A||^2 \leq 27\tilde{K}^2\varepsilon \leq \tilde{K}\nu. \tag{3.33}$$

Therefore, inequalities (3.27), (3.33) and property (6.5) on the function $\psi_{\nu,\varepsilon}$, imply that

$$\begin{aligned} \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} &\leq A + 3\varepsilon^5 A^2 \\ &\leq \frac{\tilde{K}}{\varepsilon^2}\begin{pmatrix} Id & -Id \\ -Id & Id \end{pmatrix} + 2\tilde{K}\nu Id. \end{aligned} \tag{3.34}$$

This shows that (X,Y) satisfies (3.2), with $(\tilde{\varepsilon},\tilde{\nu})$, as $\tilde{K} \geq 1$.
Let us prove now that $(p_x,-p_y)$ satisfy (3.3), with $(\tilde{\varepsilon},\tilde{\nu})$. By (6.4), we have

$$|p_x+p_y| \leq \tilde{K}\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} + \tilde{K}\nu\varepsilon. \tag{3.35}$$

And therefore, using (3.18), it is clear that, for ε small enough, $|p_x+p_y| \leq 2\tilde{K}^{\frac{3}{2}}\nu$. On another hand, by (6.2), we have for ε small enough,

$$\frac{|\bar{x}-\bar{y}|}{\varepsilon^2} \leq \tilde{K}|p_x| \wedge |p_y| + \tilde{K}^2. \tag{3.36}$$

Inequalities (3.18), (3.35) and (3.36) imply that, for ε small enough, the following inequalities hold

$$\begin{aligned} |p_x+p_y| &\leq \tilde{K}\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} + \tilde{K}\nu\varepsilon, \\ &\leq \tilde{K}^2|\bar{x}-\bar{y}|(|p_x| \wedge |p_y|) + \tilde{K}^3|\bar{x}-\bar{y}| + \tilde{K}\nu\varepsilon, \\ &\leq 2\tilde{K}\nu\varepsilon(1+|p_x| \wedge |p_y|) = \tilde{\nu}\tilde{\varepsilon}(1+|p_x| \wedge |p_y|). \end{aligned} \tag{3.37}$$

This ends the proof of Lemma 3.2.

4 Comparison Result II : the Singular Case

As in the standard case, the following conditions on the function F are naturally adaptated to the L^1-case from the conditions given by G. Barles in [3].

We denote by conditions **(H0-2)**, **(H1-2)** the conditions **(H0)**, **(H1)** where we also suppose that $p \neq 0$.

The conditions **(H2)** and **(H3)** are replaced by the following ones
(H2-2) For any $R > 0$, there exists $m_R \in \mathcal{M}$, such that, for all $0 < \nu \leq 1$

$$F(t,y,r,q,Y) - F(t,x,r,p,X) \leq m_R\Big(t, \nu + |x-y|(1+|p| \vee |q|)\Big), \tag{4.1}$$

for almost all $t \in (0,T)$, for any $x, y \in \overline{\Omega}$, $p, q \in I\!R^N \setminus \{0\}$, $r \in I\!R$, $|r| \le R$, $0 < \varepsilon$ and for any matrices $X, Y \in \mathcal{S}(N)$, satisfying the following properties

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \frac{\nu}{\varepsilon^2} \begin{pmatrix} Id & -Id \\ -Id & Id \end{pmatrix} + \nu Id, \tag{4.2}$$

$$|p-q| \le \min\Big(\varepsilon(|p| \wedge |q|), \nu\Big), \tag{4.3}$$

$$|x-y| \le \nu\varepsilon. \tag{4.4}$$

(H3-2) For every $R > 0$ and $\alpha > 0$, there exists $g_R^\alpha \in \mathcal{M}$, such that

$$|F(t,x,u,p,X) - F(t,x,v,p,X)| \le g_R^\alpha(t, |u-v|), \tag{4.5}$$

for almost every $t \in (0,T)$, for every $u, v \in I\!R$, $x \in \overline{\Omega}$, $p \in I\!R^N \setminus \{0\}$, $X \in \mathcal{S}(N)$ with $|u| \vee |v| \vee |p| \vee ||X|| \le R$ and $|p| \ge \alpha$.

We also need some information on the possible singularity of F at $p = 0$. It is the object of the following condition on F which corresponds, in the case where F is continuous in time, to the assumption (ii) of Proposition 2.3.

(H2-3) There exists a function $f : (0,T) \times \overline{\Omega} \times I\!R \to I\!R$, such that, for every $R > 0$, there exists $k_R \in \mathcal{M}$, with

$$|F(t,x,r,p,X) - f(t,x,r)| \le k_R(t, |p| + ||X||), \tag{4.6}$$

for almost all $t \in (0,T)$, for every $|r| \le R$, $p \in I\!R^N \setminus \{0\}$ and $X \in \mathcal{S}(N)$.

The function f satisfies the following property,
for every $R > 0$, there exists $f_R \in \mathcal{M}$, such that, for almost every $t \in (0,T)$, for every $x, y \in \overline{\Omega}$ and $r, z \in I\!R$, with $|r| \vee |z| \le R$, one has

$$|f(t,x,r) - f(t,y,z)| \le f_R\big(t, |x-y| + |r-z|\big). \tag{4.7}$$

For the function L, we modify the above assumptions as follows : L is independent of $r \in I\!R$ and homogeneous of degree 1 in p and we say that L satisfies **(H4-2)**, **(H5-2)**, **(H6-2)** respectively if it satisfies **(H4)**, **(H5)**, **(H6)** with ν_R, C_R, h_R independent of R (and dropping, of course, the $|u-v|$ term in the left-hand side of the inequality in **(H5)**).

The result is the following.

Theorem 4.1 (comparison result in the singular case)
*Assume that u, $v : \overline{\Omega}_T \to I\!R$ are respectively a bounded upper semicontinuous subsolution and a bounded lower semicontinuous supersolution of (1.1)-(1.2) in Ω_T. If Ω satisfies **(H7)**, F, L satisfy conditions **(H0-2)-(H6-2)** and and condition **(H2-3)** and if $u(0,.) \le v(0,.)$ in $\overline{\Omega}$, then*

$$u \le v \quad \text{in} \quad \overline{\Omega}_T. \tag{4.8}$$

As in the case of Theorem 3.1, we point out easy consequence of Theorem 4.1. Again, if $F \in C(\Gamma_T^*)$ satisfies the conditions given in [3] and L conditions **(H4-2)**-**(H6-2)**, then Theorem 4.1 applies and we still notice that **(H6-2)** is a weaker requirement on L than in [3] where it was supposed to be locally Lipschitz continuous in time.

We come back to the geometric equation (1.6): the assumptions of Theorem 4.1 hold for (1.6) if $a := \sigma\sigma^T$ with σ satisfying (3.7) with $g \in L^2(0,T)$ while b has to satisfy it with $g \in L^1(0,T)$.

As in the proof of Theorem 3.1, we are going to regularize F but we have this time to take care of the singularity for $p = 0$. The following lemma is the analogue of Lemma 3.1.

Lemma 4.1 *Assume that F satisfies **(H0-2)**, **(H2-2)** and **(H2-3)**. For every compact subset K of Γ, then (3.8)-(3.9)-(3.10) with K replaced by K^*. Moreover, if f is the function given in condition **(H2-3)**, then for every $R > 0$, we have*

$$\sup_{z \in Q_R} |f(.,z)| \in L^1(0,T),$$

where we have set $Q_R = \overline{\Omega} \times [-R, +R]$.

Proof of Theorem 4.1: The proof of Theorem 4.1 follows essentially from the same ideas as the proof of Theorem 3.1 and we use the same notations, in particular for the regularization procedure; it is based on the Lemmas 5.1, 5.2 and 6.1.
0. First, since u and v are bounded, we can set $R := \max(||u||_\infty, ||v||_\infty)$ and drop any dependence in R in all the modulus and functions appearing in assumptions **(H0-2)-(H3-2)**. Moreover, by Proposition 2.5, we can assume w.l.o.g. that $\gamma_R \equiv 0$ in $(0,T)$ in condition **(H1-2)**.
1. We argue by contradiction by assuming that

$$\max_{\overline{\Omega}_T} (u - v) = M > 0. \tag{4.9}$$

2. For $\varepsilon > 0$, we consider the following compact subsets of Γ^* and Γ respectively,

$$\begin{aligned} J_\varepsilon &= \{(x,r,p,X) \in \Gamma,\ |r| \le R+1,\ \varepsilon^6 \le |p| \le \varepsilon^{-1},\ ||X|| \le \varepsilon^{-4}\}, &(4.10)\\ K_\varepsilon &= \{(x,r,p,X) \in \Gamma,\ |r| \le R+1,\ |p| + ||X|| \le \varepsilon\}. &(4.11)\end{aligned}$$

For $n \in I\!N$, $\varepsilon > 0$, for almost every $t \in (0,T)$, we set

$$k_\varepsilon(t) = -k_{R+1}(t,\varepsilon), \quad c_n(t) = -\sup_{z \in K_{R+1}} |f(t,z) - f_n(t,z)|,$$

$$b_n^\varepsilon(t) = -\sup_{\xi \in J_\varepsilon} |F(t,\xi) - F_n(t,\xi)|. \tag{4.12}$$

By standard arguments, one shows that, for every $n \in I\!N$ and $\varepsilon > 0$, $c_n, b_n^\varepsilon \in L^1(0,T)$, with for every $\varepsilon > 0$

$$||c_n||_{L^1(0,T)} \to 0, \quad ||b_n^\varepsilon||_{L^1(0,T)} \to 0, \quad \text{as} \quad n \to +\infty. \tag{4.13}$$

Therefore, for any $\varepsilon > 0$, we can choose $n_\varepsilon \in I\!N$, such that

$$||c_{n_\varepsilon}||_{L^1(0,T)} + ||b_{n_\varepsilon}^\varepsilon||_{L^1(0,T)} \le \varepsilon. \tag{4.14}$$

If we set $F_\varepsilon = F_{n_\varepsilon}$ where F_{n_ε} is the regularization in time of F by convolution as in the step 1 of Theorem 3.1, then it is not difficult to show that F_ε is continuous on J_ε and can be extended as a continuous function on Γ_T. In the sequel we set, for $\varepsilon > 0$, $f_\varepsilon = f_{n_\varepsilon}$ and $d_\varepsilon = b^\varepsilon_{n_\varepsilon} + c_{n_\varepsilon} + k_\varepsilon$. By Lemma 4.1, we have for every $\varepsilon > 0$, $f_\varepsilon \in C(I\!R \times \overline{\Omega} \times I\!R)$. As $k_{R+1} \in \mathcal{M}$, $||k_\varepsilon||_{L^1(0,T)} \to 0$, as $\varepsilon \to 0$ and combining it with (4.14), we get

$$||d_\varepsilon||_{L^1(0,T)} \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{4.15}$$

It is clear, using that $d_\varepsilon \le b^\varepsilon_{n_\varepsilon}$ a.e. in $(0,T)$, that for almost every $t \in (0,T)$ and for every $\xi \in J_\varepsilon$, one has

$$d_\varepsilon(t) + F_\varepsilon(t,\xi) \le F(t,\xi), \quad -d_\varepsilon(t) + F_\varepsilon(t,\xi) \ge F(t,\xi). \tag{4.16}$$

Then, condition **(H2-3)** on F, implies that for every $\xi = (x,r,p,X) \in K^*_\varepsilon$, for almost every $t \in (0,T)$,

$$|F(t,\xi) - f(t,x,r)| \le k_{R+1}(t,\varepsilon) = -k_\varepsilon(t).$$

Therefore, using that $d_\varepsilon \le c_{n_\varepsilon} + k_\varepsilon$ a.e. in $(0,T)$, the following inequalities hold, for almost every $t \in (0,T)$ and for every $\xi = (x,r,p,X) \in K^*_\varepsilon$,

$$d_\varepsilon(t) + f_\varepsilon(t,x,r) \le F(t,\xi), \quad -d_\varepsilon(t) + f_\varepsilon(t,x,r) \ge F(t,\xi). \tag{4.17}$$

Now, as we have supposed $\gamma \equiv 0$ in **(H1-2)**, for every $\varepsilon > 0$, $(t,x,p,X) \in I\!R \times \overline{\Omega} \times I\!R^N \setminus \{0\} \times \mathcal{S}(N)$, $r,z \in I\!R$, with $r \le z$, the following inequality holds

$$F_\varepsilon(t,x,r,p,X) \le F_\varepsilon(t,x,z,p,X). \tag{4.18}$$

3. For $0 < \varepsilon \le \nu \le 1$, for almost every $t \in (0,T)$, we set $m_\nu(t) = m_R(t,2\nu) + f_R(t,\nu)$ and $m_{\nu,\varepsilon}(t) = m_{\nu,n_\varepsilon}(t)$. As $m_R, f_R \in \mathcal{H}$, it is clear that, $m_{\nu,\varepsilon} \in C(I\!R)$. We define the function $\Phi_{\nu,\varepsilon} \in C^2_s$, for every $(t,x,y) \in I\!R \times I\!R^N \times I\!R^N$, by

$$\Phi_{\nu,\varepsilon}(t,x,y) = \Big(\psi_{\nu,\varepsilon}(t,x,y) + 2\tilde{K}\nu\varepsilon\Big)^6 + \nu\int_0^t \tilde{h}(s)ds + \int_0^t m_{\nu,\varepsilon}(s)ds + \nu t.$$

Then, we consider the function defined, for $t \in [0,T]$ and $(x,y) \in \overline{\Omega} \times \overline{\Omega}$, by

$$\Psi_{\nu,\varepsilon}(t,x,y) = u(t,x) + \int_0^t d_\varepsilon(s)ds - \Big(v(t,y) - \int_0^t d_\varepsilon(s)ds\Big) - \Phi_{\nu,\varepsilon}(t,x,y). \tag{4.19}$$

As $\Psi_{\nu,\varepsilon} \in USC([0,T] \times \overline{\Omega} \times \overline{\Omega})$, it achieves its maximum over $[0,T] \times \overline{\Omega} \times \overline{\Omega}$ at a point $(t_{\nu,\varepsilon}, x_{\nu,\varepsilon}, y_{\nu,\varepsilon})$. To simplify, we set $(\bar{t},\bar{x},\bar{y}) = (t_{\nu,\varepsilon}, x_{\nu,\varepsilon}, y_{\nu,\varepsilon})$. In the sequel, $q_x = D_x\Phi_{\nu,\varepsilon}(\bar{t},\bar{x},\bar{y})$, $q_y = D_y\Phi_{\nu,\varepsilon}(\bar{t},\bar{x},\bar{y})$ and $A = D^2\Phi_{\nu,\varepsilon}(\bar{t},\bar{x},\bar{y})$.
4. Now we use Lemma 6.1. Since $m_R, f_R \in \mathcal{M}$, we can choose $\nu > 0$ small enough, such that

$$||m_\nu||_{L^1(0,T)} + \nu||\tilde{h}||_{L^1(0,T)} + \nu T \le \frac{M}{2}.$$

With this choice of ν, using again standard arguments, we have

$$\frac{|\bar{x}-\bar{y}|}{\varepsilon} \to 0 \quad \text{as} \quad \varepsilon \to 0, \tag{4.20}$$

$$u(\bar{t},\bar{x}) - v(\bar{t},\bar{y}) > 0, \quad \bar{t} > 0 \qquad \text{for} \quad \varepsilon \quad \text{small enough}. \tag{4.21}$$

5. Easy computations gives $q_x = \beta_{\nu,\varepsilon} D_x\psi_{\nu,\varepsilon}(\bar t,\bar x,\bar y)$ and $q_y = \beta_{\nu,\varepsilon} D_y\psi_{\nu,\varepsilon}(\bar t,\bar x,\bar y)$ with $\beta_{\nu,\varepsilon} = g_{\nu\varepsilon}(\bar t,\bar x,\bar y)$ where $g_{\nu\varepsilon}$ is given by $g_{\nu\varepsilon}(t,x,y) = 6\left(\psi_{\nu,\varepsilon}(t,x,y) + 2\tilde K\nu\varepsilon\right)^5$. Property (6.1) on F shows that $0 < \beta_{\nu\varepsilon}$, for every ε small enough. Therefore, (4.20), properties (6.9), (6.10) and the fact that L is homogeneous of degree 1 in p, show that, for ε small enough

$$L(\bar t,\bar x,u(\bar t,\bar x),q_x) \;>\; 0 \quad \text{if} \quad \bar x\in\partial\Omega, \tag{4.22}$$
$$L(\bar t,\bar y,v(\bar t,\bar y),-q_y) \;<\; 0 \quad \text{if} \quad \bar y\in\partial\Omega. \tag{4.23}$$

For almost every $t\in I\!R$ and for every $(x,y)\in I\!R^N\times I\!R^N$, we have

$$\frac{\partial\Phi_{\nu\varepsilon}}{\partial t}(t,x,y) = g_{\nu\varepsilon}(t,x,y)\;\frac{\partial\psi_{\nu\varepsilon}}{\partial t}(t,x,y) + \nu\tilde h(t) + m_{\nu,\varepsilon}(t) + \nu, \tag{4.24}$$

Using (4.20), for ε small enough $|\bar x-\bar y|\le \dfrac{\nu\varepsilon}{2}$, and therefore if $(x,y)\in B_{\frac{\nu\varepsilon}{2},\overline{\Omega}}(\bar x,\bar y)$, then $|x-y|\le 2\nu\varepsilon$. This shows at first, by property (6.1), that for ε small enough, for every $t\in I\!R$, $(x,y)\in B_{\frac{\nu\varepsilon}{2},\overline{\Omega}}(\bar x,\bar y)$, the following inequalities hold

$$0 < g_{\nu\varepsilon}(t,x,y)\le 1. \tag{4.25}$$

And then, (4.24), (4.25) and property (6.6), show that

$$\frac{\partial\Phi_{\nu\varepsilon}}{\partial t}(.,x,y)\ge m_{\nu,\varepsilon}(.)+\nu, \quad \text{in} \quad \mathcal{D}'(0,2T), \quad \forall (x,y)\in B_{\frac{\nu\varepsilon}{2},\overline{\Omega}}(\bar x,\bar y). \tag{4.26}$$

Now, we are going to consider two cases, whether $|\bar x-\bar y|\le 3\tilde K^2\nu\varepsilon^2$ along a subsequence or not.

