Université François Rabelais
Tours
École Doctorale : SST
Année universitaire 2001-2002

THÈSE POUR OBTENIR LE GRADE DE
DOCTEUR DE L'UNIVERSITÉ DE TOURS

Discipline : MATHÉMATIQUES

présentée et soutenue publiquement par

Olivier LEY

le 20 décembre 2001

Équations quasilinéaires paraboliques dégénérées et équations de Hamilton-Jacobi : équations géométriques et mouvements de fronts

Directeur de thèse : Guy BARLES

JURY

Martino BARDI	Professeur à l'université de Padoue (Italie)	Examinateur
Guy BARLES	Professeur à l'université de Tours	Directeur de thèse
Pierre CARDALIAGUET	Professeur à l'université de Bretagne Occidentale	Rapporteur
Piotr CHRUŚCIEL	Professeur à l'université de Tours	Examinateur
Francis CLARKE	Professeur, Lyon 1 et Institut universitaire de France	Examinateur
Élisabeth ROUY	Professeur à l'École centrale de Lyon	Examinatrice
Panagiotis SOUGANIDIS	Professeur à l'université du Texas (USA)	Rapporteur

Équations quasilinéaires paraboliques dégénérées et équations de Hamilton-Jacobi : équations géométriques et mouvements de fronts

Olivier Ley

Résumé

La première partie est consacrée à des équations quasilinéaires dégénérées, posées dans $\mathbb{R}^N \times (0,T)$, du type de l'équation du mouvement par courbure moyenne des graphes. Nous utilisons l'approche par lignes de niveau pour interpréter l'évolution au cours du temps des solutions non bornées comme un mouvement d'hypersurfaces dans $\mathbb{R}^{N+1}$. Nous obtenons une condition d'unicité liée au non-épaississement du front associé par cette approche géométrique et des bornes L^∞ locales qui entrainent l'existence de solutions de viscosité discontinues. Une application spectaculaire est l'existence et l'unicité d'une solution de viscosité continue pour toute donnée initiale convexe.

En travaillant directement sur les équations, nous montrons des résultats d'existence et d'unicité en dimension 1. En imposant des restrictions de type polynomial sur la croissance de la donnée initiale dans $\mathbb{R}^N$, nous prouvons qu'une grande classe d'équations est bien posée dans l'ensemble des fonctions à même croissance.

La seconde partie concerne les équations d'Hamilton-Jacobi paraboliques. En premier lieu, pour des équations posées dans tout l'espace, nous établissons des bornes inférieures de gradient pour les solutions que nous exploitons dans le cadre de l'approche par lignes de niveau. Ces bornes empêchent l'épaississement du front mais nous montrons par des contre-exemples qu'elles n'impliquent pas les propriétés plus fines espérées même pour des solutions semiconcaves.

En second lieu, nous considérons ces équations posées dans un ouvert borné régulier avec une condition de Neumann au bord. En utilisant le problème de contrôle avec réflexion au bord associé, nous prouvons que le résultat d'unicité discontinu pour l'équation posée dans $\mathbb{R}^N$ ne s'applique pas.

Quasilinear degenerate parabolic equations and Hamilton-Jacobi equations : geometrical equations and propagating fronts

Olivier Ley

Abstract

In the first part, we study quasilinear degenerate parabolic equations set in $\mathbb{R}^N \times (0,T)$ like the mean curvature equation for graphs. We use the level-set approach to interpret the time-evolution of the unbounded solutions as a propagating front in $\mathbb{R}^{N+1}$. We prove that uniqueness is equivalent to the non-fattening of the front. Existence of discontinuous viscosity solutions is obtained from a L^∞ local bound given by the level-set approach. A spectacular application is the existence of a unique continuous viscosity solution for any convex initial data.

Working directly on the equation, we get existence and uniqueness results in the one-dimensional case. By imposing some polynomial-type growth restriction on the initial data in $\mathbb{R}^N$, we prove the well-posedness of a large class of equations among functions with the same growth.

The second part concerns time-dependent Hamilton-Jacobi equations. First, for equations set in the whole space $\mathbb{R}^N$, we establish lower gradient bounds for the solutions. We exploit them to obtain regularity properties of the propagating fronts associated by the level-set approach. These bounds ensure the non-fattening but we show they are not sufficient to imply sharper regularity even for semiconcave functions.

Secondly, we consider these equations in a smooth bounded set with Neumann boundary conditions. Using the corresponding control problem with reflection, we show that the discontinuous uniqueness result which holds for such equations set in $\mathbb{R}^N$ is not true in this case.

Remerciements

Au long de ma thèse, j'ai eu l'occasion de rencontrer énormément de gens avec qui j'ai eu des discussions fructueuses et qui m'ont soutenu et encouragé. Je suis heureux de les en remercier ici.

J'ai pu compter sur la constante disponibilité et la grande compétence de Guy Barles. J'ai appris énormément à ses côtés et je réalise la chance que j'ai eue de travailler sous sa direction. Je lui exprime ici ma profonde reconnaissance.

Une grande partie des travaux présentés ici ont été réalisés en collaboration avec Samuel Biton qui est devenu au cours de ces années plus qu'un simple collègue. Je le remercie chaleureusement pour sa patience, sa disponibilité de tous les instants et sa gentillesse.

Martino Bardi m'a accueilli et encadré durant seize mois à l'université de Padoue. Il m'a ainsi offert la possibilité de continuer mon travail dans d'excellentes conditions durant mon service national. Je lui exprime toute ma gratitude pour les discussions que nous avons eues et suis particulièrement heureux de sa participation au jury.

Pierre Cardaliaguet et Panagiotis Souganidis m'ont fait l'honneur de consacrer de leur temps précieux à rapporter cette thèse et d'être présents à la soutenance. Piotr Chruściel a accepté de figurer dans le jury. Je les en remercie sincèrement.

Francis Clarke m'a initié à l'analyse non-lisse lors de mes études à Lyon ; c'est lui qui m'a fait découvrir le domaine dans lequel je travaille maintenant. Pour toutes ces raisons, j'accorde une importance particulière à sa présence dans le jury et lui fait part de ma gratitude.

J'ai pu bénéficier tout au long de ma thèse du soutien et des conseils avisés d'Élisabeth Rouy qui a, de plus, accepté de faire partie du jury : un énorme merci.

Plusieurs mathématiciens m'ont encouragé et éclairé, je voudrais remercier en particulier Piermarco Cannarsa, Franco Rampazzo, Alain Rapaport, Carlo Sinestrari et Agnès Tourin.

Mon travail a été effectué principalement au laboratoire de mathématiques de l'université de Tours et au dipartimento di matematica pura ed applicata de l'université de Padoue. Je voudrais en remercier tous les membres, sans oublier les secrétaires. J'adresse une mention spéciale aux doctorants ou anciens doctorants que j'ai cotoyés. Je ne peux les citer tous mais je pense en particulier à Mokhtar, Stéphane, Manu, Laurent, Claudia, Olivier, Benoît, Jean-Claude, Erwann, Francesca, Luca, Vitaly, Fabio, Thierry, Cyril, Ludovic et tant d'autres.

Enfin, ce travail n'aurait pas été possible sans le soutien de mes parents, ma famille, Céline et de tous mes amis.

Table des matières

Introduction

1 Présentation générale

Cette thèse regroupe différents travaux ayant en commun l'étude d'équations paraboliques dégénérées non-linéaires et plus particulièrement leurs liens avec l'évolution généralisée d'hypersurfaces ou, dans une moindre mesure, la théorie du contrôle.

Les équations auxquelles nous nous intéressons ont la forme générique

$$\frac{\partial u}{\partial t} + F(x, t, Du, D^2u) = 0 \quad \text{dans } \mathbb{R}^N \times (0, T), \tag{1}$$

avec une donnée initiale

$$u(x, 0) = u_0(x) \quad \text{pour tout } x \in \mathbb{R}^N. \tag{2}$$

Le temps final T est une constante positive arbitraire. Dans certains cas, on pourra le prendre égal à $+\infty$. La donnée initiale u_0 est une fonction définie sur $\mathbb{R}^N$; la solution u est une fonction à valeurs réelles définie sur $\mathbb{R}^N \times [0, T]$, Du représente son gradient et D^2u sa matrice hessienne. La non-linéarité F est une fonction continue (ou qui pourra présenter des discontinuités par rapport au gradient dans certains cas) de $\mathbb{R}^N \times [0, T] \times \mathbb{R}^N \times \mathcal{S}_N$ dans $\mathbb{R}$, où $\mathcal{S}_N$ est l'espace des matrices symétriques de taille $N \times N$ muni de l'ordre partiel usuel.

Dans toute la suite, l'équation sera parabolique dégénérée, c'est-à-dire que F vérifie, pour tout $x \in \mathbb{R}^N$, $t \in [0, T]$, $p \in \mathbb{R}^N$ et $X, Y \in \mathcal{S}_N$,

$$X \leq Y \quad \Longrightarrow \quad F(x, t, p, X) \geq F(x, t, p, Y). \tag{3}$$

La classe d'équations que nous venons d'introduire est extrêmement générale et intervient dans différents domaines des mathématiques ; de nombreux phénomènes physiques, biologiques, économiques, etc. se modélisent naturellement à l'aide de ce type d'équations. On peut citer entre autres des phénomènes de propagation de chaleur ou de transitions de phase, l'étude démographique d'une population, le traitement d'image, les mathématiques financières, etc. D'autre part, ces équations peuvent aussi se révéler utiles dans le traitement de problèmes où elles n'interviennent pas directement comme les problèmes de contrôle à horizon fini.

Nos principales contributions concernent les équations quasilinéaires dégénérées et les équations d'Hamilton-Jacobi. Le but de ce paragraphe est d'introduire la problématique et les résultats centraux que nous avons obtenus. Dans un souci de clarté, ils seront souvent présentés dans un cadre plus simple et moins général que dans les chapitres qui suivent. Les résultats seront décrits de façon plus précise par la suite.

La motivation de notre étude des équations quasilinéaires (la non-linéarité F est affine en la matrice hessienne) dégénérées est l'unicité des solutions non bornées de l'équation de courbure moyenne pour les graphes

$$\frac{\partial u}{\partial t} - \Delta u + \frac{\langle D^2 u Du, Du\rangle}{1+|Du|^2} = 0 \quad \text{dans } \mathbb{R}^N \times (0,\infty), \tag{4}$$

où $\langle \cdot, \cdot \rangle$ est le produit scalaire euclidien sur $\mathbb{R}^N$ et $|\cdot|$ la norme associée.

Pour situer le problème, commençons par rappeler le cas de l'équation de la chaleur

$$\frac{\partial u}{\partial t} - \Delta u = 0 \quad \text{dans } \mathbb{R}^N \times (0,T], \tag{5}$$

qui est le cas générique pour les équations linéaires uniformément paraboliques. Même hors de ce cadre, cette équation est fondamentale ; les procédures de linéarisation classiques par exemple permettent de ramener bon nombre d'équations non-linéaires à des équations de "type chaleur."

Si la condition initiale u_0 est continue et vérifie une condition de croissance de type (6), il existe une unique solution de ce problème dans la classe des fonctions u satisfaisant

$$|u(x,t)| \leq A\mathrm{e}^{B|x|^2} \quad \text{pour tout } (x,t) \in \mathbb{R}^N \times [0,T], \tag{6}$$

où $A, B > 0$ sont des constantes. De plus cette solution est alors dans $C^\infty(\mathbb{R}^N \times (0,T]) \cap C(\mathbb{R}^N \times [0,T])$. Si par contre, on permet n'importe quelle croissance pour les solutions, il peut en exister plusieurs, même pour une donnée initiale u_0 identiquement nulle. Si maintenant, on cherche à résoudre le problème avec u_0 à trop forte croissance, c'est l'existence des solutions qui peut poser problème. Pour une preuve de ces résultats, voir par exemple John [66] ou Aronson [5].

Dans cet exemple, les limitations à l'existence et à l'unicité proviennent de la croissance de la condition initiale et/ou des solutions. Le contexte est complètement différent pour l'équation (4) puisque Ecker et Huisken [45] ont obtenu le résultat extrêmement surprenant suivant.

Théorème 1.1 *Pour toute donnée initiale u_0 localement lipschitzienne, il existe une solution u de (4) dans $C^\infty(\mathbb{R}^N \times (0,T]) \cap C(\mathbb{R}^N \times [0,T])$.*

Ce résultat est bien sûr étonnant car une fonction localement lipschitzienne peut avoir un comportement quelconque à l'infini. Plusieurs questions se posent alors naturellement :

Question 1. Qu'en-est il de l'unicité sans conditions de croissance ?

Question 2. Quelle sont les propriétés de structure de ce type d'équations qui permettent de tels résultats d'existence ou d'unicité ?

Ce sont ces questions qui ont été le point de départ de notre travail sur les équations quasilinéaires. La question 1 est apparue pour la première fois dans Angenent [4]. Concernant l'existence de solutions régulières sans conditions de croissance pour ce type d'équations, un travail important a été accompli récemment par Chou et Kwong [33] pour tenter de répondre à la deuxième question.

De notre côté, nous avons obtenu de nombreux résultats partiels qui sont énoncés dans le paragraphe 4 ; dans le cas de (4), nous avons montré

Théorème 1.2 *(Réponses à la question 1 pour (4))*

(i) En dimension $N = 1$, (4) admet une unique solution $u \in C^\infty(\mathbb{R} \times (0,T]) \cap C(\mathbb{R} \times [0,T])$ pour toute donnée initiale $u_0 \in C(\mathbb{R})$.

(ii) Si u_0 est localement lipschitzienne et s'il existe des constantes $C > 0$ et $0 < \nu < (1+\sqrt{5})/2$ telle que $|Du_0(x)| \leq C(1+|x|)^\nu$ presque partout dans $\mathbb{R}^N$, alors (4) admet une unique solution $u \in C^\infty(\mathbb{R}^N \times (0,T]) \cap C(\mathbb{R}^N \times [0,T])$ dans la classe des fonctions à croissance polynômiale.

(iii) Si u_0 est convexe dans $\mathbb{R}^N$, il existe une unique solution $u \in C^\infty(\mathbb{R} \times (0,T]) \cap C(\mathbb{R} \times [0,T])$ à (4) ; de plus $u(\cdot,t)$ est convexe pour tout t.

Les résultats (i) et (ii) sont établis de "manière analytique" en travaillant directement sur l'équation (4) alors que le point (iii) apparaît comme une conséquence spectaculaire de la méthode géométrique que nous avons employée qui consiste à "voir" le graphe de $u(\cdot,t)$ comme le mouvement, en fonction du temps, d'une hypersurface dans $\mathbb{R}^{N+1}$.

Cette méthode fait appel à l'approche dite "par lignes de niveau" introduite par Osher et Sethian [76] dans un but numérique puis développée par Evans et Spruck [45] et Chen, Giga et Goto [32] pour étudier l'évolution généralisée d'hypersurfaces. Elle sera décrite en détail plus tard mais disons en tout de suite quelques mots dans le cas de l'équation (4). D'un point de vue géométrique, cette équation décrit l'évolution par courbure moyenne en fonction du temps du graphe de la donnée initiale u_0 dans $\mathbb{R}^{N+1}$. L'approche par lignes de niveau intervient quand on introduit la fonction $v : \mathbb{R}^{N+1} \times [0,+\infty) \to \mathbb{R}$ définie par $v(x,y,t) = y - u(x,t)$ où u est une solution de (4). La fonction v est alors, au moins formellement, une solution de

$$\frac{\partial v}{\partial t} - \Delta v + \frac{\langle D^2 v Dv, Dv\rangle}{|Dv|^2} = 0 \quad \text{dans } \mathbb{R}^{N+1} \times (0,\infty), \tag{7}$$

qui est l'équation correspondant au mouvement par courbure moyenne des lignes de niveau de v. Clairement, l'unicité pour (7) avec $v_0(x,y) = y - u_0(x)$ comme donnée initiale, entraîne l'unicité pour notre équation de départ.

Il faut d'abord donner un sens à la solution d'une telle équation ; nous utiliserons la notion de solution de viscosité dont nous parlerons ultérieurement. Ici, il suffit de savoir que cette notion de solutions faibles apporte l'existence et l'unicité d'une solution v bornée uniformément continue (BUC dans la suite) à (7) pour toute donnée initiale $v_0 \in BUC(\mathbb{R}^{N+1})$.

Comme l'équation (7) est de plus invariante par tout changement de fonctions croissant, on pourrait penser que le problème de l'unicité pour (4) se résout aisément au moyen de changements du type $v(x,y,t) = \tanh(y - u(x,t))$ et $v_0(x,y) = \tanh(y - u_0(x))$, quitte à étendre les résultats d'unicité pour (7) à l'ensemble des fonctions continues bornées.

Malheureusement, nous n'avons pas été capable de prouver cette extension. Néanmoins, l'approche par lignes de niveau a généré de nombreux résultats. Un résultat clé est que les graphes de toutes les solutions de (4) sont situés dans l'ensemble de niveau 0 (le "front" par la suite) de l'unique solution $v \in BUC(\mathbb{R}^{N+1} \times [0,+\infty))$ de (7) pour toute donnée initiale $v_0 \in BUC(\mathbb{R}^{N+1})$ qui vérifie $\{v_0 = 0\} = \mathrm{Graph}(u_0)$.

Comme conséquences remarquables de cette propriété, nous obtenons une borne L^∞ locale pour les solutions de (4) et l'existence de solutions de viscosité discontinues.

Mais le point sur lequel nous tenons à insister ici est celui de la nature du front. Si le front reste un graphe par exemple, notre problème d'unicité est résolu ; plus précisément, nous avons le

Théorème 1.3 *Soit $u_0 \in C(\mathbb{R}^N)$ et $v_0 \in BUC(\mathbb{R}^{N+1})$ verifiant $\{v_0 = 0\} = \mathrm{Graph}(u_0)$. Alors, l'unicité d'une solution classique pour (4) est équivalente au non-épaississement du front $\{v = 0\}$ de l'unique solution v de (7) associée à v_0.*

Il est malheureusement connu que les évolutions généralisée obtenues par l'approche par lignes de niveau peuvent générer des intérieurs, même si l'on démarre d'une hypersurface lisse (voir Ilmanen [55], Soner et Souganidis [79] et Angenent, Chopp et Ilmanen [3]) ; cependant, lorsqu'on démarre d'une hypersurface qui est un graphe, comme c'est le cas dans notre problème, le problème de savoir si le front peut développer un intérieur ou non, reste ouvert en toute généralité.

Ces problèmes de "régularité" de fronts sont aussi une des motivations de notre travail sur les équations d'Hamilton-Jacobi que nous présentons maintenant.

Considérons une équation d'Hamilton-Jacobi, c'est-à-dire une équation de type (1) complètement dégénéré,

$$\frac{\partial u}{\partial t} + H(x,t,Du) = 0 \quad \text{dans } \mathbb{R}^N \times (0,T), \tag{8}$$

où $H \in C(\mathbb{R}^N \times [0,T] \times \mathbb{R}^N)$ est appelé l'Hamiltonien en référence aux problèmes de contrôle auxquels ces équations sont intimement liées.

La première difficulté qui survient quand on travaille avec ces équations, fortement non-linéaires et complètement dégénérées, est celle de la notion même de solution. Pour bien s'en convaincre, considérons l'exemple simple d'un graphe $y = u(x,t)$ évoluant avec une vitesse normale constante égale à c. La fonction u est alors, au moins formellement, solution de l'équation de type (8),

$$\frac{\partial u}{\partial t} - c\sqrt{1+|Du|^2} = 0 \quad \text{dans } \mathbb{R}^N \times (0,T), \tag{9}$$

avec la condition initiale $u(x,0) = u_0(x)$ si on a démarré du graphe d'une fonction $u_0 : \mathbb{R}^N \to \mathbb{R}$. Cet exemple modélise la propagation d'un front de flamme comme cela est expliqué dans Barles [9]. Dans ce cas précis, une formule explicite décrivant le mouvement géométrique est connue :

$$u(x,t) = \sup_{|y-x|\le ct} \left\{ u_0(y) + \sqrt{(ct)^2 - |y-x|^2} \right\}. \tag{10}$$

Il apparaît clairement sur cette formule qu'une telle fonction u n'est en général pas lisse, même si la donnée initiale u_0 est tout ce qu'il y a de plus régulier (voir illustration sur la figure 1). Quel sens donner donc aux solutions de (9) de sorte que (10) soit l'unique solution ? La notion de solution de viscosité (voir paragraphe 2) apporte une réponse satisfaisante dans le cadre des équations de Hamilton-Jacobi :

Théorème 1.4 *(Crandall, Lions [38], Ishii [58]) Supposons que l'Hamiltonien* $H \in C(\mathbb{R}^N \times [0,T] \times \mathbb{R}^N)$ *satisfait, pour tous* $x, y \in \mathbb{R}^N$, $t \in [0,T]$ *et* $p, q \in \mathbb{R}^N$,

$$H(x,t,p) - H(y,t,p) \leq C(1+|p|)|x-y|, \tag{11}$$

$$H(x,t,p) - H(x,t,q) \leq C(1+|x|)|p-q|, \tag{12}$$

pour une certaine constante $C > 0$. *Il existe alors une unique solution de viscosité de (8) pour toute donnée initiale* $u_0 \in C(\mathbb{R}^N)$.

Il est important de remarquer que ce théorème est établi sans conditions de croissance sur la donnée initiale ou la solution. Il répond donc, pour (8), aux questions 1 et 2 posées plus haut.

Suite à ce résultat, nous nous intéressons à la régularité des solutions obtenues (les solutions de viscosité sont *a priori* seulement continues). Pour ce faire, nous établissons une version plus précise du théorème 1.4 qui met en évidence la propriété associée à l'hypothèse (12), à savoir "la vitesse finie de propagation."

Cette propriété signifie que la valeur $u(x,t)$ de la solution de l'équation en un point $(x,t) \in \mathbb{R}^N \times [0,T]$ ne dépend pas de la valeur de la condition initiale u_0 dans tout l'espace $\mathbb{R}^N$ mais seulement dans une certaine boule $\bar{B}(x, R(x,t))$ centrée en x et d'un certain rayon fini $R(x,t) > 0$. Il y a donc vitesse finie de propagation de l'information car une perturbation de la donnée initiale "loin" d'une certaine zone ne modifie pas la solution dans cette zone.

À l'aide de ce théorème, si la donnée initiale u_0 est localement lipschitzienne, nous établissons une borne supérieure de gradient pour la solution u de (8) ce qui entraîne que u est localement lipschitzienne. Mais le résultat principal consiste en une borne de gradient inférieure qui, pour des Hamiltoniens convexes et positivement homogènes de degré 1 par rapport au gradient, peut s'énoncer de la façon simplifiée suivante : si u_0 satisfait, pour un certain $\eta > 0$, $|u_0(x)| + |Du_0(x)| \geq \eta$ pour presque tout $x \in \mathbb{R}^N$, alors

$$|u| + \mathrm{e}^{-\gamma t}|Du| \geq \tilde{\eta} \quad \text{pour presque tout } (x,t) \in \mathbb{R}^N \times [0,T], \tag{13}$$

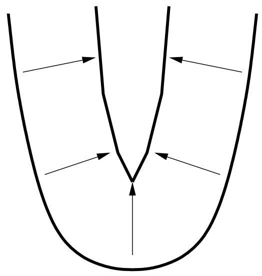

FIG. 1 – *Développement d'une singularité pour un graphe lisse évoluant à vitesse normale constante*

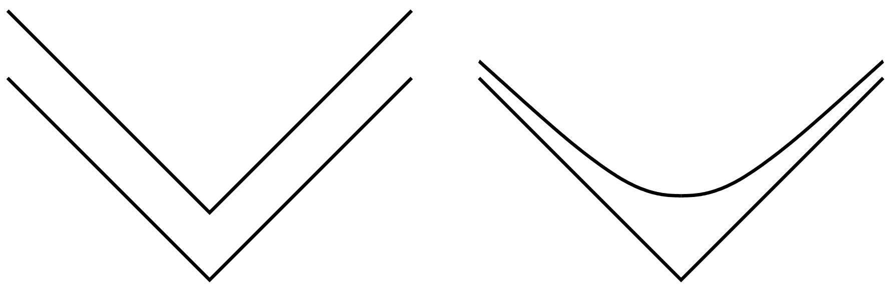

FIG. 2 – *Exemples d'évolutions avec vitesse finie de propagation et vitesse infinie de propagation*

pour des constantes $\gamma, \tilde{\eta} > 0$. Cette propriété est à relier à l'approche géométrique dont nous avons parlée plus haut, d'autant que l'hypothèse de positive homogénéité sur l'Hamiltonien est exactement celle qui permet cette approche dans le cas de (8) (voir paragraphe 3).

Précisons ce parallèle dans le cas de l'équation (9). Si l'on pose $v(x,y,t) = y - u(x,t)$, où u est une solution de (9), alors v vérifie l'équation eikonale

$$\frac{\partial v}{\partial t} - c\,|Dv| = 0 \quad \text{dans } \mathbb{R}^{N+1} \times (0,T). \tag{14}$$

Cette nouvelle équation vérifie les conditions requises pour la borne inférieure de gradient (13) et entre dans le cadre de l'approche par lignes de niveau. Sur le front, le gradient de v ne "s'annule" donc pas, le front ne s'épaissit pas et on a unicité pour (9). Ce résultat est une redite du théorème 1.4. Notre objectif ici est "d'aller plus loin" en exploitant cette borne de gradient pour obtenir des propriétés de régularité plus fines pour le front. Malgré quelques resultats dans cette direction, nous répondons surtout à ce problème par des contre-exemples montrant qu'une telle inégalité pour des fonctions non-lisses, contrairement au cas lisse, ne fournit en fait que peu de renseignements sur la régularité des lignes de niveau même en supposant la solution plus régulière, par exemple semiconcave.

Terminons le passage consacré aux bornes inférieures de gradient pour mentionner que ces bornes doivent être délicates, sinon fausses, dans le cas de (5) ou (4) (gardons à l'esprit qu'une telle borne inférieure pour (4) résoudrait le problème de l'unicité *via* le théorème 1.3). Le principal écueil est la vitesse infinie de propagation pour ces équations (cela est d'ailleurs une limitation sérieuse à la validité physique de l'équation de la chaleur). La figure 2 illustre géométriquement ce phénomène en comparant une évolution à vitesse normale constante reliée à (9) (vitesse finie de propagation) et une évolution par courbure moyenne faisant intervenir (4) (vitesse infinie de propagation ; "la brisure dans la droite modifie la courbure jusqu'a l'infini"). En fait, la vitesse finie de propagation pour (1) est étroitement reliée au caractère complètement dégénéré ou pas de l'équation comme cela est expliqué dans Biton [26].

Nous arrivons au dernier problème présenté dans cette thèse. Ce sujet est un peu à part ; il traite d'équations d'Hamilton-Jacobi de type (8) il est vrai, mais posées cette fois-ci dans un ouvert Ω borné et régulier avec des conditions de Neumann au bord,

$$\frac{\partial u}{\partial n}(x,t) = 0 \quad \text{pour } (x,t) \in \partial\Omega \times [0,T]. \tag{15}$$

Les solutions de viscosité apportent une réponse complète au problème de l'existence et de l'unicité lorsque la donnée initiale u_0 est continue (voir Lions [73]). La notion de solution de viscosité ayant été étendue aux fonctions discontinues, nous nous posons la question de l'unicité discontinue pour le problème de Neumann. En utilisant le problème de contrôle associé, qui est ici un problème de contrôle optimal avec reflexion sur le bord de Ω, nous construisons un contre-exemple montrant que les résultats naturels d'unicité dans ce cadre discontinu (ceux de Barles et Perthame [20] par exemple) ne s'étendent pas tels quels dans ce cas. Nous renvoyons au paragraphe 7 pour les détails.

2 La notion de solution de viscosité

Les solutions de viscosité ont été introduites par Crandall et Lions au début des années 80 dans [38], initialement pour pallier à l'absence de solutions régulières aux équations d'Hamilton-Jacobi (8). D'autres notions de solutions ont été introduites pour les équations aux dérivées partielles du premier ordre, citons par exemple les travaux de Kružkov [67], ceux de Subbotin [83], de Clarke [35] et de Frankowska [49] et [50]. Dans certains cas, ces solutions sont équivalentes. Nous avons choisi ici d'utiliser les solutions de viscosité qui présentent l'avantage d'avoir connu un développement spectaculaire pour les équations du second ordre depuis les travaux de Jensen [65] et Ishii [62].

Le but de cette section n'est pas de récapituler toutes les propriétés de ces solutions mais d'en donner un aperçu rapide en rappelant les principales définitions et les outils de base de cette notion de solution. Nous renvoyons aux ouvrages de Lions [72], Barles [13] et Bardi et Capuzzo Dolcetta [6] pour les équations du premier ordre. Le "User's Guide" de Crandall, Ishii et Lions [37] est l'article de référence concernant le second ordre. Comme livres sur le sujet, on pourra consulter Fleming et Soner [48] ou le recueil [7].

L'idée de base des solutions de viscosité est de remplacer les "dérivées" de la solution dans l'équation par les dérivées de fonctions-test lisses évaluées aux points de maximum ou de minimum de la différence entre la solution et la fonction-test. Cette définition s'applique donc à des fonctions pour lesquelles ces extrema sont atteints. Historiquement, les solutions de viscosité ont été définies pour des fonctions continues. Ishii [59] a étendu cette définition aux fonctions discontinues en remplaçant la solution u, dans le calculs des *extrema*, par ses enveloppes supérieure et inférieure.

Pour toute fonction $\Phi : \mathcal{A} \to \mathbb{R}$, où A est un sous-ensemble de $\mathbb{R}^M$, $M \in I\!N$, nous définissons ses enveloppes supérieure Φ^* et inférieure Φ_*, pour tout $z \in \overline{\mathcal{A}}$, par

$$\Phi^*(z) = \limsup_{z' \to z} \Phi(z') \quad \text{et} \quad \Phi_*(z) = \liminf_{z' \to z} \Phi(z').$$

La fonction u^* est semi-continue supérieurement (scs dans la suite du texte) et u_* est semi-continue inférieurement (sci) ce qui donne un sens à la

Définition 2.1 *Une fonction localement bornée u est dite sous-solution de viscosité de (1) si la condition suivante est remplie : pour toute fonction $\varphi \in C^2(\mathbb{R}^N \times [0,T])$, si $(x_0,t_0) \in \mathbb{R}^N \times (0,T]$ est un point de maximum local de $u^* - \varphi$, alors*

$$\frac{\partial \varphi}{\partial t}(x_0,t_0) + F_*(x_0,t_0,D\varphi(x_0,t_0),D^2\varphi(x_0,t_0)) \leq 0.$$

Une fonction localement bornée u est dite sur-solution de viscosité de (1) si la condition suivante est remplie : pour toute fonction $\varphi \in C^2(\mathbb{R}^N \times [0,T])$, si $(x_0,t_0) \in \mathbb{R}^N \times (0,T]$

est un point de minimum local de $u_* - \varphi$, alors

$$\frac{\partial \varphi}{\partial t}(x_0, t_0) + F^*(x_0, t_0, D\varphi(x_0, t_0), D^2\varphi(x_0, t_0)) \geq 0.$$

Enfin, u est une solution de viscosité si u est à la fois sous- et sur-solution.

À l'aide de cette définition, on peut vérifier que la fonction u donnée par (10) est une solution de viscosité de (9).

Cette définition peut aussi s'énoncer à l'aide des sur- et sous-différentiels d'ordre 2 des enveloppes de la fonction en jeu que nous introduisons maintenant.

Définition 2.2 *Soit u un fonction localement bornée sur $\mathbb{R}^N \times [0, T]$.*

Le sur-différentiel d'ordre 2 de u^* au point (x, t) est le sous-ensemble convexe de $\mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N$, noté $\mathcal{P}^{2,+}(u^*)(x, t)$, qui est constitué des triplets (a, p, X) tels que

$$u(y, s) - u(x, t) - a(s - t) - \langle p, y - x \rangle - \frac{1}{2}\langle X(y - x), y - x \rangle \leq o(|s - t| + |y - x|^2),$$

pour (y, s) voisin de (x, t).

Le sous-différentiel d'ordre 2 de u_* au point (x, t) est le sous-ensemble convexe de $\mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N$, noté $\mathcal{P}^{2,-}(u_*)(x, t)$, qui est constitué des triplets (a, p, X) tels que

$$u(y, s) - u(x, t) - a(s - t) - \langle p, y - x \rangle - \frac{1}{2}\langle X(y - x), y - x \rangle \geq o(|s - t| + |y - x|^2),$$

pour (y, s) voisin de (x, t).

Nous avons alors le

Théorème 2.1 *Une fonction localement bornée u est une sous-solution de viscosité de (1) si et seulement si, pour tout $(x, t) \in \mathbb{R}^N \times (0, T]$ et pour tout $(a, p, X) \in \mathcal{P}^{2,+}(u^*)(x, t)$,*

$$a + F_*(x, t, p, X) \leq 0.$$

Une fonction localement bornée u est une sur-solution de viscosité de (1) si et seulement si, pour tout $(x, t) \in \mathbb{R}^N \times (0, T]$ et pour tout $(a, p, X) \in \mathcal{P}^{2,-}(u_*)(x, t)$,

$$a + F^*(x, t, p, X) \geq 0.$$

Précisons à ce niveau que cette définition est une extension de la notion classique : une solution classique de (1) est solution de viscosité (utiliser la définition à l'aide des sur- et sous-différentiels pour le voir) et réciproquement, une solution de viscosité lisse est solution classique (appliquer la définition avec $\varphi := u$).

La définition équivalente que procure le théorème ci-dessus s'avère fondamentale dans le cas des équations du second ordre, en particulier dans les preuves d'unicité qui utilise le dédoublement de variables et le théorème d'Ishii [37, Theorem 8.3]. Ce résultat est fondamental pour pouvoir appliquer le principe du maximum dans le cadre des solutions de viscosité. Comme il est utilisé abondamment dans la suite, nous le rappelons sous la forme qui nous sera utile. Nous renvoyons à la littérature citée ci-dessus pour plus de détails et la preuve.

Théorème 2.2 *Soit $u_1, u_2 : \mathbb{R}^N \to \mathbb{R}$ deux fonctions localement bornées et $\varphi \in C^2(\mathbb{R}^N \times \mathbb{R}^N \times [0,T])$ telles que*

$$(x_1, x_2, t) \mapsto (u_1)^* - (u_2)_* - \varphi(x_1, x_2, t)$$

admette un maximum local $(\bar{x}_1, \bar{x}_2, \bar{t})$ dans $\mathbb{R}^N \times \mathbb{R}^N \times (0,T]$.

Supposons de plus qu'il existe $r > 0$ tel que, pour tout $M > 0$, il existe $C > 0$ tel que pour tout $(x_1, x_2, t) \in \bar{B}(x_1, r) \times \bar{B}(x_2, r) \times [\bar{t} - r, \bar{t} + r]$,

$$a_1 \leq C \quad (\textit{respectivement } a_2 \geq -C), \tag{16}$$

dès que $(a_1, p_1, X_1) \in \mathcal{P}^{2,+}((u_1)^*)(x_1, t)$, $|u_1(x_1,t)| + |p_1| + \|X_1\| \leq M$ (respectivement $(a_2, p_2, X_2) \in \mathcal{P}^{2,+}(-(u_2)_*)(x_2, t) = -\mathcal{P}^{2,-}((u_2)_*)(x_2, t)$, $|u_1(x_2,t)| + |p_2| + \|X_2\| \leq M$).

Alors, pour tout $\rho > 0$, il existe $X_1, X_2 \in \mathcal{S}_N$ telles que

$$\begin{cases} (a_1, D_{x_1}\varphi(\bar{x}_1, \bar{x}_2, \bar{t}), X_1) \in \bar{\mathcal{P}}^{2,+}((u_1)^*)(\bar{x}_1, \bar{t}) \\ \qquad\qquad \textit{et} \quad (a_2, D_{x_2}\varphi(\bar{x}_1, \bar{x}_2, \bar{t}), X_2) \in \bar{\mathcal{P}}^{2,-}((u_2)_*)(\bar{x}_2, \bar{t}), \\ -(\frac{1}{\rho} + \|A\|) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X_1 & 0 \\ 0 & -X_2 \end{pmatrix} \leq A + 2\rho A^2 \quad \textit{où} \;\; A = D^2\varphi(\bar{x}_1, \bar{x}_2, \bar{t}), \\ a_1 + a_2 = \dfrac{\partial \varphi}{\partial t}(\bar{x}_1, \bar{x}_2, \bar{t}). \end{cases}$$

Ce théorème peut sembler rébarbatif mais il faut bien voir qu'il sera utilisé lors des preuves d'unicité, u_1 étant dans ce cas une sous-solution de (1), u_2 une sur-solution et φ la fonction-test. L'hypothèse (16) est alors automatiquement satisfaite en général et l'intérêt de ce théorème est de permettre d'exprimer une condition du type "$D^2(u_1 - u_2) \leq D^2\varphi$" au point de maximum lorsque les fonctions u_1 et u_2 ne sont pas régulières et que l'on a dédoublé les variables.

Jusqu'à présent, les solutions que nous avons définies le sont dans l'ouvert $\mathbb{R}^N \times (0,T)$ sans que nous ayons précisé de conditions de bord ou de conditions aux limites (en fait, nous incluons le temps T dans les définitions ; les définitions sont équivalentes qu'on l'inclut ou pas, Cf. Crandall et Lions [38]). Dans le but d'avoir des résultats satisfaisants pour cette notion de solution, il est nécessaire de généraliser les conditions aux limites dans le sens suivant : u est sous-solution de (1)–(2) si u est sous-solution de (1) dans $\mathbb{R}^N \times (0,T]$ au sens de la définition 2.1 et si pour toute fonction $\varphi \in C^2(\mathbb{R}^N \times [0,T])$ telle que $u^* - \varphi$ atteint un maximum en $(x_0, 0) \in \mathbb{R}^N \times \{0\}$, on a

$$\max\left\{\frac{\partial \varphi}{\partial t}(x_0, 0) + F_*(x_0, 0, D\varphi(x_0,0), D^2\varphi(x_0,0)),\ u^*(x,0) - (u_0)^*(x)\right\} \leq 0. \tag{17}$$

On a une définition similaire pour les sur-solutions en remplaçant "u^*" par "u_*," "maximum" par "minimum," "max" par "min" et "$\leq$" par "$\geq$." Ces conditions sont dites "relaxées." En fait, dans le cas des conditions initiales pour des équations paraboliques de type (1), (17) revient à une condition classique, l'inégalité de viscosité dans (17) n'apportant rien en $t = 0$ (voir Barles et Perthame [21] par exemple). La condition limite pour la sous-solution s'écrit donc simplement

$$u^*(x,0) \leq (u_0)^*(x) \quad \text{pour tout } x \in \mathbb{R}^N$$

et pour la sur-solution

$$u_*(x,0) \geq (u_0)_*(x) \quad \text{pour tout } x \in \mathbb{R}^N.$$

Néanmoins, précisons que lorsque l'ouvert sur lequel nous travaillons n'est pas $\mathbb{R}^N$ tout entier mais un ouvert à bord, les conditions de bord relaxées ne sont pas artificielles. Nous avons choisi de présenter ici ce cas simple pour soulever le problème et nous renvoyons au chapitre 6 où il se pose effectivement.

La notion de solution introduite dans cette section n'aurait pas beaucoup d'intérêt si elle ne procurait pas de bons résultats d'existence, d'unicité et de stabilité. En général, les premiers résultats que l'on s'efforce d'obtenir dans cette direction sont des principes de comparaison qui prennent la forme suivante :

Si u_1, u_2 sont respectivement sous- et sur-solution de (1) et si $(u_1)^*(\cdot,0) \leq (u_2)_*(\cdot,0)$ dans $\mathbb{R}^N$, alors

$$(u_1)^* \leq (u_2)_* \quad \textit{dans } \mathbb{R}^N \times [0,T].$$

Un tel résultat entraîne aussitôt l'unicité dans la classe des solutions de viscosité discontinues pour (1)–(2) si u_0 est continue par exemple. De plus la solution sera continue. En effet, dans ce cas, $(u_0)^* = (u_0)_* = u_0$ et si u_1 et u_2 sont deux solutions, l'application du principe de comparaison à "u_1 sous-solution, u_2 sur-solution" puis en inversant les rôles, conduit à

$$(u_1)^* \leq (u_2)_* \leq (u_2)^* \leq (u_1)_* \quad \text{dans } \mathbb{R}^N \times [0,T],$$

ce qui donne le résultat.

Si maintenant la fonction u_0 est discontinue, les choses sont beaucoup plus compliquées. L'application du principe de comparaison à u_1 et u_2 donne

$$(u_1)^*(\cdot,0) \leq (u_0)^* \quad \text{et} \quad (u_0)_* \leq (u_2)_*(\cdot,0) \quad \text{dans } \mathbb{R}^N$$

et on ne peut rien en déduire. En général, on n'a même pas unicité comme la discussion relative du paragraphe 7 permet de le comprendre.

L'existence d'une solution de viscosité continue pour toute donnée initiale continue découle souvent facilement du principe de comparaison en appliquant la méthode de Perron, introduite dans le cadre des solutions de viscosité par Ishii [60]. Nous renvoyons aux sections 2.4.3, 3.3 et 4.5 où nous utilisons effectivement cette méthode. Notons également que, lorsque l'équation provient d'un problème de contrôle par exemple, la question de l'existence ne se pose pas puisque la fonction-valeur est une solution ; ceci est le cas pour (8) quand l'Hamiltonien H est convexe en le gradient. Dans d'autres cas, on a des formules explicites pour les solutions comme les formules de Hopf, de Lax-Oleinik, etc. voir par exemple [72], [10], [8] et le paragraphe 4.1 où nous parlons de formules de représentation récentes pour (7) de [29] et [80]. Enfin, notre approche géométrique (voir le paragraphe 4.1) a également procuré des résultats d'existence par une méthode peu classique.

3 Évolution d'hypersurfaces par la méthode des lignes de niveau

Nous allons maintenant décrire la méthode des lignes de niveau qui définit une évolution généralisée d'hypersurfaces. Comme rappelé ci-dessus, elle a été introduite dans [76], [45] et [32] mais ce type d'idée apparaît déjà dans Evans et Souganidis [44] et Barles [9]. Nous renvoyons aussi à Barles, Soner et Souganidis [22] dont nous nous inspirons pour la présentation qui suit, ainsi qu'à Barles et Souganidis [23].

Nous considérons l'évolution d'ensembles dans $\mathbb{R}^{N+1}$ car typiquemment ces ensembles seront par la suite les graphes ou les épigraphes de solutions de (4) ou (9) par exemple.

Commençons par une approche heuristique. Soit $(\Gamma_t)_{t\geq 0}$ une famille d'hypersurfaces lisses de $\mathbb{R}^{N+1}$ qui sont, à chaque temps $t \geq 0$, le bord d'un ouvert Ω_t^+. En chaque point z de Γ_t, on peut définir une normale $n(z,t)$. Supposons que Γ_t évolue avec une vitesse normale $\mathcal{V}_n(z,t,n,Dn)$ dépendant du point (z,t), de la normale $n(z,t)$ et de son gradient $Dn(z,t)$. Supposons de plus qu'il existe une fonction lisse $v : \mathbb{R}^{N+1}\times[0,+\infty) \to \mathbb{R}$ telle que, pour tout $t \geq 0$,

$$\Gamma_t = \{z \in \mathbb{R}^{N+1} : v(z,t) = 0\} \quad \text{et} \quad \Omega_t^+ = \{z \in \mathbb{R}^{N+1} : v(z,t) > 0\}.$$

Quelques calculs classiques permettent alors de relier la fonction v et ses différentielles avec la vitesse normale $\mathcal{V}_n$ et ses arguments. On obtient alors que v doit satisfaire, au moins formellement, l'équation de type (1)

$$\frac{\partial v}{\partial t} + F(z,t,Dv,D^2v) = 0, \tag{18}$$

posée cette fois-ci dans $\mathbb{R}^{N+1} \times (0,+\infty)$ avec, pour tout $z \in \mathbb{R}^{N+1}, t \in (0,+\infty), p \in \mathbb{R}^{N+1}$ et $X \in \mathcal{S}_{N+1}$,

$$F(z,t,p,X) = -|p|\mathcal{V}_n\left(z,t,-\frac{p}{|p|},-\frac{1}{|p|}\left(I-\frac{p\otimes p}{|p|^2}\right)X\right), \tag{19}$$

où I représente la matrice identité de taille $(N+1)\times(N+1)$ et $\otimes$ le produit tensoriel.

On vient de représenter l'evolution géométrique $(\Gamma_t)_{t\geq 0}$ par l'ensemble de niveau 0 de la solution d'une équation de type (1) dans $\mathbb{R}^{N+1} \times (0,+\infty)$. La non-linéarité F donnée par (19) n'est malheureusement plus continue puisqu'elle peut présenter des discontinuités en le gradient. Si $\mathcal{V}_n$ est décroissant en son dernier argument Dn, (19) vérifie (3) et l'équation est parabolique. De plus, par construction même, F vérifie la condition fondamentale suivante, dite *condition géométrique*,

$$F(z,t,\lambda p,\lambda X + \mu p\otimes p) = \lambda F(z,t,p,X) \quad \text{pour tous } \lambda > 0,\ \mu \in \mathbb{R}. \tag{20}$$

Cette condition est équivalente à dire que l'équation (18) est invariante par tout changement croissant de fonctions $v \mapsto \Psi(v)$, $\Psi' > 0$.

L'approche par ligne de niveau est une justification théorique de ce qui précède. Plus précisément, donnons-nous un triplet $(\Gamma_0,\Omega_0^+,\Omega_0^-)$ où Ω_0^+ et Ω_0^- sont des ouverts disjoints de $\mathbb{R}^{N+1}$ et $\Gamma_0 = (\Omega_0^+ \cup \Omega_0^-)^C$. Ce triplet forme une partition de $\mathbb{R}^{N+1}$ et le cadre typique est celui où Γ_0 est une hypersurface qui est la frontière des deux ouverts : $\Gamma_0 = \partial\Omega_0^+ = \partial\Omega_0^-$.

Considérons alors une fonction uniformément continue (on note par la suite UC) $v_0 : \mathbb{R}^{N+1} \to \mathbb{R}$ telle que

$$\{v_0 = 0\} = \Gamma_0, \quad \{v_0 > 0\} = \Omega_0^+, \quad \text{et} \quad \{v_0 < 0\} = \Omega_0^-. \tag{21}$$

La ligne de niveau 0 de v_0 est exactement Γ_0 et le choix du signe de v_0 sur les ouverts peut être vu comme une orientation de Γ_0. Notons à ce point qu'il est toujours possible de trouver une telle fonction v_0 et nous prendrons souvent la distance signée à Γ_0 qui est une fonction lipschitzienne définie par

$$\mathrm{d}(z,\Gamma_0) := \begin{cases} +\mathrm{dist}(z,\Gamma_0) & \text{si } z \in \Omega_0^+, \\ -\mathrm{dist}(z,\Gamma_0) & \text{si } z \in \Omega_0^-, \end{cases} \tag{22}$$

où “dist” est la distance positive usuelle.

On définit alors l'*évolution généralisée* de $(\Gamma_0, \Omega_0^+, \Omega_0^-)$ par la famille $(\Gamma_t, \Omega_t^+, \Omega_t^-)_{t\geq 0}$ à l'aide du théorème suivant :

Théorème 3.1 *Supposons que l'équation parabolique (18) satisfait (20) et des hypothèses convenables de telle sorte qu'elle soit bien posée dans UC.*

Soit alors $v \in UC(\mathbb{R}^{N+1} \times [0,+\infty))$ *l'unique solution de (18) avec* v_0 *comme condition initiale. De plus, si* $\tilde{v}_0 \in UC(\mathbb{R}^{N+1})$ *satisfait*

$$\{\tilde{v}_0 = 0\} = \Gamma_0, \quad \{\tilde{v}_0 > 0\} = \Omega_0^+, \quad \text{et} \quad \{\tilde{v}_0 < 0\} = \Omega_0^-,$$

et si $\tilde{v} \in UC(\mathbb{R}^{N+1})$ *est l'unique solution de (18) avec* $\tilde{v}_0$ *comme condition initiale, on a*

$$\begin{aligned} \{v(\cdot,t) > 0\} &= \{\tilde{v}(\cdot,t) > 0\} := \Omega_t^+, \\ \{v(\cdot,t) < 0\} &= \{\tilde{v}(\cdot,t) < 0\} := \Omega_t^-, \\ \{v(\cdot,t) = 0\} &= \{\tilde{v}(\cdot,t) = 0\} := \Gamma_t. \end{aligned}$$

Ce théorème définit bien une famille $(\Gamma_t, \Omega_t^+, \Omega_t^-)_{t\geq 0}$ indépendante du représentant v_0 choisi, pour peu qu'il soit uniformément continu. L'ensemble $\bigcup_{t\geq 0} \Gamma_t \times \{t\}$ sera appelé par la suite le front associé à Γ_0 par (18) et Γ_t sera le front à l'instant t.

4 Équations quasilinéaires

Dans ce paragraphe, nous décrivons, de façon plus détaillée, les résultats obtenus dans les chapitres 1, 2, 3 et 4 lorsque (1) est quasilinéaire. Les ouvrages de référence dans ce cas sont les livres de Ladyzenskaja, Solonnikov et Ural'ceva [68] et de Gilbarg et Trundinger [53] (bien que ce dernier ne s'intéresse qu'au cas stationnaire).

Les motivations et les objectifs de ce travail ont été présentés plus haut dans le cas de notre équation modèle (4). Ils sont également expliqués dans l'introduction du chapitre 1 toujours dans le cas de (4). Les résultats obtenus dans ce chapitre sont améliorés et généralisés dans les chapitres suivants. Précisons que l'introduction du chapitre 2 aborde le sujet de manière globale. Enfin, nous renvoyons aussi à l'introduction de Biton [27] en collaboration de qui ce travail a été réalisé.

4.1 L'approche géométrique

Dans le chapitre 2, nous étudions des équations quasilinéaires de la forme

$$\frac{\partial u}{\partial t} - \mathrm{Tr}\left[b(Du)D^2u\right] = 0 \quad \text{dans } \mathbb{R}^N \times (0,+\infty), \tag{23}$$

où b est continue de $\mathbb{R}^N$ dans $\mathcal{S}_N^+$, l'ensemble des matrices symétriques positives. La donnée initiale u_0 sera supposée au moins continue. Nous ne recherchons pas la plus grande généralité dans ce travail mais introduisons une méthode géométrique d'étude d'une certaine classe d'équations quasilinéaires qui pourra ensuite être généralisée à des équations du type (29).

Notre approche utilise la méthode des lignes de niveau, décrite dans le paragraphe 3. Suivant les idées de Evans [42] et Giga et Sato [52], on associe une équation géométrique à (23) de sorte que le graphe de u, solution de (23), reste dans le front. Plus précisément, en considérant une fonction $v : \mathbb{R}^{N+1} \times (0, +\infty) \to \mathbb{R}$ telle que $v(x, u(x, t), t) = 0$ et en différenciant cette égalité, on obtient l'équation que doit vérifier, au moins formellement, la fonction v :

$$\frac{\partial v}{\partial t} - \mathrm{Tr}\left[b\left(-\frac{D_x v}{D_y v}\right)\left(D^2_{xx}v - 2D^2_{xy}v \otimes \frac{D_x v}{D_y v} + D^2_{yy}v\frac{D_x v}{D_y v} \otimes \frac{D_x v}{D_y v}\right)\right] = 0, \qquad (24)$$

dans $\mathbb{R}^{N+1} \times (0, +\infty)$.

Nous commençons par donner des hypothèses sur b (voir **(H1)**–**(H6)**, chapitre 2) de sorte que (24) entre dans l'approche par lignes de niveau classique de [32] et [51]. De manière simplifiée, ces hypothèses permettent de supprimer les singularités de la non-linéarité de l'équation (24) sauf éventuellement pour $Dv = 0$. Tout se passe alors comme dans le cas de (7) et l'équation apparaît bien posée dans $UC(\mathbb{R}^{N+1} \times [0, +\infty))$. Le théorème fondamental 3.1 de l'approche par lignes de niveau s'applique alors.

Nous affaiblissons ensuite ces hypothèses en ne supposant plus que **(H1)**–**(H4)**. La non-linéarité F de (24) présente alors des singularités sur l'hyperplan

$$\mathcal{D} = \{D_y v = 0\}.$$

Mais, même dans ce cas “très singulier,” l'équation (24) apparaît bien posée dans $UC(\mathbb{R}^{N+1} \times [0, +\infty))$. Ce résultat a été montré par Ishii [63]. Nous en donnons une preuve simplifiée que permet la forme particulière de notre ensemble de singularités $\mathcal{D}$. Cette preuve présente de plus l'avantage de se généraliser au cas d'équations comme (29) que nous nous proposons d'étudier par la suite (voir [18]). Pour des exemples d'équations entrant dans notre approche, nous renvoyons à la section 2.4.4.

Il s'ensuit que le théorème 3.1 de l'approche par lignes de niveau s'applique aussi dans ce cas très singulier. Le premier résultat clé que nous obtenons alors est que les graphes de toutes les solutions de (23) sont contenus dans le front $\bigcup_{t \geq 0} \Gamma_t \times \{t\}$ associé au graphe de u_0, Graph(u_0), par (24) (voir théorème 2.6.1). Un corollaire immédiat est l'unicité pour (23) dans le cas où le front est d'intérieur vide et ceci sans restriction de croissance sur u_0 (voir théorème 2.6.2 et les discussions relatives).

L'unicité se révèle donc étroitement reliée à des propriétés de “régularité” du front associé à Graph(u_0). Nous montrons que cet épaississement est “rare” en exhibant des fonctions arbitrairement proches de u_0 (au sens de la distance de Hausdorff entre les graphes – ces fonctions ont donc la même croissance à l'infini que u_0) pour lesquelles on a unicité.

À ce niveau, faisons une petite digression. Dans le but de montrer l'unicité, au moins pour la courbure moyenne (4), nous avons essayé d'utiliser des formules de représentation des solutions de (7). Ces formules ont été établies récemment indépendamment par Soner et Touzi [80], [81] et Buckdahn, Cardaliaguet et Quincampoix [29] en considérant un problème de contrôle stochastique : on se donne une équation différentielle stochastique,

$$dX_s^{z,t} = \sqrt{2}\,\alpha(s)dW_s,\ s \in [0,t],\ t \leq T, \quad \text{et} \quad X_0^{z,t} = z \in \mathbb{R}^{N+1},$$

où W est un mouvement brownien standard $(N+1)$-dimensionnel sur une base stochastique $(\Omega, \mathcal{F}, P; (\mathcal{F}_s)_{s\in[0,T]})$ et le contrôle $\alpha(\cdot)$ est un processus $\mathcal{F}_s$-mesurable prenant ses valeurs dans le compact $\mathcal{K} = \{Y \in \mathcal{S}_{N+1} : Y \geq 0,\ I - Y^2 \geq 0 \text{ et } \mathrm{Tr}[I - Y^2] = 1\}$. La fonction-valeur définie par

$$v(z,t) = \inf_{\alpha(\cdot)} \left\{ \operatorname{ess-sup}_{\Omega} v_0(X_t^{z,t}) \right\},$$

pour tout $z \in \mathbb{R}^{N+1}$ et $t > 0$, est une solution de viscosité de (7). On a donc, par ce biais, une formule “explicite” pour l'unique solution uniformément continue de (7). Mais, même à l'aide d'une telle formule et en démarrant d'une donnée initiale v_0 dont le front est un graphe, nous n'avons pas réussi à montrer que le front ne pouvait pas s'épaissir.

Revenons aux conséquences de notre approche ; dans le théorème 2.7.1, nous établissons une borne L^∞ locale pour les solutions de (23) qui utilise des arguments géométriques d'une façon naturelle, en particulier la préservation de l'inclusion des ensembles pour l'évolution décrite par l'approche par lignes de niveau (voir Figure 2.1 pour une illustration).

Ces bornes L^∞ permettent de préciser la structure du front qu'il s'épaississe ou non : pour toute donnée initiale continue u_0, il existe une fonction scs u^+ et une fonction sci u^- telles que

$$\Gamma_t = \{(x,y) \in \mathbb{R}^{N+1} : u^-(x,t) \leq y \leq u^+(x,t)\} \quad \text{pour tout } t \geq 0.$$

De plus, ces fonctions u^+ et u^- dont les graphes délimitent le front sont respectivement la solution maximale et minimale de (23) (voir théorème 2.8.1 ainsi que la figure 2.2 pour une illustration). En particulier, on a un résultat d'existence de solutions de viscosité discontinues de (23).

Dans le cas de l'équation des graphes (4), et d'autres équations pour lesquelles les travaux de Chou et Kwong [33] s'appliquent, nous montrons que les solutions extrémales sont de plus lisses (voir section 2.9).

4.2 Solutions convexes

L'application directe la plus spectaculaire de l'approche décrite dans le paragraphe précédent est une réponse positive et complète à la question 1 du paragraphe 1 dans le cas des solutions convexes de (23).

Sous les mêmes hypothèses que celles permettant l'approche géométrique, nous prouvons l'existence et l'unicité d'une solution de viscosité continue u pour toute donnée initiale $u_0 \in C(\mathbb{R}^N)$ convexe. De plus, pour tout $t \geq 0$, $u(\cdot,t)$ est convexe (voir théorème 2.10.1). Ce théorème s'applique en particulier à (4) et la classe d'équations à laquelle il s'étend apporte une réponse à la question 2.

Ce résultat pour les solutions convexes apporte une justification frappante à l'approche par lignes de niveau (et l'utilisation de solutions de viscosité qu'elle nécessite) du fait de l'élémentarité de la preuve proposée. En effet, dans le chapitre 1 nous présentons une preuve très “analytique” dans le cas de (4) mais, d'une part, nous n'avons pas échappé à un argument géométrique nécessitant l'approche par lignes de niveau (il est sans doute possible de le contourner mais pas de façon simple) et d'autre part, nos hypothèses étaient plus fortes (on supposait u_0 coercive).

4.3 Le cas de la dimension 1

Comme dans le cas convexe, nous apportons dans le chapitre 3 une réponse positive à la question 1 pour l'équation (4) en dimension 1, qui s'écrit dans ce cas

$$\frac{\partial u}{\partial t} - \frac{u_{xx}}{1+u_x^2} = 0 \quad \text{dans } \mathbb{R}\times(0,+\infty). \tag{25}$$

La preuve consiste à intégrer (25). La fonction $v(x,t) = \int_0^x u(y,t)\,dy$ (à une fonction du temps près en fait) est alors solution de la nouvelle équation

$$\frac{\partial v}{\partial t} - \arctan(v_{xx}) = 0 \quad \text{ in } \mathbb{R}\times(0,+\infty). \tag{26}$$

Cette transformation nécessite dans un premier temps de travailler avec des solutions u au moins C^1 pour que la transformation ci-dessus ait un sens.

Nous obtenons ensuite l'unicité sans conditions de croissance pour (26) (voir ci-dessous pour des précisions) et par différentiation, l'unicité d'une solution C^1 de (25) et ceci pour toute donnée $u_0 \in C(\mathbb{R})$. Enfin, des arguments provenant de l'approche géométrique permettent d'étendre le résultat d'unicité à toutes les solutions de viscosité continues.

Dans le but d'avancer dans le problème soulevé par la question 2, nous travaillons en réalité sur des équations plus générales comme

$$\frac{\partial u}{\partial t} - (f(x,t,u,u_x))_x = 0 \quad \text{dans } \mathcal{D}'(\mathbb{R}\times(0,+\infty)), \tag{27}$$

où $f \in C(\mathbb{R}\times[0,+\infty)\times\mathbb{R}\times\mathbb{R})$ et $\mathcal{D}'(\mathbb{R}\times(0,+\infty))$ est l'espace des distributions sur $\mathbb{R}\times(0,+\infty)$. Sous une certaine condition de structure sur f (en particulier, f doit être α-hölderienne d'exposant $0<\alpha<1$ par rapport à sa dernière variable uniformément par rapport aux autres), on montre que la preuve ci-dessus s'applique : les solutions de viscosité de l'équation intégrée satisfont un principe de comparaison sans conditions de croissance. On a donc unicité pour pour les solutions C^1 de (27) (voir théorème 3.4.1).

Notons pour terminer que le principe de comparaison utilisé pour les solutions des équations intégrées est un cas particulier d'un principe de comparaison sans condition de croissance que nous avons établi pour les solutions de viscosité de (1) dans $\mathbb{R}^N\times(0,T)$ (voir théorème 3.2.1). L'hypothèse principale pour obtenir ce résultat est du type

$$F(x,t,p,X) - F(x,t,q,Y) \le C(1+|x|)|p-q| + C(\mathrm{Tr}[Y-X]^+)^\alpha, \tag{28}$$

pour $0<\alpha<1$. On peut remarquer que cette condition se réduit à l'hypothèse reliée à la vitesse finie de propagation dans le théorème 1.4 lorsque (1) est complètement dégénéré. La preuve est similaire à celle de ce dernier théorème : (28) permet d'appliquer une procédure de linéarisation pour (1) et d'utiliser ensuite une technique de sur-solutions explosives.

4.4 Unicité avec restriction sur la croissance de la donnée initiale

Dans chapitre 4, on part plutôt de la question 2 du paragraphe 1 : on considère (1) de la forme

$$\frac{\partial u}{\partial t} - \mathrm{Tr}\left[b(x,t,Du)D^2u\right] + H(x,t,Du) = 0 \quad \text{dans } \mathbb{R}^N \times (0,+\infty), \tag{29}$$

où $b \in C(\mathbb{R}^N \times [0,T] \times \mathbb{R}^N; \mathcal{S}_N^+)$ vérifie des hypothèses de même nature que (4),

Il existe une fonction continue $\sigma : \mathbb{R}^N \times [0,T] \times \mathbb{R}^N \to \mathcal{M}_{N,M}$ *et une constante* $C > 0$ *telle que*

$$\begin{aligned}
&b(x,t,p) = \sigma(x,t,p)\sigma(x,t,p)^T,\\
&\|\sigma(x,t,p) - \sigma(y,t,p)\| \le C|x-y|,\\
&\|\sigma(x,t,p) - \sigma(x,t,q)\| \le C\frac{|p-q|}{1+|p|+|q|},
\end{aligned}$$

et H vérifie les hypothèses classiques du théorème 1.4.

À partir de là, on essaie d'optimiser au mieux les techniques de comparaison pour les solutions de viscosité. Ces techniques utilisent le dédoublement de variables (voir [37] et [13] pour les détails) et l'utilisation d'une fonction-test convenable. Plus précisément, on considère

$$\max_{x,y\in\mathbb{R}^N,\ t\in[0,T]} \{(u_1)^*(x,t) - (u_2)_*(y,t) - \Phi(x,y,t)\},$$

où u_1 est une sous-solution de (29), u_2 une sur-solution et Φ la fonction-test lisse. Le meilleur choix que nous avons été capable de faire pour Φ a été de la prendre à croissance polynômiale en les variables d'espace, uniformément pour $t \in [0,T]$ (voir section 4.4 pour la forme exacte de Φ) ce qui impose de travailler dans la classe des solutions de même croissance pour que le maximum soit atteint. Enfin, nous avons abouti à des conditions de croissance pour la condition initiale qui s'expriment, dans le cas où elle est localement lipschitzienne (pour simplifier),

$$|Du_0(x)| \le C\left(1 + |x|^\nu\right) \quad \text{pour presque tout } x \in \mathbb{R}^N, \tag{30}$$

avec $0 < \nu < (1+\sqrt{5})/2$. Sous les hypothèses précédentes, avec cette condition de croissance de la donnée initiale, nous obtenons alors existence et unicité d'une solution dans la classe des fonctions à croissance polynômiale. De plus, la solution a la même croissance que u_0 (Cf. théorème 4.2.1 et corollaire 4.2.1).

Nous ne savons pas si ces conditions de croissance (assez étranges il faut le dire) sont optimales ou au contraire dues à des limitations de la méthode employée ou à la trop grande généralité de nos hypothèses ; en effet, ces hypothèses s'appliquent à une classe d'équations beaucoup plus large que celle de la courbure (4) (voir les exemples dans la section 4.2).

À nouveau, nous tenons à préciser que nous n'avons pas réussi à obtenir ces résultats en travaillant avec des solutions lisses sans utiliser les techniques des solutions de viscosité.

5 Bornes inférieures de gradient

Dans ce paragraphe, nous présentons les résultats du chapitre 5. Le résultat principal en est l'obtention de bornes inférieures de gradient généralisées pour les solutions de viscosité de (8). Nous considérons des données initiales u_0 localement lipschitziennes ; le point de départ est de montrer que la solution de viscosité est elle-même localement lipschitzienne.

Pour ce faire, nous commençons par établir une borne supérieure de gradient. Ces estimations sont habituelles dès que l'on cherche à obtenir des résultats de régularité pour les solutions. Un moyen d'obtenir ce genre d'estimation fait appel à la méthode dite de Bernstein (voir [53] pour une présentation) que nous rappelons dans le cas de l'équation de la chaleur ; soit u une solution lisse de (5)–(2) avec u_0 lipschitzienne sur $\mathbb{R}^N$. Un petit calcul montre que la fonction lisse $\chi := |Du|^2$ est sous-solution de (5). L'application du principe du maximum conduit alors à la borne recherchée,

$$\|Du(\cdot,t)\|_\infty^2 = \|\chi(\cdot,t)\|_\infty \leq \|\chi(\cdot,0)\|_\infty = \|Du_0\|_\infty^2, \tag{31}$$

pour tout $t \geq 0$, où la norme considérée est la norme sup sur $\mathbb{R}^N$. Cette borne implique que u est lipschitzienne si u_0 l'est.

La première estimation (théorème V.4.1[1]) peut être vue comme une généralisation de cette méthode aux solutions non-lisses de (8) où H satisfait les hypothèses du théorème 1.4.

Nous donnons ici une idée de la preuve qui s'inspire de techniques similaires de Barles [11] ; nous renvoyons à la section V.4 chapitre 5 pour des énoncés et des preuves précis. Introduisons la sup-convolée de la solution u (voir Lasry et Lions [69] pour l'introduction de cette notion dans le cadre des solutions de viscosité), définie par

$$u^\varepsilon(x,t) = \sup_{y\in\mathbb{R}^N} \left\{ u(y,t) - \mathrm{e}^{Kt}\frac{|x-y|^2}{\varepsilon^2} \right\}, \tag{32}$$

pour $\varepsilon, K > 0$. Cette approximation de u est encore une sous-solution de (8) (à une petite erreur près, pour K assez grand). Une application d'une version plus fine du théorème 1.4 (voir théorème V.A.1) pour comparer u^ε et u conduit à

$$u^\varepsilon - u \leq \sup_{\mathbb{R}^N}\{u_0^\varepsilon - u_0\} \quad \text{dans } \mathbb{R}^N \times [0,T]. \tag{33}$$

Mais

$$u^\varepsilon(x,t) \approx u(x,t) + \frac{1}{4}\mathrm{e}^{Kt}|Du(x,t)|^2\varepsilon^2;$$

En injectant cette estimation dans (33), on obtient la borne recherchée

$$|Du(x,t)| \leq \mathrm{e}^{KT/2}\|Du_0\|_\infty \quad \text{dans } \mathbb{R}^N,$$

ce qui entraîne que u est lipschitzienne si u_0 l'est.

Bien sûr, le calcul formel ci-dessus s'écrit rigoureusement pour des fonctions non-lisses. Il peut aussi être localisé en utilisant la vitesse finie de propagation.

[1] les références dans l'article du chapitre 5 ne suivent pas la numérotation de la thèse, nous noterons donc V... pour éviter les confusions.

La question qui se pose maintenant est de renverser cette inégalité de façon à obtenir une borne inférieure. La répétition de l'approche ci-dessus avec une inf-convolée (définie de manière similaire à (32) en remplaçant "sup" par "inf" et le "$-$" par "$+$") par exemple conduit encore à une borne supérieure. Notons que la localisation des estimations inférieures est cruciale ici ; en effet, une borne inférieure du gradient minorée par $\inf_{\mathbb{R}^N}|Du_0|$ ($= 0$ en général) n'a qu'un intérêt très limité !

La localisation ne pose pas de problème (toujours grâce au théorème V.A.1). Pour réussir à renverser l'inégalité, nous supposons que H est, en plus des hypothèses précédente, convexe en le gradient. En utilisant de manière fondamentale la caractérisation des solutions de viscosité pour (8) avec Hamiltonien convexe établie par Barron et Jensen [24] (voir aussi théorème V.3.1), nous obtenons alors que l'inf-convolée u_ε de la solution u est une sous-solution de (8) à un terme d'erreur en ε près (au lieu d'être, de façon naturelle, une sur-solution comme infimum de solutions). Le même calcul formel que ci-dessus conduit alors à la borne recherchée (Cf. théorème V.4.2 pour des énoncés précis),

$$|Du| \geq \mathrm{e}^{-KT/2} \inf_{y \in B(x_0,r)} |Du_0(y)| \quad \text{au sens de viscosité dans } B(x_0, r/2) \times [0, t_1],$$

où $r > 0$ et $t_1 > 0$ dépendent des données du problème. Si, de plus, H est positivement homogène de degré 1 (cette hypothèse correspond à (20) pour (8)), on obtient la borne inférieure globale (13) présentée plus haut.

Les applications à la régularité du front sont décrites dans le paragraphe suivant.

6 Applications à la "régularité" des fronts

Nous allons maintenant présenter les résultats de nature géométrique que nous avons obtenus. On peut les classer en deux catégories : les premiers sont obtenus comme conséquences naturelles des résultats obtenus pour les équations quasilinéaires à l'aide de notre approche géométrique et les seconds comme applications de la borne inférieure de gradient.

