# Motion by mean curvature and level-set approach

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## 1 Introduction

This text is the proceeding of a conference given at Muroran Institute of Technology, July 30th 2004. I would like to thank Professors Y. Giga and M.-H. Sato for inviting me in Japan and Professor Y. Kohsaka for giving me the opportunity to give a talk at Muroran.

The goal of this talk is to give an introduction to the mean curvature motion and the level-set approach. This presentation does not aim at being very precise or/and exhaustive. I would like to give the main ideas in a simple way and to discuss some special issues. I refer the reader who is interested in the subject to the surveys [31, 30] of Y. Giga.

The talk is divided in four parts. At first, I will give a description of the mean curvature and some examples and applications. Then, I will turn to the problem of studying evolution by mean curvature and the need of defining a generalized notion of motion. The third part is devoted to the explanation of one of the generalized motion, namely the level-set approach. Finally, I will give an example of application of the level-set approach in a theoretical context, namely the study of quasilinear partial differential equations.

## 2 Preliminaries about mean curvature motion

Given a smooth hypersurface $\Gamma$ in the Euclidean space $\mathbb{R}^N$, the mean curvature $\kappa_x$ of $\Gamma$ at the point $x$ of $\Gamma$ is the sum (or the average) of the principal curvatures at $x$. More precisely, suppose $(n_x)_{x\in\Gamma}$ be a unit normal vector field; then

$$\kappa_x = -\mathrm{div}(n_x) = \mathrm{trace}(A_x) \quad \text{for all } x \in \Gamma,$$

where div is the surface divergence and $A_x$ is the matrix of the second fundamental form at $x$.

The problem of *motion by mean curvature* of hypersurfaces or *fronts* of $\mathbb{R}^N$ is the following: consider a given initial front $\Gamma_0 \subset \mathbb{R}^N$ (for the moment, imagine that everything is smooth). We are interested in the time-evolution $(\Gamma_t)_{t\geq 0}$ of $\Gamma_0$ such that, at every time $t$, the boundary point $x \in \Gamma_t$ moves with a normal speed $V_x^{\Gamma_t}$ equal to the mean curvature of the boundary, namely,

$$\vec{V}_x^{\Gamma_t} = -\mathrm{div}(n_{x,t})\, n_{x,t} = -\mathrm{trace}(A_{x,t})\, n_{x,t} \quad \text{for all } x \in \Gamma, \tag{1}$$

and $V_x^{\Gamma_t} = \langle \vec{V}_x^{\Gamma_t}, n_{x,t} \rangle$ where $\langle \cdot, \cdot \rangle$ is the classical Euclidean inner product. See Figure 1 for an illustration.

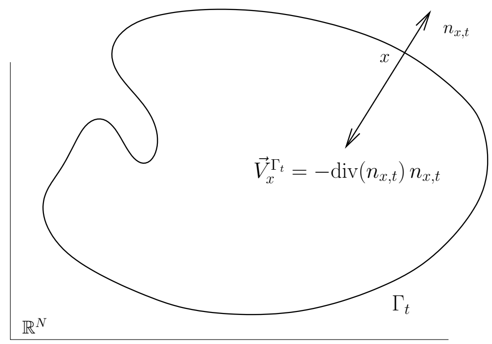


Figure 1: *Motion of a front $\Gamma_t$ by mean curvature.*

The aim is to study $\Gamma_t$ for all $t \geq 0$.

We first give some basic examples for which the behaviour of the evolution by mean curvature is well-known: circles in $\mathbb{R}^2$ or more generally spheres in $\mathbb{R}^N$ remain spheres and shrink into a point in finite time (see Figure 2). More generally, compact convex bodies in $\mathbb{R}^N$ shrink into a point in finite time and they look asymptotically like a sphere (see Figure 3). Compact subsets of $\mathbb{R}^N$ shrink into a point in finite time (see an example of behaviour in Figure 4). We refer to Huisken [36] and Gage and Hamilton [29] for proofs.

