Quasilinear Parabolic Equations, Unbounded Solutions and Geometrical equations I. A geometrical approach to the study of quasilinear parabolic equations in $I\!R^N$

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Abstract

In this article, we are interested in the existence and uniqueness of solutions for quasilinear parabolic equations set in the whole space $I\!R^N$. We consider in particular cases when there is no restriction on the growth or the behaviour of these solutions at infinity. Our model equation is the mean curvature equation for graphs for which Ecker and Huisken have shown the existence of smooth solutions for *any* locally Lipschitz continuous initial data. We use a geometrical approach which consists in seeing the evolution of the graph of a solution as a geometric motion which is then studied by the so-called "level-set approach." After determining the right class of quasilinear parabolic pdes which can be taken into account by this approach, we show how the uniqueness for the original pde is related to "fattening phenomena" in the level-set approach. Existence of solutions is proved using a local L^∞-bound obtained by using in an essential way the level-set approach. Finally we apply these results to convex initial datas and prove existence and comparison results in full generality, i.e. without restriction on their growth at infinity.

Key-words: quasilinear parabolic equations, geometrical equations, unbounded solutions, viscosity solutions, existence and comparison results, level-set approach, mean curvature equation, convex solutions.
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1 Introduction

In a serie of works (see [7] for an introductive paper, [8], [6] and [9]), we are interested in quasilinear parabolic equations set in the whole space $I\!R^N$ and, more precisely, in existence and uniqueness properties for solutions with general growth at infinity.

The starting point is this work and our main motivation comes from a result of Ecker and Huisken [16] for the so-called mean curvature equation for graphs

$$\frac{\partial u}{\partial t} - \Delta u + \frac{\langle D^2uDu, Du\rangle}{1+|Du|^2} = 0 \quad \text{in } I\!R^N \times (0,\infty), \tag{1}$$

with the initial data

$$u(x,0) = u_0(x) \quad \text{in } I\!R^N, \tag{2}$$

where $u : I\!R^N \times [0,\infty) \to I\!R$ is the solution, Du and D^2u denote respectively the gradient and the Hessian matrix of u with respect to the space variable, $u_0 : I\!R^N \to I\!R$ is a given function and $|\cdot|$ (respectively $\langle\cdot,\cdot\rangle$) stands for the classical Euclidean norm (respectively inner product) in $I\!R^N$.

Ecker and Huisken proved the following very surprising result: for any initial data $u_0 \in W^{1,\infty}_{\rm loc}(I\!R^N)$, there exists a solution u of (1)-(2) in $C^\infty(I\!R^N \times (0,\infty)) \cap C(I\!R^N \times [0,\infty))$. This result was even extended to initial data in $C(I\!R^N)$ by Angenent [1]. The intriguing point is that no assumption is made on the growth of u_0 at infinity and therefore the solution u can have also an arbitrary behavior at infinity.

This result rises up a lot of challenging questions: the first one concerns the uniqueness of the solution they build. In general, the difficulty for obtaining a uniqueness result for a pde comes from the fact that one uses a notion of weak solution: this is not at all the case here since the solutions are known to be regular, even C^∞. The difficulty is really to take into account any behaviour for the solution at infinity.

A second question is related to the existence result itself: Ecker and Huisken proved it by using differential geometry and the maximum principle and it would be interesting to have a purely analytical proof of it. Again the lack of prescribed behavior of the solutions at infinity creates an unusual difficulty. In particular, to get a local L^∞ bound on u is a priori a key point but to obtain local L^∞ bound on Du is also a rather difficult task.

Finally, one can wonder to which type of quasilinear parabolic equations the result of Ecker and Huisken can be extended. For the reader, this may seem to be a question to be investigated later but, in fact, in order to provide interesting results for (1), one has to understand the main underlying structure of the equation which allow such a strange result to hold.

Our answer to this question is the geometrical interpretation of (1) by motion by mean curvature for graphs. Motions of hypersurfaces with general curvature dependent velocities were studied recently by the so-called "level-set approach," a weak notion for the evolution which allows to define these motions past the development of singularities. The level-set approach was first introduced by Osher and Sethian [31] for numerical computations and then studied from a theoretical point of view by Evans and Spruck [18] in the case of motion by mean curvature and by Chen, Giga and Goto [13] for more general normal velocities. Later, more singular cases were investigated by Ishii [27], Ishii and Souganidis [28] and properties of the level-set approach were obtained by Barles, Soner and Souganidis [10].

In the case of equation (1), as for any suitable quasilinear parabolic equations, the level-set approach arises when we consider the motion in dimension $N+1$. To do so, one has to introduce the function $v : I\!R^{N+1} \times [0,+\infty) \to I\!R$ defined by

$$v(x,y,t) = y - u(x,t).$$

For (1), the function v is a solution of

$$\frac{\partial v}{\partial t} - \Delta v + \frac{\langle D^2 v Dv, Dv\rangle}{|Dv|^2} = 0 \quad \text{in } I\!R^{N+1} \times (0,\infty), \tag{3}$$

which is the equation in the level-set approach corresponding to motion by mean curvature.

In order to give a suitable sense of solution for this singular equation and related ones in non-divergence form, we use the notion of viscosity solutions: we refer the reader to the Users' guide of Crandall, Ishii and Lions [15] or the books of Fleming and Soner [20], Bardi and Capuzzo Dolcetta [2], Barles [5] or Bardi et al. [3] for an introduction and/or a detailed presentation of this notion of solutions.

The most classical result concerning (3) is the well-posedness in the space of bounded uniformly continuous functions (BUC in short), more precisely: for any $v_0 \in BUC(\mathbb{R}^{N+1})$, there exists a unique solution v of (3) in $BUC(\mathbb{R}^{N+1} \times [0,T])$ for all $T > 0$ such that

$$w(x,y,0) = v_0(x,y) \quad \text{in } \mathbb{R}^{N+1}.$$

At this point, it is worth remarking that boundedness is not an issue: indeed, one of the key property of (3) is to be invariant by every nondecreasing change of functions: if v is a solution of (3), then $\tanh(v)$, or more generally $\Psi(v)$ with $\Psi' > 0$, is a solution as well.

Therefore it can be thought that the study of (1)–(2) just reduces to the study of (3) through the changes $v(x,y,t) = \tanh(y - u(x,t))$ and $v_0(x,y) = \tanh(y - u_0(x))$ and that all results follow easily from an extension of the above mentioned well-posedness result to spaces of bounded continuous functions (denoted by C_{b}), v and v_0 being clearly in C_{b} but not in BUC in general. In particular, the uniqueness of a solution u of (1)–(2) is an immediate consequence of a uniqueness result for solutions of (3) in C_{b}.

Unfortunately, we are unable to prove that the problem is well-posed in C_{b} and even the extensions to the well-posedness in BUC are rather weak. The concrete consequences of this geometrical approach are, on the one hand, a local L^∞ bound for a large class of quasilinear parabolic pdes whose proof is rather simple and natural and, on the other hand, a "generic" uniqueness result for the solutions of (1)–(2) as well as for more general equations.

It is worth poiting out that the possible non-uniqueness feature for (1)–(2) is related to the so-called "fattening phenomena" or "nonempty interior difficulty" for (3); despite of the fact that it seems obvious that no interior can develop because, by the maximum principle, one has formally

$$\frac{\partial v}{\partial y}(x,y,0) > 0 \ \text{ in } \mathbb{R}^{N+1} \quad \Longrightarrow \quad \frac{\partial v}{\partial y}(x,y,t) > 0 \ \text{ in } \mathbb{R}^{N+1} \times (0,+\infty),$$

but we are unable to prove this property even in a weaker sense.

Now, we turn to a more precise description of the contents of the present paper. It is devoted to the study of the geometrical approach, explained above in the special case of the mean curvature equation, for more general pdes like

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mathrm{Tr}\left[b(Du)D^2u\right] = 0 & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{4}$$

where b is a continuous function from $I\!R^N$ into the space of the nonnegative symmetric matrices $\mathcal{S}_N^+$. The first question we adress is: when is (4) associated to a geometric pde in dimension $N+1$ to which the level-set approach applies?

In Section 2, we derive formally a geometrical equation from (4) (see (6)). We then study this equation distinguishing two cases: the "classical" one is the one to which the classical level-set approach applies and the "very singular" one for which the more sophisticated arguments of Ishii [27] are necessary. For the reader's convenience, we show how the classical results apply in the "classical" case (Section 3). In the "very singular case" (Section 4), our comparison result enters the framework of [27]. Nevetheless, the particular singular set we deal with allows a more elementary proof. It has, in particular, the advantage to use explicit test-functions which permits to extend the proof to the case when the function b in (4) depends on (x, t, Du). Here, to avoid to much technicalities, we restrict ourselves to b depending only on Du and address this more general case in the forthcoming paper [9].

Then we study the consequence of this geometrical approach for (4). At first, we prove in Section 5 that the level-set approach works. A local L^∞-bound for the solutions of (4) follows rather easily (see Section 7) and the existence of discontinuous viscosity solutions of (4) is an almost immediate consequence of it (see Theorem 8.1 in Section 8). The existence of smooth solutions requires a local gradient bound and, to prove it, we use the one of Evans and Spruck [19] for (1) and the ones of Chou and Kwong [14] for the more general equation (4) (see Section 9).

Uniqueness is an even more difficult issue and we were able to obtain it only in particular cases: of course the first "generic" uniqueness result we provide in Section 6 is not satisfactory and most of the results we obtain in this direction are proved by working directly on (4). However a striking application of the geometrical approach to uniqueness (and also existence) is the case of convex initial datas (Section 10): under suitable assumptions on b (the same as for the geometrical approach to hold), we prove that there exists a unique solution u of (4), which is convex in the space variable at each time, and this for any convex initial data u_0 without any restriction on its growth at infinity. The proof relies strongly on the convexity preserving property of Giga, Goto, Ishii and Sato [21] that we extend to our more singular case. Compared to the result we previously obtained in [7] by working directly on (1), we do not assume anymore u_0 to be coercive and we extend the result to equations like (4).

For completeness, we conclude this introduction by describing the results we obtained in the next two part of this study. Two types of uniqueness results for the non-convex case are proved: the first ones [8] concern the case $N = 1$. We show, not only for (1) but for a larger class of equations, a uniqueness result without any growth assumptions on the solutions. Unfortunately, in general, this result is valid only in the class of classical solutions; however, in the case of (1), using the argument of Section 9 and in particular Remark 9.1, this uniqueness result for smooth solutions implies a comparison result between possibly discontinuous sub- and supersolution.

The proof relies upon examining the pde obtained by integrating in x. For (1), this pde reads

$$w_t - \arctan(w_{xx}) = 0 \quad \text{in } I\!R \times (0, +\infty), \tag{5}$$

and the key point is that (5) enjoys uniqueness properties in $C(I\!R)$, essentially because one can use a "friendly giants" method, whose consequence is a general uniqueness property for (1). Of course, this method can be extended to far more general equations. We learn recently that related results were obtained independently and by rather different methods by Chou and Kwong [14].

In the second one [6], we use classical viscosity solutions arguments to prove the uniqueness for solutions of (4) and even more general equations: we obtain a comparison result for sub- and supersolutions with polynomial growth but, unfortunately, with a rather restrictive assumption on the initial data which reads in the locally Lipschitz continuous case

$$|Du_0(x)| \leq C(1 + |x|^\nu) \quad \text{in } I\!R^N,$$

for some constant $C > 0$ and $0 \leq \nu < (1 + \sqrt{5})/2$. A strange feature of this result is that it can be obtained either by working directly on (4) or on the associated geometrical pde and both proofs lead to the same condition on u_0.

As we already mention it above, in a forthcoming paper, we investigate more general equations, namely

$$\frac{\partial u}{\partial t} - \mathrm{Tr}\left[b(x, t, Du)D^2u\right] + H(x, t, Du) = 0 \quad \text{in } I\!R^N \times (0, \infty) .$$

After we obtained most of the results described above, we learn that representation formulas for the mean curvature equation (3) (and even more general geometrical equations) have been established independently by Soner and Touzi [33], [34] and by Buckdahn, Cardaliaguet and Quincampoix [12]. We tried to prove uniqueness for (1) by showing that the "non-fattening phenomena" cannot occur for (3) in the case of graphs but we failed. It is an intriguing question whether it is possible to prove such properties by using these formulas.

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2 Derivation of a geometrical pde

As explain in the introduction for the special case of equation (1), we associate a geometrical equation to the quasilinear equation (4) which allows us to use the level-set approach.

This method was already used by Evans [17] for the heat equation and by Giga and Sato [22] in the case of Hamilton-Jacobi equations.

When u is a solution of (4) we consider, for every $t \geq 0$, $\mathrm{Graph}(u(\cdot, t))$ as an hypersurface in $\mathbb{R}^{N+1}$ and, to represent it, we follow the ideas of the level-set approach, taking any function $v : \mathbb{R}^N \times \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ such that

$$v(x, u(x,t), t) = 0 \quad \text{for every } (x,t) \in \mathbb{R}^N \times (0, +\infty).$$

Note that, for all $t \geq 0$, $\mathrm{Graph}(u(\cdot,t)) \subset \Gamma_t$, where Γ_t is the 0-level-set of $v(\cdot,\cdot,t)$. Differentiating formally the previous inequality, we obtain

$$\begin{aligned}
&D_y v \frac{\partial u}{\partial t} + \frac{\partial v}{\partial t} = 0, \\
&D_x v + D_y v \, Du = 0, \\
&D^2_{xx} v + 2D^2_{xy} v \otimes Du + D^2_{yy} v Du \otimes Du + D_y v D^2 u = 0,
\end{aligned}$$

and it follows that v has to solve, at least formally

$$\frac{\partial v}{\partial t} - \mathrm{Tr}\left[b\left(-\frac{D_x v}{D_y v}\right) \left(D^2_{xx} v - 2D^2_{xy} v \otimes \frac{D_x v}{D_y v} + D^2_{yy} v \frac{D_x v}{D_y v} \otimes \frac{D_x v}{D_y v} \right) \right] = 0 \tag{6}$$

in $\mathbb{R}^{N+1} \times (0, +\infty)$.

This new equation has strong discontinuities when the gradient of the solution, $Dv = (D_x v, D_y v)$ lies in the subset

$$\mathcal{D} = \{ p = (p_1, \cdots, p_{N+1}) \in \mathbb{R}^{N+1} : p_{N+1} = 0 \}, \tag{7}$$

but (6) satisfies the following first properties which is a motivation to study (4) via the level-set approach.

Lemma 2.1 *We have*

(i) Equation (6) is degenerate parabolic outside of $\mathcal{D}$.

(ii) If $u \in C(\mathbb{R}^N \times [0, +\infty))$ is a viscosity subsolution (respectively supersolution) of (4) with initial data $u_0 \in C(\mathbb{R}^N)$, then the function $v(x,y,t) = y - u(x,t)$ defined for $(x,y,t) \in \mathbb{R}^N \times \mathbb{R} \times [0, +\infty)$ is a viscosity supersolution (respectively subsolution) of (6) with initial data $v_0(x,y) = y - u_0(x)$.

(iii) Equation (6) is invariant under every monotone change of function $v \to \Psi \circ v$, where $\Psi \in C(\mathbb{R})$ is a monotone function.

We skip the proofs of these three properties since they do not present any difficulty. Let us mention that (ii) and (iii) are obvious in the smooth case. Property (ii) is straightforward using the definition of viscosity solutions for the singular equation (6) we recall in Section 4.1. For (iii), we even prove a discontinuous version of it in Lemma 4.1. Finally it is worth pointing out that we choose to work with $v(x,y,t) = y - u(x,t)$ instead of $u(x,t) - y$ as usual.

Remark 2.1 Concerning (ii), one can wonder whether some kind of converse property is true: if v is a solution of (6) with initial data $v_0(x,y) = y - u_0(x)$, does there exist a solution u of (4) such that $v(x,y,t) = y - u(x,t)$? The answer is not clear and it is the main issue of our approach. We refer to Section 6 for related discussions and results.