Commençons par les résultats obtenus pour les évolutions géométriques gouvernées par (24). Nous avons déjà noté dans le paragraphe 4.1 l'équivalence entre l'unicité pour (23) et la condition d'intérieur vide du front associé. D'après les résultats d'unicité décrits précédemment, le front est donc d'intérieur vide dans les cas suivants :

1. sous les hypothèses de l'approche géométrique pour tout front associé par (24) au graphe d'une fonction u_0 convexe (théorème 2.10.4) ;
2. en dimension 1 pour les fronts évoluant par courbure moyenne à partir du graphe d'une fonction u_0 continue et pour des mouvements plus généraux si on démarre du graphe d'une fonction u_0 localement lipschitzienne (théorèmes 3.5.1 et 3.5.2) ;
3. si le front évolue par courbure moyenne à partir du graphe d'une fonction u_0 satisfaisant (30) (théorème 4.3.2).

Précisons que les preuves dans le cas convexe s'appuient de façon fondamentale sur un théorème de préservation de la convexité pour les solutions de (24) qui est une généralisation naturelle du théorème correspondant de [51] (voir théorème 2.10.3). Un corollaire de ce résultat (théorème 2.10.4) est, pour une évolution gouvernée par (24), la préservation de la convexité pour les ensembles Ω_t^+ et le non-épaississement des fronts

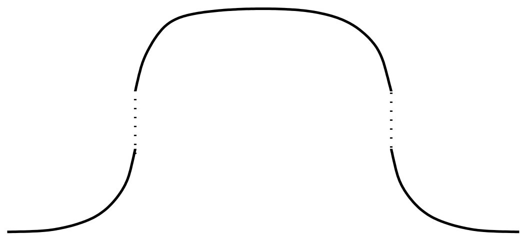

FIG. 3 – *Un front qui est un graphe discontinu mais ne peut pas être vu comme le graphe d'une fonction continue sur* $\mathbb{R}$

Γ_t jusqu'au temps d'extinction (qui peut être infini comme c'est le cas si Γ_0 est un graphe par exemple). De plus, en appliquant les résultats d'Evans et Spruck [46], le front est lisse dans le cas de la courbure moyenne. Ces résultats généralisent ce qui était connu pour le mouvement par courbure moyenne de surfaces compactes (concernant l'évolution d'hypersurfaces convexes, nous renvoyons à Evans et Spruck [45], Soner [78], Ilmanen [56], Imbert [57], ...).

Il est important de remarquer que le non-épaississement du front n'implique pas l'existence d'une solution continue à l'équation (23) comme permet de le comprendre la figure 3.

Dans [23], Barles et Souganidis introduisent une condition plus forte que celle de l'intérieur vide (voir la discussion de la section 2.6 pour des précisions) ; nous montrons que leur condition entraîne l'unicité discontinue. Comme expliqué ci-dessus, cette condition ne donne pas non plus la continuité des solutions (voir également la figure 3). D'un point de vue analytique, cela signifie que l'on n'arrive pas à obtenir un principe de comparaison pour les solutions de viscosité discontinues à partir d'arguments géométriques.

Dans le cas de l'équation de courbure (4), ou pour des équations plus générales pour lesquelles Chou et Kwong [33] ont prouvé des bornes de gradient, nous montrons que le bord du front est lisse (théorèmes 2.9.1 et 2.9.2).

Signalons enfin que, lorsque l'on a unicité d'une solution lisse pour (4), l'évolution généralisée gouvernée par (7) et décrite par l'approche par lignes de niveau correspond à l'évolution géométrique au sens de la géométrie différentielle ; c'est-à-dire que le graphe de la solution de (4) évolue avec une vitesse normale égale à la courbure moyenne.

Nous passons aux applications de la borne de gradient. La remarque de base est que, si la solution u de (8) (sous les hypothèses du théorème V.4.2) est lisse, alors, en utilisant le théorème des fonctions implicites, (13) entraîne que le front Γ_t est une sous-variété de $\mathbb{R}^N$ pour tout $t \geq 0$. Maintenant, si u est seulement localement lipschitzienne, (13) doit être pris au sens de viscosité. Cette condition, plus faible, est suffisante pour assurer le “non-épaississement” du front (Corollaire V.5.1) : pour tout $t \geq 0$, Γ_t est de mesure nulle dans $\mathbb{R}^N$.

La meilleure régularité que l'on pourrait espérer pour le front d'une solution localement lipschitzienne serait un front “localement lipschitzien” (c'est-à-dire qui est localement le graphe d'une fonction lipschitzienne – Cf. définition V.5.2). Ce résultat est faux en général. Pour améliorer nos résultats de régularité, nous considérons alors des solutions semiconcaves (pour la semiconcavité nous renvoyons à Cannarsa *et al.* [1] et [30]). Ce choix s'explique par deux raisons.

La première est que, en général, la semiconcavité est la meilleure régularité que l'on peut espérer pour les solutions des équations d'Hamilton-Jacobi (voir Lions [72], Cannarsa *et al.* [2] et [31], Rifford [77], etc.). Nous établissons dans notre cas, un théorème de préservation de la semiconcavité pour les solutions de (8), avec une hypothèse supplémentaire sur H, qui peut être vu comme une version parabolique du théorème correspondant d'Ishii [58] (Cf. théorème V.5.2 – voir section 5.2 pour la preuve).

La deuxième motivation pour s'intéresser aux fonctions concaves dans ce contexte est que de telles fonctions sont “presque C^1”; elles sont localement lipschitziennes et leurs gradients généralisés jouissent de certaines propriétés de continuité. On pourrait donc espérer généraliser le théorème des fonctions implicites. Nous montrons par un contre-exemple que ce n'est pas possible (théorème V.5.5). Néanmoins, en appliquant le théorème des fonctions implicites généralisé de Clarke [34] et la formule de la coaire (voir [43]), nous obtenons quelques résultats, plus faibles que ceux espérés : pour tout $t \geq 0$, presque tous les ensembles de niveau de u sont localement lipschitziens $\mathcal{H}^{N-1}$-presque partout, où $\mathcal{H}^{N-1}$ est la mesure de Hausdorff de dimension $N-1$ (voir corollaire V.5.2).

7 Conditions de Neumann et problème de contrôle avec réflexion

Considérons un problème de contrôle optimal de trajectoires réfléchies. On se donne un système d'équations différentielles ordinaires

$$\begin{cases} dX_s^{x,t} = -b(X_s^{x,t}, t-s, \alpha(s))ds - dk_s^{x,t} \;\text{ dans } [0,t],\; t \leq T, \\ X_0^{x,t} = x \in \overline{\Omega},\; X_s^{x,t} \in \overline{\Omega} \;\text{ pour tout } s \in [0,t], \\ k_s^{x,t} = \displaystyle\int_0^s \mathbb{1}_{\partial\Omega}(X_\tau^{x,t}) n(X_\tau^{x,t})\, d|k^{x,t}|_\tau, \end{cases} \tag{34}$$

où les inconnues sont la trajectoire $X^{x,t} \in C([0,t], \overline{\Omega})$ et le processus à variations bornées $k^{x,t} \in C([0,t], \mathbb{R}^N)$, dont la variation totale est notée $|k^{x,t}|_s$. Le contrôle $\alpha(\cdot)$ est un élément de $L^\infty([0,T], \mathcal{A})$ où $\mathcal{A}$ est un espace métrique compact et le champ de vecteurs b satisfait les hypothèses classiques des problèmes de contrôle optimal.

Lions et Sznitman [74] ont montré l'existence et l'unicité du couple solution de ce problème. L'originalité de ce système vient bien sûr du terme “de reflexion” $-dk_s^{x,t}$ qui peut être vu comme une force repoussant la trajectoire à l'intérieur de l'ouvert $\overline{\Omega}$ quand celle-ci atteint le bord. Cela se lit sur la formule explicite (voir proposition 6.3.1)

$$dk_s^{x,t} = \mathbb{1}_{\partial\Omega}(X_s^{x,t}) n(X_s^{x,t}) \langle b(X_s^{x,t}, t-s, \alpha(s)), n(X_s^{x,t})\rangle^+ \, ds,$$

sur laquelle on voit bien que ce terme compense exactement la composante normale sortante de b sur le bord et n'agit pas autrement.

Les trajectoires, dépendantes du contrôle, sont associées à un coût de la forme

$$J(x,t,\alpha(\cdot),\psi) = \int_0^t f(X_s^{x,t}, \alpha(s))\, ds + \psi(X_t^{x,t}),$$

où la fonction continue f est appelée coût instantané et ψ est le coût final. Le but est de minimiser le coût J en choisissant le contrôle optimal. On définit en conséquence la fonction-valeur u par

$$u[\psi](x,t) = \inf_{\alpha(\cdot)\in L^\infty([0,T];\mathcal{A})} J(x,t,\alpha(\cdot),\psi).$$

La connaissance de cette fonction-valeur permet de déterminer le coût minimal.

En ce qui concerne les problèmes de contrôle optimal, nous renvoyons à Lee et Markus [70], Warga [84], à Lions [72], Fleming et Soner [48], Barles [13] et Bardi et Capuzzo Dolcetta [6] pour leurs liens avec les solutions de viscosité et à Clarke *et al.* [35] pour les liens avec les solutions proximales. Dans le cas des trajectoires réfléchies, nous renvoyons à Lions et Sznitman [74] pour l'étude de (34) et à Lions [73] pour le contrôle proprement dit et ce qui va suivre.

En utilisant le principe de la programmation dynamique, sous des hypothèses classiques sur b et f, on obtient que $u[\psi]$ est une solution de viscosité de l'équation d'Hamilton-Jacobi (8) avec

$$H(x,t,p) = \sup_{\alpha\in\mathcal{K}}\{\langle b(x,\alpha),p\rangle - f(x,\alpha)\},$$

dans $\Omega\times(0,T)$ avec la donnée initiale ψ et la condition de Neumann (15) au bord (théorème 6.3.1).

Lorsque ψ est continue, la fonction-valeur est l'unique solution de viscosité du problème de Neumann. Maintenant, si ψ n'est pas continue, $u[\psi]$ est toujours solution de viscosité de (8) avec ψ comme condition initiale et la condition de Neumann au bord mais la fonction-valeur est discontinue en général. Comment comprendre l'unicité dans ce cas ? Comment caractériser la fonction-valeur ? Ce problème n'est pas nouveau ; il a été résolu par Barles et Perthame [20] dans le cas d'un problème de contrôle à temps d'arrêt et coût final discontinu posé dans $\mathbb{R}^N$. Ensuite, Barron et Jensen [24], [25] ont traité le problème correspondant à horizon fini en introduisant une nouvelle notion de solution (voir chapitre 5 à ce propos). Cette nouvelle notion de solution a été mise à profit par Barles [12], Soravia [82], Frankowska [50] et Blanc [28] pour apporter des réponses à des problème de contrôle reliés à des équations d'Hamilton-Jacobi avec des conditions de Dirichlet au bord.

Nous montrons, en suivant [20], que la fonction-valeur $u[\psi^*]$ associée au coût final ψ^* est la sous-solution maximale du problème de Neumann avec donnée initiale ψ. La fonction-valeur du problème de contrôle relaxé $\hat{u}[\psi_*]$ associée à ψ_* est elle la sur-solution maximale de la même équation (voir le théorème 6.4.1). Dans [20], ce résultat permet alors de caractériser la fonction-valeur $u[\psi]$, dans le cas où ψ est scs, comme l'unique solution sci du problème. Nous montrons par un contre-exemple que ce n'est plus le cas pour le problème avec conditions de Neumann au bord (théorème 6.4.2). Le sens naturel à donner à l'unicité ne s'étend donc pas à ce cadre.

Chapitre 1

Quelques résultats d'unicité pour l'équation du mouvement par courbure moyenne dans $\mathbb{R}^N$

Dans ce chapitre nous présentons un proceeding paru dans le Volume 8 *des* Actes du 32ème Congrès d'Analyse Numérique (juin 2000).

ESAIM: Proceedings
Actes du 32ème Congrès
d'Analyse Numérique : Canum2000
URL: http://www.emath.fr/proc/Vol.8/

Vol. 8, Septembre 2000, 1-3

Quelques résultats d'unicité pour l'équation du mouvement par courbure moyenne dans $\mathbb{R}^N$

Guy Barles[(1)], Samuel Biton[(1)] et Olivier Ley[(1),(2)]

Résumé. Récemment, Ecker et Huisken ont démontré que l'équation du mouvement par courbure moyenne des graphes dans $\mathbb{R}^N$ admet au moins une solution régulière pour toute donnée initiale dans $W^{1,\infty}_{loc}(\mathbb{R}^N)$ sans imposer aucune condition sur son comportement à l'infini. Le but de cet article est de décrire quelques résultats d'unicité pour ces solutions. La difficulté pour obtenir une telle unicité provient de l'absence de restriction sur la croissance et, plus généralement, sur le comportement des solutions à l'infini.

Mots clés. Équations quasilinéaires paraboliques, mouvement par courbure moyenne, résultat d'unicité, solutions sans comportement prescrit à l'infini, solutions de viscosité.

Abstract. *Recently, Ecker and Huisken proved that the equation for the motion by mean curvature of graphs in $\mathbb{R}^N$ has at least a classical solution for any initial data in $W^{1,\infty}_{loc}(\mathbb{R}^N)$ without imposing any condition on its behavior at infinity. The aim of this article is to describe several results concerning the uniqueness of such solutions. The main difficulty to obtain such uniqueness results comes from the lack of restriction on the growth and, more generally, the behavior of the solutions at infinity.*

Key words. *Quasilinear parabolic equations, motion by mean curvature, uniqueness results, solutions without prescribed behavior at infinity, viscosity solutions.*

AMS classifications. 35K55, 35A05, 35B50, 35K15, 53C44

1 Introduction

Dans une série de travaux récents, les géomètres Ecker et Huisken[7, 8] ont étudié les mouvements par courbure moyenne des graphes dans $\mathbb{R}^N$ et ils ont obtenu un résultat d'existence très général qui, de manière analytique, peut se formuler de la façon suivante : pour toute donnée initiale $u_0 \in W^{1,\infty}_{loc}(\mathbb{R}^N)$, il existe une solution régulière du problème :

$$\frac{\partial u}{\partial t} - \Delta u + \frac{(D^2uDu, Du)}{1+|Du|^2} = 0 \quad \text{dans } \mathbb{R}^N \times (0,+\infty) \ , \tag{1}$$

[(*)] Ce travail a été effectué avec le soutien du programme européen TMR "Viscosity Solutions and their Applications"

$$u(x,0) = u_0(x) \quad \text{dans } I\!R^N . \tag{2}$$

Ce résultat d'existence est assez surprenant car il est obtenu sans aucune hypothèse sur le comportement à l'infini de la donnée initiale et, bien entendu, il fournit des solutions qui peuvent avoir, en particulier, des croissances quelconques à l'infini.

Le but de notre travail en cours est d'étudier non seulement les propriétés d'unicité des solutions de (1)-(2) mais aussi d'équations quasilinéaires elliptiques et paraboliques plus générales posées dans $I\!R^N$ ou dans des ouverts non bornés ; nous rappelons que le cas des ouverts bornés est couvert de manière très satisfaisante par les résultats du "Users'guide" sur les solutions de viscosité de Crandall, Ishii & Lions[6].

Nous décrivons dans cet article les premiers progrès dans cette direction en nous restreignant au cas de (1)-(2) ; nous ne donnons également que des idées de preuve et nous renvoyons à [2] pour des résultats plus généraux avec des preuves complètes.

Dans l'étude des équations aux dérivées partielles non linéaires, on est plutôt habitué à ce que la difficulté pour obtenir des résultats d'unicité provienne de la prise en compte de solutions faibles. Ce n'est pas du tout le cas ici où l'obstacle majeur est l'absence de restriction sur le comportement des solutions à l'infini ; supposer que les solutions sont C^∞ en x et t ne simplifie pas vraiment le problème. En fait, cette difficulté fondamentale est agravée par la dépendance de l'équation en Du et plus précisément par l'interaction Du–D^2u qui rend délicate toute procédure de linéarisation.

Du côté des points positifs, mais encore faut-il être capable d'en tirer parti, il y a l'aspect géométrique de l'équation (1). La section 2 a pour but de décrire ces aspects géométriques dont l'intérêt est multiple : d'une part, ils conduisent à une reformulation des questions d'unicité qui a l'air plus simple mais qui conserve malheureusement bon nombre des difficultés de la formulation initiale. D'autre part, elle permet de mieux comprendre comment on peut obtenir des résultats d'existence généraux de solutions de (1)-(2) par des méthodes d'analyse.

Ces aspects géométriques utilisent "l'approche par lignes de niveaux" des mouvements d'hypersurfaces avec des vitesses normales prescrites. Cette approche qui est apparue d'abord dans le cas de mouvements à vitesse normale constante dans [1], a été utilisée de manière systématique pour la première fois par Osher & Sethian[11] pour le calcul numérique des évolutions de surfaces avec des vitesses normales quelconques. La première justification théorique est due à Evans & Spruck[9] pour le mouvement par courbure moyenne, puis Chen, Giga & Goto[5] l'ont généralisée à tous les types de vitesse.

Nous rappelons que cette approche par lignes de niveaux fournit une notion de solutions faibles pour les mouvements d'hypersurfaces de $I\!R^{N+1}$ avec des vitesses normales dépendant de la position de l'hypersurface, du temps, des directions des normales et des courbures principales.

Malheureusement, autant l'utilisation de l'interprétation géométrique du problème (1)-(2) apporte des arguments simples et efficaces pour traiter les questions d'existence, autant elle ne nous a pas permis d'obtenir des résultats d'unicité très généraux ; en fait, pas plus généraux que ceux obtenus par l'approche directe.

Nous décrivons dans les sections suivantes les quelques résultats d'unicité obtenus.

Dans la section 3, nous étudions le cas $N = 1$ qui correspond à des mouvements de graphes par courbure moyenne dans $\mathbb{R}^2$. Il est bien connu qu'au moins pour les mouvements d'hypersurfaces compactes, la dimension 2 est un cas particulier où peu de phénomènes singuliers apparaissent. Dans ce cas, nous pouvons effectivement prouver que l'unicité a lieu dans le cadre naturel, c'est-à-dire celui des solutions classiques sans comportement prescrit à l'infini.

Dans le cas $N \geq 2$, seuls des résultats partiels sont obtenus. Dans la section 4, nous considérons le cas où u_0 est convexe et coercive i.e. $u_0(x) \to +\infty$ quand $|x| \to +\infty$. On peut alors prouver l'existence d'une solution u qui est également convexe en x et coercive pour tout t. Nous montrons que cette solution est unique dans la classe la plus générale, c'est-à-dire ici celle des solutions de viscosité continues sans comportement prescrit à l'infini.

Enfin, dans la section 5 (peut-être aurions-nous dû commencer par là ?), nous nous attaquons à ce problème d'unicité à l'aide des méthodes classiques pour obtenir des résultats de comparaison pour les solutions de viscosité. En essayant de "pousser à fond" ces méthodes, nous obtenons seulement le résultat suivant : si $|Du_0(x)| \leq C(1+|x|^\nu)$ avec $\nu < 1$ et $C > 0$ alors l'unicité a lieu dans la classe des solutions à croissance polynomiale. Un résultat qui, bien entendu, est très loin d'être satisfaisant.

2 Les aspects géométriques de l'équation

Nous commençons par une remarque très simple : si u est solution de (1)-(2) alors la fonction $w : \mathbb{R}^{N+1} \to \mathbb{R}$ définie par :

$$w(x, y, t) := u(x, t) - y \ ,$$

est solution de

$$\frac{\partial w}{\partial t} - \Delta w + \frac{(D^2 w Dw, Dw)}{|Dw|^2} = 0 \quad \text{dans } \mathbb{R}^{N+1} \times (0, +\infty) \ , \tag{3}$$

$$w(x, 0) = u_0(x) - y \quad \text{dans } \mathbb{R}^{N+1} \ . \tag{4}$$

On reconnait l'équation qui intervient dans l'approche dite "par lignes de niveaux" du mouvement par courbure moyenne des articles de Chen, Giga & Goto[5] et Evans & Spruck[9]. Cette approche est basée sur les résultats suivants dans lesquels, si A est une partie d'un espace $\mathbb{R}^m$, "$UC(A)$" désignera l'espace des fonctions uniformément continues sur A ; on notera aussi X l'espace des fonctions continues qui sont dans $UC(\mathbb{R}^{N+1} \times [0, T))$ pour tout $T > 0$.

Résultat 1 : *Pour toute donnée initiale $w_0 \in UC(\mathbb{R}^{N+1})$, l'équation (3) admet une unique solution w dans X. De plus, l'équation (3) satisfait le principe du maximum pour des solutions de viscosité qui sont dans X.*

Résultat 2 : *Si $w_1, w_2 \in X$ sont deux solutions de l'équation qui vérifient :*

$$\begin{aligned}\{w_1(\cdot,0)>0\} &= \{w_2(\cdot,0)>0\}=\Omega_0,\\ \{w_1(\cdot,0)<0\} &= \{w_2(\cdot,0)<0\},\\ \{w_1(\cdot,0)=0\} &= \{w_2(\cdot,0)=0\}=\Gamma_0.\end{aligned}$$

et :

$$\liminf_{x\to+\infty}|w_1(x,0)|\ ,\ \liminf_{x\to+\infty}|w_2(x,0)|>0\ ,$$

alors, pour tout $t>0$:

$$\begin{aligned}\{w_1(\cdot,t)>0\} &= \{w_2(\cdot,t)>0\}=\Omega_t,\\ \{w_1(\cdot,t)<0\} &= \{w_2(\cdot,t)<0\},\\ \{w_1(\cdot,t)=0\} &= \{w_2(\cdot,t)=0\}=\Gamma_t.\end{aligned}$$

Les conséquences de ces résultats sont multiples. D'abord, si on se donne une hypersurface Γ_0 qui est le bord d'un ouvert Ω_0, on peut la "représenter" par une fonction uniformément continue w_0 qui sera strictement positive dans Ω_0, nulle sur Γ_0 et strictement négative ailleurs; on peut prendre par exemple la distance "signée" à Γ_0 qui vaut $+d(x,\Gamma_0)$ dans Ω_0 et $-d(x,\Gamma_0)$ sur le complémentaire. Ce choix de signes revient à orienter Γ_0, c'est-à-dire à définir "l'intérieur" et "l'extérieur" de Γ_0.

Le résultat 2 montre que l'évolution $\Gamma_0\to\Gamma_t$ ne dépend pas du choix de cette représentation, pourvu que celle-ci soit faite à l'aide d'une fonction *uniformément continue* ; elle est donc géométrique puisqu'elle ne dépend donc que de Γ_0 et de son orientation. En fait, on se rend compte dans les applications que c'est vraiment Ω_0 l'élément important (une phase dans un problème de transition de phase, par exemple).

Pour des temps petits et pour des hypersurfaces régulières, l'évolution $\Gamma_0\to\Gamma_t$ coïncide avec la propagation de Γ_0 avec une vitesse égale à la courbure moyenne suivant la définition géométrique. Et comme la solution w existe pour tous temps, cette approche par "lignes de niveaux" fournit une notion de solution faible pour le mouvement par courbure moyenne après le développement des singularités (qui est un phénomène connu pour $N\geq 2$).

Il est à noter que, même si ce mouvement peut-être considéré comme "stable" et "unique" puisqu'il hérite des propriétés de stabilité et d'unicité des solutions de viscosité de (3), "l'hypersurface" Γ_t peut être très irrégulière et même d'intérieur non vide.

La partie géométrique de ces résultats provient essentiellement du fait que l'équation est invariante par changement de variable $w\mapsto\psi(w)$, $\psi'>0$. Par exemple, dans notre cas, la fonction $\tanh(u(x,t)-y)$ est une solution de (3). Cette solution n'est pas forcément uniformément continue mais elle est au moins continue bornée.

On peut avoir l'impression (fausse) d'avoir résolu complètement le problème via la :

Proposition 2.1 : *Si le problème (3)-(4) admet une unique solution de viscosité dans $C_b(\mathbb{R}^{N+1}\times(0,+\infty))$ pour toute donnée initiale de la forme* $\tanh(u_0(x)-y)$, *alors le* *problème (1)-(2) admet une unique solution de viscosité dans $C(\mathbb{R}^N \times (0, +\infty))$ pour toute donnée initiale $u_0 \in C(\mathbb{R}^N)$.*

Mais, dans le résultat 1 ci-dessus, "l'équation (3) satisfait le principe du maximum pour des solutions de viscosité uniformément continues" signifie que l'on sait comparer des sous et sursolutions uniformément continues, oú même des sous et sursolutions dont l'une est bornée uniformément continue (BUC) et l'autre seulement continue bornée (C_b). Par contre, on n'a pas de résultat de comparaison pour des sous et sursolutions qui sont toutes les deux continues bornées et contrairement à ce que l'on peut penser, les résultats précédents, en particulier la comparaison "BUC-C_b", sont insuffisants.

Par contre, ces résultats nous permettent d'obtenir aisément une borne L^∞ sur la solution u. Notons :

$$M(x, R) = \max\{u_0(y);\ y \in B(x, R)\} \quad , \quad m(x, R) = \min\{u_0(y);\ y \in B(x, R)\}.$$

On a la :

Proposition 2.2 : *Soit u une solution de (1)-(2). Pour tout $x \in \mathbb{R}^N$ et $t > 0$, on a :*

$$m(x, \sqrt{2Nt}) - \sqrt{2Nt} \leq u(x, t) \leq M(x, \sqrt{2Nt}) + \sqrt{2Nt}\ . \tag{5}$$

Idée de la preuve : Elle repose sur un argument de comparaison que nous allons présenter de manière géométrique car il apparaitra ainsi plus simple.

En fait, le principe du maximum pour (3) peut se reformuler de la manière suivante : si les familles d'ouverts $(\Omega_t)_t$ et $(\tilde{\Omega}_t)_t$ se déplacent par courbure moyenne au sens du résultat 2 ci-dessus alors :

$$\Omega_0 \subset \tilde{\Omega}_0 \Rightarrow \Omega_s \subset \tilde{\Omega}_s, \quad \text{pour tout } s > 0\ .$$

Cette propriété de "monotonie" (préservation de l'inclusion) est fondamentale et elle peut même être utilisée comme définition non seulement dans le mouvement par courbure moyenne, mais aussi pour des mouvements plus généraux (cf. Barles & Souganidis[4]).

On se place dans $\mathbb{R}^{N+1}$ et on pose :

$$\Omega_0 := B((x, z(x,t)), \sqrt{2Nt}) \quad , \quad \tilde{\Omega}_0 = \{y > u_0(x)\}\ ,$$

où $z(x,t) = M(x, \sqrt{2Nt}) + \sqrt{2Nt}$. Dans ce cas les mouvements sont donnés par :

$$\Omega_s := B((x, z(x,t)), \sqrt{2N(t-s)}) \quad , \quad \tilde{\Omega}_s = \{y > u(x,s)\}\ ,$$

et comme $\Omega_0 \subset \tilde{\Omega}_0$, l'inclusion :

$$B((x, z(x,t)), \sqrt{2N(t-s)}) \subset \{y > u(x,s)\}\ ,$$

pour tout $s < t$ donne l'estimation (2.2) en faisant tendre s vers t.

L'étape suivante dans le processus de démonstration de l'existence serait l'estimation de gradient que nous ne détaillons pas ici. Signalons simplement que la forme de l'équation et, en particulier, la dépendance en Du permet de faire fonctionner la méthode de Bernstein d'une manière un peu spéciale. On montre que, pour une certaine fonction *bornée* χ, $\chi(|Du|^2)$ est sous-solution d'une équation linéaire ce qui permet d'obtenir l'estimation de gradient sans se préoccuper du comportement de Du à l'infini.

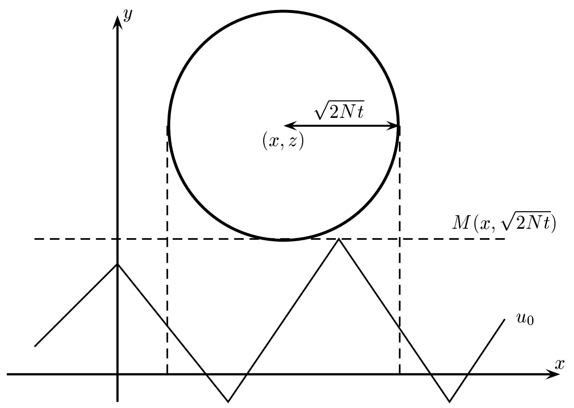

Figure 1: Au temps $t = 0$.

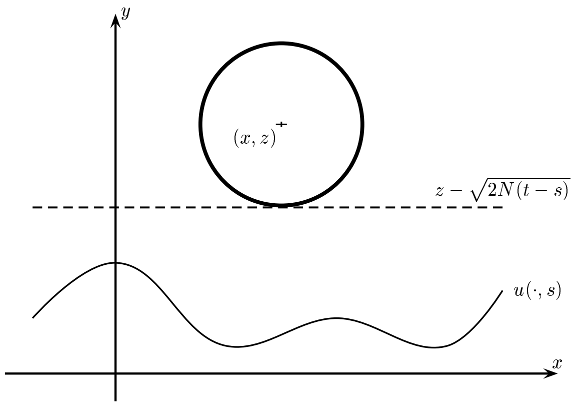

Figure 2: Au temps $0 < s < t$.

3 Le cas de la dimension 1

En dimension 1, l'équation devient :

$$(6) \qquad \frac{\partial u}{\partial t} - \frac{u_{xx}}{1+|u_x|^2} = 0 \quad \text{dans } I\!R \times (0,+\infty)\ ,$$

$$(7) \qquad u(x,0) = u_0(x) \quad \text{dans } I\!R\ .$$

On a le :

Théorème 3.1 : *Le problème (6)-(7) admet une unique solution classique pour tout $u_0 \in W^{1,\infty}_{loc}(I\!R)$.*

Preuve : L'existence provient des résultats de Ecker & Huisken[7, 8].

Pour l'unicité, on pose :

$$v(x,t) := \int_0^x u(y,t)dy\ .$$

En intégrant l'équation, il vient :

$$v_t - \mathrm{Arctg}(v_{xx}) + g(t) = 0 \quad \text{dans } I\!R \times (0,+\infty)\ ,$$

où $g(t) = \mathrm{Arctg}(v_{xx}(0,t))$. On introduit alors :

$$w(x,t) = v(x,t) + \int_0^t g(s)ds\ ,$$

qui résout le problème :

$$\frac{\partial w}{\partial t} - \mathrm{Arctg}(w_{xx}) = 0 \quad \text{dans } I\!R \times (0,+\infty)\ , \tag{8}$$

$$w(x,0) = \int_0^x u_0(y)dy \quad \text{dans } I\!R\ . \tag{9}$$

On utilise alors le :

Lemme 3.2 : *Le problème (8)-(9) admet une unique solution de viscosité pour tout $u_0 \in C(I\!R)$.*

Encore une fois c'est l'unicité qui est la partie intéressante du lemme et non l'existence qui résulte ici d'arguments très simples dès que l'on a un résultat de comparaison, ce qui sera le cas (on le verra dans la preuve du lemme).

Admettons d'abord le lemme et déduisons-en le théorème. Si u_1 et u_2 sont deux solutions classiques de (6)-(7), on construit v_1, g_1, w_1 et v_2, g_2, w_2 en procédant comme ci-dessus. Les fonctions w_1 et w_2 étant toutes les deux solutions de (8)-(9), on déduit du lemme que, pour tous x et t :

$$\int_0^x u_1(y,t)dy + \int_0^t g_1(s)ds = \int_0^x u_2(y,t)dy + \int_0^t g_2(s)ds\ ,$$

et le résultat du théorème s'obtient en dérivant cette égalité par rapport à x.

Passons maintenant à la **preuve du lemme.** Elle utilise une technique de solutions explosives ("friendly giants"). Si les fonctions w_1 et w_2 sont deux solutions de (8)-(9), on pose $\chi = w_1 - w_2$. La fonction arc-tangente étant, en particulier, höldérienne d'exposant α pour tout $\alpha \in (1/2,1)$, χ satisfait :

$$\frac{\partial \chi}{\partial t} - K_\alpha \left(\chi_{xx}^+\right)^\alpha = 0 \quad \text{dans } I\!R \times (0,+\infty)\ , \tag{10}$$

$$\chi(x,0) = 0 \quad \text{dans } I\!R\ , \tag{11}$$

pour une certaine constante positive K_α.

On introduit alors les fonctions :

$$\varphi_R(x,t) = \tilde{K}\left[(1+R^2)^{1/2}(1-ct) - (1+x^2)^{1/2}\right]^{-k}\ ,$$

où $R > 0$ et $k = (2\alpha - 1)(1 - \alpha)^{-1}$. Pour un choix convenable des constantes $\tilde{K}$ et c, les φ_R sont des sursolutions de l'équation qui explosent sur le bord du domaine borné :

$$Q_R := \{(x,t) : (1+x^2)^{1/2} < (1+R^2)^{1/2}(1-ct)\} \ .$$

La comparaison de χ et des φ_R ne pose donc aucun problème et on a :

$$\chi(x,t) \leq \varphi_R(x,t) \quad \text{dans } Q_R \ .$$

On fixe alors x et $t < c^{-1}$. Pour R assez grand, $(x,t) \in Q_R$ et on fait tendre R vers l'infini dans l'inégalité précédente : comme $\varphi_R(x,t) \to 0$ quand $R \to \infty$, on obtient $\chi(x,t) \leq 0$.

D'où $w_1(x,t) \leq w_2(x,t)$ pour tout $x \in I\!R$ et $t < c^{-1}$, puis l'égalité puisque w_1 et w_2 jouent des rôles symétriques. On répète alors l'argument en partant cette fois de $t = c^{-1}$ au lieu de $t = 0$, ce qui conduit à $w_1(x,t) = w_2(x,t)$ pour tout $x \in I\!R$ et $t < 2c^{-1}$...etc

Il est à noter enfin que ce même argument avec des modifications minimes montre que le principe du maximum a lieu pour l'équation (8) pour des sous et sursolutions de viscosité continues.

4 Sur les solutions convexes

On revient au cas de la dimension quelconque mais on suppose ici que la donnée initiale u_0 est convexe et coercive i.e. $u_0(x) \to +\infty$ quand $|x| \to +\infty$.

On a alors le :

Théorème 4.1 : *Le problème (1)-(2) admet une unique solution qui est convexe en x pour tout $t > 0$ et coercive uniformément par rapport à t. De plus, on a un résultat de comparaison pour (1)-(2) entre sur et sous-solutions de viscosité continues, sans aucune restriction sur leurs comportements à l'infini.*

Preuve : Nous commençons par démontrer **l'existence**. L'existence d'une solution classique provient des résultats de Ecker & Huisken[7, 8]. Nous allons donner une démonstration simple mais qui ne fournit qu'une solution de viscosité (et non une solution classique *a priori*).

Comme u_0 est convexe coercive, il existe une suite croissante $(u_0^k)_k$ de fonctions convexes coercives lipschitziennes telles que :

$$\sup_k u_0^k = u_0 \ .$$

On peut de plus supposer que, pour tout $k \in I\!N$ et tout $x \in I\!R^N$, on a :

$$u_0^k \geq \delta|x| - C \ ,$$

pour certaines constantes $\delta, C > 0$.

Pour tout k, comme la donnée initiale est lipschitzienne, par les résultats du "Users'guide" [6], il existe une unique solution u^k de (1)-(2) dans X avec la donnée initiale u_0^k. De plus, par un résultat de comparaison pour les solutions qui appartiennent à X, la suite $(u^k)_k$ est croissante. On applique enfin les résultats de Giga, Goto, Ishii & Sato[10] pour montrer que les u^k sont convexes en x pour tout t.

Pour obtenir des estimations sur les u^k, on utilise la Proposition 2.2 qui fournit une estimation L^∞ uniforme sur les u^k car on peut supposer que, pour tout k :

$$\min_{I\!R^N} u_0 \leq u_0^k \leq u_0 \quad \text{dans } I\!R^N \ ,$$

puis la convexité donne l'estimation uniforme de gradient :

$$||Du^k(\cdot,t)||_{L^\infty(B_R)} \leq R^{-1}\left(\max_{B_{2R}} u^k(\cdot,t) - \min_{B_{2R}} u^k(\cdot,t)\right)$$

pour tout $t \geq 0$.

Enfin, et c'est la seule estimation que nous allons utiliser pour la variable t, la convexité des u^k, conjuguée au fait qu'elles sont solutions de (1), implique qu'elles sont croissantes en temps.

Le passage à la limite utilise la méthode des semi-limites relaxées (cf. [6]) : les fonctions $\overline{u}$ et $\underline{u}$ définies par :

$$\overline{u}(x,t) = \limsup_{\substack{(y,s)\to(x,t)\\ k\to+\infty}} u^k(y,s) \quad \text{et} \quad \underline{u}(x,t) = \liminf_{\substack{(y,s)\to(x,t)\\ k\to+\infty}} u^k(y,s)\ ,$$

sont respectivement sous et sursolution de viscosité de (1)-(2). De plus, elles sont localement lipschitziennes en x pour tout t et on montre facilement que :

$$\underline{u} = \sup_k u^k \quad \text{et} \quad \overline{u} = (\underline{u})^*\ .$$

Pour conclure, il suffit de remarquer qu'étant continue en x, croissante en temps et sursolution d'une équation parabolique, $\underline{u}$ ne peut admettre de discontinuité en temps et que $u := \overline{u} = \underline{u}$ est donc une solution convexe continue. La coercivité de cette solution apparaitra dans le cours de la preuve d'unicité et résulte de l'approche géométrique.

Pour **l'unicité**, on va prouver que l'on peut comparer cette solution u avec toute sursolution continue w et toute sous-solution continue v. Curieusement les deux inégalités cherchées vont s'obtenir de manières assez différentes.

Pour la comparaison avec v, on fait le changement de variable de Kruzkov :

$$\tilde{u} = -\exp(-u) \quad , \quad \tilde{v} = -\exp(-v)\ .$$

Ces nouvelles fonctions $\tilde{u}$ et $\tilde{v}$ satisfont deux propriétés-clé : d'abord la coercivité de u implique que $\tilde{u}(x) \to 0$ quand $|x| \to +\infty$ uniformément par rapport à t ; quant à $\tilde{v}$, par sa définition même, on a $\limsup_{|x|\to+\infty} \tilde{v}(x) \leq 0$.

Ensuite $\tilde{u}$ et $\tilde{v}$ sont solutions d'une équation quasilinéaire parabolique de la forme :

$$\frac{\partial w}{\partial t} + F(w, Dw, D^2w) = 0 \quad \text{dans } I\!R^N \times (0, +\infty) \ . \tag{12}$$

De plus, la convexité de u en x va impliquer que la fonction F satisfait une propriété qui s'écrit formellement :

$$\frac{\partial F}{\partial w}(\tilde{u}, D\tilde{u}, D^2\tilde{u}) \geq -2 \ ,$$

et cette propriété permet d'adapter l'argument classique de comparaison pour les solutions de viscosité, compte-tenu du fait que l'infini ne joue plus aucun rôle. Il est néanmoins à noter que le fait que cette inégalité cruciale soit satisfaite par la fonction qui joue le rôle de la sursolution est tout à fait nécessaire dans cet argument, et ceci rend dissymétriques les rôles de u et v.

La comparaison avec w, une sursolution continue de (1)-(2), ne peut donc s'effectuer avec la même approche. On va d'abord montrer que w est coercive uniformément par rapport à t. Ceci se fait à l'aide de l'approche par ligne de niveaux : comme la donnée initiale u_0 est convexe et coercive, pour tout $p \in I\!R^N$, il existe une constante $C(p)$ telle que :

$$u_0(x) \geq p \cdot x - C(p) \quad \text{pour tout } x \in I\!R^N \ .$$

Grâce à un argument de comparaison pour l'approche par ligne de niveaux, les hyperplans restant fixes par le mouvement par courbure moyenne, on en déduit que pour tout $p \in I\!R^N$:

$$w(x,t) \geq p \cdot x - C(p) \quad \text{pour tous } (x,t) \in I\!R^N \times (0, +\infty) \ .$$

En fait, cet argument prouve, en le raffinant un peu que, outre la coercivité de w, on a :

$$w(x,t) \geq u_0(x) \quad \text{pour tous } (x,t) \in I\!R^N \times (0, +\infty) \ .$$

L'idée alors consiste à comparer w et u^k pour tout k. Une très légère adaptation des arguments de comparaison du "Users'guide" conduit à l'inégalité :

$$u^k \leq w \quad \text{dans } I\!R^N \times (0, +\infty) \ ,$$

et la conclusion s'en suit en faisant tendre k vers $+\infty$.

5 Et les méthodes classiques de solutions de viscosité?

Peut-être aurait-il été naturel de commencer par se demander : que se passe-t-il si on attaque le problème d'unicité avec des méthodes classiques de solutions de viscosité ? Typiquement cela consiste à s'intéresser à des quantités du type :

$$M^{\varepsilon,\alpha} = \max_{I\!R^N \times I\!R^N \times [0,T]} \left\{ u(x,t) - v(y,t) - \frac{|x-y|^2}{\varepsilon^2} - \alpha(|x|^2 + |y|^2) \right\} \ ,$$

où u et v sont respectivement des sous et sursolution de viscosité de l'équation considérée et ε, α sont des paramètres petits destinés à tendre vers 0. Quand tout se passe bien, on arrive à montrer d'une part que $M^{\varepsilon,\alpha}$ tend vers 0 quand ε, α tendent vers 0, ce qui donne $u \leq v$ dans $\mathbb{R}^N \times [0,T]$.

Ici on peut tenter une preuve de ce type, les termes de "pénalisation" en ε et α devant être remplacés par des substituts convenables pour tenir compte des croissances de u et v.

Le meilleur (!) résultat que nous avons obtenu avec cette stratégie est le :

Théorème 5.1 : *On suppose que u_0 est une fonction localement lipschitzienne qui satisfait :*

$$|Du_0(x)| \leq C\,(1+|x|^\nu) \quad \text{p.p. dans } \mathbb{R}^N\,,$$

pour certaines constantes $C > 0$ et $0 \leq \nu < 1$. On a alors un résultat de comparaison pour les sous et sursolutions de viscosité de (1)-(2) qui sont à croissances polynomiales.

Ce résultat est très loin d'être satisfaisant car il ne concerne que des solution à croissance strictement sous-quadratique. Il est obtenu en utilisant une "fonction-test" du type :

$$\mathcal{T}(x,y,t) = u(x,t) - v(y,t) - \mathrm{e}^{Lt} K(x+y)\left(\frac{|x-y|^p}{\varepsilon^p} + \alpha\right)\,,$$

où $L > 0$ est une constante assez grande, de même que p et $K(x+y) = 1 + |x+y|^k$ pour un certain $k \in \mathbb{N}$.

Par cette méthode, nous n'avons pas pu faire mieux ; le problème reste ouvert de savoir si la faiblesse de ce résultat est due simplement à une limitation de la méthode ou si, au contraire, on aurait des contre-exemples à l'unicité en croissance quadratique.

En fait, cette méthode de démonstration permet de traiter le cas d'équations beaucoup plus générales et elle n'utilise en aucune manière le caractère géométrique de l'équation, deux faiblesses sans doute rédhibitoires au regard de notre problème d'unicité.

Références

[1] G. Barles. Remark on a flame propagation model. *Rapport INRIA,* **464** *(1985).*

[2] G. Barles, S. Biton et O. Ley. En préparation.

[3] G. Barles, H. M. Soner et P. E. Souganidis. Front propagation and phase field theory, *SIAM J. Cont. Opt.* **31** (1993), 439–469.

[4] G. Barles et P. E. Souganidis. A new approach to front propagation problems : theory and applications. *Arch. Rat. Mech. Anal.* 141 (1998) 237-296.

[5] Y.-G. Chen, Y. Giga et S. Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, *J. Diff. Geom.* **33** (1991), 749–786.

[6] M. G. Crandall, H. Ishii et P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations, *Bull. AMS* **27** (1992), 1–67.

[7] K. Ecker et G. Huisken. Mean curvature evolution of entire graphs. *Annals of Math.* **130** (1989), 453-471.

[8] K. Ecker et G. Huisken. Interior estimates for hypersurfaces moving by mean curvature. *Invent. Math.* **105** (1991), 547-569.

[9] L. C. Evans et J. Spruck. Motion of level sets by mean curvature I. *J. Diff. Geom.* **33** (1991), 635–681.

[10] Y. Giga, S. Goto, H. Ishii et M.-H. Sato. Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains, *Indiana Univ. Math. J.* **40** (1991), no 2, 443–470.

[11] S. Osher et Sethian. Fronts moving with curvature dependent speed: algorithms based on Hamilton-Jacobi equations, *J. Comp. Phys.* **79** (1988), 12–49.

[12] P. E. Souganidis. Front propagation: Theory and Applications, CIME Course on Viscosity Solutions, *Lect. Notes in Math.* **1660**, Springer-Verlag, (1997).

(1) Laboratoire de Mathématiques et Physique Théorique. Faculté des Sciences et Techniques. Université de Tours. Parc de Grandmont. 37200 Tours, FRANCE.
(2) Dipartimento di Matematica Pura ed Applicata. Università Degli Studi Di Padova. Via Belzoni, 7. I-35131 Padova, ITALIE.

Chapitre 2

Quasilinear Parabolic Equations, Unbounded Solutions and Geometrical equations I. A geometrical approach to the study of quasilinear parabolic equations in $\mathbb{R}^N$

Guy Barles, Samuel Biton and Olivier Ley
Laboratoire de Mathématiques et Physique Théorique
Université de Tours, Parc de Grandmont, 37200 Tours, France

Abstract

In this article, we are interested in the existence and uniqueness of solutions for quasilinear parabolic equations set in the whole space $\mathbb{R}^N$. We consider in particular cases when there is no restriction on the growth or the behaviour of these solutions at infinity. Our model equation is the mean curvature equation for graphs for which Ecker and Huisken have shown the existence of smooth solutions for *any* locally Lipschitz continuous initial data. We use a geometrical approach which consists in seeing the evolution of the graph of a solution as a geometric motion which is then studied by the so-called "level-set approach." After determining the right class of quasilinear parabolic pdes which can be taken into account by this approach, we show how the uniqueness for the original pde is related to "fattening phenomena" in the level-set approach. Existence of solutions is proved using a local L^∞-bound obtained by using in an essential way the level-set approach. Finally we apply these results to convex initial datas and prove existence and comparison results in full generality, i.e. without restriction on their growth at infinity.

Key-words : quasilinear parabolic equations, geometrical equations, unbounded solutions, viscosity solutions, existence and comparison results, level-set approach, mean curvature equation, convex solutions.
AMS subject classifications : 35A05, 35B05, 35D05, 35D10, 35K15, 35K55, 53C44

[0]To appear in *Archive for Rational Mechanics and Analysis* under the title "*A Geometrical Approach to the Study of Unbounded Solutions of Quasilinear Parabolic Equations*".

Sommaire

2.1 Introduction

In a series of works (see [15] for an introductive paper, [17], [14] and [18]), we are interested in quasilinear parabolic equations set in the whole space $\mathbb{R}^N$ and, more precisely, in existence and uniqueness properties for solutions with general growth at infinity.

The starting point is this work and our main motivation comes from a result of Ecker and Huisken [41] for the so-called mean curvature equation for graphs

$$\frac{\partial u}{\partial t} - \Delta u + \frac{\langle D^2uDu, Du\rangle}{1+|Du|^2} = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty), \tag{2.1}$$

with the initial data

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N, \tag{2.2}$$

where $u : \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$ is the solution, Du and D^2u denote respectively the gradient and the Hessian matrix of u with respect to the space variable, $u_0 : \mathbb{R}^N \to \mathbb{R}$ is a given function and $|\cdot|$ (respectively $\langle\cdot,\cdot\rangle$) stands for the classical Euclidean norm (respectively inner product) in $\mathbb{R}^N$.

Ecker and Huisken proved the following very surprising result : for any initial data $u_0 \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^N)$, there exists a solution u of (2.1)-(2.2) in $C^\infty(\mathbb{R}^N \times (0,\infty)) \cap C(\mathbb{R}^N \times [0,\infty))$. This result was even extended to initial data in $C(\mathbb{R}^N)$ by Angenent [4]. The intriguing point is that no assumption is made on the growth of u_0 at infinity and therefore the solution u can have also an arbitrary behavior at infinity.

This result rises up a lot of challenging questions : the first one concerns the uniqueness of the solution they build. In general, the difficulty for obtaining a uniqueness result for a pde comes from the fact that one uses a notion of weak solution : this is not at all the case here since the solutions are known to be regular, even C^∞. The difficulty is really to take into account any behavior for the solution at infinity.

A second question is related to the existence result itself : Ecker and Huisken proved it by using differential geometry and the maximum principle and it would be interesting to have a purely analytical proof of it. Again the lack of prescribed behavior of the solutions at infinity creates an unusual difficulty. In particular, to get a local L^∞ bound on u is a priori a key point but to obtain local L^∞ bound on Du is also a rather difficult task.

Finally, one can wonder to which type of quasilinear parabolic equations the result of Ecker and Huisken can be extended. For the reader, this may seem to be a question to be investigated later but, in fact, in order to provide interesting results for (2.1), one has to understand the main underlying structure of the equation which allow such a strange result to hold.

Our answer to this question is the geometrical interpretation of (2.1) by motion by mean curvature for graphs. Motions of hypersurfaces with general curvature dependent velocities were studied recently by the so-called "level-set approach," a weak notion for the evolution which allows to define these motions past the development of singularities. The level-set approach was first introduced by Osher and Sethian [76] for numerical computations and then studied from a theoretical point of view by Evans and Spruck [45] in the case of motion by mean curvature and by Chen, Giga and Goto [32] for more general normal velocities. Later, more singular cases were investigated by Ishii [63], Ishii and Souganidis [64] and properties of the level-set approach were obtained by Barles, Soner and Souganidis [22].

In the case of equation (2.1), as for any suitable quasilinear parabolic equations, the level-set approach arises when we consider the motion in dimension $N+1$. To do so, one has to introduce the function $v : \mathbb{R}^{N+1} \times [0,+\infty) \to \mathbb{R}$ defined by

$$v(x,y,t) = y - u(x,t).$$

For (2.1), the function v is a solution of

$$\frac{\partial v}{\partial t} - \Delta v + \frac{\langle D^2 v Dv, Dv\rangle}{|Dv|^2} = 0 \quad \text{in } \mathbb{R}^{N+1} \times (0,\infty), \tag{2.3}$$

which is the equation in the level-set approach corresponding to motion by mean curvature.

In order to give a suitable sense of solution for this singular equation and related ones in non-divergence form, we use the notion of viscosity solutions : we refer the reader to the User's guide of Crandall, Ishii and Lions [37] or the books of Fleming and Soner [48], Bardi and Capuzzo Dolcetta [6], Barles [13] or Bardi et al. [7] for an introduction and/or a detailed presentation of this notion of solutions.

The most classical result concerning (2.3) is the well-posedness in the space of bounded uniformly continuous functions (BUC in short), more precisely : for any $v_0 \in BUC(\mathbb{R}^{N+1})$, there exists a unique solution v of (2.3) in $BUC(\mathbb{R}^{N+1} \times [0,T])$ for all $T > 0$ such that

$$w(x,y,0) = v_0(x,y) \quad \text{in } \mathbb{R}^{N+1}.$$

At this point, it is worth remarking that boundedness is not an issue : indeed, one of the key property of (2.3) is to be invariant by every nondecreasing change of functions : if v is a solution of (2.3), then $\tanh(v)$, or more generally $\Psi(v)$ with $\Psi' > 0$, is a solution as well.

Therefore it can be thought that the study of (2.1)–(2.2) just reduces to the study of (2.3) through the changes $v(x,y,t) = \tanh(y - u(x,t))$ and $v_0(x,y) = \tanh(y - u_0(x))$ and that all results follow easily from an extension of the above mentioned well-posedness result to spaces of bounded continuous functions (denoted by C_{b}), v and v_0 being clearly in C_{b} but not in BUC in general. In particular, the uniqueness of a solution u of (2.1)–(2.2) is an immediate consequence of a uniqueness result for solutions of (2.3) in C_{b}.

Unfortunately, we are unable to prove that the problem is well-posed in C_{b} and even the extensions to the well-posedness in BUC are rather weak. The concrete consequences of this geometrical approach are, on the one hand, a local L^∞ bound for a large class of quasilinear parabolic pdes whose proof is rather simple and natural and, on the other hand, a "generic" uniqueness result for the solutions of (2.1)–(2.2) as well as for more general equations.

It is worth pointing out that the possible non-uniqueness feature for (2.1)–(2.2) is related to the so-called "fattening phenomena" or "nonempty interior difficulty" for (2.3) ; despite of the fact that it seems obvious that no interior can develop because, by the maximum principle, one has formally

$$\frac{\partial v}{\partial y}(x,y,0) > 0 \ \text{ in } \mathbb{R}^{N+1} \quad \Longrightarrow \quad \frac{\partial v}{\partial y}(x,y,t) > 0 \ \text{ in } \mathbb{R}^{N+1} \times (0,+\infty),$$

but we are unable to prove this property even in a weaker sense.

Now, we turn to a more precise description of the contents of the present paper. It is devoted to the study of the geometrical approach, explained above in the special case of the mean curvature equation, for more general pdes like

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mathrm{Tr}\left[b(Du)D^2u\right] = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N, \end{cases} \tag{2.4}$$

where b is a continuous function from $\mathbb{R}^N$ into the space of the nonnegative symmetric matrices $\mathcal{S}_N^+$ and the initial data u_0 is any continuous function in $\mathbb{R}^N$. The first question we address is : when is (2.4) associated to a geometric pde in dimension $N+1$ to which the level-set approach applies?

In Section 2.2, we derive formally a geometrical equation from (2.4) (see (2.6)). We then study this equation distinguishing two cases : the "classical" one is the one to which the classical level-set approach applies and the "very singular" one for which the more sophisticated arguments of Ishii [63] are necessary. For the reader's convenience, we show how the classical results apply in the "classical" case (Section 2.3). In the "very singular case" (Section 2.4), our comparison result enters the framework of [63]. Nevertheless, the particular singular set we deal with allows a more elementary proof. It has, in particular, the advantage to use explicit test-functions which permits to extend the proof to the case when the function b in (2.4) depends on (x,t,Du). Here, to avoid to much technicalities, we restrict ourselves to b depending only on Du and address this more general case in the forthcoming paper [18].

Then we study the consequences of this geometrical approach for (2.4). At first, we prove in Section 2.5 that the level-set approach works. A local L^∞-bound for the solutions of (2.4) follows rather easily (see Section 2.7) and the existence of discontinuous viscosity solutions of (2.4) is an almost immediate consequence of it (see Theorem 2.8.1 in Section 2.8). The existence of smooth solutions requires a local gradient bound; we use the one of Evans and Spruck [46] for (2.1) and the ones of Chou and Kwong [33] for the more general equation (2.4) (see Section 2.9).

Uniqueness is an even more difficult issue and we were able to obtain it only in particular cases : of course the first "generic" uniqueness result we provide in Section 2.6 is not satisfactory and most of the results we obtain in this direction are proved by working directly on (2.4). However a striking application of the geometrical approach to uniqueness (and also existence) is the case of convex initial data (Section 2.10) : under suitable assumptions on b (the same as for the geometrical approach to hold), we prove that there exists a unique solution u of (2.4), which is convex in the space variable at each time, and this for any convex initial data u_0 without any restriction on its growth at infinity. The proof relies strongly on the convexity preserving property of Giga, Goto, Ishii and Sato [51] that we extend to our more singular case. Compared to the result we previously obtained in [15] by working directly on (2.1), we do not assume anymore u_0 to be coercive and we extend the result to equations like (2.4).

For completeness, we conclude this introduction by describing the results we obtained in the next two parts of this study. Two types of uniqueness results for the non-convex case are proved : the first ones [17] concern the case $N = 1$. We show, not only for (2.1) but for a larger class of equations, a uniqueness result without any growth assumptions on the solutions. Unfortunately, in general, this result is valid only in the class of classical solutions ; however, in the case of (2.1), using the argument of Section 2.9 and in particular Remark 2.9.1, this uniqueness result for smooth solutions implies a comparison result between possibly discontinuous sub- and supersolutions.

The proof relies upon examining the pde obtained by integrating in x. For (2.1), this pde reads

$$w_t - \arctan(w_{xx}) = 0 \quad \text{in } \mathbb{R} \times (0, +\infty), \tag{2.5}$$

and the key point is that (2.5) enjoys uniqueness properties in $C(\mathbb{R})$, essentially because one can use a "friendly giants" method, whose consequence is a general uniqueness property for (2.1). Of course, this method can be extended to far more general equations. We learn recently that related results were obtained independently and by rather different methods by Chou and Kwong [33].

In the second one [14], we use classical viscosity solutions arguments to prove the uniqueness for solutions of (2.4) and even more general equations : we obtain a comparison result for sub- and supersolutions with polynomial growth but, unfortunately, with a rather restrictive assumption on the initial data which reads in the locally Lipschitz continuous case

$$|Du_0(x)| \leq C(1 + |x|^\nu) \quad \text{in } \mathbb{R}^N,$$

for some constant $C > 0$ and $0 \leq \nu < (1+\sqrt{5})/2$. A strange feature of this result is that it can be obtained either by working directly on (2.4) or on the associated geometrical pde and both proofs lead to the same condition on u_0.

As we already mention it above, in a forthcoming paper, we investigate more general equations, namely

$$\frac{\partial u}{\partial t} - \mathrm{Tr}\left[b(x, t, Du)D^2u\right] + H(x, t, Du) = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty)\,.$$

After we obtained most of the results described above, we learn that representation formulas for the mean curvature equation (2.3) (and even more general geometrical equations) have been established independently by Soner and Touzi [80], [81] and by Buckdahn, Cardaliaguet and Quincampoix [29]. We tried to prove uniqueness for (2.1) by showing that the "non-fattening phenomena" cannot occur for (2.3) in the case of graphs but we failed. It is an intriguing question whether it is possible to prove such properties by using these formulas.

Acknowledgment. This work was partially supported by the TMR program "Viscosity Solutions and Their Applications." The last author would like to thank the Department of Mathematics of the university of Padova and especially Martino Bardi for their kind hospitality and fruitful exchanges.

2.2 Derivation of a geometrical pde

As explained in the introduction for the special case of equation (2.1), we associate a geometrical equation to the quasilinear equation (2.4) which allows us to use the level-set approach. This method was already used by Evans [42] for the heat equation and by Giga and Sato [52] in the case of Hamilton-Jacobi equations.

When u is a solution of (2.4) we consider, for every $t \geq 0$, $\mathrm{Graph}(u(\cdot,t))$ as an hypersurface in $\mathbb{R}^{N+1}$ and, to represent it, we follow the ideas of the level-set approach, taking any function $v : \mathbb{R}^N \times \mathbb{R} \times [0,+\infty) \to \mathbb{R}$ such that

$$v(x, u(x,t), t) = 0 \quad \text{for every } (x,t) \in \mathbb{R}^N \times [0,+\infty).$$

Note that, for all $t \geq 0$, $\mathrm{Graph}(u(\cdot,t)) \subset \Gamma_t$, where Γ_t is the 0-level-set of $v(\cdot,\cdot,t)$. Differentiating formally the previous inequality, we obtain

$$\begin{aligned}
&D_y v\,\frac{\partial u}{\partial t} + \frac{\partial v}{\partial t} = 0,\\
&D_x v + D_y v\, Du = 0,\\
&D^2_{xx} v + 2D^2_{xy} v \otimes Du + D^2_{yy} v Du \otimes Du + D_y v D^2 u = 0,
\end{aligned}$$

and it follows that v has to solve, at least formally

$$\frac{\partial v}{\partial t} - \mathrm{Tr}\left[b\left(-\frac{D_x v}{D_y v}\right)\left(D^2_{xx} v - 2D^2_{xy} v \otimes \frac{D_x v}{D_y v} + D^2_{yy} v \frac{D_x v}{D_y v} \otimes \frac{D_x v}{D_y v}\right)\right] = 0 \qquad (2.6)$$

in $\mathbb{R}^{N+1} \times (0,+\infty)$.

This new equation has strong discontinuities when the gradient of the solution, $Dv = (D_x v, D_y v)$ lies in the subset

$$\mathcal{D} = \{p = (p_1, \cdots, p_{N+1}) \in \mathbb{R}^{N+1} : p_{N+1} = 0\}, \qquad (2.7)$$

but (2.6) satisfies the following first properties which is a motivation to study (2.4) via the level-set approach.

Lemma 2.2.1 *We have*

(i) Equation (2.6) is degenerate parabolic outside of $\mathcal{D}$.

(ii) If $u \in C(\mathbb{R}^N \times [0,+\infty))$ *is a viscosity subsolution (respectively supersolution) of (2.4) with initial data* $u_0 \in C(\mathbb{R}^N)$, *then the function* $v(x,y,t) = y - u(x,t)$ *defined for* $(x,y,t) \in \mathbb{R}^N \times \mathbb{R} \times [0,+\infty)$ *is a viscosity supersolution (respectively subsolution) of (2.6) with initial data* $v_0(x,y) = y - u_0(x)$.

(iii) Equation (2.6) is invariant under every monotone change of function $v \to \Psi \circ v$, *where* $\Psi \in C(\mathbb{R})$ *is a monotone function.*

We skip the proofs of these three properties since they do not present any difficulty. Let us mention that (ii) and (iii) are obvious in the smooth case. Property (ii) is straightforward using the definition of viscosity solutions for the singular equation (2.6) we recall in Section 2.4.1. For (iii), we even prove a discontinuous version of it in Lemma 2.4.1. Finally it is worth pointing out that we choose to work with $v(x,y,t) = y - u(x,t)$ instead of $u(x,t) - y$ as usual.

Remark 2.2.1 Concerning (ii), one can wonder whether some kind of converse property is true : if v is a solution of (2.6) with initial data $v_0(x,y) = y - u_0(x)$, does there exist a solution u of (2.4) such that $v(x,y,t) = y - u(x,t)$? The answer is not clear and it is the main issue of our approach. We refer to Section 2.6 for related discussions and results.

We conclude this section by introducing some notations which are used throughout the paper. Every point z of $\mathbb{R}^{N+1}$ is written $z = (x,y)$ with $x \in \mathbb{R}^N$ and $y \in \mathbb{R}$. In a natural way, every vector p which has the meaning of a gradient is written $p = (p_x, p_y)$ with $p_x \in \mathbb{R}^N$ and $p_y = p_{N+1} \in \mathbb{R}$. We decompose every matrix $X \in \mathcal{S}_{N+1}$ in blocks in the following way

$$X = \left(\begin{array}{c|c} X_{xx} & X_{xy} \\ \hline X_{xy}^T & X_{yy} \end{array}\right),$$

where $X_{xx} \in \mathcal{S}_N, X_{xy} \in \mathbb{R}^N$, X_{xy}^T is the transpose of X_{xy} (i.e. the row vector whose coordinates are the ones of X_{xy}) and $X_{yy} \in \mathbb{R}$.

With these notations, the nonlinearity involved in (2.6) can be written, for every $p \in \mathbb{R}^{N+1} - \mathcal{D}$ and $X \in \mathcal{S}_{N+1}$,

$$F(p,X) = -\mathrm{Tr}\left[b(q)\left(X_{xx} + 2X_{xy} \otimes q + X_{yy} q \otimes q\right)\right] = -\mathrm{Tr}[\tilde{b}(p)\, X] \tag{2.8}$$

where $q = -p_x/p_y$ and

$$\tilde{b}(p) = \left(\begin{array}{c|c} b(q) & b(q)q \\ \hline (b(q)q)^T & \langle b(q)q, q\rangle \end{array}\right).$$

2.3 The geometrical equation : the classical framework

A priori the nonlinearity F is discontinuous on $\mathcal{D}$ (see (2.7)). In this section, we provide assumptions on b ensuring that we are in the "classical framework," which means that the (classical) level-set approach applies readily to (2.6) (see [45], [32] [51] and [22]). In this classical framework, F has to be continuous, except at $p = 0$. The typical example is the mean curvature equation (see Example 2.3.1).

More precisely, we start by recalling the assumptions as they appears in [51]. In the sequel, $\|\cdot\|$ is any norm on $\mathcal{S}_N$ and $S^{N-1} = \{\xi \in \mathbb{R}^N : |\xi| = 1\}$ is the unit sphere of $\mathbb{R}^N$.

(F1) $F : (\mathbb{R}^{N+1} - \{0\}) \times \mathcal{S}_{N+1} \to \mathbb{R}$ is continuous.

(F2) $F(p, X+Y) \leq F(p,X)$ for all $p \in \mathbb{R}^{N+1}$, $X, Y \in \mathcal{S}_{N+1}, Y \geq 0$.

(F3) $-\infty < F_*(0,0) = F^*(0,0) < +\infty$ where F_* and F^* are the semicontinuous envelopes of F defined by $F_*(p,X) = \liminf_{(\rho,Y)\to(p,X)} \{F(\rho,Y) : \rho \neq 0\}$ and $F^* = -(-F)_*$.

(F4) For every $R > 0$, $\sup\{|F(p,X)| : |p| \leq R, \|X\| \leq R\} < +\infty$.

We have, the following classical result.

Theorem 2.3.1 *Under assumptions **(F1)**–**(F4)**, for any initial data $v_0 \in UC(\mathbb{R}^{N+1})$, there exists a unique solution v of (2.6) which is in $UC(\mathbb{R}^{N+1} \times [0,T))$ for every $T > 0$.*

Notice that, if F is continuous, then **(F1)**–**(F4)** reduce to **(F2)** only.

We state now the assumptions on b which permit to extend the F given by (2.8) by continuity in $(\mathbb{R}^{N+1} - \{0\}) \times \mathcal{S}_{N+1}$ in order to ensure that **(F1)**–**(F4)** hold.

(H1) There is a positive constant K_1 such that $\|b(q)\| \le K_1$ for all $q \in \mathbb{R}^N$.

(H2) There is a positive constant K_2 such that $|b(q)q| \le K_2$ for every $q \in \mathbb{R}^N$.

(H3) There is a positive constant K_3 such that $|\langle b(q)q, q\rangle| \le K_3$ for every $q \in \mathbb{R}^N$.

(H4) For every $q \in S^{N-1}$, $\lim\limits_{\lambda\to+\infty} b(\lambda q)$ and $\lim\limits_{\lambda\to-\infty} b(\lambda q)$ exist and are equal. Moreover $b_\infty(q) := \lim\limits_{\lambda\to\pm\infty} b(\lambda q)$ is continuous on S^{N-1}.

(H5) For every $q \in S^{N-1}$, $\lim\limits_{\lambda\to+\infty} \lambda b(\lambda q)q$ and $\lim\limits_{\lambda\to-\infty} \lambda b(\lambda q)q$ exist and are equal. Moreover $\zeta_\infty(q) := \lim\limits_{\lambda\to\pm\infty} \lambda b(\lambda q)q$ is continuous on S^{N-1}.

(H6) For every $q \in S^{N-1}$, $\lim\limits_{\lambda\to+\infty} \lambda^2\langle b(\lambda q)q, q\rangle$ and $\lim\limits_{\lambda\to-\infty} \lambda^2\langle b(\lambda q)q, q\rangle$ exist and are equal. Moreover the function $\alpha_\infty(q) := \lim\limits_{\lambda\to\pm\infty} \lambda^2\langle b(\lambda q)q, q\rangle$ is continuous on S^{N-1}.

Proposition 2.3.1 *Let F be defined by (2.8) with $b \in C(\mathbb{R}^N; \mathcal{S}_N^+)$. Then assumptions **(H1)**–**(H6)** are equivalent to assumptions **(F1)**–**(F4)**. It follows that, under assumptions **(H1)**–**(H6)**, Theorem 2.3.1 hold.*

Proof of Proposition 2.3.1. We use the notations of Section 2.2. We start by assuming **(H1)**–**(H6)**. For every $((p_x,0),X) \in (\mathcal{D} - \{0\}) \times \mathcal{S}_{N+1}$, we extend F by setting

$$F((p_x,0),X) = -\mathrm{Tr}\left[b_\infty(p_x/|p_x|)X_{xx} + 2\langle \zeta_\infty(p_x/|p_x|), X_{xy}\rangle + X_{yy}\alpha_\infty(p_x/|p_x|)\right].$$

From the assumed continuity of b_∞, ζ_∞ and α_∞ on S^{N-1}, the extended Hamiltonian F is clearly continuous in $\mathbb{R}^{N+1} - \{0\}$; thus **(F1)** holds. Assumption **(F2)** is an immediate consequence of Lemma 2.2.1 (i). Finally, from the boundedness conditions **(H1)**, **(H2)** and **(H3)**, it is obvious that $F_*(0,0) = F^*(0,0) = 0$ and $|F(p,X)| \le K_1R + 2K_2R + K_3R$ for $|p|, \|X\| \le R$. It shows that **(F3)** and **(F4)** hold.

Conversely, suppose that **(F1)**–**(F4)** hold. **(F2)** implies easily that $b(\xi)$ is positive for all $\xi \in \mathbb{R}^N$. Let $\xi \in S^{N-1}$; for any $\lambda \ne 0$, we have

$$\tilde{b}(q, 1/\lambda) = \left(\begin{array}{c|c} b(\lambda q) & \lambda b(\lambda q)q \\ \hline (\lambda b(\lambda)q)^T & \lambda^2\langle b(\lambda q)q, q\rangle \end{array}\right). \tag{2.9}$$

From **(F4)**, we have that $\tilde{b}(q, 1/\lambda)$ is bounded for every $q \in S^{N-1}$ and $\lambda \ge 1$. It follows that $\|b(\xi)\|, |b(\xi)\xi|$ and $|\langle b(\xi)\xi, \xi\rangle|$ are bounded for every $\xi \in \{\lambda q : \lambda \ne 0, q \in S^{N-1}\} = \mathbb{R}^N - \overline{B}(0,1)$. Since these quantities are obviously bounded in $\overline{B}(0,1)$, we get **(H1)**–**(H3)**. From **(F1)**, we have that $\tilde{b}$ is continuous in $\mathbb{R}^{N+1} - \{0\}$. On the one hand, it follows easily that b is continuous in $\mathbb{R}^N$. On the other hand, sending λ to $\pm\infty$ in (2.9), we see that b_∞, ζ_∞ and α_∞ are well-defined :

$$\lim_{\lambda\to\pm\infty} \tilde{b}(q, 1/\lambda) = \tilde{b}(q, 0) = \left(\begin{array}{c|c} b_\infty(q) & \zeta_\infty(q) \\ \hline (\zeta_\infty(q))^T & \alpha_\infty(q)\rangle \end{array}\right). \tag{2.10}$$

Invoking again the continuity of $\tilde{b}$, (2.10) implies that b_∞, ζ_∞ and α_∞ are continuous on S^{N-1}; thus **(H4)**–**(H6)** hold. It ends the proof. $\square$

Proposition 2.3.1 applies of course in the case of the mean curvature equation. The computations are developed in the following example.