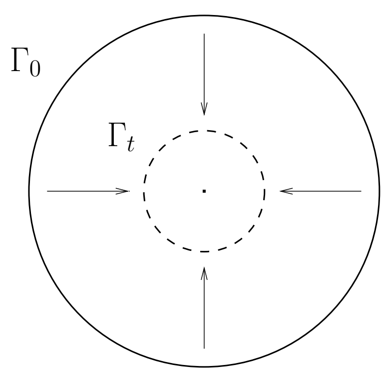


Figure 2: *Evolution by mean curvature of a sphere $\Gamma_0$ of radius $R_0$; $\Gamma_t$ is a sphere of radius $R(t) = \sqrt{R_0^2 - 2(N-1)t}$ for $0 \leq t \leq R_0^2/2(N-1)$.*

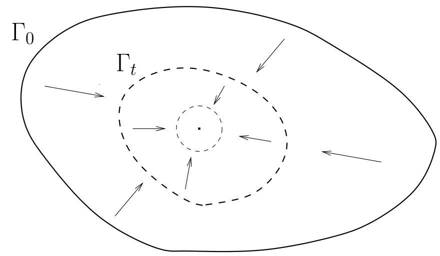


Figure 3: *Evolution by mean curvature of a convex body.*

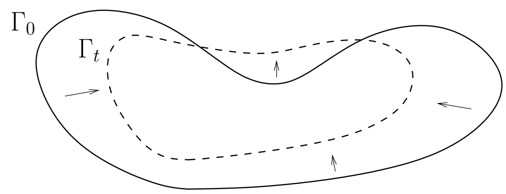


Figure 4: *Behaviour of the evolution by mean curvature of a compact set.*

Mean curvature motion appears in various fields including differential geometry ([29], [36], [22], etc.), asymptotics of reaction diffusions equations (Allen-Cahn equations, see [12], [15], [21], etc.), stochastic control and mathematical finance (see Soner and Touzi [50], Buckdahn, Cardaliaguet and Quincampoix [13]). This kind of motion has numerous applications in industrial transformation of metals, cristal growth or image processing (see for instance Mullins [44], Alvarez, Guichard, Lions and Morel [1] and references in Y. Giga [30]).

We choose here to describe an original and elegant problem in which mean curvature arises. This problem comes from the works of Kohn and Serfaty [43] and Catté, Dibos and Koeppfler [14]. Consider the following game: let $D$ be an open disk in the plane and let $M_0$ be any point in the disk. There are two players with antagonistic goals: the first player, let us call him Paul, aims at "pushing" the point outside the disk whereas the second player, Carol, wants to obstruct him by keeping the point in the disk. The rules of the game, at each step, are the following:

1. Paul chooses any direction. More precisely, he chooses a line passing through $M_0$.

2. Carol chooses a sense. She put a new point $M_1$ on the line at a fixed distance $\varepsilon$ of $M_0$ in the chosen sense.

An so on. This gives rise to a sequence $M_0, M_1, M_2, \cdots$ of points of the plane (see Figure 5).

The questions are: who is the winner? Does Carol keep $M_n$ in the disk for all $n$ or, on the contrary, could Paul be sure to push $M_n$ outside the disk after a finite number of steps?