6. We first assume that $|\bar x-\bar y|\le 3\tilde K^2\nu\varepsilon^2$ along a subsequence and, to simplify the exposure, we assume that it is true for every $0<\varepsilon\le\nu\le 1$. In the sequel, we set $A_x = D_x^2\Phi_{\nu,\varepsilon}(\bar t,\bar x,\bar y)$ and $A_y = D_y^2\Phi_{\nu,\varepsilon}(\bar t,\bar x,\bar y)$.

By the estimates on the test-function (cf. (6.1)) and using (4.17), it is easy to show that, for $\varepsilon>0$ small enough, for almost every $t\in(0,T)$, for every $\xi_1=(x_1,r_1,p_1,X_1)$, $\xi_2=(x_2,r_2,p_2,X_2)\in\Gamma^*$ in some neighborhood of $(\bar x,u(\bar t,\bar x),q_x,A_x)$ and $(\bar y,v(\bar t,\bar y),-q_y,-A_y)$ respectively, the following inequalities hold,

$$\begin{aligned} d_\varepsilon(t)+f_\varepsilon\Big(t,x_1,r_1+\big(v(\bar t,\bar y)-u(\bar t,\bar x)\big)\Big) &\le F(t,\xi_1),\\ -d_\varepsilon(t)+f_\varepsilon(t,x_2,r_2) &\ge F(t,\xi_2). \end{aligned} \tag{4.27}$$

In the sequel, we define $G_\varepsilon,\tilde G_\varepsilon\in C(\Gamma_T)$, for $(t,\xi)=(t,x,r,p,X)\in\Gamma_T$, by

$$G_\varepsilon(t,\xi) = f_\varepsilon\Big(t,x,r+\big(v(\bar t,\bar y)-u(\bar t,\bar x)\big)\Big), \quad \tilde G_\varepsilon(t,\xi)=f_\varepsilon(t,x,r),$$

for every $t\in[0,T]$ and $\xi=(x,r,p,X)\in\Gamma$. Inequalities (4.21), (4.22), (4.23) and (4.26) show that we are in position to use Lemma 5.2, with $b_1=-b_2=d_\varepsilon$, $\varphi=\Phi_{\nu,\varepsilon}$, $G_1=G_\varepsilon$,

$G_2 = \tilde{G}_\varepsilon$ and $\vartheta(t,x,y) = m_{\nu,\varepsilon}(t) + \nu$, for every $(t,x,y) \in I\!R \times I\!R^N \times I\!R^N$. Therefore, we obtain

$$m_{\nu,\varepsilon}(\bar{t}) + \nu \leq \mathcal{A}^1_\varepsilon = f_\varepsilon(\bar{t},\bar{y},v(\bar{t},\bar{y})) - f_\varepsilon(\bar{t},\bar{x},v(\bar{t},\bar{y})). \tag{4.28}$$

Then, the properties of f and (4.20) show that, for ε small enough,

$$\mathcal{A}^1_\varepsilon \leq f_{n_\varepsilon,R}(\bar{t},|\bar{x}-\bar{y}|) \leq f_{R,n_\varepsilon}(\bar{t},\nu) \leq m_{\nu,\varepsilon}(\bar{t}).$$

Which is a contradiction with (4.28).

7. Now we assume that $|\bar{x}-\bar{y}| \geq 3\tilde{K}^2\nu\varepsilon^2$ along a subsequence, and to simplify we assume that it is true for every $0 < \varepsilon \leq \nu \leq 1$.

Using again the properties of the test-function, we have, for $\varepsilon > 0$ small enough,

$$||A|| < \varepsilon^{-2}, \quad |q_x| \wedge |q_y| > 0. \tag{4.29}$$

Moreover, for almost every $t \in (0,T)$, for every $(X,Y) \in (\mathcal{S}(N))^2$, with $||X|| \vee ||Y|| \leq \varepsilon^{-4}$, we have

$$\begin{aligned} d_\varepsilon(t) + F_\varepsilon(t,x,r,p,X) &\leq F(t,x,r,p,X), \\ -d_\varepsilon(t) + F_\varepsilon(t,y,z,q,Y) &\geq F(t,y,z,q,Y), \end{aligned} \tag{4.30}$$

for every (x,r,p), (y,z,q) in Σ, in some neighborhood if $(\bar{x},u(\bar{t},\bar{x}),q_x)$ and $(\bar{y},v(\bar{t},\bar{y}),-q_y)$ respectively.

First of all, by (4.29), for ε small enough, $3||A|| < \varepsilon^{-4}$. Then, inequalities (4.21), (4.22), (4.23), (4.26) and the above properties show that we are in position to use Lemma 5.1, with $\Delta = \varepsilon^{-4}$, $b_1 = -b_2 = d_\varepsilon$, $G_1 = G_2 = F_\varepsilon$ and $\varphi = \Phi_{\nu,\varepsilon}$, $\vartheta(t,x,y) = m_{\nu,\varepsilon}(t) + \nu$, for every $(t,x,y) \in I\!R \times I\!R^N \times I\!R^N$. Therefore, there exists $(a,b) \in I\!R^2$, $(X,Y) \in (\mathcal{S}(N))^2$, such that

$$(a,q_x,X) \in \overline{\mathcal{P}}^{2,+}_{\Omega_T}\left(u(\bar{t},\bar{x}) + \int_0^{\bar{t}} d_\varepsilon(s)ds\right), \tag{4.31}$$

$$(b,-q_y,-Y) \in \overline{\mathcal{P}}^{2,-}_{\Omega_T}\left(v(\bar{t},\bar{y}) - \int_0^{\bar{t}} d_\varepsilon(s)ds\right), \tag{4.32}$$

$$a - b \geq m_{\nu,\varepsilon}(\bar{t}) + \nu, \tag{4.33}$$

$$-\left(\frac{1}{3\varepsilon^4} + ||A||\right) Id \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + 3\varepsilon^4 A^2. \tag{4.34}$$

Using Lemma 6.1 in Section 6, (4.34) and, again, the fact that for ε small enough, $3||A|| < \varepsilon^{-4}$, we get

$$||X|| \vee ||Y|| \leq ||A|| + \frac{1}{3\varepsilon^4} < \frac{2}{3\varepsilon^4}. \tag{4.35}$$

But we also have $\varepsilon^6 < |q_x| \wedge |q_y| \leq |q_x| \vee |q_y| < \varepsilon^{-1}$ and, using (4.16), we deduce that, for $\varepsilon > 0$, small enough, for almost every $t \in (0,T)$, for every $\xi_1, \xi_2 \in \Gamma^*$, in some neighborhood of $(\bar{x},u(\bar{t},\bar{x}),q_x,X)$ and $(\bar{y},v(\bar{t},\bar{y}),-q_y,Y)$ respectively, we have

$$d_\varepsilon(t) + F_\varepsilon(t,\xi_1) \leq F(t,\xi_1), \quad -d_\varepsilon(t) + F_\varepsilon(t,\xi_2) \geq F(t,\xi_2).$$

8. As u and v are respectively subsolution and supersolution of (1.1)-(1.2) in Ω_T, (4.21), (4.22), (4.23), (4.31), (4.32) and the above properties show that

$$a + F_\varepsilon(\bar t, \bar x, u(\bar t, \bar x), q_x, X) \le 0, \quad b + F_\varepsilon(\bar t, \bar y, v(\bar t, \bar y), -q_y, Y) \ge 0.$$

(We have used in the preceding inequality that $|q_x|, |q_y| > 0$ by (4.29) and that $F_\varepsilon \in C(\Gamma_T^*)$). This implies, using (4.18) and (4.33), that the following inequality holds

$$m_{\nu,\varepsilon}(\bar t) + \nu \le \mathcal{A}_\varepsilon^2 = F_\varepsilon(\bar t, \bar y, v(\bar t, \bar y), -q_y, Y) - F_\varepsilon(\bar t, \bar x, v(\bar t, \bar y), q_x, X). \tag{4.36}$$

9. An estimate of $\mathcal{A}_\varepsilon^2$. To get such an estimate, we are going to use condition **(H2-2)** on F_ε, that is the reason why, we need the following lemma.

Lemma 4.2 *For ε small enough, (X, Y), $(q_x, -q_y)$ and $(\bar x, \bar y)$ satisfy (4.2), (4.3) and (4.4) with (ε, ν), and moreover*

$$|q_x| \vee |q_y| \le \varepsilon^{-1}. \tag{4.37}$$

We postpone the proof of this lemma at the end of this subsection and end the proof of Theorem 4.1. By (4.20) and (4.37), we get for ε small enough

$$\begin{aligned} \mathcal{A}_\varepsilon^2 &\le m_R^{n_\varepsilon}\big(\bar t, \nu + |\bar x - \bar y|(1 + |q_x| \vee |q_y|)\big), \\ &\le m_R^{n_\varepsilon}\big(\bar t, \nu + \frac{2|\bar x - \bar y|}{\varepsilon}\big), \\ &\le m_R^{n_\varepsilon}(\bar t, 2\nu) \le m_{\nu,\varepsilon}(\bar t). \end{aligned}$$

This gives a contradiction with (4.36) and therefore the proof of Theorem 4.1 is complete.

To conclude, we provide the **proof of Lemma 4.2:** we first introduce some notations and give some estimates, which will be useful. For every $0 < \varepsilon \le \nu \le 1$, we set $p_x = D_x\psi_{\nu,\varepsilon}(\bar t, \bar x, \bar y)$, $p_y = D_y\psi_{\nu,\varepsilon}(\bar t, \bar x, \bar y)$, $C_{\nu,\varepsilon} = D^2\psi_{\nu,\varepsilon}(\bar t, \bar x, \bar y)$ and

$$\alpha_{\nu\varepsilon} = \psi_{\nu,\varepsilon}(\bar t, \bar x, \bar y) + 2\tilde K \nu\varepsilon, \quad \beta_{\nu\varepsilon} = 6\ \alpha_{\nu\varepsilon}^5, \quad \gamma_{\nu\varepsilon} = 30\ \alpha_{\nu\varepsilon}^4. \tag{4.38}$$

We clearly have

$$q_x = \beta_{\nu\varepsilon}\ p_x, \quad q_y = \beta_{\nu\varepsilon}\ p_y, \quad A = \beta_{\nu\varepsilon}\ C_{\nu,\varepsilon} + \gamma_{\nu\varepsilon}\ B, \tag{4.39}$$

$$\text{where} \qquad B = \begin{pmatrix} p_x \otimes p_x & p_x \otimes p_y \\ p_y \otimes p_x & p_y \otimes p_y \end{pmatrix}. \tag{4.40}$$

It is not difficult to show, that

$$||B|| \le 2(|p_x| \vee |p_y|)^2. \tag{4.41}$$

Lemma 6.1, inequalities (4.20), (4.39) and (4.41) show the following assertions

$$\alpha_{\nu\varepsilon}, \beta_{\nu\varepsilon}, \gamma_{\nu\varepsilon} \to 0, \quad \varepsilon\,(|p_x| \vee |p_y|) \to 0, \quad |p_x + p_y| \to 0, \quad \text{as} \quad \varepsilon \to 0, \tag{4.42}$$

$$||C_{\nu\varepsilon}|| \vee ||B|| = O(\varepsilon^{-2}), \quad ||A|| \le (\beta_{\nu\varepsilon} + \gamma_{\nu\varepsilon})\ O(\varepsilon^{-2}). \tag{4.43}$$

We have already proved (4.37), let us show that (X,Y), $(q_x,-q_y)$ and $(\bar x,\bar y)$ satisfy (4.2), (4.3) and (4.4), if we choose ε small enough. By (4.20), it is obvious that $(\bar x,\bar y)$ satisfies (4.4), for ε small enough. Then, inequalities (4.39), (4.42) and (4.42) show that, $|q_x+q_y|\le\nu$, for ε small enough. Therefore, to show that $(q_x,-q_y)$ satisfies (4.3), we only have to prove that $|q_x+q_y|\le\varepsilon|q_x|\wedge|q_y|$, which is equivalent to prove that $|p_x+p_y|\le\varepsilon|p_x|\wedge|p_y|$, by (4.37). By property (6.2), we have

$$\frac{|\bar x-\bar y|}{\varepsilon^2}\le\tilde K(|p_x|\wedge|p_y|)+\tilde K^2\nu\varepsilon.$$

Combining it with property (6.4) and using (4.20), we obtain

$$\begin{aligned}
|p_x+p_y| &\le \tilde K^2|\bar x-\bar y|\,(|p_x|\wedge|p_y|)+\tilde K^3\nu\varepsilon|\bar x-\bar y|+\tilde K\nu\varepsilon,\\
&\le (|p_x|\wedge|p_y|)\left(\tilde K^2\ |\bar x-\bar y|+\frac{\tilde K^2}{2}\varepsilon|\bar x-\bar y|+\frac{\varepsilon}{2}\right),\\
&\le \varepsilon\ (|p_x|\wedge|p_y|),\quad \text{for}\quad \varepsilon\quad \text{small enough.}
\end{aligned}$$

Let us show now that (X,Y) satisfies (4.2). First of all, by (4.42) and (4.43), for ε small enough, $6\varepsilon^4||A||^2\le\nu$, and therefore, by (4.34), the following inequality holds

$$\begin{pmatrix} X & 0\\ 0 & -Y\end{pmatrix}\le A+\frac{\nu}{2}Id. \tag{4.44}$$

To conclude, it is enough to show that for ε small enough, we have

$$A\le\frac{\nu}{\varepsilon^2}\begin{pmatrix} Id & -Id\\ -Id & Id\end{pmatrix}+\frac{\nu}{2}Id. \tag{4.45}$$

It is not very difficult to prove that the matrix B defined in (4.40) satisfies

$$\begin{aligned}
B &\le \left(|p_x|^2+|p_y|^2\right)\begin{pmatrix} Id & -Id\\ -Id & Id\end{pmatrix}+|p_x+p_y|^2 Id,\\
&\le \frac{\nu}{\varepsilon^2}\begin{pmatrix} Id & -Id\\ -Id & Id\end{pmatrix}+Id,
\end{aligned}\tag{4.46}$$

for ε small enough, using (4.42). Therefore, (4.39), (4.46), property (6.5) and finally (4.42), show that, for ε small enough,

$$\begin{aligned}
A &= \beta_{\nu\varepsilon}D^2\psi_{\nu\varepsilon}(\bar t,\bar x,\bar y)+\gamma_{\nu\varepsilon}B,\\
&\le \frac{\nu}{\varepsilon^2}\left(\gamma_{\nu\varepsilon}+\frac{\tilde K\beta_{\nu\varepsilon}}{\nu}\right)\begin{pmatrix} Id & -Id\\ -Id & Id\end{pmatrix}+\left(\tilde K\beta_{\nu\varepsilon}\nu+\gamma_{\nu\varepsilon}\right)Id,\\
&\le \frac{\nu}{\varepsilon^2}\begin{pmatrix} Id & -Id\\ -Id & Id\end{pmatrix}+\frac{\nu}{2}Id.
\end{aligned}$$

The proof of Lemma 4.2 is then complete.