Example 2.3.1 Mean curvature equation (2.1).
In this case,

$$b(q) = I - \frac{q \otimes q}{1 + |q|^2} \quad \text{for every } q \in \mathbb{R}^N. \tag{2.11}$$

Easy computations show that

$$F(p, X) = -\mathrm{Tr}\left[\left(I - \frac{p \otimes p}{|p|^2}\right) X\right] \quad \text{for every } (p, X) \in (\mathbb{R}^{N+1} - \{0\}) \times \mathcal{S}_{N+1}$$

and then, (2.6) associated to (2.11) is the classical geometric mean curvature equation.

The checking of **(H1)**–**(H6)** consists in straightforward computations. We obtain $b_\infty(q) = I - q \otimes q, \zeta_\infty(q) = 0$ and $\alpha_\infty(q) = 1$ for every $q \in S^{N-1}$.

2.4 The geometrical equation : the very singular case

In this section, we study the case when the discontinuities of F on $\mathcal{D}$ cannot be reduced to a discontinuity at $p = 0$. This question was addressed by many authors : Goto [54], Ishii and Souganidis [64], Ishii [63] or Ohnuma and Sato [75].

Ishii [63] deals with the worst set of singularities. Our approach is strongly inspired by his work : Ishii extends the notion of viscosity by restricting the class of test-functions. His result applies but we provide a simpler proof which relies on the special form of our set of singularities.

We refer to the end of the section for examples of pdes which are covered by our framework but which do not satisfy the assumptions of Section 2.3.

2.4.1 Definitions and first properties

We recall the definition of viscosity solutions for very singular equations as it appears in Ishii [63].

In the sequel, USC (respectively LSC) denotes the set of upper-semicontinuous (respectively lower-semicontinuous) functions. For any locally bounded function v, v^* and v_* are respectively the upper- and lower-semicontinuous envelopes of v and $\mathcal{P}^{2,+}(v^*)$ and $\mathcal{P}^{2,-}(v_*)$ are its parabolic semijets (see [37] for a definition).

We define semicontinuous envelopes for F, which are *adapted* to the set of discontinuity $\mathcal{D}$, by, for every $(p, X) \in \mathbb{R}^{N+1} \times \mathcal{S}_{N+1}$,

$$\begin{aligned} F^*(p, X) &= \limsup_{(\rho,Y)\to(p,X)} \{F(\rho, Y) : (\rho, Y) \in (\mathbb{R}^{N+1} - \mathcal{D}) \times \mathcal{S}_{N+1}\}, \\ F_*(p, X) &= \liminf_{(\rho,Y)\to(p,X)} \{F(\rho, Y) : (\rho, Y) \in (\mathbb{R}^{N+1} - \mathcal{D}) \times \mathcal{S}_{N+1}\}. \end{aligned}$$

Clearly, F^* and F_* inherit the same properties as F : they are still degenerate elliptic and geometric.

Definition 2.4.1 *A locally bounded function $v : \mathbb{R}^{N+1} \times (0, +\infty) \mapsto \mathbb{R}$ is said to be a viscosity subsolution (respectively a supersolution) of (2.6) if and only if*

$$a + F_*(p, X) \le 0 \quad \textit{for all } (x, t) \in \mathbb{R}^{N+1} \times (0, +\infty) \textit{ and } (a, p, X) \in \mathcal{P}^{2,+}(v^*)(x, t)$$

(respectively

$$a + F^*(p, X) \ge 0 \quad \textit{for all } (x, t) \in \mathbb{R}^{N+1} \times (0, +\infty) \textit{ and } (a, p, X) \in \mathcal{P}^{2,-}(v_*)(x, t) \text{).}$$

A discontinuous function v is a viscosity solution of (2.6) provided it is both a sub- and a supersolution.

With this definition, all the basic properties of viscosity solutions extend. In particular the classical stability result for viscosity solutions holds. The proof is the same than the ones in the classical references given in the introduction.

We continue with the *invariance lemma* which is a characteristic of geometric equations (Cf. Section 2.2).

Lemma 2.4.1 *If $v \in USC(\mathbb{R}^{N+1} \times [0, +\infty))$ (respectively $LSC(\mathbb{R}^{N+1} \times [0, +\infty))$) is a viscosity subsolution (respectively supersolution) of (2.6), then, for any nondecreasing function $\Psi \in USC(\mathbb{R})$ (respectively $\Psi \in LSC(\mathbb{R})$) the function $\Psi \circ v$ is a viscosity subsolution (respectively supersolution) of the same equation.*

Proof of Lemma 2.4.1. We will prove the assertion in the case of a subsolution ; the proof for supersolutions is analogous. We proceed by approximation of Ψ.

We construct a non-increasing family $(\Psi_\varepsilon)_{\varepsilon>0}$ of smooth strictly increasing functions such that

$$\inf_{\varepsilon>0} \Psi_\varepsilon = \Psi.$$

Let ϕ be a C^2 function and (x_0, t_0) be a local maximum of $\Psi_\varepsilon(v) - \phi$. Without loss of generality, we can suppose that $(\Psi_\varepsilon(v) - \phi)(x_0, t_0) = 0$. It follows that, for every $x \in \mathbb{R}^{N+1}$ and $t \in [0, +\infty)$,

$$\Psi_\varepsilon(v)(x, t) \le \phi(x, t) \Longleftrightarrow v(x, t) \le \Phi_\varepsilon \circ \phi(x, t),$$

where we set $\Phi_\varepsilon = (\Psi_\varepsilon)^{-1}$. Thus (x_0, t_0) is a local maximum of $v - \Phi_\varepsilon \circ \phi$ and since v is a subsolution of (2.6), we get

$$\Phi_\varepsilon{}' \frac{\partial \phi}{\partial t}(x_0, t_0) + F_* \left(\Phi_\varepsilon{}' D\phi(x_0, t_0), \Phi_\varepsilon{}' D^2\phi(x_0, t_0) + \Phi_\varepsilon{}'' D\phi \otimes D\phi \right) \le 0. \qquad (2.12)$$

Using that F_* is geometric and dividing (2.12) by $\Phi_\varepsilon{}' > 0$, we get

$$\frac{\partial \phi}{\partial t}(x_0, t_0) + F_* \left(D\phi(x_0, t_0), D^2\phi(x_0, t_0) \right) \le 0$$

which proves that $\Psi_\varepsilon \circ v$ is a subsolution of (2.6). From classical results about viscosity solutions, we have that

$$\limsup_{\varepsilon \to 0}{}^* \Psi_\varepsilon(v) = \inf_{\varepsilon>0} \Psi_\varepsilon \circ v = \Psi \circ v$$

is a subsolution of (2.6) which is the desired result. □

2.4.2 Comparison result

We turn now to a comparison principle for (2.6).

Theorem 2.4.1 *Suppose that **(H1)–(H4)** hold and let $v_0 \in UC(\mathbb{R}^{N+1})$. If $v_1 \in USC(\mathbb{R}^{N+1} \times [0,+\infty))$ (respectively $v_2 \in LSC(\mathbb{R}^{N+1} \times [0,+\infty))$) is a subsolution (respectively a supersolution) of (2.6), and if $v_1(\cdot,0) \le v_0 \le v_2(\cdot,0)$ in $\mathbb{R}^{N+1}$, then $v_1 \le v_2$ in $\mathbb{R}^{N+1} \times [0,+\infty)$.*

Remark 2.4.1 Note that "bounded" or "unbounded" solutions is not the point in this theorem. Since the equation is geometric, up to make a change of variable $v \mapsto \tanh(v)$ together with Lemma 2.4.1, we can suppose that the solutions are bounded. Another remark is that we are able to compare bounded continuous solutions with bounded uniformly continuous solutions. Of course, it gives uniqueness only in $UC(\mathbb{R}^{N+1} \times [0,+\infty))$.

The difficulty to prove such a result comes obviously from the unusual set of discontinuity $\mathcal{D}$. We begin by some arguments giving an idea of the proof.

Setting

$$\mathcal{S}(\mathcal{D}) = \{X \in \mathcal{S}_{N+1} : X_{xy} = 0, X_{yy} = 0\} = \left\{ \left(\begin{array}{c|c} X_{xx} & 0 \\ \hline 0 & 0 \end{array} \right) \ : \ X_{xx} \in \mathcal{S}_N \right\}, \quad (2.13)$$

we have

Lemma 2.4.2 *Assume **(H1)–(H4)**. Then, for all $p \in \mathcal{D} - \{0\}$ and $X \in \mathcal{S}(\mathcal{D})$, we have $F^*(p,X) = F_*(p,X)$. Moreover $F^*(0,0) = 0 = F_*(0,0)$.*

This lemma is proved at the end of the section. It suggests that we may use in the proof of the theorem test-functions φ such that $D^2\varphi \in \mathcal{S}(\mathcal{D})$ when $D\varphi \in \mathcal{D}$. By this way, one does not see the discontinuities of F in the proof.

Proof of Theorem 2.4.1. Without loss of generality, we can assume that v_1 and v_2 are bounded. We recall that we write z for a point $z = (x,y) \in \mathbb{R}^N \times \mathbb{R}$ and by $|z|$ we mean $(|x|^2 + y^2)^{1/2}$. We argue by contradiction, assuming that there exists $(z_0,t_0) \in \mathbb{R}^{N+1} \times [0,+\infty)$ such that $(v_1 - v_2)(z_0,t_0) > 0$. We introduce the function

$$\phi(z_1,z_2) = \frac{|x_1 - x_2|^4}{\varepsilon^4} + \frac{|y_1 - y_2|^4}{\varepsilon^4} = \varphi(z_1 - z_2)$$

which is chosen in order to ensure that $D^2\varphi(Z) \in \mathcal{S}(\mathcal{D})$ when $D\varphi(Z) \in \mathcal{D}$ (for the definitions of $\mathcal{D}$ and $\mathcal{S}(\mathcal{D})$, see (2.7) and (2.13)). We then set

$$M_{\varepsilon,\alpha,\eta} = \sup_{\mathbb{R}^{N+1}\times\mathbb{R}^{N+1}\times[0,+\infty)} \left\{ v_1(z_1,t) - v_2(z_2,t) - \phi(z_1,z_2) - \alpha(|z_1|^2 + |z_2|^2) - \eta t \right\}.$$

At first, it is clear that $M_{\varepsilon,\alpha,\eta} > 0$ for α, η sufficiently small since $\phi(z_0,z_0) = 0$. Moreover $M_{\varepsilon,\alpha,\eta}$ is achieved at some $(\bar{z}_1, \bar{z}_2, \bar{t})$ by the boundedness and the semicontinuous properties of v_1 and v_2. Actually $\bar{z}_1$ and $\bar{z}_2$ depend on $\alpha, \varepsilon, \eta$, but we omit this dependence in the notation for simplicity.

When $\bar{t} = 0$, we have

$$\begin{aligned} 0 < M_{\varepsilon,\alpha,\eta} &\leq v_1(\bar{z}_1, 0) - v_2(\bar{z}_2, 0) - \frac{|\bar{x}_1 - \bar{x}_2|^4}{\varepsilon^4} - \frac{|\bar{y}_1 - \bar{y}_2|^4}{\varepsilon^4} - \alpha(|\bar{z}_1|^2 + |\bar{z}_2|^2) \\ &\leq v_0(\bar{z}_1) - v_0(\bar{z}_2) - \frac{|\bar{x}_1 - \bar{x}_2|^4}{\varepsilon^4} - \frac{|\bar{y}_1 - \bar{y}_2|^4}{\varepsilon^4} - \alpha(|\bar{z}_1|^2 + |\bar{z}_2|^2) \end{aligned}$$

which leads to a contradiction using the uniform continuity of v_0. Thus, it cannot exist a subsequence of parameters (ε, α) going to $(0, 0)$ such that $\bar{t} = 0$. Therefore, we can suppose that $\bar{t} > 0$ for ε and α sufficiently small.

From the fundamental result of the User's guide to viscosity solutions [37, Theorem 8.3], for every $\rho > 0$, we get $a_1, a_2 \in \mathbb{R}$ and $X, Y \in \mathcal{S}_{N+1}$ such that

$$(a_1, D\varphi(\bar{z}_1 - \bar{z}_2) + 2\alpha\bar{z}_1, X + 2\alpha I) \in \bar{\mathcal{P}}^{2,+}(v_1)(\bar{z}_1, \bar{t}),$$

$$(a_2, D\varphi(\bar{z}_1 - \bar{z}_2) - 2\alpha\bar{z}_2, Y - 2\alpha I) \in \bar{\mathcal{P}}^{2,-}(v_2)(\bar{z}_2, \bar{t})$$

and

$$-(\frac{1}{\rho} + \|A\|) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \begin{pmatrix} A + 2\rho A^2 & -(A + 2\rho A^2) \\ -(A + 2\rho A^2) & A + 2\rho A^2 \end{pmatrix} \tag{2.14}$$

for some $a_1 - a_2 = \eta$ and $A = D^2\varphi(\bar{z}_1 - \bar{z}_2)$. Writing that v_1 is a subsolution and v_2 a supersolution of (2.6), we have

$$\eta + F_*(D\varphi(\bar{z}_1 - \bar{z}_2) + 2\alpha\bar{z}_1, X + 2\alpha I) - F^*(D\varphi(\bar{z}_1 - \bar{z}_2) - 2\alpha\bar{z}_2, Y - 2\alpha I) \leq 0. \tag{2.15}$$

We compute, for every $Z = (Z_x, Z_y) \in \mathbb{R}^N \times \mathbb{R}$,

$$D\varphi(Z) = \frac{4}{\varepsilon^4}(|Z_x|^2 Z_x, Z_y^3) \quad \text{and} \quad D^2\varphi(Z) = \frac{4}{\varepsilon^4}\left(\begin{array}{c|c} 2Z_x \otimes Z_x + |Z_x|^2 I & 0 \\ \hline 0 & 3Z_y^2 \end{array}\right).$$

Since $M_{\varepsilon,\alpha,\eta} > 0$, we get

$$\phi(\bar{z}_1, \bar{z}_2, \bar{t}) + \alpha(|\bar{z}_1|^2 + |\bar{z}_2|^2) \leq \|v_1\|_\infty + \|v_2\|_\infty$$

(recall that v_1 and v_2 are assumed to be bounded). It follows that

$$\lim_{\alpha \to 0^+} \alpha|\bar{z}_1|, \ \lim_{\alpha \to 0^+} \alpha|\bar{z}_2| = 0, \tag{2.16}$$

$$|\bar{z}_1 - \bar{z}_2| \text{ is bounded as } \alpha \text{ goes to } 0. \tag{2.17}$$

From (2.17) and (2.14), we obtain that X and Y are bounded when α goes to 0. Therefore, using (2.16), we can extract subsequences such that $\bar{z}_1 - \bar{z}_2 \to \bar{Z}$ and

$$(D\varphi(\bar{z}_1 - \bar{z}_2) + 2\alpha\bar{z}_1, X + 2\alpha I) \longrightarrow \left(D\varphi(\bar{Z}), \bar{X}\right),$$

$$(D\varphi(\bar{z}_1 - \bar{z}_2) - 2\alpha\bar{z}_2, Y - 2\alpha I) \longrightarrow \left(D\varphi(\bar{Z}), \bar{Y}\right),$$

when α goes to 0. Note that $\bar{X}, \bar{Y}$ satisfy also (2.14) with $A = D^2\varphi(\bar{Z})$. From (2.15), it follows

$$\eta + F_*(D\varphi(\bar{Z}), \bar{X}) - F^*(D\varphi(\bar{Z}), \bar{Y}) \leq 0. \tag{2.18}$$

Now, if $D\varphi(\bar{Z}) \notin \mathcal{D}$, then we are done since (2.14) implies that $\bar{X} \leq \bar{Y}$ and since in this case $F^* = F_* = F$ is degenerate elliptic.

But, when $D\varphi(\bar{Z}) \in \mathcal{D}$, we need more information about $\bar{X}, \bar{Y}$ in order to get the contradiction. At first, $D\varphi(\bar{Z}) \in \mathcal{D}$ implies $\bar{Z}_y = 0$; thus

$$D^2\varphi(\bar{Z}) = \left(\begin{array}{c|c} 2\bar{Z}_x \otimes \bar{Z}_x + |\bar{Z}_x|^2 I & 0 \\ \hline 0 & 0 \end{array}\right) \in \mathcal{S}(\mathcal{D}). \tag{2.19}$$

At this step, we would like to apply Lemma 2.4.2 but we need first to transfer on $\bar{X}, \bar{Y}$ the suitable property of $D^2\varphi(\bar{Z})$, namely $D^2\varphi(\bar{Z}) \in \mathcal{S}(\mathcal{D})$. To this end, we state

Lemma 2.4.3 *If $D^2\varphi(\bar{Z}) \in \mathcal{S}(\mathcal{D})$, then there exist $X', Y' \in \mathcal{S}(\mathcal{D})$ such that*

$$\bar{X} \leq X' \leq Y' \leq \bar{Y}.$$

Moreover, $X' = Y' = 0$ when $D\varphi(\bar{Z}) = 0$.

We postpone the proof and complete the one of Theorem 2.4.1. Taking advantage of the ellipticity of F^* and F_* together with Lemma 2.4.2, we get from (2.18)

$$\eta + F^*(D\varphi(\bar{Z}), X') - F^*(D\varphi(\bar{Z}), Y') \leq 0.$$

Since $X' \leq Y'$, the ellipticity of F^* leads to a contradiction. It achieves the proof of the theorem. □

We turn to the proof of the lemmas.

Proof of Lemma 2.4.2. Let us consider $((p_x, 0), X) \in \mathcal{D} \times \mathcal{S}(\mathcal{D})$. It suffices to see that $F(\rho, X + Y)$ has a limit when $(\rho, Y) \to ((p_x, 0), 0)$, $(\rho, Y) \in (\mathbb{R}^{N+1} - \mathcal{D}) \times \mathcal{S}_{N+1}$. Since $X \in \mathcal{S}(\mathcal{D})$, we have

$$\begin{aligned} F(\rho, X + Y) &= F(\rho, X) + F(\rho, Y) \\ &= -\mathrm{Tr}\left[b\left(-\frac{\rho_x}{\rho_y}\right) X_{xx}\right] - \mathrm{Tr}\left[b\left(-\frac{\rho_x}{\rho_y}\right)\left(Y_{xx} - 2Y_{xy} \otimes \frac{\rho_x}{\rho_y} + Y_{yy}\frac{\rho_x}{\rho_y} \otimes \frac{\rho_x}{\rho_y}\right)\right]. \end{aligned}$$

At first, from **(H1)**, **(H2)** and **(H3)**, when $\|Y\| \leq \varepsilon$, we obtain

$$\left| Tr\left[b\left(-\frac{\rho_x}{\rho_y}\right)\left(Y_{xx} - 2Y_{xy} \otimes \frac{\rho_x}{\rho_y} + Y_{yy}\frac{\rho_x}{\rho_y} \otimes \frac{\rho_x}{\rho_y}\right)\right]\right| \leq O(\varepsilon) \underset{\varepsilon \to 0}{\to} 0. \tag{2.20}$$

If $p_x = 0$, then $p = 0$ and (2.20) implies that $F^*(0,0) = 0 = F_*(0,0)$. If $p \in \mathcal{D} - \{0\}$, then $p_x \neq 0$ and from **(H4)**, we get

$$\lim_{\rho \to (p_x, 0)} \mathrm{Tr}\left[b\left(-\frac{\rho_x}{\rho_y}\right) X_{xx}\right] = \mathrm{Tr}\left[b_\infty(p_x/|p_x|) X_{xx}\right].$$

It achieves the proof of the lemma. □

Proof of Lemma 2.4.3. We set $A = D^2\varphi(\bar{Z})$, $B = A + 2\rho A^2$ (see (2.14)) and $B_\delta = B + \delta I$ for $\delta > 0$. Note that $B \in \mathcal{S}(\mathcal{D})$ and $B \geq 0$ as we can see it with the help of Formula (2.19). Moreover, we get from (2.14) that

$$\langle \bar{X}p, p\rangle - \langle \bar{Y}q, q\rangle \leq \langle B(p-q), p-q\rangle < \langle B_\delta(p-q), p-q\rangle. \tag{2.21}$$

It provides in particular that $\bar{X} < B_\delta$ (respectively $-\bar{Y} < B_\delta$) ; thus $(\bar{X} - B_\delta)$ (respectively $(\bar{Y} + B_\delta)$) is invertible. We then obtain X' using a sup-convolution. We set, for every $p, r \in \mathbb{R}^{N+1}$ and $k > 1$,

$$F_r(p) = \langle \bar{X}p, p\rangle - \langle kB_\delta(p-r), p-r\rangle \tag{2.22}$$

and consider $\sup\limits_{p\in\mathbb{R}^N} F_r(p)$ which is well-defined. For every $h \in \mathbb{R}^N$,

$$\langle DF_r(p), h\rangle = 2\langle \bar{X}p, h\rangle - 2\langle kB_\delta(p-r), h\rangle$$

which provides that the supremum is achieved for

$$p = (kB_\delta - \bar{X})^{-1} kB_\delta r$$

(note that $(kB_\delta - \bar{X})$ is invertible since $\bar{X} < B_\delta < kB_\delta$). Next, an explicit but tedious computation yields a matrix $X' \in \mathcal{S}_{N+1}$ such that

$$\sup_{p\in\mathbb{R}^{N+1}} F_r(p) = \langle X'r, r\rangle.$$

Taking successively the particular value $p = r$ and $p = 0$ in (2.22) we get $X' \geq \bar{X}$ and $X' \geq -kB_\delta$. Similarly, we can construct Y' by setting, for $k > 1$,

$$\inf_{p\in\mathbb{R}^{N+1}} \big\{\langle \bar{Y}p, p\rangle + \langle kB_\delta(p-s), p-s\rangle\big\} = \langle Y's, s\rangle.$$

We obtain a matrix Y' satisfying $Y' \leq \bar{Y}$ and $Y' \leq kB_\delta$.

From (2.21), we get

$$\begin{aligned}
&\langle X'r, r\rangle - \langle Y's, s\rangle \\
&\leq \sup_{p,q\in\mathbb{R}^{N+1}} \{\langle B_\delta(p-q), p-q\rangle - \langle kB_\delta(p-r), p-r\rangle - \langle kB_\delta(q-s), q-s\rangle\}
\end{aligned}$$

for any $r, s \in \mathbb{R}^{N+1}$. An explicit calculation of this supremum yields

$$\langle X'r, r\rangle - \langle Y's, s\rangle \leq \langle B_\delta(r-s), r-s\rangle.$$

By taking $r = s$, we obtain $\langle X'r, r\rangle - \langle Y'r, r\rangle \leq 0$ for every $r \in \mathbb{R}^{N+1}$. It follows $-kB_\delta \leq X' \leq Y' \leq kB_\delta$. Now, letting δ go to 0, up to extract a subsequence, we get two matrices, still denoted by X', Y', such that

$$-kB \leq X' \leq Y' \leq kB. \tag{2.23}$$

Recalling that $B \in \mathcal{S}(\mathcal{D})$, we get first that $X'_{yy} = Y'_{yy} = 0$. Then, from (2.23), for any $r \in \mathbb{R}^{N+1}$, we have

$$\langle X'r, r\rangle = \langle X'_{xx}r_x, r_x\rangle + 2\langle X'_{xy}, r_x\rangle r_y \leq k\langle Br, r\rangle \leq K|r_x|^2$$

for some positive constant K. Taking tr_x instead of r_x for $t \in \mathbb{R}$, we get

$$t^2\langle X'_{xx}r_x, r_x\rangle + 2t\langle X'_{xy}, r_x\rangle r_y \leq Kt^2|r_x|^2$$

which provides $\langle X_{xy}, r_x\rangle = 0$ dividing by t and letting t go to 0^+ or 0^-. Since it holds for any $r_x \in \mathbb{R}^N$ we are done. The same arguments hold for Y'. Finally, if $D\varphi(\bar{Z}) = 0$, then $\bar{Z}_x = 0, \bar{Z}_y = 0$; it implies $A = B = 0$. From (2.23), we get $X' = Y' = 0$ which completes the proof of the lemma. □

2.4.3 Existence of solutions

Our result is the

Theorem 2.4.2 *Assume* ***(H1)–(H4)****. For every $v_0 \in UC(\mathbb{R}^{N+1})$, there exists a unique $v \in UC\left(\mathbb{R}^{N+1} \times [0,+\infty)\right)$ solving (2.6) with initial data v_0.*

The proof uses the classical *Perron's method*, introduced in the framework of viscosity solutions by Ishii in [60] (see also [6], [13] or [37]). The application of this method in our case does not present any special difficulties. Nevertheless, we provide a proof for the readers' convenience.

Proof of Theorem 2.4.2. The uniqueness part comes immediately from Theorem 2.4.1 and because of Lemma 2.4.1, we can suppose that v_0 is bounded. We divide the proof in different steps.

Step 1. We construct a solution $v \in C(\mathbb{R}^{N+1} \times [0,+\infty))$ when the initial data is smooth. Let $v_0 \in C^2(\mathbb{R}^{N+1}) \cap W^{2,\infty}(\mathbb{R}^{N+1})$ and define, for any $C > 0$, two functions $\underline{v}, \overline{v}$ by setting

$$\underline{v}(z,t) := -Ct + v_0(z) \quad \text{and} \quad \overline{v}(z,t) = Ct + v_0(z)$$

for any $(z,t) \in \mathbb{R}^{N+1} \times [0,+\infty)$. It follows from **(H1)–(H3)** that the nonlinearity F appearing in (2.6) is bounded on bounded subsets. Therefore, C may be chosen large enough in order that $\underline{v}$ and $\overline{v}$ are respectively sub and super solution of (2.6).

Consider then the set $\mathcal{F}$ of subsolutions of (2.6) w such that $\underline{v} \le w \le \overline{v}$. Set then for every $(z,t) \in \mathbb{R}^{N+1} \times [0,+\infty)$, $v(z,t) = \sup_{w \in \mathcal{F}} w(z,t)$. The set $\mathcal{F}$ is nonempty and v is well-defined. Thus, we get from the comparison result and classical arguments of the Perron's method that v is a discontinuous solution of (2.6).

Now, we also have from the definition of v that $v^*(\cdot,0) = v_*(\cdot,0) = v_0$. Thus we deduce from the comparison result that $v^* = v_* = v$ which is the desired continuous solution.

Step 2. We show that the solution v built in Step 1 is actually in $BUC(\mathbb{R}^{N+1} \times [0,+\infty))$. First of all, since the constant functions are smooth solutions of (2.6), the comparison result provides that v is bounded, namely $|v| \le |v_0|$ in $\mathbb{R}^{N+1} \times [0,+\infty)$.

Next, from the definition of v, for all $h > 0$, we have

$$\underline{v}(\cdot,h) = v_0 - Ch \le v(\cdot,h) \le v_0 + Ch = \overline{v}(\cdot,h) \quad \text{in } \mathbb{R}^{N+1}. \tag{2.24}$$

From Theorem 2.4.1 and since the nonlinearity in (2.6) depends only on (Du, D^2u), the function $v(\cdot,\cdot+h)$ is a solution of (2.6) with initial data $v(\cdot,h)$ and $v \pm Ch$ are solutions of (2.6) with initial data $v_0 \pm Ch$. Thus Theorem 2.4.1, together with (2.24), yields

$$v - Ch \le v(\cdot,\cdot+h) \le v + Ch \quad \text{in } \mathbb{R}^{N+1} \times [0,+\infty).$$

It provides a modulus of continuity in the time variable which is independent of the space variable. Arguing in the same way with translation in space $v_0 \mapsto v_0(\cdot + \xi)$,

$\xi \in \mathbb{R}^{N+1}$, we obtain a modulus of continuity for the space variable which is independent on the time variable. It proves that $v \in BUC(\mathbb{R}^{N+1} \times [0, +\infty))$.

Step 3. The general case when $v_0 \in BUC(\mathbb{R}^{N+1})$. Using a classical convolution procedure, we construct a sequence $(v_0^n)_{n\in\mathbb{N}}$ of functions $v_0^n \in C^2(\mathbb{R}^{N+1}) \cap W^{2,\infty}(\mathbb{R}^{N+1})$ such that $|v_0 - v_0^n| \leq 1/n$ in $\mathbb{R}^{N+1}$. It follows $-2/n + v_0^n \leq v_0^m \leq v_0^n + 2/n$ for $m \geq n$. According to Steps 1 and 2, we can consider, for every $n \in \mathbb{N}$, the unique solution $v^n \in BUC(\mathbb{R}^{N+1} \times [0, +\infty))$ of (2.6) with initial data v_0^n. Proceeding as in Step 2, we deduce from the previous inequality that, $-2/n + v^n \leq v^m \leq v^n + 2/n$ for $m \geq n$. Thus $(v^n)_{n\in\mathbb{N}}$ converges uniformly in $\mathbb{R}^{N+1} \times [0, +\infty)$ to some function v which is still bounded uniformly continuous. From the stability result, v is a viscosity solution of (2.6) with initial data v_0. It achieves the proof of the Theorem. □

2.4.4 Examples

We give some examples of pdes like (2.4) which enter in the very singular case.

1. In addition to the mean curvature equation for graphs (2.1), we can deal with the non geometric mean curvature equation

$$\frac{\partial u}{\partial t} - \mathrm{div}\frac{Du}{\sqrt{1+|Du|^2}} = 0, \tag{2.25}$$

or equations like

$$\frac{\partial u}{\partial t} - \frac{\Delta u}{(1+|Du|^2)^\alpha} = 0, \quad \alpha \geq 1. \tag{2.26}$$

These equations lead to a geometrical equation like (2.6) with a singularity only at $|Dv| = 0$ and they satisfy the assumptions of the classical framework.

2. Consider a generalization of the mean curvature equation for graphs, namely

$$\frac{\partial u}{\partial t} - \mathrm{Tr}\left[(I - g(Du)Du \otimes Du)D^2u\right] = 0, \tag{2.27}$$

where g is a continuous function from $\mathbb{R}^N$ into $\mathbb{R}$. In this case, $b(q) = I - g(q)q \otimes q$ is symmetric nonnegative and satisfies **(H1)**–**(H3)** if and only if there exists a positive constant C such that

$$\frac{1}{|q|^2}\left(1 - \frac{C}{|q|^2}\right) \leq g(q) \leq \frac{1}{|q|^2} \quad \text{for every } q \in \mathbb{R}^N. \tag{2.28}$$

Using the notations of Section 2.3, for every $q \in S^{N-1}$, we have $b_\infty(q) = I - q \otimes q$ and $\zeta_\infty(q) = 0$; thus **(H4)** and **(H5)** are fulfilled and this equation falls into our study. Concerning **(H6)**, we have

$$0 \leq \langle b(\lambda q)\lambda q, \lambda q\rangle = \lambda^2(1 - \lambda^2 g(\lambda q)) \leq C.$$

We cannot conclude for a limit for all functions g; it means that the last assumption does not hold in general and this equation is not covered by the classical framework in the whole generality. We relate in detail such a situation below.

3. We turn to an explicit example of pde like (2.4) which leads to a geometrical equation whose set of singularity is exactly $\mathcal{D}$ and is not removable. Consider

$$\frac{\partial u}{\partial t} - \frac{f(Du)}{(1+|Du|^2)^2}\langle D^2uDu, Du\rangle = 0 \quad \text{in } \mathbb{R}^N \times (0,T), \tag{2.29}$$

where $f : \mathbb{R}^N \to \mathbb{R}$ is any bounded, nonnegative function. In this case,

$$b(q) = \frac{f(q)}{(1+|q|^2)^2} q \otimes q.$$

Assumptions **(H1)**–**(H3)** are clearly satisfied and, for every $q \in \mathbb{R}^N$, $b(\lambda q) \to 0$ as $\lambda \to \pm\infty$; thus **(H4)** holds. It follows that this equation is covered by "the very singular case" of this section. It leads to a geometrical equation like (2.6) with

$$F(p,X) = -\mathrm{Tr}\left[f\left(-\frac{p_x}{p_y}\right)\frac{p_x \otimes p_x}{(|p_x|^2+p_y^2)^2}(p_y^2 X_{xx} - 2p_y X_{xy} \otimes p_x + X_{yy} p_x \otimes p_x)\right],$$

for every $p = (p_x, p_y) \in \mathbb{R}^{N+1}$ and $X \in \mathcal{S}_{N+1}$.

For simplicity, set $N = 1$ and $f(q) = 1 + \cos q$. It follows $F^*((p_x,0),X) = 0$ and $F_*((p_x,0),X) = -2X_{yy}$, for every $p = (p_x, 0), p_x \neq 0$ and $X \in \mathcal{S}_2$ such that $X_{yy} > 0$. Therefore, in general

$$F^* \neq F_* \quad \text{on } \mathcal{D} = \{p : p_y = 0\}.$$

It shows that we cannot remove the singularities of F outside 0. Thus (2.29) does not satisfy the assumptions of Section 2.3.

2.5 The level-set approach

In this section, for the sake of completeness, we recall the basic ideas of the level-set approach and we apply them to Equation (2.6) both in the classical and very singular framework. We refer to [45], [32], [22], etc. for a more complete description of this approach.

We are given a triplet $(\Gamma_0, \Omega_0^+, \Omega_0^-)$, where Ω_0^+, Ω_0^- are disjoint open subsets of $\mathbb{R}^{N+1}$ and $\Gamma_0 = \left(\Omega_0^+ \cup \Omega_0^-\right)^C$. In general, one has in mind $\Gamma_0 = \partial\Omega_0^+ = \partial\Omega_0^-$. Note that these sets form a partition of $\mathbb{R}^{N+1}$ and Γ_0 can be thought as being an hypersurface.

Let v_0 be any uniformly continuous function whose 0-level-set is exactly Γ_0, namely,

$$\Gamma_0 = \{z \in \mathbb{R}^{N+1} : v_0(z) = 0\}, \tag{2.30}$$

and such that

$$\{z \in \mathbb{R}^{N+1} : v_0(z) > 0\} = \Omega_0^+ \quad \text{and} \quad \{z \in \mathbb{R}^{N+1} : v_0(z) < 0\} = \Omega_0^-. \tag{2.31}$$

This choice of signs defines an orientation of Γ_0 making possible to distinguish an "interior," Ω_0^+, and an "exterior,"Ω_0^-. Secondly, it is always possible to find such a function v_0 by taking, for example, the *signed-distance* to Γ_0 defined by

$$\mathrm{d}(z,\Gamma_0) := \begin{cases} +\mathrm{dist}(z,\Gamma_0) & \text{if } z \in \Omega_0^+, \\ -\mathrm{dist}(z,\Gamma_0) & \text{if } z \in \Omega_0^-, \end{cases} \tag{2.32}$$

where dist denotes the usual positive distance. Clearly $\mathrm{d}(\cdot,\Gamma_0)$ is Lipschitz continuous in $\mathbb{R}^{N+1}$.

We then define the *generalized evolution* of $(\Gamma_0,\Omega_0^+,\Omega_0^-)$ by the family $(\Gamma_t,\Omega_t^+,\Omega_t^-)_{t\geq 0}$, using the

Theorem 2.5.1 *Under the assumptions of Theorem 2.3.1 or 2.4.2, there exists a unique solution v of (2.6) in $UC(\mathbb{R}^{N+1}\times(0,+\infty))$ with initial data v_0. Moreover, if $\tilde{v}_0\in UC(\mathbb{R}^{N+1})$ satisfies*

$$\{\tilde{v}_0=0\}=\Gamma_0,\quad \{\tilde{v}_0>0\}=\Omega_0^+\quad \textit{and}\quad \{\tilde{v}_0<0\}=\Omega_0^-,$$

and if $\tilde{v}\in UC(\mathbb{R}^{N+1}\times(0,+\infty))$ is the viscosity solution of (2.6) with initial data $\tilde{v}_0$, then

$$\begin{aligned}\{v(\cdot,t)>0\}&=\{\tilde{v}(\cdot,t)>0\}:=\Omega_t^+,\\ \{v(\cdot,t)<0\}&=\{\tilde{v}(\cdot,t)<0\}:=\Omega_t^-,\\ \{v(\cdot,t)=0\}&=\{\tilde{v}(\cdot,t)=0\}:=\Gamma_t.\end{aligned}$$

This results implies that the family $(\Gamma_t,\Omega_t^+,\Omega_t^-)_{t\geq 0}$ exists and is uniquely defined independently of the choice of the representant $v_0\in UC(\mathbb{R}^{N+1})$ satisfying (2.30) and (2.31). The set $\bigcup_{t\geq 0}\Gamma_t\times\{t\}$ is called the *front* associated to Γ_0 by (2.6) and Γ_t is the *front at time t*. Note that, at least formally, Γ_t evolves with a normal velocity equal to

$$\mathcal{V}_n(z)=-F(D\mathrm{d}(\cdot,\Gamma_t)(z),D^2\mathrm{d}(\cdot,\Gamma_t)(z)),$$

for $z\in\Gamma_t$.

Proof of Theorem 2.5.1. We give a proof inspired by Ishii's one (see [63]). We only show that if $\{v_0>0\}\subset\{\tilde{v}_0>0\}$ then this inclusion remains true for all $t\geq 0$, i.e. $\{v(\cdot,t)>0\}\subset\{\tilde{v}(\cdot,t)>0\}$. The other inclusions are obtained by straightforward adaptations. From Theorem 2.3.1 or 2.4.2, $v,\tilde{v}\in UC(\mathbb{R}^{N+1}\times[0,+\infty))$ and we recall that the Hyperbolic Tangent function (denoted by tanh) is a bounded uniformly continuous increasing function. Then, using Lemma 2.4.1, we obtain that $\tanh(v)$ is a bounded uniformly continuous solution of (2.6) with the bounded initial data $\tanh(v_0)$. Next, we introduce the uniformly continuous increasing function $\theta^+(r):=\max(r,0)$ and claim, thanks to Lemma 2.4.1 once more, that $\theta^+\circ\tilde{v}$ and $\theta^+\circ\tanh(v)$ are both uniformly continuous solutions of (2.6). Finally, we introduce the lower-semicontinuous function

$$\theta(r)=\begin{cases}+2 & \text{if}\ \ r>0,\\ 0 & \text{if}\ \ r\leq 0,\end{cases}$$

and observe that $\theta\circ\theta^+\circ\tilde{v}$ is a lower semicontinuous supersolution of (2.6). In fact, the previous changes are made in order to obtain the suitable initial condition

$$\theta\circ\theta^+\circ\tilde{v}(\cdot,0)\geq\theta^+\circ\tanh(v(\cdot,0)),$$

which follows easily from the assumption $\{v_0 > 0\} \subset \{\tilde{v}_0 > 0\}$. Since $\theta^+ \circ \tanh(v)$ is uniformly continuous, we apply the comparison result 2.4.1 and get that, for all $t \geq 0$,

$$\theta \circ \theta^+(\tilde{v}(\cdot,t)) \geq \theta^+ \circ \tanh(v(\cdot,t)).$$

We obtain $\{v(\cdot,t) > 0\} \subset \{\tilde{v}(\cdot,t) > 0\}$ which ends the proof. □

2.6 Connection between geometrical and quasilinear pdes. Application to uniqueness

In this section, we specify the connections between (2.4) and (2.6) initiated in Section 2.2, and in particular in terms of uniqueness for (2.4). Let u be a continuous viscosity solution of (2.4) with initial data $u_0 \in C(\mathbb{R}^N)$ and v be the solution of (2.6) with initial data $\mathrm{d}(\cdot, \mathrm{Graph}(u_0))$. The main question is whether

$$\mathrm{Graph}(u(\cdot,t)) = \{(x,y) \in \mathbb{R}^{N+1} : y - u(x,t) = 0\}$$

and the front

$$\Gamma_t = \{(x,y) \in \mathbb{R}^{N+1} : v(x,y,t) = 0\}$$

coincide for all $t \geq 0$ or not. If the answer is yes, this obviously provides a uniqueness result for (2.4) since the Γ_t's are uniquely determined because of Theorem 2.5.1. And one may think that it is indeed yes by applying Theorem 2.5.1 together with Lemma 2.2.1 (ii) with initial data $\tilde{v}_0 = \tanh(y - u_0(x))$, which is a particular representation of $\mathrm{Graph}(u_0)$. Unfortunately, $\tilde{v}_0$ is not uniformly continuous if u_0 is not uniformly continuous and, as we pointed out in the introduction, we do not know how to prove Theorem 2.5.1 replacing "$\tilde{v}_0 \in UC(\mathbb{R}^{N+1})$" by "$\tilde{v}_0 \in C_\mathrm{b}(\mathbb{R}^{N+1})$"; we do not know even if such a result is true.

Nevertheless, the inclusion used in Section 2.2 to derive the geometrical pde is always true.

Theorem 2.6.1 *Suppose that **(H1)–(H4)** hold. Let u be a viscosity subsolution (respectively supersolution) of (2.4) with initial data $u_0 \in C(\mathbb{R}^N)$ and v be a viscosity solution of (2.6) with initial data* $\mathrm{d}(\cdot, \mathrm{Graph}(u_0))$. *For every $t \in [0, +\infty)$, we have*

$$\mathrm{Graph}(u(\cdot,t)) \subset \{(x,y) \in \mathbb{R}^{N+1} : v(x,y,t) \leq 0\}$$

$$(\textit{ respectively } \quad \mathrm{Graph}(u(\cdot,t)) \subset \{(x,y) \in \mathbb{R}^{N+1} : v(x,y,t) \geq 0\} \ ).$$

If u is a solution of (2.4), then

$$\mathrm{Graph}(u(\cdot,t)) \subset \Gamma_t \quad \textit{for all } t \in [0, +\infty),$$

where $(\Gamma_t)_{t\geq 0}$ is the generalized evolution associated to $\Gamma_0 = \mathrm{Graph}(u_0)$.

Proof of Theorem 2.6.1. Suppose that u is a subsolution. Define the nondecreasing function $\theta^+(r) := \max(r, 0)$. For all $z = (x, y) \in \mathbb{R}^N \times \mathbb{R}$, we have,

$$\tanh\left[\theta^+(y - u_0(x))\right] \geq \tanh\left[\mathrm{d}(z, \mathrm{Graph}(u_0))\right],$$

since, on the one hand, $|y - u_0(x)| \geq \mathrm{dist}(z, \mathrm{Graph}(u_0))$; and, on the other hand, if $y \leq u_0(x)$, then $\mathrm{d}(z, \mathrm{Graph}(u_0)) \leq 0$ (for the definition of d, see (2.32)). From Lemma 2.2.1 (ii) and from the invariance of supersolution of (2.6) under nondecreasing changes of variables (see Lemma 2.4.1), we know that the function $\tanh[\theta^+(y - u(x,t))]$ is a supersolution of (2.6) with initial data $\tanh[\theta^+(y - u_0(x))]$. Moreover, the function $\tanh(v)$ is a solution (thus a subsolution) of (2.6) with initial data $\tanh(v_0)$. Applying Theorem 2.4.1 (see Remark 2.4.1), we get

$$\tanh\left[\theta^+\big(y - u(x,t)\big)\right] \geq \tanh[v(z,t)].$$

Thus, $y = u(x,t)$ implies that $v(z,t) \leq 0$ which proves the first inclusion. If u is a supersolution, we repeat the same arguments with $\theta_-(r) := \min(r, 0)$. We get the another inclusion. To conclude for the last statement, it suffices to notice, on the one hand, that u is a solution provided that u is both a sub- and a supersolution and, on the other hand, that

$$\Gamma_t = \{z \in \mathbb{R}^{N+1} : v(z,t) \leq 0\} \cap \{z \in \mathbb{R}^{N+1} : v(z,t) \geq 0\}.$$

It achieves the proof of the theorem. □

In fact, the uniqueness for (2.4) and the so-called "fattening phenomena" for the front are closely related as shown by the

Theorem 2.6.2 *Assume that **(H1)**–**(H4)** hold and let $u_0 \in C(\mathbb{R}^N)$. Suppose that the front $\bigcup_{t\geq 0} \Gamma_t \times \{t\}$ associated to* Graph(u_0) *has empty interior in $\mathbb{R}^{N+1} \times [0, +\infty)$. Then (2.4) has at most one continuous viscosity solution with initial data u_0.*

We point out that Theorem 2.6.2 provides uniqueness only in the class of continuous functions. But discontinuous solutions may also exist if the front looks like a rake for instance. This result says nothing about the existence of solutions ; it may be impossible to put any continuous graph in the front.

In the literature, the "fattening phenomena" may have different meanings. In Theorem 2.6.2, we use the standard topological meaning. We first want to remark that assuming that Γ_t has empty interior in $\mathbb{R}^{N+1}$ for all $t \geq 0$ is stronger than assuming that $\bigcup_{t\geq 0} \Gamma_t \times \{t\}$ has empty interior in $\mathbb{R}^{N+1} \times [0, T]$. In fact, under our assumptions it turns out to be equivalent. The proof of this equivalence comes from the preservation of inclusion of sets under motions governed by (2.6) and the fact that, if we can put a ball in the front at a time t, this ball cannot shrink instantaneously. We skip the proof and refer to the one of Theorem 2.7.1 which is similar.

In Barles and Souganidis [23], a "no-interior condition" is considered, namely

$$\bigcup_{t\geq 0} \Gamma_t \times \{t\} = \partial\left(\bigcup_{t\geq 0} \Omega_t^+ \times \{t\}\right) = \partial\left(\bigcup_{t\geq 0} \Omega_t^- \times \{t\}\right). \tag{2.33}$$

This condition is stronger than the topological one. When it is satisfied, we have a better result.

Theorem 2.6.3 *Assume **(H1)–(H4)** and let $u_0 \in C(\mathbb{R}^N)$. Suppose that (2.33) holds for the front associated to* $\mathrm{Graph}(u_0)$. *If u and $\tilde{u}$ are (possibly discontinuous) viscosity solutions of (2.4), then $u_* = \tilde{u}_*$ and $u^* = \tilde{u}^*$ in $\mathbb{R}^N \times [0, +\infty)$.*

Contrarily to Theorem 2.6.2, this theorem provides a "weak" uniqueness result for discontinuous viscosity solutions of (2.4). It is worth pointing out that stronger results providing equalities like $u^* = \tilde{u}_*$ and $u_* = \tilde{u}^*$ in $\mathbb{R}^N \times [0, +\infty)$ cannot be obtained by such geometrical approach since a discontinuity of u or $\tilde{u}$ can appear or, on the contrary, be removed by a slight rotation of the axis in $\mathbb{R}^{N+1}$ and therefore such discontinuities have no real geometrical meaning.

We refer the reader to Barles, Soner and Souganidis [22], Ilmanen [56] and Soner [78] for a more complete discussion and results about the "fattening phenomena" or "non-empty interior difficulty".

We turn to the proofs.

Proof of Theorem 2.6.2. Suppose that there exist two solutions $u_1, u_2 \in C(\mathbb{R}^N \times [0, +\infty))$ of (2.4) with initial data u_0 and define v to be the solution of (2.6) with initial data $v_0 = \mathrm{d}(\cdot, \mathrm{Graph}(u_0))$. From the level-set approach, we have $\Gamma_t = \{v(\cdot, t) = 0\}$. We will see that, if u_1 and u_2 are different, then the front has nonempty interior in $\mathbb{R}^{N+1} \times [0, +\infty)$. If $u_1 \neq u_2$, we can suppose that there exists $(x_0, t_0) \in \mathbb{R}^N \times [0, +\infty)$ such that

$$u_1(x_0, t_0) - u_2(x_0, t_0) = \varepsilon > 0.$$

By continuity of u_1 and u_2, there exists some ball $B(x_0, \rho)$, $\rho > 0$ and some $\tau > 0$ such that

$$u_1(x, t) - u_2(x, t) \geq \frac{\varepsilon}{2} > 0 \quad \text{in } B(x_0, \rho) \times [t_0, t_0 + \tau]. \tag{2.34}$$

But, from Theorem 2.6.1,

$$\mathrm{Graph}(u_1(\cdot, t)), \mathrm{Graph}(u_2(\cdot, t)) \subset \{v(\cdot, t) = 0\} \quad \text{for all } t \geq 0.$$

To conclude, it suffices to show that

$$B(x_0, \rho) \times [t_0, t_0 + \tau] \subset \bigcup_{t \geq 0} \Gamma_t \times \{t\}.$$

We need a lemma whose proof is postponed.

Lemma 2.6.1 *Under Assumptions **(H1)–(H4)**, let $v \in UC(\mathbb{R}^{N+1} \times [0, +\infty))$ be a solution of (2.6) with initial data v_0. If $y \mapsto v_0(x, y)$ is nondecreasing for all $x \in \mathbb{R}^N$, then $y \mapsto v(x, y, t)$ is nondecreasing for all $(x, t) \in \mathbb{R}^N \times [0, +\infty)$. In particular, the result holds if v is the solution associated to the initial condition $v_0 = \mathrm{d}(\cdot, \mathrm{Graph}(u_0))$ where $u_0 \in C(\mathbb{R}^N)$.*

From this lemma, we obtain

$$v(x, y, t) = 0 \quad \text{for all } (x, y, t) \in B(x_0, \rho) \times [u_2(x, t_0), u_1(x, t_0)] \times [t_0, t_0 + \tau].$$

By (2.34), we obtain that $B(x_0, \rho) \times [u_2(x, t_0), u_1(x, t_0)] \times [t_0, t_0 + \tau]$ has nonempty interior in $\mathbb{R}^{N+1} \times [0, +\infty)$ which ends the proof. □

Proof of Lemma 2.6.1. By assumption, $v_0(x, y+h) \geq v_0(x, y)$ for all $x \in \mathbb{R}^{N+1}, y \in \mathbb{R}$ and $h > 0$. From the comparison result (see Theorem 2.4.1) it follows $v(\cdot, \cdot + h, t) \geq v(\cdot, \cdot, t)$ for all $t \geq 0$, since $v(\cdot, \cdot + h, \cdot)$ is a solution of (2.6) with initial data $v_0(\cdot, \cdot + h)$. It proves the first part of the lemma.

It remains to show that the function $(x, y) \mapsto \mathrm{d}((x, y), \mathrm{Graph}(u_0))$ is nondecreasing in the y variable when u_0 is continuous. To this end, consider $x \in \mathbb{R}^N$ and $y_2 \geq y_1$. We suppose that $y_2 \geq y_1 \geq u_0(x)$. Indeed, the case $u_0(x) \geq y_2 \geq y_1$ can be treated in the same way with straightforward adaptations and the case $y_2 \geq u_0(x) \geq y_1$ is obvious. Assume by contradiction that

$$0 \leq r_2 := \mathrm{d}((x, y_2), \mathrm{Graph}(u_0)) < \mathrm{d}((x, y_1), \mathrm{Graph}(u_0)) =: r_1,$$

and define $u_{0|\overline{B}(x,r_1)}$ as the restriction of u_0 to the ball $\overline{B}(x, r_1)$. From the definition of d as an infimum, the hypothesis $y_1 \geq u_0(x)$ and the continuity of u_0, it follows that

$$\mathrm{Graph}(u_{0|\overline{B}(x,r_1)}) \cap \overline{B}((x, y_1), r_1) \subset \partial^- B((x, y_1), r_1),$$

where $\partial^- B((x, y_1), r_1)$ stands for the part $\partial B((x, y_1), r_1)$ of the boundary of $\overline{B}((x, y_1), r_1)$ lying in the half-space $\{y \leq y_1\}$. Since $y_2 \geq y_1$ and $r_2 < r_1$, we get

$$\mathrm{Graph}(u_{0|\overline{B}(x,r_1)}) \cap B((x, y_2), r_2) = \emptyset$$

which gives a contradiction. □

Proof of Theorem 2.6.3. Let us show that $u_* = \tilde{u}_*$. We argue by contradiction, assuming that there exists $(\bar{x}, \bar{t}) \in \mathbb{R}^N \times [0, +\infty)$ such that $u_*(\bar{x}, \bar{t}) > \tilde{u}_*(\bar{x}, \bar{t})$. From Theorem 2.6.1, we have, for all $t \geq 0$, $\mathrm{Graph}(\tilde{u}(\cdot, t)) \subset \Gamma_t = \{v(\cdot, t) = 0\}$, where v is the solution of (2.6) with initial data $\mathrm{d}(\cdot, \mathrm{Graph}(u_0))$. It follows from (2.33) that

$$\bigcup_{t \geq 0} \mathrm{Graph}(\tilde{u}(\cdot, t)) \times \{t\} \subset \overline{\bigcup_{t \geq 0} \Omega_t^+ \times \{t\}}\ ;$$

thus, there exists a sequence $(x_n, y_n, t_n) \in \mathbb{R}^{N+1} \times [0, +\infty)$ such that $(x_n, y_n, t_n) \to (\bar{x}, \tilde{u}_*(\bar{x}, \bar{t}), \bar{t})$ when $n \to +\infty$ and $v(x_n, y_n, t_n) > 0$ for all $n \geq 0$. From the nondecrease of v in y (Lemma 2.6.1), we have $y_n > u(x_n, t_n)$ since $v(x_n, u(x_n, t_n), t_n) = 0$. It follows

$$u_*(x, t) \leq \liminf_{n \to +\infty} u(x_n, t_n) \leq \liminf_{n \to +\infty} y_n = \tilde{u}_*(x, t)$$

which is a contradiction. We prove $u^* = \tilde{u}^*$ in the same way. □

The last result of this section is related to the empty interior condition of Theorem 2.6.2 and is inspired from the related results of Evans and Spruck [46] in the mean curvature case.

If $u_0 \in C(\mathbb{R}^N)$ and if $v_0 = \mathrm{d}(\cdot, \mathrm{Graph}(u_0))$, then the subsets $\{v_0 = \lambda\}, \lambda \in \mathbb{R}$ are the graphs of functions $u_0^\lambda \in C(\mathbb{R}^N)$. More precisely, for $\lambda \geq 0$, the function $\omega(x, \lambda) := u_0^\lambda(x)$ (respectively $\omega(x, \lambda) := u_0^{-\lambda}(x)$) is the unique viscosity solution of

$$\frac{\partial \omega}{\partial \lambda} - \sqrt{1 + |D\omega|^2} = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty),$$

(respectively

$$\frac{\partial \omega}{\partial \lambda} + \sqrt{1+|D\omega|^2} = 0 \quad \text{in } \mathbb{R}^N \times (0,+\infty) \;).$$

We refer to Barles [9] for a simple proof of this claim. Our result is the

Proposition 2.6.1 *Assume* ***(H1)–(H4)****. Except for a countable subset of values of* λ*, the fronts associated to the evolution of* $\mathrm{Graph}(u_0^\lambda)$ *has empty interior in* $\mathbb{R}^{N+1} \times [0,+\infty)$*. In particular, there exists at most one continuous viscosity solution* u^λ *of (2.4) with initial data* u_0^λ*.*

We may interpret this result by saying that non uniqueness for (2.4) is a "rare" event.

Noticing that $u_0^\lambda \downarrow u_0$ in $C(\mathbb{R}^N)$ as $\lambda \downarrow 0^+$, we obtain that we can approach any $u_0 \in C(\mathbb{R}^N)$ in a monotone way by a sequence of u_0^λ for which (2.4) has at most one continuous solution. The interest of this result is that the u_0^λ's have in general the same behavior as u_0. It means that we have actually uniqueness for a large class of initial data including functions with arbitrary growth.

Proof of Proposition 2.6.1. Let v be the unique solution of (2.6) with initial data $v_0 = \mathrm{d}(\cdot, \mathrm{Graph}(u_0))$. Since, for every $\lambda \in \mathbb{R}$, $v - \lambda$ is the unique uniformly continuous solution of (2.6) with initial data $v_0 - \lambda$, at each time t, the front Γ_t^λ associated to $\mathrm{Graph}(u_0^\lambda)$ coincides with $\{v(\cdot,t) = \lambda\}$. In particular, the fronts are disjoint for different values of λ and it follows, from there, that the family of values of λ such that $\bigcup_{t\geq 0} \Gamma_t^\lambda \times \{t\}$ has nonempty interior is countable. To conclude, it is sufficient to apply Theorem 2.6.2. □

2.7 A local L^∞ a priori bound

In this section, we use the relations between (2.4) and (2.6) to provide a local L^∞-bound for the solutions of (2.4).

In order to state the main result of this section, we introduce, for any function $u_0 \in C(\mathbb{R}^N)$,

$$M_{u_0}(x,R) = \max\{u_0(y), y \in \overline{B}(x,R)\} \quad \text{and} \quad m_{u_0}(x,R) = \min\{u_0(y), y \in \overline{B}(x,R)\}.$$

We have the following

Theorem 2.7.1 *Under assumptions* **(H1)–(H4)***, if* $u \in C(\mathbb{R}^N \times [0,+\infty))$ *is a solution of (2.4) with initial data* $u_0 \in C(\mathbb{R}^N)$*, then there exists a positive constant* C *such that, for all* $x \in \mathbb{R}^N$*,* $t \geq 0$*, we have*

$$m_{u_0}(x, \sqrt{2Ct}) - \sqrt{2Ct} \leq u(x,t) \leq M_{u_0}(x, \sqrt{2Ct}) + \sqrt{2Ct}.$$

Remark 2.7.1 This local L^∞-bound is a direct consequence of the level-set approach and it justifies the fact that we need at least some kind of degeneracy on b in the gradient variable as it is implied by **(H1)–(H4)**; indeed, clearly, such a bound does not hold for the heat equation and therefore one cannot hope that such an approach applies for this equation.

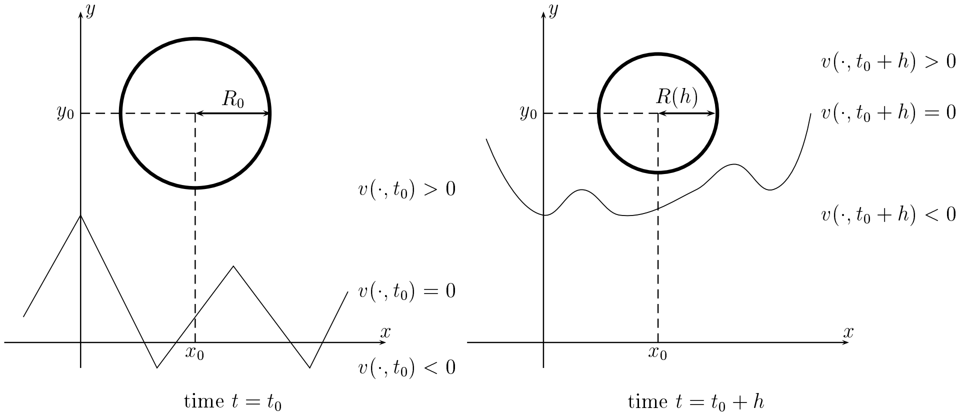

FIG. 2.1 – *Evolution of a graph with a ball which is put above it*

Proof of Theorem 2.7.1. The basic idea is that the geometrical evolution governed by (2.6) preserves the inclusion of sets. Thus one can expect that the evolution of balls initially put "under" (or "over") the graph of a solution of (2.4) will provide some control on the growth. This fact is illustrated in Figure 2.1 in the case of the mean curvature equation.

We take $v_0(z) = \mathrm{d}(z, \mathrm{Graph}(u_0)) \in UC(\mathbb{R}^{N+1})$ (where d is defined by (2.32)) and let v be the unique uniformly continuous solution of (2.6) with initial data v_0. In order to prove the result, we aim at comparing v with subsolutions like those which appear in the following Lemma whose proof is postponed.

Lemma 2.7.1 *We suppose that* **(H1), (H2)** *and* **(H3)** *hold. Fix $R_0 > 0$, $x_0 \in \mathbb{R}^N$ and $y_0 \in \mathbb{R}$. Let $\Psi : \mathbb{R} \to \mathbb{R}$ be any smooth nondecreasing function. Then the function φ, defined for every $(x, y, t) \in \mathbb{R}^N \times \mathbb{R} \times [0, +\infty)$, by*

$$\varphi(x, y, t) = \Psi(R_0^2 - 2Ct - |x - x_0|^2 - (y - y_0)^2),$$

where $C = N(K_1 + K_2 + K_3) + 1$, is a (classical) strict subsolution of (2.6).

Let $x_0 \in \mathbb{R}^N, t_0 \in (0, +\infty)$ and $y_0 = M_{u_0}(x_0, \sqrt{2Ct_0}) + \sqrt{2Ct_0}$, where C is taken as in Lemma 2.7.1. It follows

$$\overline{B}((x_0, y_0), \sqrt{2Ct_0}) \subset \{(x, y) \in \mathbb{R}^{N+1} : y > u_0(x)\},$$

which implies

$$\mathrm{d}\left(\cdot, \overline{B}((x_0, y_0), \sqrt{2Ct_0})\right) \leq \mathrm{d}(\cdot, \mathrm{Graph}(u_0)) = v_0. \tag{2.35}$$

Let us define

$$\varphi(x, y, t) = \Psi(2Ct_0 - 2Ct - |x - x_0|^2 - (y - y_0)^2),$$

with $\Psi(z) = z/2\sqrt{2Ct_0}$ for all $z \in \mathbb{R}$. The function Ψ satisfies the assumptions of Lemma 2.7.1. For clarity, we set $r = (|x - x_0|^2 + (y - y_0)^2)^{1/2}$. From (2.35), we get

$$\varphi(\cdot, \cdot, 0) = \frac{\sqrt{2Ct_0} + r}{2\sqrt{2Ct_0}}(\sqrt{2Ct_0} - r) \leq \sqrt{2Ct_0} - r = \mathrm{d}\left(\cdot, \overline{B}((x_0, y_0), \sqrt{2Ct_0})\right) \leq v_0.$$

Now, since φ is a function with quadratic growth at infinity and v is uniformly continuous, we have that

$$\min_{\mathbb{R}^{N+1}\times[0,T]}\{v-\varphi\}$$

is achieved for every $T>0$ if one assumes that it is positive. Using Lemma 2.7.1, φ is a strict smooth subsolution of (2.6) ; thus the minimum is necessarily achieved at $t=0$ which is a contradiction. Since the previous arguments hold for every $T>0$, we get finally

$$\varphi \le v \quad \text{in } \mathbb{R}^{N+1}\times[0,+\infty). \tag{2.36}$$

By Lemma 2.6.1, we have

$$\operatorname{Graph}(u(\cdot,t)) \subset \Gamma_t(v) \quad \text{for every } t\ge 0. \tag{2.37}$$

From (2.36) and (2.37), it follows that, for all $t\ge 0$,

$$\{\varphi(\cdot,t)>0\}\subset\{v(\cdot,t)>0\}\subset\{(x,y)\in\mathbb{R}^{N+1} : y\ge u(x,t)\}.$$

But

$$\varphi(x,y,t)>0 \iff (x,y)\in B((x_0,y_0),\sqrt{2C(t_0-t)}).$$

By letting $t\to t_0$ and by using the assumed continuity of u, we obtain

$$u(x_0,t_0)\le y_0 = M_{u_0}(x_0,\sqrt{2Ct_0})+\sqrt{2Ct_0}.$$

The opposite inequality is obtained with straightforward adaptations. □

We end the section with the proof of the lemma and an example.

Proof of Lemma 2.7.1. Without loss of generality, we can suppose that $x_0=0$ and $y_0=0$. Moreover, from Lemma 2.4.1, we can suppose that $\Psi(z)=z$ for every $z\in\mathbb{R}$. From **(H1)**, **(H2)** and **(H3)**, we get

$$|F_*(D\varphi,D^2\varphi)|\le N(K_1+K_2+K_3)|D^2\varphi|\le 2(C-1)$$

where we set $C=N(K_1+K_2+K_3)+1$. It follows

$$\frac{\partial\varphi}{\partial t}+F_*(D\varphi,D^2\varphi)\le -2C+2(C-1)\le -2<0,$$

which achieves the proof. □

Example 2.7.1 Evolution of balls in the case of the mean curvature equation (2.1). We recall that, in the case of (2.1), b is given by (2.11). Following the computations of Lemma 2.7.1, we have

$$\frac{\partial\varphi}{\partial t}-\operatorname{Tr}\left[b\left(-\frac{D_x\varphi}{D_y\varphi}\right)\left(D^2_{xx}\varphi-2D^2_{xy}\varphi\otimes\frac{D_x\varphi}{D_y\varphi}+D^2_{yy}\varphi\frac{D_x\varphi}{D_y\varphi}\otimes\frac{D_x\varphi}{D_y\varphi}\right)\right]=-2(C-N).$$

By taking $C = N$, we obtain that φ is in fact a classical solution of (2.6). Thus, by the level-set approach, it follows that the 0-level-set of φ evolves according to its mean curvature. An easy computation shows that

$$\Omega_0^+ = \{\varphi(\cdot,\cdot,0) > 0\} = B((x_0,y_0),R_0), \quad \Gamma_0 = \{\varphi(\cdot,\cdot,0) = 0\} = \partial B((x_0,y_0),R_0),$$

and, for every $t \geq 0$,

$$\Omega_t^+ = \{\varphi(\cdot,\cdot,t) > 0\} = B((x_0,y_0),R(t)), \quad \Gamma_t = \{\varphi(\cdot,\cdot,t) = 0\} = \partial B((x_0,y_0),R(t)),$$

where $R(t) = (R_0^2 - 2Nt)^{1/2}$. We recover by this way the well-known result of Evans and Spruck [45, Section 7.1] : balls remains balls for the mean curvature motion and they shrink into a point for $t^* = R_0^2/2N$.

2.8 The boundary of the front. Existence of discontinuous solutions

Theorem 2.6.1 provides the first connections between the front and the graphs of the solutions of (2.4) when they exist. In this section, we describe more precisely the structure of the front and obtain the existence of discontinuous solutions to (2.4).

For any continuous function u_0, we consider the generalized evolution $(\Gamma_t)_{t\geq 0}$ of Graph(u_0) and the uniformly continuous solution v of (2.6) with initial data $v_0 = \mathrm{d}(\cdot, \mathrm{Graph}(u_0))$. For every $(x,t) \in \mathbb{R}^N \times [0,+\infty)$, we define

$$u^+(x,t) := \sup\{y \in \mathbb{R} : v(x,y,t) \leq 0\} \quad \text{and} \quad u^-(x,t) := \inf\{y \in \mathbb{R} : v(x,y,t) \geq 0\}.$$

Note that the functions $u^+(\cdot,t)$ and $u^-(\cdot,t)$ are defined such that their graphs are the "upper-boundary" and the "lower-boundary" of the front Γ_t at each time t : see Figure 2.2. We have the first properties

Lemma 2.8.1 *Under assumptions **(H1)**–**(H4)**, the functions u^+ and u^- are locally bounded in $\mathbb{R}^N \times [0,+\infty))$. Moreover $u^+ \in USC(\mathbb{R}^N \times [0,+\infty))$ and $u^- \in LSC(\mathbb{R}^N \times [0,+\infty))$.*

Proof of Lemma 2.8.1. We make the proof for u^+, the one for u^- being similar. We start by proving that u^+ is well-defined and locally bounded. Looking at the proof of Theorem 2.7.1, we see that inequality (2.36) implies that, for every $(x_0,y_0,t_0) \in \mathbb{R}^N \times \mathbb{R} \times [0,+\infty)$, there exists a constant $M > 0$ such that

$$v > 0 \quad \text{in } \overline{B}((x_0,y_0),M) \times [0,t_0/2].$$

Note that M does not depend on y_0 in the sense that $v > 0$ in every $\overline{B}((x_0,y),M) \times [0,t_0/2]$ with $y \geq y_0$, by nondecrease of $y \mapsto v(x,y,t)$ for every (x,t) (see Lemma 2.6.1). It proves that $u^+ \leq y_0$ in a neighborhood of $(x_0,t_0/2)$. The same reasoning holds with straightforward adaptations to prove that u^+ is locally bounded from below.