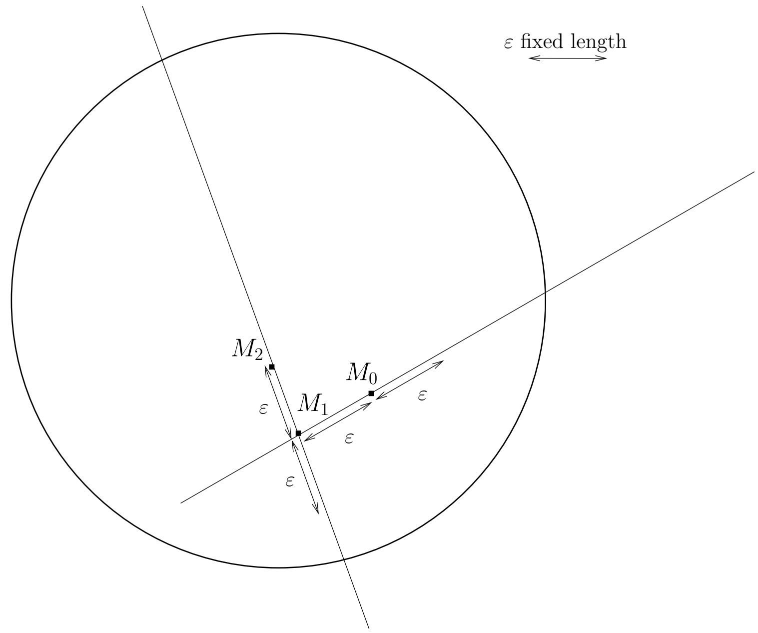


Figure 5: *Example of the first steps of a game.*

The answer is that it is possible to exit the disk in a finite number of steps. The strategy is clear: Paul must not give any choice to Carol: he has to choose at every step a line which is orthogonal to the radius of the disk (see Figure 6).

Now, what is the link with the mean curvature? It becomes clear when introducing the sets $E_{\varepsilon,i}$, $i = 1, 2, \cdots$ defined as follows: if $M \in E_{\varepsilon,i}$, then it is possible to exit the disk in exactly $i$ steps. These sets are nonempty and are annulus (see Figure 7). Introducing an auxiliary time (step 1 at time 0, step 2 at time $\tau_\varepsilon, \cdots$, step $n$ at time $(n-1)\tau_\varepsilon, \cdots$ with $\tau_\varepsilon = \varepsilon^2/2$) and letting $\varepsilon$ go to 0, we obtain that the sets $E_{\varepsilon,i}$ converge to the evolution by mean curvature $\Gamma_t$ of the disk. One can generalize this game replacing the disk by any convex set of the plane. The conclusion still holds. We refer to [43] for details and proofs.

# 3 The need of a generalized motion

Until now, we supposed that all the evolutions were smooth. But, evolution by mean curvature, even when starting with a smooth initial front $\Gamma_0$, faces the developpment of singularities. A trivial example is the case of spheres (see Figure 2): spheres remain spheres and they shrink to a point after a finite time $t^*$. At $t^*$, a singularity occurs. There are more interesting examples of course: see Figure 8 for an example and [25, 49, 35, 2, 30] for details.

A second question related to the previous issue is: how to define the evolution of a nonsmooth initial front $\Gamma_0$? Solving these problems is interesting with respect to applications. In image processing, for instance, mean curvature is used to regularize the image and the initial image $\Gamma_0$ has poor regularity. To illustrate such an issue,

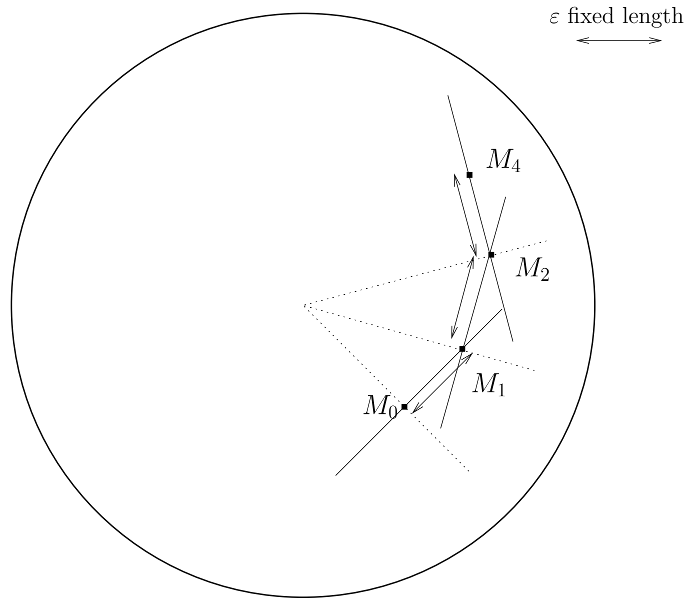


Figure 6: *Example of an optimal strategy.*

what could be the mean curvature evolution of the polygon and the "eight" of Figure 9?

Both issues imply to be able to define a notion of *generalized evolution by mean curvature* in order to deal with nonsmooth hypersurfaces. Different approaches have been proposed: level-set approach of course but not only. There are another notions of generalized solutions. Brakke [11] constructed generalized evolutions using geometrical measure theory, there are the barrier solutions of De Giorgi (see [20] and [38]) and generalized motion by mean curvature can also be obtained as limits of reaction-diffusion equations (see [12], [15], [21]).