5 The Maximum Principle for Viscosity Solutions of the Neumann Problem in the L^1-case

In this section, if $m \geq 1$, $\mathcal{O} \subset I\!R^m$, $z \in \mathcal{O}$ and $r > 0$, we set $B_{r,\mathcal{O}}(z) = B_r(z) \cap \mathcal{O}$ and $\overline{B}_{r,\mathcal{O}}(z) = \overline{B}_r(z) \cap \mathcal{O}$. In the following lemmas, $u \in USC(\Omega_T)$, $v \in LSC(\Omega_T)$ are respectively subsolution and supersolution of (1.1)-(1.2) in Ω_T. We suppose moreover that u and v are bounded over Ω_T. For $z \in \overline{\Omega} \times \overline{\Omega}$ and $r > 0$, we set, to simplify, $B_{r,\overline{\Omega}}(z) = B_{r,\overline{\Omega}\times\overline{\Omega}}(z)$ and $\overline{B}_{r,\overline{\Omega}}(z) = \overline{B}_{r,\overline{\Omega}\times\overline{\Omega}}(z)$.
The following lemma is largely inspired of the Maximum Principle for viscosity solutions proved by Crandall, Ishii and Lions in [9]. It is a key result to show the comparison results either in the standard and in the singular cases.

Lemma 5.1 *Let $(\bar{t}, \bar{x}, \bar{y}) \in (0,T] \times \overline{\Omega} \times \overline{\Omega}$, $b_1, b_2 \in L^1(0,T)$ and $\varphi \in C^2_s$ be such that $(\bar{t}, \bar{x}, \bar{y})$ is a maximum point of*

$$\Psi(t,x,y) = u(t,x) + \int_0^t b_1(s)ds - \left(v(t,y) + \int_0^t b_2(s)ds\right) - \varphi(t,x,y), \tag{5.1}$$

over $(0,T] \times \overline{\Omega} \times \overline{\Omega}$. If $A = D^2\varphi(\bar{t}, \bar{x}, \bar{y})$, $p_x = D_x\varphi(\bar{t}, \bar{x}, \bar{y})$, $q_y = -D_y\varphi(\bar{t}, \bar{x}, \bar{y})$, we assume that there exists $G_1, G_2 \in C(\Gamma_T)$ and $\Delta > 3||A||$, such that, for every $(X,Y) \in (\mathcal{S}(N))^2$, with $||X|| \vee ||Y|| \leq \Delta$, we have

$$\begin{aligned} b_1(t) + G_1(t,x,r,p,X) &\leq F(t,x,r,p,X), & (5.2)\\ b_2(s) + G_2(s,y,v,q,Y) &\geq F(s,y,v,q,Y), & (5.3) \end{aligned}$$

for almost every $t \in (0,T)$ in some neighborhood of $\bar{t}$ and for every (x,r,p), (y,v,q) in Σ^* in some neighborhood of $(\bar{x}, u(\bar{t},\bar{x}), p_x)$ and $(\bar{y}, v(\bar{t},\bar{y}), q_y)$ respectively. Finally we suppose that there exists $r > 0$, $\vartheta \in C((0,2T) \times B_{r,\overline{\Omega}}(z))$, such that

$$\frac{\partial\varphi}{\partial t}(.,z) \geq \vartheta(.,z) \quad in \quad \mathcal{D}'(0,2T), \quad \forall z \in B_{r,\overline{\Omega}}(z), \tag{5.4}$$

and that we have

$$\begin{aligned} L(\bar{t}, \bar{x}, u(\bar{t},\bar{x}), p_x) > 0 \quad if \quad \bar{x} \in \partial\Omega. & \quad (5.5)\\ L(\bar{t}, \bar{y}, v(\bar{t},\bar{y}), q_y) < 0 \quad if \quad \bar{y} \in \partial\Omega. & \quad (5.6) \end{aligned}$$

Then, there exists $(a,b) \in I\!R^2$ and $(X,Y) \in (\mathcal{S}(N))^2$, such that

$$\begin{aligned} (a, D_x\varphi(\bar{t},\bar{x},\bar{y}), X) &\in \bar{\mathcal{P}}^{2,+}_{\Omega_T}\left(u(\bar{t},\bar{x}) + \int_0^{\bar{t}} b_1(s)ds\right)\\ (b, -D_y\varphi(\bar{t},\bar{x},\bar{y}), Y) &\in \bar{\mathcal{P}}^{2,-}_{\Omega_T}\left(v(\bar{t},\bar{y}) + \int_0^{\bar{t}} b_2(s)ds\right)\\ -\left(\frac{\Delta}{3} + ||A||\right) Id &\leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \frac{3}{\Delta}A^2\\ a - b &\geq \vartheta(\bar{t},\bar{x},\bar{y}). \end{aligned} \tag{5.7}$$

We then give the following lemma, which is used to get the comparison result in the singular case.

Lemma 5.2 *Under the same conditions and with the same notations as in Lemma 5.1 but replacing (5.2)-(5.3) by : there exists $G_1, G_2 \in C(\Gamma_T)$, such that*

$$b_1(t) + G_1(t,\xi_1) \le F(t,\xi_1) \quad \text{and} \quad b_2(t) + G_2(t,\xi_2) \ge F(t,\xi_2), \tag{5.8}$$

for almost every $t \in (0,T)$ in some neighborhood of $\bar{t}$ and for every ξ_1, $\xi_2 \in \Gamma^*$ in some neighborhood of $(\bar{x}, u(\bar{t},\bar{x}), p_x, A_x)$ and $(\bar{y}, v(\bar{t},\bar{y}), q_y, B_y)$ respectively with $A_x = D_x^2\varphi(\bar{t},\bar{x},\bar{y})$, $B_y = -D_y^2\varphi(\bar{t},\bar{x},\bar{y})$. Then, we have

$$\vartheta(\bar{t},\bar{x},\bar{y}) \le G_2(\bar{t},\bar{y},v(\bar{t},\bar{y}),q_y,B_y) - G_1(\bar{t},\bar{x},u(\bar{t},\bar{x}),p_x,A_x). \tag{5.9}$$

We first give the **proof of Lemma 5.2**, the (very technical) proof of Lemma 5.1 is postponed at the end of this section.

1. We first assume that $\varphi \in C^2$ in x and t. We are going to prove that, we have

$$\frac{\partial \varphi}{\partial t}(\bar{t},\bar{x},\bar{y}) \le G_2(\bar{t},\bar{y},v(\bar{t},\bar{y}),q_y,B_y) - G_1(\bar{t},\bar{x},u(\bar{t},\bar{x}),p_x,A_x). \tag{5.10}$$

For $\nu > 0$, we consider the following function Ψ_ν defined for $(t,s,x,y) \in (0,T] \times (0,T] \times \overline{\Omega} \times \overline{\Omega}$, by

$$\Psi_\nu(t,s,x,y) = u(t,x) + \int_0^t b_1(r)dr - \left(v(s,y) + \int_0^s b_2(r)dr\right) - \varphi(t,x,y) - \frac{|s-t|^2}{2\nu} - (t-\bar{t})^2 - |x-\bar{x}|^4 - |y-\bar{y}|^4.$$

Let $r_0 > 0$, small enough, such that the function given by (5.1) achieves its maximum at $(\bar{t},\bar{x},\bar{y})$ over K_{r_0}, where $K_{r_0} = [\bar{t}-r_0, t+r_0] \times \overline{B}_{r_0,\overline{\Omega}}(\bar{x},\bar{y})$, if $\bar{t} < T$ and $K_{r_0} = [\bar{t}-r_0, T] \times \overline{B}_{r_0,\overline{\Omega}}(\bar{x},\bar{y})$, if $\bar{t} = T$. As $\Psi_\nu \in USC((0,T]\times(0,T]\times\overline{\Omega}\times\overline{\Omega})$, it achieves its maximum at $(t_\nu, s_\nu, x_\nu, y_\nu)$ over K_{r_0}. And it is not difficult to show that, as $\nu \to 0$,

$$(t_\nu, s_\nu, x_\nu, y_\nu) \to (\bar{t},\bar{t},\bar{x},\bar{y}), \; u(t_\nu,x_\nu) \to u(\bar{t},\bar{x}), \; v(s_\nu,y_\nu) \to v(\bar{t},\bar{y}). \tag{5.11}$$

If we denote by g_ν the following function, defined for $(t,x) \in \Omega_T$, by

$$g_\nu(t,x) = \varphi(t,x,y_\nu) + \frac{|t-s_\nu|^2}{2\nu} + (t-\bar{t})^2 + |x-\bar{x}|^2$$

and set $p_\nu = Dg_\nu(t_\nu,x_\nu)$ and $X_\nu = D^2 g_\nu(t_\nu,x_\nu)$, it is clear that the function $(t,x) \mapsto u(t,x) + \int_0^t b_1(s)ds - g_\nu(t,x)$ achieves a local maximum over Ω_T at (t_ν,x_ν) and (5.11) shows that, as $\nu \to 0$

$$p_\nu \to p_x \quad \text{and} \quad X_\nu \to A_x. \tag{5.12}$$

By (5.8), (5.11) and (5.12), we have, for ν small enough,

$$b_1(t) + G_1(t,\xi) \le F(t,\xi),$$

for almost every $t \in (0,T)$ in some neighborhood of t_ν and for every $\xi \in \Gamma^*$ in some neighborhood of $\xi_\nu = (x_\nu, u(t_\nu, x_\nu), p_\nu, X_\nu)$. If $\bar{x} \in \Omega$, then by (5.11), for ν small enough, $x_\nu \in \Omega$. Now, if $\bar{x}$ and $x_\nu \in \partial\Omega$, then as $L(\bar{t}, \bar{x}, u(\bar{t}, \bar{x}), p_x) > 0$, using (5.11) and the continuity of L, for ν small enough, we have $L(t_\nu, x_\nu, u(t_\nu, x_\nu), p_\nu) > 0$. As u is a subsolution of (1.1)-(1.2) in Ω_T, then in any case, for ν small enough, the following inequality holds

$$\frac{\partial \varphi}{\partial t}(t_\nu, x_\nu, y_\nu) + \frac{t_\nu - s_\nu}{\nu} + 2(t_\nu - \bar{t}) + G_1(t_\nu, \xi_\nu) \leq 0.$$

Using that the function $(s,y) \mapsto v(s,y) + \int_0^s b_2(r)dr + \varphi(t_\nu, x_\nu, y) + \frac{|s - t_\nu|^2}{2\nu} + |y - \bar{y}|^4$ has a local minimum at (s_ν, y_ν) over Ω_T, that $L(\bar{t}, \bar{y}, v(\bar{t}, \bar{y}), q_y) < 0$ if $\bar{y} \in \partial\Omega$ and that v is a supersolution of (1.1)-(1.2) in Ω_T, we can show similarly that

$$\frac{t_\nu - s_\nu}{\nu} + G_2(s_\nu, y_\nu, v(s_\nu, y_\nu), q_\nu, Y_\nu) \geq 0,$$

where, as $\nu \to 0$, $q_\nu \to q_y$ and $Y_\nu \to B_y$. Therefore, combining these two inequalitites, we have proved, that for ν small enough, the following inequality holds

$$\frac{\partial \varphi}{\partial t}(t_\nu, x_\nu, y_\nu) + 2(t_\nu - \bar{t}) \leq G_2(s_\nu, y_\nu, v(s_\nu, y_\nu), q_\nu, Y_\nu) - G_1(t_\nu, x_\nu, u(t_\nu, x_\nu), p_\nu, X_\nu),$$

which gives (5.10), using (5.11) and letting ν go to zero.

2. Assume now that $\varphi \in C^2_s$ and satisfies (5.4). Let $(\rho_n)_{n\geq 1}$ a real mollifier. For every $n \geq 1$, we define $\varphi_n \in C^2$, $\vartheta_n \in C$, for $z \in \overline{\Omega} \times \overline{\Omega}$, by

$$\varphi_n(., z) = \varphi(., z) * \rho_n, \quad \vartheta_n(., z) = \vartheta(., z) * \rho_n, \qquad \text{in} \quad I\!R.$$

By (5.4), there exists $\delta > 0$, such that, for n large enough

$$\frac{\partial \varphi_n}{\partial t} \geq \vartheta_n, \quad \text{in} \quad [\bar{t} - \delta, \bar{t} + \delta] \times B_{r,\overline{\Omega}}(\bar{x}, \bar{y}). \tag{5.13}$$

And, by classical arguments, we have, as $n \to +\infty$,

$$(\vartheta_n, \varphi_n, D\varphi_n, D^2\varphi_n) \to (\vartheta, \varphi, D\varphi, D^2\varphi), \quad \text{uniformly in} \quad K_{r_0}. \tag{5.14}$$

For every $n \in I\!N$, let $\psi_n \in C^2$, defined for every $(t, x, y) \in (0, T] \times \overline{\Omega} \times \overline{\Omega}$, by

$$\psi_n(t, x, y) = \varphi_n(t, x, y) + (t - \bar{t})^2 + |x - \bar{x}|^4 + |y - \bar{y}|^4.$$

It is not difficult to show, using (5.14), that the function

$$(t, x) \mapsto u(t, x) + \int_0^t b_1(r)dr - \left(v(t, y) + \int_0^t b_2(r)dr\right) - \psi_n(t, x, y),$$

achieves its maximum at (t_n, x_n, y_n) over K_{r_0}, with as $n \to +\infty$

$$\begin{aligned} &(t_n, x_n, y_n) \to (\bar{t}, \bar{x}, \bar{y}), \ u(t_n, x_n) \to u(\bar{t}, \bar{x}), \ v(t_n, y_n) \to v(\bar{t}, \bar{y}), \\ &p_n = D_x\psi_n(t_n, x_n, y_n) \to p_x, \quad q_n = -D_y\psi_n(t_n, x_n, y_n) \to q_y, \\ &X_n = D^2_x\psi_n(t_n, x_n, y_n) \to A_x, \quad Y_n = -D^2_y\psi_n(t_n, x_n, y_n) \to B_y. \end{aligned} \tag{5.15}$$

Therefore, (5.8) is satisfied for almost every $t \in (0,T)$ in some neighborhood of t_n and for every $\xi_1, \xi_2 \in \Gamma^*$ in some neighborhood of $(x_n, u(t_n,x_n), p_n, X_n)$ and $(y_n, v(t_n,y_n), q_n, Y_n)$ respectively. Then, by the preceding step, we obtain that, for n large enough,

$$\frac{\partial \psi_n}{\partial t}(t_n,x_n,y_n) = \frac{\partial \varphi_n}{\partial t}(t_n,x_n,y_n) + 2(t_n - \bar{t}) \\ \leq G_2(t_n,y_n,v(t_n,y_n),q_n,Y_n) - G_1(t_n,x_n,u(t_n,x_n),p_n,X_n). \tag{5.16}$$

Using (5.13), (5.15), (5.16) and finally (5.14), we get (5.9) by letting n tend to infinity. And the proof of Lemma 5.2 is complete.