We turn to the proof of the upper-semicontinuity of u^+. Consider any sequence of points $((x_n,y_n,t_n))_{n\in\mathbb{N}}$ such that $(x_n,y_n,t_n) \in \mathcal{H} := \{(x,y,t) \in \mathbb{R}^{N+1} \times [0,+\infty) : y \leq u^+(x,t)\}$ and $(x_n,y_n,t_n) \to (x,y,t)$ as $n \to +\infty$. For every n, we have $v(x_n,y_n,t_n) \leq 0$. Since v is continuous, by sending n to infinity, we get $v(x,y,t) \leq 0$ which proves that $(x,y,t) \in \mathcal{H}$; thus $\mathcal{H}$ is closed. It ends the proof. □

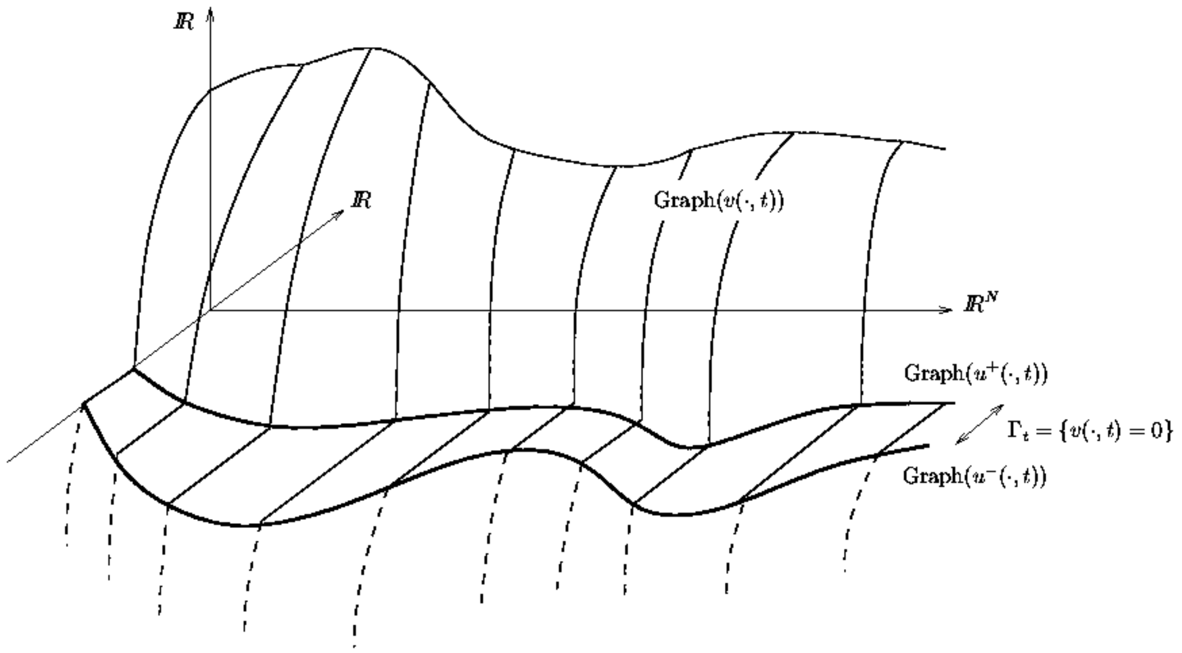

FIG. 2.2 – *front which fattens at time* $t > 0$

Theorem 2.8.1 *Suppose that **(H1)**–**(H4)** hold. Let* $u_0 \in C(\mathbb{R}^N)$ *and* v *be the solution of (2.6) associated to the initial data* $v_0 = \mathrm{d}(\cdot, \mathrm{Graph}(u_0))$*. Then* u^+ *and* u^- *are (possibly discontinuous) viscosity solutions of (2.4) with initial data* u_0*. Moreover,* u^+ *and* u^- *are respectively the maximal subsolution and the minimal supersolution of (2.4) with initial data* u_0*.*

We refer to Figure 2.2 for an illustration of this theorem. Let us point out that this result provides only the existence of a discontinuous viscosity solution to (2.4) for any continuous initial data. We refer to Section 2.9 for optimal results of regularity of u^+ and u^-. We give a geometrical proof of the theorem using the fact that characteristic functions of sets which evolve are discontinuous solutions of (2.6).

We first introduce some notations. For any subset $A \subset \mathbb{R}^{N+1}$, $\mathrm{int}(A)$ denotes the interior of A in $\mathbb{R}^{N+1}$ and $\mathbb{1}_A$ is the characteristic function of A defined, for any $(x,y) \in \mathbb{R}^{N+1}$, by $\mathbb{1}_A(x,y) = 1$ if $(x,y) \in A$ and 0 otherwise. For the sake of simplicity of notations, when the set $A = A_t$ depends on t, we will denote by $\mathbb{1}_{A_t}$ the function $(x,y,t) \mapsto \mathbb{1}_{A_t}(x,y)$.

We need the following lemma due to Barles, Soner and Souganidis [22].

Lemma 2.8.2 *Let* u_0 *and* v *as in Theorem 2.8.1 and consider*

$$\Omega_t^+ = \{v(\cdot,t) > 0\} \quad \text{and} \quad \Upsilon_t = \mathrm{int}\big(\{v(\cdot,t) \geq 0\}\big).$$

Then the functions $\mathbb{1}_{\Omega_t}$ *and* $\mathbb{1}_{\Upsilon_t}$ *are (discontinuous) viscosity solutions of (2.6).*

Proof of Theorem 2.8.1. We make the proof for u^+ calling it u for clarity of notations. The same reasoning holds with easy adaptations for u^-. Let us start by showing that u is a subsolution. Remembering that u is an upper-semicontinuous function by Lemma 2.8.1, we consider a smooth function $\phi(x,t)$ such that $u - \phi$ achieves a global maximum of 0 at $(\bar{x}, \bar{t}) \in \mathbb{R}^N \times (0, +\infty)$. It follows that $u \leq \phi$ and $u(\bar{x}, \bar{t}) = \phi(\bar{x}, \bar{t})$. Set $\psi(x, y, t) = \tanh(y - \phi(x, t))$.

We claim that $(1\!\!1_{\Omega_t})_* - \psi$ achieves a global minimum 0 at $(\bar{x}, u(\bar{x}, \bar{t}), \bar{t})$. By continuity of v, we have $v(\bar{x}, u(\bar{x}, \bar{t}), \bar{t}) = 0$; thus $(1\!\!1_{\Omega_t})_*(\bar{x}, u(\bar{x}, \bar{t}), \bar{t}) = 0$. It implies $((1\!\!1_{\Omega_t})_* - \psi)(\bar{x}, u(\bar{x}, \bar{t}), \bar{t}) = 0$. It remains to check that $((1\!\!1_{\Omega_t})_* - \psi)(x, y, t) \geq 0$ for every (x, y, t). If $(1\!\!1_{\Omega_t})_*(x, y, t) = 0$, then $(x, y, t) \in \{v \leq 0\}$; thus, from Lemma 2.6.1, we have $y \leq u(x, t) \leq \phi(x, y, t)$. We obtain $y - \phi(x, t) \leq 0$ and $((1\!\!1_{\Omega_t})_* - \psi)(x, y, t) \geq 0$ in this case. Now, if $(1\!\!1_{\Omega_t})_*(x, y, t) = 1$, then the same inequality holds since $\tanh \leq 1$. It proves the claim.

We compute the derivatives of ψ and get,

$$\frac{\partial \psi}{\partial t} = -\tanh' \cdot \frac{\partial \phi}{\partial t}, \quad D_x \psi = -\tanh' \cdot D_x \phi, \quad D_y \psi = \tanh',$$

$$D^2_{xx} \psi = \tanh'' \cdot D_x \phi \otimes D_x \phi - \tanh' \cdot D^2_{xx} \phi, \quad D^2_{yy} \psi = \tanh'', \quad D^2_{xy} \psi = -\tanh'' \cdot D_x \phi.$$

By Proposition 2.8.2, $(1\!\!1_{\Omega_t})_*$ is a supersolution of (2.6); writing the viscosity inequality at the point $(\bar{x}, u(\bar{x}, \bar{t}), \bar{t})$, a calculation leads to

$$\frac{\partial \phi}{\partial t} - \operatorname{Tr}\left[b(D_x \phi) D^2_{xx} \phi\right] \leq 0,$$

which shows that u is a viscosity subsolution.

We continue by proving that u_* is a supersolution. Consider a smooth function ϕ such that $u_* - \phi$ achieves a global minimum of 0 at $(\bar{x}, \bar{t}) \in \mathbb{R}^N \times (0, +\infty)$. We claim first that $(1\!\!1_{\Omega_t})^*(\bar{x}, u_*(\bar{x}, \bar{t}), t) = 1$. Otherwise, $(\bar{x}, u_*(\bar{x}, \bar{t}), t)$ lies in the interior of $\bigcup_{t \geq 0} \Gamma_t \times \{t\}$; it means that there exists $\varepsilon > 0$ such that

$$v(x, y, t) = 0 \quad \text{for every } (x, y, t) \in \overline{B}(\bar{x}, \varepsilon) \times [u_*(\bar{x}, \bar{t}) - \varepsilon, u_*(\bar{x}, \bar{t}) + \varepsilon] \times [\bar{t} - \varepsilon, \bar{t} + \varepsilon].$$

By definition of u, it follows that $u(x, t) \geq u_*(\bar{x}, \bar{t}) + \varepsilon$ for every $x \in \overline{B}(\bar{x}, \varepsilon)$ and $t \in [\bar{t} - \varepsilon, \bar{t} + \varepsilon]$. It leads to a contradiction and proves the claim.

Defining ψ as above, we observe that the function $(1\!\!1_{\Omega_t})^* - \psi$ achieves a global maximum point at $\big((\bar{x}, u_*(\bar{x}, \bar{t})), \bar{t}\big)$. Indeed, if $(1\!\!1_{\Omega_t})^*(x, y, t) = 0$, then $(1\!\!1_{\Omega_t})^* - \psi \leq 1$. If $(1\!\!1_{\Omega_t})^*(x, y, t) = 1$, then $(x, y, t) \in \overline{\{v > 0\}} = \{y \geq u_*(x, t)\}$, since u_* is lower-semicontinuous. It follows $y \geq \phi(x, t)$ and $\tanh(y - \phi(x, t)) \geq 0$; thus $(1\!\!1_{\Omega_t})^* - \psi \leq 1$ and we are done in any case. Using that $(1\!\!1_{\Omega_t})^*$ is a subsolution of (2.6) by Proposition 2.8.2, we conclude as above.

It remains to check that the initial condition holds. On the one hand, from the continuity of v, we have $v(x, u(x, 0), 0) = 0$. It implies $u(x, 0) = u_0(x)$ since Γ_0 is exactly the graph of the continuous function u_0. On the other hand, looking at the proof of the supersolution, we see $(x, u_*(x, t), t) \in \overline{\Omega}_t$ for every $(x, t) \in \mathbb{R}^N \times [0, +\infty)$. By continuity of v, we get $v(x, u_*(x, t), t) = 0$. For $t = 0$, it means $u_*(x, 0) = u_0(x)$.

Finally, note that from Theorem 2.6.1, the graphs of all subsolutions of (2.6) lie in $\{v \leq 0\}$ and in the same way the graphs of all supersolutions of (2.6) lie in $\{v \geq 0\}$. Therefore, u^- is the minimal subsolution and u^+ is the maximal supersolution of (2.6); and the proof of the theorem is complete. □

Remark 2.8.1 If the "no-interior condition" (2.33) holds, then Theorem 2.6.3 implies $(u^+)_* = u^-$ and $(u^-)^* = u^+$ in $\mathbb{R}^N \times [0, +\infty)$. But even in this case, we cannot conclude to the existence of a continuous viscosity solution since the front may look like a Heaviside function for instance.

2.9 Fronts with more regularity

As mentioned before, the extremal solutions u^+ and u^- have no regularity in general. We give below some conditions under which they are smooth. It is the case, when the front is associated to (2.4) and a locally Lipschitz initial data u_0, as soon as the solutions u of this equation satisfy local L^∞ and gradient bounds. On one hand, these bounds allows the construction of smooth solutions for any continuous initial data. On the other hand, using approximation methods, we see that this regularity holds for the extremal solutions. We start with a more precise result in the case of the mean curvature equation.

Theorem 2.9.1 *Let $u_0 \in C(\mathbb{R}^N)$. Then the extremal solutions u^+ and u^- of (2.1) with initial data u_0 are in $C^\infty(\mathbb{R}^N \times (0,+\infty)) \cap C(\mathbb{R}^N \times [0,+\infty))$.*

We recall that the smooth existence for $u_0 \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^N)$ is proved in Ecker and Huisken [41] (see also Chou and Kwong [33]) using a gradient estimate. Here, following Angenent [4], we take advantage of an interior gradient estimate of Evans and Spruck [45] to prove the result for initial data u_0 which are merely continuous.

In the general case, we have

Theorem 2.9.2 *Assume that b satisfies assumptions **(H1)–(H4)** and*

$$b(q) \geq \Lambda(|q|)Id \tag{2.38}$$

for some nonnegative continuous function Λ in $\mathbb{R}^N$. Suppose that for any $u_0 \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^N)$ there exists a smooth solution of (2.4) with initial data u_0 satisfying a local gradient bound, namely

$$\|Du\|_{\infty,\Omega_{R,T}} \leq K,$$

where $\Omega_{R,T} := \overline{B}(0,R) \times [0,T]$ and K is a positive constant which depends only on R, T, $\|u\|_{\infty,\bar{\Omega}_{R,T}}$ and $\|Du_0\|_{\infty,\overline{B}(0,R')}$, with $R' = R'(R,T) > 0$. Then the extremal solution u^+ and u^- are smooth.

The above theorem applies to more general quasilinear equations than (2.1) (see examples at the end of the section) but it requires the initial data to be locally Lipschitz continuous.

Remark 2.9.1 Under the assumptions of Theorem 2.9.1 or 2.9.2, if Γ_t has empty interior in $\mathbb{R}^{N+1}$ for all $t \geq 0$ (or equivalently the front $\bigcup_{t\geq 0} \Gamma_t \times \{t\}$ has empty interior in $\mathbb{R}^{N+1} \times [0,T]$, see Section 2.6), then the smoothness of the extremal solutions together with Theorem 2.6.2 implies $u^+ = u^-$ in $\mathbb{R}^N \times [0,+\infty)$. It follows $\Gamma_t = \partial\Omega_t^+ = \partial\Omega_t^-$ for all $t \geq 0$. In this case, we have in particular uniqueness and comparison for the discontinuous solutions of (2.4). Moreover, the weak notion of propagation given by the level-set approach coincides with the classical notion in differential geometry.

Before turning to the proof of the theorems, we state a lemma concerning the time regularity of solutions for which the space regularity is already known. We recall that a function $m : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a modulus of continuity if $m(0^+) := \lim_{s\to 0^+} = 0$ and $m(s+t) \leq m(s) + m(t)$ for any $s, t \geq 0$.

Lemma 2.9.1 *Let $R > 0$, $0 \le t_0 < T$, $x_0 \in \mathbb{R}^N$ and $u \in C(\overline{B}(x_0, R) \times [t_0, T])$ be a viscosity solution of the equation*

$$\frac{\partial u}{\partial t} + G(x, t, Du, D^2u) = 0 \quad \text{in} \quad \Omega_{R,t_0,T} = B(x_0, R) \times (t_0, T), \tag{2.39}$$

where $G \in C(\overline{B}(x_0, R) \times [t_0, T] \times \mathbb{R}^N \times \mathcal{S}_N)$ is degenerate elliptic. If m denotes a modulus of continuity of $u(\cdot, t_0)$, i.e. if, for every $x, y \in \overline{B}(x_0, R)$, we have

$$|u(y, t_0) - u(x, t_0)| \le m(|y - x|),$$

then there exists a modulus of continuity $\tilde{m}$ depending only on G, m and $\|u\|_{\infty,\overline{\Omega}_{R,t_0,T}}$ such that, for every $t \in [t_0, T]$ and $x \in \overline{B}(x_0, R/2)$,

$$|u(x, t) - u(x, t_0)| \le \tilde{m}(|t - t_0|). \tag{2.40}$$

Moreover, if $m(r) = Lr$ for some $L \ge 0$ and if

$$|G(x, t, p, X)| \le M(1 + |X|) \quad \text{on} \quad \overline{B}(x_0, R) \times [t_0, T] \times \overline{B}(0, L) \times \mathcal{S}_N \tag{2.41}$$

for some constant $M \ge 0$, then there exists a positive constant $\tilde{L} = \tilde{L}(L, M, \|u\|_{\infty,\overline{\Omega}_{R,t_0,T}})$ such that $\tilde{m}(r) = \tilde{L}r^{1/2}$.

Of course, the key point in Lemma 2.9.1 is the fact that $\tilde{m}$ depends only on G, m and $\|u\|_{\infty,\overline{\Omega}_{R,t_0,T}}$. As a by-product of this result, it is clear that a local L^∞–bound together with a time-uniform space-local modulus of continuity for the solutions of equations like (2.39) implies a uniform local modulus of continuity in time. In the statement of Lemma 2.9.1, for the sake of simplicity of formulation, we do make precise the dependence with respect to G, except in the second part of the result ; this dependence will appear clearly in the proof.

Proof of Theorem 2.9.1. We divide the proof in two steps.

Step 1. We construct a smooth solution for any continuous initial data. Let $u_0 \in C(\mathbb{R}^N)$ and $(u_0^R)_{R>0}$ be a sequence of uniformly continuous functions converging to u_0, uniformly on every compact subset. Thanks to classical results for viscosity solutions (see [32] and references therein), we associate to each u_0^R a continuous viscosity solution u^R of (2.1) with initial data u_0^R. But the u^R satisfy the L^∞ local bound of Theorem 2.7.1, and, from Evans and Spruck [46], we learn that the u^R are in fact smooth and satisfy the interior local gradient bounds proved in [46]. From Lemma 2.9.1 we get then interior local modulus of continuity for the u^R; therefore, up to an extraction argument, we can suppose that the family $(u^R)_{R>0}$ converge locally uniformly in $\mathbb{R}^N \times (0, +\infty)$ to a function $u \in C(\mathbb{R}^N \times (0, +\infty))$ which is, by a classical stability result, a viscosity solution of (2.1) in $\mathbb{R}^N \times (0 + \infty)$.

It remains to check that the initial condition is continuously satisfied. In view of Lemma 2.9.1, the u^R admit the same modulus of continuity at time $t = 0$ and it follows that u is continuous at time $t = 0$ with $u(\cdot, 0) = u_0$. Finally, from [46] we get that, $u \in C^\infty(\mathbb{R}^N \times (0, +\infty))$ as a continuous solution of (2.1).

Step 2. We show that u^+ is smooth ; the proof for u^- is the same with straightforward adaptations. Let $u_0 \in C(\mathbb{R}^N)$. Consider, for any $\lambda > 0$, the function u_0^λ defined by

$$\mathrm{Graph}(u_0^\lambda) = \{\mathrm{d}(\cdot, \mathrm{Graph}(u_0)) = \lambda\},$$

and the unique uniformly continuous solution v of (2.3) with initial data $\mathrm{d}(\cdot, \mathrm{Graph}(u_0))$ (we recall that (2.3) is the geometrical equation associated to (2.1)). By Step 1, we associate to each $\lambda > 0$ a smooth solution u^λ of (2.1) which satisfies, from Theorem 2.6.1, that, for all $t \geq 0$,

$$\mathrm{Graph}(u^\lambda(\cdot, t)) \subset \{v(\cdot, t) = \lambda\} \subset \{v(\cdot, t) > 0\}. \tag{2.42}$$

Now, as in Step 1, the family $(u^\lambda)_{\lambda>0}$ satisfies the interior gradient bound of [46] ; thus, using Lemma 2.9.1 and the same arguments as above, we can assume that u^λ converges locally uniformly to a solution u of (2.1) with initial data u_0. From (2.42), we get that $u \geq u^+$ and thus $u = u^+$. It follows that u^+ is continuous and therefore smooth thanks again to [46]. □

Proof of Theorem 2.9.2. Since the proof is close to the previous one, we only give a sketch of the proof. We use arguments of Step 2 in the proof of Theorem 2.9.1. The only change is that, using the ellipticity condition (2.38), we get, in addition to the gradient bound, local bounds for high order derivatives of the u_λ (see Ladyzenskaja, Solonnikov and Ural'ceva [68]). It follows that, up to an extraction, we can assume that the family $(u_\lambda)_{\lambda>0}$ converges locally uniformly to a smooth function u which is also a solution of (2.4). We conclude as in the proof of Theorem 2.9.1, that $u = u^+$ is actually smooth. □

It remains to give the proof of the lemma.

Proof of Lemma 2.9.1. The main step in the proof consists in showing that, for any $\eta > 0$, one can find positive constants $C, K > 0$ large enough, depending only on η, G, m and $\|u\|_{\infty,\bar\Omega_{R,t_0,T}}$ such that, for any $x \in B(x_0, R/2)$

$$u(y,t) - u(x,t_0) \leq \eta + C|y-x|^2 + K(t-t_0) \ \text{ for every } (y,t) \in \bar\Omega_{R,t_0,T}, \tag{2.43}$$

and

$$u(y,t) - u(x,t_0) \geq -\eta - C|y-x|^2 - K(t-t_0) \ \text{ for every } (y,t) \in \bar\Omega_{R,t_0,T}. \tag{2.44}$$

We only prove (2.43), (2.44) being proved in an analogous way. In the sequel, x is fixed in $B(x_0, R/2)$.

First, if we take

$$C \geq \frac{8\|u\|_{\infty,\bar\Omega_{R,t_0,T}}}{R^2}, \tag{2.45}$$

then (2.43) is clearly fulfilled on $\partial B(x_0, R) \times [t_0, T]$, for every $\eta, K > 0$ and for every $x \in B(x_0, R/2)$. It is worth noticing that C may be taken independent of x.

Next, we would like to ensure that (2.43) holds for $t = t_0$. To this end, we argue by contradiction assuming there exists $\eta > 0$ such that, for every $C > 0$, there exists $y_C \in \overline{B}(x_0, R)$ such that

$$u(y_C, t_0) - u(x, t_0) > \eta + C|y_C - x|^2. \tag{2.46}$$

It follows

$$|y_C - x| \leq \sqrt{\frac{2\|u\|_{\infty,\Omega_{R,t_0,T}}}{C}}. \tag{2.47}$$

Thus $|y_C - x| \to 0$ when $C \to \infty$. Coming back to (2.46), we get

$$m(|y_C - x|) \geq u(y_C, t_0) - u(x, t_0) > \eta + C|y_C|^2 \geq \eta.$$

Using (2.47), the inequality $m(|y_C - x|) \geq \eta$ leads to a contradiction as soon as we choose C large enough and this choice depends only on η, $\|u\|_{\infty,\Omega_{R,t_0,T}}$ and m. Therefore, by choosing C large enough, we have that (2.43) is satisfied on the parabolic boundary $(\partial B(x_0, R) \times [t_0, T]) \cup (\overline{B}(x_0, R) \times \{t_0\})$.

Finally, using the continuity of G, we can take K large enough in order that the function $(y, t) \mapsto u(x, t_0) + \eta + C|y - x|^2 + K(t - t_0) := \chi(y, t)$ is a (smooth) strict supersolution of (2.39). Thus, since u is a viscosity subsolution of (2.39), by using only the definition of viscosity subsolution, it is clear that $\max_{\bar{\Omega}_{R,t_0,T}}\{u - \chi\}$ is necessarily achieved on the parabolic boundary of $\Omega_{R,t_0,T}$. And (2.43) follows.

The first part of the lemma follows by observing that all our constants depend only, when η is fixed, on G, m and $\|u\|_{\infty,\bar{\Omega}_{R,t_0,T}}$ but not on $x \in B(x_0, R/2)$.

If we assume that $m(r) = Lr$ for some positive constant L, the condition (2.43) at time $t = t_0$ reads

$$u(y, t_0) - u(x, t_0) \leq L|y - x| \leq \eta + C|y - x|^2,$$

for every $y \in \overline{B}(x_0, R)$. Writing that the discriminant of $C|y - x|^2 - L|y - x| + \eta$ is nonpositive, we get that it holds if

$$C \geq \frac{L^2}{4\eta}.$$

Using (2.41), χ is a supersolution if $K \geq M(1 + 2C)$. Introducing these estimates in (2.43), we finally obtain for $y = x$

$$u(x, t) - u(x, t_0) \leq \eta + M\left(1 + \frac{L^2}{2\eta}\right)(t - t_0),$$

for all $t \in [t_0, T]$. An easy optimization with respect to η of the right-hand side term of the previous inequality gives that, for all $t \in [t_0, T]$,

$$u(x, t) - u(x, t_0) \leq \tilde{L}\sqrt{t - t_0},$$

for some positive constant $\tilde{L}$ depending on M and L. This concludes the proof of the Lemma. □

We conclude this section with examples of equations satisfying the assumptions of Theorem 2.9.2. The following equations come from the paper of Chou and Kwong [33] (see Section 2.4.4 for the precise statement of the equations).

1. The non geometric mean curvature equation (2.25) and equation (2.26) are associated to a front with smooth boundary when u_0 is locally Lipschitz continuous.

2. Consider equation (2.27) with $g(q) = g(|q|)$ continuous in $\mathbb{R}^N$. Suppose that

$$\frac{1}{r^2}\left(1-\frac{C}{r^2}\right) \le g(r) < \frac{1}{r^2} \quad \text{for every } r>0, \tag{2.48}$$

and, in addition, that g is a C^1 function such that

$$2g + rg' \le \frac{C}{(1+r^2)^{3/2}},$$

for every $r \ge 0$. Then Chou and Kwong [33] gives the gradient bound the conclusion of Theorem 2.9.2 holds. Note that (2.48) is nothing but (2.28) with a strict inequality on the right-hand side. This strict inequality ensures that the condition of ellipticity (2.38) is satisfied.

2.10 Application to convex solutions

In this section, we are interested in convex solutions of (2.4). We also derive some properties for the generalized evolution of convex sets.

Our main result is

Theorem 2.10.1 *Assume* ***(H1)–(H4)*** *and let u_0 be a convex function in $\mathbb{R}^N$.*

(i) Suppose that $u_1 \in USC(\mathbb{R}^N \times [0,+\infty))$ (respectively $u_2 \in LSC(\mathbb{R}^N \times [0,+\infty))$) is a viscosity subsolution (respectively supersolution) of (2.4). If $u_1(\cdot,0) \le u_0 \le u_2(\cdot,0)$ in $\mathbb{R}^N$, then $u_1 \le u_2$ in $\mathbb{R}^N \times [0,+\infty)$.

(ii) There exists a unique continuous viscosity solution u to (2.4) with initial data u_0. Moreover $u(\cdot,t)$ is convex for all $t \ge 0$.

It is worth pointing out that, in the previous theorem, the existence and comparison properties hold without any restriction on the growth at infinity of the solutions or the initial data. Moreover, they hold both in the classical and very singular framework. Therefore, we have a complete answer in this case.

In the particular case of the mean curvature equation, the solution is in addition smooth.

Theorem 2.10.2 *If the initial data $u_0 \in C(\mathbb{R}^N)$ is convex, then there exists a unique continuous solution u of the mean curvature equation for graphs (2.1); moreover $u \in C^\infty(\mathbb{R}^N \times (0,+\infty)) \cap C(\mathbb{R}^N \times [0,+\infty))$ and $u(\cdot,t)$ is convex for any $t \ge 0$.*

Theorem 2.10.1 and 2.10.2 strongly justify the geometrical approach to study (2.4) : in the case of convex solutions, the existence of solutions follows (rather) easily from the L^∞-bound of Theorem 2.7.1 since it implies also a gradient bound ; the existence proof can be done either using Theorem 2.8.1 or directly on (2.4) as in [15]. For the comparison result, we point out that working on (2.6) as we do it here, in particular for the mean curvature equation, provides better results : in [15], we obtain a comparison result working directly on (2.4) by using a Kružkov change ($u \mapsto -\exp(-u)$) but with stronger assumptions on b and u_0 which was assumed to be coercive.

Below, we will give a proof which is simpler and essentially based on the preservation of convexity for geometric motions governed by (2.6) ; more precisely, we have

Theorem 2.10.3 *Suppose that **(H1)**–**(H4)** hold. Let $v_0 \in UC(\mathbb{R}^N)$ be a convex (respectively concave) function and v be the associated solution of (2.6). Then $v(\cdot, t)$ is convex (respectively concave) for any $t \geq 0$.*

Proof of Theorem 2.10.3. This result is a consequence of the one established by Giga, Goto, Ishii and Sato in [51] that we extend to the very singular case by using an approximation argument.

Step 1. We define, for any $\varepsilon > 0$ and $(p, M) \in \mathbb{R}^{N+1} \times \mathcal{S}_{N+1}$,

$$F_\varepsilon(p, M) = \begin{cases} \varphi_\varepsilon(1/p_y)F(p, M) & \text{if} \quad p_y \neq 0 \\ 0 & \text{if} \quad p_y = 0 \end{cases}$$

where F appears in (2.6) and φ_ε is a smooth nonnegative real valued function with compact support in $[-2/\varepsilon, 2/\varepsilon]$ and such that $\varphi_\varepsilon(r) = 1$ for $r \in [-1/\varepsilon, 1/\varepsilon]$. The F_ε's satisfy assumptions **(F1)**–**(F4)** and thus we can apply for each ε the results of Giga, Goto, Ishii and Sato [51] and get for any $T > 0$ a solution $v_\varepsilon \in UC(\mathbb{R}^{N+1} \times (0, T))$ of

$$\frac{\partial v_\varepsilon}{\partial t} + F_\varepsilon(Dv_\varepsilon, D^2 v_\varepsilon) = 0 \quad \text{in} \quad \mathbb{R}^{N+1} \times (0, T) \tag{2.49}$$

with initial data v_0. Moreover, since v_0 is convex and F_ε remains linear in the Hessian, we learn also from [51] that $v_\varepsilon(\cdot, t)$ is convex for any $t \geq 0$.

Step 2. Our aim is now to show that the family $(v_\varepsilon)_{\varepsilon>0}$ is locally bounded. To this end, we introduce

$$\chi(z, t) = a|z|^2 + b + Ct.$$

Since $v_0 \in UC(\mathbb{R}^{N+1})$ there exist $a, b \in \mathbb{R}$ such that $v_0 \leq \chi(\cdot, 0)$ and since v_ε has at most linear growth, χ is greater than v_ε at infinity for all $\varepsilon > 0$.

Moreover, if follows from **(H1)** that F is bounded on bounded set and so are the F_ε's uniformly in ε. Then, an easy computation of the derivatives of χ shows that, up to take C sufficiently large independent of ε, the function $\chi(z, t) = a|z|^2 + b + Ct$ is a smooth supersolution of (2.49). It follows that $v_\varepsilon \leq \chi$ for all $\varepsilon > 0$. Reasoning in the same way with a subsolution, we get that the family $(v_\varepsilon)_{\varepsilon>0}$ is locally bounded independently of ε.

Step 3. From the previous step we are able to introduce the "half-relaxed-limits" $\overline{v}$ and $\underline{v}$ of the family $(v_\varepsilon)_{\varepsilon>0}$. Since (F_ε) tends to F locally uniformly on $(\mathbb{R}^{N+1} - \mathcal{D}) \times \mathcal{S}_{N+1}$, we obtain, from the stability result, that they are respectively sub- and supersolution of (2.6) with initial data v_0.

Finally, from the comparison result of Theorem 2.4.1, we have that $\overline{v} = \underline{v} = v$ and therefore that the family $(v_\varepsilon)_{\varepsilon>0}$ converges locally uniformly to v as ε tends to 0. It follows that $v(\cdot,t)$ is convex for any $t \in [0,T)$. Since we can repeat the arguments for any $T \geq 0$ it completes the proof. $\square$

We continue with

Lemma 2.10.1 *Let $\Gamma_0 = \mathrm{Graph}(u_0)$. If u_0 is convex in $\mathbb{R}^N$, then the signed-distance* $\mathrm{d}(\cdot,\Gamma_0)$ *(see (2.32) for a definition) is concave.*

Proof of Lemma 2.10.1 It suffice to shown that, for any $z_i = (x_i, y_i) \in \mathbb{R}^{N+1}$, $i \in \{1,2\}$, we have

$$v_0\left(\frac{z_1+z_2}{2}\right) \geq \frac{v_0(z_1)+v_0(z_2)}{2}. \tag{2.50}$$

To this end, we set $z = (z_1+z_2)/2$ and denote by $\mathcal{P}_z$ the hyperplane which contains $z' \in \Gamma_0$ such that $v_0(z) = \pm|z-z'|$ and is orthogonal to $z-z'$. At this stage, one has to distinguish many cases depending on the position of z_1, z_2 relatively to Γ_0. Since their study are similar, we provide the proof of (2.50) only in the case $z_1 \in \{v_0 \geq 0\}, z_2 \in \{v_0 \leq 0\}$ and $v_0(z) = -|z-z'|$.

In this case, since u_0 is assumed to be convex, $\mathcal{P}_z \subset \{v_0 \leq 0\}$; thus we have

$$v_0(z_1) = \mathrm{d}(z_1,\Gamma_0) \leq \mathrm{dist}(z_1,\mathcal{P}_z), \quad v_0(z_2) = \mathrm{d}(z_2,\Gamma_0) \leq -\mathrm{dist}(z_2,\mathcal{P}_z). \tag{2.51}$$

But, using the orthogonal projection on $\mathcal{P}_z$, one sees that

$$|z-z'| = \frac{\mathrm{d}(z_2,\mathcal{P}_z) - \mathrm{d}(z_1,\mathcal{P}_z)}{2} = -v_0(z) \tag{2.52}$$

and combining (2.51) and (2.52), one gets (2.50) $\square$

We are now able to give the proof of the main result.

Proof of Theorem 2.10.1. We begin with (i). Let u_0 be a convex function on $\mathbb{R}^N$ and $v_0 = d(\cdot, \mathrm{Graph}(u_0))$. From Lemma 2.10.1, v_0 is concave. Therefore, applying Theorem 2.10.3, the associated solution v of (2.6) is also concave with respect to the space variable at any time $t \geq 0$.

Assume then by contradiction that $y_2 = u_2(x,t) < u_1(x,t) = y_1$ for some $(x,t) \in \mathbb{R}^N \times [0,+\infty)$. It follows from Theorem 2.6.1 that $v(x,y_2,t) \geq 0$ and $v(x,y_1,t) \leq 0$. Thus from Lemma 2.6.1, $v(x,\cdot,t) \equiv 0$ on $[y_2,y_1]$. Since it is a concave function, it implies that $v(x,\cdot,t) \equiv 0$ on $[y_2,+\infty)$ which is a contradiction with Lemma 2.8.1.

We turn to the proof of (ii). Applying the previous comparison result to the extremal solutions u^+ and u^- we obtain that they are equal to the same continuous function u, such that $\{v(\cdot,t) = 0\} = \mathrm{Graph}(u(\cdot,t))$ for all $t \geq 0$, which turns out to be the unique continuous viscosity solution of (2.4). Since $v(\cdot,t)$ is concave, $u(\cdot,t)$ is convex for any $t \geq 0$. It completes the proof of (ii) $\square$

We conclude this section with some consequences of Theorem 2.10.3 for the geometrical evolution of sets in the convex case.

Theorem 2.10.4 *Let Ω_0^+ be any open convex subset of $\mathbb{R}^{N+1}$ with boundary Γ_0 and let $(\Omega_t^+, \Omega_t^-, \Gamma_t)_{t\geq 0}$ be the generalized evolution of $(\Omega_0^+, \Omega_0^-, \Gamma_0)$ in the sense of Section 2.5. Then, while $\Omega_t^+ \neq \emptyset$, it remains convex and Γ_t is its boundary. In particular, Γ_t has empty interior in $\mathbb{R}^{N+1}$.*

This result is known in the case of motion by mean curvature of compact convex sets (see Evans and Spruck [45], Soner [78], Ilmanen [56]). Here, the result holds for possibly noncompact hypersurfaces (like graphs for instance) and for general motions governed by (2.6). Note that we get immediate properties of regularity of the front at each time t before extinction : in the general case the front is locally a Lipschitz continuous graph ; in the case of the mean curvature equation, the front is even a smooth hypersurface (for the regularity issue, see Evans and Spruck [46] and Imbert [57]).

Proof of Theorem 2.10.4. Let v be the unique solution of (2.6) with initial data $v_0 = \mathrm{d}(\cdot, \Gamma_0)$ associated to Γ_0 via the level-set approach. Let $t \geq 0$ such that $\Omega_t^+ \neq \emptyset$. From Theorem 2.10.3, $v(\cdot, t)$ is concave ; thus Ω_t^+ is convex. To prove that $\partial\Omega_t^+ = \Gamma_t$, we argue by contradiction, assuming there exists $z_0 \in \Gamma_t$ and $r > 0$ such that $B(z_0, r) \cap \Omega_t^+ = \emptyset$. We have $v(z_0, t) = 0$ and $v(\cdot, t) \leq 0$ on $B(z_0, r)$. Since $v(\cdot, t)$ is concave, it follows $v(\cdot, t) \leq 0$ in $\mathbb{R}^{N+1}$ which is a contradiction with $\Omega_t^+ \neq \emptyset$. $\square$

Chapitre 3

Quasilinear parabolic equations, unbounded solutions and geometrical equations II. Uniqueness without growth conditions and applications to the mean curvature flow in $\mathbb{R}^2$

Guy Barles, Samuel Biton and Olivier Ley
Laboratoire de Mathématiques et Physique Théorique
Université de Tours,
Parc de Grandmont, 37200 Tours, France

Abstract

In this article, we prove a comparison result for viscosity solutions of a certain class of fully nonlinear, possibly degenerate, parabolic equations ; the main new feature of this result is that it holds for any, possibly discontinuous, solutions without imposing any restrictions on their growth at infinity. The main application of this result which was also our main motivation to prove it, is the uniqueness of solutions to one-dimensional equations including the mean curvature equation for graphs without assuming any restriction on their behavior at infinity.

Key-words : quasilinear parabolic equations, mean curvature equations, uniqueness without growth conditions, viscosity solutions.
AMS subject classifications : 35A05, 35B05, 35D05, 35K15, 35K55, 53C44

[0]This work is a preprint of the University of Tours which is submitted to publication.

Sommaire

3.1 Introduction

The main motivation of this paper comes from the following, rather surprising, result of Ecker and Huisken [41] : for any initial data $u_0 \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N)$, there exists a smooth solution of the equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \dfrac{\langle D^2uDu, Du\rangle}{1+|Du|^2} = 0 \quad \text{in } \mathbb{R}^N \times (0,+\infty), \\ u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N\,. \end{cases} \tag{3.1}$$

Here and below the solution u is a real-valued function, Du and D^2u denote respectively its gradient and Hessian matrix while $|\cdot|$ and $\langle \cdot,\cdot\rangle$ stand for the classical Euclidean norm and inner product in $\mathbb{R}^N$.

In order to be complete, it is also worth mentioning that, following Angenent [4], one can extend this result to merely continuous initial data using the interior gradient estimates of Evans and Spruck [46] (see [16]).

The very non-standard feature of this kind of results is that no assumption on the behavior of the initial data at infinity is imposed and therefore the solutions may also have any possible behavior at infinity. A natural and intriguing question is then whether such a solution is unique or not.

This question leads us to study of the uniqueness properties for unbounded viscosity solutions of quasilinear, possibly degenerate, parabolic equations set in $\mathbb{R}^N$ and this article is a continuation of this program started in [16] (see also [15], [14] and [18]).

In this paper, we are more particularly interested in the one-dimensional case for (3.1) where this equation reduces to

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{u_{xx}}{1+u_x^2} = 0 \qquad \text{in } \mathbb{R} \times (0,T), \\ u(\cdot,0) = u_0 \qquad\qquad\quad \text{in } \mathbb{R}. \end{cases} \tag{3.2}$$

One of the main contribution of this work is a uniqueness result for smooth solutions of (3.2) (and even for more general equations), without any restriction on their growth at infinity (see Theorem 3.4.2) ; it immediately yields a comparison result for, possibly discontinuous, viscosity solutions of (3.2) by the geometrical approach of [16] (cf. Corollary 3.5.1), i.e. a complete answer to the question we address. After this work was completed, we learn that Chou and Kwong [33] proved general uniqueness results for a certain class of quasilinear parabolic equations in the one-dimensional case : their hypothesis are of different nature from ours and their methods are quite different but their result, which is also valid for solutions without any restriction on their behavior at infinity, includes also the mean curvature equation.

This kind of uniqueness results is non-standard : it is well-known that uniqueness fails, in general, if we do not impose growth conditions on the solutions at infinity, like, for example, the heat equation. The only (general) case where similar results hold is the one of first-order equations where they are a consequence of "finite speed of propagation" type properties (see Crandall and Lions [38], Ishii [58] and Ley [71]) but they are very unusual for second-order equation.

Our uniqueness proof for (3.2) consists in integrating the equation, which leads to consider the pde

$$\frac{\partial v}{\partial t} - \arctan(v_{xx}) = 0 \quad \text{in } \mathbb{R} \times (0, +\infty). \tag{3.3}$$

And this is this new equation which is shown to satisfy a comparison principle for viscosity solutions without growth condition at infinity.

We refer the reader to Crandall, Ishii and Lions [37], Fleming and Soner [48], Bardi and Capuzzo Dolcetta [6] or Barles [13], etc. for a presentation of the notion of viscosity solutions and for the key basic results.

In fact, the comparison result we are going to prove and apply to (3.3) holds in $\mathbb{R}^N$ (and not only in dimension 1) and for more general equations of the type

$$\begin{cases} \dfrac{\partial u}{\partial t} + F(x, t, Du, D^2u) = 0 & \text{in } \mathbb{R}^N \times (0, T), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{3.4}$$

where F is a continuous function from $\mathbb{R}^N \times [0, T] \times \mathbb{R}^N \times \mathcal{S}_N$ into $\mathbb{R}$ and satisfies suitable assumptions (see **(H1)** and **(H2)** in Section 3.2). Our proof relies essentially on the use of a "friendly giant method." This result implies the uniqueness for smooth solutions of quite general equations in the one-dimension space including (3.2) as we mention it above (see Section 3.4).

Finally we investigate the consequences of the existence and uniqueness of solutions for (3.2) from the geometrical point of view. The geometrical interpretation of (3.2) is that the graph of u at every time $t \geq 0$, namely

$$\Gamma_t := \mathrm{Graph}(u(\cdot, t)),$$

is a smooth hypersurface of $\mathbb{R}^2$ which evolves by its mean curvature. Such evolution has been studied recently by the so-called "level-set approach" : introduced for numerical purposes by Osher and Sethian [76], this approach was developed by Evans and Spruck [45] and Chen, Giga and Goto [32] ; the generalized motion of hypersurfaces is described through the evolution of the level-sets of solutions of a suitable geometrical pde. This approach offers a lot of advantages : it is possible to deal with nonsmooth hypersurfaces, it allows numerical computations, etc. The main issue is the agreement with the classical motion defined in differential geometry. These questions were addressed by a lot of authors (see especially Evans and Spruck [45], [46] and Ilmanen [56]).

Our uniqueness result allows us to show that, in the case of the mean curvature motion of any entire continuous graph in $\mathbb{R}^2$, the level-set approach agrees with the classical motion defined in differential geometry. As a by-product, we have a comparison result between possibly discontinuous viscosity sub and supersolutions of (3.2) and therefore the uniqueness holds even in the class of discontinuous solutions. We refer to Section 3.5 for the complete statement of these results.

The paper is divided as follows. In Section 3.2, we prove the strong comparison principle for viscosity solutions of equation (3.4). Section (3.3) is devoted to the proof of the existence of a solution for (3.4). In Section 3.4, we apply the previous result to uniqueness for some one-dimensional equations including (3.2) as a particular case. In the last section, we state some geometrical consequences for the mean curvature flow in $\mathbb{R}^2$ and more general geometrical motions.

Acknowledgment. This work was partially supported by the TMR program "Viscosity Solutions and Their Applications." The last author would like to thank the Department of Mathematics of the university of Padova and especially Martino Bardi for their kind hospitality and fruitful exchanges.

3.2 A comparison result for viscosity solutions without growth conditions

In order to state the comparison principle for viscosity solutions of equation (3.4), we use the following assumptions in which $\bar{B}(x_0, R) = \{x \in \mathbb{R}^N : |x - x_0| \leq R\}$ denotes the ball of radius $R > 0$ centered in $x_0 \in \mathbb{R}^N$ and $\mathcal{S}_N$ the space of $N \times N$ symmetric matrices; moreover, for every symmetric matrix $X = P(\mathrm{diag}(\lambda_i)_{1 \leq i \leq N})P^T$, we define $X^+ = P(\mathrm{diag}(\lambda_i^+)_{1 \leq i \leq N})P^T$, where the λ_i's are the eigenvalues of X, P is orthogonal with transpose P^T and, for any $\lambda \in \mathbb{R}$, $\lambda^+ = \max\{\lambda, 0\}$.

(H1) For any $R > 0$, there exists a function $m_R : \mathbb{R}_+ \to \mathbb{R}_+$ such that $m_R(0^+) = 0$ and

$$F(y, t, \eta(x-y), Y) - F(x, t, \eta(x-y), X) \leq m_R\left(\eta|x-y|^2 + |x-y|\right),$$

for all $x, y \in \bar{B}(0, R)$, $t \in [0, T], X, Y \in \mathcal{S}_N$, and $\eta > 0$ such that

$$-3\eta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\eta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

(H2) There exists $0 < \alpha < 1$ and constants $K_1 > 0$ and $K_2 > 0$ such that

$$F(x, t, p, X) - F(x, t, q, Y) \leq K_1|p - q|(1 + |x|) + K_2(\mathrm{Tr}[Y - X]^+)^\alpha,$$

for every $(x, t, p, X, Y) \in \mathbb{R}^N \times \mathbb{R}^N \times [0, T] \times \mathcal{S}_N \times \mathcal{S}_N$.

Our result is the following

Theorem 3.2.1 *Assume that **(H1)** and **(H2)** hold and let u (respectively v) be an upper-semicontinuous viscosity subsolution (respectively lower-semicontinuous viscosity supersolution) of (3.4). If $u(\cdot, 0) \leq v(\cdot, 0)$ in $\mathbb{R}^N$, then $u \leq v$ in $\mathbb{R}^N \times [0, +\infty)$.*

Before proving this result, let us comment the assumptions. Assumption **(H1)** is the classical one to ensure uniqueness for this type of fully nonlinear parabolic equations (see Condition (3.14) of [37]) : here we just state it in a more local way. The assumption **(H2)** contains all the restriction which allows such a result to hold : the first term of the right-hand side is the classical one to obtain "finite speed of propagation" type results while the second one concerns the behavior of F in D^2u, and in particular at infinity as (3.3) shows it. Obviously such an assumption or a related one is needed since such a result cannot be true for the heat equation. A nonlinearity F which satisfies **(H2)** is necessarily of the form $F_1(x, t, p) + F_2(x, t, X)$. We also point out that **(H2)** implies that F is degenerate elliptic. We do not know if these assumptions are close or far to be optimal.

As an example of pde which satisfies **(H1)**–**(H2)**, we have

$$\frac{\partial u}{\partial t} - \left[(a(x,t)\Delta u)^+\right]^\alpha + b(x,t)|Du| = f(x,t) \quad \text{in } \mathbb{R}^N \times (0, T),$$

where a, b are continuous functions on $\mathbb{R}^N \times (0,T)$, Lipschitz continuous in x uniformly with respect to t, $a \geq 0$, $f \in C(\mathbb{R}^N \times (0,T))$ and $0 < \alpha < 1$.

Now we turn to the proof of the theorem.

Proof of Theorem 3.2.1. The proof is done in two steps : the first one is a kind of linearization procedure which yields a new pde for $u - v$. The second one consists in the construction of a suitable smooth supersolution of this new pde which blows up at the boundary of balls.

The first step is described in the

Lemma 3.2.1 *Under the assumptions of Theorem 3.2.1, the upper-semicontinuous function $\omega := u - v$ is a viscosity subsolution of*

$$A[\omega] = \frac{\partial \omega}{\partial t} - K_1(1+|x|)|D\omega| - K_2\big[Tr(D^2\omega)^+\big]^\alpha = 0 \quad \text{in } \mathbb{R}^N \times (0,+\infty). \tag{3.5}$$

Moreover $\omega(\cdot,0) \leq 0$ in $\mathbb{R}^N$.

Then the second step relies on the

Lemma 3.2.2 *There exists $c > 0$ and $k > \beta > 0$ such that, for $R > 0$ large enough, the smooth function*

$$\chi_R(x,t) = \varphi\Big((1+R^2)^{\frac{1}{2}}(1-ct) - (1+|x|^2)^{\frac{1}{2}}\Big) \quad \text{with } \varphi(r) = R^\beta/r^k, \tag{3.6}$$

is a strict supersolution of (3.5) in the domain

$$\mathcal{D}(c,R) = \left\{(x,t) \in \mathbb{R}^N \times [0,T] :\ ct < \frac{1}{2},\ (1+|x|^2)^{\frac{1}{2}} < (1+R^2)^{\frac{1}{2}}(1-ct)\right\}. \tag{3.7}$$

We postpone the proof of the lemmas and we first conclude the proof of the theorem.

We consider $\sup\limits_{\mathcal{D}(c,R)} \{\omega - \chi_R\}$; since χ_R goes to $+\infty$ on the lateral side of $\mathcal{D}(c,R)$, this supremum is achieved at some point $(\bar{x},\bar{t}) \in \mathcal{D}(c,R)$.

We cannot have $\bar{t} > 0$ because, otherwise since by Lemma 3.2.1 ω is a viscosity subsolution of (3.5), we would have $A[\chi_R(\bar{x},\bar{t})] \leq 0$, which would contradict the fact that χ_R is a strict supersolution of this pde by Lemma 3.2.2.

Thus, necessarily $\bar{t} = 0$ and the maximum is nonpositive since $\omega(\cdot,0) \leq 0$ and $\chi_R(\cdot,0) > 0$ in $\mathbb{R}^N$. It follows that, for every $(x,t) \in \mathcal{D}(c,R)$,

$$(u-v)(x,t) \leq \frac{R^\beta}{[(1+R^2)^{\frac{1}{2}}(1-ct) - (1+|x|^2)^{\frac{1}{2}}]^k}\,. \tag{3.8}$$

In order to conclude, we let R go to $+\infty$ in this inequality for $t < 1/2c$: since $k > \beta > 0$, we obtain $(u-v)(x,t) \leq 0$ for any $x \in \mathbb{R}^N$. To prove the same property for all $t \in [0,T]$, we iterate in time and the proof of the theorem is complete. □

We turn to the proof of the lemmas.

Proof of Lemma 3.2.1. Let $\phi \in C^2(\mathbb{R}^N \times (0,+\infty))$ and suppose that $(\bar{x},\bar{t}) \in \mathbb{R}^N \times (0,+\infty)$ is a strict local maximum point of $\omega - \phi = u - v - \phi$ and more precisely a strict maximum point of this function in $\bar{B}(\bar{x},\rho)^2 \times [\bar{t}-\rho, \bar{t}+\rho]$ for some $\rho > 0$.

We consider, for $\varepsilon > 0$,

$$\max_{\bar{B}(\bar{x},\rho)^2 \times [\bar{t}-\rho,\bar{t}+\rho]} \left\{ u(x,t) - v(y,t) - \phi(x,t) - \frac{|x-y|^2}{\varepsilon^2} \right\} . \tag{3.9}$$

By classical arguments, it is easy to prove that the maximum in (3.9) is achieved at points $(x_\varepsilon, y_\varepsilon, t_\varepsilon)$ such that

$$(x_\varepsilon, y_\varepsilon, t_\varepsilon) \to (\bar{x},\bar{x},\bar{t}) \quad \text{and} \quad \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.10}$$

Hence, for ε small enough, $(x_\varepsilon, y_\varepsilon, t_\varepsilon) \in B(\bar{x},\rho) \times B(\bar{x},\rho) \times (\bar{t}-\rho, \bar{t}+\rho)$ and following [37], there exist $a, b \in \mathbb{R}$, $X, Y \in \mathcal{S}_N$, such that, if we set $p_\varepsilon := \dfrac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2}$, we have

$$(a, D\phi(x_\varepsilon,t_\varepsilon) + p_\varepsilon, X + D^2\phi(x_\varepsilon,t_\varepsilon)) \in \bar{\mathcal{P}}^{2,+}u(x_\varepsilon,t_\varepsilon) \ ,$$

$$(b, p_\varepsilon, Y) \in \bar{\mathcal{P}}^{2,-}v(x_\varepsilon,t_\varepsilon)$$

and such that $a - b = \dfrac{\partial\phi}{\partial t}(x_\varepsilon,t_\varepsilon)$ and

$$-\frac{6}{\varepsilon^2}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \frac{6}{\varepsilon^2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Using that u and v are respectively viscosity sub- and supersolution of (3.4), we have

$$\frac{\partial\phi}{\partial t}(x_\varepsilon,t_\varepsilon) + F\left(x_\varepsilon, D\phi(x_\varepsilon,t_\varepsilon) + p_\varepsilon, X + D^2\phi(x_\varepsilon,t_\varepsilon)\right) - F\left(y_\varepsilon, p_\varepsilon, Y\right) \le 0.$$

From **(H1)** applied with $R = |\bar{x}| + \rho$, we have

$$-F\left(y_\varepsilon, p_\varepsilon, Y\right) \ge -F\left(x_\varepsilon, p_\varepsilon, X\right) - m_R\left(2\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} + |x_\varepsilon - y_\varepsilon|\right)$$

and from **(H2)**, we get

$$\begin{aligned} &F\left(x_\varepsilon, D\phi(x_\varepsilon,t_\varepsilon) + p_\varepsilon, X + D^2\phi(x_\varepsilon,t_\varepsilon)\right) - F\left(x_\varepsilon, p_\varepsilon, X\right) \\ &\qquad \ge -K_1|D\phi(x_\varepsilon,t_\varepsilon)|(1+|x_\varepsilon|) - K_2\left[\mathrm{Tr}(D^2\phi(x_\varepsilon,t_\varepsilon))^+\right]^\alpha . \end{aligned}$$

Finally, we obtain

$$\begin{aligned} &\frac{\partial\phi}{\partial t}(x_\varepsilon,t_\varepsilon) - K_1|D\phi(x_\varepsilon,t_\varepsilon)|(1+|x_\varepsilon|) - K_2\left[\mathrm{Tr}(D^2\phi(x_\varepsilon,t_\varepsilon))^+\right]^\alpha \\ &\qquad\qquad \le m_R\left(2\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} + |x_\varepsilon - y_\varepsilon|\right). \end{aligned}$$

Letting ε go to 0 and using (3.10) yields

$$\frac{\partial\phi}{\partial t}(\bar{x},\bar{t}) - K_1|D\phi(\bar{x},\bar{t})|(1+|\bar{x}|) - K_2\left(\mathrm{Tr}[D^2\phi(\bar{x},\bar{t})]^+\right)^\alpha \le 0$$

which is exactly the inequality showing that ω is a subsolution of (3.5). $\square$

Proof of Lemma 3.2.2. In order to check that χ is a strict supersolution for a suitable choice of constants $k > \beta > 0$ and $c > 0$ independent of R, we set $r = (1+R^2)^{1/2}(1-ct) - (1+|x|^2)^{1/2}$ and note that, for $(x,t) \in \mathcal{D}(c,R)$, we have $r \in [0,\bar{R}]$ where $\bar{R} = \sqrt{1+R^2}$. Moreover, the function φ is decreasing convex, i.e.

$$\varphi' < 0, \quad \varphi'' \geq 0 \quad \text{on } (0,\bar{R}].$$

The computation of the derivatives of χ gives

$$\frac{\partial \chi}{\partial t} = -c(1+R^2)^{\frac{1}{2}}\varphi', \qquad D\chi = -\varphi' \frac{x}{(1+|x|^2)^{\frac{1}{2}}},$$

$$D^2\chi = -\varphi' \left(\frac{I}{(1+|x|^2)^{\frac{1}{2}}} - \frac{x \otimes x}{(1+|x|^2)^{\frac{3}{2}}} \right) + \varphi'' \frac{x \otimes x}{1+|x|^2} .$$

Substituting the derivatives in $A[\cdot]$ (see Lemma 3.2.1), we have

$$\begin{aligned} A[\chi](x,t) = &-c(1+R^2)^{1/2}\varphi'(r) - K_1(1+|x|)|\varphi'(r)|\frac{|x|}{(1+|x|^2)^{1/2}} \\ &-K_2\left(\mathrm{Tr}\left[-\varphi'(r)\left(\frac{I}{(1+|x|^2)^{\frac{1}{2}}} - \frac{x\otimes x}{(1+|x|^2)^{\frac{3}{2}}}\right) + \varphi''(r)\frac{x\otimes x}{1+|x|^2}\right]^+\right)^\alpha \end{aligned} \tag{3.11}$$

for all $(x,t) \in \mathcal{D}(c,R)$.

We estimate (3.11) from below. First

$$-K_1(1+|x|)|\varphi'(r)|\frac{|x|}{(1+|x|^2)^{\frac{1}{2}}} \geq 2K_1\varphi'(r)\bar{R}, \tag{3.12}$$

for every $(x,t) \in \mathcal{D}(c,R)$ since $|\varphi'(r)| = -\varphi'(r) \geq 0$. Noticing that the matrices

$$X = -\varphi'(r)\left(\frac{I}{(1+|x|^2)^{\frac{1}{2}}} - \frac{x\otimes x}{(1+|x|^2)^{\frac{3}{2}}}\right) \quad \text{and} \quad Y = \varphi''(r)\frac{x\otimes x}{1+|x|^2}$$

are nonnegative and, using the inequality $(a+b)^\alpha \leq a^\alpha + b^\alpha$ for $a,b \geq 0$ and $0 < \alpha < 1$, we obtain

$$\begin{aligned} (\mathrm{Tr}[X+Y]^+)^\alpha = (\mathrm{Tr}X + \mathrm{Tr}Y)^\alpha &\leq (\mathrm{Tr}X)^\alpha + (\mathrm{Tr}Y)^\alpha \\ &\leq N^\alpha(-\varphi'(r))^\alpha + N^\alpha(\varphi''(r))^\alpha. \end{aligned} \tag{3.13}$$

From (3.11), (3.12) and (3.13), it follows

$$\begin{aligned} A[\chi](x,t) &\geq -\varphi'(r)(c-2K_1)\bar{R} - K_2N^\alpha\big(-\varphi'(r)\big)^\alpha - K_2N^\alpha\big(\varphi''(r)\big)^\alpha \\ &\geq k(c-2K_1)\bar{R}\frac{R^\beta}{r^{k+1}} - K'_2\frac{R^{\alpha\beta}}{r^{\alpha(k+1)}} - K'_3\frac{R^{\alpha\beta}}{r^{\alpha(k+2)}} \\ &\geq \frac{R^\beta}{r^{k+1}}\Big(\underbrace{k(c-2K_1)\bar{R} - K'_2R^{(\alpha-1)\beta}r^{(k+1)(1-\alpha)} - K'_3R^{(\alpha-1)\beta}r^{(k+1)-\alpha(k+2)}}_{S(r)}\Big) \end{aligned}$$

where K_2' and K_3' are positive constants which depend only on N, K_2, k and α.

We want to choose c, β and k in order that the quantity $S(r)$ in the previous inequalities is positive. We first choose k such that the quantity $S(r)$ is nonincreasing on $[0, \bar{R}]$: since the exponent $(k+1)(1-\alpha)$ of the first term in r is already positive for every choice of $k>0$, it is enough to take

$$k \geq \frac{2\alpha - 1}{1-\alpha} \quad \Longrightarrow \quad (k+1) - \alpha(k+2) \geq 0\ , \tag{3.14}$$

in order to ensure that the exponent of the second term in r is also positive.

We are then left to choose parameters in order to have $S(\bar{R}) > 0$; to do so, a necessary condition is clearly

$$c > 2K_1\ . \tag{3.15}$$

Using the fact that $R \leq \sqrt{1+R^2} = \bar{R}$, we have

$$\begin{aligned} S(\bar{R}) &= k(c-2K_1)\bar{R} - K_2' R^{(\alpha-1)\beta}\bar{R}^{(1-\alpha)(k+1)} - K_3' R^{(\alpha-1)\beta}\bar{R}^{(k+1)-\alpha(k+2)} \\ &\geq k(c-2K_1)\bar{R} - K_2'\bar{R}^{(1-\alpha)(k+1-\beta)} - K_3'\bar{R}^{(\alpha-1)\beta+(k+1)-\alpha(k+2)}. \end{aligned}$$

Now it is clear that, if we can take the exponents of $\bar{R}$ in the two last terms strictly less than 1, then we are done since for large R, the right-hand side of the inequality would be strictly positive.

This yields the following conditions

$$(1-\alpha)(k+1-\beta) < 1, \tag{3.16}$$
$$(\alpha-1)\beta + (k+1) - \alpha(k+2) < 1. \tag{3.17}$$

We recall that $0 < \alpha < 1$; on one hand, an easy computation shows that Condition (3.17) is automatically satisfied when (3.16) holds. On the other hand, Condition (3.16) holds if we choose β such that

$$k > \beta > k - \frac{\alpha}{1-\alpha}. \tag{3.18}$$

Finally, we can fix all the constants in order to fulfill (3.14), (3.15) and (3.18) and thus all the required properties of χ are satisfied. □

In fact, a close look at the proof of Theorem 3.2.1 shows that the existence of the “friendly giants” χ_R allows to get some local estimate on the difference between two solutions. It leads to the following kind of stability result which will be useful later.

Proposition 3.2.1 *Let $(F_\varepsilon)_{\varepsilon>0}$ be a family of continuous functions satisfying assumptions **(H1)** and **(H2)** uniformly with respect to ε. Assume that there exists continuous viscosity solutions u^ε and v^ε of equation (3.4) with F replaced by F_ε such that $(u^\varepsilon - v^\varepsilon)(\cdot, 0)$ converges locally uniformly to 0 in $\mathbb{R}^N$. Then $u^\varepsilon - v^\varepsilon$ converges locally uniformly to 0 in $\mathbb{R}^N \times [0,T]$.*

Proof of Proposition 3.2.1. Since the general case will follow by iterating in time, we only prove the result for $t \leq 1/2c$ recalling that c is the constant appearing in the definition of the “friendly giants” χ_R (see (3.6)).

We first observe that, for any $R>0$, the function $u^\varepsilon - \sup_{B(0,R)}\{(u^\varepsilon - v^\varepsilon)(\cdot,0)\}$ is still a solution of (3.4) with F replaced by F_ε and that we have

$$u^\varepsilon(\cdot,0) - \sup_{B(0,R)}\{(u^\varepsilon - v^\varepsilon)(\cdot,0)\} \le v^\varepsilon(\cdot,0) \quad \text{in } B(0,R)\,.$$

Since F_ε satisfies **(H2)** independently of ε, we get from inequality (3.8) that, for all $(x,t)\in\mathcal{D}(c,R)$,

$$u^\varepsilon(x,t) - v^\varepsilon(x,t) \le \sup_{B(0,R)}\{(u^\varepsilon - v^\varepsilon)(\cdot,0)\} + \chi_R(x,t).$$

In order to prove that $u^\varepsilon - v^\varepsilon$ converges locally uniformly to 0, we take an arbitrary $\eta>0$ and (x,t) in a compact subset K of $\mathbb{R}^N\times[0,1/2c]$. For R sufficiently large, we have $\chi_R(x,t)\le\eta/2$ in K. Then, for ε sufficiently small, we have $\sup_{B(0,R)}\{(u^\varepsilon - v^\varepsilon)(\cdot,0)\}\le\eta/2$. Therefore $u^\varepsilon(x,t)-v^\varepsilon(x,t)\le\eta$ in K. And thus $\limsup\limits_{\varepsilon\to0}\sup\limits_{K}\,(u^\varepsilon - v^\varepsilon)\le 0$.

By exchanging the roles of u^ε and v^ε, we obtain the lower estimate and the proof is complete. □

3.3 Existence result for equation (3.4)

In this section we use the comparison theorem to get the following existence result.

Theorem 3.3.1 *Suppose that **(H1)** and **(H2)** hold. Then, for every initial datum $u_0\in C(\mathbb{R}^N)$, there exists a unique continuous viscosity solution of (3.4).*

Proof of Theorem 3.3.1. The uniqueness part is given by Theorem 3.2.1. For the existence, we set $\varphi_\varepsilon(r):=\min\{\max\{r,-1/\varepsilon\},1/\varepsilon\}$ for every $r\in\mathbb{R}$ and $\varepsilon>0$ and consider truncations of pde (3.4) of the form

$$\frac{\partial u}{\partial t} + F_\varepsilon(x,t,Du,D^2u) = 0, \tag{3.19}$$

where $F_\varepsilon = \varphi_\varepsilon\circ F$. Noticing that $\varphi_\varepsilon(a)-\varphi_\varepsilon(b)\le\max\{a-b,0\}$, we see that the F_ε satisfy assumptions **(H1)** and **(H2)**, uniformly with respect to ε.

We first get an existence result for (3.19) using Perron's method. Since $|F_\varepsilon(x,t,p,M)|\le 1/\varepsilon$ for all (x,t,p,M), the function $\bar u_\varepsilon(x,t):=u_0(x)+t/\varepsilon$ (respectively $\underline{u}_\varepsilon(x,t):=u_0(x)-t/\varepsilon$) is a supersolution (respectively a subsolution) of (3.19). Theorem 3.2.1 provides a strong comparison result for (3.19) and therefore Perron's method applies readily giving the existence of a continuous solution u_ε of (3.19) with initial datum u_0.

The next step consists in deriving a local L^∞-bound for the family $(u_\varepsilon)_{\varepsilon>0}$ we built above. To do so, we use of the "friendly giants" introduced in Section 3.2. Let $R,c>0$ and $\mathcal{D}(c,R)$ defined by formula (3.7). We set

$$C_R = 2\sup_{B(0,R)}|u_0|,\quad K_R = 2\sup_{\mathcal{D}(c,R)}|F(\cdot,\cdot,0,0)| \ge 2\sup_{\mathcal{D}(c,R)}|F_\varepsilon(\cdot,\cdot,0,0)|,$$

and we consider the functions $(x,t)\mapsto -C_R-K_Rt-\chi_R(x,t)$ and $(x,t)\mapsto C_R+K_Rt+\chi_R(x,t)$ in $\mathcal{D}(c,R)$. Tedious but straightforward computations shows that these functions are respectively sub- and supersolution of (3.19) in $\mathcal{D}(c,R)$ and it is clear that we have $-C_R - \chi_R(x,0) \le u_\varepsilon(x,0) \le C_R + \chi_R(x,0)$ in $B(0,R)$. Then, easy comparison arguments show that

$$-C_R - K_R t - \chi_R \le u_\varepsilon \le C_R + K_R t + \chi_R \quad \text{in } \mathcal{D}(c,R)\ .$$

It follows that the family $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $D(c,R/2)$ independently of ε. Iterating in time, we get the local boundedness of $(u_\varepsilon)_{\varepsilon>0}$ in $\mathbb{R}^N \times [0,+\infty)$.

Finally we apply the "half-relaxed-limits" method which consists in introducing

$$\bar{u} = \limsup_{\varepsilon\to 0^+} {}^* u_\varepsilon \quad \text{and} \quad \underline{u} = \liminf_{\varepsilon\to 0^+} {}_* u_\varepsilon$$

which are well-defined because of the local L^∞-bound on the u_ε's. Moreover, they are respectively discontinuous sub- and super-solution of (3.4) with initial datum u_0 since (F_ε) converges locally uniformly to F. The strong comparison result for (3.4) (Theorem 3.2.1) shows that $\bar{u} = \underline{u} := u$ and u is the desired continuous solution of (3.4) we wanted to build. □

3.4 Uniqueness for one-dimensional equations

In this section, we provide some applications of the previous result in the case of one-dimensional quasilinear parabolic equations. We address here only uniqueness questions.

We consider the equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \left(f(x,t,u,u_x)\right)_x = 0 & \text{in } \mathcal{D}'(\mathbb{R}\times(0,+\infty)), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}, \end{cases} \tag{3.20}$$

where $u_0 \in C(\mathbb{R})$, the nonlinearity $f \in C(\mathbb{R}\times[0,+\infty)\times\mathbb{R}\times\mathbb{R})$ and $\mathcal{D}'(\mathbb{R}\times(0,+\infty))$ is the space of distributions on $\mathbb{R}\times(0,+\infty)$. For reasons which will be clear below, we consider only cases when the solution u is in $C^1(\mathbb{R}\times(0,+\infty))$.

To state our result, we use the following assumption on f

(H3) f is locally Lipschitz continuous in $\mathbb{R}\times(0,+\infty)\times\mathbb{R}\times\mathbb{R}$ and there exists constants $C>0$ and $0<\alpha<1$ such that, for any $t>0$ and $x,y,u,v,p,q\in\mathbb{R}$, we have

$$\begin{aligned} &f(x,t,u,p) - f(y,t,v,q) \\ &\qquad\qquad \le C\left[(1+|p|+|q|)|x-y| + (1+|x|+|y|)|u-v| + \left((p-q)^+\right)^\alpha\right]\ . \end{aligned}$$

Note that this assumption implies **(H1)** and **(H2)** in the one-dimensional case. Equation (3.20) makes sense in the space of distributions since the assumed regularity for the solution ensures that both u and $f(x,t,u,u_x)$ belong to $L^1_{\mathrm{loc}}(\mathbb{R}\times(0,+\infty))$.

Our result is the

Theorem 3.4.1 *Under assumptions **(H3)**, Equation (3.20) has at most one solution in $C^1(\mathbb{R}\times(0,+\infty))\cap C(\mathbb{R}\times[0,+\infty))$ for each initial datum $u_0\in C(\mathbb{R})$.*

Proof of Theorem 3.4.1. We suppose, by contradiction, that we have two solutions $u, v \in C^1(\mathbb{R}\times(0,+\infty)) \cap C(\mathbb{R}\times[0,+\infty))$ of Equation (3.20). For every $\varepsilon > 0$, we define

$$\tilde{u}^\varepsilon(x,t) = \int_0^x u(y,t+\varepsilon)\,dy + \int_\varepsilon^{t+\varepsilon} f(0,\tau,u(0,\tau),u_x(0,\tau))\,d\tau$$

and $\tilde{v}^\varepsilon$ in the same way replacing u by v in the right-hand side of this equality.

By standard arguments in the theory of distributions, one proves easily that $\tilde{u}^\varepsilon$ and $\tilde{v}^\varepsilon$ are classical solutions of

$$\frac{\partial\omega}{\partial t} - f(x,t+\varepsilon,\omega_x,\omega_{xx}) = 0 \quad \text{in } \mathbb{R}\times(0,+\infty)\,. \tag{3.21}$$

The functions $f(\cdot,\cdot+\varepsilon,\cdot,\cdot)$ satisfy assumptions **(H1)** and **(H2)** uniformly in ε, and

$$(\tilde{u}^\varepsilon - \tilde{v}^\varepsilon)(x,0) = \int_0^x (u(y,\varepsilon) - v(y,\varepsilon))dy$$

converges locally uniformly to 0 in $\mathbb{R}$ since $u(x,0) = v(x,0) = u_0(x)$ in $\mathbb{R}$. Therefore, by applying Proposition 3.2.1, we deduce that $\tilde{u}^\varepsilon - \tilde{v}^\varepsilon$ converges to 0 locally uniformly in $\mathbb{R}\times[0,+\infty)$.