We conclude this section by introducing some notations which are used throughout the paper. Every point z of $\mathbb{R}^{N+1}$ is written $z = (x,y)$ with $x \in \mathbb{R}^N$ and $y \in \mathbb{R}$. In a natural way, every vector p which has the meaning of a gradient is written $p = (p_x, p_y)$ with $p_x \in \mathbb{R}^N$ and $p_y = p_{N+1} \in \mathbb{R}$. We decompose every matrix $X \in \mathcal{S}_{N+1}$ in blocks in the following way

$$X = \left(\begin{array}{c|c} X_{xx} & X_{xy} \\ \hline X_{xy}^T & X_{yy} \end{array}\right),$$

where $X_{xx} \in \mathcal{S}_N, X_{xy} \in \mathbb{R}^N$, X_{xy}^T is the transpose of X_{xy} (i.e. the row vector whose coordinates are the ones of X_{xy}) and $X_{yy} \in \mathbb{R}$.

With these notations, the nonlinearity involved in (6) can be written, for every $p \in \mathbb{R}^{N+1} - \mathcal{D}$ and $X \in \mathcal{S}_{N+1}$,

$$F(p,X) = -\mathrm{Tr}\left[b(q)\left(X_{xx} + 2X_{xy} \otimes q + X_{yy} q \otimes q\right)\right] = -\mathrm{Tr}[\tilde{b}(p)\, X] \tag{8}$$

where $q = -p_x/p_y$ and

$$\tilde{b}(p) = \left(\begin{array}{c|c} b(q) & b(q)q \\ \hline (b(q)q)^T & \langle b(q)q, q\rangle \end{array}\right).$$

3 The geometrical equation: the classical framework

A priori the nonlinearity F is discontinuous on $\mathcal{D}$ (see (7)). In this section, we provide assumptions on b ensuring that we are in the "classical framework," which means that the (classical) level-set approach applies readily to (6) (see [18], [13] [21] and [10]). In this classical framework, F has to be continuous, except at $p=0$. The typical example is the mean curvature equation (see Example 3.1).

More precisely, we start by recalling the assumptions as they appears in [21]. In the sequel, $\|\cdot\|$ is any norm on $\mathcal{S}_N$ and $S^{N-1}=\{\xi\in I\!R^N : |\xi|=1\}$ is the unit sphere of $I\!R^N$.

(F1) $F:(I\!R^{N+1}-\{0\})\times\mathcal{S}_{N+1}\to I\!R$ is continuous.

(F2) $F(p,X+Y)\le F(p,X)$ for all $p\in I\!R^{N+1}$, $X,Y\in\mathcal{S}_{N+1}, Y\ge 0$.

(F3) $-\infty<F_*(0,0)=F^*(0,0)<+\infty$ where F_* and F^* are the semicontinuous envelopes of F defined by $F_*(p,X)=\liminf_{(\rho,Y)\to(p,X)}\{F(\rho,Y):\rho\neq 0\}$ and $F^*=-(-F)_*$.

(F4) For every $R>0$, $\sup\{|F(p,X)| : |p|\le R, \|X\|\le R\}<+\infty$.

We have, the following classical result.

Theorem 3.1 *Under assumptions* ***(F1)–(F4)****, for any initial data* $v_0\in UC(I\!R^{N+1})$*, there exists a unique solution* v *of (6) which is in* $UC(I\!R^{N+1}\times[0,T))$ *for every* $T>0$*.*

Notice that, if F is continuous, then **(F1)–(F4)** reduce to **(F2)** only.

We state now the assumptions on b which permit to extend the F given by (8) by continuity in $(I\!R^{N+1}-\{0\})\times\mathcal{S}_{N+1}$ in order to ensure that **(F1)–(F4)** hold.

(H1) There is a positive constant K_1 such that $\|b(q)\|\le K_1$ for all $q\in I\!R^N$.

(H2) There is a positive constant K_2 such that $|b(q)q|\le K_2$ for every $q\in I\!R^N$.

(H3) There is a positive constant K_3 such that $|\langle b(q)q,q\rangle|\le K_3$ for every $q\in I\!R^N$.

(H4) For every $q\in S^{N-1}$, $\lim_{\lambda\to+\infty}b(\lambda q)$ and $\lim_{\lambda\to-\infty}b(\lambda q)$ exist and are equal. Moreover $b_\infty(q):=\lim_{\lambda\to\pm\infty}b(\lambda q)$ is continuous on S^{N-1}.

(H5) For every $q\in S^{N-1}$, $\lim_{\lambda\to+\infty}\lambda b(\lambda q)q$ and $\lim_{\lambda\to-\infty}\lambda b(\lambda q)q$ exist and are equal. Moreover $\zeta_\infty(q):=\lim_{\lambda\to\pm\infty}\lambda b(\lambda q)q$ is continuous on S^{N-1}.

(H6) For every $q\in S^{N-1}$, $\lim_{\lambda\to+\infty}\lambda^2\langle b(\lambda q)q,q\rangle$ and $\lim_{\lambda\to-\infty}\lambda^2\langle b(\lambda q)q,q\rangle$ exist and are equal. Moreover the function $\alpha_\infty(q):=\lim_{\lambda\to\pm\infty}\lambda^2\langle b(\lambda q)q,q\rangle$ is continuous on S^{N-1}.

Proposition 3.1 *Let* F *be defined by (8) with* $b\in C(I\!R^N;\mathcal{S}_N^+)$*. Then assumptions* ***(H1)–(H6)*** *are equivalent to assumptions* ***(F1)–(F4)****. It follows that, under assumptions* ***(H1)–(H6)****, Theorem 3.1 hold.*

Proof of Proposition 3.1. We use the notations of Section 2. We start by assuming **(H1)**–**(H6)**. For every $((p_x, 0), X) \in (\mathcal{D} - \{0\}) \times \mathcal{S}_{N+1}$, we extend F by setting

$$F((p_x,0),X) = -\mathrm{Tr}\left[b_\infty(p_x/|p_x|)X_{xx} + 2\langle \zeta_\infty(p_x/|p_x|), X_{xy}\rangle + X_{yy}\alpha_\infty(p_x/|p_x|)\right].$$

From the assumed continuity of b_∞, ζ_∞ and α_∞ on S^{N-1}, the extended Hamiltonian F is clearly continuous in $\mathbb{R}^{N+1} - \{0\}$; thus **(F1)** holds. Assumption **(F2)** is an immediate consequence of Lemma 2.1 (i). Finally, from the boundedness conditions **(H1)**, **(H2)** and **(H3)**, it is obvious that $F_*(0,0) = F^*(0,0) = 0$ and $|F(p,X)| \le K_1 R + 2K_2 R + K_3 R$ for $|p|, \|X\| \le R$. It shows that **(F3)** and **(F4)** hold.

Conversely, suppose that **(F1)**–**(F4)** hold. **(F2)** implies easily that $b(\xi)$ is positive for all $\xi \in \mathbb{R}^N$. Let $\xi \in S^{N-1}$; for any $\lambda \neq 0$, we have

$$\tilde{b}(q, 1/\lambda) = \left(\begin{array}{c|c} b(\lambda q) & \lambda b(\lambda q) q \\ \hline (\lambda b(\lambda) q)^T & \lambda^2 \langle b(\lambda q) q, q\rangle \end{array}\right). \tag{9}$$

From **(F4)**, we have that $\tilde{b}(q,1/\lambda)$ is bounded for every $q \in S^{N-1}$ and $\lambda \geq 1$. It follows that $\|b(\xi)\|, |b(\xi)\xi|$ and $|\langle b(\xi)\xi, \xi\rangle|$ are bounded for every $\xi \in \{\lambda q : \lambda \neq 0, q \in S^{N-1}\} = \mathbb{R}^N - \overline{B}(0,1)$. Since these quantities are obviously bounded in $\overline{B}(0,1)$, we get **(H1)**–**(H3)**. From **(F1)**, we have that $\tilde{b}$ is continuous in $\mathbb{R}^{N+1} - \{0\}$. On the one hand, it follows easily that b is continuous in $\mathbb{R}^N$. On the other hand, sending λ to $\pm\infty$ in (9), we see that b_∞, ζ_∞ and α_∞ are well-defined:

$$\lim_{\lambda \to \pm\infty} \tilde{b}(q, 1/\lambda) = \tilde{b}(q, 0) = \left(\begin{array}{c|c} b_\infty(q) & \zeta_\infty(q) \\ \hline (\zeta_\infty(q))^T & \alpha_\infty(q)\rangle \end{array}\right). \tag{10}$$

Invoking again the continuity of $\tilde{b}$, (10) implies that b_∞, ζ_∞ and α_∞ are continuous on S^{N-1}; thus **(H4)**–**(H6)** hold. It ends the proof. $\square$

Proposition 3.1 applies of course in the case of the mean curvature equation. The computations are developed in the following example.

Example 3.1 Mean curvature equation (1).
In this case,

$$b(q) = I - \frac{q \otimes q}{1 + |q|^2} \quad \text{for every } q \in \mathbb{R}^N. \tag{11}$$

Easy computations show that

$$F(p,X) = -\mathrm{Tr}\left[\left(I - \frac{p \otimes p}{|p|^2}\right) X\right] \quad \text{for every } (p,X) \in (\mathbb{R}^{N+1} - \{0\}) \times \mathcal{S}_{N+1}$$

and then, (6) associated to (11) is the classical geometric mean curvature equation.

The checking of **(H1)**–**(H6)** consists in straightforward computations. We obtain $b_\infty(q) = I - q \otimes q, \zeta_\infty(q) = 0$ and $\alpha_\infty(q) = 1$ for every $q \in S^{N-1}$.

4 The geometrical equation: the very singular case

In this section, we study the case when the discontinuities of F on $\mathcal{D}$ cannot be reduced to a discontinuity at $p = 0$. This question was adressed by many authors: Goto [23], Ishii and Souganidis [28], Ishii [27] or Ohnuma and Sato [30].

Ishii [27] deals with the worst set of singularities. Our approach is strongly inspired by his work: Ishii extends the notion of viscosity by restricting the class of test-functions. His result applies but we provide a simpler proof which relies on the special form of our set of singularities.

We refer to the end of the section for examples of pde which is covered by our framework but which does not satisfies the assumptions of Section 3.

4.1 Definitions and first properties

We recall the definition of viscosity solutions for very singular equations as it appears in Ishii [27].

In the sequel, USC (respectively LSC) denotes the set of upper-semicontinuous (respectively lower-semicontinuous) functions. For any locally bounded function v, v^* and v_* are respectively the upper- and lower-semicontinuous envelopes of v and $\mathcal{P}^{2,+}(v^*)$ and $\mathcal{P}^{2,-}(v_*)$ are its parabolic semijets (see [15] for a definition).

We define semicontinuous envelopes for F, which are *adapted* to the set of discontinuity $\mathcal{D}$, by, for every $(p, X) \in I\!\!R^{N+1} \times \mathcal{S}_{N+1}$,

$$\begin{aligned} F^*(p,X) &= \limsup_{(\rho,Y)\to(p,X)} \{F(\rho,Y) : (\rho,Y) \in (I\!\!R^{N+1} - \mathcal{D}) \times \mathcal{S}_{N+1}\}, \\ F_*(p,X) &= \liminf_{(\rho,Y)\to(p,X)} \{F(\rho,Y) : (\rho,Y) \in (I\!\!R^{N+1} - \mathcal{D}) \times \mathcal{S}_{N+1}\}. \end{aligned}$$

Clearly, F^* and F_* inherit the same properties as F: they are still degenerate elliptic and geometric.

Definition 4.1 *A locally bounded function $v : I\!\!R^{N+1} \times (0, +\infty) \mapsto I\!\!R$ is said to be a viscosity subsolution (respectively a supersolution) of (6) if and only if*

$$a + F_*(p, X) \le 0 \quad \textit{for all } (x,t) \in I\!\!R^{N+1} \times (0,+\infty) \textit{ and } (a,p,X) \in \mathcal{P}^{2,+}(v^*)(x,t)$$

(respectively

$$a + F^*(p, X) \ge 0 \quad \textit{for all } (x,t) \in I\!\!R^{N+1} \times (0,+\infty) \textit{ and } (a,p,X) \in \mathcal{P}^{2,-}(v_*)(x,t) \text{).}$$

A discontinuous function v is a viscosity solution of (6) provided it is both a sub- and a supersolution.

With this definition, all the basic properties of viscosity solutions extend. In particular the classical stability result for viscosity solutions holds. The proof is the same than the ones in the classical references given in the introduction.

We continue with the *invariance lemma* which is a characteristic of geometric equations (Cf. Section 2).

Lemma 4.1 *If $u \in USC(\mathbb{R}^N \times [0,+\infty))$ (respectively $v \in LSC(\mathbb{R}^N \times [0,+\infty))$) is a viscosity subsolution (respectively supersolution) of (6), then, for any nondecreasing function $\Psi \in USC(\mathbb{R})$ (respectively $\Psi \in LSC(\mathbb{R})$) the function $\Psi \circ u$ (respectively $\Psi \circ v$) is a viscosity subsolution (respectively supersolution) of the same equation.*

Proof of Lemma 4.1. We will prove the assertion in the case of a subsolution; the proof for supersolutions is analogous. We proceed by approximation of Ψ.

Consider first the case $\Psi \in C(\mathbb{R})$. We construct a increasing familly $(\Psi_\varepsilon)_{\varepsilon>0}$ of smooth strictly increasing functions converging to Ψ. Let ϕ be a C^2 function and (x_0,t_0) be a local maximum of $\Psi_\varepsilon(v)-\phi$. Without loss of generality, we can suppose that $(\Psi_\varepsilon(v)-\phi)(x_0,t_0)=0$. It follows that, for every $x \in \mathbb{R}^{N+1}$ and $t \in [0,+\infty)$,

$$\Psi_\varepsilon(v)(x,t) \le \phi(x,t) \Longleftrightarrow v(x,t) \le \Phi_\varepsilon \circ \phi(x,t),$$

where we set $\Phi_\varepsilon = (\Psi_\varepsilon)^{-1}$. Thus (x_0,t_0) is a local maximum of $v - \Phi_\varepsilon \circ \phi$ and since v is a subsolution of (6), we get

$${\Phi_\varepsilon}' \frac{\partial \phi}{\partial t}(x_0,t_0) + F_*\left({\Phi_\varepsilon}' D\phi(x_0,t_0), {\Phi_\varepsilon}' D^2\phi(x_0,t_0) + {\Phi_\varepsilon}'' D\phi \otimes D\phi\right) \le 0. \tag{12}$$

Using that F_* is geometric and dividing (12) by ${\Phi_\varepsilon}' > 0$, we get

$$\frac{\partial \phi}{\partial t}(x_0,t_0) + F_*\left(D\phi(x_0,t_0), D^2\phi(x_0,t_0)\right) \le 0$$

which proves that $\Psi_\varepsilon \circ v$ is a subsolution of (6). From classical results about viscosity solutions, we have that

$$\limsup_{\varepsilon\to 0}{}^{*}\, \Psi_\varepsilon(v) = \sup_{\varepsilon>0} \Psi_\varepsilon \circ v = \Psi \circ v$$

is a subsolution of (6) which is the desired result.

Next, if one has $\Psi \in USC(\mathbb{R})$, we define an increasing family $(\Psi_\varepsilon)_{\varepsilon>0}$ of continuous increasing functions converging to $\Psi^* = \Psi$, by setting

$$\Psi_\varepsilon(x) = \inf_{|y-x|\le 1} \left\{ \Psi(y) + \frac{|x-y|^2}{\varepsilon^2} \right\}$$

for every $x \in \mathbb{R}$. From the continuous case studied above, we get that $\Psi_\varepsilon \circ v$ is a subsolution of (6) for every $\varepsilon > 0$ and conclude in the same way. □

4.2 Comparison result

We turn now to a comparison principle for (6).