Now we turn to the **proof of Lemma 5.1** . It relies on the following proposition, which long and very technical proof is postponed. In the sequel, $k \in I\!N$, $k \geq 1$ and for every $i = 1 \dots k$, $N_i \in I\!N$, $N_i \geq 1$. We set $\overline{N} = N_1 + \dots + N_k$.

Proposition 5.1 *Let I a bounded segment of $I\!R$, $\mathcal{O}_i$ a locally compact subset of R^{N_i}, $u_i \in USC(I \times \mathcal{O}_i)$, for every $i = 1 \dots k$. We set $\mathcal{O} = \mathcal{O}_1 \times \dots \times \mathcal{O}_k$ and define the following function for $t \in I$, $x = (x_1, \dots, x_k) \in \mathcal{O}$, by $w(t,x) = u_1(t,x) + \dots + u_k(t,x)$. Let $\varphi \in C^2_s$, $\vartheta \in C$, $r > 0$, $J \supset I$ an open subset of $I\!R$, such that the function $w - \varphi$ has a maximum point at $(\hat{t}, \hat{x})$ over $I \times \mathcal{O}$ and such that*

$$\frac{\partial \varphi(.,x)}{\partial t} \geq \vartheta(.,x) \quad in \quad \mathcal{D}'(J), \quad \forall x \in B_{r,\mathcal{O}}(\hat{x}), \tag{5.17}$$

We set $A = D^2\varphi(\hat{t},\hat{x})$ and we say that $\varepsilon > 0$ satisfies $P(\hat{t},\hat{x},w,\varphi)$ if $\varepsilon||A|| < 1$ and if there exists $r_\varepsilon > 0$ and $C_\varepsilon > 0$ such that, for every $i = 1, \dots, k$

$$\begin{aligned} a_i \leq C_\varepsilon, \qquad & \textit{whenever} \quad (a_i, p_i, X_i) \in \mathcal{P}^{2,+}_{I \times \mathcal{O}_i} u_i(t_i, x_i,) \\ & \textit{with} \quad |t_i - \hat{t}| + |x_i - \hat{x}_i| + |u_i(t_i,x_i) - u_i(\hat{t},\hat{x}_i)| + |p_i - D_{x_i}\varphi(\hat{t},\hat{x})| \leq r_\varepsilon \\ & \textit{and} \quad ||X_i|| \leq \frac{2}{\varepsilon} + ||A||. \end{aligned} \tag{5.18}$$

Then, if ε satisfies $P(\hat{t},\hat{x},w,\varphi)$, there exists $(a_1, \dots, a_k) \in I\!R^k$ and $(X_1, \dots, X_k) \in \mathcal{S}(N_1) \times \dots \times \mathcal{S}(N_k)$, such that

$$(a_i, D_{x_i}\varphi(\hat{t},\hat{x}), X_i) \in \bar{\mathcal{P}}^{2,+}_{I \times \mathcal{O}_i} u_i(\hat{t},\hat{x}_i), \quad \textit{for every} \quad i = 1 \dots k, \tag{5.19}$$

$$-\left(\frac{1}{\varepsilon} + ||A||\right) I \leq \begin{pmatrix} X_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & X_k \end{pmatrix} \leq A + \varepsilon A^2, \tag{5.20}$$

$$a_1 + \dots + a_k \geq \vartheta(\hat{t},\hat{x}). \tag{5.21}$$

Proof of Lemma 5.1: We are going to use Proposition 5.1, with $I = (0,T]$, $J = (0,2T)$, $k = 2$, $N_1 = N_2 = N$, $\mathcal{O}_1 = \mathcal{O}_2 = \overline{\Omega}$, $(\hat{t}, \hat{x_1}, \hat{x_2}) = (\bar{t}, \bar{x}, \bar{y})$, and $u_1(t,x) = u(t,x) + \int_0^t b_1(s)ds$, $u_2(t,x) = -(v(t,x) + \int_0^t b_2(s)ds)$, for every $(t,x) \in \Omega_T$. To show Lemma 5.1, we just have to prove that $\varepsilon = \dfrac{3}{\Delta}$ satisfies $P(\bar{t},(\bar{x},\bar{y}),w,\varphi)$. First of all, by hypothesis

$\Delta > 3||A||$ and therefore $\varepsilon||A|| < 1$. Next, for $(t_1, x_1), (t_2, x_2) \in \Omega_T$, $\bar{r} > 0$, we consider

$$(a, p, X) \in \mathcal{P}^{2,+}_{\Omega_T} u_1(t_1, x_1), \quad (-b, -q, -Y) \in \mathcal{P}^{2,+}_{\Omega_T} u_2(t_2, x_2), \tag{5.22}$$

$$|t_1 - \bar{t}| + |x_1 - \bar{x}| + |u_1(t_1, x_1) - u_1(\bar{t}, \bar{x})| + |p - p_x| \le \bar{r}, \tag{5.23}$$

$$|t_2 - \bar{t}| + |x_2 - \bar{y}| + |u_2(t_2, x_2) - u_2(\bar{t}, \bar{y})| + |q - q_y| \le \bar{r}, \tag{5.24}$$

$$||X|| \vee ||Y|| \le \frac{2\Delta}{3} + ||A||. \tag{5.25}$$

It is worth noticing that $(b, q, Y) \in \mathcal{P}^{2,-}_{\Omega_T}(v(t, x) + \int_0^t b_2(s)ds) = -\mathcal{P}^{2,+}_{\Omega_T} u_2(t_2, x_2)$.

Using again the fact that $\Delta > 3||A||$ and inequality (5.25) imply that $||X|| \vee ||Y|| < \Delta$. Inequalities (5.23) and (5.24) show the following

$$|u(t_1, x_1) - u(\bar{t}, \bar{x})| \le |u_1(t_1, x_1) - u_1(\bar{t}, \bar{x})| + |\int_t^{\bar{t}} b_1(s)ds| \le \bar{r} + |\int_t^{\bar{t}} b_1(s)ds| \to 0,$$

$$|v(t_2, x_2) - v(\bar{t}, \bar{y})| \le |u_2(t_2, x_2) - u_2(\bar{t}, \bar{x})| + |\int_t^{\bar{t}} b_2(s)ds|$$

$$\le \bar{r} + |\int_t^{\bar{t}} b_2(s)ds| \to 0, \tag{5.26}$$

as $\bar{r} \to 0$, using that $b_1, b_2 \in L^1(0, T)$ and $|t - \bar{t}| \le \bar{r}$. Therefore, (5.24), (5.25), (5.26), the fact that $||X|| \vee ||Y|| < \Delta$, show that (5.2) and (5.3) hold, for almost every $t, s \in (0, T)$ in some neighborhood of t_1, t_2 respectively and $\xi, \xi' \in \Gamma^*$ respectively in some neighborhood of $\xi_1 = (x_1, u(t_1, x_1), p, X)$ and $\xi_2 = (x_2, v(t_2, x_2), q, Y)$ respectively.

If $\bar{x}$ (resp. $\bar{y}$) $\in \Omega$, then by (5.23) (resp. (5.24)), x_1 (resp. x_2) $\in \Omega$, for $\bar{r}$ small enough. Now, if $\bar{x}$ and x_1 (resp. $\bar{y}$ and x_2) $\in \partial\Omega$, then by (5.5) (resp. (5.6)), using the continuity of L and (5.23) (resp. (5.24)), we have, for $\bar{r}$ small enough,

$$L(t_1, x_1, u(t_1, x_1), p) > 0 \quad (\text{resp.} \quad L(t_2, x_2, v(t_2, x_2), q) < 0).$$

Therefore, as u (resp. v) is a subsolution (resp. supersolution) of (1.1)- (1.2) in Ω_T, (5.22) and the preceding remarks, we have shown, for $\bar{r}$ small enough,

$$a + G_1(t_1, \xi_1) \le 0 \quad \text{and} \quad b + G_2(t_2, \xi_2) \ge 0.$$

This shows the desired estimates on a and $-b$ since $G_1, G_2 \in C(\Gamma_T)$ and Lemma 5.1 follows from Proposition 5.1.

Now we give the **proof of Proposition 5.1**. It is based on the following result which is largely inspired of the analogous lemma given in [9].

As defined and used in our paper, for $m \ge 1$, Θ a subset of $I\!R^m$, $USC(\Theta)$ consists of the upper semicontinuous functions mapping Θ into $I\!R$; however it is convenient to allow the value $-\infty$, so in the sequel, we define $USC(\Theta)$ as the subset of upper semicontinuous functions over Θ taking values in $I\!R \cup \{-\infty\}$.

Theorem 5.1 *For every $i = 1, \dots, k$, let $u_i \in USC(I\!R \times I\!R^{N_i})$. For $t \in I\!R, x = (x_1, \dots, x_k) \in I\!R^{N_1} \times \dots \times I\!R^{N_k}$, we set $w(t, x) = u_1(t, x_1) + \dots + u_k(t, x_k)$. Assume that $u_i(0, 0) = 0$, for every $i = 1, \dots, k$. Let $A \in \mathcal{S}(\overline{N})$, such that the function*

$$(t, x) \mapsto w(t, x) - \frac{1}{2}Ax \cdot x,$$

has a strict maximum point over $I\!R \times I\!R^{\overline{N}}$ *at* $(0,0)$.

We say that $\varepsilon > 0$ *satisfies* $\bar{P}(w,A)$, *if* $\varepsilon||A|| < 1$ *and if there exists* $r_\varepsilon > 0$ *and* $C_\varepsilon > 0$ *such that, for every* $i = 1 \dots k$,

$$a_i \le C_\varepsilon, \qquad \textit{whenever} \quad (a_i, p_i, X_i) \in \mathcal{P}^{2,+}u_i(t_i, x_i), \tag{5.27}$$

$$\textit{with} \qquad |t_i| + |x_i| + |u_i(t_i,x_i)| + |p_i| \le r_\varepsilon, \tag{5.28}$$

$$\textit{and} \quad ||X_i|| \le \frac{2}{\varepsilon} + ||A||. \tag{5.29}$$

Then, if ε *satisfies* $\bar{P}(w,A)$, *there exists* $(a_1, \dots, a_k) \in I\!R^k$ *and* $(X_1, \dots, X_k)$ $\in \mathcal{S}(N_1) \times \dots \times \mathcal{S}(N_k)$, *such that*

$$(a_i, 0, X_i) \in \bar{\mathcal{P}}^{2,+}u_i(0,0), \quad \textit{for every} \quad i = 1 \dots k, \tag{5.30}$$

$$-\left(\frac{1}{\varepsilon} + ||A||\right) I \le \begin{pmatrix} X_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & X_k \end{pmatrix} \le A + \varepsilon A^2, \tag{5.31}$$

$$a_1 + \dots + a_k = 0. \tag{5.32}$$

Proof of Proposition 5.1 : the proof of Proposition 5.1 is divided into two steps. In the first one, we suppose that $\varphi \in C^2$ and use Taylor's formula while, in the second one, we regularize the function φ in order to be in position to use the first step.

First Step. Suppose that $\varphi \in C^2$. We are going to prove that there exists $(a_1, \dots, a_k) \in I\!R^k$ and $(X_1, \dots, X_k) \in \mathcal{S}(N_1) \times \dots \times \mathcal{S}(N_k)$, satisfying (5.19), (5.20) and

$$a_1 + \dots + a_k = \frac{\partial \varphi}{\partial t}(\bar{t}, \bar{x}, \bar{y}). \tag{5.33}$$

1. In order to simplify matters, we make some reductions. We may as well assume that $I = I\!R$, for every $i = 1 \dots k$, $\mathcal{O}_i = R^{N_i}$ and that $(\hat{t}, \hat{x}) = (0,0)$. Indeed, let $\tilde{I}$ a compact neighborhood of $\hat{t}$ in I and for every $i = 1 \dots k$, K_i a compact neighborhood of $\hat{x_i}$ in $\mathcal{O}_i$. For $i = 1 \dots k$, we denote by $\tilde{u_i}$ the function which is equal to u_i in $\tilde{I} \times K_i$ and to $-\infty$ otherwise. For every $i = 1 \dots k$, the closeness of $\tilde{I} \times K_i$ guarantees that $\tilde{u_i} \in USC(I\!R \times I\!R^{N_i})$. Now, it is not difficult to show, that for $(t, x_i) \in \tilde{I} \times K_i$, one has $\overline{\mathcal{P}}^{2,+}_{I \times \mathcal{O}_i} u_i(t,x_i) = \overline{\mathcal{P}}^{2,+} \tilde{u_i}(t,x_i)$, as $u_i(t,x_i) > -\infty$. It is clear that, if we set $\tilde{w} = \tilde{u_1} + \dots + \tilde{u_k}$, the function $\tilde{w} - \varphi$ has a maximum point over $I\!R \times I\!R^{\overline{N}}$ at $(\hat{t}, \hat{x})$. One can then easily ckeck that ε satisfies $P(\hat{t}, \hat{x}, \hat{w}, \hat{\varphi})$, (where we have replaced $\mathcal{P}^{2,+}_{I \times \mathcal{O}_i}$ by $\mathcal{P}^{2,+}$) is equivalent to ε satisfies $P(\hat{t}, \hat{x}, w, \varphi)$. Then translations put $(\hat{t}, \hat{x})$ at the origin.

In the sequel, to simplify, we set $P(0,0,w,\varphi) = P(w,\varphi)$.

2. Now we use Theorem 5.1. For every $\gamma > 0$, we set $A_\gamma = A + \gamma I$, where $A = D^2\varphi(0,0)$. For $i = 1 \dots k$, we define the following functions, for $t \in I\!R$, $x_i \in I\!R^{N_i}$ and $x = (x_1, \dots, x_k)$, by

$$\begin{aligned} v_{i,\gamma}(t,x_i) &= u_i(t,x_i) - u_i(0,0) - D_{x_i}\varphi(0,0) \cdot x_i - \frac{1}{k}\frac{\partial \varphi}{\partial t}(0,0)t \\ &\qquad -\frac{t}{2}D^2_{(t,x_i)}\varphi(0,0) \cdot x_i - \frac{1}{2k}\frac{\partial^2 \varphi}{\partial t^2}(0,0)t^2 - \gamma t^2, \\ w_\gamma(t,x) &= v_{1,\gamma}(t,x_1) + \dots + v_{k,\gamma}(t,x_k). \end{aligned} \tag{5.34}$$

By Taylor's formula, it is obvious that the function $(t,x) \mapsto w_\gamma(t,x) - \frac{1}{2}(A_\gamma x)\cdot x$ has a strict local maximum at (0,0) over $I\!R \times I\!R^N$ and identical arguments as those used in (i), allow us to consider that it is a strict global one. We then need the following technical lemma, the proof of which is postponed at the end of this section.