In this talk, we will focus on the level-set approach.

# 4 Level-set approach and viscosity solutions

Level-set approach is extensively studied in front propagation. For the mean curvature flow, this method was introduced for numerical purposes by Osher and Sethian [46]. Rigorous justifications of this approach are due, independently, to Chen, Giga and Goto [16] and Evans and Spruck [25, 26, 27, 28]. Their proofs are based on the notion of viscosity solutions which were introduced by Crandall and Lions [19] and then extensively studied (see Crandall, Ishii and Lions [18] and the references therein). After these pioneering works, many mathematicians developed the subject. Among them, we can quote for instance Barles, Soner and Souganidis [8], Soner [48], Souganidis [51, 52], Ishii and Souganidis [41], Ohnuma and Sato [45], Goto [34], Ishii [40], Barles, Biton and Ley [5], Giga, Ishimura and Kohsaka [33] and many others. We refer to Giga [31] for a complete bibliography, details and proofs of what follows.

The idea of the level-set approach is to define a generalized evolution $(\Gamma_t)_{t\geq 0}$ of

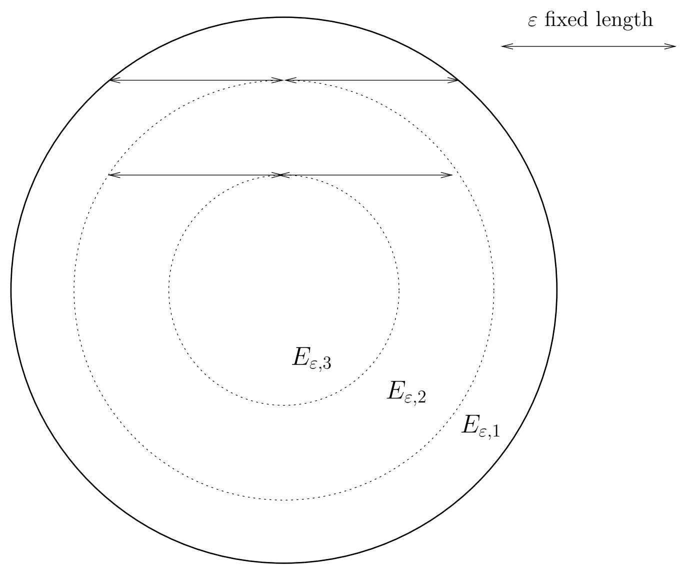


Figure 7: $E_{\varepsilon,i} = \{M$ such that it is possible to exit in $i$ steps$\}$.

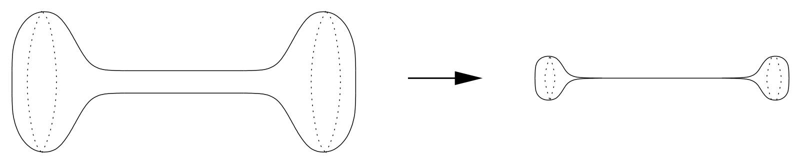

Figure 8: *Evolution of a smooth hypersurface leading to the formation of singularities in* $\mathbb{R}^3$.

an initial front $\Gamma_0 \in \mathbb{R}^N$ by representing $\Gamma_t$, for all $t \geq 0$, as the zero level-set of an auxiliary function $v : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$, precisely,

$$\Gamma_t = \{x \in \mathbb{R}^N : v(x,t) = 0\} \quad \text{for all } t \geq 0.$$

Then, at least formally, the function $v$ has to satisfy the *mean curvature equation*

$$\frac{\partial v}{\partial t} - \Delta v + \frac{\langle D^2 v Dv, Dv\rangle}{|Dv|^2} = 0 \quad \text{on } \bigcup_{t\geq 0} \Gamma_t \times \{t\}. \tag{2}$$

Note that this equation can also be written

$$\frac{\partial v}{\partial t} - |Dv| \operatorname{div}\left(\frac{Dv}{|Dv|}\right) = 0$$

or

$$\frac{\partial v}{\partial t} - \operatorname{trace}(\left(Id_{\mathbb{R}^N} - \frac{Dv \otimes Dv}{|Dv|^2}\right) D^2 v) = 0.$$