Theorem 4.1 *Suppose that **(H1)**–**(H4)** hold and let* $v_0 \in UC(I\!R^{N+1})$. *If* $v_1 \in USC(I\!R^{N+1} \times [0, +\infty))$ *(respectively* $v_2 \in LSC(I\!R^{N+1} \times [0, +\infty))$*) is a subsolution (respectively a supersolution) of (6), and if* $v_1(\cdot, 0) \le v_0 \le v_2(\cdot, 0)$ *in* $I\!R^{N+1}$, *then* $v_1 \le v_2$ *in* $I\!R^{N+1} \times [0, +\infty)$.

Remark 4.1 Note that "bounded" or "unbounded" solutions is not the point in this theorem. Since the equation is geometric, up to make a change of variable $v \mapsto \tanh(v)$ together with Lemma 4.1, we can suppose that the solutions are bounded. Another remark is that we are able to compare bounded continuous solutions with bounded uniformly continuous solutions. Of course, it gives uniqueness only in $UC(I\!R^{N+1} \times [0, +\infty))$.

The difficulty to prove such a result comes obviously from the unusual set of discontinuity $\mathcal{D}$. We begin by some arguments giving an idea of the proof.

Setting

$$\mathcal{S}(\mathcal{D}) = \{X \in \mathcal{S}_{N+1} : X_{xy} = 0, X_{yy} = 0\} = \left\{ \left(\begin{array}{c|c} X_{xx} & 0 \\ \hline 0 & 0 \end{array} \right) \; : \; X_{xx} \in \mathcal{S}_N \right\}, \tag{13}$$

we have

Lemma 4.2 *Assume **(H1)**–**(H4)**. Then, for all* $p \in \mathcal{D} - \{0\}$ *and* $X \in \mathcal{S}(\mathcal{D})$, *we have* $F^*(p, X) = F_*(p, X)$. *Moreover* $F^*(0,0) = 0 = F_*(0,0)$.

This lemma is proved at the end of the section. It suggests that we may use in the proof of the theorem test-functions φ such that $D^2\varphi \in \mathcal{S}(\mathcal{D})$ when $D\varphi \in \mathcal{D}$. By this way, one does not see the discontinuities of F in the proof.

Proof of Theorem 4.1. Without loss of generality, we can assume that v_1 and v_2 are bounded. We recall that we write z for a point $z = (x, y) \in I\!R^N \times I\!R$ and by $|z|$ we mean $(|x|^2 + y^2)^{1/2}$. We argue by contradiction, assuming that there exists $(z_0, t_0) \in I\!R^{N+1} \times [0, T)$ such that $(v_1 - v_2)(z_0, t_0) > 0$. We introduce the function

$$\phi(z_1, z_2) = \frac{|x_1 - x_2|^4}{\varepsilon^4} + \frac{|y_1 - y_2|^4}{\varepsilon^4} = \varphi(z_1 - z_2)$$

which is chosen in order to ensure that $D^2\varphi(Z) \in \mathcal{S}(\mathcal{D})$ when $D\varphi(Z) \in \mathcal{D}$ (for the definitions of $\mathcal{D}$ and $\mathcal{S}(\mathcal{D})$, see (7) and (13)). We then set

$$M_{\varepsilon,\alpha,\eta} = \sup_{I\!R^{N+1} \times I\!R^{N+1} \times [0,+\infty)} \left\{ v_1(z_1, t) - v_2(z_2, t) - \phi(z_1, z_2) - \alpha(|z_1|^2 + |z_2|^2) - \eta t \right\}.$$

At first, it is clear that $M_{\varepsilon,\alpha,\eta} > 0$ for α, η sufficiently small since $\phi(z_0, z_0) = 0$. Moreover $M_{\varepsilon,\alpha,\eta}$ is achieved at some $(\bar{z}_1, \bar{z}_2, \bar{t})$ by the boundedness and the semicontinuous properties of v_1 and v_2. Actually $\bar{z}_1$ and $\bar{z}_2$ depend on $\alpha, \varepsilon, \eta$, but we omit this dependence in the notation for simplicity.

When $\bar{t} = 0$, we have

$$\begin{aligned} 0 < M_{\varepsilon,\alpha,\eta} &\leq v_1(\bar{z}_1, 0) - v_2(\bar{z}_2, 0) - \frac{|\bar{x}_1 - \bar{x}_2|^4}{\varepsilon^4} - \frac{|\bar{y}_1 - \bar{y}_2|^4}{\varepsilon^4} - \alpha(|\bar{z}_1|^2 + |\bar{z}_2|^2) \\ &\leq v_0(\bar{z}_1) - v_0(\bar{z}_2) - \frac{|\bar{x}_1 - \bar{x}_2|^4}{\varepsilon^4} - \frac{|\bar{y}_1 - \bar{y}_2|^4}{\varepsilon^4} - \alpha(|\bar{z}_1|^2 + |\bar{z}_2|^2) \end{aligned}$$

which leads to a contradiction using the uniform continuity of v_0. Thus, it cannot exist a subsequence of parameters (ε, α) going to $(0, 0)$ such that $\bar{t} = 0$. Therefore, we can suppose that $\bar{t} > 0$ for ε and α sufficiently small.

From the fundamental result of the Users' guide to viscosity solutions [15, Theorem 8.3], for every $\rho > 0$, we get $a_1, a_2 \in I\!R$ and $X, Y \in \mathcal{S}_{N+1}$ such that

$$(a_1, D\varphi(\bar{z}_1 - \bar{z}_2) + 2\alpha\bar{z}_1, X + 2\alpha I) \in \bar{\mathcal{P}}^{2,+}(v_1)(\bar{z}_1, \bar{t}),$$

$$(a_2, D\varphi(\bar{z}_1 - \bar{z}_2) - 2\alpha\bar{z}_2, Y - 2\alpha I) \in \bar{\mathcal{P}}^{2,-}(v_2)(\bar{z}_2, \bar{t})$$

and

$$-(\frac{1}{\rho} + \|A\|) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \begin{pmatrix} A + 2\rho A^2 & -(A + 2\rho A^2) \\ -(A + 2\rho A^2) & A + 2\rho A^2 \end{pmatrix} \tag{14}$$

for some $a_1 - a_2 = \eta$ and $A = D^2\varphi(\bar{z}_1 - \bar{z}_2)$. We compute, for every $Z = (Z_x, Z_y) \in I\!R^N \times I\!R$,

$$D\varphi(Z) = \frac{4}{\varepsilon^4}(|Z_x|^2 Z_x, Z_y^3) \quad \text{and} \quad A = D^2\varphi(Z) = \frac{4}{\varepsilon^4}\left(\begin{array}{c|c} 2Z_x \otimes Z_x + |Z_x|^2 I & 0 \\ \hline 0 & 3Z_y^2 \end{array}\right).$$

Since $M_{\varepsilon,\alpha,\eta} > 0$, we get

$$\phi(\bar{z}_1, \bar{z}_2, \bar{t}) + \alpha(|\bar{z}_1|^2 + |\bar{z}_2|^2) \leq \|v_1\|_\infty + \|v_2\|_\infty$$

(recall that v_1 and v_2 are assumed to be bounded). It follows that

$$\lim_{\alpha \to 0^+} \alpha|\bar{z}_1|, \lim_{\alpha \to 0^+} \alpha|\bar{z}_2| = 0, \tag{15}$$

$$|\bar{z}_1 - \bar{z}_2| \text{ is bounded as } \alpha \text{ goes to } 0. \tag{16}$$

From (16), (15) and (14), we obtain that X and Y are bounded when α goes to 0. It follows easily that a_1, a_2 are also bounded; thus we can extract subsequences such that $\bar{z}_1 - \bar{z}_2 \to Z$ and

$$(a_1, D\varphi(\bar{z}_1 - \bar{z}_2) + 2\alpha\bar{z}_1, X + 2\alpha I) \longrightarrow \big(\bar{a}_1, D\varphi(Z), \bar{X}\big) \in \bar{\mathcal{P}}^{2,+}(v_1)(\bar{z}_1, \bar{t}),$$

$$(a_2, D\varphi(\bar{z}_1 - \bar{z}_2) - 2\alpha\bar{z}_2, Y - 2\alpha I) \longrightarrow \big(\bar{a}_2, D\varphi(Z), \bar{Y}\big) \in \bar{\mathcal{P}}^{2,-}(v_2)(\bar{z}_2, \bar{t}),$$

when α goes to 0. Note that $\bar{X}, \bar{Y}$ satisfy also (14) with $A = D^2\varphi(Z)$. Writing that v_1 is a subsolution and v_2 a supersolution of (6), we get

$$\eta + F_*(D\varphi(Z), \bar{X}) - F^*(D\varphi(Z), \bar{Y}) \leq 0. \tag{17}$$

Now, if $D\varphi(Z) \notin \mathcal{D}$, then we are done since (14) implies that $\bar{X} \leq \bar{Y}$ and since in this case $F^* = F_* = F$ is degenerate elliptic.

But, when $D\varphi(Z) \in \mathcal{D}$, we need more information about $\bar{X}, \bar{Y}$ in order to get the contradiction. At first, $D\varphi(Z) \in \mathcal{D}$ implies $Z_y = 0$; thus

$$D^2\varphi(Z) = \left(\begin{array}{c|c} 2Z_x \otimes Z_x + |Z_x|^2 I & 0 \\ \hline 0 & 0 \end{array} \right) \in \mathcal{S}(\mathcal{D}). \tag{18}$$

At this step, we would like to apply Lemma 4.2 but we need first to transfer on $\bar{X}, \bar{Y}$ the suitable property of $D^2\varphi(Z)$, namely $D^2\varphi(Z) \in \mathcal{S}(\mathcal{D})$. To this end, we state

Lemma 4.3 *If $D^2\varphi(Z) \in \mathcal{S}(\mathcal{D})$, then there exist $X', Y' \in \mathcal{S}(\mathcal{D})$ such that*

$$\bar{X} \leq X' \leq Y' \leq \bar{Y}.$$

Moreover, $X' = Y' = 0$ when $D\varphi(Z) = 0$.

We postpone the proof and complete the one of Theorem 4.1. Taking advantage of the ellipticity of F^* and F_* together with Lemma 4.2, we get from (17)

$$\eta + F^*(D\varphi(Z), X') - F^*(D\varphi(Z), Y') \leq 0.$$

Since $X' \leq Y'$, the ellipticity of F^* leads to a contradiction. It achieves the proof of the theorem. $\square$

We turn to the proof of the lemmas.

Proof of Lemma 4.2. Let us consider $((p_x, 0), X) \in \mathcal{D} \times \mathcal{S}(\mathcal{D})$. It suffices to see that $F(\rho, X + Y)$ has a limit when $(\rho, Y) \to ((p_x, 0), 0)$, $(\rho, Y) \in (\mathbb{R}^{N+1} - \mathcal{D}) \times \mathcal{S}_{N+1}$. Since $X \in \mathcal{S}(\mathcal{D})$, we have

$$\begin{aligned} F(\rho, X + Y) &= F(\rho, X) + F(\rho, Y) \\ &= -\mathrm{Tr}\left[b\left(-\frac{\rho_x}{\rho_y}\right) X_{xx} \right] - \mathrm{Tr}\left[b\left(-\frac{\rho_x}{\rho_y}\right) \left(Y_{xx} - 2Y_{xy} \otimes \frac{\rho_x}{\rho_y} + Y_{yy} \frac{\rho_x}{\rho_y} \otimes \frac{\rho_x}{\rho_y} \right) \right]. \end{aligned}$$

At first, from **(H1)**, **(H2)** and **(H3)**, when $\|Y\| \leq \varepsilon$, we obtain

$$\left| Tr\left[b\left(-\frac{\rho_x}{\rho_y}\right) \left(Y_{xx} - 2Y_{xy} \otimes \frac{\rho_x}{\rho_y} + Y_{yy} \frac{\rho_x}{\rho_y} \otimes \frac{\rho_x}{\rho_y} \right) \right] \right| \leq O(\varepsilon) \underset{\varepsilon \to 0}{\to} 0. \tag{19}$$

If $p_x = 0$, then $p = 0$ and (19) implies that $F^*(0,0) = 0 = F_*(0,0)$. If $p \in \mathcal{D} - \{0\}$, then $p_x \neq 0$ and from **(H4)**, we get

$$\lim_{\rho \to (p_x, 0)} \operatorname{Tr}\left[b\left(-\frac{\rho_x}{\rho_y} \right) X_{xx} \right] = \operatorname{Tr}\left[b_\infty(p_x/|p_x|) X_{xx} \right].$$

It achieves the proof of the lemma. $\square$

Proof of Lemma 4.3. We set $A = D^2\varphi(Z)$, $B = A + 2\rho A^2$ (see (14)) and $B_\delta = B + \delta I$ for $\delta > 0$. Note that $B \in \mathcal{S}(\mathcal{D})$ and $B \geq 0$ as we can see it with the help of Formula (18). Moreover, we get from (14) that

$$\langle \bar{X}p, p\rangle - \langle \bar{Y}q, q\rangle \leq \langle B(p-q), p-q\rangle < \langle B_\delta(p-q), p-q\rangle. \tag{20}$$

It provides in particular that $\bar{X} < B_\delta$ (respectively $-\bar{Y} < B_\delta$); thus $(\bar{X} - B_\delta)$ (respectively $(\bar{Y} + B_\delta)$) is invertible. We then obtain X' using a sup-convolution. We set, for every $p, r \in I\!R^{N+1}$ and $k > 1$,

$$F_r(p) = \langle \bar{X}p, p\rangle - \langle kB_\delta(p-r), p-r\rangle \tag{21}$$

and consider $\sup\limits_{p \in I\!R^N} F_r(p)$ which is well-defined. For every $h \in I\!R^N$,

$$\langle DF_r(p), h\rangle = 2\langle \bar{X}p, h\rangle - 2\langle kB_\delta(p-r), h\rangle$$

which provides that the supremum is achieved for

$$p = (kB_\delta - \bar{X})^{-1} kB_\delta r$$

(note that $(kB_\delta - \bar{X})$ is invertible since $\bar{X} < B_\delta < kB_\delta$). Next, an explicite but tedious computation yields a matrix $X' \in \mathcal{S}_{N+1}$ such that

$$\sup_{p \in I\!R^{N+1}} F_r(p) = \langle X'r, r\rangle.$$

Taking successively the particular value $p = r$ and $p = 0$ in (21) we get $X' \geq \bar{X}$ and $X' \geq -kB_\delta$. Similarly, we can construct Y' by setting, for $k > 1$,

$$\inf_{p \in I\!R^{N+1}} \left\{ \langle \bar{Y}p, p\rangle + \langle kB_\delta(p-s), p-s\rangle \right\} = \langle Y's, s\rangle.$$

We obtain a matrix Y' satisfying $Y' \leq \bar{Y}$ and $Y' \leq kB_\delta$.

From (20), we get

$$\begin{aligned} &\langle X'r, r\rangle - \langle Y's, s\rangle \\ &\leq \sup_{p,q \in I\!R^{N+1}} \left\{ \langle B_\delta(p-q), p-q\rangle - \langle kB_\delta(p-r), p-r\rangle - \langle kB_\delta(q-s), q-s\rangle \right\} \end{aligned}$$

for any $r, s \in I\!R^{N+1}$. An explicit calculation of this supremum yields

$$\langle X'r, r\rangle - \langle Y's, s\rangle \leq \langle B_\delta(r-s), r-s\rangle.$$

By taking $r = s$, we obtain $\langle X'r, r\rangle - \langle Y'r, r\rangle \leq 0$ for every $r \in I\!R^{N+1}$. It follows $-kB_\delta \leq X' \leq Y' \leq kB_\delta$. Now, letting δ go to 0, up to extract a subsequence, we get two matrices, still denoted by X', Y', such that

$$-kB \leq X' \leq Y' \leq kB. \tag{22}$$

Recalling that $B \in \mathcal{S}(\mathcal{D})$, we get first that $X'_{yy} = Y'_{yy} = 0$. Then, from (22), for any $r \in I\!R^{N+1}$, we have

$$\langle X'r, r\rangle = \langle X'_{xx}r_x, r_x\rangle + 2\langle X'_{xy}, r_x\rangle r_y \leq k\langle Br, r\rangle \leq K|r_x|^2$$

for some positive constant K. Taking tr_x instead of r_x for $t \in I\!R$, we get

$$t^2\langle X'_{xx}r_x, r_x\rangle + 2t\langle X'_{xy}, r_x\rangle r_y \leq Kt^2|r_x|^2$$

which provides $\langle X_{xy}, r_x\rangle = 0$ dividing by t and letting t go to 0^+ or 0^-. Since it holds for any $r_x \in I\!R^N$ we are done. The same arguments hold for Y'. Finally, if $D\varphi(Z) = 0$, then $Z_x = 0, Z_y = 0$; it implies $A = B = 0$. From (22), we get $X' = Y' = 0$ which completes the proof of the lemma. □

4.3 Existence of solutions

Our result is the

Theorem 4.2 *Assume* ***(H1)–(H4)****. For every $v_0 \in UC(I\!R^{N+1})$, there exists a unique $v \in UC\left(I\!R^{N+1} \times [0, +\infty)\right)$ solving (6) with initial data v_0.*

The proof uses the classical *Perron's method*, introduced in the framework of viscosity solutions by Ishii in [26] (see also [2], [5] or [15]). The application of this method in our case does not present any special difficulties. Nevertheless, we provide a proof for the readers' convenience.