Lemma 5.3
(i) If ε satisfies $P(w,\varphi)$, then, for γ small enough, $\varepsilon_\gamma = \dfrac{\varepsilon}{1-\varepsilon\gamma}$ satisfies $\bar{P}(w_\gamma, A_\gamma)$.
(ii) For every $i = 1\dots k$ and $\gamma > 0$, the following equality holds

$$\overline{\mathcal{P}}^{2,+} v_{i,\gamma}(0,0) = \overline{\mathcal{P}}^{2,+} u_i(0,0) - \left(\frac{1}{k}\frac{\partial\varphi}{\partial t}(0,0), D_{x_i}\varphi(0,0), 0\right). \tag{5.35}$$

We first admit this lemma and continue the proof of Proposition 5.1. Suppose that ε satisfies $P(w,\varphi)$. Theorem 5.1 and Lemma 5.3 give the existence, for $\gamma > 0$ small enough, of $(a_{1,\gamma},\dots,a_{k,\gamma}) \in I\!R^k$, $(X_{1,\gamma},\dots,X_{k,\gamma}) \in \mathcal{S}(N_1)\times\dots\times\mathcal{S}(N_k)$, such that

$$(a_{i,\gamma}, 0, X_{i,\gamma}) \in \bar{\mathcal{P}}^{2,+} v_{i,\gamma}(0,0), \quad \text{for every} \quad i = 1\dots k, \tag{5.36}$$

$$-\left(\frac{1}{\varepsilon_\gamma} + ||A_\gamma||\right) I \le \begin{pmatrix} X_{1,\gamma} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & X_{k,\gamma} \end{pmatrix} \le A_\gamma + \varepsilon_\gamma A_\gamma^2, \tag{5.37}$$

$$a_{1,\gamma} + \dots + a_{k,\gamma} = 0. \tag{5.38}$$

We first prove that, for every $i = 1\dots k$, the sequences $(a_{i,\gamma})_\gamma$, $(X_{i,\gamma})_\gamma$ are bounded. Indeed, (5.37) and the fact that for γ small enough, $\varepsilon_\gamma ||A_\gamma|| < 1$ and $\varepsilon_\gamma^{-1} + ||A_\gamma|| \le \varepsilon^{-1} + ||A||$ show that, for every $i = 1\dots k$ and γ small enough, we have

$$||X_{i,\gamma}|| \le \varepsilon_\gamma^{-1} + ||A_\gamma|| \le \varepsilon^{-1} + ||A||. \tag{5.39}$$

By (5.35), we have, for every $i = 1\dots k$, $\gamma > 0$,

$$\left(a_{i,\gamma} + \frac{1}{k}\,\frac{\partial\varphi}{\partial t}(0,0), D_{x_i}\varphi(0,0), X_{i,\gamma}\right) \in \overline{\mathcal{P}}^{2,+} u_i(0,0). \tag{5.40}$$

(5.39) and (5.40) show that $a_{i,\gamma} \le C_\varepsilon$, for every $i = 1\dots k$, for γ small enough. Combining it with (5.38), it is clear that for every $i = 1\dots k$, the sequence $(a_{i,\gamma})_\gamma$ is bounded. By extracting if necessary subsequences, there exists $(b_i)_{1\le i\le k} \in R^k$, $(X_i)_{1\le i\le k} \in \mathcal{S}(N_1)\times\dots\times\mathcal{S}(N_k)$, such that, for every $1 \le i \le k$, as $\gamma \to 0$,

$$a_{i,\gamma} \to b_i, \quad X_{i,\gamma} \to X_i. \tag{5.41}$$

By (5.37), we get (5.20), by letting γ tend to zero. Then the estimates on the X_i coming from the matrix inequality, (5.40) and (5.41) show that, for every $i = 1\dots k$,

$$\left(b_i + \frac{1}{k}\,\frac{\partial\varphi}{\partial t}(0,0), D_{x_i}\varphi(0,0), X_i\right) \in \overline{\mathcal{P}}^{2,+} u_i(0,0).$$

And, if we set, for every $i = 1\dots k$, $a_i = b_i + \dfrac{1}{k}\,\dfrac{\partial\varphi}{\partial t}(0,0)$, we get, by (5.38), (5.33), by letting γ tend to zero. This ends the proof of the first step.

Second Step. Assume now that $\varphi \in C^2_s$ and satisfies (5.17). Let $(\rho_n)_{n\geq 1}$ a real mollifier. For $n \geq 1$, we define the functions $\varphi_n \in C^2$ and $\vartheta_n \in C$, for $(t,x) \in I\!R \times I\!R^N$, by

$$\varphi_n(t,x) = (\varphi(.,x) * \rho_n)(t) \quad \text{and} \quad \vartheta_n(t,x) = (\vartheta(.,x) * \rho_n)(t).$$

By restricting if necessary I and J, we can assume that (5.17) holds with φ_n, ϑ_n. For n, we consider the function $\Phi_n \in C^2$ defined for $(t,x) \in I \times \mathcal{O}$, by

$$\Phi_n(t,x) = \varphi_n(t,x) + (t-\hat{t})^2 + |x-\hat{x}|^4.$$

It is not difficult to show that the function $w - \Phi_n$ achieves its maximum over $I \times \mathcal{O}$ at a point (t_n, x_n), with as $n \to +\infty$,

$$\begin{aligned} &(t_n,x_n) \to (\hat{t},\hat{x}), \quad u_i(t_n,x_n) \to \; u_i(\hat{t},\hat{x}), \quad \forall i = 1\ldots k,\\ &p_n = D\Phi_n(t_n,x_n) \to \; p = D\varphi(\hat{t},\hat{x}),\\ &A_n = D^2\Phi_n(t_n,x_n) \to \; A = D^2\varphi(\hat{t},\hat{x}). \end{aligned} \tag{5.42}$$

Let ε satisfying $P(\hat{t},\hat{x},w,\varphi)$. For $n \in I\!N$, we set $\varepsilon_n = 2\varepsilon(2 + ||A|| - ||A_n||)^{-1}$. It it not difficult to prove, using (5.42) that for n large enough, ε_n satisfies $P(t_n,x_n,w,\Phi_n)$. Therefore, we are in position to use the first step, and we know that, for every n large enough, there exists $(a_1^n,\ldots,a_k^n) \in I\!R^k$, $(X_1^n,\ldots,X_k^n) \in \mathcal{S}(N_1)\times\ldots\times\mathcal{S}(N_k)$ such that

$$(a_i^n, D_{x_i}\Phi_n(t_n,x_n), X_i^n) \in \bar{\mathcal{P}}^{2,+}_{I\times\mathcal{O}_i} u_i(t_n,x_n), \quad \text{for every} \quad i = 1\ldots k, \tag{5.43}$$

$$-\left(\frac{1}{\varepsilon_n} + ||A_n||\right) Id \leq \begin{pmatrix} X_1^n & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & X_k^n \end{pmatrix} \leq A_n + \varepsilon_n A_n^2, \tag{5.44}$$

$$a_1^n + \ldots + a_k^n = \frac{\partial \Phi_n}{\partial t}(t_n,x_n). \tag{5.45}$$

Similarly as in the first step, we prove that for every $i = 1\ldots k$, the sequences $(a_i^n)_n$, $(X_i^n)_n$ are bounded. Therefore, by extracting if necessary subsequences, there exists $(a_1,\ldots,a_k) \in I\!R^k$, $(X_1,\ldots,X_k) \in \mathcal{S}(N_1)\times\ldots\times\mathcal{S}(N_k)$, such that, as $n \to +\infty$, for every $i = 1\ldots k$,

$$a_i^n \to a_i, \quad X_i^n \to X_i.$$

Then the estimates on the X_i coming from the matrix inequality, (5.42), (5.43), (5.44) and finally (5.45), show that $(a_i)_{1\leq i\leq k}, (X_i)_{1\leq i\leq k}$ satisfy (5.19), (5.20) and (5.21). To end the proof of this proposition, it is enough to prove Lemmas 5.3.

Proof of Lemma 5.3 : For (i) : as $\varepsilon_\gamma ||A_\gamma|| \underset{\gamma\to 0}{\to} \varepsilon ||A||$, it is clear that, for γ small enough, $\varepsilon_\gamma ||A_\gamma|| < 1$. It is easy to show, that for every $i = 1\ldots k$, for $(t,x_i) \in I\!R \times I\!R^{N_i}$, one has

$$\mathcal{P}^{2,+} v_{i,\gamma}(t,x_i) = \mathcal{P}^{2,+} u_i(t,x_i) - \left(\frac{1}{k}\frac{\partial \varphi}{\partial t}(0,0) + 2t\gamma + \frac{t}{k}\frac{\partial^2 \varphi}{\partial t^2}(0,0) + \frac{1}{2}D^2_{(t,x_i)}\varphi(0,0)\cdot x_i, D_{x_i}\varphi(0,0) + \frac{t}{2}D^2_{(t,x_i)}\varphi(0,0), 0\right). \tag{5.46}$$

For $i = 1 \dots k$, let $(a_i, p_i, X_i) \in \mathcal{P}^{2,+} v_{i,\gamma}(t_i, x_i)$, with

$$|t_i| + |x_i| + |p_i| + |v_{i,\gamma}(t_i, x_i)| \leq r, \quad ||X_i|| \leq \frac{2}{\varepsilon_\gamma} + ||A_\gamma||.$$

By the choose of ε_γ, one has, $\dfrac{2}{\varepsilon_\gamma} + ||A_\gamma|| \leq \dfrac{2}{\varepsilon} + ||A||$, for every $\gamma > 0$. Then, using also (5.46), it is not difficult to show, that taking r small enough, we get, for every $i = 1 \dots k$,

$$\begin{array}{l} (b_i, q_i, X_i) \in \mathcal{P}^{2,+} u_i(t_i, x_i), \quad ||X_i|| \leq \dfrac{2}{\varepsilon} + ||A||, \\ |t_i| + |x_i| + |u_i(t_i, x_i)| + |q_i - D_{x_i}\varphi(0,0)| \leq r_\varepsilon, \end{array}$$

$$\begin{array}{rcl} \text{where} \quad b_i &=& a_i + \dfrac{1}{k}\dfrac{\partial \varphi}{\partial t}(0,0) + \dfrac{t}{k}\dfrac{\partial^2 \varphi}{\partial t^2}(0,0) + 2\gamma t + D^2_{(t,x_i)}\varphi(0,0) \cdot x_i, \\ q_i &=& D_{x_i}\varphi(0,0) + \dfrac{t}{2} D^2_{(t,x_i)}\varphi(0,0). \end{array}$$

As ε satisfies $P(w, \varphi)$, we have, for every $i = 1 \dots k$, $b_i \leq C_\varepsilon$, which shows that $a_i \leq \bar{C}_\varepsilon$, with $\bar{C}_\varepsilon$ depends on C_ε, γ, $|\dfrac{\partial \varphi}{\partial t}(0,0)|$, $|\dfrac{\partial^2 \varphi}{\partial t^2}(0,0)|$, $|D^2_{(t,x_i)}\varphi(0,0)|$ and on k. This ends the proof of i).

(ii) is a direct consequence of (5.46), noticing that, by the definition of $v_{i,\gamma}$, $u_i(t_n, x_n) \to u_i(0,0)$, as $n \to +\infty$, if $(t_n, x_n, v_{i,\gamma}(t_n, x_n)) \to (0, 0, v_{i,\gamma}(0,0))$, as $n \to +\infty$.

Now, we provide the **Proof of Theorem 5.1:** let $\varepsilon > 0$ satisfying $\bar{P}(w, A)$.

1. For every $\nu > 0$, $x = (x_1, \dots, x_k) \in I\!R^{N_1} \times \dots \times I\!R^{N_k}$ and $(t_1, \dots, t_k) \in I\!R^k$, we define the following function, by

$$f_\nu(t_1, \dots, t_k, x) = u_1(t_1, x_1) + \dots + u_k(t_k, x_k) - \frac{1}{2}Ax \cdot x - \frac{1}{4\nu}\sum_{i=1}^{k} |t_i - t_{i+1}|^2,$$

(where we have set $t_{k+1} = t_1$). It is not very difficult to show that f_ν has a local maximum at some point (ξ^ν, x^ν) over $I\!R^k \times I\!R^{\overline{N}}$, which satisfies, for every $i = 1 \dots k$, as $\nu \to 0$,

$$(\xi^\nu, x^\nu) \to (0,0), \quad \frac{|\xi_i^\nu - \xi_{i+1}^\nu|^2}{\nu} \to 0, \quad u_i(\xi_i^\nu, x_i^\nu) \to 0. \tag{5.47}$$

For every $i = 1 \dots k$, $(t, x_i) \in I\!R \times I\!R^{N_i}$, we set

$$v_i(t, x_i) = u_i(t + \xi_i^\nu, x_i + x_i^\nu) - u_i(\xi_i^\nu, x_i^\nu) - (Ax^\nu)_i \cdot x_i - \frac{t}{2\nu}(2\xi_i^\nu - \xi_{i+1}^\nu - \xi_{i-1}^\nu),$$

where we have set $\xi_0^\nu = \xi_k^\nu$. Easy computations show that

$$v_1(t_1, x_1) + \dots + v_k(t_k, x_k) - \frac{1}{2}Ax \cdot x - \frac{1}{4\nu}\sum_{i=1}^{k} |t_i - t_{i+1}|^2 \leq 0, \tag{5.48}$$

for $(t_1, \ldots, t_k) \in I\!R^k$ and $x \in I\!R^{\overline{N}}$, small enough. Using standard arguments, we can suppose that (5.48) holds in whole $I\!R^k \times I\!R^{\overline{N}}$. In the sequel, for $\xi = (\xi_1, \ldots, \xi_k) \in I\!R^k$ and $x = (x_1, \ldots, x_k) \in I\!R^{N_1} \times \ldots \times I\!R^{N_k}$, we set

$$\bar{w}(\xi, x) = v_1(\xi_1, x_1) + \ldots + v_k(\xi_k, x_k).$$

2. We introduce sup convolutions procedures. The Cauchy-Schwarz inequality yields

$$Ax \cdot x \le (A + \varepsilon A^2)z \cdot z + \left(\frac{1}{\varepsilon} + ||A||\right)|x - z|^2 \quad \text{for every} \quad x, z \in I\!R^{\overline{N}}, \tag{5.49}$$

and, for every $(t_1, \ldots, t_k), (s_1, \ldots, s_k) \in I\!R^k$,

$$\sum_{i=1}^{k} |t_i - t_{i+1}|^2 \le 6 \sum_{i=1}^{k} |t_i - s_i|^2 + 3 \sum_{i=1}^{k} |s_i - s_{i+1}|^2. \tag{5.50}$$

Setting $\lambda = \dfrac{1}{\varepsilon} + ||A||$, then using (5.48), (5.49) and (5.50), we get, for every $x, z \in I\!R^{\overline{N}}$ and $(t_1, \ldots, t_k), \xi \in I\!R^k$

$$\left(v_1(t_1, x_1) - \frac{\lambda}{2}|x_1 - z_1|^2 - \frac{3}{2\nu}|\xi_1 - t_1|^2\right) + \ldots + \left(v_k(t_k, x_k) - \frac{\lambda}{2}|x_k - z_k|^2 - \frac{3}{2\nu}|\xi_k - t_k|^2\right)$$
$$\le \frac{1}{2}(A + \varepsilon A^2)z \cdot z + \frac{3}{4\nu} \sum_{i=1}^{N} |\xi_i - \xi_{i+1}|^2. \tag{5.51}$$

For every $(\xi, z) \in I\!R^k \times I\!R^{\overline{N}}$, we set

$$\begin{aligned} \hat{v}_i(\xi_i, z_i) &= \operatorname*{Sup}_{t \in I\!R, x_i \in I\!R^{N_i}} \left(v_i(t, x_i) - \frac{\lambda}{2}|x_i - z_i|^2 - \frac{3}{2\nu}|\xi_i - t|^2\right), \\ g(\xi, z) &= \operatorname*{Sup}_{\xi' \in I\!R^k, x \in I\!R^{\overline{N}}} \left(\bar{w}(\xi, x) - \frac{\lambda}{2}|x - z|^2 - \frac{3}{2\nu}|\xi - \xi'|^2\right), \end{aligned} \tag{5.52}$$

so that,

$$g(\xi, z) = \hat{v}_1(\xi_1, z_1) + \ldots + \hat{v}_k(\xi_k, z_k). \tag{5.53}$$

Inequality (5.51) implies that

$$g(\xi, z) \le (A + \varepsilon A^2)z \cdot z + \frac{3}{4\nu} \sum_{i=1}^{k} |\xi_i - \xi_{i+1}|^2, \tag{5.54}$$

for every $(\xi, z) \in I\!R^k \times I\!R^{\overline{N}}$. This shows in particular that $g(0, 0) = \hat{v}_1(0, 0) + \ldots + \hat{v}_k(0, 0) \le 0$. On another hand, by definition, for every $i = 1 \ldots k$, we have $\hat{v}_i(0, 0) \ge v_i(0, 0) = 0$. Therefore, we have proved that $\hat{v}_i(0, 0) = 0$, for every $i = 1 \ldots k$.