Let us describe more precisely the level-set approach. The given data is the initial front $\Gamma_0 \subset \mathbb{R}^N$ which is supposed to be the boundary of an open set $\Omega_0 \subset \mathbb{R}^N$

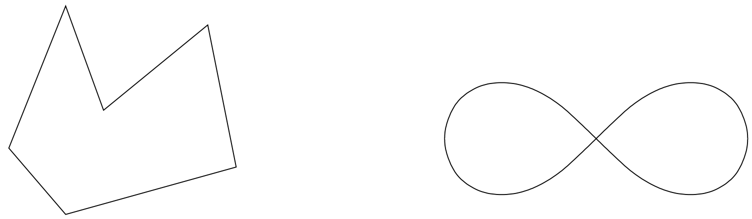

Figure 9: *What are the evolutions of these "singular" hypersurfaces of* $\mathbb{R}^2$*?*

(note that no more regularity is assumed on $\Gamma_0$; in particular $\Gamma_0$ can be the sets of Figure 9).

1. Consider the mean curvature equation (2) (on the whole space and not only on the front)

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v + \dfrac{\langle D^2 v Dv, Dv\rangle}{|Dv|^2} = 0 & \text{in } \mathbb{R}^N \times (0, +\infty), \\ v(\cdot, 0) = v_0 & \text{in } \mathbb{R}^N. \end{cases} \tag{3}$$

   where $v_0 : \mathbb{R}^N \to \mathbb{R}$ is any uniformly continuous ($UC$ in short) function such that

$$\{v_0 = 0\} = \Gamma_0, \quad \{v_0 > 0\} = \Omega_0 \quad \text{and} \quad \{v_0 < 0\} = \mathbb{R}^N \backslash (\Gamma_0 \cup \Omega_0) \tag{4}$$

   (take for instance the signed distance to $\Gamma_0$ which is positive in $\Omega_0$).

2. Solve (3) with viscosity solutions using

   **Theorem 4.1** *([16, 25, 18]) For any UC $v_0$ there exists a unique UC viscosity solution $v$ of (3).*

3. Define

$$\Gamma_t := \{v(\cdot, t) = 0\} \quad \text{for all } t \geq 0$$

   and use the fundamental theorem of the level-set approach

   **Theorem 4.2** *([16, 25]) The generalized evolution by mean curvature $(\Gamma_t)_{t\geq 0}$ of the couple $(\Gamma_0, \Omega_0)$ is well-defined in the sense that $\Gamma_t$ does not depend on the choice of the UC function $v_0$ but only on the initial data $(\Gamma_0, \Omega_0)$.*

   It means that, when replacing $v_0$ by another function $\tilde{v}_0$ satisfying (4), the solution $v$ can change but not its zero level-set!

Comparing to other approaches of generalized evolutions, there are advantages and disadvantages. Let us start with disadvantages. The generalized evolution is defined as the zero level-set of the solution $v$ of (3) which is merely continuous. In particular, $\Gamma_t$ is not a smooth hypersurface in general, it has poor regularity. Sometimes, $\Gamma_t$ is even a "fat set", see the fattening phenomenon in [8] and Figures 11, 12. Such a fattening phenomenon can be seen as a nonuniqueness of the evolution (see for instance Corollary 5.5).

Fortunately, this approach has a lot of advantages: at first, it can be computed numerically (see Sethian [47] and the references theirein). Secondly, it is defined for all times, even when starting with a front not regular enough and/or past singularities. Finally, the generalized evolution fits with the classical one of differential geometry as long as the latter exists (for links between the different notions of evolutions, see [24, 37, 28]).