Proof of Theorem 4.2. The uniqueness part comes immediately from Theorem 4.1 and because of Lemma 4.1, we can suppose that v_0 is bounded. We divide the proof in different steps.

Step 1. We construct a solution $v \in C(I\!R^{N+1} \times [0, +\infty))$ when the initial data is smooth. Let $v_0 \in C^2(I\!R^{N+1}) \cap W^{2,\infty}(I\!R^{N+1})$ and define, for any $C > 0$, two functions $\underline{u}, \overline{v}$ by setting

$$\underline{v}(z, t) := -Ct + v_0(z) \quad \text{and} \quad \overline{v}(z, t) = Ct + v_0(z)$$

for any $(z,t) \in I\!R^{N+1} \times [0,+\infty)$. It follows from **(H1)–(H3)** that the nonlinearity F appearing in (6) is bounded on bounded subsets. Therefore, C may be chosen large enough in order that $\underline{v}$ and $\overline{v}$ are respectively sub and super solution of (6).

Consider then the set $\mathcal{F}$ of subsolutions of (6) w such that $\underline{v} \le w \le \overline{v}$. Set then for every $(z,t) \in I\!R^{N+1} \times [0,+\infty)$, $v(z,t) = \sup_{w\in\mathcal{F}} w(z,t)$. The set $\mathcal{F}$ is nonempty and v is well-defined. Thus, we get from the comparison result and classical arguments of the Perron's method that v is a discontinuous solution of (6).

Now, we also have from the definition of v that $v^*(\cdot,0) = v_*(\cdot,0) = v_0$. Thus we deduce from the comparison result that $v^* = v_* = v$ which is the desired continous solution.

Step 2. We show that the solution v built in Step 1 is actually in $BUC(I\!R^{N+1}\times[0,+\infty))$. First of all, since the constant functions are smooth solutions of (6), the comparison result provides that v is bounded, namely $\|v\| \le \|v_0\|$, where here and below, $\|\cdot\|$ will denote the sup norm on continuous functions removing the set where they are defined when there is no ambiguity.

Next, from the definition of v, for all $h>0$, we have

$$\underline{v}(\cdot,h) = v_0 - Ch \le v(\cdot,h) \le v_0 + Ch = \overline{v}(\cdot,h) \quad \text{in } I\!R^{N+1}. \tag{23}$$

From Theorem 4.1 and since the nonlinearity in (6) depends only on (Du, D^2u), the function $v(\cdot,\cdot+h)$ is a solution of (6) with initial data $v(\cdot,h)$ and $v\pm Ch$ are solutions of (6) with initial data $v_0 \pm Ch$. Thus Theorem 4.1, together with (23), yields

$$v - Ch \le v(\cdot,\cdot+h) \le v + Ch \quad \text{in } I\!R^{N+1} \times [0,+\infty).$$

It provides a modulus of continuity in the time variable which is independent of the space variable. Arguing in the same way with translation in space $u_0 \mapsto u_0(\cdot+h)$ we obtain a modulus of continuity for the space variable which is independent on the time variable. It proves that $v \in BUC(I\!R^{N+1}\times[0,+\infty))$.

Step 3. The general case when $v_0 \in BUC(I\!R^{N+1})$. Using a classical convolution procedure, we construct a sequence $(v_0^n)_{n\in I\!N}$ of functions $v_0^n \in C^2(I\!R^{N+1}) \cap W^{2,\infty}(I\!R^{N+1})$ such that $\|v_0 - v_0^n\| \le 1/n$. It follows $-2/n + v_0^n \le v_0^m \le v_0^n + 2/n$ for $m \ge n$. According to Steps 1 and 2, we can consider, for every $n\in I\!N$, the unique solution $v^n \in BUC(I\!R^{N+1}\times[0,+\infty))$ of (6) with initial data v_0^n. Proceeding as in Step 2, we deduce from the previous inequality that, $-2/n + v^n \le v^m \le v^n + 2/n$ for $m \ge n$, Thus $(v^n)_{n\in I\!N}$ converges uniformly in $I\!R^{N+1}\times[0,+\infty)$ to some function v which is still bounded uniformly continuous. From the stability result, v is a viscosity solution of (6) with initial data v_0. It achieves the proof of the Theorem. $\square$

4.4 Examples

We give some examples of pdes like (4) which enter in the very singular case.

1. In addition to the mean curvature equation for graphs (1), we can deal with the non geometric mean curvature equation

$$\frac{\partial u}{\partial t} - \mathrm{div}\frac{Du}{\sqrt{1+|Du|^2}} = 0, \tag{24}$$

or equations like

$$\frac{\partial u}{\partial t} - \frac{\Delta u}{(1+|Du|^2)^\alpha} = 0, \quad \alpha \geq 1. \tag{25}$$

These equations lead to a geometrical equation like (6) with a singularity only at $|Dv| = 0$ and they satisfy the assumptions of the classical framework.

2. Consider a generalization of the mean curvature equation for graphs, namely

$$\frac{\partial u}{\partial t} - \mathrm{Tr}\left[(I - g(Du)Du \otimes Du)D^2u\right] = 0, \tag{26}$$

where g is a continuous function from $I\!R^N$ into $I\!R$. In this case, $b(q) = I - g(q)q \otimes q$ is symmetric nonnegative and satisfies **(H1)**–**(H3)** if and only if there exists a positive constant C such that

$$\frac{1}{|q|^2}\left(1 - \frac{C}{|q|^2}\right) \leq g(q) \leq \frac{1}{|q|^2} \quad \text{for every } q \in I\!R^N. \tag{27}$$

Using the notations of Section 3, for every $q \in S^{N-1}$, we have $b_\infty(q) = I - q \otimes q$ and $\zeta_\infty(q) = 0$; thus **(H4)** and **(H5)** are fulfilled and this equation falls into our study. Concerning **(H6)**, we have

$$0 \leq \langle b(\lambda q)\lambda q, \lambda q\rangle = \lambda^2(1 - \lambda^2 g(\lambda q)) \leq C.$$

We cannot conclude for a limit for all functions g; it means that the last assumption does not hold in general and this equation is not covered by the classical framework in the whole generality. We relate in detail such a situation below.

3. We turn to an explicit example of pde like (4) which leads to a geometrical equation whose set of singularity is exactly $\mathcal{D}$ and is not removable. Consider

$$\frac{\partial u}{\partial t} - \frac{f(Du)}{(1+|Du|^2)^2}\langle D^2uDu, Du\rangle \quad \text{in } I\!R^N \times (0,T), \tag{28}$$

where $f : I\!R^N \to I\!R$ is any bounded, nonnegative function. In this case,

$$b(q) = \frac{f(q)}{(1+|q|^2)^2} q \otimes q.$$

Assumptions **(H1)**–**(H3)** are clearly satisfied and, for every $q \in I\!\!R^N$, $b(\lambda q) \to 0$ as $\lambda \to \pm\infty$; thus **(H4)** holds. It follows that this equation is covered by "the very singular case" of this section. It leads to a geometrical equation like (6) with

$$F(p,X) = -\mathrm{Tr}\left[f\left(-\frac{p_x}{p_y}\right)\frac{p_x \otimes p_x}{(|p_x|^2+p_y^2)^2}(p_y^2 X_{xx} - 2p_y X_{xy} \otimes p_x + X_{yy} p_x \otimes p_x)\right],$$

for every $p = (p_x, p_y) \in I\!\!R^{N+1}$ and $X \in \mathcal{S}_{N+1}$.

For simplicity, set $N = 1$ and $f(q) = 1 + \cos q$. It follows $F^*((p_x, 0), X) = 0$ and $F_*((p_x, 0), X) = -2X_{yy}$, for every $p = (p_x, 0), p_x \neq 0$ and $X \in \mathcal{S}_2$ such that $X_{yy} > 0$. Therefore, in general

$$F^* \neq F_* \quad \text{on } \mathcal{D} = \{p : p_y = 0\}.$$

It shows that we cannot remove the singularities of F oustide 0. Thus (28) does not satisfy the assumptions of Section 3.

5 The level-set approach

In this section, for the sake of completeness, we recall the basic ideas of the level-set approach and we apply them to Equation (6) both in the classical and very singular framework. We refer to [18], [13], [10], etc. for a more complete description of this approach.

We are given a triplet $(\Gamma_0, \Omega_0^+, \Omega_0^-)$, where Ω_0^+, Ω_0^- are disjoint open subsets of $I\!\!R^{N+1}$ and $\Gamma_0 = \left(\Omega_0^+ \cup \Omega_0^-\right)^C$. In general, one has in mind $\Gamma_0 = \partial\Omega_0^+ = \partial\Omega_0^-$. Note that these sets form a partition of $I\!\!R^{N+1}$ and Γ_0 can be thought as being an hypersurface.

Let v_0 be any uniformly continuous function whose 0-level-set is exactly Γ_0, namely,

$$\Gamma_0 = \{z \in I\!\!R^{N+1} : v_0(z) = 0\}, \tag{29}$$

and such that

$$\{z \in I\!\!R^{N+1} : v_0(z) > 0\} = \Omega_0^+ \quad \text{and} \quad \{z \in I\!\!R^{N+1} : v_0(z) < 0\} = \Omega_0^-. \tag{30}$$

This choice of signs defines an orientation of Γ_0 making possible to distinguish an "interior," Ω_0^+, and an "exterior,"Ω_0^-. Secondly, it is always possible to find such a function v_0 by taking, for example, the *signed-distance* to Γ_0 defined by

$$\mathrm{d}(z,\Gamma_0) := \begin{cases} +\mathrm{dist}(z,\Gamma_0) & \text{if } z \in \Omega_0^+, \\ -\mathrm{dist}(z,\Gamma_0) & \text{if } z \in \Omega_0^-, \end{cases} \tag{31}$$

where dist denotes the usual positive distance. Clearly $\mathrm{d}(\cdot,\Gamma_0)$ is Lipschitz continuous in $I\!\!R^{N+1}$.

We then define the *generalized evolution* of $(\Gamma_0, \Omega_0^+, \Omega_0^-)$ by the family $(\Gamma_t, \Omega_t^+, \Omega_t^-)_{t\geq 0}$, using the

Theorem 5.1 *Under the assumptions of Theorem 3.1 or 4.2, there exists a unique solution v of (6) in $UC(I\!\!R^{N+1} \times (0,+\infty))$ with initial data v_0. Moreover, if $\tilde{v}_0 \in UC(I\!\!R^{N+1})$ satisfies*

$$\{\tilde{v}_0 = 0\} = \Gamma_0, \quad \{\tilde{v}_0 > 0\} = \Omega_0^+ \quad \text{and} \quad \{\tilde{v}_0 < 0\} = \Omega_0^-,$$

and if $\tilde{v} \in UC(I\!\!R^{N+1} \times (0,+\infty))$ is the viscosity solution of (6) with initial data $\tilde{v}_0$, then

$$\{v(\cdot,t) > 0\} = \{\tilde{v}(\cdot,t) > 0\} := \Omega_t^+,$$
$$\{v(\cdot,t) < 0\} = \{\tilde{v}(\cdot,t) < 0\} := \Omega_t^-,$$
$$\{v(\cdot,t) = 0\} = \{\tilde{v}(\cdot,t) = 0\} := \Gamma_t.$$

This results implies that the family $(\Gamma_t, \Omega_t^+, \Omega_t^-)_{t\geq 0}$ exists and is uniquenely defined independently of the choice of the representant $v_0 \in UC(I\!\!R^{N+1})$ satisfying (29) and (30). The set $\bigcup_{t\geq 0} \Gamma_t \times \{t\}$ is called the *front* associated to Γ_0 by (6) and Γ_t is the *front at time* t. Note that, at least formally, Γ_t evolves with a normal velocity equal to

$$\mathcal{V}_n(z) = -F(D\mathrm{d}(\cdot,\Gamma_t)(z), D^2\mathrm{d}(\cdot,\Gamma_t)(z)),$$

for $z \in \Gamma_t$.

Proof of Theorem 5.1. We give a proof inspired by Ishii's one (see [27]). We only show that if $\{v_0 > 0\} \subset \{\tilde{v}_0 > 0\}$ then this inclusion remains true for all $t \geq 0$, i.e. $\{v(\cdot,t) > 0\} \subset \{\tilde{v}(\cdot,t) > 0\}$. The other inclusions are obtained by straighforward adaptations. From Theorem 3.1 or 4.2, $v, \tilde{v} \in UC(I\!\!R^{N+1} \times [0,+\infty))$ and we recall that the Hyperbolic Tangent function (denoted by tanh) is a bounded uniformly continuous increasing function. Then, using Lemma 4.1, we obtain that $\tanh(v)$ is a bounded uniformly continuous solution of (6) with the bounded initial data $\tanh(v_0)$. Next, we introduce the uniformly continuous increasing function $\theta^+(r) := \max(r,0)$ and claim, thanks to Lemma 4.1 once more, that $\theta^+ \circ \tilde{v}$ and $\theta^+ \circ \tanh(v)$ are both uniformly continuous solutions of (6). Finally, we introduce the lower-semicontinuous function

$$\theta(r) = \begin{cases} +2 & \text{if } \ r > 0, \\ 0 & \text{if } \ r \leq 0, \end{cases}$$

and observe that $\theta \circ \theta^+ \circ \tilde{v}$ is a lower semicontinuous supersolution of (6). In fact, the previous changes are made in order to obtain the suitable initial condition

$$\theta \circ \theta^+ \circ \tilde{v}(\cdot,0) \geq \theta^+ \circ \tanh(v(\cdot,0)),$$

which follows easily from the assumption $\{v_0 > 0\} \subset \{\tilde{v}_0 > 0\}$. Since $\theta^+ \circ \tanh(v)$ is uniformly continuous, we apply the comparison result 4.1 and get that, for all $t \geq 0$,

$$\theta \circ \theta^+(\tilde{v}(\cdot,t)) \geq \theta^+ \circ \tanh(v(\cdot,t)).$$

We obtain $\{v(\cdot,t) > 0\} \subset \{\tilde{v}(\cdot,t) > 0\}$ which ends the proof. $\square$

6 Connection between geometrical and quasilinear pdes. Application to uniqueness

In this section, we specify the connections between (4) and (6) initiated in Section 2, and in particular in terms of uniqueness for (4). Let u be a continuous viscosity solution of (4) with initial data $u_0 \in C(I\!R^N)$ and v be the solution of (6) with initial data $\mathrm{d}(\cdot, \mathrm{Graph}(u_0))$. The main question is whether

$$\mathrm{Graph}(u(\cdot,t)) = \{(x,y) \in I\!R^{N+1} : y - u(x,t) = 0\}$$

and the front

$$\Gamma_t = \{(x,y) \in I\!R^{N+1} : v(x,y,t) = 0\}$$

coincide for all $t \geq 0$ or not. If the answer is yes, this obviously provides a uniqueness result for (4) since the Γ_t's are uniquely determined because of Theorem 5.1. And one may think that it is indeed yes by applying Theorem 5.1 together with Lemma 2.1 (ii) with initial data $\tilde{v}_0 = \tanh(y - u_0(x))$, which is a particular representation of $\mathrm{Graph}(u_0)$. Unfortunately, $\tilde{v}_0$ is not uniformly continuous if u_0 is not uniformly continuous and, as we pointed out in the introduction, we do not know how to prove Theorem 5.1 replacing "$\tilde{v}_0 \in UC(I\!R^{N+1})$" by "$\tilde{v}_0 \in C_{\mathrm{b}}(I\!R^{N+1})$"; we do not know even if such a result is true.