It follows, on one hand, that the integral

$$\int_0^t [f(0,\tau,u(0,\tau),u_x(0,\tau)) - f(0,\tau,v(0,\tau),v_x(0,\tau))]d\tau$$

is well-defined (notice that it was not the case a priori for $\int_0^t f(0,\tau,u(0,\tau),u_x(0,\tau))d\tau$ and $\int_0^t f(0,\tau,v(0,\tau),v_x(0,\tau))d\tau$) and letting ε go to 0, we obtain that, for all $(x,t) \in \mathbb{R}\times[0,+\infty)$,

$$\int_0^x (u-v)(y,t)dy + \int_0^t [f(0,\tau,u(0,\tau),u_x(0,\tau)) - f(0,\tau,v(0,\tau),v_x(0,\tau))]d\tau = 0.$$

To prove the result, we have just to differentiate this equality with respect to x. □

Remark 3.4.1 : It would be very interesting to be able to prove the above result by assuming only the solutions to be in $W^{1,\infty}_{\mathrm{loc}}$. The difficulty to do that would be in the integration of equation (3.20) to get equation (3.21). Only few results exists in this direction and mainly for first-order equations (cf. Corrias [36]).

A key application of the above result concerns the mean curvature equation for graphs in $\mathbb{R}$.

Theorem 3.4.2 *The mean curvature equation for graphs (3.2) has a unique smooth solution $u \in C(\mathbb{R}\times[0,+\infty)) \cap C^\infty(\mathbb{R}\times(0,+\infty))$ for every initial datum $u_0 \in C(\mathbb{R})$.*

Proof of Theorem 3.4.2. The existence part of the theorem comes from the result of Ecker and Huisken [41] (see also [16]). The uniqueness part is an immediate consequence of Theorem 3.4.1 by taking $f(p) = \arctan(p)$. In this case (3.20) reads exactly (3.2) and, since the function arctan is a smooth nondecreasing bounded function which lies in $W^{1,\infty}(\mathbb{R})$, thus in $C^{0,\alpha}(\mathbb{R})$ for every $\alpha \in (0,1)$, **(H3)** holds and therefore Theorem 3.4.1 applies. □

3.5 Application to geometrical motions in the plane

We consider, in this section, applications for the mean curvature motion of graphs in the plane $\mathbb{R}^2$. We first recall briefly some basic facts about the level-set approach in the case of the mean curvature motion.

In this framework, the graph of the initial datum $u_0 \in C(\mathbb{R})$ of (3.2) is represented as the hypersurface $\Gamma_0 = \{(x,y) \in \mathbb{R}^2 : u_0(x) = y\}$ in the plane. We also define $\Omega_0 = \{(x,y) \in \mathbb{R}^2 : u_0(x) < y\}$ and we take any uniformly continuous function $v_0 : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\Gamma_0 = \{(x,y) \in \mathbb{R}^2 : v_0(x,y) = 0\} \quad \text{and} \quad \Omega_0 = \{(x,y) \in \mathbb{R}^2 : v_0(x,y) > 0\}. \quad (3.22)$$

If $\mathcal{X}$ is the space of functions $w : \mathbb{R}^2 \times (0,+\infty) \to \mathbb{R}$ which are uniformly continuous in $\mathbb{R}^2 \times (0,+\infty)$ for all $T > 0$, by results of Evans and Spruck [45] and Chen, Giga and Goto [32], there exists a unique viscosity solution $v \in \mathcal{X}$ of the geometrical equation

$$\begin{cases} \dfrac{\partial v}{\partial t} - \dfrac{1}{v_x^2 + v_y^2}(v_{xx}v_y^2 - 2v_{xy}v_xv_y + v_{yy}v_x^2) = 0 & \text{in } \mathbb{R}^2 \times (0,+\infty), \\ v(\cdot,\cdot,0) = v_0 \quad \text{in } \mathbb{R}^2. \end{cases}$$

Moreover, if we define, for every $t \geq 0$,

$$\Gamma_t = \{(x,y) \in \mathbb{R}^2 : v(x,y,t) = 0\} \quad \text{and} \quad \Omega_t = \{(x,y) \in \mathbb{R}^2 : v(x,y,t) > 0\}, \quad (3.23)$$

then the sets $(\Gamma_t)_{t\geq 0}$ and $(\Omega_t)_{t\geq 0}$ depend only on Γ_0 and Ω_0 but not on the choice of their representation through v_0.

The family $(\Gamma_t)_{t\geq 0}$ is called *the generalized evolution by mean curvature of the graph* Γ_0. A natural issue is the connection between this generalized evolution and the classical motion by mean curvature. We recall that in general Γ_t is just defined as the 0-level set of a continuous function and therefore it may be nonsmooth and even fatten.

In our context, we have

Theorem 3.5.1 *If $u_0 \in C(\mathbb{R})$, then, for every $t \geq 0$, the set Γ_t is a entire smooth graph, namely*

$$\Gamma_t = \{(x,y) \in \mathbb{R}^2 : y = u(x,t)\},$$

where u is the unique smooth solution of (3.2) with initial datum u_0. Moreover, the evolution of Γ_t agrees with the classical motion by mean curvature in the sense of differential geometry.

Proof of Theorem 3.5.1. From [16], we know that, if we start with an hypersurface Γ_0 which is an entire continuous graph in $\mathbb{R} \times \mathbb{R}$, then, for every $t \geq 0$,

$$\Gamma_t = \{(x,y) \in \mathbb{R}^2 : u^-(x,t) \leq y \leq u^+(x,t)\},$$

where u^- and u^+ are respectively the minimal and the maximal (possibly discontinuous) viscosity solution of (3.2). In the special case of the mean curvature equation, we proved that the boundary of the front Γ_t is smooth. It follows that u^- and u^+ are smooth. Uniqueness for (3.2) in the class of smooth functions (see Theorem 3.4.2) implies that $u^- = u^+ = u$ where u is the unique solution to (3.2). Finally, $\Gamma_t = \mathrm{Graph}(u(\cdot,t))$ is a smooth submanifold of $\mathbb{R}^2$ (in particular, Γ_t never fattens). In this case, the generalized evolution coincides with the classical evolution by mean curvature (see Evans and Spruck [45] and [47] for the agreement with an alternative generalized motion). □

Using the previous geometrical approach, we can state a refinement of Theorem 3.4.2, namely a comparison result which holds for any viscosity sub- and supersolutions of (3.2).

Corollary 3.5.1 *Let $u_0 \in C(\mathbb{R})$. If u_1 (respectively u_2) is an upper-semicontinuous viscosity subsolution (respectively lower-semicontinuous supersolution) of (3.2) and if $u_1(x,0) \le u_0(x) \le u_2(x,0)$ in $\mathbb{R}$, then $u_1 \le u_2$ in $\mathbb{R}\times[0,+\infty)$.*

Proof of Corollary 3.5.1. This result is an immediate consequence of a more general result in the case of the mean curvature equation : uniqueness for smooth solutions implies comparison in the class of discontinuous viscosity solutions (see [16] for a proof). To be self-contained, we provide a short proof which emphasizes the main ideas of the proof.

By Theorem 6.1 in [16], since u_1 and u_2 are respectively an upper-semicontinuous viscosity subsolution and a lower-semicontinuous supersolution of (3.2), we have

$$\mathrm{Graph}(u_1(\cdot,t)) \subset \{(x,y)\in\mathbb{R}^{N+1} : v(x,y,t) \le 0\}\ ,$$

and

$$\mathrm{Graph}(u_2(\cdot,t)) \subset \{(x,y)\in\mathbb{R}^{N+1} : v(x,y,t) \ge 0\}\ .$$

Essentially, this comes from the preservation of inclusion for sets moving by their mean curvature in the level set approach (see Evans and Spruck [45]).

We recall that v is an increasing function of y but the above inclusions do not give any information about the relative position of $\mathrm{Graph}(u_1(\cdot,t))$ and $\mathrm{Graph}(u_2(\cdot,t))$ when the fronts $\Gamma_t(u_0)$ develop interior. But, thanks to Theorem 3.5.1, we know that $\Gamma_t(u_0)$ is exactly the graph of $u(\cdot,t)$, where u is the unique smooth solution of (3.2) with initial datum u_0. The front does not fatten and is the boundary of $\Omega_t(u_0)$. It follows that

$$u_1(\cdot,t) \le u(\cdot,t) \le u_2(\cdot,t) \quad \text{in } \mathbb{R}^N\ ,$$

which ends the proof. □

We conclude this section with an extension of the previous results to some more general equations associated to more general geometric motions. We consider

$$\begin{cases} \dfrac{\partial u}{\partial t} - (f(u_x))_x = 0 & \text{in } \mathbb{R}\times(0,+\infty),\\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}, \end{cases} \tag{3.24}$$

where $f \in C^1(\mathbb{R})$.

Theorem 3.5.2 *Let $u_0 \in W^{1,\infty}_{\rm loc}(\mathbb{R})$. Suppose that*

$$f \in C^1(\mathbb{R}) \quad \text{and} \quad 0 < f'(p) \le \frac{C}{1+p^2} \quad \text{for every } p \in \mathbb{R}. \tag{3.25}$$

Then (3.24) admits a unique solution $u \in C^2(\mathbb{R} \times (0,+\infty)) \cap C(\mathbb{R} \times [0,+\infty))$. The generalized evolution of $\Gamma_0 = \mathrm{Graph}(u_0)$ is $\Gamma_t = \mathrm{Graph}(u(\cdot,t))$ for $t \ge 0$. It evolves with normal velocity equal to

$$\mathcal{V}((x,y),t) = f'(-\mathrm{cotan}(\theta))\frac{\kappa}{\sin^2(\theta)}, \tag{3.26}$$

where θ and κ denote respectively the angle between the y-axis and the normal outward vector and the curvature to Γ_t at the point (x,y).

Proof of Theorem 3.5.2. We consider as above an uniformly continuous function $v_0 : \mathbb{R}^2 \to \mathbb{R}$ such that $\{v_0 = 0\} = \Gamma_0$ and $\{v_0 > 0\} = \Omega_0 := \{y > u_0(x)\}$. Following [16, Section 4], if (3.25) holds, then

$$\begin{cases} \dfrac{\partial v}{\partial t} - f'\left(-\dfrac{v_x}{v_y}\right)(v_{xx} - 2v_{xy}\left(\dfrac{v_x}{v_y}\right) + v_{yy}\left(\dfrac{v_x}{v_y}\right)^2) = 0 \quad \text{in } \mathbb{R}^2 \times (0,+\infty), \\ v(\cdot,\cdot,0) = v_0 \quad \text{in } \mathbb{R}^2 \end{cases} \tag{3.27}$$

admits a unique solution $v \in \mathcal{X}$. The level set approach applies for the generalized evolution $(\Gamma_t)_{t\ge 0}$ of Γ_0 and evolves formally with normal velocity given by (3.26).

On a other hand, using Chou and Kwong [33], we learn from [16] that, again because of (3.25), the boundary of Γ_t is made of the graphs of two smooth solutions of (3.24), namely u^+ and u^-. Noticing that (3.25) implies **(H3)**, we obtain that (3.24) has a unique smooth solution; thus $u^+ = u^- := u$ and $\Gamma_t = \mathrm{Graph}(u(\cdot,t))$. Finally, note that, since Γ_t is smooth, (3.26) holds in a classical sense. □

Chapitre 4

Quasilinear parabolic equations, unbounded solutions and geometrical equations III. Uniqueness through classical viscosity solutions' methods

Guy Barles, Samuel Biton, Mariane Bourgoing and Olivier Ley
Laboratoire de Mathématiques et Physique Théorique
Université de Tours,
Parc de Grandmont, 37200 Tours, France

Abstract

In this article, we are interested in uniqueness results for viscosity solutions of a general class of quasilinear, possibly degenerate, parabolic equations set in $\mathbb{R}^N$. Using classical viscosity solutions' methods, we obtain a general comparison result for solutions with polynomial growths but with a restriction on the growth of the initial data. The main application is the uniqueness of solutions for the mean curvature equation for graphs which was only known in the class of uniformly continuous functions. An application to the mean curvature flow is given.

Key-words : Quasilinear parabolic equations, solutions with polynomial growth, comparison results, mean curvature equation, viscosity solutions.
AMS subject classifications : 35A05, 35B05, 35D05, 35K15, 35K55, 53C44

[0]This work is a preprint of the University of Tours which is submitted to publication.

Sommaire

4.1 Introduction

This article is a continuation of the program started in [16] (see [15] for an introductive paper) which aim is the study of the uniqueness properties for unbounded viscosity solutions of quasilinear, possibly degenerate, parabolic equations set in $\mathbb{R}^N$. This program was motivated by the following surprising result of Ecker and Huisken [41] : for any initial data $u_0 \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N)$, there exists a smooth solution of the equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \dfrac{\langle D^2uDu, Du\rangle}{1+|Du|^2} = 0 \quad \text{in } \mathbb{R}^N \times (0,+\infty)\,, \\ u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N\,. \end{cases} \tag{4.1}$$

Here and below the solution u is a real-valued functions, Du and D^2u denote respectively its gradient and Hessian matrix while $|\cdot|$ and $\langle\cdot,\cdot\rangle$ stand for the classical Euclidean norm and inner product in $\mathbb{R}^N$.

The very non-standard feature of this result is that no assumption on the behavior of the initial data at infinity is imposed and therefore the solutions may have also any possible behavior at infinity.

A natural question is then whether such a solution is unique or not. It is a very intriguing and challenging question since one has to take in account the lack of restriction on the behavior of the solutions at infinity, an unusual fact. As far as we know, this question in its full generality is still open in $\mathbb{R}^N$ for $N > 1$ while for $N = 1$, the result was proved independently and by different methods by Chou and Kwong [33] and Barles, Biton and Ley [17].

In a series of papers ([16], [17] and [18]), we address the more general question of the uniqueness of unbounded viscosity solutions, not only for (4.1) but also for more general quasilinear degenerate parabolic equations like

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mathrm{Tr}\left[b(x,t,Du)D^2u\right] + H(x,t,u,Du) = 0 \quad \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N, \end{cases} \tag{4.2}$$

where $T > 0$ is any positive constant and b is a function taking values in the set of nonnegative symmetric matrix and $H \in C(\mathbb{R} \times [0,T] \times \mathbb{R} \times \mathbb{R}^N)$.

When one wants to prove such uniqueness results, the first key difficulty is the one we point out above i.e. the a priori unboundedness of solutions or more generally the fact that solutions may have any behavior at infinity. But the gradient dependence in the diffusion matrix b above is also a major difficulty : because of that, it does not seem possible to obtain uniqueness results through some kind of linearization procedure and even the uniqueness of smooth solutions is far from being obvious, except if one imposes restrictions on D^2u at infinity, a type of assumptions that we want to avoid. For results obtained by linearization procedure, we refer the reader for example to Crandall and Lions [38], Ishii [58] or Ley [71] where the uniqueness of solutions of first-order equations were obtained using "finite speed of propagation" type properties, to Barles [13] where optimal uniqueness results for solutions with exponential growth of (stationary) first-order equations were proved or to Barles, Buckdahn and Pardoux [19] for solutions with exponential growths of a system of Hamilton-Jacobi-Bellman type Equations.

To the best of our knowledge, the most general uniqueness results for quasilinear equations – i.e. for equations involving the above mentioned difficulty on the Du-dependence – concern only uniformly continuous viscosity solutions : we refer the reader the "Users'guide of viscosity solutions" of Crandall, Ishii and Lions [37] and to Giga, Goto, Ishii and Sato [51] for results in this direction.

The aim of this article is to push as far as possible the classical arguments used for proving comparison results for viscosity solutions in order to obtain such results for the largest possible class of quasilinear equations and initial datas.

In the general comparison theorem we are able to prove by using this approach (see Theorem 4.2.1), the conditions we have to impose on the equation and the behavior of solution at infinity, namely to have a polynomial growth, seem rather reasonable. Surprisingly the main restriction concerns the initial data which has to satisfy the following, rather unnatural, condition : there exists a modulus of continuity[2] m and $0 \leq \nu < (1+\sqrt{5})/2$ such that, for any $x, y \in \mathbb{R}^N$,

$$|u_0(x) - u_0(y)| \leq m\left((1+|x|+|y|)^\nu |x-y|\right) . \tag{4.3}$$

Unfortunately such type of restriction seems to be an unavoidable artefact of this method. We do not know how optimal is this result and in particular the limiting exponent $(1+\sqrt{5})/2$. We think that the result is true for any ν but we were unable to prove it.

Crandall and Lions [39] obtained related results under similar assumptions but for fully nonlinear pdes whose Hamiltonians depend only on the second derivatives of the solution u. Their techniques do not apply to our case. Comparing to the existence result for the mean curvature mentioned above, condition (4.3) may appear restrictive but we point out that our proof works under a general structure assumption on b and not only for the mean curvature equation. We refer to Section 4.2 for a precise statement of the assumptions on b and H and to Section 4.3 for a detailed treatment of the mean curvature equation for graphs.

Unfortunately the proof of this comparison result is very technical (see Section 4.4) and relies on the use of a tricky test-function that, as we already mentioned it above, we tried to build in an optimal way.

This result yields a comparison principle between semicontinuous sub- and supersolutions which is a key tool for obtaining existence results through the Perron's method (see [60], [37], etc.). It therefore allows us to show such an existence result of solutions with a suitable growth for (4.2).

The paper is divided as follows : in the next section, we start by setting the problem and the assumptions we will use. Then we state a comparison principle (Theorem 4.2.1) which is the main result of the paper. As an immediate consequence, we obtain the existence and uniqueness of a continuous viscosity solution to our problem (Corollary 4.2.1). We end this section with some examples of equations which are included in our study. Section 4.3 deals with the fundamental example of the mean curvature equation for graphs. We provide an application of the uniqueness theorem to the mean curvature flow of entire graphs. The last sections 4.4 and 4.5 are devoted to the proofs of the main theorems.

[2] A function $m : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a modulus of continuity if $m(0_+) := \lim_{s\to 0_+} m(s) = 0$ and $m(t+s) \leq m(t) + m(s)$ for any $s, t \in \mathbb{R}^+$.

Acknowledgment. This work was partially supported by the TMR program "Viscosity Solutions and Their Applications." The last author would like to thank the Department of Mathematics of the university of Padova and especially Martino Bardi for their kind hospitality and fruitful exchanges.

4.2 Statement of the results and examples

Before stating the problem and our results, we introduce some notations. In the sequel, $\mathcal{M}_{N,M}$ is the set of $N \times M$–matrices and $\mathcal{S}_N$ (respectively $\mathcal{S}_N^+$) denotes the set of the symmetric (respectively symmetric nonnegative) matrices. For every $A \in \mathcal{M}_{N,M}$, A^T denotes the transpose of A. Finally, we introduce the space $\mathcal{C}_{\text{poly}}$ of locally bounded, possibly discontinuous, functions on $\mathbb{R}^N \times [0,T]$ which have a polynomial growth with respect to the space variable. More precisely, $u \in \mathcal{C}_{\text{poly}}$ if there exists a constant $k > 0$ such that

$$\frac{u(x,t)}{1+|x|^k} \underset{|x|\to+\infty}{\longrightarrow} 0, \text{ uniformly with respect to } t \in [0,T].$$

We will use the following assumptions for equation (4.2) :

(H1) There exists a continuous function $\sigma : \mathbb{R}^N \times [0,T] \times \mathbb{R}^N \to \mathcal{M}_{N,M}$ and some constant $\tilde{C}$ such that

$$b(x,t,p) = \sigma(x,t,p)\sigma(x,t,p)^T,$$
$$\|\sigma(x,t,p) - \sigma(y,t,p)\| \le \tilde{C}|x-y|,$$
$$\|\sigma(x,t,p) - \sigma(x,t,q)\| \le \tilde{C}\frac{|p-q|}{1+|p|+|q|}.$$

(H2) The function H is continuous on $\mathbb{R}^N \times [0,T] \times \mathbb{R} \times \mathbb{R}^N$, $u \mapsto H(x,t,u,p)$ is nondecreasing for every (x,t,p) and there exists a modulus of continuity $\tilde{m}$ such that, for every $x,y,p,q \in \mathbb{R}^N$, $u \in \mathbb{R}$ and $t \in [0,+\infty)$,

$$|H(x,t,u,p) - H(y,t,u,p)| \le \tilde{m}\left((1+|p|)|x-y|\right) ,$$

and

$$|H(x,t,u,p) - H(x,t,u,q)| \le \tilde{m}\left((1+|x|)|p-q|\right) .$$

Finally we say that a function $\omega : \mathbb{R}^N \to \mathbb{R}$ satisfies the assumption **(H3-ω)(ν)** if there exists a modulus of continuity m such that, for every $x, y \in \mathbb{R}^N$,

$$|\omega(x) - \omega(y)| \le m\left((1+|x|+|y|)^\nu|x-y|\right) .$$

Our main result is the following

Theorem 4.2.1 *Assume that **(H1)** and **(H2)** hold and let $u \in \mathcal{C}_{\text{poly}}$ (respectively $v \in \mathcal{C}_{\text{poly}}$) be a upper-semicontinuous (respectively lower-semicontinuous) viscosity sub-solution (respectively viscosity supersolution) of (4.2). If*

$$u(x,0) \le u_0(x) \le v(x,0) \quad \text{in } \mathbb{R}^N ,$$

*where u_0 is a function which satisfies **(H3-u_0)(ν)** with $0 \le \nu < (1+\sqrt{5})/2$, then*

$$u \le v \quad \text{in } \mathbb{R}^N \times [0,T].$$

The proof is postponed to Section 4.4. An immediate consequence of Theorem 4.2.1 is the

Corollary 4.2.1 *Assume that **(H1)** and **(H2)** hold and let u_0 be a function which satisfies*
***(H3-**u_0**)**(ν), with $0 \le \nu < (1+\sqrt{5})/2$. In $\mathcal{C}_{\rm poly}$, there exists a unique continuous viscosity solution u of (4.2). Moreover, there exists $C > 0$ such that*

$$|u(x,t)| \le C(1+|x|^{\nu+1}).$$

The existence of a solution is a consequence of the comparison result by using Perron's method. A priori, this existence results takes place in the class of functions with polynomial growth but we prove that the solution inherits the growth of the initial datum. The proof will be given in Section 4.5.

Remark 4.2.1 : Concerning the existence of classical solutions to these equations, we refer to Chou and Kwong [33] who provided $W^{1,\infty}$ local bounds for the solutions of a large class of quasilinear equations without growth restriction on the initial data. The existence of a smooth solution to the mean curvature equation for graphs, also in the case when there is no restriction on the initial data, was first shown by Ecker and Huisken [41] and Evans and Spruck [47] (see also [16]).

Before giving examples of equations to which these results apply, let us make some comments about the assumption **(H1)** and **(H3-**ω**)**(ν), **(H2)** being a natural and classical assumption. In **(H1)**, the two first conditions are classical; note that the first one implies that the equation is degenerate parabolic. Of course, the third one is the most interesting since it concerns the behavior of b in the gradient variable and we recall that it is a key difficulty here. We have chosen this type of assumption for σ in p because this is the type of dependence we have for (4.1). We have decided not to consider in this paper different choices of p–dependence in order to keep the length of this paper reasonable. They would lead to results with, in particular, different limitation on the growth of the initial data.

It is anyway worth pointing out that, with such type of assumptions on σ, there is no hope to prove a better result than a comparison principle in $\mathcal{C}_{\rm poly}$ or for solutions with exponential growth : indeed the heat equation satisfies **(H1)**. In order to go further, one has to take in account some degeneracy of the equation (typically $\sigma(p)p \to 0$ as $|p| \to +\infty$) but our proof does not see at all this kind of property.

Notice that the assumptions on σ yield the existence of a positive constant C such that, for every $(x,t,p) \in \mathbb{R}^N \times [0,T] \times \mathbb{R}^N$,

$$\|\sigma(x,t,p)\| \le C(1+|x|)\,, \tag{4.4}$$

a key fact in the proof.

A more readable version of assumption **(H3-**ω**)**(ν) on the initial data of the equation is when ω is locally Lipschitz continuous, then **(H3-**ω**)** holds if

$$|D\omega(x)| \le C\,(1+|x|^{\nu}) \quad \text{for almost every } x \in \mathbb{R}^N.$$

We end this section by a list of equations for which the previous results apply.

1. Equations of the form

$$\frac{\partial u}{\partial t} - a(x,t)\frac{\Delta u}{(1+|Du|^2)^\alpha} + H(x,t,u,Du) = 0.$$

The above results apply when $\alpha > 0$, H satisfies **(H2)** and $a \in C(\mathbb{R}^N \times [0,+\infty))$ is a bounded nonnegative function such that $\sqrt{a}$ is Lipschitz continuous in x, uniformly with respect to t. Since it is not obvious, we check the third statement of **(H1)**. We note first that, if the statement holds for positive α_1 and α_2, then it holds for $\alpha = \alpha_1 + \alpha_2$. Therefore, it is sufficient to prove it for $0 < \alpha \le 1$. Without loss of generality, we can suppose $a \equiv 1$, the dimension of the space $N = 1$ and $q \ge p \ge 0$. We have

$$\sigma(p) - \sigma(q) = \frac{1}{(1+p^2)^\alpha}\left(1 - \left(1 + \frac{p^2-q^2}{1+q^2}\right)^\alpha\right).$$

From the inequality $1-(1+x)^\alpha \le -x$ for $-1 \le x \le 0$, we get

$$\sigma(p) - \sigma(q) \le \frac{1}{(1+p^2)^\alpha}\frac{(q-p)(p+q)}{1+q^2} \le \frac{2q}{1+q^2}(q-p).$$

Since we assumed $0 \le p \le q$, we can find a constant C independent of p, q such that $2q/(1+q^2) \le C/(1+p+q)$. It gives the conclusion.

2. Non geometric curvature flow :

$$\frac{\partial u}{\partial t} - \mathrm{div}\frac{Du}{\sqrt{1+|Du|^2}} = 0.$$

3. Consider

$$\frac{\partial u}{\partial t} - \mathrm{Tr}[A(x,t)\left(I - \frac{Du \otimes Du}{1+|Du|^2}\right)A(x,t)D^2u] + H(x,t,u,Du) = 0,$$

If A is a bounded continuous function from $\mathbb{R}^N \times [0,+\infty)$ into $\mathcal{M}_N$, Lipschitz in x (uniformly in t) and H satisfies **(H2)**, then our results apply. In particular,

$$\frac{\partial u}{\partial t} - a(x,t)\sqrt{1+|Du|^2}\left(\mathrm{div}\frac{Du}{\sqrt{1+|Du|^2}} + c(x,t)\right) = 0$$

fulfills the assumptions as soon as the continuous functions a, c are bounded, a is nonnegative and $\sqrt{a}, c$ are Lipschitz continuous in x uniformly in t. If $a \equiv 1$ and c is constant, then we recognize the geometric eikonal equation. The mean curvature equation for graphs ($a \equiv 1$ and $c \equiv 0$) is studied in great details in the following section.

4.

$$\frac{\partial u}{\partial t} - \sup_{\alpha\in\mathcal{A}} \mathrm{Tr}[b_\alpha(x,t,Du)D^2u] + \sup_{\beta\in\mathcal{B}} H_\beta(x,t,u,Du) = 0,$$

when **(H1)** holds for b_α and **(H2)** holds for H_β with constants independent of α, β.

5. With minor adaptations in the proof of the comparison result we can deal with

$$\frac{\partial u}{\partial t} - f\left(\mathrm{Tr}[b(x,t,Du)D^2u]\right) + H(x,t,u,Du) = 0,$$

under **(H1)** and **(H2)** for a nondecreasing nonnegative function f which satisfies, for every $z, z' \in \mathbb{R}$,

$$f(z) - f(z') \le C|z'-z|^\alpha,$$

with $\alpha \in [0,1]$.

4.3 Application to the mean curvature equation

Our main motivation is to prove some uniqueness results for the mean curvature equation for graphs. In this section, we recall some facts about this equation and show that it enters the framework of Theorem 4.2.1. We then apply these results to the mean curvature flow for entire graphs in $\mathbb{R}^{N+1}$.

We can write the mean curvature equation in different forms. To follow the notations of the previous section, we will write it as

$$\frac{\partial u}{\partial t} - \mathrm{Tr}\left[b_{\mathrm{c}}(Du)\ D^2 u\right] = 0, \tag{4.5}$$

where, for every $p \in \mathbb{R}^N$,

$$b_{\mathrm{c}}(p) = I - \frac{p \otimes p}{1+|p|^2}\ . \tag{4.6}$$

The same equation is often written

$$\frac{\partial u}{\partial t} - \Delta u + \frac{\langle D^2 u Du, Du\rangle}{1+|Du|^2} = 0 \quad \text{in } \mathbb{R}^N \times (0,+\infty), \tag{4.7}$$

or in divergence form

$$\frac{\partial u}{\partial t} - \sqrt{1+|\nabla u|^2}\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0. \tag{4.8}$$

The operator b_{c} defined by (4.6) maps $\mathbb{R}^N$ into $\mathcal{S}_N^+$. For every $p \in \mathbb{R}^N$, we can define the positive symmetric square root $b_{\mathrm{c}}^{1/2}(p)$ of $b_{\mathrm{c}}(p) \in \mathcal{S}_N^+$. The following lemma provides some properties of $b_{\mathrm{c}}^{1/2}$.

Lemma 4.3.1 *For every $p \in \mathbb{R}^N$,*

$$b_{\mathrm{c}}^{1/2}(p) = I - \frac{1}{\sqrt{1+|p|^2}(1+\sqrt{1+|p|^2})} p \otimes p.$$

Moreover, $b_{\mathrm{c}}^{1/2}$ is bounded and Lipschitz continuous in $\mathbb{R}^N$; there exists a positive constant C such that, for every $p, q \in \mathbb{R}^N$,

$$\|b_{\mathrm{c}}^{1/2}(p) - b_{\mathrm{c}}^{1/2}(q)\| \leq \frac{C|p-q|}{1+|p|+|q|}. \tag{4.9}$$

Proof of Lemma 4.3.1. We first compute $b_{\mathrm{c}}^{1/2}$. For every $q \in (\mathrm{Span}\, p)^\perp$, $b_{\mathrm{c}}(p)q = q$ and $b_{\mathrm{c}}(p)p = p/(1+|p|^2)$. It follows that we can look for $b_{\mathrm{c}}^{1/2}$ in the form $b_{\mathrm{c}}^{1/2}(p) = I - f(p)p \otimes p$. Then, an easy calculation gives the result. Looking at the formula, it is clear that $b_{\mathrm{c}}^{1/2}$ is continuous and bounded.

To check (4.9), we write

$$\begin{aligned}
\|b_{\mathrm{c}}^{1/2}(p) - b_{\mathrm{c}}^{1/2}(q)\| &= \|f(q)q \otimes q - f(p)p \otimes p\| \\
&= \|q \otimes (q-p)f(q) + (q-p) \otimes pf(q) + p \otimes p(f(q) - f(p))\| \\
&\leq (|q|f(q) + |p|f(q))|p-q| + |p|^2|f(p) - f(q)|.
\end{aligned}$$

Without loss of generality, we can take $|q| \geq |p|$. On the one hand,

$$\begin{aligned} |q|f(q) + |p|f(q) &\leq 2|q|f(q) \leq \frac{2|q|}{\sqrt{1+|q|^2}(1+\sqrt{1+|q|^2})} \leq \frac{2}{1+\sqrt{1+|q|^2}} \\ &\leq \frac{2}{1+|q|} \leq \frac{2}{1+\frac{1}{2}(|p|+|q|)} \leq \frac{4}{1+|p|+|q|}. \end{aligned} \tag{4.10}$$

On the other hand, by setting $g(p) = \sqrt{1+|p|^2}$,

$$\begin{aligned} |p|^2|f(q) - f(p)| &\leq |p|^2 f(p)f(q)\,|g(p)(1+g(p)) - g(q)(1+g(q))| \\ &\leq |p|^2 f(p)f(q)\,|1+g(p)+g(q)|\,|g(p)-g(q)|. \end{aligned}$$

But a straightforward calculation shows that $|g(p) - g(q)| \leq |p - q|$ and that the application

$$(p,q) \mapsto |p|^2 f(p)f(q)|1+g(p)+g(q)|(1+|p|+|q|)$$

is continuous and bounded on the set $\{(p,q) \in \mathbb{R}^{2N} : |p| \leq |q|\}$ thus there exists a constant $C_1 > 0$ such that for every $p,q \in \mathbb{R}^N$, $|p| \leq |q|$, we have

$$|p|^2 f(p)f(q)\,|1+g(p)+g(q)|\,|g(p)-g(q)| \leq \frac{C_1|p-q|}{1+|p|+|q|}. \tag{4.11}$$

Combining (4.10) and (4.11), we obtain the result with $C = 4 + C_1$. □

Thanks to the previous lemma and the existence of a smooth solution to (4.7) (cf. [41], [47] and [16]) we get

Theorem 4.3.1 *Assume that $u_0 \in C(\mathbb{R}^N)$ satisfies **(H3-u_0)**(ν), with $0 \leq \nu < (1+\sqrt{5})/2$. Then (4.7) (or equivalently (4.5) or (4.8)) has a unique solution $u \in C(\mathbb{R}^N \times [0,+\infty)) \cap C^\infty(\mathbb{R}^N \times (0,+\infty))$ in $\mathcal{C}_{\mathrm{poly}}$.*

We turn to a geometrical application to the so-called level-set approach to the generalized evolution of hypersurfaces by their mean curvature. This method, introduced for numerical computations by Osher and Sethian [76], was developed theoretically by Evans and Spruck [45] and Chen, Giga and Goto [32].

Let us recall briefly the level-set approach in the case of the mean curvature motion of entire graphs. We consider the graph of the initial datum $u_0 \in C(\mathbb{R})$ of (4.7) as an hypersurface $\Gamma_0 = \{(x,y) \in \mathbb{R}^N \times \mathbb{R} : u_0(x) = y\}$ of $\mathbb{R}^{N+1}$. Define $\Omega_0 = \{(x,y) \in \mathbb{R}^{N+1} : u_0(x) < y\}$. We take a uniformly continuous function $v_0 : \mathbb{R}^{N+1} \to \mathbb{R}$ such that

$$\Gamma_0 = \{(x,y) \in \mathbb{R}^{N+1} : v_0(x,y) = 0\} \quad \text{and} \quad \Omega_0 = \{(x,y) \in \mathbb{R}^{N+1} : v_0(x,y) > 0\} \tag{4.12}$$

(choose the signed-distance to Γ_0 for instance). Next, we consider a function $v : \mathbb{R}^{N+1} \times (0,+\infty) \to \mathbb{R}$ such that $v(x,u(x,t),t) = 0$ where u is a solution of (4.7). Formally, v has to satisfy the well-known geometrical mean curvature equation

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v + \dfrac{\langle D^2 v Dv, Dv\rangle}{|Dv|^2} & \text{in } \mathbb{R}^{N+1} \times (0,T), \\ v(\cdot,\cdot,0) = v_0 \quad \text{in } \mathbb{R}^{N+1}. \end{cases}$$

This equation admits a unique viscosity solution $v \in UC(\mathbb{R}^{N+1} \times [0,+\infty))$ for every initial datum $v_0 \in UC(\mathbb{R}^{N+1})$, where UC denotes the uniformly continuous functions. Moreover, the level-set approach works : it means that we can define, for every $t \in [0,T]$,

$$\Gamma_t = \{(x,y) \in \mathbb{R}^{N+1} : v(x,y,t) = 0\} \quad \text{and} \quad \Omega_t = \{(x,y) \in \mathbb{R}^{N+1} : v(x,y,t) > 0\};$$

the sets $(\Gamma_t)_t$ and $(\Omega_t)_t$ depend only on the initial sets Γ_0 and Ω_0 but not on the choice of v_0. The family $(\Gamma_t)_t$ is called *the generalized evolution by mean curvature of the graph* Γ_0 and Γ_t is called the *front.* A natural issue is the connection between this generalized evolution and the classical motion by mean curvature of the graph of u_0. Note that Γ_t is defined as the 0-level-set of a uniformly continuous function ; it may be very irregular in general and can even develop an interior in $\mathbb{R}^{N+1}$. In our context, we have

Theorem 4.3.2 *If $u_0 \in C(\mathbb{R}^N)$ satisfies **(H3-**u_0**)**(ν), with $0 \leq \nu < (1+\sqrt{5})/2$, then, for every $t \in [0,T]$, the set Γ_t is a entire smooth graph, namely*

$$\Gamma_t = \{(x,y) \in \mathbb{R}^{N+1} : y = u(x,t)\},$$

where u is the unique smooth solution of (4.7) with initial datum u_0. Moreover, the evolution of Γ_t agrees with the classical motion by mean curvature in the sense of the differential geometry.

Proof of Theorem 4.3.2. From [16], we know that, if we start with an hypersurface Γ_0 which is an entire continuous graph in $\mathbb{R}^N \times \mathbb{R}$, then, for every $t \in [0,T]$,

$$\Gamma_t = \{(x,y) \in \mathbb{R}^{N+1} : u^-(x,t) \leq y \leq u^+(x,t)\},$$

where u^- and u^+ are respectively the minimal and the maximal (discontinuous) viscosity solution of (4.7). In the special case of the mean curvature equation, we proved that the boundary of the front Γ_t is smooth. It follows that u^- and u^+ are smooth. At this step, we aim at applying Theorem 4.3.1 to show that $u^- = u^+$. To do it, we need to know that $u^-, u^+ \in \mathcal{C}_{\text{poly}}$. Note that Corollary 4.2.1 is not sufficient because we suppose, a priori, that the solution has a polynomial growth. To overcome this difficulty, we invoke a L^∞ local bound for (4.7) (see [16]) : there exists a constant C such that, for every $(x,t) \in \mathbb{R}^N \times [0,T]$,

$$|u(x,t)| \leq \max\{|u_0(y)| + \sqrt{2Ct} : y \in \bar{B}(x, \sqrt{2Ct})\}.$$

From **(H3-**u_0**)**(ν), there exists a positive constant $\tilde{C}$ such that $|u_0(y)| \leq \tilde{C}(1+|y|^{1+\nu})$. It follows

$$|u(x,t)| \leq \tilde{C}(1 + (|x| + \sqrt{2Ct})^{1+\nu}) + \sqrt{2CT},$$

which proves the claim.

It implies $u^-, u^+ \in \mathcal{C}_{\text{poly}}$ and from Theorem 4.3.1, we obtain $u^- = u^+ := u$. Finally, $\Gamma_t = \text{Graph}(u(\cdot,t))$ is a smooth submanifold of $\mathbb{R}^{N+1}$ (in particular, Γ_t never fattens). In this case, the generalized evolution fits in with the classical evolution by mean curvature (see Evans and Spruck [45] and [47] for the agreement with an alternative generalized motion). $\square$

We refer to [16] and [18] for some more general geometrical motions. In these works, we associate a geometrical motion to some quasilinear equations of the type (4.2) for which the above techniques apply.

4.4 Proof of Theorem 4.2.1

1. We argue by contradiction assuming that there exists $(\tilde{x}, \tilde{t}) \in \mathbb{R}^N \times [0, T]$ such that

$$u(\tilde{x}, \tilde{t}) > v(\tilde{x}, \tilde{t}). \tag{4.13}$$

For $\varepsilon, \alpha > 0, p \geq k > \max\{2, \nu+1\}$ to be chosen later on, we introduce the functions defined by : for every $x', y' \in \mathbb{R}^N$,

$$K(x') := 1 + |x'|^k, \quad \varphi(y') := \frac{|y'|^p}{\varepsilon^p} \quad \text{and} \quad \Psi(x', y') := K(x')(\varphi(y') + \alpha).$$

Then we consider the test-function given by : for every $x, y \in \mathbb{R}^N$,

$$\Phi(x, y, t) := \mathrm{e}^{Lt}\, \Psi(x + y, x - y) + \eta t,$$

where $L, \eta > 0$.

2. The first step of the proof is the

Lemma 4.4.1 *Under the assumptions of Theorem 4.2.1, there exists a constant $C > 0$ such that, for any $x \in \mathbb{R}^N$ and $t \in [0, T]$,*

$$u(x, t) \leq C(1 + |x|^{\nu+1}) \quad \textit{and} \quad v(x, t) \geq -C(1 + |x|^{\nu+1})\ . \tag{4.14}$$

Moreover, if $p \geq k > \nu + 1$, the

$$\sup_{(x,y,t) \in (\mathbb{R}^N)^2 \times [0,T]} \left\{ u(x, t) - v(y, t) - \Phi(x, y, t) \right\}$$

is finite and is achieved at a point $(\bar{x}, \bar{y}, \bar{t})$. Finally, we can choose the parameters η, α and ε small enough in order to have a positive supremum and $|\bar{x} - \bar{y}| \leq 1$.

We postpone the proof of this lemma to the end of the section. From now on, we suppose that the supremum is positive and that $|\bar{x} - \bar{y}| \leq 1$.

3. The idea of the proof is the following : the seventh next steps are devoted to show that we can fix the parameters in Φ in order to force the maximum to be achieved at time $t = 0$. The last step deals with the case $t = 0$. We prove that the particular form of the test-function and the assumption about the modulus of continuity of the initial data lead to a contradiction with (4.13) which will be the end of the proof.

4. We first consider the case when the supremum is achieved at a point such that $\bar{t} > 0$.

By applying the fundamental result of the Users' guide to viscosity solutions ([37, Theorem 8.3]), we know that, for every $\rho > 0$, there exist $a_1, a_2 \in \mathbb{R}$ and $X, Y \in \mathcal{S}_N$ such that

$$(a_1, D_x\Phi(\bar{x}, \bar{y}, \bar{t}), X) \in \bar{\mathcal{P}}^{2,+}(u)(\bar{x}, \bar{t})\ ,$$

$$(a_2, -D_y\Phi(\bar{x}, \bar{y}, \bar{t}), Y) \in \bar{\mathcal{P}}^{2,-}(v)(\bar{y}, \bar{t})\ ,$$

$$a_1 - a_2 = \frac{\partial \Phi}{\partial t}(\bar{x}, \bar{y}, \bar{t})\ ,$$

and

$$\left(\begin{array}{cc} X & 0 \\ 0 & -Y \end{array} \right) \leq A + \rho A^2 \quad \text{where} \quad A = D^2\Phi(\bar{x}, \bar{y}, \bar{t}) \ . \tag{4.15}$$

Therefore, since u and v are respectively sub- and supersolution of (4.2), we have

$$a_1 - \mathrm{Tr}\,[b(\bar{x}, \bar{t}, D_x\Phi(\bar{x}, \bar{y}, \bar{t}))X] + H(\bar{x}, \bar{t}, u(\bar{x}, \bar{t}), D_x\Phi(\bar{x}, \bar{y}, \bar{t})) \leq 0 \tag{4.16}$$

and

$$a_2 - \mathrm{Tr}\,[b(\bar{y}, \bar{t}, -D_y\Phi(\bar{x}, \bar{y}, \bar{t}))Y] + H(\bar{y}, \bar{t}, v(\bar{y}, \bar{t}), -D_y\Phi(\bar{x}, \bar{y}, \bar{t})) \geq 0. \tag{4.17}$$

By subtracting (4.17) from (4.16), we obtain

$$\begin{aligned} \eta + L\Phi(\bar{x}, \bar{y}, \bar{t}) \ \leq \ & \mathrm{Tr}\,[b(\bar{x}, \bar{t}, D_x\Phi(\bar{x}, \bar{y}, \bar{t}))X] - \mathrm{Tr}\,[b(\bar{y}, \bar{t}, -D_y\Phi(\bar{x}, \bar{y}, \bar{t}))Y] \\ & + H(\bar{y}, \bar{t}, v(\bar{y}, \bar{t}), -D_y\Phi(\bar{x}, \bar{y}, \bar{t})) - H(\bar{x}, \bar{t}, u(\bar{x}, \bar{t}), D_x\Phi(\bar{x}, \bar{y}, \bar{t})) \end{aligned} \tag{4.18}$$

In order to show that the maximum cannot be achieved for $\bar{t} > 0$, we have to prove that this inequality cannot hold for a suitable choice of the parameters and, to do so, we have to estimate the right-hand side of this inequality.

5. For the sake of notational simplicity, we write

$$\sigma_x = \sigma(\bar{x}, \bar{t}, D_x\Phi(\bar{x}, \bar{y}, \bar{t})) \quad \text{and} \quad \sigma_y = \sigma(\bar{y}, \bar{t}, -D_y\Phi(\bar{x}, \bar{y}, \bar{t}))$$

and omit in the following computations the dependence on $(\bar{x}, \bar{y}, \bar{t})$ when there is no ambiguity. For any orthonormal basis $(e_i)_{1 \leq i \leq N=1}$ of $\mathbb{R}^N$, we have

$$\begin{aligned} \mathrm{Tr}\,[b(\bar{x}, \bar{t}, D_x\Phi(\bar{x}, \bar{y}, \bar{t}))X - b(\bar{y}, \bar{t}, -D_y\Phi(\bar{x}, \bar{y}, \bar{t}))Y] &= \mathrm{Tr}\left[\sigma_x^T X \sigma_x - \sigma_y^T Y \sigma_y\right] \\ &= \sum_{i=1}^{N} \langle X\sigma_x e_i, \sigma_x e_i\rangle - \langle Y\sigma_y e_i, \sigma_y e_i\rangle \ . \end{aligned} \tag{4.19}$$

On an other hand, we can write (4.15) as, for all $\zeta, \xi \in \mathbb{R}^N$,

$$\begin{aligned} \langle X\zeta, \zeta\rangle - \langle Y\xi, \xi\rangle \leq e^{L\bar{t}}\langle D^2_{xx}\Psi(\zeta+\xi), \zeta+\xi\rangle + 2e^{L\bar{t}}\langle D^2_{xy}\Psi(\zeta+\xi), \zeta-\xi\rangle \\ + e^{L\bar{t}}\langle D^2_{yy}\Psi(\zeta-\xi), \zeta-\xi\rangle + \rho\langle A^2(\zeta,\xi), (\zeta,\xi)\rangle. \end{aligned}$$

Using this last inequality with $\zeta = \sigma_x e_i$ and $\xi = \sigma_y e_i$ in (4.19) and letting ρ go to 0, (4.18) becomes

$$\eta + L\Phi \leq \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4, \tag{4.20}$$

where

$$\left\{ \begin{array}{l} \mathcal{A}_1 = e^{L\bar{t}}\|D^2_{xx}\Psi\| \ \|\sigma_x + \sigma_y\|^2, \\ \mathcal{A}_2 = 2e^{L\bar{t}}\|D^2_{xy}\Psi\| \ \|\sigma_x + \sigma_y\| \ \|\sigma_x - \sigma_y\|, \\ \mathcal{A}_3 = e^{L\bar{t}}\|D^2_{yy}\Psi\| \ \|\sigma_x - \sigma_y\|^2, \\ \mathcal{A}_4 = H(\bar{y}, \bar{t}, v(\bar{y}, \bar{t}), -D_y\Phi) - H(\bar{x}, \bar{t}, u(\bar{x}, \bar{t}), D_x\Phi). \end{array} \right.$$

In the fourth next steps, we estimate the $\mathcal{A}_i, 1 \le i \le 4$. These estimates are heavy to obtain but the basic idea is easy : we want to prove that, for a suitable choice of parameters, each $|\mathcal{A}_i|$ is bounded by $\bar{C}\Phi + \beta$, where $\bar{C}$ is a fixed constant and β is sufficiently small. It will lead to a contradiction with (4.20) if we take L large enough.

In all the estimates below, C will denote a constant which may vary from line to line, may depend on k, p but not on ε, α and L.

6. Estimate of $\mathcal{A}_1$.
We have

$$\|D_{xx}\Psi\| \le k(k-1)|\bar{x}+\bar{y}|^{k-2}(\varphi+\alpha).$$

From (4.4), we get

$$\mathcal{A}_1 \le 2e^{L\bar{t}}C^2 k(k-1)|\bar{x}+\bar{y}|^{k-2}((1+|\bar{x}|)^2+(1+|\bar{y}|)^2)(\varphi+\alpha).$$

But, since $|\bar{x}-\bar{y}| \le 1$, we can compare both $|\bar{x}|$ and $|\bar{y}|$ with $|\bar{x}+\bar{y}|$. It follows that there exists a constant $C = C(k)$ such that

$$\mathcal{A}_1 \le Ce^{L\bar{t}}K(\varphi+\alpha). \tag{4.21}$$

7. Estimate of $\mathcal{A}_2$.
An explicit computation gives $D^2_{xy}\Psi = DK \otimes D\varphi$. We may assume without loss of generality that $\bar{x} \ne \bar{y}$ or equivalently $D\varphi \ne 0$ since otherwise $\mathcal{A}_2$ would be 0 and causes no problem.

From inequalities (4.4) and $|\bar{x}-\bar{y}| \le 1$, we then obtain

$$\mathcal{A}_2 \le 2Ce^{L\bar{t}}|DK||D\varphi|(3+|\bar{x}+\bar{y}|)\|\sigma_x-\sigma_y\|. \tag{4.22}$$

Using **(H1)**, we have

$$\begin{aligned}\|\sigma_x-\sigma_y\| &\le \|\sigma(\bar{x},\bar{t},D_x\Phi)-\sigma(\bar{y},\bar{t},D_x\Phi)\| + \|\sigma(\bar{y},\bar{t},D_x\Phi)-\sigma(\bar{y},\bar{t},-D_y\Phi)\| \\ &\le C|\bar{x}-\bar{y}| + C\frac{|D_x\Phi+D_y\Phi|}{1+|D_x\Phi|+|D_y\Phi|}.\end{aligned}$$

But $D_x\Phi = e^{L\bar{t}}(DK(\varphi+\alpha)+KD\varphi)$ and $D_y\Phi = e^{L\bar{t}}(DK(\varphi+\alpha)-KD\varphi)$. We obtain

$$\begin{aligned}\|\sigma_x-\sigma_y\| &\le C|\bar{x}-\bar{y}| + C\frac{2e^{L\bar{t}}|DK|(\varphi+\alpha)}{1+\frac{1}{2}(|D_x\Phi+D_y\Phi|+|D_x\Phi-D_y\Phi|)} \\ &\le C|\bar{x}-\bar{y}| + 2C\frac{|DK|(\varphi+\alpha)}{K|D\varphi|}.\end{aligned} \tag{4.23}$$

Combining this last inequality with (4.22) and using that

$$|DK| = k|\bar{x}+\bar{y}|^{k-1} \quad \text{and} \quad |D\varphi| = p\,\frac{|\bar{x}-\bar{y}|^{p-1}}{\varepsilon^p},$$

show that there exists a positive constant $C = C(k,p)$ such that

$$\mathcal{A}_2 \le Ce^{L\bar{t}}K(\varphi+\alpha). \tag{4.24}$$

8. Estimate of $\mathcal{A}_3$.
This step is the most technical one. First, the computation of $D^2_{yy}\Psi$ gives

$$\mathcal{A}_3 \leq e^{L\bar{t}} K \|D^2\varphi\| \, \|\sigma_x - \sigma_y\|^2$$

and we have

$$\begin{aligned}
\|\sigma_x - \sigma_y\|^2 &= \|\sigma(\bar{x},\bar{t},D_x\Phi) - \sigma(\bar{y},\bar{t},D_x\Phi)\|^2 \\
&\quad +2\,\|\sigma(\bar{x},\bar{t},D_x\Phi) - \sigma(\bar{y},\bar{t},D_x\Phi)\|\,\|\sigma(\bar{y},\bar{t},D_x\Phi) - \sigma(\bar{y},\bar{t},-D_y\Phi)\| \\
&\quad + \|\sigma(\bar{y},\bar{t},D_x\Phi) - \sigma(\bar{y},\bar{t},-D_y\Phi)\|^2 .
\end{aligned}$$

It follows

$$\begin{aligned}
|\mathcal{A}_3| &\leq 2K\,\mathrm{e}^{L\bar{t}}\,\|\sigma(\bar{x},\bar{t},D_x\Phi) - \sigma(\bar{y},\bar{t},D_x\Phi)\|^2\,\|D^2\varphi\| \\
&\quad +2K\,\mathrm{e}^{L\bar{t}}\,\|\sigma(\bar{y},\bar{t},D_x\Phi) - \sigma(\bar{y},\bar{t},-D_y\Phi)\|^2\,\|D^2\varphi\|. \qquad (4.25)
\end{aligned}$$

We estimate separately the two terms which appear in the right-hand side (4.25). For the first one, using **(H1)**, we obtain

$$2K\,\mathrm{e}^{L\bar{t}}\,\|\sigma(\bar{x},\bar{t},D_x\Phi) - \sigma(\bar{y},\bar{t},D_x\Phi)\|^2\,\|D^2\varphi\| \leq 2C^2p(p-1)\,\mathrm{e}^{L\bar{t}}K\varphi \qquad (4.26)$$

because $|\bar{x}-\bar{y}|^2\|D^2\varphi\| \leq p(p-1)\varphi$.

The estimate of the second term is the hardest one. Again we may assume without loss of generality that $\bar{x} \neq \bar{y}$ or equivalently $D\varphi \neq 0$; we have

$$\begin{aligned}
\text{second term } &= 2K\,\mathrm{e}^{L\bar{t}}\,\|\sigma(\bar{y},\bar{t},D_x\Phi) - \sigma(\bar{y},\bar{t},-D_y\Phi)\|^2\,\|D^2\varphi\| \\
&\leq 2CK\mathrm{e}^{L\bar{t}}\,\frac{|\bar{x}-\bar{y}|^{p-2}}{\varepsilon^p}\left(\frac{C|D_x\Phi + D_y\Phi|}{1+|D_x\Phi|+|D_y\Phi|}\right)^2 \\
&\leq 2CK\mathrm{e}^{L\bar{t}}\,\frac{|\bar{x}-\bar{y}|^{p-2}}{\varepsilon^p}\left(\frac{|D_x\Phi + D_y\Phi|}{1+\frac{1}{2}\left(|D_x\Phi+D_y\Phi|+|D_x\Phi-D_y\Phi|\right)}\right)^2 \\
&\leq 2CK\mathrm{e}^{L\bar{t}}\,\frac{|\bar{x}-\bar{y}|^{p-2}}{\varepsilon^p}\left(\frac{2\mathrm{e}^{L\bar{t}}(\varphi+\alpha)|DK|}{1+\mathrm{e}^{L\bar{t}}(\varphi+\alpha)|DK|+\mathrm{e}^{L\bar{t}}K|D\varphi|}\right)^2 \\
&\leq 2CK\mathrm{e}^{L\bar{t}}\,\frac{|\bar{x}-\bar{y}|^{p-2}}{\varepsilon^p}\,\frac{\left(\mathrm{e}^{L\bar{t}}(\varphi+\alpha)|DK|\right)^2}{\left(\mathrm{e}^{L\bar{t}}(\varphi+\alpha)|DK|\right)^{2\lambda}\left(\mathrm{e}^{L\bar{t}}K|D\varphi|\right)^{2\mu}}
\end{aligned}$$

for every positive λ, μ such that $\lambda + \mu \leq 1$.

We now distinguish two cases.
— 1[st] case. If $\varphi \geq \alpha$, then taking $\lambda = 0$ and $\mu = 1$, we get

$$\begin{aligned}
\text{second term } &\leq 2CK\mathrm{e}^{L\bar{t}}\,\frac{|x-y|^{p-2}}{\varepsilon^p}\,\frac{\mathrm{e}^{2L\bar{t}}\varphi^2|DK|^2}{\mathrm{e}^{2L\bar{t}}K^2|D\varphi|^2} \\
&\leq 2CK\mathrm{e}^{L\bar{t}}\varphi\left(\frac{|DK|}{K}\right)^2 \\
&\leq 2CK\mathrm{e}^{L\bar{t}}\varphi. \qquad (4.27)
\end{aligned}$$

— 2^{nd} case. Otherwise, $\alpha \geq \varphi$. Then

$$\begin{array}{rcl} \text{second term} & \leq & 2CK\mathrm{e}^{L\bar{t}}\dfrac{|\bar{x}-\bar{y}|^{p-2}}{\varepsilon^p}\,\dfrac{\left(\mathrm{e}^{L\bar{t}}\alpha|DK|\right)^2}{(\mathrm{e}^{L\bar{t}}\alpha|DK|)^{2\lambda}\left(\mathrm{e}^{L\bar{t}}K\frac{p|\bar{x}-\bar{y}|^{p-1}}{\varepsilon^p}\right)^{2\mu}} \\ & \leq & 2CK^{1-2\mu}\mathrm{e}^{L\bar{t}(3-2\lambda-2\mu)}|\bar{x}-\bar{y}|^{p-2-2\mu(p-1)}\ \varepsilon^{-p+2\mu p}\quad (\alpha|DK|)^{2-2\lambda}\ . \end{array}$$

We choose $\mu = \dfrac{p-2}{2(p-1)}$ and this yields

$$\text{second term} \leq 2C\mathrm{e}^{3L\bar{t}}\varepsilon^{-\frac{p}{p-1}}\alpha^{2-2\lambda}K^{\frac{1}{p-1}}|DK|^{2-2\lambda}.$$

From now on, we take

$$p > k+1 \tag{4.28}$$

and we choose $\lambda = \dfrac{1}{2} - \dfrac{\theta}{2}$ with

$$\theta = \frac{p-k-1}{(p-1)(k-1)}. \tag{4.29}$$

Using that $|DK| \leq kK^{\frac{k-1}{k}}$, we obtain

$$\begin{array}{rcl} \text{second term} & \leq & 2C\mathrm{e}^{3L\bar{t}}\varepsilon^{-\frac{p}{p-1}}\alpha^{2-2\lambda}K^{\frac{1}{p-1}+\frac{(2-2\lambda)(k-1)}{k}} \\ & \leq & 2C\mathrm{e}^{3L\bar{t}}\varepsilon^{-\frac{p}{p-1}}\alpha\alpha^{\theta}K\ . \end{array} \tag{4.30}$$

Combining (4.27) and (4.30), we obtain

$$\text{second term} \leq \max\left\{2C\mathrm{e}^{L\bar{t}}K\varphi,\ 2C\mathrm{e}^{3L\bar{t}}\varepsilon^{-p/(p-1)}\ \alpha^{\theta}\ \alpha K\right\}, \tag{4.31}$$

and from (4.26) and (4.31), we get

$$\begin{array}{rcl} |\mathcal{A}_3| & \leq & C\mathrm{e}^{L\bar{t}}K\varphi + CK\mathrm{e}^{L\bar{t}}(\varphi+\alpha) \\ & & +\max\left\{2C\mathrm{e}^{L\bar{t}}K\varphi,\ 2C\mathrm{e}^{3L\bar{t}}\ \varepsilon^{-p/(p-1)}\ \alpha^{\theta}\ \alpha K\right\} \\ & \leq & C\max\left\{\mathrm{e}^{L\bar{t}}K(\varphi+\alpha),\ \mathrm{e}^{3L\bar{t}}\ \varepsilon^{-p/(p-1)}\ \alpha^{\theta}\ \alpha K\right\}, \end{array} \tag{4.32}$$

where C is a positive constant independent of ε, α, L and η.

9. Estimate of $\mathcal{A}_4$.
Since we have

$$0 < \Phi(\bar{x},\bar{y},\bar{t}) \leq u(\bar{x},\bar{t}) - v(\bar{y},\bar{t})$$

and since $u \mapsto H(\cdot,\cdot,u,\cdot)$ is nondecreasing, we first get

$$\begin{array}{rcl} \mathcal{A}_4 & = & H(\bar{y},\bar{t},v(\bar{y},\bar{t}),-D_y\Phi) - H(\bar{x},\bar{t},u(\bar{x},\bar{t}),D_x\Phi) \\ & \leq & H(\bar{y},\bar{t},u(\bar{x},\bar{t}),-D_y\Phi) - H(\bar{x},\bar{t},u(\bar{x},\bar{t}),D_x\Phi). \end{array}$$

Then, using **(H2)**, it follows that

$$
\begin{aligned}
\mathcal{A}_4 &\leq |H(\bar{y},\bar{t},u(\bar{x},\bar{t}),-D_y\Phi) - H(\bar{y},\bar{t},u(\bar{x},\bar{t}),D_x\Phi)| \\
&\qquad\qquad + |H(\bar{y},\bar{t},u(\bar{x},\bar{t}),D_x\Phi) - H(\bar{x},\bar{t},u(\bar{x},\bar{t}),D_x\Phi)| \\
&\leq m\left((1+|\bar{y}|)|D_x\Phi + D_y\Phi|\right) + m\left((1+|D_x\Phi|)|\bar{x}-\bar{y}|\right).
\end{aligned}
$$

Because of the properties of a modulus of continuity, for every $r>0$ and every $\gamma>0$, there exists a positive constant $C(\gamma)$ such that $m(r) \leq \gamma + C(\gamma)r$. Taking $\gamma = \eta/2$ and up to replace at each step $C(\eta)$ by a larger one which depends only on η, we obtain

$$
\begin{aligned}
\mathcal{A}_4 &\leq \frac{\eta}{2} + C(\eta)\Big((1+|\bar{y}|)|D_x\Phi + D_y\Phi| + (1+|D_x\Phi|)|\bar{x}-\bar{y}|\Big) \\
&\leq \frac{\eta}{2} + C(\eta)\left(\mathrm{e}^{L\bar{t}}(1+|\bar{y}|)|DK|(\varphi+\alpha) + (1+\mathrm{e}^{L\bar{t}}(|DK|(\varphi+\alpha)+K|D\varphi|))|\bar{x}-\bar{y}|\right) \\
&\leq \frac{\eta}{2} + C(\eta)\left(\mathrm{e}^{L\bar{t}}K(\varphi+\alpha) + |\bar{x}-\bar{y}|\right).
\end{aligned}
$$

But from Young's inequality, we have

$$
|\bar{x}-\bar{y}| \leq \frac{|\bar{x}-\bar{y}|}{\varepsilon^p} + \varepsilon^q \leq K(\varphi+\alpha) + \varepsilon^q
$$

and finally, we obtain

$$
\mathcal{A}_4 \leq \eta + C(\eta)\left(\mathrm{e}^{L\bar{t}}K(\varphi+\alpha) + \varepsilon^q\right), \tag{4.33}
$$

where $C(\eta)$ is a positive constant independent of α, ε and L.

10. End of the case $\bar{t} > 0$.
Plugging estimates (4.21), (4.24), (4.32) and (4.33) in (4.20) yields

$$
\eta + L\Phi \leq \frac{\eta}{2} + (C + C(\eta))\Phi + C\,\mathrm{e}^{3L\bar{t}}\,\varepsilon^{-p/(p-1)}\,\alpha^\theta\,\alpha K + C(\eta)\varepsilon^q,
$$

where C is a positive constant which is independent of ε, α, η and L. Thus, we can first choose a constant L large enough, namely $L(\eta) \geq C + C(\eta) + 1$, such that the previous inequality becomes

$$
\frac{\eta}{2} + \Phi \leq C\,\mathrm{e}^{3L\bar{t}}\,\varepsilon^{-p/(p-1)}\,\alpha^\theta\,\alpha K + C(\eta)\,\varepsilon^q, \tag{4.34}
$$

for every positive ε, α and η. Then we take $\alpha = \varepsilon^\ell$, choosing ℓ large enough such that

$$
\theta\ell > \frac{p}{p-1}. \tag{4.35}
$$

Thus (4.34) reads

$$
\frac{\eta}{2} + \mathrm{e}^{L\bar{t}}K(\varphi+\alpha) \leq C\,\mathrm{e}^{3LT}\,\varepsilon^{\theta\ell - p/(p-1)}\,\alpha K + C(\eta)\varepsilon^q.
$$

Taking ε small enough such that $C(\eta)\varepsilon^q < \eta/2$ and $C\,\mathrm{e}^{2LT}\,\varepsilon^{\theta\ell-p/(p-1)} \leq 1$, we obtain a contradiction. Finally, the conclusion of the preceding steps is : if we choose L, p

and k as above and take $\alpha = \varepsilon^\ell$ (where ℓ satisfies (4.35)), then necessarily $\bar{t} = 0$ for sufficiently small ε.

11. From the previous step, we know that the maximum of the function $u - v - \Phi$ is achieved at a point $(\bar{x}, \bar{y}, 0)$ which implies, from (4.13) and Lemma 4.4.1, that there exists $\delta > 0$ such that

$$0 < \delta \leq u(\tilde{x}, \tilde{t}) - v(\tilde{x}, \tilde{t}) - \varepsilon^\ell \, \mathrm{e}^{L\tilde{t}} K(2\tilde{x}) - \eta\tilde{t} \leq u(\bar{x}, 0) - v(\bar{y}, 0) - K(\bar{x} + \bar{y})(\varphi + \varepsilon^\ell),$$

for ε and η small enough. Since $u(\bar{x}, 0) \leq u_0 \leq v(\bar{y}, 0)$, this inequality leads to

$$\delta \leq u_0(\bar{x}) - u_0(\bar{y}) - K(\bar{x} + \bar{y})(\varphi + \varepsilon^\ell).$$

Now, from **(H3-u_0)**(ν), there exists $C(\delta) > 0$ such that

$$\begin{aligned} \delta &\leq m\left((1 + |\bar{x}| + |\bar{y}|)^\nu |\bar{x} - \bar{y}|\right) - K(\bar{x} + \bar{y})(\varphi + \varepsilon^\ell) \\ &\leq \frac{\delta}{2} + C(\delta)(1 + |\bar{x}| + |\bar{y}|)^\nu |\bar{x} - \bar{y}| - K(\bar{x} + \bar{y})(\varphi(\bar{x} - \bar{y}) + \varepsilon^\ell). \end{aligned} \tag{4.36}$$

Since $|\bar{x} - \bar{y}| \leq 1$, there exists $\tilde{C}$ such that $(1 + |\bar{x}| + |\bar{y}|)^\nu \leq \tilde{C}(1 + |\bar{x} + \bar{y}|^\nu)$ and therefore (4.36) yields

$$\frac{\delta}{2} \leq C(\delta)\tilde{C}(1 + |\bar{x} + \bar{y}|^\nu)|\bar{x} - \bar{y}| - (1 + |\bar{x} + \bar{y}|^k)\left(\frac{|\bar{x} - \bar{y}|^p}{\varepsilon^p} + \varepsilon^\ell\right). \tag{4.37}$$

In order to study the right-hand side of this inequality, we introduce the function g defined by

$$g(r, s) = C(\delta)\tilde{C}(1 + r^\nu)s - (1 + r^k)\left(\frac{s^p}{\varepsilon^p} + \varepsilon^\ell\right),$$

for every $r, s \geq 0$. One easily shows that there exists a constant C depending only on $C(\delta), \tilde{C}$ and p such that

$$g(r, s) \leq C \frac{(1 + r^\nu)^{\frac{p}{p-1}}}{(1 + r^k)^{\frac{1}{p-1}}} \varepsilon^{\frac{p}{p-1}} - \varepsilon^\ell (1 + r^k). \tag{4.38}$$

Then, using that, for every $r > 0$, $\dfrac{(1 + r^\nu)^{\frac{p}{p-1}}}{(1 + r^k)^{\frac{1}{p-1}}} \leq \tilde{C}(1 + r^{\frac{\nu p - k}{p-1}})$, with $\tilde{C} = \tilde{C}(N, \nu, p, k)$, we obtain from (4.38)

$$g(r, s) \;\leq\; f(r) := C\tilde{C}\left(1 + r^{\frac{\nu p - k}{p-1}}\right)\varepsilon^{\frac{p}{p-1}} - (1 + r^k)\varepsilon^\ell, \tag{4.39}$$

for every $r, s \geq 0$. We take p large enough such that

$$\nu p - k > 0. \tag{4.40}$$

Since, in addition, we have already chosen $k > \nu + 1$ at the beginning, the function f achieves a maximum at

$$r = C' \varepsilon^{\frac{\ell(p-1) - p}{p(\nu - k)}},$$

where the C' depends only on $C, \tilde{C}, p, \nu, k$. Replacing r by this value, we obtain that for every $r \geq 0$,

$$f(r) \leq C\tilde{C}\varepsilon^{\frac{p}{p-1}} + \bar{C}\varepsilon^{\gamma}, \tag{4.41}$$

where $\bar{C}$ is a constant depending on $C, \tilde{C}, p, \nu, k$ and

$$\gamma = \frac{kp - \ell(\nu p - k)}{p(k-\nu)}.$$

From (4.37),(4.39) and (4.41), we obtain

$$\frac{\delta}{2} \leq C\left(\varepsilon^{\frac{p}{p-1}} + \varepsilon^{\gamma}\right)$$

with C is a constant depending only on δ, ν, p, k and N.

In order to conclude through the above inequality by letting ε tend to 0, we need to have $\gamma > 0$ i.e. to be able to choose the parameters in such a way to have $\ell < \dfrac{kp}{\nu p - k}$ since $\nu p - k > 0$ from (4.40). To do so, we recall that we have chosen $k > \max\{2, \nu+1\}$ at the beginning, $p > k+1$ (see (4.28)) and $\ell > \dfrac{p}{\theta(p-1)}$ (see (4.35)) where $\theta = \dfrac{p-k-1}{(p-1)(k-1)}$ (see (4.29)). Thus, we need to find ℓ, k and p such that

$$k > \max\{2, \nu+1\}, \quad p > k+1, \quad \nu p > k \quad \text{and} \quad \frac{p(k-1)}{p-k-1} < \ell < \frac{kp}{\nu p - k}. \tag{4.42}$$

To fulfill the last condition in (4.42), it is sufficient to have $p(\nu - k(\nu-1)) > 2k$. It follows that, if we can find a suitable k, up to take p large enough, then all the conditions would be satisfied. We distinguish two cases :

— 1^{st} case. If $\nu \leq 1$, then we take $k > 2$ and we are done.

— 2^{nd} case. If $\nu > 1$, then we have to find k such that

$$\nu + 1 < k < \frac{\nu}{\nu - 1}.$$

It leads to a condition on ν, namely $\nu^2 < \nu + 1$ which is automatically satisfied provided $\nu < \dfrac{1+\sqrt{5}}{2}$ as we supposed it.

Finally, in any case, it is possible to fulfill conditions (4.42) ; thus the proof of the theorem is complete. □

Now we turn to the proof of the Lemma.

Proof of Lemma 4.4.1. We first prove (4.14). We are going to do it only for the subsolution u, the proof for the supersolution v being analogous.