Let us turn to some examples of generalized evolutions by mean curvature obtained with the level-set approach. Most of the ones we present appear in the papers mentioned above (especially [25, 48, 39]) and we refer to them for further details. At first, for spheres, we obtain the classical evolution and $\Gamma_t = \emptyset$ for $t$ greater than the extinction time. Evolution of every compact convex body of $\mathbb{R}^N$ is well-defined, even if the convex set is nonsmooth, and they shrink into a point. If everything is smooth, we get the classical evolution. The generalized evolution of the smooth set $\Gamma_0$ of Figure 8 is represented in Figure 10: the evolution coincides with the classical one until singularities occur and is defined in an expected way past the singularities. The evolution of the polygon of Figure 9 is well-defined too and we obtain the

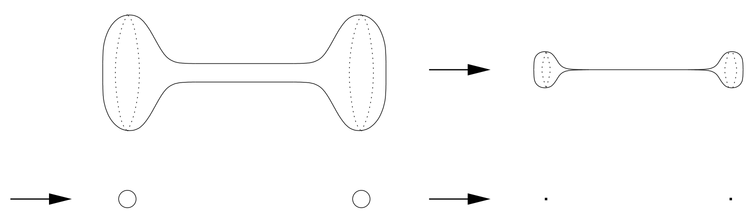

Figure 10: *Generalized evolution using level-set approach for the set of Figure 8.*

expected evolution: there is an immediate regularization effect and $\Gamma_t$ shrinks into a point in finite time. As announced, there are cases of fattening. In Figures 11 and 12, we give two examples of sets $\Gamma_0$ which fatten instantaneously.

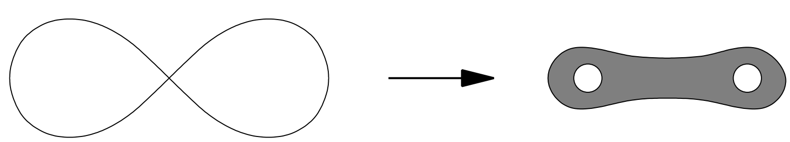

Figure 11: *Instantaneous fattening of the "eight".*

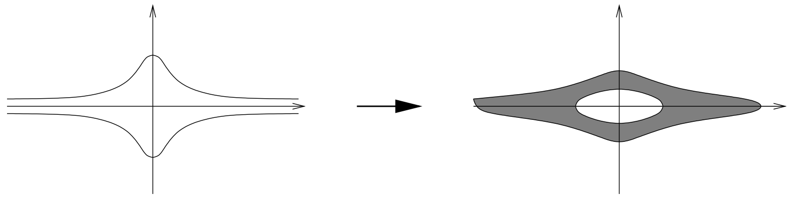

Figure 12: *Instantaneous fattening of the union of two asymptotic graphs.*

The question whether an initial hypersurface (smooth or not) fattens is a difficult question. We refer to [8] and [9] for some discussions. In the next section, we focus on the graph case: can a graph evolving by mean curvature fatten?

# 5 An example of application: uniqueness results for quasilinear pde's

The fattening of a graph evolving by mean curvature is an important issue since it is related to the uniqueness of the solution of the *mean curvature equation for graphs.* It was the subject of the series of works [6, 5, 7, 4, 10, 9]. The motivation comes from the following theorem:

**Theorem 5.1** *(Ecker and Huisken [22, 23]) If $u_0 : \mathbb{R}^N \to \mathbb{R}$ is locally Lipschitz continuous then*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \dfrac{\langle D^2 u Du, Du\rangle}{1+|Du|^2} = 0 & \text{in } \mathbb{R}^N \times (0,+\infty), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{5}$$

*admits at least one smooth solution $u \in C^\infty(\mathbb{R}^N \times (0,+\infty)) \cap C(\mathbb{R}^N \times [0,+\infty))$.*

This theorem is rather surprising since a locally Lipschitz continuous function can have an arbitrary growth at infinity (for example $u_0(x) = \exp(\exp(\exp(|x|^2)))$ is locally Lipschitz continuous!). The situtation is very different from the case of the heat equation for instance where one has to impose restriction on the growth to obtain existence theorems (see John [42]).

The natural question is: is there uniqueness for (5)? ([3])

To our knowledge, the question is open in the whole generality. There are partial positive answers:

(i) It is true in dimension $N = 1$ (see Chou and Kwong [17] and [7]);

(ii) It is true in any dimension when imposing some polynomial-type restrictions on the growth of the initial data $u_0$ (see [4] for details);

(iii) It is true for in any dimension in the radial case, i.e. when $u_0$ is radial (see [10]);

(iv) It is true in any dimension when the initial data $u_0$ is convex ([5]) or at least *convex at infinity* (when $u_0$ is a compactly supported continuous perturbation of a convex function, [9]). In particular, the uniqueness result is true for $u_0(x) = \exp(\exp(\exp(|x|^2)))$.