Nevertheless, the inclusion used in Section 2 to derive the geometrical pde is always true.

Theorem 6.1 *Suppose that **(H1)**–**(H4)** hold. Let u be a viscosity subsolution (respectively supersolution) of (4) with initial data $u_0 \in C(I\!R^N)$ and v be a viscosity solution of (6) with initial data* $\mathrm{d}(\cdot, \mathrm{Graph}(u_0))$. *For every $t \in [0, +\infty)$, we have*

$$\mathrm{Graph}(u(\cdot,t)) \subset \{(x,y) \in I\!R^{N+1} : v(x,y,t) \leq 0\}$$

$$(\ \textit{respectively} \quad \mathrm{Graph}(u(\cdot,t)) \subset \{(x,y) \in I\!R^{N+1} : v(x,y,t) \geq 0\}\ ).$$

If u is a solution of (4), then

$$\mathrm{Graph}(u(\cdot,t)) \subset \Gamma_t \quad \textit{for all } t \in [0,+\infty),$$

where $(\Gamma_t)_{t \geq 0}$ is the generalized evolution associated to $\Gamma_0 = \mathrm{Graph}(u_0)$.

Proof of Theorem 6.1. Suppose that u is a subsolution. Define the nondecreasing function $\theta^+(r) := \max(r, 0)$. For all $z = (x,y) \in I\!R^N \times I\!R$, we have,

$$\tanh\left[\theta^+(y - u_0(x))\right] \geq \tanh\left[\mathrm{d}(z, \mathrm{Graph}(u_0))\right],$$

since, on the one hand, $|y - u_0(x)| \geq \mathrm{dist}(z, \mathrm{Graph}(u_0))$; and, on the other hand, if $y \leq u_0(x)$, then $\mathrm{d}(z, \mathrm{Graph}(u_0)) \leq 0$ (for the definition of d, see (31)). From Lemma 2.1 (ii) and from the invariance of supersolution of (6) under nondecreasing changes of variables (see Lemma 4.1), we know that the function $\tanh[\theta^+(y - u(x,t))]$ is a supersolution of (6) with initial data $\tanh[\theta^+(y - u_0(x))]$. Moreover, the function $\tanh(v)$ is a solution (thus a subsolution) of (6) with initial data $\tanh(v_0)$. Applying Theorem 4.1 (see Remark 4.1), we get

$$\tanh\left[\theta^+\big(y - u(x,t)\big)\right] \geq \tanh[v(z,t)].$$

Thus, $y = u(x,t)$ implies that $v(z,t) \leq 0$ which proves the first inclusion. If u is a supersolution, we repeat the same arguments with $\theta_-(r) := \min(r, 0)$. We get the another inclusion. To conclude for the last statement, it suffices to notice, on the one hand, that u is a solution provided that u is both a sub- and a supersolution and, on the other hand, that

$$\Gamma_t = \{z \in I\!\!R^{N+1} : v(z,t) \leq 0\} \cap \{z \in I\!\!R^{N+1} : v(z,t) \geq 0\}.$$

It achieves the proof of the theorem. $\square$

In fact, the uniqueness for (4) and the so-called "fattening phenomena" for the front are closely related as shown by the

Theorem 6.2 *Assume that **(H1)–(H4)** hold and let $u_0 \in C(I\!\!R^N)$. Suppose that the front $\bigcup_{t\geq 0} \Gamma_t \times \{t\}$ associated to* $\mathrm{Graph}(u_0)$ *has empty interior in $I\!\!R^{N+1} \times [0, +\infty)$. Then (4) has at most one continuous viscosity solution with initial data u_0.*

We point out that Theorem 6.2 provides uniqueness only in the class of continuous functions. But discontinuous solutions may also exist if the front looks like a rake for instance. This result says nothing about the existence of solutions; it may be impossible to put any continuous graph in the front.

In the litterature, the "fattening phenomena" may have different meanings. In Theorem 6.2, we use the standard topological meaning. We first want to remark that assuming that Γ_t has empty interior in $I\!\!R^{N+1}$ for all $t \geq 0$ is stronger than assuming that $\bigcup_{t\geq 0} \Gamma_t \times \{t\}$ has empty interior in $I\!\!R^{N+1} \times [0, T]$. In fact, under our assumptions it turns out to be equivalent. The proof of this equivalence comes from the preservation of inclusion of sets under motions governed by (6) and the fact that, if we can put a ball in the front at a time t, this ball cannot shrink instantaneously. We skip the proof and refer to the one of Theorem 7.1 which is similar.

In Barles and Souganidis [11], a "no-interior condition" is considered, namely

$$\bigcup_{t\geq 0} \Gamma_t \times \{t\} = \partial\left(\bigcup_{t\geq 0} \Omega_t^+ \times \{t\}\right) = \partial\left(\bigcup_{t\geq 0} \Omega_t^- \times \{t\}\right). \tag{32}$$

This condition is stronger than the topological one. When it is satisfied, we have a better result.

Theorem 6.3 *Assume **(H1)**–**(H4)** and let $u_0 \in C(I\!\!R^N)$. Suppose that (32) holds for the front associated to* $\mathrm{Graph}(u_0)$. *If u and $\tilde{u}$ are (possibly discontinous) viscosity solutions of (4), then $u_* = \tilde{u}_*$ and $u^* = \tilde{u}^*$ in $I\!\!R^N \times [0, +\infty)$.*

Contrarily to Theorem 6.2, this theorem provides a "weak" uniqueness result for discontinuous viscosity solutions of (4). It is worth pointing out that stronger results providing equalities like $u^* = \tilde{u}_*$ and $u_* = \tilde{u}^*$ in $I\!\!R^N \times [0, +\infty)$ cannot be obtained by such geometrical approach since a discontinuity of u or $\tilde{u}$ can appear or, on the contrary, be removed by a slight rotation of the axis in $I\!\!R^{N+1}$ and therefore such discontinuities have no real geometrical meaning.

We refer the reader to Barles, Soner and Souganidis [10] and Ilmanen [24] for a more complete discussion and results about the "fattening phenomena" or "non-empty interior difficulty".

We turn to the proofs.

Proof of Theorem 6.2. Suppose that there exist two solutions $u_1, u_2 \in C(I\!\!R^N \times [0, +\infty))$ of (4) with initial data u_0 and define v to be the solution of (6) with initial data $v_0 = \mathrm{d}(\cdot, \mathrm{Graph}(u_0))$. From the level-set approach, we have $\Gamma_t = \{v(\cdot, t) = 0\}$. We will see that, if u_1 and u_2 are different, then the front has nonempty interior in $I\!\!R^{N+1} \times [0, +\infty)$. If $u_1 \neq u_2$, we can suppose that there exists $(x_0, t_0) \in I\!\!R^N \times [0, +\infty)$ such that

$$u_1(x_0, t_0) - u_2(x_0, t_0) = \varepsilon > 0.$$

By continuity of u_1 and u_2, there exists some ball $B(x_0, \rho)$, $\rho > 0$ and some $\tau > 0$ such that

$$u_1(x, t) - u_2(x, t) \geq \frac{\varepsilon}{2} > 0 \quad \text{in } B(x_0, \rho) \times [t_0, t_0 + \tau]. \tag{33}$$

But, from Theorem 6.1,

$$\mathrm{Graph}(u_1(\cdot, t)), \mathrm{Graph}(u_2(\cdot, t)) \subset \{v(\cdot, t) = 0\} \quad \text{for all } t \geq 0.$$

To conclude, it suffices to show that

$$B(x_0, \rho) \times [t_0, t_0 + \tau] \subset \bigcup_{t \geq 0} \Gamma_t \times \{t\}.$$

We need a lemma whose proof is postponed.

Lemma 6.1 *Under Assumptions **(H1)**–**(H4)**, let $v \in UC(I\!\!R^{N+1} \times [0, +\infty))$ be a solution of (6) with initial data v_0. If $y \mapsto v_0(x, y)$ is nondecreasing for all $x \in I\!\!R^N$, then $y \mapsto v(x, y, t)$ is nondecreasing for all $(x, t) \in I\!\!R^N \times [0, +\infty)$. In particular, the result holds if v is the solution associated to the initial condition $v_0 = \mathrm{d}(\cdot, \mathrm{Graph}(u_0))$ where $u_0 \in C(I\!\!R^N)$.*

From this lemma, we obtain

$$v(x,y,t)=0 \quad \text{for all } (x,y,t)\in B(x_0,\rho)\times[u_2(x,t_0),u_1(x,t_0)]\times[t_0,t_0+\tau].$$

By (33), we obtain that $B(x_0,\rho)\times[u_2(x,t_0),u_1(x,t_0)]\times[t_0,t_0+\tau]$ has nonempty interior in $\mathbb{R}^{N+1}\times[0,+\infty)$ which ends the proof. $\square$

Proof of Lemma 6.1. By assumption, $v_0(x,y+h)\geq v_0(x,y)$ for all $x\in\mathbb{R}^{N+1}, y\in\mathbb{R}$ and $h>0$. From the comparison result (see Theorem 4.1) it follows $v(\cdot,\cdot+h,t)\geq v(\cdot,\cdot,t)$ for all $t\geq 0$, since $v(\cdot,\cdot+h,\cdot)$ is a solution of (6) with initial data $v_0(\cdot,\cdot+h)$. It proves the first part of the lemma.

It remains to show that the function $(x,y)\mapsto \mathrm{d}((x,y),\mathrm{Graph}(u_0))$ is nondecreasing in the y variable when u_0 is continuous. To this end, consider $x\in\mathbb{R}^N$ and $y_2\geq y_1$. We suppose that $y_2\geq y_1\geq u_0(x)$. Indeed, the case $u_0(x)\geq y_2\geq y_1$ can be treated in the same way with straightforward adaptations and the case $y_2\geq u_0(x)\geq y_1$ is obvious. Assume by contradiction that

$$0\leq r_2:=\mathrm{d}((x,y_2),\mathrm{Graph}(u_0))<\mathrm{d}((x,y_1),\mathrm{Graph}(u_0))=:r_1,$$

and define $u_{0|\overline{B}(x,r_1)}$ as the restriction of u_0 to the ball $\overline{B}(x,r_1)$. From the definition of d as an infimum, the hypothesis $y_1\geq u_0(x)$ and the continuity of u_0, it follows that

$$\mathrm{Graph}(u_{0|\overline{B}(x,r_1)})\cap\overline{B}((x,y_1),r_1)\subset\partial^-B((x,y_1),r_1),$$

where $\partial^-B((x,y_1),r_1)$ stands for the part $\partial B((x,y_1),r_1)$ of the boundary of $\overline{B}((x,y_1),r_1)$ lying in the half-space $\{y\leq y_1\}$. Since $y_2\geq y_1$ and $r_2<r_1$, we get

$$\mathrm{Graph}(u_{0|\overline{B}(x,r_1)})\cap B((x,y_2),r_2)=\emptyset$$

which gives a contradiction. $\square$

Proof of Theorem 6.3. Let us show that $u_*=\tilde{u}_*$. We argue by contradiction, assuming that there exists $(\bar{x},\bar{t})\in\mathbb{R}^N\times[0,+\infty)$ such that $u_*(\bar{x},\bar{t})>\tilde{u}_*(\bar{x},\bar{t})$. From Theorem 6.1, we have, for all $t\geq 0$, $\mathrm{Graph}(\tilde{u}(\cdot,t))\subset\Gamma_t=\{v(\cdot,t)=0\}$, where v is the solution of (6) with initial data $\mathrm{d}(\cdot,\mathrm{Graph}(u_0))$. It follows from (32) that

$$\bigcup_{t\geq 0}\mathrm{Graph}(\tilde{u}(\cdot,t))\times\{t\}\subset\overline{\bigcup_{t\geq 0}\Omega_t^+\times\{t\}}\ ;$$

thus, there exists a sequence $(x_n,y_n,t_n)\in\mathbb{R}^{N+1}\times[0,+\infty)$ such that $(x_n,y_n,t_n)\to(\bar{x},\tilde{u}_*(\bar{x},\bar{t}),\bar{t})$ when $n\to+\infty$ and $v(x_n,y_n,t_n)>0$ for all $n\geq 0$. From the nondecrease of v in y (Lemma 6.1), we have $y_n>u(x_n,t_n)$ since $v(x_n,u(x_n,t_n),t_n)=0$. It follows

$$u_*(x,t)\leq\liminf_{n\to+\infty}u(x_n,t_n)\leq\liminf_{n\to+\infty}y_n=\tilde{u}_*(x,t)$$

which is a contradiction. We prove $u^* = \tilde{u}^*$ in the same way. □

The last result of this section is related to the empty interior condition of Theorem 6.2 and is inspired from the related results of Evans and Spruck [19] in the mean curvature case.

If $u_0 \in C(I\!R^N)$ and if $v_0 = \mathrm{d}(\cdot, \mathrm{Graph}(u_0))$, then the subsets $\{v_0 = \lambda\}, \lambda \in I\!R$ are the graphs of functions $u_0^\lambda \in C(I\!R^N)$. More precisely, for $\lambda \geq 0$, the function $\omega(x, \lambda) := u_0^\lambda(x)$ (respectively $\omega(x, \lambda) := u_0^{-\lambda}(x)$) is the unique viscosity solution of

$$\frac{\partial \omega}{\partial \lambda} - \sqrt{1 + |D\omega|^2} = 0 \quad \text{in } I\!R^N \times (0, +\infty),$$

(respectively

$$\frac{\partial \omega}{\partial \lambda} + \sqrt{1 + |D\omega|^2} = 0 \quad \text{in } I\!R^N \times (0, +\infty) \;),$$

We refer to Barles [4] for a simple proof of this claim. Our result is the

Proposition 6.1 *Assume **(H1)**–**(H4)**. Except for a countable subset of values of λ, the fronts associated to the evolution of* $\mathrm{Graph}(u_0^\lambda)$ *has empty interior in $I\!R^{N+1} \times [0, +\infty)$. In particular, there exists at most one continuous viscosity solution u^λ of (4) with initial data u_0^λ.*

We may interpret this result by saying that nonuniqueness for (4) is a "rare" event.

Noticing that $u_0^\lambda \downarrow u_0$ in $C(I\!R^N)$ as $\lambda \downarrow 0^+$, we obtain that we can approach any $u_0 \in C(I\!R^N)$ in a monotone way by a sequence of u_0^λ for which (4) has at most one continuous solution. The interest of this result is that the u_0^λ's have in general the same behaviour as u_0. It means that we have actually uniqueness for a large class of initial datas including functions with arbitrary growth.

Proof of Proposition 6.1. For the proof of the first part, we refer the reader to Evans and Spruck [19]. For the second one, we just remark that if v is the unique solution of (6) with initial data $v_0 = \mathrm{d}(\cdot, \mathrm{Graph}(u_0))$. Since, for every $\lambda \in I\!R$, $v - \lambda$ is the unique uniformly continuous solution of (6) with initial data $v_0 - \lambda$, at each time t, the front Γ_t^λ associated to $\mathrm{Graph}(u_0^\lambda)$ coincides with $\{v(\cdot, t) = \lambda\}$. In particular, the fronts are disjoint for different values of λ and it follows, from there, that the family of values of λ such that $\bigcup_{t \geq 0} \Gamma_t^\lambda \times \{t\}$ has nonempty interior is countable. To conclude, it suffices to apply Theorem 6.2. □

7 A local L^∞ a priori bound

In this section, we use the relations between (4) and (6) to provide a local L^∞-bound for the solutions of (4).