To do so, we introduce for C, L, $\varepsilon > 0$, $k \geq 2$ and the sequence of smooth functions $(\chi_\varepsilon)_\varepsilon$ defined by

$$\chi_\varepsilon(x,t) = \mathrm{e}^{Lt}\left(C(1+|x|^2)^{\frac{\nu+1}{2}} + \varepsilon(1+|x|^k)\right).$$

Tedious but straightforward computations show that, for any C, ν and k, if L is chosen large enough, then χ_ε is a strict supersolution of (4.2) for any ε small enough.

On an other hand, since $u(x,0) \leq u_0(x)$ in $\mathbb{R}^N$ and since u_0 satisfies **(H3-**u_0**)**(ν), it is clear that if C is chosen large enough, then $u(x,0) \leq \chi_\varepsilon(x,0)$ in $\mathbb{R}^N$.

Finally since u is in $\mathcal{C}_{\text{poly}}$, we have, for k large enough $u(x,t)/(1+|x|)^k \to 0$ when $|x| \to +\infty$ uniformly for $t \in [0,T]$ and therefore $u(x,t) \leq \chi_\varepsilon(x,t)$ for $|x|$ large enough.

Using these three properties, one shows easily that $u(x,t) \leq \chi_\varepsilon(x,t)$ in $\mathbb{R}^N \times [0,T]$: indeed, because of the last one, $u - \chi_\varepsilon$ achieves its maximum on $\mathbb{R}^N \times [0,T]$ but since u is a viscosity subsolution of (4.2) and χ_ε is a strict smooth supersolution of this equation, this maximum cannot be achieved for $t > 0$. Therefore it is achieved for $t = 0$ and the maximum is therefore negative, proving the claim. Letting ε tend to 0 yields (4.14).

Now we prove the second part of the lemma. Since $u(x,t) - v(y,t) - \Phi(x,y,t)$ is upper-semicontinuous, in order to prove that the supremum is achieved, it suffices to prove that this function tends to $-\infty$ when $|x|, |y| \to +\infty$ (uniformly with respect to $t \in [0,T]$). But

$$\begin{aligned}
\Phi(x,y,t) &= \mathrm{e}^{Lt} K(x+y)\left(\varphi(x-y) + \alpha\right) + \eta t \\
&\geq \frac{|x-y|^p}{\varepsilon^p} + \alpha(1 + |x+y|^k) \\
&\geq \min\left\{\alpha, \frac{1}{\varepsilon^p}\right\} \left(|x-y|^p + 1 + |x+y|^k\right). \qquad (4.43)
\end{aligned}$$

Since $p \geq k > \max\{2, \nu+1\}$, using the convexity of $r \mapsto r^k$, we have

$$|x-y|^p + 1 + |x+y|^k \geq |x-y|^k + |x+y|^k \geq |x|^k + |y|^k.$$

From (4.43) and (4.14), it follows

$$\begin{aligned}
u(x,t) - v(y,t) - \Phi(x,y,t) &\leq u(x,t) - v(y,t) - \min\left\{\alpha, \frac{1}{\varepsilon^p}\right\}\left(|x|^k + |y|^k\right) \\
&\leq C\left(1 + |x|^{\nu+1} + |y|^{\nu+1}\right) - \min\left\{\alpha, \frac{1}{\varepsilon^p}\right\}\left(|x|^k + |y|^k\right)
\end{aligned}$$

which proves the claim since $\nu + 1 < k$.

On an other hand, we have

$$u(\bar{x},\bar{t}) - v(\bar{y},\bar{t}) - \Phi(\bar{x},\bar{y},\bar{t}) \geq u(\tilde{x},\tilde{t}) - v(\tilde{x},\tilde{t}) - \mathrm{e}^{L\tilde{t}} K(2\tilde{x})\alpha - \eta\tilde{t}.$$

From (4.13), the right-hand side is positive if α and η are sufficiently small.

For α and η sufficiently small, we have

$$0 < u(\bar{x},\bar{t}) - v(\bar{y},\bar{t}) - \Phi(\bar{x},\bar{y},\bar{t}) \leq C\left(1 + |\bar{x}|^{\nu+1} + |\bar{y}|^{\nu+1}\right) - \mathrm{e}^{L\bar{t}} K(\bar{x}+\bar{y})(\varphi + \alpha) - \eta\bar{t}.$$

By using the convexity of $r \mapsto r^{\nu+1}$, it follows

$$\left(1 + |\bar{x}+\bar{y}|^k\right)\left(\frac{|\bar{x}-\bar{y}|^p}{\varepsilon^p} + \alpha\right) \leq C\left(1 + |\bar{x}+\bar{y}|^{\nu+1} + |\bar{x}-\bar{y}|^{\nu+1}\right).$$

Since $\nu + 1 < k$, we obtain

$$\frac{|\bar{x}-\bar{y}|^p}{\varepsilon^p} \leq C\left(1 + |\bar{x}-\bar{y}|^{\nu+1}\right)$$

which implies that $|\bar{x} - \bar{y}| \leq 1$ for ε small enough since $p > \nu + 1$. And the proof of the lemma is complete. □

4.5 Proof of Corollary 4.2.1

The uniqueness part in the statement of Corollary 4.2.1 is an immediate consequence of Theorem 4.2.1. In this section, we prove the existence result using Perron's method. For the description of this method in the context of viscosity solutions, we refer to Ishii [60] and Crandall, Ishii and Lions [37].

A straightforward computation shows that, if $C > 0$ and L are large enough, then $\underline{u}(x,t) = -C\,\mathrm{e}^{Lt}(1+|x|^2)^{\frac{\nu+1}{2}}$ and $\overline{u}(x,t) = C\,\mathrm{e}^{Lt}(1+|x|^2)^{\frac{\nu+1}{2}}$ are respectively viscosity sub- and supersolution of (4.2).

We define the set $\mathcal{S}$ in the following way : a locally bounded, possibly discontinuous, function w defined on $\mathbb{R}^N \times [0,T]$ is in $\mathcal{S}$ if $\underline{u} \le w \le \overline{u}$ on $\mathbb{R}^N \times [0,T]$ and if w^* is a viscosity subsolution of (4.2) satisfying the initial condition in the viscosity sense, i.e.

$$\min\left\{\frac{\partial w}{\partial t} - \mathrm{Tr}\left[b(x,0,Dw)D^2w\right] + H(x,0,w,Dw), \quad w^*(x,0) - u_0(x)\right\} \le 0 \quad (4.44)$$

in $\mathbb{R}^N \times \{0\}$. Then we set

$$u(x,t) = \sup_{w\in\mathcal{S}} w(x,t).$$

Classical arguments show that u is still a subsolution. Moreover, when a subsolution w satisfies the initial condition (4.44), it is well-known (see [13] for instance) that $w^*(x,0) \le u_0(x)$ for all $x \in \mathbb{R}^N$. From the definition of u, it follows $u^*(x,0) \le u_0(x)$ for all $x \in \mathbb{R}^N$.

From Perron's method, u is a supersolution which satisfies the initial condition (4.44) where we replace "min" by "max", "w^*" by "w_*" and "$\le$" by "$\ge$". As above, it follows $u_*(x,0) \ge u_0(x)$ for all $x \in \mathbb{R}^N$.

Finally, we have $u^*(\cdot,0) \le u_0 \le u_*(\cdot,0)$ in $\mathbb{R}^N$. Applying our comparison result (Theorem 4.2.1) to u^* and u_*, we obtain $u^* \le u_*$ which means that u is a continuous viscosity solution of (4.2) with initial datum u_0. $\square$

CHAPITRE 5

Bornes inférieures de gradient pour des équations d'Hamilton-Jacobi

Sommaire

L'article qui constitue ce chapitre est paru dans la revue Advances in Differential Equations *en 2001. Nous reproduisons l'article tel quel. À la fin du chapitre nous présentons la preuve du théorème 5.2 de semiconcavité, qui est seulement esquissée dans l'article, ainsi que quelques compléments.*

5.1 Lower-bound gradient estimates and propagating fronts

Advances in Differential Equations Volume 6, Number 5, May 2001, Pages 547–576

LOWER-BOUND GRADIENT ESTIMATES FOR FIRST-ORDER HAMILTON-JACOBI EQUATIONS AND APPLICATIONS TO THE REGULARITY OF PROPAGATING FRONTS

Olivier Ley
Université de Tours, Parc de Grandmont, 37200 Tours, France

(Submitted by: P.L. Lions)

Abstract. This paper is concerned with first-order time-dependent Hamilton-Jacobi equations. Exploiting some ideas of Barron and Jensen [9], we derive lower-bound estimates for the gradient of a locally Lipschitz-continuous viscosity solution u of equations with a convex Hamiltonian. Using these estimates in the context of the level-set approach to front propagation, we investigate the regularity properties of the propagating front of u, namely $\Gamma_t = \{x \in \mathbb{R}^n : u(x,t) = 0\}$ for $t \geq 0$. We show that, contrary to the smooth case, such estimates do not guarantee, in general, any expected regularity for Γ_t even if u is semiconcave.

1. Introduction

We consider the first-order time-dependent Hamilton-Jacobi Equation

$$\frac{\partial \omega}{\partial t} + H(x,t,D_x\omega) = 0 \quad \text{in } \mathbb{R}^n \times (0,T) \tag{1.1}$$

with the initial condition

$$\omega(x,0) = u_0(x) \quad \text{in } \mathbb{R}^n \tag{1.2}$$

where $H \in C(\mathbb{R}^n \times [0,T] \times \mathbb{R}^n)$, $u_0 \in C(\mathbb{R}^n)$ and the solution ω is a real-valued function. Under suitable assumptions on the Hamiltonian H, if the initial condition u_0 is in $W^{1,\infty}(\mathbb{R}^n)$, it is well-known that there exists a unique viscosity solution $u \in W^{1,\infty}(\mathbb{R}^n \times [0,T])$ of (1.1)–(1.2). Many works are related to this problem: Crandall and Lions [14], Lions [22], Ishii [19], Barles [5, 7], Bardi and Capuzzo-Dolcetta [4], ...and references therein.

Here, we are given a Hamiltonian $H(x,t,p)$ which is assumed to be Lipschitz continuous with respect to x and p and which satisfies

(H1-β) $\left|\frac{\partial H}{\partial x}(x,t,p)\right| \leq C_1(\beta + |p|)$ for almost every $(x,t,p) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n$

Accepted for publication May 1999.
AMS Subject Classifications: 49L25, 35F25, 35B37, 26B25, 26B10.

(in the sequel, β will be chosen equal to 0 or 1) and

(H2) $\left|\frac{\partial H}{\partial p}(x,t,p)\right| \leq A_2|x| + B_2$ for almost every $(x,t,p) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n$.

Assumption **(H2)** implies a property of "finite speed of propagation": the value of the solution u at the point (x,t) does not depend on the initial condition u_0 in the whole space $\mathbb{R}^n$ but only in a ball $B(x, r(x,t))$ where $r(x,t) = \mathrm{e}^{(A_2+B_2+A_2|x|)t} - 1$, the base of the so-called *domain of dependence* $\mathcal{D}(x,r) \subset \mathbb{R}^n \times [0,T]$ (see Remark 6.1). This property allows us to localize many arguments and to compare unbounded solutions, regardless of their growth at infinity. This fact was first proved by Ishii in [19]. To be as self-contained as possible, we give in the Appendix a simplified proof of this comparison theorem (Theorem 6.1).

As an application, if the initial data u_0 is locally Lipschitz continuous in $\mathbb{R}^n$, we obtain upper-bound estimates for the gradient of a solution u of (1.1)–(1.2) which imply that u is locally Lipschitz continuous in $\mathbb{R}^n \times [0,T]$ (Theorem 4.1). In the same way, if u_0 is semiconcave in $\mathbb{R}^n$, then so is $u(\cdot,t)$ for $t \in [0,T]$ (Theorem 5.2).

On the other hand, when H satisfies **(H1-β)** and

(H3) $H(x,t,p)$ is convex in the p-variable,

Barron and Jensen showed in [9] that a continuous viscosity solution of (1.1) satisfies stronger properties than the ones needed to be a viscosity solution. In particular, the inf-convolution of a solution u of (1.1) is an approximate subsolution of (1.1) in addition to being, as is well-known, an approximate supersolution (see Lemma 3.2). Exploiting this fact with the help of the comparison theorem, we prove a lower-bound estimate for the gradient which is an unusual result in the theory of partial differential equations: suppose that, for some positive $\eta > 0$ and some ball $B(x_0, r) \subset \mathbb{R}^n$, we have

$$|Du_0| \geq \eta \quad \text{in } B(x_0,r) \quad \text{in the viscosity sense.} \tag{1.3}$$

Then the solution u of (1.1)–(1.2) satisfies the following similar lower bound in the viscosity sense, namely

$$|D_x u| \geq \tilde{\eta}\,\mathrm{e}^{-\gamma t/2} \quad \text{in the domain of dependence } \mathcal{D}(x_0,r). \tag{1.4}$$

If u_0 is smooth, from the implicit function theorem, (1.3) implies that $\Gamma_0 \cap B(x_0,r)$ is a smooth hypersurface where $\Gamma_0 = \{x \in \mathbb{R}^n \,:\, u_0(x) = 0\}$ is the 0-level set of u_0. In the same way, if the solution u is smooth, (1.4) implies that, for any $t \in [0,T]$, $\Gamma_t \cap \mathcal{D}(x_0,r)$ is a smooth hypersurface where

$$\Gamma_t = \{x \in \mathbb{R}^n \,:\, u(x,t) = 0\} \tag{1.5}$$

is the 0-level set of u. Now, the question is, what can we say about the regularity of Γ_t if u_0 and u are only Lipschitz continuous or if they are semiconcave?

This question is especially relevant in the theory of the level-set approach to weak propagation of fronts developed by Evans and Spruck [17] and Chen, Giga and Goto [11] (see also Barles, Soner and Souganidis [8]). It can be described in the following way. We consider equation (1.1) with a Hamiltonian which is homogeneous of degree 1 in the gradient variable; i.e.,

(H4) $H(x,t,\lambda p) = \lambda H(x,t,p)$ for all $x \in \mathbb{R}^n$, $t \in [0,T]$, $p \in \mathbb{R}^n$ and $\lambda \geq 0$.

(Note that it implies that the equation is invariant by changes $u \to \varphi(u)$, $\varphi' > 0$). Given an open set Ω_0 whose boundary is Γ_0, we first choose a function u_0 such that

$$\Gamma_0 = \{x \in \mathbb{R}^n \; : \; u_0(x) = 0\} \quad \text{and} \quad \Omega_0 = \{x \in \mathbb{R}^n \; : \; u_0(x) < 0\}. \tag{1.6}$$

We define the front Γ_t at time t by (1.5) where u is the unique viscosity solution of (1.1)–(1.2). Then it is shown in [17] and [11] that the propagation of Γ_t depends only on the sets Γ_0 and Ω_0 but not on the choice of u_0 satisfying (1.6).

In this context, we consider a Hamiltonian which satisfies **(H1-β)**, **(H2)**, **(H3)** and **(H4)**. Then Γ_t can be seen as a front which propagates in the sense of the level-set approach described above. The most typical example is the well-known Eikonal equation

$$\frac{\partial \omega}{\partial t} + c\,|D_x\,\omega| = 0,$$

which is related to a front propagating with a constant normal velocity c (see Examples 4.1 and 5.3).

In order to investigate the regularity properties of the front Γ_t, we have to assume some regularity properties of the initial hypersurface Γ_0. We suppose that

(H5) there exists a positive constant η such that

$$|u_0| + |Du_0| \geq 2\eta > 0 \quad \text{in } \mathbb{R}^n \quad \textit{in the viscosity sense.}$$

If u_0 is smooth, **(H5)** is nothing but saying that the graph $\{z = u_0(x)\}$ intersects the hyperplane $\{z = 0\}$ transversely. If u_0 is only Lipschitz continuous, this condition means that, for every $x \in \mathbb{R}^n$, either $|u_0(x)| \geq \eta$ or $|p| \geq \eta$ where $p \in D^-u_0(x)$. Let us mention that if $D^-u_0(x) = \emptyset$, then **(H5)** is fulfilled at x even if $u_0(x) = 0$. We refer to Examples 5.3, 5.4 and Theorem 5.5 for some illustrations.

With these assumptions on H and on the initial data u_0, we obtain the lower-bound estimate

$$|u| + \frac{e^{\gamma t}}{4}|D_x u|^2 \geq C(\eta) > 0 \quad \text{in } \mathbb{R}^n \times [0,T] \text{ in the viscosity sense} \tag{1.7}$$

for some positive constants $C(\eta)$ and γ (Theorem 4.2).

Now, we want to investigate the regularity of a front associated to a semiconcave function. Semiconcave functions arise naturally in the theory of viscosity solutions (see Lions [22], Ishii [19], Cannarsa *et al.* [10, 3, 1] and Theorem 5.2). In general, they are the most regular solutions one can hope for time-dependent Hamilton-Jacobi equations. Roughly speaking, they are "nearly C^1." If the semiconcave function u satisfies (1.7), which can be seen as an expected generalization of a transversality condition for nonsmooth functions, a natural question arises: does estimate (1.7) imply regularity for Γ_t?

We unfortunately answer in a negative way. On the one hand, surprisingly, even if u is semiconcave, the front can be nasty (see the example developed in Theorem 5.5). On the other hand, if we start with a smooth set Γ_0, "non-Lipschitz" singularities can occur (see Example 5.3).

Nevertheless, we obtain some partial positive results. As an immediate corollary of Theorem 4.2, we recover a known result of Barles *et al.* [8] about the so-called *nonempty interior difficulty*: the front has zero Lebesgue measure in this case.

For Lipschitz-continuous solutions, the sharpest regularity we can hope would be a front which is locally Lipschitz (i.e., locally the graph of a Lipschitz-continuous function; see Definition 5.2). In fact, the front for semiconcave solutions is "regular from one side," which means that the front has locally smooth support hypersurfaces which lie in $\bar{\Omega}_0 = \{x \in \mathbb{R}^n : u_0(x) \leq 0\}$ (see Theorem 5.3).

A sufficient condition for Γ_t to be locally Lipschitz at x is

$$0 \notin \partial_C u(x,t) \tag{1.8}$$

where $\partial_C u(\cdot,t)$ denotes the Clarke subdifferential. In this case, we can apply a nonsmooth inversion Theorem of Clarke [12] (see Theorem 5.4). But condition (1.8) does not characterize such points of regularity (cf. Example 5.4) and we cannot hope for regularity everywhere.

The $(n-1)$-Hausdorff measure $\mathcal{H}^{n-1}$ can be viewed as a measure on the level sets. The co-area formula implies that for almost every level set of a semiconcave function, the $\mathcal{H}^{n-1}$-measure of the set of the singular points is 0 (Corollary 5.2). But this gives no result to particular level sets like the front. Theorem 5.5 gives an example of a semiconcave function whose front contains a positive $(n-1)$-Hausdorff measure set of singular points, which shows that we cannot have more regularity.

The paper is organized as follows. In Section 3, we give a simplified proof of a theorem of [9], using some ideas of Barles [6]. In Section 4, we prove the upper and lower-bound estimates for the gradient of a solution u of (1.1)–(1.2). Finally, Section 5 is devoted to the study of the regularity of the front Γ_t.

Acknowledgments. I am grateful to Guy Barles, who brought this problem to my attention. I wish to thank Piermarco Cannarsa for helpful discussions and valuable suggestions and Francis Clarke, who pointed out an error to me in a previous version of this paper. Moreover, I am indebted to Stéphane Munier for his help with the pictures.

2. The notion of viscosity solution

Throughout the paper, Ω will denote an open subset of $\mathbb{R}^n$ and $|\cdot|$ is the standard Euclidean norm.

We recall for convenience the definition of viscosity solutions for equations like (1.1) (see Crandall, Ishii and Lions [15], Barles [7], Bardi and Capuzzo-Dolcetta [4], Fleming and Soner [18], ...).

Definition 2.1. A function $u \in C(\Omega \times (0,T))$ is a viscosity solution of (1.1) if and only if u is a viscosity subsolution; i.e., for all $\varphi \in C^1(\Omega \times (0,T))$, at each local maximum point $(\bar{x},\bar{t}) \in \Omega \times (0,T)$ of $u - \varphi$, we have

$$\frac{\partial \varphi}{\partial t}(\bar{x},\bar{t}) + H(\bar{x},\bar{t},D_x\varphi(\bar{x},\bar{t})) \leq 0,$$

and u is a viscosity supersolution; i.e., for all $\varphi \in C^1(\Omega \times (0,T))$, at each local minimum point $(\bar{x},\bar{t}) \in \Omega \times (0,T)$ of $u - \varphi$, we have

$$\frac{\partial \varphi}{\partial t}(\bar{x},\bar{t}) + H(\bar{x},\bar{t},D_x\varphi(\bar{x},\bar{t})) \geq 0.$$

If one considers initial conditions, $u \in C(\Omega \times [0,T])$ is a viscosity solution of (1.1)–(1.2) if and only if u is a viscosity subsolution of (1.1) and $u(x,0) \leq u_0(x)$ for all $x \in \Omega$, and u is a supersolution of (1.1) and $u(x,0) \geq u_0(x)$ for all $x \in \Omega$. Therefore, in the case of continuous viscosity solutions, the initial conditions are taken in a classical sense: $u(x,0) = u_0(x)$ for all $x \in \Omega$.

Viscosity solutions can also be characterized in terms of sub- and superdifferential, the definitions of which are recalled.

Definition 2.2. Let $u \in C(\Omega \times (0,T))$. The superdifferential (respectively the subdifferential) of u at the point $(x,t) \in \Omega \times (0,T)$ is the closed convex set

$$D^+_{x,t}u(x,t) = \Big\{(p,q) \in \mathbb{R}^n \times (0,T) : \limsup_{\substack{y \to x, y \in \Omega \\ \tau \to t, \tau \in (0,T)}} \frac{u(y,\tau) - u(x,t) - \langle p, y-x\rangle - q\cdot(\tau - t)}{|y-x| + |\tau - t|} \leq 0\Big\},$$

respectively,

$$D^-_{x,t}u(x,t) = \Big\{(p,q) \in \mathbb{R}^n \times (0,T) : \liminf_{\substack{y \to x, y \in \Omega \\ \tau \to t, \tau \in (0,T)}} \frac{u(y,\tau) - u(x,t) - \langle p, y-x\rangle - q\cdot(\tau - t)}{|y-x| + |\tau - t|} \geq 0\Big\}.$$

3. Barron-Jensen solutions to first-order Hamilton-Jacobi equations

We recall the concept of Barron-Jensen viscosity solutions which are defined for equations with a convex Hamiltonian.

Definition 3.1. Let u be lower-semicontinuous in $\Omega \times (0,T)$. We say that u is a Barron-Jensen solution of (1.1) if it satisfies the condition that for all $\varphi \in C^1(\Omega \times (0,T))$, at each local minimum point $(\bar{x},\bar{t}) \in \Omega \times (0,T)$ of $u - \varphi$, we have

$$\frac{\partial \varphi}{\partial t}(\bar{x},\bar{t}) + H(\bar{x},\bar{t}, D_x\varphi(\bar{x},\bar{t})) = 0.$$

This concept of solution was introduced in 1990 by Barron and Jensen [9]. It was also studied for example in Barles [6, 7] and Bardi and Capuzzo-Dolcetta [4] (in the last reference they call such solutions *bilateral supersolutions*). The link with the classical notion of viscosity solution is given by the following theorem:

Theorem 3.1. *Assume that* **(H1-β)** *and* **(H3)** *hold. Let $u \in C(\Omega \times (0,T))$. Then u is a viscosity solution of* (1.1) *if and only if u is a Barron-Jensen solution of* (1.1).

This theorem was first stated by Barron and Jensen in [9]. It says that in the continuous case, if the Hamiltonian is convex in p and Lipschitz continuous in x, then viscosity solutions and Barron-Jensen solutions coincide. We present here a simplified proof due to Barles [6] which we adapt in the case of an evolution equation. We point out that contrary to the original proof, we are not required to have a Hamiltonian which is Lipschitz continuous in the t-variable.

We start by giving two fundamental lemmas, proving them and using them to prove Theorem 3.1. We introduce some notation: for $(x_0,t_0)\in\Omega\times(0,T)$, we define $\mathcal{A}=B(x_0,\sigma)\times(t_0-\sigma,t_0+\sigma)$ and $\mathcal{A}_\rho=B(x_0,\sigma-\rho)\times(t_0-\sigma+\rho,t_0+\sigma-\rho)$ where $\sigma>0$ is taken small enough such that $\bar{\mathcal{A}}\subset\Omega\times(0,T)$ and $0<\rho<\sigma$. If u is continuous we define $M_{\mathcal{A}}=(2\sup_{(y,s)\in\bar{\mathcal{A}}}|u(y,s)|)^{1/2}$.

Lemma 3.1. *Assume* **(H3)**. *If $u\in C(\Omega\times(0,T))$ is a locally Lipschitz-continuous subsolution of* (1.1) *in $\Omega\times(0,T)$, then u is a viscosity supersolution of*

$$-\frac{\partial\omega}{\partial t}-H(x,t,D_x\,\omega)=0\quad \textit{in } \Omega\times(0,T).$$

Proof of Lemma 3.1. Let $(x_0,t_0)\in\Omega\times(0,T)$ and define $\mathcal{A},\mathcal{A}_\rho$ as above, taking $\sigma>0$ small enough such that u is Lipschitz continuous in $\bar{\mathcal{A}}$ with Lipschitz constant ℓ. By Rademacher's theorem, u is differentiable almost everywhere in $\mathcal{A}$, and then, by Corollary 2.1 of Barles [7], we have

$$\frac{\partial u}{\partial t}(x,t)+H(x,t,D_x\,u(x,t))\le 0\quad\text{almost everywhere in }\ \mathcal{A}. \tag{3.1}$$

Let $\phi\in C^\infty(\mathbb{R}^n\times\mathbb{R}),\phi\ge 0$, such that $\operatorname{supp}\phi\subset B(0,1)\times(0,1)$ and $\int_{\mathbb{R}^n\times\mathbb{R}}\phi(y,s)\,dy\,ds=1$. We set $\phi_\tau(x,t)=\tau^{-n-1}\phi\left(x/\tau,t/\tau\right)$. Let $\tau<\rho/2$. Then the classical convolution

$$u_\tau(x,t)=(u*\phi_\tau)(x,t)=\int_{\mathbb{R}^n\times\mathbb{R}}u(y,s)\,\phi_\tau(x-y,t-s)\,dy\,ds$$

is well-defined for every (x,t) in $\mathcal{A}_\rho$ since $\phi_\tau(x-y,t-s)=0$ for (y,s) outside $\mathcal{A}$. From (3.1), we get

$$\frac{\partial u_\tau}{\partial t}(x,t)+\int_{\mathbb{R}^n\times\mathbb{R}}H(y,s,D_xu(y,s))\phi_\tau(x-y,t-s)\,dy\,ds\le 0.$$

To deal with the integral, we define a modulus of continuity $\mu_{\mathcal{A}}$ of H in the compact set $\bar{\mathcal{A}}\times\bar{B}(0,\ell)$. Recall that $\phi_\tau(x-y,t-s)=0$ for (y,s) outside $\mathcal{A}$ and that $\int_{\mathbb{R}^n\times\mathbb{R}}\phi(y,s)\,dy\,ds=1$. We have

$$\frac{\partial u_\tau}{\partial t}(x,t)+\int_{\mathbb{R}^n\times\mathbb{R}}H(x,t,D_xu(y,s))\phi_\tau(x-y,t-s)\,dy\,ds\le\mu_{\mathcal{A}}(\tau). \tag{3.2}$$

Since H is convex in p, from Jensen's inequality, we get

$$H\Big(x,t,\underbrace{\int_{\mathbb{R}^n\times\mathbb{R}}D_x\,u(y,s)\phi_\tau(x-y,t-s)\,dy\,ds}_{D_xu_\tau(x,t)}\Big) \tag{3.3}$$

$$\leq \int_{\mathbb{R}^n\times\mathbb{R}} H(x,t,D_x\, u(y,s))\phi_\tau(x-y,t-s)\,dy\,ds.$$

Combining (3.2) and (3.3), we have

$$\frac{\partial u_\tau}{\partial t}(x,t) + H\left(x,t,D_x u_\tau(x,t)\right) \leq \mu_{\mathcal{A}}(\tau). \tag{3.4}$$

Now, since $u_\tau \in C^\infty(\mathcal{A}_\rho)$, (3.4) means that u_τ is a classical subsolution of the equation $\partial\omega/\partial t + H(x,t,D_x\,\omega) - \mu_{\mathcal{A}}(\tau) = 0$ in $\mathcal{A}_\rho$; thus it is a classical supersolution of the equation $-\partial\omega/\partial t - H(x,t,D_x\,\omega) = -\mu_{\mathcal{A}}(\tau)$ in $\mathcal{A}_\rho$. Applying the discontinuous stability result (see Theorem 4.1 of Barles [7]), we obtain that $(\liminf_{\tau\to 0}{}_* \, u_\tau)(x,t) = u(x,t)$ is a viscosity supersolution of

$$-\frac{\partial\omega}{\partial t} - H(x,t,D_x\,\omega) \geq 0 \quad \text{in } \mathcal{A}_\rho. \tag{3.5}$$

Since $\cup_{\Omega\times(0,T)}\mathcal{A}_\rho = \Omega\times]0,T[$, we conclude that u is a viscosity supersolution of (3.5) in $\Omega\times(0,T)$ as desired. $\square$

Lemma 3.2. *Assume* **(H1-β)** *and* **(H3)** *and take $\gamma \geq 5C_1/2$. Let $u \in C(\Omega\times[0,T])$ be a Barron-Jensen solution of (1.1) and $(x_0,t_0)\in\Omega\times(0,T)$. If $\varepsilon \leq \rho/2M_{\mathcal{A}}$ and $\alpha \leq \rho/2M_{\mathcal{A}}$, then the inf-convolution*

$$u_{\varepsilon,\alpha}(x,t) = \inf_{(y,s)\in\bar{\mathcal{A}}} \left\{ u(y,s) + \mathrm{e}^{-\gamma t}\frac{|x-y|^2}{\varepsilon^2} + \frac{|t-s|^2}{\alpha^2} \right\}$$

is a viscosity subsolution of

$$\frac{\partial\omega}{\partial t} + H(x,t,D_x\,\omega) - \mu_{\varepsilon,\mathcal{A}}(M_{\mathcal{A}}\alpha) - \frac{\beta C_1 \mathrm{e}^{\gamma T}}{2}\varepsilon^2 = 0 \quad \text{in } \mathcal{A}_\rho$$

where $\mu_{\varepsilon,\mathcal{A}}(\cdot)$ is a modulus of continuity for H in the compact set $\bar{\mathcal{A}}\times\bar{B}(0,2M_{\mathcal{A}}/\varepsilon)$.

Proof of Lemma 3.2. Let $\varphi \in C^1(\Omega\times(0,T))$, $(\bar{x},\bar{t})\in\mathcal{A}_\rho$ be a local minimum of $u_{\varepsilon,\alpha}-\varphi$ and $(\bar{y},\bar{s})\in\bar{\mathcal{A}}$ be a point where the infimum in $u_{\varepsilon,\alpha}(\bar{x},\bar{t})$ is achieved. Define

$$f(x,t,y,s) = u(y,s) + \mathrm{e}^{-\gamma t}\frac{|x-y|^2}{\varepsilon^2} + \frac{|t-s|^2}{\alpha^2} - \varphi(x,t).$$

We see easily that $(\bar{x},\bar{t},\bar{y},\bar{s})$ is a minimum for f in $\mathcal{A}_\rho\times\bar{\mathcal{A}}$. From classical estimates for inf-convolutions, we have that $|\bar{y}-\bar{x}| \leq M_{\mathcal{A}}\varepsilon$ and $|\bar{s}-\bar{t}| \leq M_{\mathcal{A}}\alpha$. Hence, if $\varepsilon \leq \rho/2M_{\mathcal{A}}$ and $\alpha\leq\rho/2M_{\mathcal{A}}$, then it forces $(\bar{y},\bar{s})$ to lie in the open set $\mathcal{A}$; thus $(\bar{x},\bar{t},\bar{y},\bar{s})$ is a local minimum for f.

On the one side, $(\bar{y},\bar{s})$ is a local minimum of $f(\bar{x},\bar{t},y,s)$; hence taking $-\mathrm{e}^{-\gamma\bar{t}}|\bar{x}-y|^2/\varepsilon^2-|\bar{t}-s|^2/\alpha^2+\varphi(\bar{x},\bar{t})$ as a test function, since u is a Barron-Jensen of (1.1), we get

$$2\,\frac{\bar{t}-\bar{s}}{\alpha^2}+H\Big(\bar{y},\bar{s},2\,\mathrm{e}^{-\gamma\bar{t}}\frac{\bar{x}-\bar{y}}{\varepsilon^2}\Big)=0. \tag{3.6}$$

On the other side, $(\bar{x},\bar{t})$ is a local minimum of the C^1-function $f(x,t,\bar{y},\bar{s})$; hence

$$\frac{\partial\varphi}{\partial t}(\bar{x},\bar{t})=-\gamma\mathrm{e}^{-\gamma\bar{t}}\frac{|\bar{y}-\bar{x}|^2}{\varepsilon^2}+2\,\frac{\bar{t}-\bar{s}}{\alpha^2}\quad\text{and}\quad D_x\varphi(\bar{x},\bar{t})=2\,\mathrm{e}^{-\gamma\bar{t}}\frac{\bar{x}-\bar{y}}{\varepsilon^2}. \tag{3.7}$$

From (3.6) and (3.7), we have

$$\frac{\partial\varphi}{\partial t}(\bar{x},\bar{t})+\gamma\mathrm{e}^{-\gamma\bar{t}}\frac{|\bar{y}-\bar{x}|^2}{\varepsilon^2}+H\left(\bar{y},\bar{s},D_x\varphi(\bar{x},\bar{t})\right)=0. \tag{3.8}$$

We want to replace $(\bar{y},\bar{s})$ by $(\bar{x},\bar{t})$ in H in the equality (3.8). Since

$$|D_x\varphi(\bar{x},\bar{t})|=2\,\mathrm{e}^{-\gamma\bar{t}}\frac{|\bar{y}-\bar{x}|}{\varepsilon^2}\leq\frac{2M_{\mathcal{A}}}{\varepsilon},$$

we define a modulus of continuity $\mu_{\varepsilon,\mathcal{A}}(\cdot)$ for H in the compact set $\bar{\mathcal{A}}\times\bar{B}\left(0,2M_{\mathcal{A}}/\varepsilon\right)$. We have

$$H\left(\bar{y},\bar{s},D_x\varphi(\bar{x},\bar{t})\right)\geq H\left(\bar{y},\bar{t},D_x\varphi(\bar{x},\bar{t})\right)-\mu_{\varepsilon,\mathcal{A}}(M_{\mathcal{A}}\alpha). \tag{3.9}$$

Note that we do not replace $\bar{y}$ by $\bar{x}$ using the uniform continuity of H not to get a term $\mu_{\varepsilon,\mathcal{A}}(M_{\mathcal{A}}\varepsilon)$ we cannot control; we need **(H1-β)**, which implies

$$H\left(\bar{y},\bar{t},D_x\varphi(\bar{x},\bar{t})\right)\geq H\left(\bar{x},\bar{t},D_x\varphi(\bar{x},\bar{t})\right)-C_1\left(\beta+|D_x\varphi(\bar{x},\bar{t})|\right)|\bar{x}-\bar{y}| \tag{3.10}$$

and

$$\beta C_1|\bar{x}-\bar{y}|+C_1|D_x\varphi(\bar{x},\bar{t})||\bar{x}-\bar{y}| \leq\frac{\beta C_1\mathrm{e}^{\gamma\bar{t}}\varepsilon^2}{2}+\beta C_1\mathrm{e}^{-\gamma\bar{t}}\frac{|\bar{y}-\bar{x}|^2}{2\varepsilon^2}+2C_1\mathrm{e}^{-\gamma\bar{t}}\frac{|\bar{y}-\bar{x}|^2}{\varepsilon^2}.$$

Using (3.9) and (3.10) in (3.8), we get

$$\frac{\partial\varphi}{\partial t}(\bar{x},\bar{t})+\big(\gamma-\frac{\beta C_1}{2}-2C_1\big)\mathrm{e}^{-\gamma\bar{t}}\frac{|\bar{y}-\bar{x}|^2}{\varepsilon^2}+H\left(\bar{x},\bar{t},D_x\varphi(\bar{x},\bar{t})\right) \leq\mu_{\varepsilon,\mathcal{A}}(M_{\mathcal{A}}\alpha)+\frac{\beta C_1\mathrm{e}^{\gamma T}\varepsilon^2}{2}. \tag{3.11}$$

If $\gamma\geq 5C_1/2$, the "bad term" $\big(\gamma-\frac{\beta C_1}{2}-2C_1\big)\mathrm{e}^{-\gamma\bar{t}}\frac{|\bar{y}-\bar{x}|^2}{\varepsilon^2}$ is positive then we can get rid of it in (3.11). Since the inf-convolution $u_{\varepsilon,\alpha}$ is Lipschitz continuous in $\mathcal{A}_\rho$, the inequality (3.11) holds almost everywhere in $\mathcal{A}_\rho$. Then we proceed as in the proof of Lemma 3.1, regularizing $u_{\varepsilon,\alpha}$ by classical convolution, using the uniform continuity of H, Jensen's inequality and the discontinuous stability result. We obtain that $u_{\varepsilon,\alpha}$ is a subsolution of the desired equation, which completes the proof. □

Remark 3.1. The fact that we can write an equality in (3.6) is essential for the proof; if we use only that u is a viscosity supersolution, we will obtain that $u_{\varepsilon,\alpha}$ is a supersolution (instead of a subsolution as desired) of a perturbed Hamilton-Jacobi equation, which is a classical result (for classical results about sup-convolution, we refer to Lasry and Lions [21], Barles [7] and Bardi and Capuzzo-Dolcetta [4]). Here, in a sense, the concept of Barron-Jensen solutions allows us to "reverse the inequality."

Remark 3.2. In fact, we can conclude directly from (3.11) by using a result of Lions [22] since H is convex.

We now turn to the proof of the theorem.

Proof of Theorem 3.1. We start by showing that a continuous viscosity solution of (1.1) is a Barron-Jensen solution of the same equation. Let $(x_0, t_0) \in \Omega \times (0, T)$ and $\mathcal{A}, \mathcal{A}_\rho$ be defined as above. The sup-convolution

$$u^{\varepsilon,\alpha}(x,t) = \sup_{(y,s)\in\bar{\mathcal{A}}} \left\{ u(y,s) - \mathrm{e}^{Kt}\frac{|x-y|^2}{\varepsilon^2} - \frac{|t-s|^2}{\alpha^2} \right\}$$

is a locally Lipschitz-continuous solution of $\partial\omega/\partial t + H(x,t,D_x\omega) - \beta C_1\varepsilon^2/2 - \mu_{\varepsilon,\mathcal{A}}(M_\mathcal{A}\alpha) = 0$ in $\mathcal{A}_\rho$ for $K \geq 5C_1/2$, $\varepsilon \leq \rho/2M_\mathcal{A}$ and $\alpha \leq \rho/2M_\mathcal{A}$. This result is classical and does not require H to be convex. Applying Lemma 3.1, $u^{\varepsilon,\alpha}$ is a viscosity supersolution of

$$-\frac{\partial\omega}{\partial t} - H(x,t,D_x\,\omega) + \frac{\beta C_1\varepsilon^2}{2} + \mu_{\varepsilon,\mathcal{A}}(M_\mathcal{A}\alpha) = 0 \quad \text{in } \mathcal{A}_\rho.$$

Letting α go to 0 first, then letting ε go to 0, we obtain, from the discontinuous stability result, that u is a supersolution of $-\partial\omega/\partial t - H(x,t,D_x\,\omega) = 0$ in $\mathcal{A}_\rho$ thus in $\Omega \times (0,T)$ since $\cup_{\Omega\times(0,T)}\mathcal{A}_\rho = \Omega \times (0,T)$. Adding the fact that u is a viscosity solution–thus a supersolution of (1.1), we can conclude that u is a Barron-Jensen solution of (1.1) in $\Omega \times (0,T)$.

Conversely, if u is a Barron-Jensen solution of (1.1) in $\Omega \times (0,T)$, we have to show that u is a viscosity solution. It is clear that u is a supersolution; it suffices to prove that u is a subsolution of (1.1). Let $(x_0,t_0) \in \Omega \times (0,T)$. From Lemma 3.2, the inf-convolution $u_{\varepsilon,\alpha}$ is a subsolution of $\partial\omega/\partial t + H(x,t,D_x\omega) - \mu_{\varepsilon,\mathcal{A}}(M_\mathcal{A}\alpha) - \beta C_1\mathrm{e}^{\gamma T}\varepsilon^2/2 = 0$ in $\mathcal{A}_\rho$. Letting $\alpha \to 0$ first, then letting $\varepsilon \to 0$, and thanks to the discontinuous stability theorem once more, we get that u is a viscosity subsolution of (1.1) in every $\mathcal{A}_\rho$ thus in $\Omega \times (0,T)$, which ends the proof. $\square$

Remark 3.3. Theorem 3.1 is true for a slightly more general equation

$$\frac{\partial \omega}{\partial t} + H(x,t,\omega,D_x\,\omega) = 0 \quad \text{in } \Omega \times (0,T)$$

if we assume, as in Barron and Jensen [9], that for all $x \in \mathbb{R}^n$, $t \in [0,T]$, $u, v \in \mathbb{R}$ and $p \in \mathbb{R}^n$, $|H(x,t,u,p) - H(x,t,v,p)| \le \mu_H(|u-v|)$ where $\mu_H(\alpha) \to 0$ when $\alpha \to 0$.

4. A lower-bound estimate for the gradient

We start by stating a result which provides a locally Lipschitz-continuous solution.

Theorem 4.1. (*Upper bound for the gradient*) *Assume* **(H1-β)**, **(H2)** *and that u_0 is locally Lipschitz continuous in $\mathbb{R}^n$. Let $u \in C(\mathbb{R}^n \times [0,T])$ be a viscosity solution of* (1.1)–(1.2). *Then u is locally Lipschitz continuous in $\mathbb{R}^n \times [0,T]$. Moreover, if u_0 is Lipschitz continuous in $\mathbb{R}^n$, then u is Lipschitz continuous in the x-variable in $\mathbb{R}^n$, uniformly with respect to $t \in [0,T]$, and*

$$\|D_x u(\cdot,t)\|_\infty \le \ell \quad \textit{where} \quad \ell = 2\mathrm{e}^{5C_1T/4}\Big(\|Du_0\|_\infty^2 + \frac{\beta C_1 T}{2}\Big)^{1/2}.$$

Upper bounds for the gradient are classical estimates for the solutions of PDEs. Now, if the Hamiltonian is convex, we obtain a surprising result: some lower-bound estimates also hold.

Theorem 4.2. (*Lower bound for the gradient*) *We suppose that the assumptions of Theorem 4.1 and* **(H3)** *hold.*

(*i*) *Let $x_0 \in \mathbb{R}^n$ and $r > 0$. If $|Du_0| \ge \eta$ in $B(x_0,r)$ in the viscosity sense for some positive η, then there exist some positive constants $\tilde\eta, \gamma$ and $0 < t_0 \le T$ such that*

$$|D_x u| \ge \mathrm{e}^{-\gamma t/2}\tilde\eta \quad \textit{in} \quad \mathcal{D}(x_0,r) \cap (\mathbb{R}^n \times (0,t_0)) \quad \textit{in the viscosity sense,}$$

where $\mathcal{D}(x_0,r) = \{(x,t) \in B(x_0,r) \times (0,T) : e^{(A_2+B_2+A_2|x_0|)t}(1+|x-x_0|) \le r+1\}$.

(*ii*) *If $\beta = 0$ in* **(H1-β)**, **(H5)** *and* **(H4)** *hold, then there exist positive constants C and γ such that*

$$|u| + \frac{\mathrm{e}^{\gamma t}}{4}|D_x u|^2 \ge C > 0 \quad \textit{in} \quad \mathbb{R}^n \times (0,T) \quad \textit{in the viscosity sense.}$$

Before giving the proofs of the theorems, we give some examples of Hamiltonians which satisfy the assumptions of Theorems 4.1 and 4.2.

Example 4.1. The most typical example in our context is $H(x,t,p) = a(x,t)|p|$ with $a(x,t) \geq 0$ since it leads to equations which are related to a front propagating with a normal velocity $a(x,t)$. If $a \in C(\mathbb{R}^n \times [0,T])$ is Lipschitz continuous in x (uniformly with respect to t), then H satisfies **(H1-0)** and **(H2)**. Moreover, H satisfies **(H3)** and **(H4)**. An example of a front propagating with constant normal velocity is developed in Example 5.3.

Example 4.2. From optimal control problems, we get some Hamiltonians of the form $H(x,t,p) = \sup_{v\in V}\{\langle b(x,t,v),p\rangle + f(x,t,v)\}$ where V is a compact set. H is clearly convex in p. If b, f are continuous and Lipschitz continuous in x (uniformly with respect to (t,v)) and b is bounded (which is usually supposed in classical control theory), then H satisfies **(H1-0)** and **(H2)**. If $f \equiv 0$, then H satisfies **(H4)**.

Proof of Theorem 4.1. Let $x_0 \in \mathbb{R}^n$ and $r > 0$. We define

$$M_r = \sup_{\bar{B}(x_0,r)\times[0,T]} (2|u|)^{1/2} \quad \text{and}$$

$$||Du_0||_r = \sup\left\{\frac{|u_0(x) - u_0(y)|}{|x-y|} : x, y \in \bar{B}(x_0,r), x \neq y\right\}.$$

Notice that $||Du_0||_r < +\infty$ since u_0 is locally Lipschitz continuous. We start by showing that u is locally Lipschitz continuous in the x variable, uniformly with respect to $t \in [0,T]$. From classical results about sup-convolutions, for $0 < \rho < r, \varepsilon \leq \rho/2M_r$ and $K \geq 5C_1/2$, the sup-convolution

$$u^\varepsilon(x,t) = \sup_{y\in\bar{B}(x_0,2R)} \left\{u(y,t) - \mathrm{e}^{Kt}\frac{|x-y|^2}{\varepsilon^2}\right\}$$

is a viscosity subsolution of $\partial\omega/\partial t + H(x,t,D_x\omega) - \beta C_1\varepsilon^2/2 = 0$ in $B(x_0, r-\rho)\times(0,T)$. From now on, we fix $L = A_2 + B_2 + A_2|x_0|$ and we take $r - \rho \geq 2\mathrm{e}^{LT}$. Then, since u is a supersolution of (1.1), the comparison theorem 6.1 yields

$$u^\varepsilon(x,t) - u(x,t) \leq \sup_{y\in\bar{B}(x_0,r)} \{u^\varepsilon(y,0) - u(y,0)\} + \int_0^t \frac{\beta C_1}{2}\varepsilon^2\,ds \tag{4.1}$$

for all $(x,t) \in \bar{B}(x_0, \mathrm{e}^{-LT}(r-\rho)/2) \times [0,T]$. We estimate the supremum $\sup_{y\in\bar{B}(x_0,r)}\{u^\varepsilon(y,0) - u(y,0)\}$. Let $y \in \bar{B}(x_0,r)$ and suppose that $\bar{y} \in \bar{B}(x_0,r)$ is a point where the supremum in $u^\varepsilon(y,0)$ is achieved. Then

$$0 \leq u^\varepsilon(y,0) - u(y,0) = u_0(\bar{y}) - \frac{|y-\bar{y}|^2}{\varepsilon^2} - u_0(y); \tag{4.2}$$

hence, $\frac{|y-\bar{y}|^2}{\varepsilon^2} \leq u_0(\bar{y}) - u_0(y) \leq \|Du_0\|_r |y-\bar{y}|$. This implies

$$|y-\bar{y}| \leq \|Du_0\|_r \varepsilon^2. \tag{4.3}$$

Using (4.2) and (4.3) we obtain the desired estimate: $u^\varepsilon(y,0) - u(y,0) \leq \|Du_0\|_r^2\, \varepsilon^2$ for all $y \in \bar{B}(x_0,r)$. From (4.1), it follows that

$$u^\varepsilon(x,t) - u(x,t) \leq \lambda\varepsilon^2 \quad \text{where} \quad \lambda = \|Du_0\|_r^2 + \frac{\beta C_1 T}{2}.$$

Hence, for every $x, y \in \bar{B}(x_0, \mathrm{e}^{-LT}(r-\rho)/2)$ and $t \in [0,T]$, we have

$$u(y,t) - u(x,t) \leq \lambda\varepsilon^2 + \mathrm{e}^{KT}\frac{|y-x|^2}{\varepsilon^2}.$$

The C^1-function $f(\varepsilon) = \lambda\varepsilon^2 + \mathrm{e}^{KT}|y-x|^2/\varepsilon^2$ achieves its minimum at $\bar{\varepsilon} = \mathrm{e}^{KT/4}\lambda^{-1/4}|y-x|^{1/2}$. Taking $|y-x|$ small enough such that $\bar{\varepsilon} < \rho/2M_r$, we obtain that

$$u(y,t) - u(x,t) \leq f(\bar{\varepsilon}) = 2\mathrm{e}^{KT/2}\lambda^{1/2}|y-x|,$$

which implies that u is locally Lipschitz continuous in x in $\bar{B}(x_0, \mathrm{e}^{-LT}(r-\rho)/2)$, uniformly with respect to $t \in [0,T]$, with Lipschitz constant

$$\ell_r = 2\mathrm{e}^{KT/2}\Big(\|Du_0\|_r^2 + \frac{\beta C_1 T}{2}\Big)^{1/2}.$$

This implies that u is Lipschitz continuous in x in $\bar{B}(x_0, \mathrm{e}^{-LT}(r-\rho)/2)$ with the same Lipschitz constant as desired. Moreover, if u_0 is Lipschitz continuous in $\mathbb{R}^n$, then u is clearly Lipschitz continuous in x with Lipschitz constant $\ell = 2\mathrm{e}^{KT/2}\Big(\|Du_0\|_\infty^2 + \frac{\beta C_1 T}{2}\Big)^{1/2}$.

It remains to prove that u is Lipschitz continuous in the t variable in $[0,T]$, uniformly with respect to $x \in \bar{B}(x_0,r)$. Define

$$S_r = \sup_{\substack{(y,s) \in \bar{B}(x_0,r) \times [0,T] \\ p \in \bar{B}(0,\ell_r)}} |H(y,s,p)|.$$

Let $x \in \bar{B}(x_0, r-\rho)$ where $0 < \rho < r$ and $t \in [0,T)$. The function

$$\Psi(y,s) = u(y,s) - C(s-t) - \frac{|y-x|^2}{\varepsilon^2}$$

achieves its maximum in $\bar{B}(x_0,r) \times [t,T]$ at a point $(\bar{y},\bar{s})$. If $\varepsilon \leq \rho/2M_r$, then $\bar{y} \in B(x_0,r)$. If $C > 2M_r/(T-t)$, then the maximum cannot be achieved at $\bar{s} = T$; otherwise, we have $\Psi(x,t) \leq \Psi(\bar{y},T)$, and it follows that

$$-2M_r \leq u(x,t) - u(\bar{y},T) \leq -C(T-t),$$

which leads to a contradiction. If $t < \bar{s} < T$, since u is a viscosity subsolution of (1.1), we have

$$C + H\Big(\bar{y}, \bar{s}, 2\frac{\bar{y} - x}{\varepsilon^2}\Big) \leq 0.$$

But $2|\bar{y} - x|/\varepsilon^2 \leq \ell_r$ since u is locally Lipschitz continuous in x, and if $C > S_r$, this inequality cannot hold. Finally, if $C = C_{r,t} = \max\{2M_r/(T - t), S_r\} + 1$, then the maximum has to be achieved at $\bar{s} = t$. Letting ε go to 0, we obtain

$$u(x, s) - u(x, t) \leq C_{r,t}(s - t) \tag{4.4}$$

for every $s \geq t$. In the same way, using the function $u(y, s) + C_{r,t}(s - t) + |y - x|^2/\varepsilon^2$ and the fact that u is a supersolution of (1.1), we obtain

$$u(x, s) - u(x, t) \geq -C_{r,t}(s - t) \tag{4.5}$$

for every $s \geq t$. From (4.4) and (4.5), it follows that u is locally Lipschitz continuous in t in $[0, T)$. To conclude, let us show that the Lipschitz constant is independent of t. Since u is locally Lipschitz continuous in $\bar{B}(x_0, r) \times [0, T)$, from Rademacher's theorem, u is differentiable almost everywhere; thus,

$$\frac{\partial u}{\partial t}(x, t) + H(x, t, D_x\, u(x, t)) = 0 \quad \text{almost everywhere in } B(x_0, r) \times [0, T).$$

Hence,

$$\Big|\frac{\partial u}{\partial t}(x, t)\Big| \leq S_R \quad \text{almost everywhere in } B(x_0, R) \times [0, T),$$

which implies that u is Lipschitz continuous in the t variable in $[0, T]$, uniformly with respect to $x \in \bar{B}(x_0, r)$, with Lipschitz constant S_r, which completes the proof of the theorem. □

Proof of Theorem 4.2. We start by (i); we aim at showing that u is a viscosity supersolution of $|D_x u| - \tilde{\eta}e^{-\gamma t/2} = 0$ in $\mathcal{D}(x_0, r) \cap (\mathbb{R}^n \times (0, t_0))$ where $\gamma, \tilde{\eta}$ and t_0 have to be specified. We use the notation introduced in the proof of Theorem 4.1.

From Lemma 3.2, if $\varepsilon \leq \rho/2M_r$ and $\gamma = 5C_1/2$, then the inf-convolution

$$u_\varepsilon(x, t) = \inf_{y \in \bar{B}(x_0, 2R)} \Big\{u(y, t) + e^{-\gamma t}\frac{|x - y|^2}{\varepsilon^2}\Big\}$$

is a viscosity subsolution of

$$\frac{\partial \omega}{\partial t} + H(x, t, D_x\omega) - \frac{\beta C_1 e^{\gamma T}}{2}\varepsilon^2 = 0 \quad \text{in } B(x_0, r - \rho) \times (0, T).$$

Since u is a supersolution of (1.1), from Theorem 6.1, we get

$$u_\varepsilon(x,t) - u(x,t) \le \sup_{y\in\bar{B}(x_0,r)} \{u_\varepsilon(x,0) - u(x,0)\} + \int_0^t \frac{\beta C_1 \mathrm{e}^{\gamma T}}{2}\varepsilon^2\, ds \tag{4.6}$$

for all $(x,t) \in \{(x,t) \in \bar{B}(x_0, r-\rho) \times [0,T] : \mathrm{e}^{Lt}(1+|x-x_0|) \le r+1\} = \bar{\mathcal{D}}(x_0,r) \cap (\bar{B}(x_0,r-\rho)\times[0,T])$. The estimate of the supremum in (4.6) is a fundamental step in the proof of this theorem and is given in the following proposition:

Proposition 4.1. *If $|Du_0| \ge \eta$ in $B(x_0,r)$ in the viscosity sense, then*

$$\sup_{y\in\bar{B}(x_0,r)} \{u_\varepsilon(x,0) - u(x,0)\} \le -\frac{\eta^2}{4}\varepsilon^2.$$

The proof of the proposition is postponed for clarity and uses as a main tool the following lemma:

Lemma 4.1. (*Viscosity decrease principle*) *Let Ω be an open subset of $\mathbb{R}^n$ and v be a lower-semicontinuous function satisfying $|Dv| \ge \eta > 0$ in Ω in the viscosity sense. Then, for all $x_0 \in \mathbb{R}^n$ and for all $r > 0$ such that $\bar{B}(x_0,r) \subset \Omega$, we have*

$$\inf_{y\in\bar{B}(x_0,r)} v(y) \le v(x_0) - \eta r.$$

The proof of this lemma and some comments about it are postponed.

We now return to the proof of the theorem. Using the above proposition, we get

$$u_\varepsilon(x,t) - u(x,t) \le -\frac{\eta^2}{4}\varepsilon^2 + \frac{\beta C_1 \mathrm{e}^{\gamma T} t}{2}\varepsilon^2 \tag{4.7}$$

for all $(x,t) \in \bar{\mathcal{D}}(x_0,r) \cap (\bar{B}(x_0,r-\rho)\times[0,T])$. Let $(\bar{x},\bar{t}) \in \mathcal{D}(x_0,r) \cap (B(x_0,r-\rho)\times[0,T])$ and $(p,q) \in D^-_{x,t}u(\bar{x},\bar{t})$. By definition of the subdifferential, $u(y,\bar{t}) \ge u(\bar{x},\bar{t}) + \langle p, y-\bar{x}\rangle + |y-\bar{x}|\mu(y-\bar{x})$ for all $y\in\mathbb{R}^n$ where $\mu(\cdot)$ is a function with limit 0 at 0. Writing this inequality at a point $\bar{y}$ where the infimum is achieved in $u_\varepsilon(\bar{x},\bar{t})$, we obtain

$$\begin{aligned} u_\varepsilon(\bar{x},\bar{t}) &= u(\bar{y},\bar{t}) + \mathrm{e}^{-\gamma\bar{t}}\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} \\ &\ge u(\bar{x},\bar{t}) + \langle p, \bar{y}-\bar{x}\rangle + \mathrm{e}^{-\gamma\bar{t}}\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} + |\bar{y}-\bar{x}|\mu(\bar{y}-\bar{x}). \end{aligned} \tag{4.8}$$

On the one side, the C^1-function $g(y) = \langle p, y - \bar{x}\rangle + \mathrm{e}^{-\gamma \bar{t}}|\bar{x} - y|^2/\varepsilon^2$ achieves its minimum at the point $\hat{y} = \bar{x} - p\mathrm{e}^{\gamma \bar{t}}\varepsilon^2/2$ thus,

$$g(\bar{y}) \geq g(\hat{y}) = -\mathrm{e}^{\gamma \bar{t}}\frac{|p|^2}{4}\varepsilon^2. \tag{4.9}$$

On the other side, we want to obtain an estimate for the term $|y - \bar{x}|\mu(y - \bar{x})$ in (4.8). We have

$$u_\varepsilon(\bar{x}, \bar{t}) = u(\bar{y}, \bar{t}) + \mathrm{e}^{-\gamma \bar{t}}\frac{|\bar{x} - \bar{y}|^2}{\varepsilon^2} \leq u(\bar{x}, \bar{t}),$$

which implies

$$|\bar{x} - \bar{y}|^2 \leq \mathrm{e}^{\gamma T}\varepsilon^2\left(u(\bar{x}, \bar{t}) - u(\bar{y}, \bar{t})\right).$$

But, from Theorem 4.1, u is locally Lipschitz continuous, uniformly with respect to $t \in [0, T]$; thus,

$$u(\bar{x}, \bar{t}) - u(\bar{y}, \bar{t}) \leq \ell_r|\bar{x} - \bar{y}|$$

where ℓ_r is the Lipschitz constant of $u(\cdot, t)$ in $\bar{B}(x_0, r)$. Defining $M = \mathrm{e}^{\gamma T}\ell_r$ (note that M does not depend on $\bar{y}$), it follows that

$$|\bar{y} - \bar{x}|\mu(\bar{y} - \bar{x}) \leq M\varepsilon^2\mu(M\varepsilon^2). \tag{4.10}$$

Now, using the information (4.9) and (4.10) in (4.8), we obtain

$$u_\varepsilon(\bar{x}, \bar{t}) \geq u(\bar{x}, \bar{t}) - \frac{\mathrm{e}^{\gamma \bar{t}}}{4}|p|^2\varepsilon^2 - M\varepsilon^2\mu(M\varepsilon^2). \tag{4.11}$$

Finally, combining (4.11) with the inequality (4.7) and dividing by ε^2, we get

$$\mathrm{e}^{\gamma \bar{t}}|p|^2 \geq \eta^2 - 2\beta C_1\mathrm{e}^{\gamma T}\bar{t} - 4M\mu(M\varepsilon^2).$$

Letting ε go to 0 and fixing $t_0 \in (0, T]$ small enough such that $\eta^2 - 2\beta C_1\mathrm{e}^{\gamma T}t_0 = \tilde{\eta}^2 > 0$ (note that we can take $\tilde{\eta} = \eta$ and $t_0 = T$ if $\beta = 0$), we obtain

$$|p| \geq \mathrm{e}^{-\gamma \bar{t}/2}\tilde{\eta} \qquad \text{for every } (p, q) \in D^-_{x,t}(\bar{x}, \bar{t}).$$

Since $\tilde{\eta}$ does not depend on ρ, we can let ρ go to 0, which shows that u is a viscosity supersolution of $|D_x u| - \mathrm{e}^{-\gamma t/2}\tilde{\eta} = 0$ in $\mathcal{D}(x_0, r) \cap (B(x_0, r) \times (0, t_0))$.

Now, we turn to the proof of (ii). We have to show that u is a viscosity supersolution of

$$|\omega| + \frac{\mathrm{e}^{\gamma t}}{4}|D_x\omega|^2 - C = 0 \quad \text{in } \mathbb{R}^n \times (0, T),$$

where γ and C have to be specified. Let $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$ and $r > 0$. As in the proof of (i), for any $0 < \rho < r$, if $\gamma = 5C_1/2$ and $\varepsilon \leq \rho/2M_r$, then the inf-convolution u_ε is a viscosity subsolution of $\partial\omega/\partial t + H(x, t, D_x\omega) = 0$ in $B(x_0, r - \rho) \times (0, T)$ (here $\beta = 0$). We need the following lemma.

Lemma 4.2. *Assume* **(H4)**. *Let* $0 < \varepsilon < 1$ *and* $\Psi_\varepsilon(u)(x,t) = u(x,t) + \varepsilon^2|u(x,t)|$. *If* u *is a viscosity solution of* (1.1), *then* $\Psi_\varepsilon(u)$ *is still a viscosity solution of* (1.1).

This lemma is classical in the theory of viscosity solutions (see Barles [7] for example). The proof is based on **(H4)** since, under this assumption, the equation (1.1) is invariant by increasing changes of variable.

From now on, we take $r - \rho \geq 2\mathrm{e}^{LT}$ where $L = A_2 + B_2 + A_2|x_0|$. Applying the lemma and the comparison theorem 6.1 for the subsolution u_ε and the supersolution $\Psi_\varepsilon(u)$, we get

$$u_\varepsilon(x,t) - \Psi_\varepsilon(u)(x,t) \leq \sup_{y \in \bar{B}(x_0,r)} \left\{ u_\varepsilon(x,0) - u(x,0) - \varepsilon^2|u(x,0)| \right\}.$$

The estimate of the supremum is given by

Proposition 4.2. *Assume* **(H5)**. *Then*

$$\sup_{y \in \bar{B}(x_0,r)} \{u_\varepsilon(x,0) - u(x,0) - \varepsilon^2|u(x,0)|\} \leq -\bar{C}\varepsilon^2 \quad \textit{where} \quad \bar{C} = \min\left(\frac{\eta}{2}, \frac{\eta^2}{4}\right).$$

The proof of this proposition is a straightforward adaptation of the proof of Proposition 4.1 so we skip it. Applying this proposition, the end of the proof can be obtained using the same arguments as in the proof of (i). □

We turn to the proof of Proposition 4.1 and Lemma 4.1.

Proof of Proposition 4.1. Let $\bar{x} \in \bar{B}(x_0,r)$ and $0 < \tilde{r} \leq r$. Using the viscosity decrease principle (Lemma 4.1) in $B(x_0,\tilde{r})$, we have

$$u_\varepsilon(\bar{x},0) = \inf_{y \in \bar{B}(x_0,r)} \{u_0(y) + \frac{|y-\bar{x}|^2}{\varepsilon^2}\} \leq \inf_{y \in \bar{B}(\bar{x},\tilde{r})} u_0(y) + \frac{\tilde{r}^2}{\varepsilon^2} \leq u_0(\bar{x}) - \tilde{r}\eta + \frac{\tilde{r}^2}{\varepsilon^2}.$$

Taking $\tilde{r} = \varepsilon^2\eta/2$, it follows that $u_\varepsilon(\bar{x},0) - u(\bar{x},0) \leq -\eta^2\varepsilon^2/4$, which ends the proof. □

Proof of Lemma 4.1. We argue by contradiction. Suppose that

$$\exists \tilde{r} > 0, \quad \inf_{y \in \bar{B}(x_0,\tilde{r})} v(y) > v(x_0) - \eta\tilde{r}. \tag{4.12}$$

Since the inequality in (4.12) is strict, we can find $\tilde{\eta}, \hat{\eta} > 0$ such that $\tilde{\eta} < \hat{\eta} < \eta$ and (4.12) holds with $\tilde{\eta}$. For $\theta > 0$, define $f(y) = v(y) + \hat{\eta}|y - x_0|^{1+\theta}/\tilde{r}^\theta$. If $S(x_0,\tilde{r})$ is the sphere $\{x \in \mathbb{R}^n : |x - x_0| = \tilde{r}\}$, then from (4.12) with $\tilde{\eta}$, we have

$$\inf_{y \in S(x_0,\tilde{r})} f(y) \geq v(x_0) - \tilde{\eta}\tilde{r} + \hat{\eta}\tilde{r} > v(x_0). \tag{4.13}$$

Hence, the minimum of the lower-semicontinuous function f in the compact set $\bar{B}(x_0,\tilde{r})$ is achieved at a point $\bar{y}$ which lies in the open ball $B(x_0,\tilde{r})$ since

(4.13) shows that it cannot be achieved on the boundary $S(x_0, \tilde{r})$. Thus $\bar{y}$ is a local minimum of f. Since v is a viscosity supersolution of $|D\omega| - \eta = 0$ in Ω, it follows that

$$\left|D_y\Big(-\frac{\hat{\eta}}{r^\theta}|y-x_0|^{1+\theta}\Big)\right| \geq \eta.$$

But

$$\eta \leq \left|D_y\Big(-\frac{\hat{\eta}}{r^\theta}|y-x_0|^{1+\theta}\Big)\right| = \frac{\hat{\eta}(1+\theta)}{r^\theta}|\bar{y}-x_0|^\theta \leq \hat{\eta}(1+\theta),$$

which leads to a contradiction for θ small enough since $\hat{\eta} < \eta$. This ends the proof. □

Remark 4.1. The *viscosity decrease principle* is an adaptation of a result of Clarke [12] in the context of lower-semicontinuous viscosity solutions (see Clarke *et al.* [13] for a similar result with proximal gradients). Roughly speaking, it means that, if the gradient of a function is large enough, then the function has to decrease enough if we move away enough from x_0 in the right direction.

5. Regularity properties of the level sets for semiconcave solutions

5.1. **Semiconcave solutions.** For properties of semiconcave functions, we refer to Cannarsa *et al.* [10, 2, 3].

Definition 5.1. A function $v \in C(\mathbb{R}^n)$ is said to be semiconcave if for every convex, compact subset $\mathcal{C}$ of $\mathbb{R}^n$, there exists $\theta \in (0,1]$ and $\Lambda > 0$ such that $v(y+h)-2v(y)+v(y-h) \leq \Lambda|h|^{1+\theta}$ for every y, h such that $y, y-h, y+h \in \mathcal{C}$.

We say that v is uniformly semiconcave if θ and Λ can be chosen independent of $\mathcal{C}$.

Note that the definition implies that a semiconcave function can be seen locally as the sum of a concave and a smooth function.

We give below some results which provide that a solution u of (1.1)–(1.2) is semiconcave under an additional assumption on H. For a convex, compact subset $\mathcal{C} \subset \mathbb{R}^n$ and $l > 0$, we state

(H6-$\mathcal{C}$-l) There exist some constants $\ell' = \ell'(l) > l$, $\theta = \theta(l) \in (0,1]$ and $a = a(\mathcal{C}, l)$, $b = b(\mathcal{C}, l) > 0$ such that $H(x+h,t,p+k) - 2H(x,t,p) + H(x-h,t,p-k) \geq -a|h|^{1+\theta} - b|h||k|$ for all x, h such that $x, x+h, x-h \in \mathcal{C}$, $t \in [0,T]$ and p, k in the ball $\bar{B}(0,\ell')$.

Remark 5.1. If H is smooth and convex in p, then we see easily that **(H6-$\mathcal{C}$-l)** is fulfilled for every convex, compact set $\mathcal{C} \subset \mathbb{R}^n$ and every $l > 0$. This example shows that **(H6-$\mathcal{C}$-l)** is not too restrictive.

Theorem 5.1. *Let u be a solution of* (1.1)–(1.2) *which is Lipschitz continuous in the x variable with a Lipschitz constant ℓ. Assume* **(H2)** *and* **(H6-$\mathcal{C}$-ℓ)** *for every convex, compact subset $\mathcal{C}$. If u_0 is uniformly semiconcave in $\mathbb{R}^n$, then $u(\cdot,t)$ is semiconcave in $\mathbb{R}^n$, uniformly with respect to $t \in [0,T]$.*

Theorem 5.2. *Assume* **(H1-β)**, **(H2)** *and* **(H6-$\mathcal{C}$-l)** *for every convex, compact subset $\mathcal{C}$ and every $l > 0$. If u_0 is semiconcave in $\mathbb{R}^n$, then $u(\cdot,t)$ is semiconcave in $\mathbb{R}^n$, uniformly with respect to $t \in [0,T]$.*

Theorem 5.1 is an adaptation of a theorem of Ishii [19] in the case of an evolution equation. The proof is similar, so we skip it. Theorem 5.2 is a straightforward application of previous results. At first, notice that u_0, being semiconcave, is locally Lipschitz continuous in $\mathbb{R}^n$. Thus the solution u of (1.1)–(1.2) is locally Lipschitz continuous from Theorem 4.1. Moreover, Theorem 6.1 allows us to localize the arguments of the proof of Theorem 5.1. This implies the result.

Let us give some examples of Hamiltonians which satisfy **(H6-$\mathcal{C}$-l)**:

Example 5.1. Coming back to Example 4.1, if a is locally Lipschitz continuous and uniformly semiconvex (i.e., $-a$ is uniformly semiconcave), then the Hamiltonian $H(x,t,p) = a(x,t)|p|$ with $a(x,t) \geq 0$ satisfies **(H6-$\mathcal{C}$-l)** for every convex compact subset $\mathcal{C}$ and for every $l > 0$.