In the remaining part of this talk, we will describe the convex result (iv) which relies on a geometrical approach using the level-set approach. This link is based, at least formally, on the following lemma

**Lemma 5.2** *([6]) If $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ is a solution of (5) in $\mathbb{R}^N \times (0, +\infty)$ with a locally Lipschitz continuous initial data $u_0$, then $v : \mathbb{R}^{N+1} \times [0, +\infty) \to \mathbb{R}$ defined by $v(x, y, t) = y - u(x, t)$ for every $(x, y) \in \mathbb{R}^N \times \mathbb{R}$ is a continuous viscosity solution of (3) in $\mathbb{R}^{N+1} \times (0, +\infty)$ with initial data $v_0(x, y) = y - u_0(x)$.*

This result means that, if $u$ is a solution of the mean curvature equation for graphs, then the graph of $u$ (seen as an hypersurface in $\mathbb{R}^{N+1}$) evolves by mean curvature (see Figure 13).

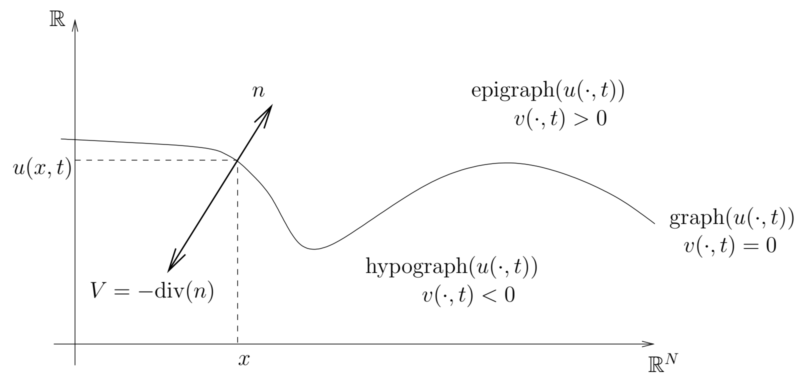


Figure 13: *Evolution of a graph by mean curvature and level-set approach (time $t$ is fixed).*

We aim at applying the level-set approach to the graph of $u$. Thus, we set

$$\begin{aligned} \Gamma_0 &:= \mathrm{graph}(u_0) = \{(x, y) \in \mathbb{R}^{N+1} : y = u_0(x)\} \\ \text{and} \quad \Omega_0 &:= \mathrm{epigraph}(u_0) = \{(x, y) \in \mathbb{R}^{N+1} : y > u_0(x)\}. \end{aligned} \tag{6}$$

Noticing that $v_0$ satisfies (4), a question arises: is $v$ the solution of (3) with initial data $v_0$? We know that $v$ is a solution but the uniqueness part of Theorem 4.1 holds only in the class of $UC$ and $v_0$ is $UC$ if and only if so is $u_0$. In the same way, $\Gamma_t$ is defined as the level-set of the solution of (3) by Theorem 4.2 and is independent of $v_0$ if $v_0$ is $UC$. The extension of these theorem to the class of continuous functions is an open problem. This is the main difficulty which is directly related (via some changes of functions) to the original question of uniqueness for (5). Therefore, level-set approach does not apply directly using $v_0$.

Nevertheless, using a $UC$ function $v_0$ satisfying (6), we have some results which are proved in [5]. At first, we know that the graphs of all solutions of (5) are contained in the generalized evolution of graph$(u_0)$ (see Figure 14).