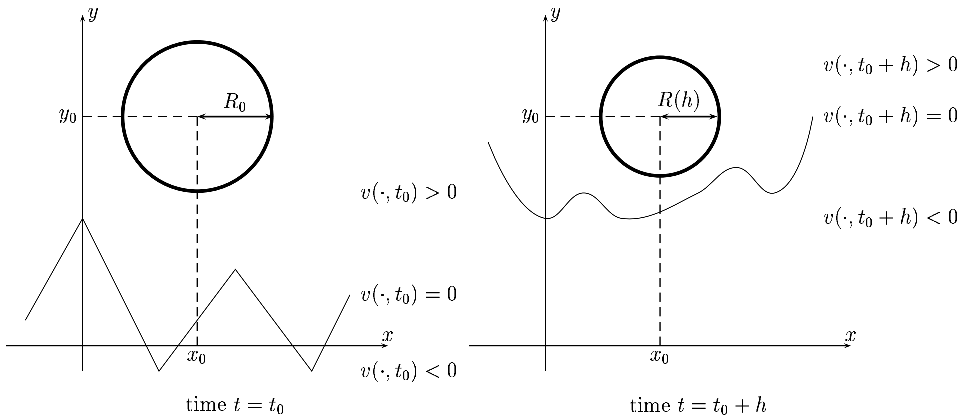

Figure 1: *Evolution of a graph with a ball which is put above it*

In order to state the main result of this section, we introduce, for any function $u_0 \in C(I\!R^N)$,

$$M_{u_0}(x,R)=\max\{u_0(y), y\in \overline{B}(x,R)\} \quad \text{and} \quad m_{u_0}(x,R)=\min\{u_0(y), y\in \overline{B}(x,R)\}.$$

We have the following

Theorem 7.1 *Under assumptions* **(H1)–(H4)**, *if $u\in C(I\!R^N\times[0,+\infty))$ is a solution of (4) with initial data $u_0\in C(I\!R^N)$, then there exists a positive constant C such that, for all $x\in I\!R^N$, $t\geq 0$, we have*

$$m_{u_0}(x,\sqrt{2Ct})-\sqrt{2Ct}\leq u(x,t)\leq M_{u_0}(x,\sqrt{2Ct})+\sqrt{2Ct}.$$

Remark 7.1 This local L^∞-bound is a direct consequence of the level-set approach and it justifies the fact that we need at least some kind of degeneracy on b in the gradient variable as it is implied by **(H1)**–**(H4)**; indeed, clearly, such a bound does not hold for the heat equation and therefore one cannot hope that such an approach applies for this equation.

Proof of Theorem 7.1. The basic idea is that the geometrical evolution governed by (6) preserves the inclusion of sets. Thus one can expect that the evolution of balls initially put "under" (or "over") the graph of a solution of (4) will provide some control on the growth. This fact is illustrated in Figure 1 in the case of the mean curvature equation.

We take $v_0(z)=\mathrm{d}(z,\mathrm{Graph}(u_0))\in UC(I\!R^{N+1})$ (where d is defined by (31)) and let v be the unique uniformly continuous solution of (6) with initial data v_0. In order to prove the result, we aim at comparing v with subsolutions like those which appear in the following Lemma whose proof is postponed.

Lemma 7.1 *We suppose that* **(H1)**, **(H2)** *and* **(H3)** *hold. Fix* $R_0 > 0$, $x_0 \in I\!R^N$ *and* $y_0 \in I\!R$. *Let* $\Psi : I\!R \to I\!R$ *be any smooth nondecreasing function. Then the function* φ, *defined for every* $(x,y,t) \in I\!R^N \times I\!R \times [0,+\infty)$, *by*

$$\varphi(x,y,t) = \Psi(R_0^2 - 2Ct - |x-x_0|^2 - (y-y_0)^2),$$

where $C = N(K_1 + K_2 + K_3) + 1$, *is a (classical) strict subsolution of (6).*

Let $x_0 \in I\!R^N, t_0 \in (0,+\infty)$ and $y_0 = M(x_0, \sqrt{2Ct_0}) + \sqrt{2Ct_0}$, where C is taken as in Lemma 7.1. It follows

$$\overline{B}((x_0,y_0), \sqrt{2Ct_0}) \subset \{(x,y) \in I\!R^{N+1} : y > u_0(x)\},$$

which implies

$$\mathrm{d}\left(\cdot, \overline{B}((x_0,y_0), \sqrt{2Ct_0})\right) \le \mathrm{d}(\cdot, \mathrm{Graph}(u_0)) = v_0. \tag{34}$$

Let us define

$$\varphi(x,y,t) = \Psi(2Ct_0 - 2Ct - |x-x_0|^2 - (y-y_0)^2),$$

with $\Psi(z) = z/2\sqrt{2Ct_0}$ for all $z \in I\!R$. The function Ψ satisfies the assumptions of Lemma 7.1. For clarity, we set $r = (|x-x_0|^2 + (y-y_0)^2)^{1/2}$. From (34), we get

$$\varphi(\cdot,\cdot,0) = \frac{\sqrt{2Ct_0} + r}{2\sqrt{2Ct_0}}(\sqrt{2Ct_0} - r) \le \sqrt{2Ct_0} - r = \mathrm{d}\left(\cdot, \overline{B}((x_0,y_0), \sqrt{2Ct_0})\right) \le v_0.$$

Now, since φ is a function with quadratic growth at infinity and v is uniformly continuous, we have that

$$\min_{I\!R^{N+1}\times[0,T]} \{v - \varphi\}$$

is achieved for every $T > 0$ if one assumes that it is positive. Using Lemma 7.1, φ is a strict smooth subsolution of (6); thus the minimum is necessarily achieved at $t = 0$ which is a contradiction. Since the previous arguments hold for every $T > 0$, we get finally

$$\varphi \le v \quad \text{in } I\!R^{N+1} \times [0,+\infty). \tag{35}$$

By Lemma 6.1, we have

$$\mathrm{Graph}(u(\cdot,t)) \subset \Gamma_t(v) \quad \text{for every } t \ge 0. \tag{36}$$

From (35) and (36), it follows that, for all $t \ge 0$,

$$\{\varphi(\cdot,t) \ge 0\} \subset \{v(\cdot,t) \ge 0\} \subset \{(x,y) \in I\!R^{N+1} : y \ge u(x,t)\}.$$

But

$$\varphi(x,y,t) \geq 0 \Longleftrightarrow (x,y) \in \overline{B}((x_0,y_0),\sqrt{2C(t_0-t)}).$$

By letting $t \to t_0$ and by using the assumed continuity of u, we obtain

$$u(x_0,t_0) \leq y_0 = M(x_0,\sqrt{2Ct_0}) + \sqrt{2Ct_0}.$$

The opposite inequality is obtained with straightforward adaptations. $\square$

We end the section with the proof of the lemma and an example.

Proof of Lemma 7.1. Without loss of generality, we can suppose that $x_0 = 0$ and $y_0 = 0$. Moreover, from Lemma 4.1, we can suppose that $\Psi(z) = z$ for every $z \in I\!R$. From **(H1)**, **(H2)** and **(H3)**, we get

$$|F_*(D\varphi, D^2\varphi)| \leq N(K_1+K_2+K_3)|D^2\varphi| \leq 2(C-1)$$

where we set $C = N(K_1+K_2+K_3)+1$. It follows

$$\frac{\partial\varphi}{\partial t} + F_*(D\varphi, D^2\varphi) \leq -2C + 2(C-1) \leq -2 < 0,$$

which achieves the proof. $\square$

Example 7.1 Evolution of balls in the case of the mean curvature equation (1).
We recall that, in the case of (1), b is given by (11). Following the computations of Lemma 7.1, we have

$$\frac{\partial\varphi}{\partial t} - \mathrm{Tr}\left[b\left(-\frac{D_x\varphi}{D_y\varphi}\right)\left(D^2_{xx}\varphi - 2D^2_{xy}\varphi \otimes \frac{D_x\varphi}{D_y\varphi} + D^2_{yy}\varphi\frac{D_x\varphi}{D_y\varphi}\otimes\frac{D_x\varphi}{D_y\varphi}\right)\right] = -2(C-N).$$

By taking $C = N$, we obtain that φ is in fact a classical solution of (6). Thus, by the level-set approach, it follows that the 0-level-set of φ evolves according to its mean curvature. An easy computation shows that

$$\Omega_0^+ = \{\varphi(\cdot,\cdot,0) > 0\} = B((x_0,y_0),R_0), \quad \Gamma_0 = \{\varphi(\cdot,\cdot,0) = 0\} = \partial B((x_0,y_0),R_0),$$

and, for every $t \geq 0$,

$$\Omega_t^+ = \{\varphi(\cdot,\cdot,t) > 0\} = B((x_0,y_0),R(t)), \quad \Gamma_t = \{\varphi(\cdot,\cdot,t) = 0\} = \partial B((x_0,y_0),R(t)),$$

where $R(t) = (R_0^2 - 2Nt)^{1/2}$. We recover by this way the well-known result of Evans and Spruck [18, Section 7.1]: balls remains balls for the mean curvature motion and they shrink into a point for $t^* = R_0^2/2N$.

8 The boundary of the front. Existence of discontinuous solutions

Theorem 6.1 provides the first connections between the front and the graphs of the solutions of (4) when they exist. In this section, we describe more precisely the structure of the front and obtain the existence of discontinuous solutions to (4).

For any continuous function u_0, we consider the generalized evolution $(\Gamma_t)_{t\geq 0}$ of $\mathrm{Graph}(u_0)$ and the uniformly continuous solution v of (6) with initial data $v_0 = \mathrm{d}(\cdot, \mathrm{Graph}(u_0))$. For every $(x,t) \in I\!\!R^N \times [0,+\infty)$, we define

$$u^+(x,t) := \sup\{y \in I\!\!R : v(x,y,t) \leq 0\} \quad \text{and} \quad u^-(x,t) := \inf\{y \in I\!\!R : v(x,y,t) \geq 0\}.$$

Note that the functions $u^+(\cdot,t)$ and $u^-(\cdot,t)$ are defined such that their graphs are the "upper-boundary" and the "lower-boundary" of the front Γ_t at each time t: see Figure 2. We have the first properties

Lemma 8.1 *Under assumptions **(H1)**–**(H4)**, the functions u^+ and u^- are locally bounded in $I\!\!R^N \times [0,+\infty))$. Moreover $u^+ \in USC(I\!\!R^N \times [0,+\infty))$ and $u^- \in LSC(I\!\!R^N \times [0,+\infty))$.*

Proof of Lemma 8.1. We make the proof for u^+, the one for u^- being similar. We start by proving that u^+ is well-defined and locally bounded. Looking at the proof of Theorem 7.1, we see that inequality (35) implies that, for every $(x_0,y_0,t_0) \in I\!\!R^N \times I\!\!R \times [0,+\infty)$, there exists a constant $M > 0$ such that

$$v > 0 \quad \text{in } \overline{B}((x_0,y_0),M) \times [0,t_0/2].$$

Note that M does not depend on y_0 in the sense that $v > 0$ in every $\overline{B}((x_0,y),M) \times [0,t_0/2]$ with $y \geq y_0$, by nondecrease of $y \mapsto v(x,y,t)$ for every (x,t) (see Lemma 6.1). It proves that $u^+ \leq y_0$ in a neighborhood of $(x_0,t_0/2)$. The same reasoning holds with straightforward adaptations to prove that u^+ is locally bounded from below.

We turn to the proof of the upper-semicontinuity of u^+. Consider any sequence of points $((x_n,y_n,t_n))_{n\in I\!\!N}$ such that $(x_n,y_n,t_n) \in \mathcal{H} := \{(x,y,t) \in I\!\!R^{N+1} \times [0,+\infty) : y \leq u^+(x,t)\}$ and $(x_n,y_n,t_n) \to (x,y,t)$ as $n \to +\infty$. For every n, we have $v(x_n,y_n,t_n) \leq 0$. Since v is continuous, by sending n to infinity, we get $v(x,y,t) \leq 0$ which proves that $(x,y,t) \in \mathcal{H}$; thus $\mathcal{H}$ is closed. It ends the proof. $\square$

Theorem 8.1 *Suppose that **(H1)**–**(H4)** hold. Let $u_0 \in C(I\!\!R^N)$ and v be the solution of (6) associated to the initial data $v_0 = \mathrm{d}(\cdot, \mathrm{Graph}(u_0))$. Then u^+ and u^- are (possibly discontinuous) viscosity solutions of (4) with initial data u_0. Moreover, u^+ and u^- are respectively the maximal subsolution and the minimal supersolution of (4) with initial data u_0.*

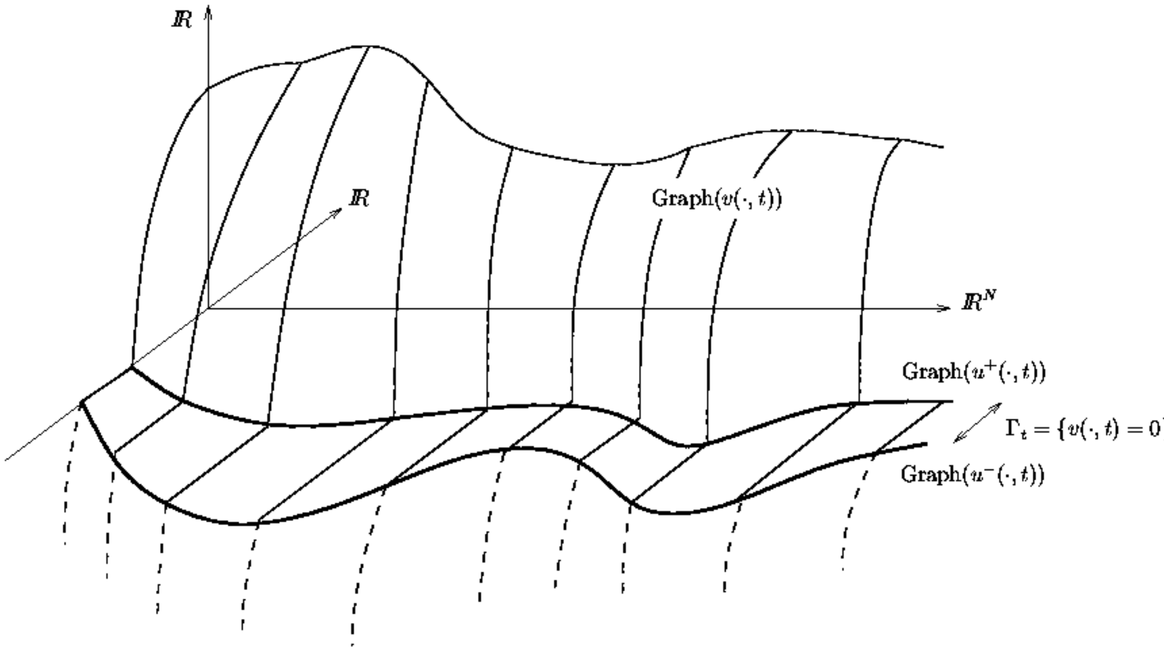

Figure 2: *front which fattens at time $t > 0$*

We refer to Figure 2 for an illustration of this theorem. Let us point out that this result provides only the existence of a discontinuous viscosity solution to (4) for any continuous initial data. We refer to Section 9 for optimal results of regularity of u^+ and u^-. We give a geometrical proof of the theorem using the fact that characteristic functions of sets which evolve are discontinuous solutions of (6).

We first introduce some notations. For any subset $A \subset I\!R^{N+1}$, $\mathrm{int}(A)$ denotes the interior of A in $I\!R^{N+1}$ and $1\!\!1_A$ is the characteristic function of A defined, for any $(x,y) \in I\!R^{N+1}$, by $1\!\!1_A(x,y) = 1$ if $(x,y) \in A$ and 0 otherwise. For the sake of simplicity of notations, when the set $A = A_t$ depends on t, we will denote by $1\!\!1_{A_t}$ the function $(x,y,t) \mapsto 1\!\!1_{A_t}(x,y)$.