Example 5.2. If b is $C^{1,\theta}$ (in (x,t) uniformly with respect to v) for some $\theta \in (0,1]$ and $-f$ is uniformly semiconcave in (x,t), then the Hamiltonian H of Example 4.2 satisfies **(H6-$\mathcal{C}$-l)** for every convex, compact subset $\mathcal{C}$ and for every $l > 0$.

5.2. **Regularity of the level sets.** We start by introducing some notation and giving some definitions. In this section, $\mathcal{L}^p$ denotes the Lebesgue measure in $\mathbb{R}^p$ and $\mathcal{H}^p$ is the Hausdorff measure (for a definition, see Morgan [23] or Evans and Gariepy [16]). Let $u \in C(\mathbb{R}^n \times [0,T])$. We denote by $\Gamma_t^\alpha = \{x \in \mathbb{R}^n : u(x,t) = \alpha\}$ the α-level set of u at time t. To simplify the notation, we write Γ_t for the 0-level set and we call it a *front of u at time t*; finally, if u does not depend on t, we denote by Γ^α the α-level set and Γ the front. If $(e_1, e_2, \dots, e_n)$ is an orthonormal basis of $\mathbb{R}^n$, we denote by $(y_1, y_2, \dots, y_n) = (\hat{y}, y_n)$ the coordinates of a vector y in this basis.

Definition 5.2. We say that Γ_t^α is locally Lipschitz at $x \in \Gamma_t^\alpha$ if Γ_t^α is the graph of a Lipschitz-continuous function near x; i.e., there exists a neighborhood $V(x)$ of x in $\mathbb{R}^n$, an orthonormal basis $(e_1, e_2, \dots, e_n)$ and a Lipschitz-continuous function $\zeta : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\Gamma_t^\alpha \cap V(x) = \{(\hat{y}, \zeta(\hat{y})) : \hat{y} \in V(\hat{x})\}$ where $V(\hat{x}) = V(x) \cap \mathrm{Span}(e_1, e_2, \dots, e_{n-1})$ is a neighborhood of $\hat{x}$.

From now on, we consider a solution u of (1.1)–(1.2) under the assumptions of Theorem 4.2 (ii). Then for any $t \in [0,T]$,

$$|D_x u(\cdot,t)| \geq c > 0 \quad \text{in } \Omega \text{ in the viscosity sense,} \tag{5.1}$$

where $c = \min\{\eta, (2C)^{1/2} e^{-\gamma T/2}\}$ and $\Omega = \{x \in \mathbb{R}^n : |u(x,t)| < \min\{\eta, C/2\}\}$ is a neighborhood of Γ_t. Notice that Ω depends on t, but for the sake of simplicity of notation, we will omit writing, here and below, the subscript t as much as possible. We aim at proving some properties of regularity of the front provided by (5.1).

Corollary 5.1. *Under the assumptions of Theorem* 4.2, $\mathcal{L}^n(\Gamma_t) = 0$ *for any* $t \geq 0$.

Proof of Corollary 5.1. We use a classical theorem which is given in Kavian [20], page 85 (another reference is Evans and Gariepy [16], page 84): for any function $v \in W^{1,p}_{loc}(\Omega)$, $1 \leq p \leq \infty$, we have $\mathbf{1}_{\{v=0\}} \cdot Dv = 0$ almost everywhere in Ω where $\mathbf{1}_A$ is the characteristic function of the set A. We argue by contradiction, supposing that $\mathcal{L}^n(\Gamma_t) > 0$. Since u is locally Lipschitz continuous, $u \in W^{1,\infty}_{loc}$ and $\mathbf{1}_{\Gamma_t} \cdot D_x u = 0$ almost everywhere in Ω. There exists a point $x_0 \in \Gamma_t$ such that u is differentiable at x_0 and $D_x u(x_0,t) = 0$. But (5.1) has to be satisfied in the classical sense at such a point; hence, $|D_x u(x_0,t)| > 0$, which leads to a contradiction. □

Note that the above result is an immediate corollary of Theorem 4.2 and is true even if u is not semiconcave. Now, suppose in addition that u_0 is semiconcave and that **(H6-$\mathcal{C}$-l)** holds for every convex, compact subset $\mathcal{C}$ and for every $l > 0$. Then from Theorem 5.2, $u(\cdot,t)$ is semiconcave and Γ_t satisfies the following property for any $t \in [0,T]$:

Theorem 5.3. *Let* $x = (\hat{x}, x_n) \in \Gamma_t$. *There exists an orthonormal basis* $(e_1, e_2, \ldots, e_n)$, *a neighborhood* $V(x) = V(\hat{x}) \times V(x_n)$ *of* x *and a* C^1*-function* $\zeta : V(\hat{x}) \to \mathbb{R}$ *such that* $D\zeta(\hat{x}) = 0$ *and* $\{(\hat{y}, z) \in \Omega : z \geq \zeta(\hat{y})\} \subset \{y \in \Omega : u(y,t) \leq 0\}$.

Remark 5.2. This theorem means that the front has locally C^1-support hypersurfaces "to one side." This comes from the fact that a semiconcave function admits at each point C^1 upper-support functions.

Proof of Theorem 5.3. We start by a general fact: let x be in Ω; there exists p in $D^+_x u(x,t)$ such that $|p| \geq c > 0$. Let us prove this assertion: the function u is differentiable almost everywhere in Ω; thus, from (5.1) we get a sequence $x_k \to x$ of points of Ω such that u is differentiable at x_k and $|D_x u(x_k,t)| \geq c$. We conclude, letting $k \to \infty$, by using the upper-semicontinuity of $D^+_x u(\cdot,t)$ (see Cannarsa and Soner [10]).

Now, we turn to the proof of the theorem. Let $x \in \Gamma_t$. There exists $p \in D_x^+ u(x,t)$ such that $|p| \geq c$. As u is a viscosity subsolution of (1.1), there exists $\varphi \in C^1(\Omega \times (0,T))$ such that $u - \varphi$ has a local maximum at (x,t) and $D_x\varphi(x,t) = p$ (see Barles [7], page 18). Moreover, we can assume that $u(x,t) - \varphi(x,t) = 0$, which implies $\varphi(x,t) = 0$ since $x \in \Gamma_t$. We fix an orthonormal basis $(e_1, e_2, \ldots, e_n)$ with $e_n = -p/|p|$. From the implicit function theorem, there exists a neighborhood $V(\hat{x})$ of $\hat{x}$ and a C^1-function $\zeta : V(\hat{x}) \to \mathbb{R}$ such that $\varphi(\hat{y}, \zeta(\hat{y}), t) = 0$ for all $\hat{y} \in V(\hat{x})$. It follows that $u(\hat{y}, \zeta(\hat{y}), t) \leq 0$ for all $\hat{y} \in V(\hat{x})$ and $D\zeta(\hat{x}) = 0$. Moreover,

$$\begin{aligned} u(\hat{y}, y_n, t) \leq \varphi(\hat{y}, y_n, t) &= \varphi(x,t) + \langle D_x\varphi(x,t), y - x\rangle + o(y-x) \\ &= -|p| \cdot (y_n - x_n) + o(y-x) < 0 \end{aligned}$$

for y sufficiently close to x and $y_n - x_n > 0$. The proof ends by noting that $\{(\hat{y}, z) \in \Omega : z \geq \zeta(\hat{y})\} \subset \{y \in \Omega : u(y,t) \leq 0\}$. □

Remark 5.3. The regularity property of Theorem 5.3 does not hold for the front if u is not semiconcave. Let us give a counterexample: let $w(x,y) = |x| - y$. The function w is Lipschitz continuous and is convex but not semiconcave. The estimate (5.1) is clearly satisfied. We compute the front $\Gamma = \{(x,y) \in \mathbb{R}^2 : y = |x|\}$ and $\{(x,y) \in \mathbb{R}^2 : w(x,y) \leq 0\} = \{(x,y) \in \mathbb{R}^2 : y \geq |x|\}$ and, from this, it is clear that the regularity property of Theorem 5.3 is not fulfilled at $(0,0)$.

We need some definitions. The *Clarke subdifferential* or *Clarke generalized gradient* (see Clarke [12]) is well adapted for Lipschitz continuous functions:

Definition 5.3. Let u be a Lipschitz-continuous function in $\mathbb{R}^n$ and denote by $\Sigma(u)$ the subset of $\mathbb{R}^n$ of Lebesgue measure zero where u is not differentiable. The Clarke subdifferential of u at the point x in $\mathbb{R}^n$ is the nonempty, compact, convex set

$$\begin{aligned} \partial_C u(x) &= \left\{\xi \in \mathbb{R}^n : \limsup_{y \to x,\, t \downarrow 0} \frac{u(y+tp) - u(y)}{t} \geq \langle \xi, p\rangle \ \text{ for all } p \in \mathbb{R}^n\right\} \\ &= \operatorname{co}\left\{\lim_{k \to +\infty} Du(x_k) \ : \ x_k \to x,\, x_k \notin \Sigma(u)\right\}. \end{aligned}$$

Definition 5.4. We set $\mathcal{S} = \{x \in \Omega : \Gamma_t^{u(x,t)}$ is not locally Lipschitz at $x\}$ and $\mathcal{S}_0 = \{x \in \Omega : 0 \in \partial_C u(x,t)\}$.

Theorem 5.4. *Let $u \in C(\mathbb{R}^n \times [0,T])$ be locally Lipschitz continuous. If $x \notin \mathcal{S}_0$, then $\Gamma_t^{u(x,t)}$ is locally Lipschitz at x.*

Remark 5.4. This theorem means that for every $x \in \mathbb{R}^n$ such that the Clarke subdifferential of the function does not contain 0, the $u(x,t)$-level set is locally Lipschitz at x. The converse is not true. The function defined in Example 5.4 satisfies $0 \in \partial_C u(x)$ for every x in the 0-level set of u though the front is locally Lipschitz.

Proof of Theorem 5.4. The proof consists in showing that we can use the nonsmooth inversion theorem of Clarke [12]. Let $x \in \Omega$ such that $0 \notin \partial_C u(x,t)$ and let $\alpha = u(x,t)$. The set $\partial_C u(x)$ is convex and compact and does not contain 0; thus, by the separation theorem of convex sets we have

$$\exists q \in \mathbb{R}^n,\ |q| = 1 \text{ such that } \forall p \in \partial_C u(x),\ \langle q, p\rangle \geq \nu > 0. \tag{5.2}$$

From now on, we fix an orthonormal basis $(e_1, e_2, \ldots, e_n)$ of $\mathbb{R}^n$ such that $e_n = q$. Let $F_t : \mathbb{R}^n \to \mathbb{R}^n$ be the function $F_t(\hat{y}, y_n) = (\hat{y}, u(\hat{y}, y_n, t))$. Since F_t is locally Lipschitz continuous, we have

$$DF_t(\hat{y}, y_n) = \left(\begin{array}{c|c} & \frac{\partial u}{\partial y_1}(\hat{y}, y_n, t) \\ Id_{n-1} & \vdots \\ & \frac{\partial u}{\partial y_{n-1}}(\hat{y}, y_n, t) \\ \hline 0 & \frac{\partial u}{\partial y_n}(\hat{y}, y_n, t) \end{array}\right)$$

for almost every $y \in \mathbb{R}^n$. The generalized Jacobian of F_t at x is

$$\begin{aligned}\partial_C F_t(\hat{x}, x_n) &= \mathrm{co}\left\{\lim_{k\to+\infty} DF_t(y^{(k)}),\ y^{(k)} \underset{k\to+\infty}{\longrightarrow} x \text{ and } F_t \text{ is differentiable at } y^{(k)}\right\} \\ &= \left\{\left(\begin{array}{c|c} & p_1 \\ Id_{n-1} & \vdots \\ & p_{n-1} \\ \hline 0 & p_n \end{array}\right),\ p = (p_1, \ldots, p_n) \in \partial_C u(x)\right\}.\end{aligned}$$

By definition, $\partial_C F_t(\hat{x}, x_n)$ is of maximal rank provided every matrix in $\partial_C F_t(\hat{x}, x_n)$ is of maximal rank. This condition is clearly fulfilled since, from (5.2), $p_n = \langle p, e_n\rangle \geq \nu > 0$. Thus we can apply the nonsmooth inverse function theorem: there exists a neighborhood $V = V(\hat{x}) \times V(x_n)$ of $(\hat{x}, x_n)$, a neighborhood $V' = V'(\hat{x}) \times V'(\alpha)$ of $F_t(\hat{x}, x_n) = (\hat{x}, u(x,t)) = (\hat{x}, \alpha)$ and a Lipschitz-continuous function $G_t : V' \to V$ such that $\forall (\hat{y}', y_n') \in V'$, $F_t \circ G_t(\hat{y}', y_n') = (\hat{y}', y_n')$ (and $\forall (\hat{y}, y_n) \in V$, $G_t \circ F_t(\hat{y}, y_n) = (\hat{y}, y_n)$). We obtain that for all $y' = (\hat{y}', y_n') \in V'$,

$F_t \circ G_t(\hat{y}', y_n')$

$$= \Big(\underbrace{\langle e_1, G_t(y')\rangle, \ldots, \langle e_{n-1}, G_t(y')\rangle}_{\hat{y}'}, \underbrace{u\left(\langle e_1, G_t(y')\rangle, \ldots, \langle e_n, G_t(y')\rangle, t\right)}_{y'_n} \Big);$$

hence, for all $(\hat{y}', y'_n) \in V'(\hat{x}) \times V'(\alpha)$, $u\left(\hat{y}', \langle e_n, G_t(\hat{y}', y'_n)\rangle, t\right) = y'_n$. We take $y'_n = \alpha$ and define $\zeta(\hat{y}') = \langle e_n, G_t(\hat{y}', \alpha)\rangle$. The function ζ is Lipschitz continuous since G_t is Lipschitz continuous. Note that ζ has the same Lipschitz constant as G_t, which is $1/\delta$, where δ is the distance from 0 to the convex compact set $\partial_C u(x,t)$. We get the desired result: for all $\hat{y}' \in V'(\hat{x})$,

$$u(\hat{y}', \zeta(\hat{y}'), t) = \alpha$$

and $\zeta(\hat{y}')$ is the unique solution of $u(\hat{y}', \cdot, t) = \alpha$ in V'. □

Corollary 5.2. *Under the assumptions of Theorem* 4.2 *and Theorem* 5.2,
(*i*) $\mathcal{L}^n(\mathcal{S}) = 0$;
(*ii*) *for any* $t \geq 0$, $\mathcal{H}^{n-1}(\mathcal{S} \cap \Gamma_t^\alpha) = 0$ *for* $\mathcal{L}^1$*-almost-every* $\alpha \in \mathbb{R}$.

Remark 5.5. (i) means that for a semiconcave function which satisfies (5.1), the front is locally Lipschitz almost everywhere. $\mathcal{H}^{n-1}$ can be seen as a measure on the α-level set Γ_t^α. Hence, (ii) means that almost every α-level set is locally Lipschitz almost everywhere in the sense of the $\mathcal{H}^{n-1}$ measure. But, note that this theorem gives no information about any special level set, particularly the front (see the counter-example developed in Theorem 5.5).

Proof of Corollary 5.2. We start by proving (i). If $u(\cdot,t)$ is differentiable at a point $x_0 \in \mathbb{R}^n$, then $D_x^+ u(x_0,t) = D_x^- u(x_0,t) = \{D_x u(x_0,t)\}$ and $|D_x u(x_0,t)| \geq c > 0$ by (5.1). Thus $0 \notin D_x^+ u(x_0,t)$. Moreover, since $u(\cdot,t)$ is semiconcave, $D_x^+ u(\cdot,t) = \partial_C u(\cdot,t)$ for every $t \in [0,T]$ (see Cannarsa and Soner [10] for instance). Since $u(\cdot,t)$ is differentiable almost everywhere, it follows that $\mathcal{L}^n(\mathcal{S}_0) = 0$, which implies the result since $\mathcal{S} \subset \mathcal{S}_0$.

Now, we turn to the proof of (ii). We apply the coarea formula (see Evans and Gariepy [16]) with the set $\mathcal{S}$ and the locally Lipschitz-continuous function $u(\cdot,t), t \geq 0$:

$$\int_{\mathcal{S}} |D_x u(x,t)|\, dx = \int_{\mathbb{R}} \mathcal{H}^{n-1}(\mathcal{S} \cap \Gamma_t^\alpha)\, d\alpha.$$

Since $\mathcal{L}^n(\mathcal{S}) = 0$, we obtain the desired result. □

We end by giving some examples and a theorem which develops an example of a semiconcave function, satisfying (5.1), whose front is however very nasty.

Example 5.3. *A smooth front which develops a "non-Lipschitz singularity."* Let $u_0 : \mathbb{R}^2 \to \mathbb{R}$ be defined by $u_0 = \min\{u_1, u_2\}$ where $u_1(x,y) = (x^2 + (y-5)^2)^{1/2} - 4$ and $u_2(x,y) = (x^2 + (y+5)^2)^{1/2} - 4$. On the one side, the function

u_1 (respectively u_2) is Lipschitz continuous and its graph is a cone centered at the point $(0,5)$ (respectively $(0,-5)$). Thus u_0 is Lipschitz continuous. On the other side u_0 satisfies **(H5)**: $|u_0|+|Du_0|\geq 1$ since

— for every $(x,y)\notin\{(0,-5),(0,5)\}$ such that $u_1(x,y)\neq u_2(x,y)$, u_0 is differentiable and $|Du_0(x,y)|=1$;

— at each point (x,y) such that $u_1(x,y)=u_2(x,y)$, $D^-u_0(x,y)=\emptyset$;

— $|u_0(0,-5)|\geq 1$ and $|u_0(0,5)|\geq 1$.

We consider the Eikonal equation whose associated front propagates with a normal constant velocity equal to 1:

$$\begin{cases}\frac{\partial\omega}{\partial t}+|D_x\omega|=0 & \text{in } \mathbb{R}^2\times(0,2]\\ \omega(x,0)=u_0(x) & \text{in } \mathbb{R}^2.\end{cases}\tag{5.3}$$

The Hamiltonian $H(p)=|p|$ satisfies the assumptions of Theorem 4.2 (with **(H1-0)**); thus, the lower-bound inequality (5.1) holds for $t\in[0,2]$. Moreover, the solution of (5.3) is unique and given by the Oleinik-Lax formula (to go into the details, see e.g. Lions [22] or Barles [7]):

$$u(x,y,t)=\inf\{u_0(x',y'),\ ((x-x')^2+(y-y')^2)^{1/2}\leq t\}.$$

Then, one computes easily $\Gamma_t=\partial(B_{1,t}\cup B_{2,t})$ for $t\in[0,2]$ where $\partial(B_{1,t}\cup B_{2,t})$ is the boundary of the union of the balls $B_{1,t}=\{(x,y)\in\mathbb{R}^2 : x^2+(y-5)^2<(4+t)^2\}$ and $B_{2,t}=\{(x,y)\in\mathbb{R}^2 : x^2+(y+5)^2<(4+t)^2\}$. The front is smooth for $t\in[0,1)$ (union of 2 disjoint circles), is not locally Lipschitz at $(0,0)$ for $t=1$ (the circles are tangent at $(0,0)$) and is locally Lipschitz for $t\in(1,2]$. This illustrates the case of a front which is initially smooth but develops a non-Lipschitz singularity. Note that from the result of uniqueness of the propagation of the front (see the Introduction), up to regularize u_0, we can even start with a smooth function u_0 (thus a semiconcave function) whose front Γ_0 is $\partial(B_{1,0}\cup B_{2,0})$. The propagation of the front will be the same.

Example 5.4. *The "roof."* Let $v:\mathbb{R}^2\to\mathbb{R}$ be defined by $v(x,y)=-|y|$. The function v is Lipschitz continuous, concave and $\Gamma=\{(x,y)\in\mathbb{R}^2 : y=0\}$. The function v satisfies the lower-bound estimate of Theorem 4.2 since $D^-v(x,0)=\emptyset$ and $|D^-v(x,y)|=1$ if $y\neq 0$. And $D^+v(x,0)=\partial_C v(x,0)=\{0\}\times[-1,1]$. In this case, the front is locally Lipschitz everywhere but $0\in\partial_C v(x,y)$ for every $(x,y)\in\Gamma$; thus, $\mathcal{S}$ is not equal to $\mathcal{S}_0$. Moreover, coming back to Example 5.3, we see that $(0,0)\in\Gamma_1(u)$, $D_x^-u(0,0,1)=D^-v(0,0)$ and $D_x^+u(0,0,1)=D^+v(0,0)$ while $\Gamma(v)$ is smooth and $\Gamma_1(u)$ is singular at the point $(0,0)$. It shows that sub- and superdifferentials do not characterize the points of the front which are not locally Lipschitz.

Theorem 5.5. *Let K be a Cantor set in $[0,1]$ such that $\mathcal{L}^1(K) > 0$. Define $v : [0,1] \times [-1,1] \to \mathbb{R}$ by $v(x,y) = -|y| + d_K^2(x)$ where $d_K(x)$ is the distance of x to K. Then*

(i) v is Lipschitz continuous, semiconcave and satisfies the lower-bound estimate (5.1);

(ii) $\mathcal{H}^1(\Gamma \cap \mathcal{S}) > 0$ where $\Gamma = \{(x,y) \in [0,1] \times [-1,1] : |y| = d_K^2(x)\}$ is the front of v.

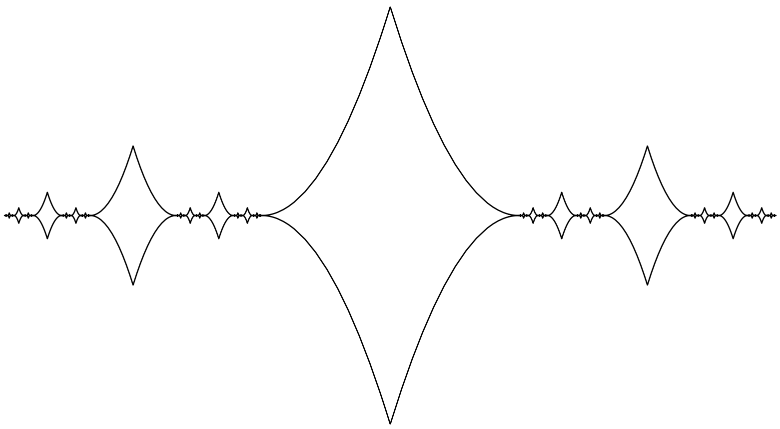

Figure 1. $\Gamma = \{(x,y) \in [0,1] \times [-1,1] : |y| = d_K^2(x)\}$

Proof of Theorem 5.5. The function v is clearly Lipschitz continuous since $[0,1] \times [-1,1]$ is bounded. Writing

$$v(x,y) = -|y| + \inf_{x' \in K} \{|x'|^2 - 2\langle x, x'\rangle\} + |x|^2,$$

we see that v is semiconcave. The function v satisfies (5.1) since for every $p \in D^- v(x,y)$, $|p| \geq |\langle p, (0,1)\rangle| \geq 1$ (notice that if $D^- v(x,y) = \emptyset$, (5.1) is trivially fulfilled).

Now we look more precisely at the front Γ of v (depicted in Figure 1). We define $\tilde{\Gamma}$ by

$$\tilde{\Gamma} = \Gamma \cap ([0,1] \times \{0\}) = \{(x,0) \in [0,1] \times \{0\} : d_K^2(x) = 0\} = K \times \{0\}$$

since K is closed. We want to prove that every point of $\tilde{\Gamma}$ is singular (i.e., lies in $\mathcal{S}$). We start by an important remark: if $(x_0, 0) \in \tilde{\Gamma}$ (i.e., $x_0 \in K$), then $\Gamma \cap \{(x_0, y) : y \in [-1,1]\} = \{(x_0, 0)\}$ since in this case, a point $(x_0, y), y \neq 0$ cannot satisfy $|y| = d_K^2(x_0) = 0$ (roughly speaking, if $(x_0, 0) \in \Gamma$, then there cannot be another point of Γ above or below). Recalling that there is no isolated point in the Cantor set K, we can divide the points of $\tilde{\Gamma}$ in two sets:

— the first set is countable and contains every point $(x_1, 0) \in \tilde{\Gamma}$ which is an extremity of a segment $((x_1, 0), (x_1 + \lambda, 0))$, $\lambda > 0$, which was removed in order to construct the Cantor set. An easy computation shows that, in this case, the piece of curve $\{(x, y) \in [x_1, x_1 + \lambda] \times [-1, 1] : x = \sqrt{|y|} + x_1\}$ is contained in Γ and is not locally Lipschitz at $(x_1, 0)$; thus, Γ is not locally Lipschitz at $(x_1, 0)$;

— the second set consists of points $(x_2, 0) \in \tilde{\Gamma}$ such that there exist two sequences $(x_k^+, 0) \to_{k\to\infty} (x_2, 0)$, $x_k^+ \in K$, $x_k^+ > x_2$ and $(x_k^-, 0) \to_{k\to\infty} (x_2, 0)$, $x_k^- \in K$, $x_k^- < x_2$.

From the remark, there cannot exist any piece of continuous curve which passes through $(x_2, 0)$ and lies in Γ. Thus the connected component of $(x_2, 0)$ in Γ is $\{(x_2, 0)\}$, and Γ is not locally Lipschitz at $(x_2, 0)$. It follows that $\mathcal{H}^1(\Gamma \cap \mathcal{S}) \geq \mathcal{H}^1(\tilde{\Gamma}) = \mathcal{L}^1(K) > 0$. $\square$

6. Appendix

We state here a comparison result for unbounded, continuous viscosity solutions. This result was proved by Ishii [19]. We give a simplified proof which avoids the classical step-by-step argument.

Theorem 6.1. (*Comparison result for unbounded solutions*) *Assume* **(H1-β)** *and* **(H2)** *and let $f, g \in C(\mathbb{R}^n \times [0,T])$. Let $x_0 \in \mathbb{R}^n$ and $r > 0$, and define $L = A_2 + B_2 + A_2|x_0|$. If $u \in C(\bar{B}(x_0, r) \times [0,T])$ is a viscosity subsolution of*

$$\frac{\partial \omega}{\partial t} + H(x, t, D_x\omega) = f \text{ in } B(x_0, r) \times (0,T) \text{ and } \omega(x, 0) = u_0(x) \text{ in } B(x_0, r) \quad (6.1)$$

and v is a viscosity supersolution of

$$\frac{\partial \omega}{\partial t} + H(x, t, D_x\omega) = g \text{ in } B(x_0, r) \times (0,T) \text{ and } \omega(x, 0) = v_0(x) \text{ in } B(x_0, r), \quad (6.2)$$

then

$$u(x,t) - v(x,t) \leq \sup_{y \in \bar{B}(x_0, r)} \{u_0(y) - v_0(y)\} + \int_0^t \sup_{y \in \bar{B}(x_0, r)} \{f(y,s) - g(y,s)\}\, ds \quad (6.3)$$

for every $(x, t) \in \bar{\mathcal{D}}(x_0, r)$ where

$$\mathcal{D}(x_0, r) = \{(x,t) \in B(x_0, r) \times (0,T) : \mathrm{e}^{LT}(1 + |x - x_0|) - 1 \leq r\}.$$

In particular, if $r \geq 2\mathrm{e}^{LT}$, then (6.3) *holds in $\bar{B}(x_0, \mathrm{e}^{-LT} r/2) \times [0,T]$.*

Remark 6.1. The main point is that assumption **(H2)** implies a property of "finite speed of propagation," which can be described from two points of view:

- the values of the solution u at the point (x,t) do not depend on the initial data u_0 in the whole space $\mathbb{R}^n$ but only in a ball $\bar{B}(x, r(x,t))$ where $r(x,t) = \mathrm{e}^{(A_2 + B_2 + A_2|x_0|)t} - 1$;

- the values of the initial data in some ball $B(x_0, r)$ completely determine the values of the solution u of the equation in the domain of dependence $\bar{\mathcal{D}}(x_0, r)$ (cf. Figure 2).

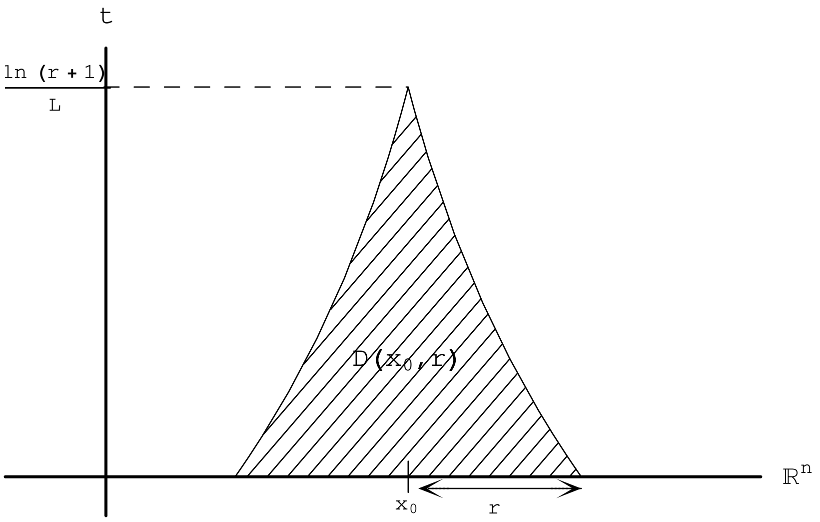

Figure 2. $\mathcal{D}(x_0, r) = \{(x, t) \in \bar{B}(x_0, r) \times [0, T] : \mathrm{e}^{LT}(1 + |x - x_0|) - 1 \le r\}$.

We start by a lemma, proving it and using it to prove the theorem.

Lemma 6.1. *Under the assumptions of Theorem 6.1, the function $w = u - v$ is a viscosity subsolution of*

$$\frac{\partial \omega}{\partial t} - (A_2|x| + B_2)\,|D_x\omega| \le f - g \quad \text{in } B(x_0, r) \times (0, T). \tag{6.4}$$

Proof of Lemma 6.1. Let $\varphi \in C^1(B(x_0, r) \times (0, T))$, and let $(\bar{x}, \bar{t}) \in B(x_0, r) \times (0, T)$ be a local maximum of $w - \varphi$. Without loss of generality, we can suppose that $(\bar{x}, \bar{t})$ is a strict local maximum in $\bar{B}(\bar{x}, \sigma) \times [\bar{t} - \sigma, \bar{t} + \sigma]$ for $\sigma > 0$ small enough. We define

$$\Psi_{\varepsilon,\alpha}(x, y, t, s) = u(x, t) - v(y, s) - \frac{|x - y|^2}{\varepsilon^2} - \frac{|t - s|^2}{\alpha^2} - \varphi(x, t)$$

and

$$M_{\varepsilon,\alpha} = \max_{\bar{B}(\bar{x},\sigma)^2 \times [\bar{t}-\sigma, \bar{t}+\sigma]^2} \Psi_{\varepsilon,\alpha} = \Psi_{\varepsilon,\alpha}(x_{\varepsilon,\alpha}, y_{\varepsilon,\alpha}, t_{\varepsilon,\alpha}, s_{\varepsilon,\alpha}).$$

Since $(\bar{x}, \bar{t})$ is a strict maximum, we know (see Barles [7]) that

$$M_{\varepsilon,\alpha} \to \max_{\bar{B}(\bar{x},\sigma) \times [\bar{t}-\sigma, \bar{t}+\sigma]} w - \varphi = (w - \varphi)(\bar{x}, \bar{t}),$$

$x_{\varepsilon,\alpha}, y_{\varepsilon,\alpha} \to \bar{x}$ and $t_{\varepsilon,\alpha}, s_{\varepsilon,\alpha} \to \bar{t}$ when $\varepsilon, \alpha \to 0$. As u is a subsolution of (6.1) and v is a supersolution of (6.2), by a straightforward calculation, we obtain

$$2\frac{t_{\varepsilon,\alpha} - s_{\varepsilon,\alpha}}{\alpha^2} + \frac{\partial \varphi}{\partial t}(x_{\varepsilon,\alpha}, t_{\varepsilon,\alpha}) + H\Big(x_{\varepsilon,\alpha}, t_{\varepsilon,\alpha}, 2\frac{x_{\varepsilon,\alpha} - y_{\varepsilon,\alpha}}{\varepsilon^2} + D_x\varphi(x_{\varepsilon,\alpha}, t_{\varepsilon,\alpha})\Big)$$

$$\leq f(x_{\varepsilon,\alpha}, t_{\varepsilon,\alpha}) \tag{6.5}$$

and

$$2\frac{t_{\varepsilon,\alpha} - s_{\varepsilon,\alpha}}{\alpha^2} + H\Big(y_{\varepsilon,\alpha}, s_{\varepsilon,\alpha}, 2\frac{x_{\varepsilon,\alpha} - y_{\varepsilon,\alpha}}{\varepsilon^2}\Big) \geq g(y_{\varepsilon,\alpha}, s_{\varepsilon,\alpha}). \tag{6.6}$$

Subtracting (6.6) from (6.5) and letting α go to 0, we get that $x_{\varepsilon,\alpha} \to x_\varepsilon$, $y_{\varepsilon,\alpha} \to y_\varepsilon$, $t_{\varepsilon,\alpha} \to t_\varepsilon$ and

$$\frac{\partial \varphi}{\partial t}(x_\varepsilon, t_\varepsilon) + H\big(x_\varepsilon, t_\varepsilon, 2\frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2} + D_x\varphi(x_\varepsilon, t_\varepsilon)\big) - H\big(y_\varepsilon, t_\varepsilon, 2\frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2}\big) \leq (f-g)(x_\varepsilon, t_\varepsilon).$$

From **(H1-β)** and **(H2)**, we have

$$\begin{aligned}
&H\big(x_\varepsilon, t_\varepsilon, 2\frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2} + D_x\varphi(x_\varepsilon, t_\varepsilon)\big) - H\big(y_\varepsilon, t_\varepsilon, 2\frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2}\big) \\
&\geq -\big(A_2|x_\varepsilon| + B_2\big)|D_x\varphi(x_\varepsilon, t_\varepsilon)| - C_1\big(\beta + 2\frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon^2}\big)|x_\varepsilon - y_\varepsilon| \\
&\geq -\big(A_2|x_\varepsilon| + B_2\big)|D_x\varphi(x_\varepsilon, t_\varepsilon)| - \frac{\beta C_1 \varepsilon^2}{2} - \big(\frac{\beta C_1}{2} + 2C_1\big)\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2}.
\end{aligned}$$

It follows that

$$\frac{\partial \varphi}{\partial t}(x_\varepsilon, t_\varepsilon) - (A_2|x_\varepsilon| + B_2)\,|D_x\varphi(x_\varepsilon, t_\varepsilon)| \leq \frac{C_1\varepsilon^2}{2} + \frac{5C_1}{2}\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} + (f-g)(x_\varepsilon, t_\varepsilon).$$

Letting ε go to 0, we end the proof since $|x_\varepsilon - y_\varepsilon|^2/\varepsilon^2 \to 0$, x_ε, $y_\varepsilon \to \bar{x}$ and $t_\varepsilon \to \bar{t}$. ☐

Proof of Theorem 6.1. Let $0 < \varepsilon < r$ and $\chi_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ be a C^1 function in $[0, r)$ such that $\chi_\varepsilon \equiv 0$ in $[0, r-\varepsilon]$, χ_ε is increasing in $[0, r)$ with limit $+\infty$ at r and $\chi_\varepsilon \equiv +\infty$ in $[r, +\infty)$. Let $\eta > 0$. The function $w(x,t) - \eta t - \frac{\eta}{T-t} - \int_0^t \sup_{y\in\bar{B}(x_0,r)}\{f(y,s) - g(y,s)\}\,ds - \chi_\varepsilon(\mathrm{e}^{Lt}(1 + (|x-x_0|^2 + \varepsilon^2)^{\frac{1}{2}}) - 1)$ achieves its maximum at a point $(\bar{x}, \bar{t}) \in B(x_0, r) \times [0, T)$. If $\bar{t} > 0$, then we have a local maximum and, using that w is a subsolution of (6.4) in $B(x_0, r) \times (0, T)$, we obtain from Lemma 6.1,

$$\begin{aligned}
&\eta + \frac{\eta}{(T-\bar{t})^2} + \sup_{y\in\bar{B}(x_0,r)}\{f(y,t) - g(y,t)\} - (f(\bar{x},\bar{t}) - g(\bar{x},\bar{t})) \\
&+ \chi'_\varepsilon \cdot \mathrm{e}^{L\bar{t}}\left(L(1 + (|\bar{x} - x_0|^2 + \varepsilon^2)^{1/2}) - (A_2|\bar{x}| + B_2)\,\frac{|\bar{x} - x_0|}{(|\bar{x} - x_0|^2 + \varepsilon^2)^{1/2}}\right) \leq 0.
\end{aligned} \tag{6.7}$$

But

$$(A_2|\bar{x}| + B_2)\frac{|\bar{x} - x_0|}{(|\bar{x} - x_0|^2 + \varepsilon^2)^{1/2}} \leq A_2|\bar{x} - x_0| + A_2|x_0| + B_2,$$

which implies that (6.7) leads to a contradiction since $L = A_2 + B_2 + |x_0|$. Thus $\bar{t} = 0$ and we obtain that, for every $(x,t) \in \mathbb{R}^n \times [0, T]$,

$$w(x,t) - \eta t - \frac{\eta}{T-t} - \int_0^t \sup_{y\in\bar{B}(x_0,r)}\{f(y,s) - g(y,s)\}\,ds$$

$$- \chi_\varepsilon \left(\mathrm{e}^{Lt}(1 + (|x - x_0|^2 + \varepsilon^2)^{\frac{1}{2}}) - 1\right) \le w(\bar{x}, 0) \le \sup_{y \in \bar{B}(x_0, r)} w(y, 0).$$

Letting η go to 0, we have the desired comparison (6.3) for every (x, t) such that

$$\chi_\varepsilon\left(\mathrm{e}^{Lt}(1 + (|x - x_0|^2 + \varepsilon^2)^{\frac{1}{2}}) - 1\right) = 0 \iff \mathrm{e}^{Lt}(1 + (|x - x_0|^2 + \varepsilon^2)^{\frac{1}{2}}) - 1 \le r - \varepsilon.$$

Letting ε go to 0, we get the comparison (6.3) for every $(x, t) \in \bar{\mathcal{D}}(x_0, r)$. Moreover, if $r \ge 2\mathrm{e}^{LT}$, a straightforward computation (taking $\varepsilon = 1$) shows that (6.3) is fulfilled for every $(x, t) \in \bar{B}(x_0, \mathrm{e}^{-LT} r/2) \times [0, T]$, which completes the proof. $\square$

References

[1] P. Albano and P. Cannarsa, *Singularities of semiconcave functions in Banach spaces,* "Stochastic analysis, control, optimization and applications," pp. 171–190, Systems Control Found. Appl., Birkhäuser Boston, Boston, MA, 1999.

[2] G. Alberti, L. Ambrosio, and P. Cannarsa, *On the singularities of convex functions,* Manuscripta Math., 76 (1992), 421–435.

[3] L. Ambrosio, P. Cannarsa, and H. M. Soner, *On the propagation of singularities of semi-convex functions,* Annali Scuola Norm. Sup. Pisa (IV), 20 (1993), 597–616.

[4] M. Bardi and I. Capuzzo-Dolcetta, "Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations," Birkhäuser, Boston, 1997.

[5] G. Barles, *Uniqueness and regularity results for first-order Hamilton-Jacobi equations,* Indiana Univ. Math. J., 39 (1990), 443–466.

[6] G. Barles, *Discontinuous viscosity solutions of first-order Hamilton-Jacobi equations: a guided visit,* Nonlinear Analysis, TMA, 20 No 9 (1993), 1123–1134.

[7] G. Barles, "Solutions de Viscosité des Équations de Hamilton-Jacobi," Collection "Mathématiques et Applications" of SMAI, No 17. Springer-Verlag, Paris, 1994.

[8] G. Barles, H. M. Soner, and P. E. Souganidis, *Front propagation and phase field theory,* SIAM J. Control and Optimization, 31 (1993), 439–469.

[9] E. N. Barron and R. Jensen, *Semicontinuous viscosity solutions of Hamilton-Jacobi equations with convex Hamiltonians,* Comm. in Partial Differential Equations, 15 (1990), 1713–1742.

[10] P. Cannarsa and H. M. Soner, *On the singularities of the viscosity solutions to Hamilton-Jacobi-Bellman equations,* Indiana Univ. Math. J., 36 (1987), 501–524.

[11] Y.-G. Chen, Y. Giga, and S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations,* J. Differential Geom., 33 (1991), 749–786.

[12] F. H. Clarke, "Optimization and Nonsmooth Analysis," Wiley Interscience, New-York, 1983.

[13] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, and P. R. Wolenski, "Nonsmooth Analysis and Control Theory," GTM, 178, Springer, New-York, 1–42. 1998.

[14] M. G. Crandall and P.-L. Lions, *Viscosity solutions of Hamilton-Jacobi equations,* Trans. Amer. Math. Soc., 277 (1983), 1–42.

[15] M. G. Crandall, H. Ishii, and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations,* Bull. Amer. Soc., 27 (1992), 1–67.

[16] L. C. Evans and R. F. Gariepy, "Measure Theory and Fine Properties of Functions," Studies in Advanced Mathematics, CRC Press, Boca Raton, 1992.

[17] L. C. Evans and J. Spruck, *Motion of level sets by mean curvature,* J. Differential Geom., 33 (1991), 635–681.

[18] W. H. Fleming and H. M. Soner, "Controlled Markov Processes and Viscosity Solutions," Applications of Mathematics, Springer-Verlag, New-York, 1993.

[19] H. Ishii, *Uniqueness of unbounded viscosity solution of Hamilton-Jacobi equations,* Indiana Univ. Math. J., 33 (1984), 721–748.

[20] O. Kavian, "Introduction à la Théorie des Points Critiques," Collection "Mathématiques et Applications" of SMAI, No 13. Springer-Verlag, Paris, 1993.

[21] J.-M. Lasry and P.-L. Lions, *A remark on regularization in Hilbert spaces,* Israel J. Math., 55 (1986), 257–266.

[22] P.-L. Lions, "Generalized Solutions of Hamilton-Jacobi Equations," Research Notes in Mathematics, 69, Pitman, London, 1982.

[23] F. Morgan, "Geometric Measure Theory–A Beginner's Guide," Academic Press, Boston, 1988.

5.2 Complément sur la semiconcavité

Proof of Theorem 5.1. This proof is inspired by the similar one in Ishii [58].

For $x, y, z \in \mathbb{R}^n$ and $t \in [0, T]$, let

$$\phi(x, y, z, t) = u(x, t) - 2u(y, t) + u(z, t) - \chi(x, y, z, t),$$

where

$$\chi(x, y, z, t) = \frac{1}{\mu}|x - 2y + z|^{1+\beta} + \frac{\tilde{C}\mathrm{e}^{K_1 t}}{1+\alpha}\left(|x - y|^{1+\alpha} + |y - z|^{1+\alpha}\right) + \mathrm{e}^{K_2 t}\varepsilon\langle y\rangle$$

for every $\mu, \beta, \tilde{C}, \alpha, K_1, K_2, \varepsilon > 0$ and $\langle y\rangle = (1 + |y|^2)^{1/2}$.

We consider

$$\max_{\mathbb{R}^{3n}\times[0,T]} \phi, \tag{5.1}$$

and claim that this maximum is achieved. Indeed, since u is Lipschitz continuous in x with constant ℓ, we have

$$\phi(x, y, z, t) \leq \ell\,(|x - y| + |y - z|) - \frac{\tilde{C}\mathrm{e}^{K_1 t}}{1+\alpha}\left(|x - y|^{1+\alpha} + |y - z|^{1+\alpha}\right) - \mathrm{e}^{K_2 t}\varepsilon\langle y\rangle.$$

We see that ϕ tends to $-\infty$ if $|x - y|, |y - z|$ or $|y|$ tends to $+\infty$. It follows that ϕ tends to $-\infty$ if $|x|$, $|y|$ or $|z|$ tends to $+\infty$ and finally ϕ achieves its global maximum in at a point $(x_0, y_0, z_0, t_0) \in \mathbb{R}^{3n} \times [0, T]$.

We claim that we can choose the constants which appear in ϕ in order to ensure that the maximum is attained for $t_0 = 0$. We argue by contradiction, assuming on the contrary that $t_0 > 0$ for every choice of parameters.

We compute, for the smooth function χ,

$$\begin{aligned}
\frac{\partial\chi}{\partial t} &= K_1\frac{\tilde{C}\mathrm{e}^{K_1 t_0}}{1+\alpha}\left(|x_0 - y_0|^{1+\alpha} + |y_0 - z_0|^{1+\alpha}\right) + K_2\,\mathrm{e}^{K_2 t_0}\varepsilon\langle y_0\rangle,\\
D_x\chi &= \underbrace{\frac{1+\beta}{\mu}\frac{(x_0 - 2y_0 + z_0)}{|x_0 - 2y_0 + z_0|^{1-\beta}}}_{p_0} + \underbrace{\tilde{C}\mathrm{e}^{K_1 t_0}\frac{x_0 - y_0}{|x_0 - y_0|^{1-\alpha}}}_{\mathcal{X}},\\
D_y\chi &= -2p_0 - \mathcal{X} - \underbrace{\tilde{C}\mathrm{e}^{K_1 t_0}\frac{z_0 - y_0}{|z_0 - y_0|^{1-\alpha}}}_{\mathcal{Z}} + \varepsilon\,\mathrm{e}^{K_2 t_0}\frac{y_0}{\langle y_0\rangle},\\
D_z\chi &= p_0 + \mathcal{Z}
\end{aligned}$$

Using that u is a viscosity solution of (1.1) with $t_0 > 0$, we obtain three differential inequalities, one for each spatial variable,

$$\begin{aligned}
&\frac{\partial\chi}{\partial t}(x_0, y_0, z_0, t_0) + H\left(x_0, t_0, p_0 + \mathcal{X}\right) \leq 0,\\
&-\frac{1}{2}\frac{\partial\chi}{\partial t}(x_0, y_0, z_0, t_0) + H\left(y_0, t_0, p_0 + \frac{1}{2}\left(\mathcal{X} + \mathcal{Z}\right) - \varepsilon\,\mathrm{e}^{K_2 t_0}\frac{\varepsilon y_0}{2\langle y_0\rangle}\right) \geq 0,\\
&\frac{\partial\chi}{\partial t}(x_0, y_0, z_0, t_0) + H\left(z_0, t_0, p_0 + \mathcal{Z}\right) \leq 0.
\end{aligned}$$

Summing the previous inequalities, it follows

$$
\begin{aligned}
&3K_1\,\frac{\tilde{C}e^{K_1 t_0}}{1+\alpha}\left(|x_0-y_0|^{1+\alpha}+|y_0-z_0|^{1+\alpha}\right)+3K_2\,e^{K_2 t_0}\varepsilon\langle y_0\rangle \qquad (5.2)\\
\leq\;& -H\left(x_0,t_0,p_0+\mathcal{X}\right)+2H\left(y_0,t_0,p_0+\frac{1}{2}\left(\mathcal{X}+\mathcal{Z}\right)-\varepsilon\,e^{K_2 t_0}\frac{\varepsilon y_0}{2\langle y_0\rangle}\right)\\
& -H\left(z_0,t_0,p_0+\mathcal{Z}\right)\\
\leq\;& -H\left(x_0,t_0,p_0+\mathcal{X}\right)+2H\left(y_0,t_0,p_0+\frac{1}{2}\left(\mathcal{X}+\mathcal{Z}\right)\right)-H\left(z_0,t_0,p_0+\mathcal{Z}\right)\\
& +\left(A_2|y_0|+B_2\right)\varepsilon\,e^{K_2 t_0},
\end{aligned}
$$

from **(H4)** and since $|y_0|\leq\langle y_0\rangle$.

The next step is to send μ to 0. In this aim, we need to control x_0, y_0 and z_0. Furthermore we will need a bound for $|p_0|$. These results are given in the following lemma whose proof is postponed.

Lemma 5.2.1 *The following properties hold true.*

(i) There exists a compact subset $\mathcal{K}(\varepsilon)$ of $\mathbb{R}^n$ which depends on ε (but is independent of μ) and contains x_0, y_0 and z_0 for all $\mu>0$.

(ii) $\dfrac{1}{\mu}\left|x_0-2y_0+z_0\right|^{\beta}\leq\dfrac{\varepsilon\,e^{K_2 T}}{2}+\ell.$

On the one side, we apply **(H6-$\mathcal{K}(\varepsilon)$-l)** and obtain some constants $\ell'=\ell'(\ell)>\ell$, $\theta_\ell\in(0,1]$ and $a(\varepsilon), b(\varepsilon)>0$. Since ℓ' does not depend on ε, we can choose ε_0 and β_0 small enough such that $(1+\beta_0)(\varepsilon_0\,e^{K_2T}/2+\ell)<\ell'$ (it is possible since $\ell<\ell'$) ; and we fix ε and β such that $0<\varepsilon<\varepsilon_0$ and $0<\beta<\beta_0$. We have

$$
|p_0|=\frac{1+\beta}{\mu}|x-2y+z|^{\beta}\leq(1+\beta)\left(\frac{\varepsilon\,e^{K_2T}}{2}+\ell\right)<\ell'. \qquad (5.3)
$$

On the other side, Lemma 5.2.1 (ii) implies that

$$
\frac{1}{\mu}\left|x_0-2y_0+z_0\right|^{1+\beta}\underset{\mu\to 0}{\to}0. \qquad (5.4)
$$

Now, since x_0, y_0 and z_0 are in the compact set $\mathcal{K}(\varepsilon)$, up to extracting some subsequences, we can suppose that

$$
x_0\underset{\mu\to0}{\to}\bar{x_0}\,,\quad y_0\underset{\mu\to0}{\to}\bar{y_0}\,,\quad z_0\underset{\mu\to0}{\to}\bar{z_0}\,,\quad t_0\underset{\mu\to0}{\to}\bar{t_0}\in[0,T]\ \text{ and }\ p_0\underset{\mu\to0}{\to}\bar{p_0}\in B(0,\ell').
$$

Note that ε_0 and β_0 have been chosen such that we get $|\bar{p_0}|<\ell'$. This property will be important to be able to use **(H6-$\mathcal{C}$-l)** below. From (5.4), we have in particular that $|x_0-2y_0+z_0|\to 0$ as $\mu\to 0$; thus we can define

$$
\bar{h_0}=\bar{x_0}-\bar{y_0}=\bar{y_0}-\bar{z_0},\quad \bar{\mathcal{X}}=\tilde{C}e^{K_1\bar{t_0}}\frac{\bar{h_0}}{|\bar{h_0}|^{1-\alpha}}\ \text{ and }\ \bar{\mathcal{Z}}=\tilde{C}e^{K_1\bar{t_0}}\frac{-\bar{h_0}}{|\bar{h_0}|^{1-\alpha}}.
$$

At this step, we write the inequality (5.2) in its new form and apply **(H6-$\mathcal{C}$-l)**

$$\begin{aligned}
&6K_1\,\mathrm{e}^{K_1\bar{t_0}}\frac{\tilde{C}}{1+\alpha}|\bar{h_0}|^{1+\alpha}+3K_2\,\mathrm{e}^{K_2\bar{t_0}}\varepsilon\langle\bar{y_0}\rangle \qquad (5.5)\\
\leq\;& -H\left(\bar{y_0}+\bar{h_0},\bar{t_0},\bar{p_0}+\bar{\mathcal{X}}\right)+2H\left(\bar{y_0},\bar{t_0},\bar{p_0}\right)-H\left(\bar{y_0}-\bar{h_0},\bar{t_0},\bar{p_0}+\bar{\mathcal{Z}}\right)\\
&+A_2|\bar{y_0}|\varepsilon\,\mathrm{e}^{K_2\bar{t_0}}+B_2\varepsilon\,\mathrm{e}^{K_2\bar{t_0}}\\
\leq\;& a(\varepsilon)|\bar{h_0}|^{1+\theta_\ell}+b(\varepsilon)|\bar{h_0}||\bar{\mathcal{X}}|+A_2|\bar{y_0}|\varepsilon\,\mathrm{e}^{K_2\bar{t_0}}+B_2\varepsilon\,\mathrm{e}^{K_2\bar{t_0}}.
\end{aligned}$$

We need the lemma

Lemma 5.2.2 *Taking* $\tilde{C}\geq 2(\ell+\mathrm{e}^{K_2T}/2)\geq(1+\alpha)(\ell+\varepsilon\mathrm{e}^{K_2T}/2)$, *we have* $|\bar{h_0}|\leq 1$.

We take $\tilde{C}\geq 2(\ell+\mathrm{e}^{K_2T}/2)$ as in the lemma and $\alpha\leq\theta_\ell$. Then, from Lemma 5.2.2, we obtain that $|\bar{h_0}|^{1+\theta_\ell}\leq|\bar{h_0}|^{1+\alpha}$. Finally, from (5.5) we have

$$\left(6K_1\,\mathrm{e}^{K_1\bar{t_0}}\frac{\tilde{C}}{1+\alpha}-b(\varepsilon)\,\tilde{C}\,\mathrm{e}^{K_1\bar{t_0}}-a(\varepsilon)\right)|\bar{h_0}|^{1+\theta_\ell}+\left(3K_2\langle\bar{y_0}\rangle-A_2|\bar{y_0}|-B_2\right)\varepsilon\,\mathrm{e}^{K_2\bar{t_0}}\leq 0.$$

Now we can fix K_1 and K_2 big enough such that

$$\left(6K_1\,\mathrm{e}^{K_1\bar{t_0}}\frac{\tilde{C}}{1+\alpha}-b(\varepsilon)\,\tilde{C}\,\mathrm{e}^{K_1\bar{t_0}}-a(\varepsilon)\right)>0\ \text{ and }\ \left(3K_2\langle\bar{y_0}\rangle-A_2|\bar{y_0}|-B_2\right)>0.$$

Taking $K_1>2(a(\varepsilon)+b(\varepsilon))/6\geq(1+\alpha)(a(\varepsilon)/\tilde{C}+b(\varepsilon))/6$ (without any loss of generality we can suppose $\tilde{C}>1$) and $K_2>(A_2+B_2)/3$, it leads to a contradiction. Thus, the constants in χ can be taken to force t_0 to be 0.

It implies that for all y,h in $\mathbb{R}^n$ and t in $[0,T]$, we have

$$\begin{aligned}
&u(y+h,t)-2u(y,t)+u(y+h,t)-\mathrm{e}^{K_1t}\frac{\tilde{C}}{1+\alpha}|h|^{1+\alpha}-\varepsilon\,\mathrm{e}^{K_2t}\langle y\rangle \qquad (5.6)\\
\leq\;& \sup_{\substack{y,h\,\in\,\mathbb{R}^n\\ t\,\in\,[0,T]}}\phi(y+h,y,y-h,t)=\phi(x_0,y_0,z_0,0).
\end{aligned}$$

Letting μ going to 0 in (5.6) as above, we obtain

$$\begin{aligned}
&u(y+h,t)-2u(y,t)+u(y+h,t)-\mathrm{e}^{K_1t}\frac{\tilde{C}}{1+\alpha}|h|^{1+\alpha}-\varepsilon\,\mathrm{e}^{K_2t}\langle y\rangle \qquad (5.7)\\
\leq\;& u_0(\bar{y_0}+\bar{h_0})-2u_0(\bar{y_0})+u_0(\bar{y_0}-\bar{h_0})-\frac{\tilde{C}}{1+\alpha}|\bar{h_0}|^{1+\alpha}.
\end{aligned}$$

Now, recall that u_0 is uniformly semiconcave and call $\theta_0\in(0,1]$ and $\Lambda_0>0$ the constants which appear in Definition 5.1. Formula (5.7) gives

$$\begin{aligned}
&u(y+h,t)-2u(y,t)+u(y+h,t) \qquad (5.8)\\
&\leq\mathrm{e}^{K_1t}\frac{\tilde{C}}{1+\alpha}|h|^{1+\alpha}+\Lambda_0|\bar{h}_0|^{1+\alpha_0}-\frac{\tilde{C}}{1+\alpha}|\bar{h}_0|^{1+\alpha}+\varepsilon\,\mathrm{e}^{K_2t}\langle y\rangle.
\end{aligned}$$

Taking at the beginning $\alpha \leq \min\{\theta_0, \theta_\ell\}$ and $\tilde{C} > \max\{2(\ell + \mathrm{e}^{K_2 T}/2), 2\Lambda_0\}$ (note that α and $\tilde{C}$ do not depend on ε), we define $\hat{C} = (\tilde{C}\mathrm{e}^{K_1 T})/(1+\alpha)$ and get from (5.8)

$$u(y+h,t) - 2u(y,t) + u(y+h,t) \leq \hat{C}\,|h|^{1+\alpha} + \varepsilon\,\mathrm{e}^{K_2 t}\langle y\rangle.$$

Finally, we can let ε go to 0 and conclude that $u(\cdot,t)$ is (uniformly) semiconcave. □

Proof of Theorem 5.2. The proof of the "local version" is now straightforward using Theorem 5.1.

Let $x_0 \in \mathbb{R}^n$ and $r > 0$. The function u_0 is semiconcave thus locally Lipschitz continuous in $\mathbb{R}^n$ (see Bardi and Capuzzo Dolcetta [6]). It follows that u_0 is uniformly semiconcave (with constants $\Lambda_0 > 0$ and $\theta_0 \in (0,1]$) and Lipschitz continuous (with constant $\ell_0 > 0$) in the compact subset $\bar{B}(x_0, r)$. Note that the constants Λ_0, θ_0 and ℓ_0 depend on x_0 and r. At this step a standard regularization procedure implies the existence of $\tilde{u}_0 \in C(\mathbb{R}^n)$ which is Lipschitz continuous in $\mathbb{R}^n$ with constant ℓ_0 and uniformly semiconcave in $\mathbb{R}^n$ with constants Λ_0 and θ_0, and such that

$$\tilde{u}_0 \equiv u_0 \quad \text{in } \bar{B}(x_0, r). \tag{5.9}$$

Consider the solution $\tilde{u}$ of (1.1) with initial datum $\tilde{u}_0$. From Theorem 4.1, we have that $\tilde{u}$ is Lipschitz continuous w.r.t. x in $\mathbb{R}^n \times [0,T]$. It follows that we can apply Theorem 5.1 to get that $\tilde{u}$ is uniformly semiconcave in $\mathbb{R}^n \times [0,T]$.

Now, if we take at the beginning $r > 2\mathrm{e}^{LT}$ where $L = A_2 + B_2 + A_2|x_0|$, then Theorem A.1 together with (5.9) implies that

$$u \equiv \tilde{u} \quad \text{in } \bar{B}(x_0, \mathrm{e}^{-LT} r/2) \times [0,T].$$

Thus u is semiconcave in $\bar{B}(x_0, \mathrm{e}^{-LT} r/2) \times [0,T]$. Since the choice of x_0 and r was arbitrary, we have the desired result. □

We turn to the proof of the lemmas and end by giving the details of the computations in the examples.

Proof of Lemma 5.2.1. We start by proving (i). Since (x_0, y_0, z_0, t_0) is a maximum for ϕ, we have

$$\begin{aligned} -\varepsilon\,\mathrm{e}^{K_2 t} &= \phi(0,0,0,t_0) \leq \phi(x_0,y_0,z_0,t_0) \\ &\leq \ell\,(|x_0 - y_0| + |y_0 - z_0|) - \varepsilon\,\mathrm{e}^{K_2 t_0}\langle y_0\rangle - \frac{\tilde{C}\mathrm{e}^{K_1 t_0}}{1+\alpha}\left(|x_0 - y_0|^{1+\alpha} + |y_0 - z_0|^{1+\alpha}\right). \end{aligned}$$

It follows

$$\begin{aligned} \varepsilon\,\mathrm{e}^{K_2 t_0}\langle y_0\rangle + \frac{\tilde{C}\mathrm{e}^{K_1 t_0}}{1+\alpha}\left(|x_0 - y_0|^{1+\alpha} + |y_0 - z_0|^{1+\alpha}\right) \\ \leq \ell\,(|x_0 - y_0| + |y_0 - z_0|) + \varepsilon\,\mathrm{e}^{K_2 t}, \end{aligned} \tag{5.10}$$

which implies that there exists a constant R (which depends on $\tilde{C}, \alpha$ and ℓ) such that the quantities $\varepsilon\,\langle y_0\rangle$, $|x_0 - y_0|$ and $|y_0 - z_0|$ are bounded by R. Thus, there exists a compact subset $\mathcal{K}(\varepsilon)$ which contains (x_0, y_0, z_0).

We turn to the proof of (ii). Writing $\phi\left(x_0, \dfrac{x_0+z_0}{2}, z_0, t_0\right) \le \phi(x_0, y_0, z_0, t_0)$, we obtain

$$\frac{1}{\mu}\left|x_0 - 2y_0 + z_0\right|^{1+\beta} \le \underbrace{2\left(u\left(\frac{x_0+z_0}{2}, t_0\right) - u(y_0, t_0)\right)}_{\mathcal{A}} + \underbrace{\varepsilon\, \mathrm{e}^{K_2 t_0}\left(\left\langle \frac{x_0+z_0}{2}\right\rangle - \langle y_0\rangle\right)}_{\mathcal{B}}$$
$$+ \underbrace{\frac{\tilde{C}\mathrm{e}^{K_1 t_0}}{1+\alpha}\left(\frac{1}{2^\alpha}|x_0 - z_0|^{1+\alpha} - |x_0 - y_0|^{1+\alpha} - |y_0 - z_0|^{1+\alpha}\right)}_{\mathcal{C}}.$$

We will deal successively with $\mathcal{A}, \mathcal{C}$ and $\mathcal{B}$.

The estimate of $\mathcal{A}$ is easy since u is Lipschitz continuous. We have

$$\mathcal{A} \le 2\ell\left|\frac{x_0+z_0}{2} - y_0\right| = \ell\,|x_0 - 2y_0 + z_0|.$$

To show that $\mathcal{C} \le 0$, it suffices to see that $|x_0 - z_0|^{1+\alpha}/2^\alpha \le |x_0 - y_0|^{1+\alpha} + |y_0 - z_0|^{1+\alpha}$ which is easy achieved by a convex inequality.

To estimate $\mathcal{B}$, we want to prove that

$$\varepsilon\left(\left\langle \frac{x_0+z_0}{2}\right\rangle - \langle y_0\rangle\right) \le \frac{\varepsilon}{2}|x_0 - 2y_0 + z_0|$$

By easy changes of variables, remembering that $\langle y\rangle = (1+|y|^2)^{1/2}$, it is sufficient to show that

$$\left(1+|p+q|^2\right)^{1/2} - \left(1+|p|^2\right)^{1/2} \le |q|,$$

for p, q in $\mathbb{R}^n$. It is done by studying the variations of the function $f(x) = (1+|p+x|^2)^{1/2}$.

Finally, combining the different estimates we have

$$\frac{1}{\mu}\left|x_0 - 2y_0 + z_0\right|^{1+\beta} \le \ell|x_0 - 2y_0 + z_0| + \frac{\varepsilon\,\mathrm{e}^{K_2 t_0}}{2}|x_0 - 2y_0 + z_0|$$

which implies the desired result. □

Proof of Lemma 5.2.2. Letting μ decreasing to 0 in the inequality (5.10) (see the proof of Lemma 5.2.1), we obtain

$$\frac{2\tilde{C}\mathrm{e}^{K_1 \bar{t_0}}}{1+\alpha}|\bar{h_0}|^{1+\alpha} \le 2\ell|\bar{h_0}| + \varepsilon\,\mathrm{e}^{K_2 \bar{t_0}},$$

which implies

$$|\bar{h_0}|^{1+\alpha} \le \frac{(1+\alpha)\ell}{\tilde{C}}|\bar{h_0}| + \frac{\varepsilon\,(1+\alpha)\mathrm{e}^{K_2 \bar{t_0}}}{2\tilde{C}}.$$

To force $|\bar{h_0}|$ to be smaller than 1, it suffices to have

$$\frac{(1+\alpha)\ell}{\tilde{C}} + \frac{\varepsilon\,(1+\alpha)\mathrm{e}^{K_2 \bar{t_0}}}{2\tilde{C}} \le 1.$$

Finally, we can take $\tilde{C} \geq (1+\alpha)(\ell + \varepsilon\, \mathrm{e}^{K_2 T}/2)$. Note that $(1+\alpha)(\ell + \varepsilon\, \mathrm{e}^{K_2 T}/2) \leq 2(\ell + \mathrm{e}^{K_2 T}/2)$ since $\alpha \leq 1$ and since we can suppose that $\varepsilon \leq 1$. It ends the proof. □

About Remark 5.1. If $H \in C^2(\mathbb{R}^n \times [0,T] \times \mathbb{R}^n)$, then we can write its Taylor expansion at any point (x,t,p). It follows

$$\begin{aligned} &H(x+h,t,p+k) - 2H(x,t,p) + H(x-h,p-k) \\ &\quad = \; 2\left(H_{xx}\cdot h^2 + 2H_{xp}\cdot h\cdot k + H_{pp}\cdot k^2\right) + o(h^2,k^2) \\ &\quad \geq \; -2|H_{xx}||h|^{1+\theta} - 4|H_{xp}||h||k| \end{aligned}$$

for every $\theta \in (0,1]$, when $|h| \leq 1$ and H is convex in p. We see that **(H6-$\mathcal{C}$-l)** is satisfied for every convex compact subset $\mathcal{C}$ and every $l > 0$ since $|H_{xx}|$ and $|H_{xx}|$ are bounded on bounded sets.

About Example 5.1. We consider $H(x,t,p) = a(x,t)|p|$, where a is locally Lipschitz continuous in x (uniformly w.r.t. t) and uniformly semiconcave. Let $\mathcal{C}$ be a convex compact subset and let $l > 0$. We denote by $C_{a,\mathcal{C}}$, Λ_a and θ_a respectively, the Lipschitz and semiconcave constants of a. Using the convexity of the Euclidean norm $|\cdot|$, we have

$$\begin{aligned} &a(x+h,t)|p+k| - 2a(x,t)|p| + a(x-h,t)|p-k| \\ &\quad \geq \; a(x+h,t)|p+k| - a(x,t)\left(|p+k| + |p-k|\right) + a(x-h,t)|p-k| \\ &\quad \geq \; -\Lambda_a|p+k||h|^{1+\theta_a} + \left(|p+k|+|p-k|\right)\left(a(x,t) - a(x-h,t)\right) \\ &\quad \geq \; -\Lambda_a(2+l)|h|^{1+\theta_a} - 2C_{a,\mathcal{C}}|h||k| \end{aligned}$$

for $|k| \leq 1+l$ and $|h| \leq 1$. Thus **(H6-$\mathcal{C}$-l)** is satisfied for every convex compact subset $\mathcal{C}$ and every $l > 0$.

About Example 5.2. We consider $H(x,t,p) = \sup_{v\in U}\{\langle b(x,t,v),p\rangle + f(x,t,v)\}$. Let $l > 0$ and $\mathcal{C}$ be any convex, compact subset of $\mathbb{R}^n$. Consider $x,h \in \mathbb{R}^n$ such that $|h| \leq 1$ and $x, x-h, x+h \in \mathcal{C}$. Finally, let $l < \ell' \leq 2l$ and $p,k \in \bar{B}(0,\ell')$. We compute

$$\begin{aligned} &H(x+h,t,p+k) - 2H(x,t,p) + H(x-h,t,p-k) \\ &\quad = \sup_v\{\langle b(x+h,t,v),p+k\rangle + f(x+h,t,v)\} - 2\sup_v\{\langle b(x,t,v),p\rangle + f(x,t,v)\} \\ &\qquad\qquad + \sup_v\{\langle b(x-h,t,v),p-k\rangle + f(x-h,t,v)\} \\ &\quad \geq \inf_v\{\langle b(x+h,t,v) - 2b(x,t,v) + b(x-h,t,v),p\rangle \\ &\quad +\langle b(x+h,t,v) - b(x-h,t,v),k\rangle + f(x+h,t,v) - 2f(x,t,v) + f(x-h,t,v)\} \\ &\quad \geq \inf_v\left\{-C|h|^{1+\theta}|p| - \mathrm{Lip}(b,\mathcal{C})|h||k| - C_f|h|^{1+\theta_f}\right\}, \end{aligned}$$

using that b is $C^{1,\theta}$ in x (uniformly with respect to (t,v)) and f is uniformly semiconvex. It follows

$$H(x+h,t,p+k) - 2H(x,t,p) + H(x-h,t,p-k) \geq -A|h|^{1+\theta'} - B|h||k|,$$

where

$$\theta' = \min\{\theta,\theta_f\}, \quad A = 2Cl + C_f \quad \text{and} \quad B = \mathrm{Lip}(b,\mathcal{C}).$$

Chapitre 6

Discontinuous viscosity solutions of Hamilton-Jacobi equations with a Neumann boundary condition : a counter-example to uniqueness

Olivier Ley
Laboratoire de Mathématiques et Physique Théorique
Université de Tours,
Parc de Grandmont, 37200 Tours, France

Abstract

We consider a first-order time-dependent Hamilton-Jacobi equation set in a bounded open subset Ω of $\mathbb{R}^N$ with Neumann boundary conditions. When the initial data u_0 is continuous, Lions [73] proved that there exists a unique viscosity solution of the problem. An application is the characterization of the value function of an optimal control problem of reflected trajectories with a final cost equal to u_0. Here we address the question of a discontinuous final cost. We prove that the value function $u[(u_0)^*]$ associated to the final cost $(u_0)^*$ is the maximal subsolution of the Neumann problem and the relaxed value function $\hat{u}[(u_0)_*]$ associated to $(u_0)_*$ is the minimal supersolution. The main result of this paper is to show by a counter-example that, contrary to the control problem set in the whole space, their lower-semicontinuous envelopes do not coincide in general, making impossible to characterize the value function of the problem by this way.

[0]This work was partially supported by the TMR program "Viscosity Solutions and Their Applications."