**Theorem 5.3** *Let $u_0 \in C(\mathbb{R}^N)$ and set $\Gamma_0 = \mathrm{graph}(u_0)$. If $u$ is a solution of (5) with initial data $u_0$, then, for all $t \geq 0$, we have $\mathrm{graph}(u(\cdot,t)) \subset \Gamma_t$, where $\Gamma_t$ is the generalized evolution by mean curvature of $\Gamma_0$ obtained by the level-set approach.*

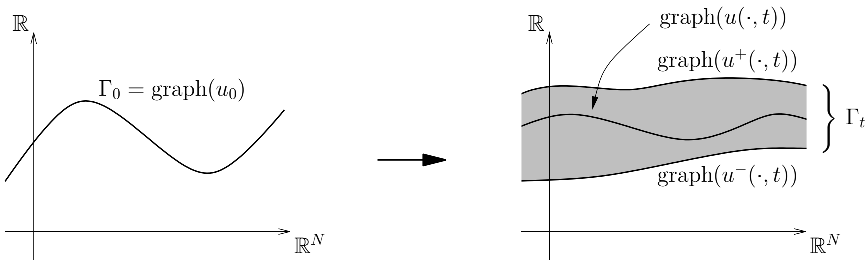


Figure 14: *Evolution by mean curvature of a graph which fattens a time $t$.*

The second result deals with the structure of the front. As above, let $u_0 \in C(\mathbb{R}^N)$ and set $\Gamma_0 = \mathrm{graph}(u_0)$. Consider any $UC$ function $v_0$ satisfying (6) and let $v$ be the unique $UC$ solution of (3). Define the functions

$$u^+(x,t) = \sup\{y \in \mathbb{R} : v(x,y,t) \leq 0\} \quad \text{and} \quad u^-(x,t) = \inf\{y \in \mathbb{R} : v(x,y,t) \geq 0\}.$$

The functions $u^+$ and $u^-$ are respectively the "upper-boundary" and the "lower-boundary" of the generalized evolution by mean curvature $\Gamma_t$ of $\Gamma_0$ (see Figure 14).

**Theorem 5.4** *The functions $u^+, u^- \in C^\infty(\mathbb{R}^N \times (0,+\infty)) \cap C(\mathbb{R}^N \times [0,+\infty))$. Moreover $u^+$ is the maximal solution of (5) and $u^-$ is the minimal solution of (5).*

An immediate consequence of this result is a reformulation of our uniqueness question for (5).

**Corollary 5.5** *Let $u_0 \in C(\mathbb{R}^N)$ and set $\Gamma_0 = \mathrm{graph}(u_0)$. Then (5) has a unique solution with initial data $u_0$ if and only if the generalized evolution $\Gamma_t$ of $\Gamma_0$ does not fatten.*

We conclude with the announced application to the convex case.

**Theorem 5.6** *If $u_0 \in C(\mathbb{R}^N)$ is convex in $\mathbb{R}^N$ and $\Gamma_0 = \mathrm{graph}(u_0)$, then the generalized evolution $\Gamma_t$ of $\Gamma_0$ does not fatten. As a consequence, for any convex continuous data $u_0$, (5) has a unique solution $u \in C^\infty(\mathbb{R}^N \times (0,+\infty)) \cap C(\mathbb{R}^N \times [0,+\infty))$. Moreover $u(\cdot,t)$ is convex in $\mathbb{R}^N$ for all $t \geq 0$.*

**Sketch of proof.** Let $v_0(x,y) = d_s((x,y),\Gamma_0)$ where $d_s(\cdot,\Gamma_0) : \mathbb{R}^{N+1} \to \mathbb{R}$ is the signed distance to the graph of $u_0$

$$d_s((x,y),\Gamma_0) = \begin{cases} \inf\{|(x',y') - (x,y)| : y' = u_0(x')\} & \text{if } y \geq u_0(x), \\ -\inf\{|(x',y') - (x,y)| : y' = u_0(x')\} & \text{if } y \leq u_0(x). \end{cases}$$

Since $u_0$ is convex, $d_s(\cdot, \Gamma_0)$ is a $UC$ concave function in $\mathbb{R}^N$. Using a preservation of concavity theorem of Giga, Goto, Ishii and Sato [32], we obtain that $v(\cdot, t)$ is convex in $\mathbb{R}^{N+1}$ for all $t \geq 0$, where $v$ is the unique $UC$ of (3) with initial data $v_0$. But $\Gamma_t = \{v(\cdot, t) = 0\}$ is the zero level-set of a nonconstant concave function. It follows that $\Gamma_t$ has empty interior. $\square$

**Acknowledgment.** I would like to thank C. Imbert and E. Chasseigne for their comments on the first version of this paper.

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