We need the following lemma due to Barles, Soner and Souganidis [10].

Lemma 8.2 *Let u_0 and v as in Theorem 8.1 and consider*

$$\Omega_t^+ = \{v(\cdot,t) > 0\} \quad \text{and} \quad \Upsilon_t = \mathrm{int}\big(\{v(\cdot,t) \geq 0\}\big).$$

Then the functions $1\!\!1_{\Omega_t}$ and $1\!\!1_{\Upsilon_t}$ are (discontinuous) viscosity solutions of (6).

Proof of Theorem 8.1. We make the proof for u^+ calling it u for clarity of notations. The same reasoning holds with easy adaptations for u^-. Let us start by showing that u is a subsolution. Remembering that u is an upper-semicontinuous function by Lemma 8.1, we consider a smooth function $\phi(x,t)$ such that $u - \phi$ achieves a global maximum of

0 at $(\bar{x},\bar{t}) \in I\!R^N \times (0,+\infty)$. It follows that $u \le \phi$ and $u(\bar{x},\bar{t}) = \phi(\bar{x},\bar{t})$. Set $\psi(x,y,t) = \tanh(y-\phi(x,t))$. We claim that $(1\!\!1_{\Omega_t})_* - \psi$ achieves a global minimum 0 at $(\bar{x}, u(\bar{x},\bar{t}), \bar{t})$. By continuity of v, we have $v(\bar{x}, u(\bar{x},\bar{t}), \bar{t}) = 0$; thus $(1\!\!1_{\Omega_t})_*(\bar{x}, u(\bar{x},\bar{t}), \bar{t}) = 0$. It implies $((1\!\!1_{\Omega_t})_* - \psi)(\bar{x}, u(\bar{x},\bar{t}), \bar{t}) = 0$. It remains to check that $((1\!\!1_{\Omega_t})_* - \psi)(x,y,t) \ge 0$ for every (x,y,t). If $(1\!\!1_{\Omega_t})_*(x,y,t) = 0$, then $(x,y,t) \in \{v \le 0\}$; thus, from Lemma 6.1, we have $y \le u(x,t) \le \phi(x,y,t)$. We obtain $y - \phi(x,t) \le 0$ and $((1\!\!1_{\Omega_t})_* - \psi)(x,y,t) \ge 0$ in this case. Now, if $(1\!\!1_{\Omega_t})_*(x,y,t) = 1$, then the same inequality holds since $\tanh \le 1$. It proves the claim.

We compute the derivatives of ψ and get,

$$\frac{\partial \psi}{\partial t} = -\tanh' \cdot \frac{\partial \phi}{\partial t}, \quad D_x\psi = -\tanh' \cdot D_x\phi, \quad D_y\psi = \tanh',$$

$$D^2_{xx}\psi = \tanh'' \cdot D_x\phi \otimes D_x\phi - \tanh' \cdot D^2_{xx}\phi, \quad D^2_{yy}\psi = \tanh'', \quad D^2_{xy}\psi = -\tanh'' \cdot D_x\phi.$$

By Proposition 8.2, $(1\!\!1_{\Omega_t})_*$ is a supersolution of (6); writing the viscosity inequality at the point $(\bar{x}, u(\bar{x},\bar{t}), \bar{t})$, a calculation leads to

$$\frac{\partial \phi}{\partial t} - \mathrm{Tr}\left[b(D_x\phi) D^2_{xx}\phi\right] \le 0,$$

which shows that u is a viscosity subsolution.

We continue by proving that u_* is a supersolution. Consider a smooth function ϕ such that $u_* - \phi$ achieves a global minimum of 0 at $(\bar{x},\bar{t}) \in I\!R^N \times (0,+\infty)$. We claim first that $(1\!\!1_{\Omega_t})^*(\bar{x}, u_*(\bar{x},\bar{t}), t) = 1$. Otherwise, $(\bar{x}, u_*(\bar{x},\bar{t}), t)$ lies in the interior of $\bigcup_{t\ge 0} \Gamma_t \times \{t\}$; it means that there exists $\varepsilon > 0$ such that

$$v(x,y,t) = 0 \quad \text{for every } (x,y,t) \in \overline{B}(\bar{x},\varepsilon) \times [u_*(\bar{x},\bar{t}) - \varepsilon, u_*(\bar{x},\bar{t}) + \varepsilon] \times [\bar{t}-\varepsilon, \bar{t}+\varepsilon].$$

By definition of u, it follows that $u(x,t) \ge u_*(\bar{x},\bar{t}) + \varepsilon$ for every $x \in \overline{B}(\bar{x},\varepsilon)$ and $t \in [\bar{t}-\varepsilon, \bar{t}+\varepsilon]$. It leads to a contradiction and proves the claim.

Defining ψ as above, we observe that the function $(1\!\!1_{\Omega_t})^* - \psi$ achieves a global maximum point at $\big((\bar{x}, u_*(\bar{x},\bar{t})), \bar{t}\big)$. Indeed, if $(1\!\!1_{\Omega_t})^*(x,y,t) = 0$, then $(1\!\!1_{\Omega_t})^* - \psi \le 1$. If $(1\!\!1_{\Omega_t})^*(x,y,t) = 1$, then $(x,y,t) \in \overline{\{v > 0\}} = \{y \ge u_*(x,t)\}$, since u_* is lower-semicontinuous. It follows $y \ge \phi(x,t)$ and $\tanh(y-\phi(x,t)) \ge 0$; thus $(1\!\!1_{\Omega_t})^* - \psi \le 1$ and we are done in any case. Using that $(1\!\!1_{\Omega_t})^*$ is a subsolution of (6) by Proposition 8.2, we conclude as above.

It remains to check that the initial condition holds. On the one hand, from the continuity of v, we have $v(x, u(x,0), 0) = 0$. It implies $u(x,0) = u_0(x)$ since Γ_0 is exactly the graph of the continuous function u_0. On the other hand, looking at the proof of the supersolution, we see $(x, u_*(x,t), t) \in \overline{\Omega}_t$ for every $(x,t) \in I\!R^N \times [0,+\infty)$. By continuity of v, we get $v(x, u_*(x,t), t) = 0$. For $t = 0$, it means $u_*(x,0) = u_0(x)$.

Finally, note that from Theorem 6.1, the graphs of all subsolutions of (6) lie in $\{v \le 0\}$ and in the same way the graphs of all supersolutions of (6) lie in $\{v \ge 0\}$. Therefore, u^-

is the minimal subsolution and u^+ is the maximal supersolution of (6); and the proof of the theorem is complete. $\square$

Remark 8.1 If the "no-interior condition" (32) holds, then Theorem 6.3 implies $(u^+)_* = u^-$ and $(u^-)^* = u^+$ in $I\!R^N \times [0, +\infty)$. But even in this case, we cannot conclude to the existence of a continuous viscosity solution since the front may look like a Heaviside function.

9 Fronts with more regularity

As mentioned before, the extremal solutions u^+ and u^- have no regularity in general. We give below some conditions under which they are smooth. It is the case, when the front is associated to (4) and a locally Lipschitz initial data u_0, as soon as the solutions u of this equation satisfy local L^∞ and gradient bounds. On one hand, these bounds allows the construction of smooth solutions for any continuous initial data. On the other hand, using approximation methods, we see that this regularity holds for the extremal solutions. We start with a more precise result in the case of the mean curvature equation.

Theorem 9.1 *Let $u_0 \in C(I\!R^N)$. Then the extremal solutions u^+ and u^- of (1) with initial data u_0 are in $C^\infty(I\!R^N \times (0, +\infty)) \cap C(I\!R^N \times [0, +\infty))$.*

We recall that the smooth existence for $u_0 \in W^{1,\infty}_{\text{loc}}(I\!R^N)$ is proved in Ecker and Huisken [16] (see also Chou and Kwong [14]) using a gradient estimate. Here, following Angenent [1], we take advantage of an interior gradient estimate of Evans and Spruck [18] to prove the result for initial data u_0 which are merely continuous.

In the general case, we have

Theorem 9.2 *Assume that b satisfies assumptions **(H1)**–**(H4)** and*

$$b(q) \geq \Lambda(|q|)Id \tag{37}$$

for some nonnegative continuous function Λ in $I\!R^N$. Suppose that for any $u_0 \in W^{1,\infty}_{\text{loc}}(I\!R^N)$ there exists a smooth solution of (4) with initial data u_0 satisfying a local gradient bound, namely

$$\|Du\|_{\infty, \Omega_{R,T}} \leq K,$$

where $\Omega_{R,T} := \overline{B}(0, R) \times [0, T]$ and K is a positive constant which depends only on R, T, $\|u\|_{\infty, \bar{\Omega}_{R,T}}$ and $\|Du_0\|_{\infty, \overline{B}(0,R')}$, with $R' = R'(R, T) > 0$. Then the extremal solution u^+ and u^- are smooth.

The above theorem applies to more general quasilinear equations than (1) (see examples at the end of the section) but it requires the initial data to be locally Lipschitz continuous.

Remark 9.1 Under the assumptions of Theorem 9.1 or 9.2, if Γ_t has empty interior in $I\!R^{N+1}$ for all $t \geq 0$ (or equivalently the front $\bigcup_{t\geq 0} \Gamma_t \times \{t\}$ has empty interior in $I\!R^{N+1} \times [0,T]$, see Section 6), then the smoothness of the extremal solutions together with Theorem 6.2 implies $u^+ = u^-$ in $I\!R^N \times [0,+\infty)$. It follows $\Gamma_t = \partial\Omega_t^+ = \partial\Omega_t^-$ for all $t \geq 0$. In this case, we have in particular uniqueness and comparison for the discontinuous solutions of (4). Moreover, the weak notion of propagation given by the level-set approach coincides with the classical notion in differential geometry.

Before turning to the proof of the theorems, we state a lemma concerning the time regularity of solutions for which the space regularity is already known. We recall that a function $m : I\!R^+ \to I\!R^+$ is said to be a modulus of continuity if $m(0^+) := \lim_{s\to 0^+} = 0$ and $m(s+t) \leq m(s) + m(t)$ for any $s,t \geq 0$.

Lemma 9.1 *Let $R > 0$, $0 \leq t_0 < T$, $x_0 \in I\!R^N$ and $u \in C(\overline{B}(x_0,R)\times[t_0,T])$ be a viscosity solution of the equation*

$$\frac{\partial u}{\partial t} + G(x,t,Du,D^2u) = 0 \quad \text{in} \quad \Omega_{R,t_0,T} = B(x_0,R) \times (t_0,T), \tag{38}$$

where $G \in C(\overline{B}(x_0,R)\times[t_0,T]\times I\!R^N \times \mathcal{S}_N)$ is degenerate elliptic. If m denotes a modulus of continuity of $u(\cdot,t_0)$, i.e. if, for every $x,y \in \overline{B}(x_0,R)$, we have

$$|u(y,t_0) - u(x,t_0)| \leq m(|y-x|),$$

then there exists a modulus of continuity $\tilde{m}$ depending only on G, m and $\|u\|_{\infty,\overline{\Omega}_{R,t_0,T}}$ such that, for every $t \in [t_0,T]$ and $x \in \overline{B}(x_0,R/2)$,

$$|u(x,t) - u(x,t_0)| \leq \tilde{m}(|t-t_0|). \tag{39}$$

Moreover, if $m(r) = Lr$ for some $L \geq 0$ and if

$$|G(x,t,p,X)| \leq M(1+|X|) \quad \text{on} \quad \overline{B}(x_0,R) \times [t_0,T] \times \overline{B}(0,L) \times \mathcal{S}_N \tag{40}$$

for some constant $M \geq 0$, then there exists a positive constant $\tilde{L} = \tilde{L}(L,M,\|u\|_{\infty,\overline{\Omega}_{R,t_0,T}})$ such that $\tilde{m}(r) = \tilde{L}r^{1/2}$.

Of course, the key point in Lemma 9.1 is the fact that $\tilde{m}$ depends only on G, m and $\|u\|_{\infty,\overline{\Omega}_{R,t_0,T}}$. As a by-product of this result, it is clear that a uniform local L^∞–bound together with a uniform local modulus of continuity in space for the solutions of equations like (38) implies a uniform local modulus of continuity in time if the equations satisfy also uniform properties. In the statement of Lemma 9.1, for the sake of simplicity of formulation, we do make precise the dependence with respect to G, except in the second part of the result; this dependence will appear clearly in the proof.

Proof of Theorem 9.1. We divide the proof in two steps.

Step 1. We construct a smooth solution for any continuous initial data. Let $u_0 \in C(I\!R^N)$ and $(u_0^R)_{R>0}$ be a sequence of uniformly continuous functions converging to u_0, uniformly on every compact subset. Thanks to classical results for viscosity solutions (see [13] and references therein), we associate to each u_0^R a continuous viscosity solution u^R of (1) with initial data u_0^R. But the u^R satisfy the L^∞ local bound of Theorem 7.1, and, from Evans and Spruck [19], we learn that the u^R are in fact smooth and satisfy the interior local gradient bounds proved in [19]. From Lemma 9.1 we get then interior local modulus of continuity for the u^R; therefore, up to an extraction argument, we can suppose that the family $(u^R)_{R>0}$ converge locally uniformly in $I\!R^N \times (0, +\infty)$ to a function $u \in C(I\!R^N \times (0, +\infty))$ which is, by a classical stability result, a viscosity solution of (1) in $I\!R^N \times (0 + \infty)$.

It remains to check that the initial condition is continuously satisfied. In view of Lemma 9.1, the u^R admit the same modulus of continuity at time $t = 0$ and it follows that u is continuous at time $t = 0$ with $u(\cdot, 0) = u_0$. Finally, from [19] we get that, $u \in C^\infty(I\!R^N \times (0, +\infty))$ as a continuous solution of (1).

Step 2. We show that u^+ is smooth; the proof for u^- is the same with straightforward adaptations. Let $u_0 \in C(I\!R^N)$. Consider, for any $\lambda > 0$, the function u_0^λ defined by

$$\mathrm{Graph}(u_0^\lambda) = \{\mathrm{d}(\cdot, \mathrm{Graph}(u_0)) = \lambda\},$$

and the unique uniformly continuous solution v of (3) with initial data $\mathrm{d}(\cdot, \mathrm{Graph}(u_0))$ (we recall that (3) is the geometrical equation associated to (1)). By Step 1, we associate to each $\lambda > 0$ a smooth solution u^λ of (1) which satisfies, from Theorem 6.1, that, for all $t \geq 0$,

$$\mathrm{Graph}(u^\lambda(\cdot, t)) \subset \{v(\cdot, t) = \lambda\} \subset \{v(\cdot, t) > 0\}. \tag{41}$$

Now, as in Step 1, the family $(u^\lambda)_{\lambda>0}$ satisfies the interior gradient bound of [19]; thus, using Lemma 9.1 and the same arguments as above, we can assume that u^λ converges locally uniformly to a solution u of (1) with initial data u_0. From (41), we get that $u \geq u^+$ and thus $u = u^+$. It follows that u^+ is continuous and therefore smooth thanks again to [19]. □

Proof of Theorem 9.2. Since the proof is close to the previous one, we only give a sketch of the proof. We use arguments of Step 2 in the proof of Theorem 9.1. The only change is that, using the ellipticity condition (37), we get, in addition to the gradient bound, local bounds for high order derivatives of the u_λ (see Ladyzenskaja, Solonnikov and Ural'ceva [29]). It follows that, up to an extraction, we can assume that the family $(u_\lambda)_{\lambda>0}$ converges locally uniformly to a smooth function u which is also a solution of (4). We conclude as in the proof of Theorem 9.1, that $u = u^+$ is actually smooth. □

It remains to give the proof of the lemma.