Sommaire

6.1 Introduction

In this work, we are interested in the solutions of the first-order time-dependent Hamilton-Jacobi equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + H(x,t,Du) = 0 \quad \text{in } \Omega \times (0,T), \\ u(\cdot,0) = u_0 \quad \text{in } \overline{\Omega}, \end{cases} \tag{6.1}$$

with the Neuman boundary condition

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{6.2}$$

The set Ω is a bounded smooth domain of $\mathbb{R}^N$ and n is the unit outward normal. The initial data u_0 is a given function from $\overline{\Omega}$ into $\mathbb{R}$, $u : \overline{\Omega} \times [0,T] \to \mathbb{R}$ is the solution, and $H \in C(\overline{\Omega} \times [0,T] \times \mathbb{R}^N)$.

It is well-known that such an equation with such a boundary condition cannot have a classical solution in general, even when the initial data and H are regular enough. We have to consider weak solutions. In this work, we use viscosity solutions. We refer the classical literature on this subject [38], [72], [13], [6], [48] and [37].

Lions [73] (see also Ishii [61]) solves completely (6.1)–(6.2) when the initial data u_0 is continuous. He proves the existence and the uniqueness of a continuous viscosity solution. We mention that he works with less regular domains and more general Neumann boundary conditions, like oblique derivatives boundary conditions for example. The regularity of the boundary $\partial\Omega$ is not an issue here, we do not search the best generality and suppose that $\partial\Omega$ is $W^{2,\infty}$ in order to be able to define the unit outward normal n.

In [59], Ishii extended the notion of viscosity solutions to discontinuous functions (see the references above for further details). The issue we address here is : can we extend in a natural way the uniqueness results of [73] when u_0 is merely a locally bounded function ?

We first have to explain what we mean by "in a natural way." The problem of uniqueness for discontinuous viscosity solutions has been treated by many authors. Barles and Perthame [20] studied the stationary case set in the whole space $\mathbb{R}^N$. Their work was motivated by characterizing the value function of a deterministic optimal stopping time problem with a discontinuous stopping cost. They prove that the lower-semicontinuous (LSC in short) envelope of the value function is the unique LSC viscosity solution of (6.1) in $\mathbb{R}^N$. Therefore, "uniqueness for (6.1)" has to be understood by : all the discontinuous viscosity solutions of (6.1) have the same lsc envelope.

Then, Barron and Jensen [24], [25] dealt with the initial time problem in $\mathbb{R}^N \times (0,T)$. They introduce the so-called Barron-Jensen concept of viscosity solution for (6.1) with convex Hamiltonians in p (i.e. for equations which correspond to a control problem). This new definition requires only one of the two conditions needed in the classical one (but in a slightly stronger sense) and both definitions coincide when the solutions are continuous. By this way, under a suitable taking into account of the initial condition, they prove uniqueness for LSC viscosity solutions of (6.1) and obtained the characterization of the value function of the related control problem with upper-semicontinuous (USC) terminal data.

Other works are related to this problem of discontinuous viscosity solutions for (6.1). We refer to Barles [13], Soravia [82], Frankowska [50] and Blanc [28] for the Dirichlet problem with convex Hamiltonians. All these works are based on Barron-Jensen solutions

Here, we address the same kind of issue for (6.1)–(6.2). We show by a counter-example that a uniqueness result as the one of Barles and Perthame [20] does not hold. It means that the viscosity "relaxed" Neumann boundary conditions are not sufficient to ensure satisfactory results and one has to impose a new – and acceptable – definition of boundary conditions for this problem. To our knowledge, this problem is still open.

Our counter-example relies on the associated control problem which takes the form, in this case, of an optimal control problem with reflection at the boundary. When u_0 is continuous, the value function is the unique continuous viscosity solution of (6.1)–(6.2), where H is the Hamiltonian which comes from the control problem (see [73]). When, u_0 is discontinuous, following [20], we prove that the relaxed value function associated to the terminal cost $(u_0)_*$ is the minimal discontinuous supersolution of (6.1)–(6.2) and the value function associated to $(u_0)^*$ is the maximal discontinuous subsolution. We then show that, contrary to [20], the LSC envelopes of the extremal solutions do not coincide in general.

The paper is organized as follows. In Section 6.2, we state a comparison result for the discontinuous solutions of (6.1)–(6.2) whose an immediate consequence is the results of [73], namely the well-posedness of (6.1)–(6.2) in the class of the continuous viscosity solutions. In Section 6.3, we introduce the reflected control problem. Section 6.4 is devoted to the counter-example which is the main result of this paper.

Acknowledgment. I am grateful to Guy Barles, who brought this problem to my attention. I would like to thank Élisabeth Rouy for helpful discussions and valuable suggestions.

6.2 Hamilton-Jacobi equation with the Neumann boundary condition

For any locally bounded function $u : \overline{\Omega} \times [0,T] \to \mathbb{R}$, we define the USC and LSC semicontinuous envelopes respectively, by

$$u^*(x,t) = \limsup_{(y,s)\to(x,t)} u(y,s) \quad \text{and} \quad u_*(x,t) = \liminf_{(y,s)\to(x,t)} u(y,s).$$

We then recall the definition of viscosity solutions of (6.1)–(6.2).

Definition 6.2.1 *(i) A locally bounded function $u : \overline{\Omega} \times [0,T] \to \mathbb{R}$ is a (discontinuous) viscosity subsolution of (6.1)–(6.2) if for every $\varphi \in C^1(\overline{\Omega} \times [0,T])$ and $(x_0,t_0) \in \overline{\Omega} \times [0,T]$ such that $u^* - \varphi$ achieves a local maximum, we have*

$$\begin{cases} \dfrac{\partial \varphi}{\partial t}(x_0,t_0) + H(x_0,t_0,D\varphi(x_0,t_0)) \leq 0 & \text{if } (x_0,t_0) \in \Omega \times (0,T], \\ \min\left\{\dfrac{\partial \varphi}{\partial t}(x_0,t_0) + H,\ \dfrac{\partial \varphi}{\partial n}(x_0,t_0)\right\} \leq 0 & \text{if } (x_0,t_0) \in \partial\Omega \times (0,T], \\ u^*(x_0,0) \leq (u_0)^*(x_0) & \text{if } (x_0,t_0) \in \overline{\Omega} \times \{0\}. \end{cases}$$

(ii) A locally bounded function $u : \overline{\Omega} \times [0,T] \to \mathbb{R}$ is a (discontinuous) viscosity supersolution of (6.1)–(6.2) if the previous definition holds when replacing "u^*" by "u_*," "maximum" by "minimum," "$\leq$" by "$\geq$," "$\min$" by "$\max$" and "$(u_0)^*$" by "$(u_0)_*$."

(iii) u is a (discontinuous) viscosity solution of (6.1)–(6.2) if u is both a sub- and a supersolution.

In this definition, the boundary condition on $\partial\Omega \times (0,T]$ is said to be "relaxed." Normally, one has to consider similar relaxed conditions on $\partial\Omega \times \{0\}$ and $\Omega \times \{0\}$ but, for parabolic equations, the "classical condition" always holds at $t = 0$ (see Barles [13]).

We use the following assumptions for the Hamiltonian H. They are classical to obtain uniqueness and existence results for viscosity solutions. A function $m : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a modulus of continuity if $\lim_{s\to 0^+} m(s) = 0$ and $m(s+t) \leq m(s) + m(t)$ for all $s, t \geq 0$.

(H1) There exists a modulus of continuity m_1 such that, for all $(x,t), (y,s) \in \overline{\Omega} \times [0,T]$, and $\lambda \geq 1$, we have

$$H(x,t,\lambda(x-y)) - H(y,s,\lambda(x-y)) \leq m_1(\lambda|x-y|^2 + |x-y|).$$

(H2) There exists a modulus of continuity m_2 such that, for every $(x,t) \in \overline{\Omega} \times [0,T]$ and $p, q \in \mathbb{R}^N$, we have

$$H(x,t,p) - H(x,t,q) \leq m_2(|p-q|).$$

We state the results.

Theorem 6.2.1 *Assume that $\partial\Omega$ is $W^{2,\infty}$ and **(H1)** and **(H2)** hold. Let u (respectively v) be a subsolution (respectively a supersolution) of (6.1)–(6.2). If $u^*(\cdot,0) \leq v_*(\cdot,0)$ in $\overline{\Omega}$, then $u^* \leq v_*$ in $\overline{\Omega} \times [0,T]$.*

Theorem 6.2.2 *Under the assumptions of Theorem 6.2.1, if $u_0 \in C(\overline{\Omega})$, then, there exists a unique viscosity solution $u \in C(\overline{\Omega} \times [0,T])$ of (6.1)–(6.2).*

Remark 6.2.1 To avoid technicality here, we consider equations (6.1) for which H does not depend on u. Both above results are true with easy adaptations if $H = H(x,t,u,Du)$ and H satisfy **(H1)** and **(H2)** for $|u| \leq R$ and for every $R > 0$, and

(H3) For every $R > 0$, there exists $\gamma_R > 0$ such that, for all $(x,t) \in \overline{\Omega} \times [0,T]$, $p \in \mathbb{R}^N$ and $-R \leq w_2 \leq w_1 \leq R$, we have

$$H(x,t,w_1,p) - H(x,t,w_2,p) \geq \gamma_R(w_1 - w_2).$$

These results appear in Lions [73] (see also Ishii [61]). To be self-contained, we provide a modern proof of the first one. The second one turns out to be an immediate corollary of Theorem 6.2.1 by applying the classical Perron's method. The proof is done in [61], so we skip it (see also Da Lio [40] for Perron's method in the context of generalized boundary conditions).

We suppose that the boundary of Ω is regular enough. In particular, it implies that the signed-distance (which one takes positive in the interior of Ω and negative outside) is $C^{1,1}$ on a neighborhood of $\partial\Omega$. We denote by d_Ω a function which is equal to the signed-distance in a neighborhood of $\partial\Omega$ and which is extended to the whole space $\mathbb{R}^N$ in order to have $d_\Omega \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$. Note that, for every $x \in \partial\Omega$, $Dd_\Omega(x) = -n(x)$, where $n(x)$ is the unit outward normal at the point x. Moreover, the assumed regularity on $\partial\Omega$ gives the following estimate.

Lemma 6.2.1 *If $\partial\Omega$ is $W^{2,\infty}$, then there exists $C > 0$ such that, for all $x \in \partial\Omega$ and $y \in \overline{\Omega}$,*

$$\langle x - y, n(x)\rangle \geq -C|x-y|^2.$$

We turn to the proofs.

Poof of Theorem 6.2.1. We argue by contradiction, assuming

$$\max_{\overline{\Omega}\times[0,T]} \{u^*(x,t) - v_*(x,t)\} = L > 0.$$

For every $\varepsilon, \beta, \alpha, \eta > 0$, we introduce

$$f_{\varepsilon,\beta,\alpha,\eta}(x,t,y,s) = u^*(x,t) - v_*(y,s) - \frac{|x-y|^2}{\varepsilon^2} - \frac{|t-s|^2}{\beta^2} + \alpha d_\Omega(x) + \alpha d_\Omega(y) - \eta t.$$

for $x, y \in \overline{\Omega}$ and $s, t \in [0,T]$. The maximum

$$M_{\varepsilon,\beta,\alpha,\eta} = \max_{\overline{\Omega}\times[0,T]\times\overline{\Omega}\times[0,T]} f_{\varepsilon,\beta,\alpha,\eta}$$

is achieved at a point $(\bar{x}, \bar{t}, \bar{y}, \bar{s})$ and we have the classical following results.

Lemma 6.2.2 *If α and η are sufficiently small, then*

$$M_{\varepsilon,\beta,\alpha,\eta} \underset{\varepsilon,\beta\to 0}{\longrightarrow} \max_{\overline{\Omega}\times[0,T]} \{u^*(x,t) - v_*(x,t) + 2\alpha d_\Omega(x) - \eta t\} \geq \frac{L}{2}.$$

Moreover, if $\varepsilon, \beta > 0$ are small enough, then $\bar{t}, \bar{s} > 0$.

From now on, we take ε, β such that the conclusion of the lemma holds. From the definition of $(\bar{x}, \bar{t}, \bar{y}, \bar{s})$, $(\bar{x}, \bar{t})$ is a maximum of $f_{\varepsilon,\beta,\alpha}(\cdot,\cdot,\bar{y},\bar{s})$ in $\overline{\Omega}\times(0,T]$. We write that u is a viscosity subsolution distinguishing two cases :
— if $\bar{x} \in \Omega, \bar{t} > 0$, then

$$\eta + 2\frac{\hat{t}-\hat{s}}{\beta^2} + H\Big(\bar{x}, \bar{t}, 2\frac{\bar{x}-\bar{y}}{\varepsilon^2} + \alpha n(\bar{x})\Big) \leq 0. \tag{6.3}$$

— if $\bar{x} \in \partial\Omega, \bar{t} > 0$, then

$$\min\left\{\eta + 2\frac{\bar{t}-\bar{s}}{\beta^2} + H\Big(\bar{x}, \bar{t}, 2\frac{\bar{x}-\bar{y}}{\varepsilon^2} + \alpha n(\bar{x})\Big),\ 2\langle\frac{\bar{x}-\bar{y}}{\varepsilon^2}, n(\bar{x})\rangle + \alpha\right\} \leq 0. \tag{6.4}$$

We claim that we can choose ε small enough in order that (6.3) holds in (6.4). From Lemma 6.2.1, we have

$$2\langle\frac{\bar{x}-\bar{y}}{\varepsilon^2}, n(\bar{x})\rangle + \alpha \geq \alpha - 2C\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2}.$$

By Lemma 6.2.2, $|\bar{x}-\bar{y}|^2/\varepsilon^2 \to 0$ as $\varepsilon \to 0$; thus, for ε small enough, we get

$$2\langle \frac{\bar{x}-\bar{y}}{\varepsilon^2}, n(\bar{x})\rangle + \alpha > 0,$$

which proves the claim.

We exploit the fact that v is a supersolution. The point $(\bar{y},\bar{s})$ is a minimum of $-f_{\varepsilon,\beta,\alpha}(\bar{x},\bar{t}\cdot,\cdot)$ in $\overline{\Omega}\times(0,T]$. Using similar arguments as above, in any case, the boundary condition cannot hold if ε is small enough. It follows

$$-2\frac{\bar{s}-\bar{t}}{\beta^2} + H\Big(\bar{y},\bar{s},-2\frac{\bar{y}-\bar{x}}{\varepsilon^2} - \alpha n(\bar{y})\Big) \geq 0. \tag{6.5}$$

Subtracting (6.5) to (6.3), we obtain

$$\eta + H\Big(\bar{x},\bar{t},2\frac{\bar{x}-\bar{y}}{\varepsilon^2} + \alpha n(\bar{x})\Big) - H\Big(\bar{y},\bar{s},-2\frac{\bar{y}-\bar{x}}{\varepsilon^2} - \alpha n(\bar{y})\Big) \leq 0.$$

From **(H1)** and **(H2)**, it follows

$$0 < \eta \leq 2m_2(\alpha) + m_1\Big(\frac{2}{\varepsilon^2}|\bar{x}-\bar{y}|^2 + |\bar{x}-\bar{y}|\Big).$$

Sending ε, β to 0 first and letting then α go to 0, it leads to a contradiction. It completes the proof of the theorem. □

Proof of Lemma 6.2.1. Let $x \in \partial\Omega$ and $y \in \overline{\Omega}$. We define $f(\lambda) = d_\Omega(x+\lambda(y-x)) - \lambda\langle Dd_\Omega(x), y-x\rangle$ for $\lambda \in [0,1]$. Since $d_\Omega \in C^{1,1}_{\mathrm{loc}}$ and Ω is bounded, there exists a constant $C = C(\Omega) > 0$ such that

$$\begin{aligned} |f'(\lambda)| &= |\langle Dd_\Omega(x+\lambda(y-x)), y-x\rangle - \langle Dd_\Omega(x), y-x\rangle| \\ &\leq C|\lambda(y-x)||y-x| \leq C|y-x|^2. \end{aligned}$$

Applying the mean value inequality, we obtain

$$f(1) - f(0) = d_\Omega(y) - \langle Dd_\Omega(x), y-x\rangle - d_\Omega(x) \leq C|y-x|^2$$

which gives the desired result since $Dd_\Omega(x) = -n(x)$ and $d_\Omega(y) \geq 0$. □

6.3 Optimal control with reflection at the boundary

In this section, we study a control problem with reflection at the boundary which leads to (6.1)–(6.2) (see Lions [73], Ishii [61] and Barles [13]).

Let $\mathcal{A}$ be a compact metric space. For $M \in I\!N$ and $g \in C(\overline{\Omega}\times[0,T]\times\mathcal{A},\mathbb{R}^M)$, we introduce the following assumption.

(H4) There exists a constant $C > 0$ such that, for every $x, y \in \overline{\Omega}, t \in [0,T]$ and $\alpha \in \mathcal{A}$,

$$|g(x,t,\alpha)| \leq C \quad \text{and} \quad |g(x,t,\alpha) - g(y,t,\alpha)| \leq C|x-y|.$$

We consider the controlled reflected trajectories

$$\begin{cases} dX_s^{x,t} = b(X_s^{x,t}, t-s, \alpha(s))ds - dk_s^{x,t} \text{ in } [0,t],\ t \le T, \\ X_0^{x,t} = x \in \overline{\Omega},\ X_s^{x,t} \in \overline{\Omega} \text{ for every } s \in [0,t], \\ k_s^{x,t} = \int_0^s \mathbb{I}_{\partial\Omega}(X_\tau^{x,t}) n(X_\tau^{x,t})\, d|k^{x,t}|_\tau, \end{cases} \tag{6.6}$$

where the unknowns are the trajectory $X^{x,t} \in C([0,t], \overline{\Omega})$ and the bounded variation process $k^{x,t} \in C([0,t], \mathbb{R}^N)$; $|k^{x,t}|_s$ stands for the total variation of $k^{x,t}$ on $[0,s]$, the control $\alpha(\cdot) \in L^\infty([0,T], \mathcal{A})$ and the vector field $b \in C(\overline{\Omega} \times [0,T] \times \mathcal{A}, \mathbb{R}^N)$ satisfies **(H4)**.

Under the previous assumptions, we know from Lions and Sznitman [74] that, for every $(x,t) \in \overline{\Omega} \times [0,T]$ and $\alpha(\cdot) \in L^\infty([0,T], \mathcal{A})$, the system (6.6) admits a unique solution $(X^{x,t}, k^{x,t}) \in C([0,t], \mathbb{R}^N) \times BV([0,t], \mathbb{R}^N)$. Note that, without the process $k^{x,t}$, (6.6) is a classical dynamical system for optimal control problems.

To force the trajectory to remain in $\overline{\Omega}$, there are different ways. One way is to restrict the class of the admissible controls : it leads to a state constraint problem. Here, we choose to add the term "$-dk_s^{x,t}$" in the equation which governs the trajectory. Roughly speaking, this term acts like a force pushing back the trajectory when it crosses the boundary of Ω.

In the case of (6.6), we have an explicit formula for the bounded variation process.

Proposition 6.3.1 *For every $(x,t) \in \overline{\Omega} \times [0,T]$, The solution $k^{x,t}$ is given by*

$$dk_s^{x,t} = \mathbb{I}_{\partial\Omega}(X_s^{x,t}) n(X_s^{x,t}) \langle b(X_s^{x,t}, t-s, \alpha(s)), n(X_s^{x,t}) \rangle^+ \, ds. \tag{6.7}$$

Proof of Proposition 6.3.1. Let $(x,t) \in \overline{\Omega} \times [0,T]$ and suppose that $X_s^{x,t} \in \partial\Omega$. Since, the trajectory remains in $\overline{\Omega}$, necessarily $d(d_\Omega(X_s^{x,t})) \ge 0$. But

$$\begin{aligned} d(d_\Omega(X_s^{x,t})) &= \langle Dd_\Omega(X_s^{x,t}), dX_s^{x,t} \rangle \\ &= -\langle n(X_s^{x,t}), b(X_s^{x,t}, t-s, \alpha(s)) \rangle \, ds + \langle n(X_s^{x,t}), dk_s^{x,t} \rangle \end{aligned}$$

We then set (6.7) and check that it satisfies (6.6). We conclude by uniqueness. □

We state some estimates for the trajectories we need in the sequel.

Lemma 6.3.1 *(Estimates for the trajectories) There exist a constant $C > 0$ and a modulus of continuity m which depend only on Ω, T and b such that, for every $x, x' \in \overline{\Omega}$, $t, t', s \in [0,T]$, we have*

(i) $|k^{x,t}|_s \le C$;
(ii) $|X_s^{x,t} - x| \le C\sqrt{s}$;
(iii) $|X_s^{x,t} - X_s^{x',t'}| \le C|x-x'| + m(|t-t'|)$.

We skip the proof since it is close to some arguments of [73].

Remark 6.3.1 We recover classical estimates for systems governed by ordinary differential equations. In particular, we have a continuous dependence of the trajectory

$X^{x,t}$ with respect to the data x, t. Fix any control $\alpha(\cdot)$ and consider some associated trajectories $X^{x,t}$ and X^{x_n,t_n}, $n \in I\!N$. Without the term $dk_s^{x,t}$ in (6.6), we know that, if $X_{t_n}^{x_n,t_n} \to X_t^{x,t}$ as $n \to +\infty$, then $(x_n, t_n) \to (x, t)$. We point out that it is not the case anymore for (6.6). Roughly speaking, we lose the memory of the starting point (in fact we cannot solve this system backward). We refer to Theorem 6.4.2 and Figure 6.1 for an illustration.

Let $f \in C(\overline{\Omega} \times [0,T] \times \mathcal{A})$ satisfying **(H4)**. For any locally bounded $\psi : \mathbb{R}^N \to \mathbb{R}$, for every $(x,t) \in \overline{\Omega} \times [0,T]$ and $\alpha(\cdot) \in L^\infty([0,T], \mathcal{A})$, we define the cost function

$$J(x, t, \psi, \alpha(\cdot)) = \int_0^t f(X_s^{x,t}, t - s, \alpha(s))\, ds + \psi(X_t^{x,t}),$$

and the value function

$$u[\psi](x,t) = \inf_{\alpha(\cdot) \in L^\infty([0,T],\mathcal{A})} J(x, t, \psi, \alpha(\cdot)).$$

We have

Proposition 6.3.2 *(Dynamical programming principle) For every $(x,t) \in \overline{\Omega} \times [0,T]$ and $\varepsilon > 0$,*

$$u[\psi](x,t) = \inf_{\alpha(\cdot) \in L^\infty([0,T],\mathcal{A})} \left\{ \int_0^\varepsilon f(X_s^{x,t}, t - s, \alpha(s))\, ds + u[\psi](X_\varepsilon^{x,t}, t - \varepsilon) \right\}.$$

The proof does not present any difficulty in comparison with the classical one, when handing the existence and uniqueness of a solution of (6.6). So, we skip it and refer to the classical literature for control problems (Lee and Markus [70], Warga [84], Lions [72], Bardi and Capuzzo Dolcetta [6], Barles [13], Fleming and Soner [48], etc.).

The following theorem makes the link between the control problem and the Hamilton-Jacobi equation with the Neumann boundary condition.

Theorem 6.3.1 *For any locally bounded function ψ, the value function $u[\psi]$ is a viscosity solution of (6.1)–(6.2) with*

$$H(x,t,p) = \sup_{\alpha \in \mathcal{A}} \left\{ -\langle p, b(x,t,\alpha) \rangle - f(x,t,\alpha) \right\}, \tag{6.8}$$

for any $(x,t,p) \in \overline{\Omega} \times [0,T] \times \mathbb{R}^N$.

The proof of this theorem, in the case of control problems with reflection, is less classical, so we provide a proof.

Proof of Theorem 6.3.1. For the sake of notational simplicity, we write u instead of $u[\psi]$ in the proof.

We begin by showing that u is a subsolution. Let $\varphi \in C^1(\overline{\Omega} \times [0,T])$ and $(x_0, t_0) \in \overline{\Omega} \times [0,T]$ such that (x_0, t_0) is a local maximum of $u^* - \varphi$. Without loss of generality, we can assume that $u^*(x_0,t_0) = \varphi(x_0,t_0)$ and $u^* \le \varphi$ in $\overline{\Omega} \times [0,T]$ (see Barles [13] for instance).

Suppose first $t_0 = 0$. For every $(x,t) \in \overline{\Omega} \times [0,T]$ and $\alpha \in \mathcal{A}$, we have

$$u(x,t) \le \int_0^t f(X_s^{x,t}, t-s, \alpha)\, ds + \psi(X_t^{x,t}).$$

Thus, using that f is bounded and the continuity of $t \mapsto X_t^{x,t}$ (see Lemma 6.3.1), we get

$$\begin{aligned} u^*(x_0,0) = \limsup_{x\to x_0, t\to 0} u(x,t) &\le \limsup_{x\to x_0, t\to 0} \int_0^t f(X_s^{x,t}, t-s, \alpha)\, ds + \limsup_{x\to x_0, t\to 0} \psi(X_t^{x,t}) \\ &\le \limsup_{x\to x_0} \psi(x) = \psi^*(x_0). \end{aligned}$$

It proves the result for $t_0 = 0$.

Now, suppose that $t_0 > 0$. From Proposition 6.3.2, for every $\varepsilon > 0$, $(x,t) \in \overline{\Omega} \times [0,T]$ and $\alpha \in \mathcal{A}$, we have

$$u(x,t) \le \int_0^t f(X_s^{x,t}, t-s, \alpha)\, ds + u(X_\varepsilon^{x,t}, t-\varepsilon).$$

It follows

$$\begin{aligned} u^*(x_0,t_0) &= \limsup_{x\to x_0, t\to t_0} u(x,t) \\ &\le \limsup_{x\to x_0, t\to t_0} \int_0^\varepsilon f(X_s^{x,t}, t-s, \alpha)\, ds + \limsup_{x\to x_0, t\to t_0} u(X_\varepsilon^{x,t}, t-\varepsilon) \\ &\le \int_0^\varepsilon f(X_s^{x_0,t_0}, t_0-s, \alpha)\, ds + \limsup_{x\to X_\varepsilon^{x_0,t_0}, t\to t_0-\varepsilon} u(x,t) \\ &\le \int_0^\varepsilon f(X_s^{x_0,t_0}, t_0-s, \alpha)\, ds + u^*(X_\varepsilon^{x_0,t_0}, t_0-\varepsilon), \end{aligned} \tag{6.9}$$

by continuity of f and $(x,t,s) \mapsto X_s^{x,t}$ (see Lemma 6.3.1). Since φ is C^1, we get

$$\begin{aligned} &\varphi(X_\varepsilon^{x_0,t_0}, t_0-\varepsilon) \qquad\qquad (6.10) \\ &= \varphi(x_0,t_0) - \int_0^\varepsilon \frac{\partial \varphi}{\partial t}(X_s^{x_0,t_0}, t_0-s)\, ds + \int_0^\varepsilon \langle D\varphi(X_s^{x_0,t_0}, t_0-s), dX_s^{x_0,t_0}\rangle \\ &= u^*(x_0,t_0) - \int_0^\varepsilon \frac{\partial \varphi}{\partial t}(X_s^{x_0,t_0}, t_0-s)\, ds + \int_0^\varepsilon \langle D\varphi(X_s^{x_0,t_0}, t_0-s), b(X_s^{x_0,t_0}, t_0-s,\alpha)\rangle\, ds \\ &\qquad - \int_0^\varepsilon \mathbb{1}_{\partial\Omega}(X_s^{x_0,t_0}) \frac{\partial \varphi}{\partial n}(X_s^{x_0,t_0}, t_0-s)\, d|k^{x_0,t_0}|_s, \end{aligned}$$

using (6.6) and the assumption $\varphi(x_0,t_0) = u^*(x_0,t_0)$. Recalling that $u^*(X_\varepsilon^{x_0,t_0}, t_0-\varepsilon) \leq \varphi(X_\varepsilon^{x_0,t_0}, t_0-\varepsilon)$, from (6.9) and (6.10), we obtain

$$\int_0^\varepsilon h(X_s^{x_0,t_0}, t_0-s,\alpha)\,ds + \int_0^\varepsilon \mathbb{1}_{\partial\Omega}(X_s^{x_0,t_0})\frac{\partial\varphi}{\partial n}(X_s^{x_0,t_0}, t_0-s)\,d|k^{x_0,t_0}|_s \leq 0, \qquad (6.11)$$

where we set

$$h(x,t,\alpha) := \frac{\partial\varphi}{\partial t}(x,t) - \langle b(x,t,\alpha), D\varphi(x,t)\rangle - f(x,t,\alpha). \qquad (6.12)$$

Using the continuity of f, b, the derivatives of φ and $s \mapsto X_s^{x_0,t_0}$, we have

$$\int_0^\varepsilon h(X_s^{x_0,t_0}, t_0-s,\alpha)\,ds = \varepsilon h(x_0,t_0,\alpha) + o(\varepsilon)$$

and

$$\int_0^\varepsilon \mathbb{1}_{\partial\Omega}(X_s^{x_0,t_0})\frac{\partial\varphi}{\partial n}(X_s^{x_0,t_0}, t_0-s)\,d|k^{x_0,t_0}|_s = K_\varepsilon\frac{\partial\varphi}{\partial n}(x_0,t_0) + K_\varepsilon\, m_\varphi(\varepsilon),$$

where

$$K_\varepsilon = \int_0^\varepsilon \mathbb{1}_{\partial\Omega}(X_s^{x_0,t_0})\,d|k^{x_0,t_0}|_s \qquad (6.13)$$

and

$$m_\varphi(\varepsilon) = \sup_{s\in[0,\varepsilon]}\left|\frac{\partial\varphi}{\partial n}(x_0,t_0) - \frac{\partial\varphi}{\partial n}(X_s^{x_0,t_0}, t_0-s)\right|.$$

Note that $m_\varphi(\varepsilon) \to 0$ as $\varepsilon \to 0$ because φ is C^1 and $s \to X_s^{x_0,t_0}$ is continuous from Lemma 6.3.1.

If $x_0 \in \Omega$, then K_ε is identically 0 for ε small enough (since, in this case, the trajectory lies in the interior of Ω on $[0,\varepsilon]$). Dividing (6.11) by ε and passing to the limit $\varepsilon \to 0$, we obtain $h(x_0,t_0,\alpha) \leq 0$. Since α was chosen arbitrary in $\mathcal{A}$, this implies

$$\sup_{\alpha\in\mathcal{A}} h(x_0,t_0,\alpha) = \frac{\partial\varphi}{\partial t}(x_0,t_0) + H(x_0,t_0,D\varphi(x_0,t_0)) \leq 0,$$

which gives the conclusion for $(x_0,t_0) \in \Omega\times(0,T]$.

It remains to deal with the case $(x_0,t_0) \in \partial\Omega\times(0,T]$. Dividing (6.11) by $\varepsilon + K_\varepsilon$, we have

$$\frac{\varepsilon h(x_0,t_0,\alpha) + K_\varepsilon\frac{\partial\varphi}{\partial n}(x_0,t_0)}{\varepsilon+K_\varepsilon} + \frac{o(\varepsilon) + K_\varepsilon m_\varphi(\varepsilon)}{\varepsilon+K_\varepsilon} \leq 0. \qquad (6.14)$$

At this step, we use a classical convex inequality : for every $A, B \in \mathbb{R}$ and $\lambda,\mu \geq 0$, we have

$$\min\{A,B\} \leq \frac{\lambda A + \mu B}{\lambda+\mu} \leq \max\{A,B\}. \qquad (6.15)$$

Therefore, using the first inequality and then sending ε to 0, from (6.14), we get

$$\min\left\{h(x_0,t_0,\alpha), \frac{\partial\varphi}{\partial n}(x_0,t_0)\right\} \le 0.$$

Taking the supremum over $\alpha \in \mathcal{A}$, we obtain exactly the "relaxed" viscosity inequality for $(x_0,t_0) \in \partial\Omega \times (0,T]$. It proves that u is a subsolution.

We prove that u is a supersolution, taking as above a C^1 function φ and a point $(x_0,t_0) \in \overline{\Omega} \times [0,T]$. We suppose that (x_0,t_0) is a minimum of $u_* - \varphi$ whose value is 0 and that $u_* \ge \varphi$ in $\overline{\Omega} \times [0,T]$.

Suppose that $t_0 = 0$. For $\eta > 0$, we consider an η-optimal control $\alpha_\eta(\cdot)$. We have, for every (x,t),

$$\begin{aligned} u(x,t) + \eta &\ge \int_0^t f(X_s^{x,t}, t-s, \alpha_\eta(s))\,ds + \psi(X_t^{x,t}) \\ &\ge \psi(X_t^{x,t}) - Ct, \end{aligned}$$

since f is bounded by C. Passing to the lim inf with $x \to x_0, t \to 0$, we get

$$u_*(x_0,0) + \eta \ge \liminf_{x\to x_0, t\to 0} \psi(X_t^{x,t}) \ge \liminf_{x \to x_0} \psi(x) = \psi_*(x_0).$$

Sending η to 0, we obtain the desired inequality for $t_0 = 0$.

We suppose now that $t_0 > 0$ and $x_0 \in \partial\Omega$ (which is the most difficult case). For every $\varepsilon > 0$ and every control $\alpha(\cdot) \in L^\infty([0,T],\mathcal{A})$, we have

$$\begin{aligned} &\int_0^\varepsilon f(X_s^{x_0,t_0}, t_0 - s, \alpha(s))\,ds + u_*(X_\varepsilon^{x_0,t_0}, t_0 - \varepsilon) \\ &\ge \inf_{\alpha(\cdot)\in L^\infty([0,T],\mathcal{A})} \left\{ \int_0^\varepsilon f(X_s^{x_0,t_0}, t_0 - s, \alpha(s))\,ds + \varphi(X_\varepsilon^{x_0,t_0}, t_0 - \varepsilon) \right\}. \end{aligned}$$

Taking the infimum over $\alpha(\cdot)$ in the previous inequality and applying Proposition 6.3.2, we obtain

$$u_*(x_0,t_0) = \varphi(x_0,t_0) \ge \inf_{\alpha(\cdot)\in L^\infty([0,T],\mathcal{A})} \left\{ \int_0^\varepsilon f(X_s^{x_0,t_0}, t_0 - s, \alpha(s))\,ds + \varphi(X_\varepsilon^{x_0,t_0}, t_0 - \varepsilon) \right\}.$$

Using that φ is C^1 as in (6.10), the previous inequality yields

$$\begin{aligned} 0 \ \ge \ \inf_{\alpha(\cdot)\in L^\infty([0,T],\mathcal{A})} \Big\{ &- \int_0^\varepsilon h(X_s^{x_0,t_0}, t_0 - s, \alpha(s))\,ds \\ &- \int_0^\varepsilon \frac{\partial\varphi}{\partial n}(X_s^{x_0,t_0}, t_0 - s) \mathbb{1}_{\partial\Omega}(X_s^{x_0,t_0})\, d|k^{x_0,t_0}|_s \Big\}, \end{aligned} \tag{6.16}$$

where h is defined by (6.12).

On the one hand, we have

$$
\begin{aligned}
& h(X_s^{x_0,t_0}, t_0 - s, \alpha(s)) \\
& \leq \frac{\partial \varphi}{\partial t}(X_s^{x_0,t_0}, t_0 - s) + H(X_s^{x_0,t_0}, t_0 - s, D\varphi(X_s^{x_0,t_0}, t_0 - s)) := \bar{h}(X_s^{x_0,t_0}, t_0 - s).
\end{aligned}
$$

On the second hand, we set

$$
\begin{aligned}
\mu(s) \quad = \quad & \left| H(X_s^{x_0,t_0}, t_0 - s, D\varphi(X_s^{x_0,t_0}, t_0 - s)) - H(x_0, t_0, D\varphi(x_0, t_0)) \right| \\
& + \left| \frac{\partial \varphi}{\partial t}(X_s^{x_0,t_0}, t_0 - s) - \frac{\partial \varphi}{\partial t}(x_0, t_0) \right|
\end{aligned}
$$

and

$$
\nu(s) = \left| \frac{\partial \varphi}{\partial n}(X_s^{x_0,t_0}, t_0 - s) - \frac{\partial \varphi}{\partial n}(x_0, t_0) \right|,
$$

The continuous functions $\mu(s)$ and $\nu(s)$ converge to 0 as $s \to 0$ because all the functions which appear in their definition are continuous. A priori, they depend on the control $\alpha(\cdot)$ through X^{x_0,t_0} but the estimate (ii) of Lemma 6.3.1 proves they converge to 0 uniformly with respect to the control.

From (6.16), we obtain

$$
\begin{aligned}
0 \quad \geq \quad & \inf_{\alpha(\cdot) \in L^\infty([0,T],\mathcal{A})} \Big\{ -\varepsilon \bar{h}(x_0, t_0) - \int_0^\varepsilon \mu(s)\, ds \\
& \qquad - K_\varepsilon \frac{\partial \varphi}{\partial n}(x_0, t_0) - \int_0^\varepsilon \nu(s)\, \mathbb{1}_{\partial\Omega}(X_s^{x_0,t_0})\, d|k^{x_0,t_0}|_s \Big\}, \\
\geq \quad & \inf_{\alpha(\cdot) \in L^\infty([0,T],\mathcal{A})} \Big\{ (\varepsilon + K_\varepsilon) \Big(- \frac{\varepsilon \bar{h}(x_0, t_0) + K_\varepsilon \frac{\partial \varphi}{\partial n}(x_0, t_0)}{\varepsilon + K_\varepsilon} \\
& \qquad - \frac{\varepsilon \sup_{[0,\varepsilon]} \mu + K_\varepsilon \sup_{[0,\varepsilon]} \nu}{\varepsilon + K_\varepsilon} \Big) \Big\} \\
\geq \quad & \inf_{\alpha(\cdot) \in L^\infty([0,T],\mathcal{A})} \{\varepsilon + K_\varepsilon\} \Big(- \max \left\{ \bar{h}(x_0, t_0), \frac{\partial \varphi}{\partial n}(x_0, t_0) \right\} \\
& \qquad - \max\{ \sup_{[0,\varepsilon]} \mu, \sup_{[0,\varepsilon]} \nu \} \Big),
\end{aligned}
$$

where K_ε is defined by (6.13) and by using the fact that μ and ν are independent of the control. Dividing the inequality by $\inf_{\alpha(\cdot) \in L^\infty([0,T],\mathcal{A})}\{\varepsilon + K_\varepsilon\} > 0$, and letting $\varepsilon \to 0$, we obtain

$$
\max \left\{ \bar{h}(x_0, t_0), \frac{\partial \varphi}{\partial n}(x_0, t_0) \right\} \geq 0,
$$

which ends the proof. □

When ψ is continuous, we can characterize the value function as the unique viscosity solution of the Neumann problem (this theorem is proved in [73] in the case of an infinite horizon problem).

Corollary 6.3.1 *If $\psi \in C(\overline{\Omega})$, then $u[\psi]$ is the unique viscosity solution of (6.1)–(6.2) with H defined by (6.8).*

Proof of Corollary 6.3.1. From Theorem 6.3.1, $u[\psi]$ is a viscosity solution of (6.1)–(6.2). Let v be another solution. Note at this point that the Hamiltonian H defined by (6.8) satisfies the assumptions **(H1)** and **(H2)** provided b and f satisfy **(H4)**. Therefore, since $\psi^* = \psi_* = \psi$ when $\psi \in C(\overline{\Omega})$, we can apply Theorem 6.2.1 twice to compare $u[\psi]$ and v and we obtain

$$u^*[\psi] \leq v_* \quad \text{and} \quad v^* \leq u_*[\psi] \quad \text{ in } \overline{\Omega} \times [0,T].$$

Therefore $u_*[\psi] \leq u^*[\psi]$ and $v^* \leq v_*$. It follows that $u[\psi]$ and v are continuous and equal in $\overline{\Omega} \times [0,T]$. □

We end this section by introducing the relaxed control problem associated to the control problem with reflection. For relaxed control problems, we refer for example to [70], [84], [13] and [6]. We introduce $P(\mathcal{A})$ the space of the probability measures on $\mathcal{A}$. The system is governed by (6.6) where the first ordinary differential equation is replaced by

$$d\hat{X}_s^{x,t} = \int_{\mathcal{A}} b(\hat{X}_s^{x,t}, t-s, \alpha)\, d\mu_s(\alpha) ds - d\hat{k}_s^{x,t} \ \text{ in } [0,t],\ t \leq T, \tag{6.17}$$

$$\tag{6.18}$$

where the control $(\mu_s)_{s\in[0,T]} \in L^\infty([0,T], P(\mathcal{A}))$. The results of Lions and Sznitman apply : for every $(\mu_s)_{s\in[0,T]} \in L^\infty([0,T], P(\mathcal{A}))$, there is a unique relaxed solution $(\hat{X}^{x,t}, \hat{k}^{x,t}) \in C([0,t], \mathbb{R}^N) \times BV([0,t], \mathbb{R}^N)$.

The interest of using relaxed control consists in the compactness properties of $L^\infty([0,T], P(\mathcal{A}))$ for the weak*-topology. For every $a \in \mathcal{A}$, we denote by δ_a the Dirac measure at a. Using this measure, we identify $L^\infty([0,T], \mathcal{A})$ with the set $\{(\delta_{\alpha(s)})_{s\in[0,T]} : \alpha(\cdot) \in L^\infty([0,T], \mathcal{A})\}$.

Theorem 6.3.2 *We have*

(i) the space $L^\infty([0,T], P(\mathcal{A}))$ is compact for the weak-topology and $L^\infty([0,T], \mathcal{A})$ is a dense subset ;*

(ii) if $(\alpha^n)_{n\in I\!N}$ is a sequence of classical controls, there exists a subsequence (still denotes by $(\alpha^n)_{n\in I\!N}$) which converges in $L^\infty([0,T], P(\mathcal{A}))$ for the weak-topology to a relaxed control $(\mu_s)_{s\in[0,T]}$. Then, for every $(x,t) \in \bar{\Omega} \times [0,T]$, the associated trajectories $X^{n,x,t}$ converge uniformly in $[0,T]$ to the relaxed trajectory $\hat{X}^{x,t}$ associated to $(\mu_s)_{s\in[0,T]}$ and*

$$\int_0^t f(X_s^{n,x,t}, t-s, \alpha^n(s))\, ds \underset{n\to\infty}{\longrightarrow} \int_0^t \int_{\mathcal{A}} f(\hat{X}_s^{x,t}, t-s, \alpha)\, d\mu_s(\alpha) ds.$$

This theorem is classical when dealing with classical control problem (see [70] and [84]).

We then define the relaxed value function

$$\hat{u}[\psi](x,t) = \inf_{\mu \in L^\infty([0,T], P(\mathcal{A}))} \left\{ \int_0^t \int_{\mathcal{A}} f(\hat{X}_s^{x,t}, t-s, \alpha)\, d\mu_s(\alpha) ds + \psi(\hat{X}_t^{x,t}) \right\}.$$

The dynamic programming principle (Proposition 6.3.2) holds true and the relaxed value $\hat{u}[\psi]$ is a viscosity solution of (6.1)–(6.2) (Theorem 6.3.1 is valid for the relaxed control problem). In particular, note that Corollary 6.3.1 implies

$$u[\psi] = \hat{u}[\psi]$$

when ψ is continuous.

6.4 Properties of the value functions and counter-example to uniqueness

In this section, we address the question of uniqueness for (6.1)–(6.2) when ψ is discontinuous.

We begin with a result which was proved by Barles and Perthame [20] in the case of a optimal stopping time problem with discontinuous stopping cost which corresponds to a time-independent Hamilton-Jacobi set in the whole space $\mathbb{R}^N$.

Theorem 6.4.1 *Let ψ be any locally bounded function and v be a viscosity solution of (6.1)–(6.2) with initial data ψ, where H is given by (6.8). We have*

$$v^* \leq u[\psi^*] \quad \textit{and} \quad \hat{u}[\psi_*] \leq v_* \quad \textit{in } \overline{\Omega} \times [0,T].$$

Therefore, $u[\psi^*]$ is the maximal subsolution and $\hat{u}[\psi_*]$ is the minimal supersolution.

Proof of Theorem 6.4.1. We begin by proving the first inequality. Let $(\psi_n)_n$ be a nonincreasing sequence of bounded uniformly continuous functions which converges to ψ^* ; for instance, we can choose

$$\psi^n(x) = \sup_{y \in \overline{\Omega}} \{\psi^*(y) - n^2|x-y|^2\}.$$

On the one hand, since $\psi_n \geq \psi_{n+1} \geq \psi^*$, v is a subsolution of (6.1)–(6.2) with initial data ψ^n. On the other hand, $u[\psi^n]$ (respectively $u[\psi^{n+1}]$) is the unique continuous viscosity solution of (6.1)–(6.2) with initial data ψ^n (respectively ψ_{n+1}). From Theorem 6.2.1, it follows

$$v^* \leq u_*[\psi^{n+1}] = u[\psi^{n+1}] \leq u[\psi^n] = u_*[\psi^n] \quad \text{in } \overline{\Omega} \times [0,T]. \tag{6.19}$$

We need the following lemma.

Lemma 6.4.1
$$\lim_{n\to+\infty} u[\psi^n] = \inf_{n \in I\!N} u[\psi^n] = u[\psi^*].$$

Sending n to $+\infty$ in (6.19) and applying the lemma gives the first inequality of the theorem.

We turn to the proof of the second inequality. We introduce a nondecreasing sequence of bounded uniformly continuous functions $(\phi_n)_{n\in I\!N}$ converging to ψ_* ; As above, using Theorem 6.2.1, we see that $(u[\phi_n])_n$ is nondecreasing and $u[\phi_n] \leq v_*$ in $\overline{\Omega} \times [0,T]$. We then have

Lemma 6.4.2

$$\lim_{n\to+\infty} u[\phi^n] = \sup_{n\in I\!N} u[\phi^n] = \hat{u}[\psi_*].$$

The conclusion is an immediate consequence of the lemma. It completes the proof of the theorem. □

Proof of Lemma 6.4.1. For every $n \in I\!N$, $\psi^n \geq \psi^{n+1} \geq \psi^*$ in $\overline{\Omega}$. Applying Theorem 6.2.1, we obtain

$$u[\psi^n] \geq u[\psi^{n+1}] \geq u^*[\psi^*] \quad \text{in } \overline{\Omega}\times[0,T]. \tag{6.20}$$

Therefore $(u[\psi^n])$ is nonincreasing and it proves the first equality of the lemma.

On an other hand, for every $\alpha(\cdot) \in L^\infty([0,T],\mathcal{A})$ and $(x,t) \in \overline{\Omega}\times[0,T]$, we have

$$\int_0^t f(X_s^{x,t}, t-s, \alpha(s))\,ds + \psi^n(X_t^{x,t}) \geq u[\psi^n](x,t) \geq \inf_{n\in I\!N} u[\psi^n](x,t),$$

where $X^{x,t}$ is the trajectory associated to $\alpha(\cdot)$. Taking the infimum over n and then over $\alpha(\cdot)$, we get

$$u[\psi^*](x,t) \geq \inf_{n\in I\!N} u[\psi^n](x,t). \tag{6.21}$$

From (6.20) and (6.21), we get the desired result. Note that we proved in addition $u[\psi^*] \geq u^*[\psi^*]$ which means that $u[\psi^*]$ is USC. □

Proof of Lemma 6.4.2. For every relaxed control $(\mu_s)_{s\in[0,T]} \in L^\infty([0,T],P(\mathcal{A}))$ and $(x,t) \in \overline{\Omega}\times[0,T]$, we have

$$\int_0^t \int_{\mathcal{A}} f(\hat{X}_s^{x,t}, t-s, \alpha)\,d\mu_s(\alpha)ds + \phi^n(\hat{X}_t^{x,t}) \geq \hat{u}[\phi^n](x,t) = u[\phi^n](x,t),$$

where $\hat{X}^{x,t}$ is the relaxed trajectory associated to $(\mu_s)_{s\in[0,T]}$. Taking the supremum over n and the infimum over $(\mu_s)_{s\in[0,T]}$, we obtain

$$\hat{u}[\psi_*] \geq \sup_{n\in I\!N} u[\phi^n].$$

We prove the opposite inequality. For every $n \in I\!N$, there exists a $(1/n)$-optimal control $\alpha^n \in L^\infty([0,T],\mathcal{A})$ such that

$$\frac{1}{n} + \int_0^t f(X_s^{n,x,t}, t-s, \alpha^n(s))\,ds + \phi^n(X_t^{n,x,t}) \leq u[\phi^n](x,t),$$

where $X^{n,x,t}$ is the trajectory associated to α^n. From Theorem 6.3.2, we can extract a subsequence (still denoted by $(\alpha^n)_{n\in I\!N}$) which converges to $(\mu_s)_{s\in[0,T]} \in L^\infty([0,T],P(\mathcal{A}))$

and such that $X^{n,x,t}$ converges uniformly in $[0,T]$ to a trajectory $\hat{X}^{x,t}$ which is associated to $(\mu_s)_{s\in[0,T]}$. Recall that, since $(u[\phi^n])_{n\in\mathbb{N}}$ is nondecreasing, we have (see [13] and [6]),

$$\sup_{n\in\mathbb{N}} u[\phi^n] = \liminf_{n\to+\infty}{}_* u[\phi^n]. \tag{6.22}$$

It follows

$$\begin{aligned}
\hat{u}[\psi_*] &\le \int_0^t \int_{\mathcal{A}} f(\hat{X}_s^{x,t}, t-s, \alpha)\, d\mu_s(\alpha) ds + \psi_*(\hat{X}_t^{x,t}) \\
&\le \liminf_{n\to+\infty}{}_* \left(\int_0^t f(X_s^{n,x,t}, t-s, \alpha^n(s))\, ds + \phi^n(X_t^{n,x,t}) \right) = \sup_{n\in\mathbb{N}} u[\phi^n].
\end{aligned}$$

It completes the proof. Note that (6.22) implies that $\hat{u}[\psi_*]$ is LSC. □

From this result, Barles and Perthame [20] obtain the following uniqueness result : if the final cost ψ satisfies a "regularity" condition, namely

$$(\psi^*)_* = \psi_*,$$

then all the discontinuous viscosity solution have the same LSC envelope. It means that the LSC envelope of the value function $u[\psi]$ is the unique LSC viscosity solution of (6.1)–(6.2).

The question we address is : is it possible to prove this result for the Neumann problem ? The main result of this paper is the following counter-example showing that this uniqueness result does not hold in the case of the control problem with reflection. To our knowledge, the problem of the characterization of the value function of this problem is still open.

Theorem 6.4.2 *Under the assumptions of Theorem 6.4.1, even if ψ is "regular", namely*

$$(\psi^*)_* = \psi_* \quad \textit{and} \quad (\psi_*)^* = \psi^* \quad \textit{on } \overline{\Omega}, \tag{6.23}$$

then the LSC envelope $u_*[\psi^*]$ and $\hat{u}[\psi_*]$ do not coincide in general.

Proof of Theorem 6.4.2. We construct a control problem with reflection for which $u_*[\psi^*] > \hat{u}[\psi_*]$.

Let $\Omega = (0,1)$ and for (6.6), we take the simple system

$$\begin{cases}
dX_s^{x,t} = b(t-s)ds - dk_s^{x,t} \;\; \text{in } [0,t], \;\; t \le T, \\
X_0^{x,t} = x \in [0,1], \;\; X_s^{x,t} \in [0,1] \;\; \text{for every } s \in [0,t], \\
k_s^{x,t} = \displaystyle\int_0^s \mathbb{1}_{\partial[0,1]}(X_\tau^{x,t}) n(X_\tau^{x,t})\, d|k^{x,t}|_\tau,
\end{cases} \tag{6.24}$$

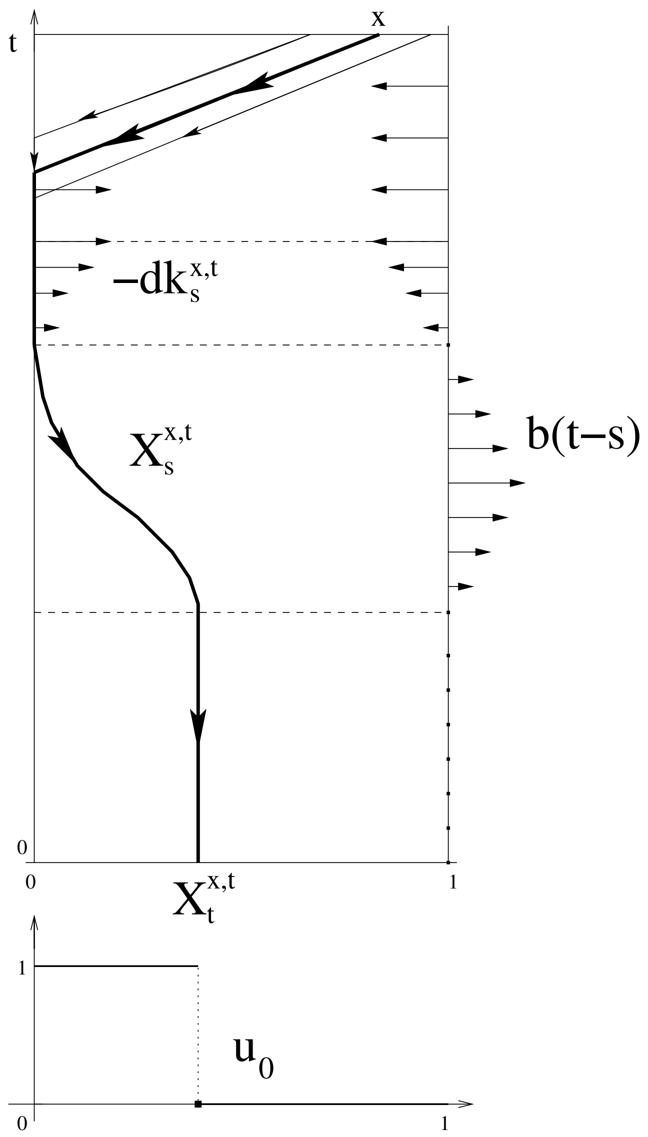

FIG. 6.1 – *Reflected trajectories of the system (6.24)*

We set

$$b(\tau) = \begin{cases} 0 & \text{for } \tau \in [0,1], \\ \frac{\pi}{4}\sin[\pi(2-\tau)] & \text{for } \tau \in [1,2], \\ 2-\tau & \text{for } \tau \in [2,3], \\ -1 & \text{for } \tau \in [3,+\infty), \end{cases}$$

(see Figure 6.1 for an illustration).

From Proposition 6.3.1, we have

$$dk_s^{x,t} = \mathbb{1}_{\{0\}}(X_s^{x,t})\min\{0, b(t-s)\}ds + \mathbb{1}_{\{1\}}(X_s^{x,t})\max\{0, b(t-s)\}ds. \tag{6.25}$$

It follows that we can compute explicitly the trajectories of (6.24). We claim that we chose the vector field b such that,

$$X_t^{x,t} = 1/2 \quad \text{for any } (x,t) \in [0,1] \times [3,+\infty).$$

Indeed, let $x \in [0,1]$ and $t \geq 3$. For $s \in [0, t-3]$, $dX_s^{x,t} = -ds$ if $X_s^{x,t} \in (0,1]$ and $dX_s^{x,t} = 0$ if $X_s^{x,t} = 0$. In any case, $X_{t-3}^{x,t} = 0$. For $s \in [t-3, t-2]$, from (6.25), $dX_s^{x,t} = 0$ and $X_{t-2}^{x,t} = 0$. For $s \in [t-2, t-1]$, we have to integrate

$$dX_s^{x,t} = \frac{\pi}{4}\sin[\pi(2-t+s)], \quad X_{t-2}^{x,t} = 0,$$

which gives $X_{t-1}^{x,t} = 1/2$. And for $s \in [t-1, t]$, $dX_s^{x,t} = 0$. It proves the claim. Such trajectories are drawn on Figure 6.1.

We then consider the control problem governed by (6.24) with the running cost $f \equiv 0$ and the final cost ψ such that $\psi(y) = 1$ if $y \in [0, 1/2)$ and $\psi(y) = 0$ if $y \in [1/2, 1]$. The function ψ is LSC in $[0,1]$ and satisfies (6.23). Since (6.24) is independent of the control, the value function is

$$u[\psi](x,t) = \psi(X_t^{x,t}).$$

On the one hand, $\hat{u}[\psi_*](x,t) = u[\psi_*](x,t) = \psi_*(X_t^{x,t}) = \psi_*(1/2)$. It follows

$$\hat{u}_*[\psi_*](x,t) = 0.$$

On the second hand, $u[\psi^*](x,t) = \psi^*(X_t^{x,t})$. For any sequence (x_n, t_n) which converges to (x,t), there exists n_0 such that $t_n \geq 3$ for $n \geq n_0$. It follows $u[\psi^*](x_n, t_n) = \psi^*(X_{t_n}^{x_n,t_n}) = \psi^*(1/2) = 1$; Taking the infimum over all such sequences, we get

$$u_*[\psi^*](x,t) = 1.$$

It ends the proof. □

Bibliographie générale

[1] G. Alberti, L. Ambrosio, and P. Cannarsa. On the singularities of convex functions. *Manuscripta Math.*, 76 :421–435, 1992.

[2] L. Ambrosio, P. Cannarsa, and H. M. Soner. On the propagation of singularities of semi-convex functions. *Ann. Scuola Norm. Sup. Pisa (4)*, 20 :597–616, 1993.

[3] S. Angenent, D. L. Chopp, and T. Ilmanen. A computed example of nonuniqueness of mean curvature flow in $\mathbb{R}^3$. *Comm. Partial Differential Equations*, 20(11-12) :1937–1958, 1995.

[4] S. B. Angenent. Some recent results on mean curvature flow. *RAM Res. Appl. Math.*, 30 :1–18, 1994.

[5] D. G. Aronson. Widder's inversion theorem and the initial distribution problem. *SIAM J. Math. Anal.*, 12 :639–651, 1981.

[6] M. Bardi and I. Capuzzo Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations.* Birkhäuser Boston Inc., Boston, MA, 1997.

[7] M. Bardi, M. G. Crandall, L. C. Evans, , H. M. Soner, and P. E. Souganidis. Viscosity solutions and applications. I. Capuzzo Dolcetta and P.-L. Lions Eds. In *Lecture Notes in Mathematics*, volume 1660. Springer, Berlin, 1997.

[8] M. Bardi and L. C. Evans. On Hopf's formulas for solutions of Hamilton-Jacobi equations. *Nonlinear Anal.*, 8 :1373–1381, 1984.

[9] G. Barles. Remark on a flame propagation model. *Rapport INRIA*, 464, 1985.

[10] G. Barles. Uniqueness for first-order Hamilton-Jacobi equations and Hopf formula. *J. Differential Equations*, 69(3) :346–367, 1987.

[11] G. Barles. Uniqueness and regularity results for first-order Hamilton-Jacobi equations. *Indiana Univ. Math. J.*, 39(2) :443–466, 1990.

[12] G. Barles. Discontinuous viscosity solutions of first-order Hamilton-Jacobi equations : a guided visit. *Nonlinear Anal.*, 20(9) :1123–1134, 1993.

[13] G. Barles. *Solutions de viscosité des équations de Hamilton-Jacobi.* Springer-Verlag, Paris, 1994.

[14] G. Barles, S. Biton, M. Bourgoing, and O. Ley. Quasilinear parabolic equations, unbounded solutions and geometrical equations III. Uniqueness through classical viscosity solutions' methods. *Preprint*, 2001.

[15] G. Barles, S. Biton, and O. Ley. Quelques résultats d'unicité pour l'équation de mouvement par courbure moyenne dans $\mathbb{R}^n$. *ESAIM : Proceedings, Actes du 32ème Congrès d'Analyse Numérique : Canum 2000*, 8, 2000.

[16] G. Barles, S. Biton, and O. Ley. Quasilinear parabolic equations, unbounded solutions and geometrical equations I. A geometrical approach to the study of quasilinear parabolic equations in $I\!R^N$. *Preprint*, 2001.

[17] G. Barles, S. Biton, and O. Ley. Quasilinear parabolic equations, unbounded solutions and geometrical equations II. Uniqueness without growth conditions and applications to the mean curvature flow in $I\!R^2$. *Preprint*, 2001.

[18] G. Barles, S. Biton, and O. Ley. Quasilinear parabolic equations, unbounded solutions and geometrical equations IV. The (x,t)-dependent case. *In preparation*, 2001.

[19] G. Barles, R. Buckdahn, and E. Pardoux. Backward stochastic differential equations and integral-partial differential equations. *Stoch. and Stoch. Rep.*, 60 :57–83, 1997.

[20] G. Barles and B. Perthame. Discontinuous solutions of deterministic optimal stopping time problems. *RAIRO Modél. Math. Anal. Numér.*, 21(4) :557–579, 1987.

[21] G. Barles and B. Perthame. Exit time problems in optimal control and vanishing viscosity method. *SIAM J. Control Optim.*, 26(5) :1133–1148, 1988.

[22] G. Barles, H. M. Soner, and P. E. Souganidis. Front propagation and phase field theory. *SIAM J. Control Optim.*, 31(2) :439–469, 1993.

[23] G. Barles and P. E. Souganidis. A new approach to front propagation problems : theory and applications. *Arch. Rational Mech. Anal.*, 141(3) :237–296, 1998.

[24] E. N. Barron and R. Jensen. Semicontinuous viscosity solutions of Hamilton-Jacobi equations with convex hamiltonians. *Comm. Partial Differential Equations*, 15(12) :1713–1740, 1990.

[25] E. N. Barron and R. Jensen. Optimal control and semicontinuous viscosity solutions. *Proc. Amer. Math. Soc.*, 113(2) :397–402, 1991.

[26] S. Biton. Nonlinear monotone semigroups and viscosity solutions. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 18(3) :383–402, 2001.

[27] S. Biton. Semi-groupes non-linéaires monotones, équations géométriques et solutions de viscosité non bornées d'équations quasilinéaires paraboliques. *Thèse de doctorat, université de Tours*, 2001.

[28] A.-P. Blanc. Deterministic exit time control problems with discontinuous exit costs. *SIAM J. Control Optim.*, 35(2) :399–434, 1997.

[29] R. Buckdahn, P. Cardaliaguet, and M. Quincampoix. A representation formula for the mean curvature motion. *To appear in SIAM J. Math. Anal.*, 2000.

[30] P. Cannarsa and C. Sinestrari. *Semiconcave functions, Hamilton-Jacobi equations, and optimal control.* Birkhäuser Boston Inc., Boston, MA, to appear.

[31] P. Cannarsa and H. M. Soner. On the singularities of the viscosity solutions to Hamilton-Jacobi-Bellman equations. *Indiana Univ. Math. J.*, 36(3) :501–524, 1987.

[32] Y. G. Chen, Y. Giga, and S. Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differential Geom.*, 33(3) :749–786, 1991.

[33] K.-S. Chou and Y.-C. Kwong. On quasilinear parabolic equations which admit global solutions for initial data with unrestricted growth. *Calc. Var.*, 12 :281–315, 2001.

[34] F. H. Clarke. *Optimization and nonsmooth analysis.* John Wiley & Sons Inc., New York, 1983. A Wiley-Interscience Publication.

[35] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, and P. R. Wolenski. *Nonsmooth analysis and control theory.* Springer-Verlag, New York, 1998.

[36] L. Corrias. Fast Legendre-Fenchel transform and applications to Hamilton-Jacobi equations and conservation laws. *SIAM J. Numer. Anal.*, 33(4) :1534–1558, 1996.

[37] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1) :1–67, 1992.

[38] M. G. Crandall and P.-L. Lions. Viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.*, 277(1) :1–42, 1983.

[39] M. G. Crandall and P.-L. Lions. Quadratic growth of solutions of fully nonlinear second order equations in $I\!R^n$. *Differential Integral Equations*, 3(4) :601–616, 1990.

[40] F. Da Lio. Strong comparison results fot quasilinear equations in annular domains and applications. *Preprint*, 2001.

[41] K. Ecker and G. Huisken. Interior estimates for hypersurfaces moving by mean curvature. *Invent. Math.*, 105(3) :547–569, 1991.

[42] L. C. Evans. A geometric interpretation of the heat equation with multivalued initial data. *SIAM J. Math. Anal.*, 27 :932–958, 1996.

[43] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions.* CRC Press, Boca Raton, FL, 1992.

[44] L. C. Evans and P. E. Souganidis. Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations. *Indiana Univ. Math. J.*, 33(5) :773–797, 1984.

[45] L. C. Evans and J. Spruck. Motion of level sets by mean curvature. I. *J. Differential Geom.*, 33(3) :635–681, 1991.

[46] L. C. Evans and J. Spruck. Motion of level sets by mean curvature. III. *J. Geom. Anal.*, 2(2) :121–150, 1992.

[47] L. C. Evans and J. Spruck. Motion of level sets by mean curvature. IV. *J. Geom. Anal.*, 5(1) :77–114, 1995.

[48] W. H. Fleming and H. M. Soner. *Controlled Markov processes and viscosity solutions.* Springer-Verlag, New York, 1993.

[49] H. Frankowska. Équations d'Hamilton-Jacobi contingents. *C. R. Acad. Sci., Sér. I*, 304 :295–298, 1987.

[50] H. Frankowska. Lower semicontinuous solutions of Hamilton-Jacobi-Bellman equations. *SIAM J. Control Optim.*, 31(1) :257–272, 1993.

[51] Y. Giga, S. Goto, H. Ishii, and M.-H. Sato. Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains. *Indiana Univ. Math. J.*, 40(2) :443–470, 1991.

[52] Y. Giga and M.-H. Sato. On semicontinuous solutions for general Hamilton-Jacobi equations. *Proc. Japan Acad. Ser. A Math. Sci.*, 75(9) :159–162, 1999.

[53] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Springer-Verlag, Berlin, second edition, 1983.

[54] S. Goto. Generalized motion of hypersurfaces whose growth speed depends superlinearly on the curvature tensor. *Differential Integral Equations*, 7 :323–343, 1994.

[55] T. Ilmanen. Generalized flow of sets by mean curvature on a manifold. *Indiana Univ. Math. J.*, 41 :671–705, 1992.

[56] T. Ilmanen. Elliptic regularization and partial regularity for motion by mean curvature. *Mem. Amer. Math. Soc.*, 108(520), 1994.

[57] C. Imbert. Some regularity results for anisotropic motion of fronts. *Preprint*, 2001.

[58] H. Ishii. Uniqueness of unbounded viscosity solution of Hamilton-Jacobi equations. *Indiana Univ. Math. J.*, 33(5) :721–748, 1984.

[59] H. Ishii. Hamilton-Jacobi equations with discontinuous Hamiltonians on arbitrary open sets. *Bull. Fac. Sci. Eng. Chuo Univ.*, 28 :33–77, 1985.

[60] H. Ishii. Perron's method for Hamilton-Jacobi equations. *Duke Math. J.*, 55(2) :369–384, 1987.

[61] H. Ishii. *Lectures at Brown University*, 1988.

[62] H. Ishii. On uniqueness and existence of viscosity solutions of fully nonlinear second order elliptic pde's. *Comm. Pure Appl. Math.*, 42 :14–45, 1989.

[63] H. Ishii. Degenerate parabolic PDEs with discontinuities and generalized evolutions of surfaces. *Adv. Differential Equations*, 1(1) :51–72, 1996.

[64] H. Ishii and P. E. Souganidis. Generalized motion of non compact hypersurfaces with velocity having arbitrary growth on the curvature tensor. *Tôhuko Math. J.*, 47 :227–250, 1995.

[65] R. Jensen. The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations. *Arch. Rational Mech. Anal.*, 101 :1–27, 1988.

[66] F. John. *Partial differential equations.* Springer-Verlag, New York, fourth edition, 1991.

[67] S. N. Kružkov. First order quasilinear equations in several independent variables. *Math. USSR-Sb.*, 10(2) :217–243, 1970.

[68] O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Ural'ceva. *Linear and quasilinear equations of parabolic type.* American Mathematical Society, Providence, R.I., 1967. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23.

[69] J.-M. Lasry and P.-L. Lions. A remark on regularization in Hilbert spaces. *Isr. J. Math.*, 55 :257–266, 1986.

[70] E. B. Lee and L. Markus. *Foundations of optimal control theory.* J. Wiley, New York, 1967.

[71] O. Ley. Lower-bound gradient estimates for first-order Hamilton-Jacobi equations and applications to the regularity of propagating fronts. *Adv. Differential Equations*, 6(5) :547–576, 2001.

[72] P.-L. Lions. *Generalized solutions of Hamilton-Jacobi equations.* Pitman (Advanced Publishing Program), Boston, Mass., 1982.

[73] P.-L. Lions. Neumann type boundary conditions for Hamilton-Jacobi equations. *Duke Math. J.*, 52 :793–820, 1985.

[74] P.-L. Lions and A. S. Sznitman. Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math.*, 37 :511–537, 1984.

[75] M. Ohnuma and K. Sato. Singular degenerate parabolic equations with applications to geometric evolutions. *Differential Integral Equations*, 6(5) :1265–1280, 1993.

[76] S. Osher and J. Sethian. Fronts propagating with curvature dependent speed : algorithms based on Hamilton-Jacobi formulations. *J. Comp. Physics*, 79 :12–49, 1988.

[77] L. Rifford. Existence of Lipschitz and semiconcave control-Lyapunov functions. *SIAM J. Control Optim.*, 39(4) :1043–1064, 2000.

[78] H. M. Soner. Motion of a set by the curvature of its boundary. *J. Differential Equations*, 101 :313–372, 1993.

[79] H. M. Soner and P. E. Souganidis. Singularities and uniqueness of cylindrically symmetric surfaces moving by mean curvature. *Comm. Partial Differential Equations*, 18(5-6) :859–894, 1993.

[80] H. M. Soner and N. Touzi. Dynamic programming for stochastic target problems and geometric flows. *preprint*, 2000.

[81] H. M. Soner and N. Touzi. A stochastic representation for mean curvature type geometric flows. *preprint*, 2001.

[82] P. Soravia. Discontinuous viscosity solutions to Dirichlet problems for Hamilton-Jacobi equations with convex Hamiltonians. *Comm. Partial Differential Equations*, 18(9-10) :1493–1514, 1993.

[83] A. I. Subbotin. *Generalized solutions of first-order PDEs.* Birkhäuser Boston Inc., Boston, MA, 1995.

[84] J. Warga. *Optimal control of differential and functional equations.* Academic Press, New York, 1972.