3. We now recall Alexandrov's Theorem and Jensen's Lemma.

Theorem 5.2 *Let $\varphi : I\!R^n \to I\!R$ be semiconvex. Then φ is twice differentiable almost every where on $I\!R^n$.*

Lemma 5.4 *Let $\varphi : I\!R^n \to I\!R$ be semiconvex and $\hat{x}$ be a strict maximum point of φ. For $p \in I\!R^n$, we set $\varphi_p(x) = \varphi(x) + p \cdot x$. Then, for $r, \delta > 0$,*

$$K = \{x \in B(\hat{x}, r) : \text{there exists } p \in B_\delta(0) \text{ for which } \varphi_p \text{ has a local maximum at } x\}$$

has a positive measure.

For the proof of these Theorem and Lemma, we refer to [9].

For every $i = 1 \dots k$, the functions g and $\hat{v}_i$ are semiconvex. Indeed, the supremum of convex functions is convex and clearly, the functions $(\xi, z) \mapsto g(\xi, z) + \frac{\lambda}{2}|z|^2 + \frac{3}{2\nu}|\xi|^2$ and $(t, x_i) \mapsto \hat{v}_i(t, x_i) + \frac{\lambda}{2}|x_i|^2 + \frac{3}{2\nu}t^2$ are convex.
Inequality (5.54) shows that the following function defined for $\xi \in I\!R^k$ and $z \in I\!R^{\overline{N}}$, by

$$\Psi(\xi, z) = g(\xi, z) - \frac{1}{2}(A + \varepsilon A^2)z \cdot z - \frac{1}{2\nu}\sum_{i=1}^{k}|\xi_i - \xi_{i+1}|^2 - |z|^4 - |\xi|^4$$

has a strict global maximum at (0,0) and by the semiconvexity of g, Ψ is clearly semiconvex. Therefore Theorem 5.2 and Lemma 5.4 show that, for every $\alpha > 0$, there exists $(p_\alpha, z_\alpha) \in (I\!R^{\overline{N}})^2$, $(q_\alpha, \xi_\alpha) \in (I\!R^k)$, with $|p_\alpha|, |z_\alpha|, |q_\alpha|, |\xi_\alpha| \le \alpha$ and such that the following function

$$(\xi, z) \mapsto \Psi(\xi, z) + p_\alpha \cdot z + q_\alpha \cdot \xi,$$

has a maximum at (ξ_α, z_α) over $I\!R^k \times I\!R^{\overline{N}}$, with Ψ twice differentiable at (ξ_α, z_α).
This implies in one hand that $g(\xi_\alpha, z_\alpha) = \hat{v}_1(\xi_{1,\alpha}, z_{1,\alpha}) + \dots + \hat{v}_k(\xi_{k,\alpha}, z_{k,\alpha}) \ge O(\alpha)$. Now, for every $1 \dots i \dots k$, using the upper semicontinuity of the fucntion $\hat{v}_{i,\alpha}$, we have $\limsup_{\alpha \to 0} \hat{v}_i(\xi_{i,\alpha}, z_{i,\alpha}) \le \hat{v}_i(0,0) = 0$. This shows finally, that for every $1 \le i \le k$, we have,

$$\hat{v}_i(\xi_{i,\alpha}, z_{i,\alpha}) \to 0, \quad \text{as} \quad \alpha \to 0. \tag{5.55}$$

In a second hand, that shows that g is twice differentiable at (ξ_α, z_α) and therefore, for every $i = 1 \dots k$, $\hat{v}_i$ is twice differentiable at $(\xi_{i,\alpha}, z_{i,\alpha})$. Then, using the properties of maximum, we show easily that

$$|D_\xi g(\xi_\alpha, z_\alpha)| + |D_z g(\xi_\alpha, z_\alpha)| = O(\alpha), \tag{5.56}$$

$$D_z^2 g(\xi_\alpha, z_\alpha) \le A + \varepsilon A^2 + O(\alpha^2). \tag{5.57}$$

For every $\alpha > 0$, $i = 1 \dots k$, we set $a_{i,\alpha} = \frac{\partial \hat{v}_i}{\partial t}(\xi_{i,\alpha}, z_{i,\alpha})$, $p_{i,\alpha} = D_{z_i}\hat{v}_i(\xi_{i,\alpha}, z_{i,\alpha})$ and $X_{i,\alpha} = D^2_{z_i}\hat{v}_i(\xi_{i,\alpha}, z_{i,\alpha})$. Then, (5.53) and (5.56), show that, for every $i = 1 \dots k$ and α, we have

$$(a_{i,\alpha}, p_{i,\alpha}, X_{i,\alpha}) \in \mathcal{P}^{2,+}\hat{v}_i(\xi_{i,\alpha}, z_{i,\alpha}), \tag{5.58}$$

$$|a_{i,\alpha}| + |p_{i,\alpha}| = O(\alpha), \tag{5.59}$$

$$-\lambda Id \le \begin{pmatrix} X_{1,\alpha} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & X_{k,\alpha} \end{pmatrix} \le A + \varepsilon A^2 + O(\alpha^2), \tag{5.60}$$

the left equality in (5.60) coming from the semiconvexity of g.

Next we use the following result which proof is easy and left to the reader.

Lemma 5.5 *Let $0 < \varepsilon$ satisfying $P(\hat{t}, \hat{x}, w, \varphi)$ and $r_\varepsilon, C_\varepsilon$, such that (5.18) holds. Suppose that, for every $1 \leq i \leq k$,*

$$\begin{aligned}
&(a_i, p_i, X_i) \in \overline{\mathcal{P}}^{2,+}_{I\times\mathcal{O}_i} u_i(t_i, x_i),\\
&\textit{with} \quad |t_i - \hat{t}| + |x_i - \hat{x}_i| + |u_i(t_i, x_i) - u_i(\hat{t}, \hat{x}_i)| + |p_i - D_{x_i}\varphi(\hat{t}, \hat{x})| \leq \frac{r_\varepsilon}{2},\\
&\textit{and} \quad ||X_i|| \leq \frac{1}{\varepsilon} + ||A||. \qquad (5.61)
\end{aligned}$$

Then $a_i \leq C_\varepsilon$.

Inequality (5.55), (5.59), (5.60 and Lemma 5.5, show that, by extracting if necessary subsequences and letting α tend to zero, there exists $(X_1, \ldots, X_k) \in \mathcal{S}(N_1) \times \ldots \mathcal{S}(N_k)$, such that

$$(0, 0, X_i) \in \overline{\mathcal{P}}^{2,+} \hat{v}_i(0, 0), \quad \text{for every } i = 1 \ldots k$$

$$\text{and} \qquad -\lambda Id \leq \begin{pmatrix} X_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & X_k \end{pmatrix} \leq A + \varepsilon A^2. \qquad (5.62)$$

4. Lemma 5.6, which is provided and proved below shows that, for every $i = 1 \ldots k$, $(0, 0, X_i) \in \overline{\mathcal{P}}^{2,+} v_i(0, 0)$. In fact, as v_i depends on ν, the matrices X_i depend on ν. This is the reason why, in the sequel, we will denote them by $X_{i,\nu}$. By the definition of v_i, we show easily that, for every $i = 1 \ldots k$, we have

$$(a_{i,\nu}, p_{i,\nu}, X_{i,\nu}) \in \overline{\mathcal{P}}^{2,+} u_i(\xi_{i\nu}, x_{i,\nu}) \qquad (5.63)$$

$$\text{where} \qquad a_{i,\nu} = \frac{1}{2\nu}\big(2\xi_{i,\nu} - \xi_{i+1,\nu} - \xi_{i-1,\nu}\big) \quad \text{and} \quad p_{i,\nu} = (Ax_\nu)_i.$$

As a consequence of the matrix inequality, we have, for every ν, $||X_{i,\nu}|| \leq \frac{1}{\varepsilon} + ||A||$. Then, as ε satifies $P(w, A)$, (5.47), (5.63) and Remark 5.5, show that, for ν small enough, $a_{i,\nu} \leq C_\varepsilon$, for every $i = 1 \ldots k$. On another hand, it is easy to verify that $a_{1,\nu} + \ldots + a_{k,\nu} = 0$, which implies, with the preceding inequality that, for every $i = 1 \ldots k$, the sequence $(a_{i,\nu})_\nu$ is bounded.

Therefore, by extracting subsequences if necessary, using the estimates on the $X_{i,\nu}$ and (5.47), we obtain Theorem 5.1.
It remains to state and prove Lemma 5.6.

Lemma 5.6 *Let $p \in I\!N, p \geq 1$, and v defined in $I\!R \times I\!R^p$. For $(t, x) \in I\!R \times I\!R^p$, we set $\hat{v}(t, x) = \operatorname*{Sup}_{(s,y)\in I\!R\times I\!R^p} \big(v(s, y) - \lambda_1|x - y|^2 - \lambda_2|s - t|^2\big)$.*
If $(a, p, X) \in \mathcal{P}^{2,+}\hat{v}(s_0, y_0)$, with $(p, y_0) \in I\!R^p \times I\!R^p$, $(a, s_0) \in I\!R^2$, $X \in \mathcal{S}(p)$, then

$$(a, p, X) \in \mathcal{P}^{2,+} v(s_0 + \frac{1}{2\lambda_2}\ a, y_0 + \frac{1}{2\lambda_1}\ p), \qquad (5.64)$$

$$\hat{v}(s_0, y_0) = v(s_0 + \frac{1}{2\lambda_2}\ a, y_0 + \frac{1}{2\lambda_1}\ p) - \frac{|p|^2}{4\lambda_1} - \frac{a^2}{4\lambda_2}. \qquad (5.65)$$

Therefore if $(0,0,X) \in \overline{\mathcal{P}}^{2,+}\hat{v}(0,0)$, *then* $(0,0,X) \in \overline{\mathcal{P}}^{2,+}v(0,0)$.

Proof of Lemma 5.6: Let $(t_0, x_0) \in I\!R \times I\!R^p$ such that

$$\hat{v}(s_0, y_0) = v(t_0, x_0) - \lambda_1|x_0 - y_0|^2 - \lambda_2|s_0 - t_0|^2.$$

For every $(x, y) \in I\!R^{2p}$ and $(s, t) \in I\!R^2$, the following inequalities hold

$$\begin{aligned} v(t,x) - \lambda_1|x-y|^2 - \lambda_2|t-s|^2 &\le \hat{v}(s,y) \\ &= \hat{v}(s_0,y_0) + a(s-s_0) + p\cdot(y-y_0) + \frac{1}{2}X(y-y_0)\cdot(y-y_0) + o(|s-s_0| + |y-y_0|^2) \\ &= v(t_0,x_0) - \lambda_1|x_0-y_0|^2 - \lambda_2|s_0-t_0|^2 + a(s-s_0) + p\cdot(y-y_0) \\ &\qquad + \frac{1}{2}X(y-y_0)\cdot(y-y_0) + o(|s-s_0| + |y-y_0|^2). \end{aligned} \tag{5.66}$$

First, taking $y = x + y_0 - x_0$ and $s = t + s_0 - t_0$, we show that $(a, p, X) \in \mathcal{P}^{2,+}v(t_0, x_0)$. Then taking $x = x_0$ and $t = t_0$, we get

$$\begin{aligned} 0 \le \lambda_1\big(|x_0-y|^2 - |x_0-y_0|^2\big) + \lambda_2((t_0-s)^2 - (t_0-s_0)^2) + a(s-s_0) \\ + p\cdot(y-y_0) + \frac{1}{2}X(y-y_0)\cdot(y-y_0) + o(|s-s_0| + |y-y_0|^2). \end{aligned} \tag{5.67}$$

Then choosing, for $\beta \in I\!R$, $y = y_0 + \beta(2\lambda_1(y_0 - x_0) + p)$ and $s = s_0$, we obtain

$$0 \le \beta|2\lambda_1(y_0 - x_0) + p|^2 + O(\beta^2).$$

Taking $\beta > 0$ (then $\beta < 0$) and letting it tend to zero, we get $x_0 = y_0 + \dfrac{1}{2\lambda_1}\, p$.
Now, setting $y = y_0$ and $s = s_0 + \beta$, with $\beta \in I\!R$, we get

$$0 \le \beta\,(a + 2\lambda_2(s_0 - t_0)) + o(\beta).$$

And we obtain that $t_0 = s_0 + \dfrac{1}{2\lambda_2}\, a$.
Therefore, we have shown (5.64) or (5.65). Let $(0,0,X) \in \overline{\mathcal{P}}^{2,+}\hat{v}(0,0)$, there exists

$$(a_n, p_n, X_n) \to (0,0,X), \quad (t_n, x_n) \to (0,0) \quad \text{and} \quad \hat{v}(t_n, x_n) \to \hat{v}(0,0),$$

such that for every $n \in I\!N$, $(a_n, p_n, X_n) \in \mathcal{P}^{2,+}\hat{v}(t_n, x_n)$. By (5.64) and (5.65), we have for every $n \in I\!N$, $(a_n, p_n, X_n) \in \mathcal{P}^{2,+}v(t_n + \frac{a_n}{2\lambda_2}, x_n + \frac{p_n}{2\lambda_1})$ and $v(t_n + \frac{a_n}{2\lambda_2}, x_n + \frac{p_n}{2\lambda_1}) = \hat{v}(t_n, x_n) + \frac{|p_n|^2}{4\lambda_1} + \frac{a_n^2}{4\lambda_2}$. To conclude, we only have to prove that $v(t_n + \frac{a_n}{2\lambda_2}, x_n + \frac{p_n}{2\lambda_1}) \to v(0,0)$, as $n \to +\infty$. By the definition, $\hat{v}(0,0) \ge v(0,0)$, then using the upper semicontinuity of v, we get

$$\begin{aligned} v(0,0) \ge \limsup_{n\to+\infty} v(t_n + \frac{a_n}{2\lambda_2}, x_n + \frac{p_n}{2\lambda_1}) \ge \liminf_{n\to+\infty} v(t_n + \frac{a_n}{2\lambda_2}, x_n + \frac{p_n}{2\lambda_1}) \\ \ge \liminf_{n\to+\infty} \hat{v}(t_n, x_n) + \frac{|p_n|^2}{4\lambda_1} + \frac{a_n^2}{4\lambda_2} = \hat{v}(0,0) \ge v(0,0). \end{aligned}$$

This ends the proof of Lemma 5.6.

6 The test-function

The following lemma is an adaptation of Lemma 5.1 proved by G. Barles in [3]. The difficulty to construct a suitable test-function, in our case, comes from the weak dependence of L in time; more precisely, the main difference is that L is not assumed to be locally Lipschitz continuous in t. In the sequel, $u \in USC(\overline{\Omega}_T)$, $v \in LSC(\overline{\Omega}_T)$ and u and v are bounded over $\overline{\Omega}_T$. Let $R > 0$ such that $|u|, |v| \leq R$ in $\overline{\Omega}_T$. We assume that Ω satisfies **(H7)**.