Proof of Lemma 9.1. The main step in the proof consists in showing that, for any $\eta > 0$, one can find positive constants $C, K > 0$ large enough, depending only on η, G, m and $\|u\|_{\infty,\bar{\Omega}_{R,t_0,T}}$ such that, for any $x \in B(x_0, R/2)$

$$u(y,t) - u(x,t_0) \leq \eta + C|y-x|^2 + K(t-t_0) \text{ for every } (y,t) \in \bar{\Omega}_{R,t_0,T}, \tag{42}$$

and

$$u(y,t) - u(x,t_0) \geq -\eta - C|y-x|^2 - K(t-t_0) \text{ for every } (y,t) \in \bar{\Omega}_{R,t_0,T}. \tag{43}$$

We only prove (42), (43) being proved in an analogous way. In the sequel, x is fixed in $B(x_0, R/2)$.

First, if we take

$$C \geq \frac{8\|u\|_{\infty,\bar{\Omega}_{R,t_0,T}}}{R^2}, \tag{44}$$

then (42) is clearly fulfilled on $\partial B(x_0,R) \times [t_0,T]$, for every $\eta, K > 0$ and for every $x \in B(x_0, R/2)$. It is worth noticing that C may be taken independent of x.

Next, we would like to ensure that (42) holds for $t = t_0$. To this end, we argue by contradiction assuming there exists $\eta > 0$ such that, for every $C > 0$, there exists $y_C \in \overline{B}(x_0, R)$ such that

$$u(y_C, t_0) - u(x,t_0) > \eta + C|y_C - x|^2. \tag{45}$$

It follows

$$|y_C - x| \leq \sqrt{\frac{2\|u\|_{\infty,\Omega_{R,t_0,T}}}{C}}. \tag{46}$$

Thus $|y_C - x| \to 0$ when $C \to \infty$. Coming back to (45), we get

$$m(|y_C - x|) \geq u(y_C,t_0) - u(x,t_0) > \eta + C|y_C|^2 \geq \eta.$$

Using (46), the inequality $m(|y_C - x|) \geq \eta$ leads to a contradiction as soon as we choose C large enough and this choice depends only on η, $\|u\|_{\infty,\Omega_{R,t_0,T}}$ and m. Therefore, by choosing C large enough, we have that (42) is satisfied on the parabolic boundary $(\partial B(x_0,R) \times [t_0,T]) \cup (\overline{B}(x_0,R) \times \{t_0\})$.

Finally, using the continuity of G, we can take K large enough in order that the function $(y,t) \mapsto u(x,t_0) + \eta + C|y-x|^2 + K(t-t_0) := \chi(y,t)$ is a (smooth) strict supersolution of (38). Thus, since u is a viscosity subsolution of (38), by using only the definition of viscosity subsolution, it is clear that $\max_{\bar\Omega_{R,t_0,T}}\{u-\chi\}$ is necessarily achieved on the parabolic boundary of $\Omega_{R,t_0,T}$. And (42) follows.

The first part of the lemma follows by observing that all our constants depend only, when η is fixed, on G, m and $\|u\|_{\infty,\bar\Omega_{R,t_0,T}}$ but not on $x \in B(x_0, R/2)$.

If we assume that $m(r) = Lr$ for some positive constant L, the condition (42) at time $t = t_0$ reads

$$u(y,t_0) - u(x,t_0) \le L|y-x| \le \eta + C|y-x|^2,$$

for every $y \in \overline{B}(x_0,R)$. Writing that the discriminant of $C|y-x|^2 - L|y-x| + \eta$ is nonpositive, we get that it holds if

$$C \ge \frac{L^2}{4\eta}.$$

Using (40), χ is a supersolution if $K \ge M(1+2C)$. Introducing these estimates in (42), we finally obtain for $y = x$

$$u(x,t) - u(x,t_0) \le \eta + M\left(1 + \frac{L^2}{2\eta}\right)(t-t_0),$$

for all $t \in [t_0, T]$. An easy optimization with respect to η of the right-hand side term of the previous inequality gives that, for all $t \in [t_0,T]$,

$$u(x,t) - u(x,t_0) \le \tilde{L}\sqrt{t-t_0},$$

for some positive constant $\tilde{L}$ depending on M and L. This concludes the proof of the Lemma. □

We conclude this section with examples of equations satisfying the assumptions of Theorem 9.2. The following equations come from the paper of Chou and Kwong [14] (see Section 4.4 for the precise statement of the equations).

1. The non geometric mean curvature equation (24) and equation (25) are associated to a front with smooth boundary when u_0 is locally Lipschitz continuous.

2. Consider equation (26) with $g(q) = g(|q|)$ continuous in $I\!R^N$. Suppose that

$$\frac{1}{r^2}\left(1 - \frac{C}{r^2}\right) \le g(r) < \frac{1}{r^2} \quad \text{for every } r > 0, \tag{47}$$

and, in addition, that g is a C^1 function such that

$$2g + rg' \le \frac{C}{(1+r^2)^{3/2}},$$

for every $r \ge 0$. Then Chou and Kwong [14] gives the gradient bound the conclusion of Theorem 9.2 holds. Note that (47) is nothing but (27) with a strict inequality on the right-hand side. This strict inequality ensures that the condition of ellipticity (37) is satisfied.

10 Application to convex solutions

In this section, we are interested in convex solutions of (4). We also derive some properties for the generalized evolution of convex sets.

Our main result is

Theorem 10.1 *Assume **(H1)**–**(H4)** and let u_0 be a convex function in $\mathbb{R}^N$.*

(i) Suppose that $u_1 \in USC(\mathbb{R}^N \times [0,+\infty))$ (respectively $u_2 \in LSC(\mathbb{R}^N \times [0,+\infty))$) is a viscosity subsolution (respectively supersolution) of (4). If $u_1(\cdot,0) \le u_0 \le u_2(\cdot,0)$ in $\mathbb{R}^N$, then $u_1 \le u_2$ in $\mathbb{R}^N \times [0,+\infty)$.

(ii) There exists a unique continuous viscosity solution u to (4) with initial data u_0. Moreover $u(\cdot,t)$ is convex for all $t \ge 0$.

It is worth pointing out that, in the previous theorem, the existence and comparison properties hold without any restriction on the growth at infinity of the solutions or the initial datas. Moreover, they hold both in the classical and very singular framework. Therefore, we have a complete answer in this case.

In the particular case of the mean curvature equation, the solution is in addition smooth.

Theorem 10.2 *If the initial data $u_0 \in C(\mathbb{R}^N)$ is convex, then there exists a unique continuous solution u of the mean curvature equation for graphs (1); moreover $u \in C^\infty(\mathbb{R}^N \times (0,+\infty)) \cap C(\mathbb{R}^N \times [0,+\infty))$ and $u(\cdot,t)$ is convex for any $t \ge 0$.*

Theorem 10.1 and 10.2 strongly justify the geometrical approach to study (4): in the case of convex solutions, the existence of solutions follows (rather) easily from the L^∞-bound of Theorem 7.1 since it implies also a gradient bound; the existence proof can be done either using Theorem 8.1 or directly on (4) as in [7]. For the comparison result, we point out that working on (6) as we do it here, in particular for the mean curvature equation, provides better results: in [7], we obtain a comparison result working directly on (4) by using a Kružkov change ($u \mapsto -\exp(-u)$) but with stronger assumptions on b and u_0 which was assumed to be coercive.

Below, we will give a proof which is simpler and essentially based on the preservation of convexity for geometric motions governed by (6); more precisely, we have

Theorem 10.3 *Suppose that **(H1)**–**(H4)** hold. Let $v_0 \in UC(\mathbb{R}^N)$ be a convex (respectively concave) function and v be the associated solution of (6). Then $v(\cdot,t)$ is convex (respectively concave) for any $t \ge 0$.*

Proof of Theorem 10.3. This result is a consequence of the one established by Giga, Goto, Ishii and Sato in [21] that we extend to the very singular case by using an approximation argument.

Step 1. We define, for any $\varepsilon > 0$ and $(p, M) \in I\!\!R^{N+1} \times \mathcal{S}_{N+1}$,

$$F_\varepsilon(p, M) = \begin{cases} \varphi_\varepsilon(1/p_y)F(p, M) & \text{if} \quad p_y \neq 0 \\ 0 & \text{if} \quad p_y = 0 \end{cases}$$

where F appears in (6) and φ_ε is a smooth nonnegative real valued function with compact support in $[-2/\varepsilon, 2/\varepsilon]$ and such that $\varphi_\varepsilon(r) = 1$ for $r \in [-1/\varepsilon, 1/\varepsilon]$. The F_ε's satisfy assumptions **(F1)**–**(F4)** and thus we can apply for each ε the results of Giga, Goto, Ishii and Sato [21] and get for any $T > 0$ a solution $v_\varepsilon \in UC(I\!\!R^{N+1} \times (0, T))$ of

$$\frac{\partial v_\varepsilon}{\partial t} + F_\varepsilon(Dv_\varepsilon, D^2 v_\varepsilon) = 0 \;\text{ in }\; I\!\!R^{N+1} \times (0, T) \tag{48}$$

with initial data v_0. Moreover, since v_0 is convex and F_ε remains linear in the Hessian, we learn also from [21] that $v_\varepsilon(\cdot, t)$ is convex for any $t \geq 0$.

Step 2. Our aim is now to show that the family $(v_\varepsilon)_{\varepsilon>0}$ is locally bounded. To this end, we introduce

$$\chi(z, t) = a|z|^2 + b + Ct.$$

Since $v_0 \in UC(I\!\!R^{N+1})$ there exist $a, b \in I\!\!R$ such that $v_0 \leq \chi(\cdot, 0)$ and since v_ε has at most linear growth, χ is greater than v_ε at infinity for all $\varepsilon > 0$.

Moreover, if follows from **(H1)** that F is bounded on bounded set and so are the F_ε's uniformly in ε. Then, an easy computation of the derivatives of χ shows that, up to take C sufficiently large independent of ε, the function $\chi(z, t) = a|z|^2 + b + Ct$ is a smooth supersolution of (48). It follows that $v_\varepsilon \leq \chi$ for all $\varepsilon > 0$. Reasoning in the same way with a subsolution, we get that the family $(v_\varepsilon)_{\varepsilon>0}$ is locally bounded independently of ε.

Step 3. From the previous step we are able to introduce the "half-relaxed-limits" $\overline{v}$ and $\underline{v}$ of the family $(v_\varepsilon)_{\varepsilon>0}$. Since (F_ε) tends to F locally uniformly on $(I\!\!R^{N+1} - \mathcal{D}) \times \mathcal{S}_{N+1}$, we obtain, from the stability result, that they are respectively sub- and supersolution of (6) with initial data v_0.

Finally, from the comparison result of Theorem 4.1, we have that $\overline{v} = \underline{v} = v$ and therefore that the family $(v_\varepsilon)_{\varepsilon>0}$ converges locally uniformly to v as ε tends to 0. It follows that $v(\cdot, t)$ is convex for any $t \in [0, T)$. Since we can repeat the arguments for any $T \geq 0$ it completes the proof. $\square$

We continue with

Lemma 10.1 *Let* $\Gamma_0 = \mathrm{Graph}(u_0)$. *If* u_0 *is convex in* $I\!\!R^N$, *then the signed-distance* $\mathrm{d}(\cdot, \Gamma_0)$ *(see (31) for a definition) is concave.*

Proof of Lemma 10.1 It suffice to shown that, for any $z_i = (x_i, y_i) \in I\!\!R^{N+1}$, $i \in \{1,2\}$, we have

$$v_0\left(\frac{z_1+z_2}{2}\right) \geq \frac{v_0(z_1)+v_0(z_2)}{2}. \tag{49}$$

To this end, we set $z = (z_1+z_2)/2$ and denote by $\mathcal{P}_z$ the hyperplane which contains $z' \in \Gamma_0$ such that $v_0(z) = \pm|z-z'|$ and is orthogonal to $z-z'$. At this stage, one has to distinguish many cases depending on the position of z_1, z_2 relatively to Γ_0. Since their study are similar, we provides the proof of (49) only in the case $z_1 \in \{v_0 \geq 0\}, z_2 \in \{v_0 \leq 0\}$ and $v_0(z) = -|z-z'|$.

In this case, since u_0 is assumed to be convex, $\mathcal{P}_z \subset \{v_0 \leq 0\}$; thus we have

$$v_0(z_1) = \mathrm{d}(z_1, \Gamma_0) \leq \mathrm{dist}(z_1, \mathcal{P}_z), \quad v_0(z_2) = \mathrm{d}(z_2, \Gamma_0) \leq -\mathrm{dist}(z_2, \mathcal{P}_z). \tag{50}$$

But, using the orthogonal projection on $\mathcal{P}_z$, one sees that

$$|z-z'| = \frac{\mathrm{d}(z_2, \mathcal{P}_z) - \mathrm{d}(z_1, \mathcal{P}_z)}{2} = -v_0(z) \tag{51}$$

and combining (50) and (51), one gets (49) □

We are now able to give the proof of the main result.

Proof of Theorem 10.1. We begin with (i). Let u_0 be a convex function on $I\!\!R^N$ and $v_0 = d(\cdot, \mathrm{Graph}(u_0))$. From Lemma 10.1, v_0 is concave. Therefore, applying Theorem 10.3, the associated solution v of (6) is also concave with respect to the space variable at any time $t \geq 0$.

Assume then by contradiction that $y_2 = u_2(x,t) < u_1(x,t) = y_1$ for some $(x,t) \in I\!\!R^N \times [0,+\infty)$. It follows from Theorem 6.1 that $v(x, y_2, t) \geq 0$ and $v(x, y_1, t) \leq 0$. Thus from Lemma 6.1, $v(x,\cdot,t) \equiv 0$ on $[y_2, y_1]$. Since it is a concave function, it implies that $v(x,\cdot,t) \equiv 0$ on $[y_2, +\infty)$ which is a contradiction with Lemma 8.1.

We turn to the proof of (ii). Applying the previous comparison result to the extremal solutions u^+ and u^- we obtain that they are equal to the same continuous function u, such that $\{v(\cdot,t) = 0\} = \mathrm{Graph}(u(\cdot,t))$ for all $t \geq 0$, which turns out to be the unique continuous viscosity solution of (4). Since $v(\cdot,t)$ is concave, $u(\cdot,t)$ is convex for any $t \geq 0$. It completes the proof of (ii) □

We conclude this section with some consequences of Theorem 10.3 for the geometrical evolution of sets in the convex case.

Theorem 10.4 *Let Ω_0^+ be any open convex subset of $I\!\!R^{N+1}$ with boundary Γ_0 and let $(\Omega_t^+, \Omega_t^-, \Gamma_t)_{t\geq 0}$ be the generalized evolution of $(\Omega_0^+, \Omega_0^-, \Gamma_0)$ in the sense of Section 5. Then, while $\Omega_t^+ \neq \emptyset$, it remains convex and Γ_t is its boundary. In particular, Γ_t has empty interior in $I\!\!R^{N+1}$.*

This result is known in the case of motion by mean curvature of compact convex sets (see Evans and Spruck [18], Soner [32], Ilmanen [24]). Here, the result holds for possibly noncompact hypersurfaces (like graphs for instance) and for general motions governed by (6). Note that we get immediate properties of regularity of the front at each time t before extinction: in the general case the front is locally a Lispchitz continuous graph; in the case of the mean curvature equation, the front is even a smooth hypersurface (for the regularity issue, see Evans and Spruck [19] and Imbert [25]).

Proof of Theorem 10.4. Let v be the unique solution of (6) with initial data $v_0 = \mathrm{d}(\cdot, \Gamma_0)$ associated to Γ_0 via the level-set approach. Let $t \geq 0$ such that $\Omega_t^+ \neq \emptyset$. From Theorem 10.3, $v(\cdot, t)$ is concave; thus Ω_t^+ is convex. To prove that $\partial\Omega_t^+ = \Gamma_t$, we argue by contradiction, assuming there exists $z_0 \in \Gamma_t$ and $r > 0$ such that $B(z_0, r) \cap \Omega_t^+ = \emptyset$. We have $v(z_0, t) = 0$ and $v(\cdot, t) \leq 0$ on $B(z_0, r)$. Since $v(\cdot, t)$ is concave, it follows $v(\cdot, t) \leq 0$ in $I\!R^{N+1}$ which is a contradiction with $\Omega_t^+ \neq \emptyset$. □

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