Lemma 6.1
*(i) If L satisfies conditions **(H4)**, **(H5)** and **(H6)** and if*

$$M = \max_{\overline{\Omega}_T} \big(u(t,x) - v(t,x)\big) > 0,$$

then, there exists $\tilde{K} > 1$, $0 < \tilde{\nu} < 1$, such that if $0 < \nu < \tilde{\nu}$ and $0 < \varepsilon \leq \nu$ small enough compared to ν, there exists $\psi_{\nu,\varepsilon} \in C^2_s$, with, for every $t \in I\!R$, $x, y \in \overline{\Omega}$,

$$-\tilde{K}\nu\varepsilon + \tilde{K}^{-1}\frac{|x-y|^2}{\varepsilon^2} \leq \psi_{\nu,\varepsilon}(t,x,y) \leq \tilde{K}\frac{|x-y|^2}{\varepsilon^2} + \tilde{K}\nu\varepsilon. \tag{6.1}$$

Moreover, for every $t \in I\!R$, $x, y \in \overline{\Omega}$ such that $|x-y| \leq \nu\varepsilon$, one has

$$-\tilde{K}\beta + \tilde{K}^{-1}\frac{|x-y|}{\varepsilon^2} \leq |D_x\psi_{\nu,\varepsilon}(t,x,y)| \wedge |D_y\psi_{\nu,\varepsilon}(t,x,y)| \tag{6.2}$$

$$|D_x\psi_{\nu,\varepsilon}(t,x,y)| \vee |D_y\psi_{\nu,\varepsilon}(t,x,y)| \leq \tilde{K}\frac{|x-y|}{\varepsilon^2} + \tilde{K}\beta, \tag{6.3}$$

with $\beta = 1$, and

$$|D_x\psi_{\nu,\varepsilon}(t,x,y) + D_y\psi_{\nu,\varepsilon}(t,x,y)| \leq \tilde{K}\frac{|x-y|^2}{\varepsilon^2} + \tilde{K}\nu\varepsilon, \tag{6.4}$$

$$-\frac{\tilde{K}}{\varepsilon^2}Id \leq D^2\psi_{\nu,\varepsilon}(t,x,y) \leq \frac{\tilde{K}}{\varepsilon^2}\begin{pmatrix} Id & -Id \\ -Id & Id \end{pmatrix} + \tilde{K}\nu Id. \tag{6.5}$$

There exists $\tilde{h} \in L^1_{loc}(I\!R)$, such that, for every $x, y \in \overline{\Omega}$, with $|x-y| \leq 2\nu\varepsilon$, one has

$$\frac{\partial\psi_{\nu,\varepsilon}(.,x,y)}{\partial t} \geq -\nu\tilde{h}, \quad in \quad \mathcal{D}'(0,2T). \tag{6.6}$$

The constants $\tilde{K}, \tilde{\nu}$ depends on $T, R, C_R, \nu_R, ||h_R||_{L^1(0,T)}$ and on $diam(\Omega)$, where we set $diam(\Omega) = \sup_{(x,y)\in\Omega^2} |x-y|$ and $\tilde{h}$ is equal to $h_R + 1$ on $(0,T)$ and to 1 otherwise.

Finally, there exists $\delta > 0$, such that if $t \in [0,T]$, $x, y \in \overline{\Omega}$, with $|x-y| \leq \nu\varepsilon (\leq \delta)$ are such that $u(t,x) - v(t,y) \geq M - \delta$, then we have

$$L\big(t,x,u(t,x),D_x\psi_{\nu,\varepsilon}(t,x,y)\big) > 0, \qquad if \quad x \in \partial\Omega, \tag{6.7}$$

$$L(t,y,v(t,y),-D_y\psi_{\nu,\varepsilon}(t,x,y)) < 0, \qquad if \quad y \in \partial\Omega. \tag{6.8}$$

*(ii) If L satisfies **(H4-2)**,**(H5-2)** and **(H6-2)**, then there exists a constant $\tilde{K} > 1$, $0 < \tilde{\nu} < 1$, such that, if $0 < \nu < \tilde{\nu}$ and $0 < \varepsilon < \nu$, small enough compared to ν, there exists $\psi_{\nu,\varepsilon} \in C^2_s$ satisfying conditions (6.1) to (6.7), with $\beta = \nu\varepsilon$, and $\tilde{h}$ is the function equal to $h+1$ in $(0,T)$ and to 1 otherwise. The constants $\tilde{K}, \tilde{\nu}$ depends on $T, R, C, \nu, ||h||_{L^1(0,T)}$ and on diam(Ω).*
Finally, for every $t \in [0,T]$, for every $x, y \in \overline{\Omega}$, with $|x-y| \le \nu\varepsilon$, one has

$$L(t, x, D_x\psi_{\nu,\varepsilon}(t,x,y)) > 0, \qquad \text{if} \quad x \in \partial\Omega, \tag{6.9}$$

$$L(t, y, -D_y\psi_{\nu,\varepsilon}(t,x,y)) < 0, \qquad \text{if} \quad y \in \partial\Omega. \tag{6.10}$$

Proof of Lemma 6.1 : We are going to prove i), the proof of (ii) being similar and even simpler. We are only pointing out the differences between the proof of Lemma 6.1 and Lemma 5.1 in [3]; we refer to the paper of [3] for additional details.

We consider M and R defined as above. Using the regularity of Ω, there exists $d \in W^{3,\infty}(I\!R^N)$ be a function, which agrees with the sign-distance to $\partial\Omega$ in a neighborhood of $\partial\Omega$, with $d > 0$ in Ω, which satisfies $|Dd(x)| \le 1$ in $I\!R^N$ and Dd has a compact support. We will denote below $n(x) = -Dd(x)$ even if $x \notin \partial\Omega$. As G. Barles remarks, we can suppose that $n \in C^2$, indeed it is enough to use regularization arguments, noticing that only the L^∞ norm of D^2n is playing a role in the proof.

0. The same way as in Lemma 5.1 of [3], we extend the function L to $I\!R \times \mathcal{W} \times [-R,R] \times I\!R^N$, where $\mathcal{W}$ is a neighborhood of $\partial\Omega$, in order that properties **(H4)**, **(H5)** and **(H6)** are still available. We can show quite easily that there exists a function $C : I\!R \times I\!R^N \times I\!R \times I\!R^N \to I\!R$, such that

$$L(t, x, u, p + C(t,x,u,p)n(x)) = 0,$$

for every $t \in I\!R, x \in \mathcal{W}, |u| \le R$ and $p \in I\!R^N$, and which satisfies moreover,

$$|C(t,x,u,0)| \le C_1, \tag{6.11}$$

$$|C(t,x,u,p) - C(t,y,v,q)| \le C_2\Big((1+|p|+|q|)|x-y| + |u-v| + |p-q|\Big), \tag{6.12}$$

$$|C(t,x,u,p) - C(s,x,u,p)| \le C_3\ (1+|p|)\int_t^s \bar{h}(w)dw, \tag{6.13}$$

for every $t, s, u, v \in I\!R$, with $t \le s$, $x, y, p, q \in I\!R^N$ and $\bar{h}$ is equal to h_R in $(0,T)$ and to *zero* otherwise. In the sequel, we say that a constant C depends on the data if it depends on $R, C_R, \nu_R, ||h_R||_{L^1(0,T)}$ and on diam(Ω). The constants C_1, C_2 and C_3 depend on the data.

1. Regularization of the function C in the variables $(x,u,p) \in I\!R^N \times I\!R \times I\!R^N$.
For $\alpha > 0$, we define the functions C_α, for every $(t,x,u,p) \in I\!R \times I\!R^N \times I\!R \times I\!R^N$, by

$$C_\alpha(t,x,u,p) = \iiint_{I\!R^N \times I\!R \times I\!R^N} C(t,y,v,q)\rho\left((x-y)\frac{\Gamma}{\Lambda}\right)\tilde{\rho}(\frac{u-v}{\Lambda})\rho\left(\frac{p-q}{\Lambda}\right)\frac{\Gamma^N}{\Lambda^{2N+1}}dydvdq,$$

where $\rho \in \mathcal{D}(I\!R^N)$, $\rho \ge 0$, supp$(\rho) \subset B(0,1)$, with $\int_{I\!R^N} \rho(y)dy = 1$ and where $\tilde{\rho}$ satisfies the same properties as ρ except for $N = 1$. Finally

$$\Lambda = \left(\alpha^2 + p\cdot n(x)^2\right)^{\frac{1}{2}} \quad \text{and} \quad \Gamma = (1+|p|^2)^{\frac{1}{2}}.$$

The estimations on the function C_α, its first and second derivates with respect to the variables (x, u, p), given in [3], are still available, for every $(t, x, u, p) \in I\!R \times I\!R^N \times I\!R \times I\!R^N$, with K is a constant depending on the data. In particular, we have, for every $(t, x, u, p) \in I\!R \times I\!R^N \times I\!R \times I\!R^N$ and $\alpha > 0$,

$$|D_u C_\alpha(t, x, u, p)| \leq K. \tag{6.14}$$

2. The dependence of L in u.
Using Lemma 5.2 in [3] and looking carefully at its proof, we show that there exists a C^∞ function $\xi : I\!R^{N+1} \to I\!R$ and a constant $\delta > 0$, such that, if $t \in [0, T]$, $x, y \in \overline{\Omega}$ are such that $u(t, x) - v(t, y) \geq M - \delta$ and $|x - y| \leq \delta$, then

$$u(t, x) - \xi(t, \frac{x+y}{2}) \geq 0 \quad \text{and} \quad \xi(t, \frac{x+y}{2}) - v(t, y) \geq 0.$$

Moreover, we can choose ξ, such that $|\xi| \leq R + 1$, in $I\!R \times I\!R^N$.
3. The test function.
For $0 < \varepsilon \leq \nu$, we introduce the function $\psi_{\nu,\varepsilon} \in C^2_s$ defined for $(t, x, y) \in I\!R \times I\!R^N \times I\!R^N$, by

$$\begin{aligned}\psi_{\nu,\varepsilon}(t, x, y) =& \exp\Big(-\tilde{K}_1[d(x) + d(y)]\Big) \frac{|x-y|^2}{\varepsilon^2} - C_{\nu\varepsilon}\left(t, \frac{x+y}{2}, \xi\left(t, \frac{x+y}{2}\right), p\right) \Big(d(x) - d(y)\Big) \\ &+ \frac{A(d(x) - d(y))^2}{\varepsilon^2} - \tilde{K}_2 \nu\varepsilon\big(d(x) + d(y)\big),\end{aligned}$$

where $p = \exp\Big(-\tilde{K}_1[d(x) + d(y)]\Big) \dfrac{2(x-y)}{\varepsilon^2}$.

By choosing the constant $A, \tilde{K}_1, \tilde{K}_2$ as in [3], we get (6.1) to (6.5), (6.7) and (6.8).
4. An estimate on the derivate in the sense of distribution of the test function in t.
By the definition of $C_{\nu\varepsilon}$, we show easily that $C_{\nu\varepsilon}$ satisfies (6.13) with $2C_3$ instead of C_3. Then, for every $t, s \in [0, 2T]$, with $t \leq s$ and $x, y \in \overline{\Omega} \times \overline{\Omega}$, setting $z = \dfrac{x+y}{2}$, we have

$$\begin{aligned}|\psi_{\nu,\varepsilon}(t, x, y) - \psi_{\nu,\varepsilon}(s, x, y)| &= |d(x) - d(y)|\Big(C_{\nu\varepsilon}\left(t, z, \xi(t, z), p\right) - C_{\nu\varepsilon}\left(s, z, \xi(s, z), p\right)\Big), \\ &\leq |x - y|\Big(2C_3(1 + |p|)\int_s^t \bar{h} + K \int_s^t |\frac{\partial \xi}{\partial t}(., z)|\Big), \quad \text{by} \quad (6.14) \\ &\leq \bar{C}\left(\frac{|x-y|^2}{\varepsilon^2} + |x - y|\right)\int_s^t (1 + \bar{h}), \quad \text{by the definition of } p \\ &\leq \bar{C}(\nu^2 + \varepsilon^2)\int_s^t (1 + \bar{h}) \leq \nu \int_s^t (1 + \bar{h}), \end{aligned} \tag{6.15}$$

for ν small enough then ε small enough compared to ν, the constant $\bar{C}$ depends on the data. To get (6.6), it is enough to use the following lemma, which proof, based on Lebesgue and Fubini's Theorems is left to the reader.

Lemma 6.2 *Let $a < b$, f, $g \in L^1(a, b)$, such that, for almost every $a < s \leq t < b$, $|f(s) - f(t)| \leq \int_s^t g(r)dr$. Then, we have, $-g \leq \dfrac{\partial f}{\partial t} \leq g$ in $\mathcal{D}'(a, b)$.*

7 Proof of the Lemmas and Propositions given in Sections 2,3,4

7.1 Proofs of Lemmas and Propositions given in the second section

In the sequel, we will only consider the case of subsolutions, the supersolutions one being treated similarly.

Proof of Proposition 2.3: We only prove it in the case when F satisfies (ii) and L is homogeneous of degree 1 in p, the case when $F \in C(\Gamma_T)$ being simpler

At first, let us show that a L^1 subsolution is a classical one. To this end, consider $(t_0, x_0) \in \Omega_T$, $\varphi \in C^\infty$, such that $u - \varphi$ has a local maximum over Ω_T. If F_* is the lower semicontinuous envelope of F, by classical arguments, there exists a sequences $(F_n)_{n \in I\!N} \in C(\Gamma_T)$ such that $F_n \leq F_*$ in Γ_T, for every n and

$$F_n(t_0, \xi_0) \to F_*(t_0, \xi_0), \quad \text{as} \quad n \to +\infty.$$

Therefore, using Definition 2.1, with $b \equiv 0$ and $G = F_n$, we get the result by letting n tend to infinity.

Conversely, assume that u is a classical subsolution and show that it is a L^1 one. Let $(t_0, x_0) \in \Omega_T$, $b \in L^1(0, T)$, $\varphi \in C^\infty$, $G \in C(\Gamma_T)$, such that the function

$$(t, x) \mapsto u(t, x) + \int_0^t b(s)ds - \varphi(t, x),$$

has a strict maximum point at (t_0, x_0) over Q_{t_0} and such that (2.1) holds.

Recalling the notations of Definition 2.1, namely $p_0 = D\varphi(t_0, x_0)$ and $X_0 = D^2\varphi(t_0, x_0)$, we have the following technical lemma, largely inspired of Proposition 2.2 in G. Barles and Ch. Georgelin [4], which is proved at the end of the present proof.

Lemma 7.1 *If L is homogeneous of degree 1 in p, we can assume in Definition 2.1 that either $p_0 \neq 0$ or $p_0 = 0$ and $X_0 = 0$.*

By Lemma 7.1, we can assume either that $p_0 \neq 0$ or that $p_0 = 0$ and $X_0 = 0$. And as F satisfies (ii), it implies in particular, that $t \mapsto F_*(t, \xi_0) \in C([0, T])$.

Inequality (2.1) implies the following inequality, for almost every t close enough to t_0

$$b(t) \leq k(t) = F(t, \xi_0) - G(t, \xi_0), \tag{7.1}$$

and using the above remark, we know that $k \in C([0, T])$. Then, we still denote by k and b their extensions to $I\!R$, which are equal to $k(0)$ in $]-\infty, 0]$ and to $k(T)$ in $[0, +\infty[$. Let $(\rho_n)_{n \geq 1}$ a real mollifier. For every $n \geq 1$, we set $b_n = b * \rho_n$ and $k_n = k * \rho_n$, which are continuous on $I\!R$. It is not difficult to show that the function $\psi_n : (t, x) \mapsto \varphi(t, x) - \int_0^t b_n(s)ds$ achieves its maximum over Q_{t_0} at (t_n, x_n), with as $n \to +\infty$,

$$(t_n, x_n) \to (t_0, x_0), \quad u(t_n, x_n) \to u(t_0, x_0). \tag{7.2}$$

Using that u is a classical subsolution of (1.1)-(1.2) in Ω_T, we have

$$\begin{aligned} \frac{\partial \psi_n}{\partial t}(t_n, x_n) + F_*(t_n, \xi_n) &\leq 0 \quad \text{if} \quad x_n \in \Omega, \\ \min\left(\frac{\partial \psi_n}{\partial t}(t_n, x_n) + F_*(t_n, \xi_n), L(t_n, x_n, u(t_n, x_n), D\varphi(t_n, x_n))\right) &\leq 0 \quad \text{if} \quad x_n \in \partial\Omega, \end{aligned}$$

with $\xi_n = (x_n, u(t_n, x_n), D\varphi(t_n, x_n), D^2\varphi(t_n, x_n))$. But, by (7.1), we have, for every $n \geq 1$,

$$\frac{\partial \psi_n}{\partial t}(t_n, x_n) \geq \frac{\partial \varphi}{\partial t}(t_n, x_n) - k_n(t_n).$$

Using (7.2) and the fact that $k_n(t_n) \to k(t_0) = F_*(t_0, \xi_0) - G(t_0, \xi_0)$, we get the wanting result by letting n to infinity. This ends the proof of Proposition 2.3.

We now give the **proof of Lemma 7.1:** Let $(t_0, x_0) \in \Omega_T$, $b \in L^1(0, T)$, $\varphi \in C^\infty(\Omega_T)$ and $G \in C(\Gamma_T)$, such that $(t, x) \mapsto u(t, x) + \int_0^t b(s)ds - \varphi(t, x)$, has a strict maximum at (t_0, x_0) over Q_{t_0}, and such that (2.1) holds. Assume that $p_0 = 0$. As L is homogeneous of degree 1, (2.3) is satisfied if $x_0 \in \partial\Omega$. Therefore, in the sequel, we can assume that $x_0 \in \Omega$. For $\varepsilon > 0$, we define the following function, by

$$\vartheta_\varepsilon(t, x, y) = u(t, x) + \int_0^t b(s)ds - \frac{|x - y|^4}{\varepsilon} - \varphi(t, y), \quad \forall (t, x, y) \in (0, T] \times \overline{\Omega} \times \overline{\Omega}.$$

It is not difficult to show that it achieves its maximum over $[\frac{t_0}{2}, T] \times \overline{\Omega} \times \overline{\Omega}$, at $(t_\varepsilon, x_\varepsilon, y_\varepsilon)$,

$$\text{with} \qquad (t_\varepsilon, x_\varepsilon, y_\varepsilon) \to (t_0, x_0, x_0), \quad u(t_\varepsilon, x_\varepsilon) \to u(t_0, x_0), \quad \text{as} \quad \varepsilon \to 0. \tag{7.3}$$

$$\text{Thus} \quad p_\varepsilon = D\varphi(t_\varepsilon, y_\varepsilon) \to p_0 = 0 \quad \text{and} \quad X_\varepsilon = D^2\varphi(t_\varepsilon, y_\varepsilon) \to X_0. \tag{7.4}$$

As $x_0 \in \Omega$, for ε small enough, $x_\varepsilon, y_\varepsilon \in \Omega$. Therefore, the function $y \mapsto \vartheta_\varepsilon(t_\varepsilon, x_\varepsilon, y) \in C^\infty$ and has a local minimum at y_ε over $\mathbb{R}^N$, for ε small enough, thus

$$p_\varepsilon = \frac{4(x_\varepsilon - y_\varepsilon)|x_\varepsilon - y_\varepsilon|^2}{\varepsilon}, \quad X_\varepsilon \geq -\frac{4|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} Id - 8\frac{(x_\varepsilon - y_\varepsilon) \otimes (x_\varepsilon - y_\varepsilon)}{\varepsilon}. \tag{7.5}$$

Now, we consider two cases, whereas there is an infinity of ε such that $x_\varepsilon = y_\varepsilon$ or not.
(i) Assume that there exists an infinity of ε such that $x_\varepsilon = y_\varepsilon$. To simplify, we can assume that it is true for every ε. This implies in particular, by (7.5), that $p_\varepsilon = 0$ and $X_\varepsilon \geq 0$, for every ε. This shows, by (7.4), that $X_0 \geq 0$. In the sequel, we set for $(t, \xi) = (t, x, r, p, X) \in \Gamma_T^*$, $G_0(t, \xi) = G(t, x, r, p, X + X_0) \in C(\Gamma_T)$. By (2.1), as F is degenerate elliptic, we have

$$b(t) + G_0(t, \xi) \leq F(t, \xi), \tag{7.6}$$

for almost every $t \sim t_0$ and for every $\xi \sim (x_0, u(t_0, x_0), 0, 0)$ and therefore, by (7.3), for almost every $t \sim t_\varepsilon$ and for every $\xi \sim (x_\varepsilon, u(t_\varepsilon, x_\varepsilon), 0, 0)$. Now, the function

$(t,x) \mapsto u(t,x) + \int_0^t b(s)ds - \psi_\varepsilon(t,x)$, where $\psi_\varepsilon(t,x) = \dfrac{|x-y_\varepsilon|^4}{\varepsilon} + \varphi(t,y_\varepsilon)$ achieves its maximum over Q_{t_0} at $(t_\varepsilon, x_\varepsilon)$. And as $x_\varepsilon = y_\varepsilon$, for every ε, then $D\psi_\varepsilon(t_\varepsilon,x_\varepsilon) = 0$ and $D^2\psi_\varepsilon(t_\varepsilon,x_\varepsilon) = 0$. Therefore, by (7.6), as $x_\varepsilon \in \Omega$ for ε small enough, we get

$$\frac{\partial\psi_\varepsilon}{\partial t}(t_\varepsilon,x_\varepsilon) + G_0(t_\varepsilon,x_\varepsilon,u(t_\varepsilon,x_\varepsilon),0,0) \le 0.$$

The result follows using the definition of G_0, (7.3), (7.4) by letting ε tend to zero.
(ii) Assume now that there exists an infinity of ε, such that $x_\varepsilon \ne y_\varepsilon$ and to simplify, we suppose that it is true, for every ε. By (7.5), we have in particular $p_\varepsilon \ne 0$, for every ε. The following function, defined for $(t,x) \in \Omega_T$, by

$$\vartheta_\varepsilon(t,x,x-(x_\varepsilon-y_\varepsilon)) = u(t,x) + \int_0^t b(s)ds - \varphi_\varepsilon(t,x), \quad \text{where} \quad \varphi_\varepsilon(t,x) = \varphi(t,x-(x_\varepsilon-y_\varepsilon)),$$

has a local maximum at $(t_\varepsilon,x_\varepsilon)$ over Ω_T. And we have clearly,

$$D\varphi_\varepsilon(t_\varepsilon,x_\varepsilon) = p_\varepsilon, \quad D^2\varphi_\varepsilon(t_\varepsilon,x_\varepsilon) = X_\varepsilon, \quad \frac{\partial\varphi_\varepsilon}{\partial t}(t_\varepsilon,x_\varepsilon) = \frac{\partial\varphi}{\partial t}(t_\varepsilon,y_\varepsilon). \tag{7.7}$$

By (7.3) and (7.4), (2.1) holds for almost every $t \sim t_\varepsilon$ and for every $\xi \sim \xi_\varepsilon = (x_\varepsilon, u(t_\varepsilon,x_\varepsilon), p_\varepsilon, X_\varepsilon)$. As $p_\varepsilon \ne 0$, using (7.7) and as $x_\varepsilon \in \Omega$, for ε small enough, we have the following inequality

$$\frac{\partial\varphi}{\partial t}(t_\varepsilon,y_\varepsilon) + G(t_\varepsilon,\xi_\varepsilon) \le 0.$$

And the result follows, using (7.3) and (7.4) and letting ε tend to zero.

Proof of Proposition 2.4: Let u a subsolution of (1.1)-(1.2) in Ω_T^- and $0 < h < T$, we are going to prove that u is a subsolution of (1.1)-(1.2) in Ω_h. The only difficulty is when the maximum point is achieved at time h. Therefore, let $b \in L^1(0,h)$, $\varphi \in C^\infty$, $x_0 \in \overline{\Omega}$, $G \in C(\Gamma_h)$, such that the function $\Psi : (t,x) \mapsto u(t,x) + \int_0^t b(r)dr - \varphi(t,x)$ has a strict maximum point at (h,x_0) over $[\frac{h}{2},h] \times \overline{\Omega}$, and such that (2.1) holds for almost every $0 < t < h$, $t \sim h$ and for every $\xi \sim \xi_0$. We define $G_h \in C(\Gamma_T)$, for every $\xi \in \Gamma$, by $G_h(t,\xi) = G(t,\xi)$ if $0 \le t \le h$ and $G_h(t,\xi) = G(h,\xi)$ if $h \le t \le T$ and we denote by K_0 the following compact subset of Γ

$$K_0 = \{(x,r,p,X) \in \Gamma, \quad |r| \le |u(h,x_0)| + 1,\ |p| \le |p_0| + 1,\ ||X|| \le ||X_0|| + 1\}.$$

We set, for almost every $t \in (0,T)$, $a(t) = \sup_{\xi \in K_0^*} |F(t,\xi) - G_h(t,\xi)|$ and we define $c \in L^1(0,T)$, as follows $c(t) = b(t)$ if $0 < t < h$ and $c(t) = -a(t)$ if $h < t < T$. For $\varepsilon > 0$, we consider the following function defined, for $(t,x) \in \Omega_T^-$, by

$$\Psi_\varepsilon(t,x) = u(t,x) + \int_0^t c(s)ds - \varphi(t,x) - \frac{(h-t)^2}{\varepsilon^2}.$$

It achieves its maximum over $[\frac{h}{2}, \frac{T+h}{2}] \times \overline{\Omega}$ at a point $(t_\varepsilon,x_\varepsilon)$, with as $\varepsilon \to 0$,

$$\Big(t_\varepsilon, x_\varepsilon, u(t_\varepsilon,x_\varepsilon)\Big) \to \Big(h, x_0, u(h,x_0)\Big). \tag{7.8}$$

$$\text{and such that} \qquad h \le t_\varepsilon < \frac{T+h}{2}, \quad \text{for} \quad \varepsilon \quad \text{small enough.} \tag{7.9}$$

The right-hand of this last inequality is a direct consequence of (7.8), as $h < T$. Let us show the left-hand. Assume that $t_\varepsilon < h$, then we get,

$$\Psi(t_\varepsilon, x_\varepsilon) - \frac{(h-t_\varepsilon)^2}{\varepsilon^2} = \Psi_\varepsilon(t_\varepsilon, x_\varepsilon) \ge \Psi_\varepsilon(h, x_0) = \Psi(h, x_0). \tag{7.10}$$

Now, by (7.8), $t_\varepsilon > \frac{h}{2}$, for ε small enough, and therefore (7.10) contradicts the fact that Ψ has a strict maximum point at (h, x_0) over $[\frac{h}{2}, h] \times \overline{\Omega}$.
By (2.1), the definitions of G_h, a and c and (7.8), we clearly have, for ε small enough,

$$c(t) + G_h(t, \xi) \le F(t, \xi), \tag{7.11}$$

for almost every $t \sim t_\varepsilon$ and for every $\xi \sim \xi_\varepsilon = (x_\varepsilon, u(t_\varepsilon, x_\varepsilon), p_\varepsilon, X_\varepsilon)$, where $p_\varepsilon = D\varphi(t_\varepsilon, x_\varepsilon)$ and $X_\varepsilon = D^2\varphi(t_\varepsilon, x_\varepsilon)$. As u is a subsolution of (1.1)-(1.2) in Ω_T^-, the following holds

$$\begin{aligned} \alpha_\varepsilon = \frac{\partial \varphi}{\partial t}(t_\varepsilon, x_\varepsilon) + \frac{2(t_\varepsilon - h)}{\varepsilon^2} + G_h(t_\varepsilon, \xi_\varepsilon) \quad &\le \quad 0 \quad \text{if} \quad x_\varepsilon \in \Omega, \\ \min\Big(\alpha_\varepsilon, L(t_\varepsilon, x_\varepsilon, u(t_\varepsilon, x_\varepsilon), p_\varepsilon)\Big) \quad &\le \quad 0 \quad \text{if} \quad x_\varepsilon \in \partial\Omega. \end{aligned}$$

The result follows, using (7.8), (7.9), the definition of G_h and letting ε tend to zero.

7.2 Proofs of Lemmas 3.1 and 4.1

We do it at the same time for Lemma 3.1 and Lemma 4.1. We first show that F is degenerate elliptic.

Let $X, Y \in \mathcal{S}(N)$, with $Y \ge X$, $t \in (0,T)$ and $(x, r, p) \in \Sigma$ (resp. Σ^*). First, we have, for every $\lambda > 0$,

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \frac{1}{2\lambda} \begin{pmatrix} Id & -Id \\ -Id & Id \end{pmatrix} + \lambda ||Y||^2 \, Id. \tag{7.12}$$

Using (7.12), with $\lambda = \varepsilon^2$ (resp. $\lambda = \varepsilon$), (X, Y) satisfies (3.2), (resp. (4.2)), for every $\varepsilon > 0$ and $\nu = \nu_\varepsilon = \varepsilon^2 (||Y||^2 + 1)$ (resp. $\nu = \nu_\varepsilon = \varepsilon (||Y||^2 + 1)$). Therefore, for ε small enough, we are in position to use condition **(H2)** (resp. **(H2-2)**) on F and we get,

$$F(t, x, r, p, Y) - F(t, x, r, p, X) \le m_R(t, \nu_\varepsilon), \tag{7.13}$$

where $R = |r| + 1$. As $m_R \in \mathcal{M}$, $m_R(t, \nu_\varepsilon) \underset{\varepsilon \to 0}{\to} 0$ in $L^1(0,T)$, and therefore by extracting a subsequence if necessary, $m_R(t, \nu_\varepsilon) \underset{\varepsilon \to 0}{\to} 0$, for almost every $t \in (0,T)$. This shows that F is degenerate elliptic.

Now we prove (3.9)-(3.10). Let K be a compact subset of Γ (resp. Γ^*) and $R > 0$ such that $K \subset \bar{B}_R(0)$. (resp. with moreover $\alpha > 0$, such that $|p| \ge \alpha$, whenever

$\xi = (x, r, p, X) \in K$). Let $t \in (0, T)$, $\xi_1 = (x_1, r_1, p_1, X_1)$ and $\xi_2 = (x_2, r_2, p_2, X_2) \in K$, with $|\xi_1 - \xi_2| \leq r$. We have

$$\begin{array}{rcl} |F(t,\xi_1) - F(t,\xi_2)| & \leq & \mathcal{A}_1(t) + \mathcal{A}_2(t), \\ \text{where } \mathcal{A}_1(t) & = & |F(t,\xi_1) - F(t,x_1,r_2,p_1,X_1)|, \\ \text{and } \mathcal{A}_2(t) & = & |F(t,x_1,r_2,p_1,X_1) - F(t,\xi_2)|. \end{array} \quad (7.14)$$

By **(H3)** (resp. **(H3-2)**), then $\mathcal{A}_1(t) \leq g(t, r)$, with $g \in \mathcal{M}$, $g = g_R$ (resp. $g = g_R^\alpha$).
To give an estimate of $\mathcal{A}_2$, we are going to use condition **(H2)** (resp. **(H2-2)**) on F. At first, it is not very difficult to prove that for every $(X, Y) \in (\mathcal{S}(N))^2$ and $\lambda > 0$, one has

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \lambda \begin{pmatrix} Id & -Id \\ -Id & Id \end{pmatrix} + \left(\frac{1}{2}||X - Y|| + \frac{1}{\lambda}||X + Y||^2\right) Id. \quad (7.15)$$

Therefore, (X_1, X_2) satisfies, for every $\lambda > 0$,

$$\begin{pmatrix} X_1 & 0 \\ 0 & -X_2 \end{pmatrix} \leq \lambda \begin{pmatrix} Id & -Id \\ -Id & Id \end{pmatrix} + \left(\frac{r}{2} + \frac{2R^2}{\lambda}\right) Id. \quad (7.16)$$

This implies that, for r small enough, $(X_1, X_2), (p_1, p_2), (x_1, x_2)$ satisfy (3.2), (3.3) and (3.4), with $\nu = 2R^{\frac{2}{3}} r^{\frac{2}{3}}$, $\varepsilon = 2^{-1} r^{\frac{1}{3}} R^{-\frac{2}{3}}$ and $\lambda = \varepsilon^{-2}$ (resp. they satisfy (4.2), (4.3) and (4.4), with $\nu = (2Rr)^{1/2}$, $\varepsilon = r^{1/2}(2R)^{-1/2}$ and $\lambda = \nu\varepsilon^{-2}$. Therefore, we obtain, by (3.1) (resp. (4.1)), $\mathcal{A}_2(t) \leq m_R(t, \alpha_r)$, where $\alpha_r \underset{r \to 0}{\longrightarrow} 0$. The result follows, using that $g, m_R \in \mathcal{M}$.

Now we prove (3.8). Let K a compact subset of Γ and r small enough fixed, such that $h_K^r \in L^1(0, T)$. There exists $m \in I\!N$, $(\xi_i)_{1 \leq i \leq m} \subset K$, such that $K \subset \bigcup_{i=1}^{m} B_r(\xi_i)$. Thus, for almost every $t \in (0, T)$,

$$\sup_{\xi \in K} |F(t, \xi)| \leq h_K^r(t) + \sup_{1 \leq i \leq m} |F(t, \xi_i)|.$$

This proves (3.8), by (3.9)-(3.10) and condition **(H0)** on F.

Finally we point out that the assertion on f in Lemma 4.1 is proved in the same way as above.

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