A counter-example to the characterization of the discontinuous value function of a reflected control problem

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Abstract

We consider an optimal control problems of reflected trajectories with a discontinuous terminal cost. We follow the the discontinuous approach of Barles and Perthame to study this problem. By a counter-example, we prove that this approach does not apply in order to characterize the value function.

Key-words: Optimal control, reflected trajectories, viscosity solutions, Hamilton-Jacobi equations with Neumann boundary conditions

1 The optimal control problem with reflection

We are interested in a deterministic optimal control problem of reflected trajectories at the boundary of an open bounded subset $\Omega \subset I\!R^N$ whose boundary $\partial\Omega$ is $W^{2,\infty}$. We use the framework of Lions [11] (see also [9], [2]); the reflected trajectories are governed by the system of ordinary differential equations

$$\left\{\begin{array}{l} dX_s^{x,t} = b(X_s^{x,t}, t-s, \alpha(s))ds - dk_s^{x,t} \ \text{ in } [0,t],\ t \leq T, \\ X_0^{x,t} = x \in \overline{\Omega}, \quad X_s^{x,t} \in \overline{\Omega} \ \text{ for every } s \in [0,t], \\ k_s^{x,t} = \displaystyle\int_0^s 1\!\!1_{\partial\Omega}(X_\tau^{x,t}) n(X_\tau^{x,t})\, d|k^{x,t}|_\tau, \end{array}\right. \tag{1}$$

This work was partially supported by the TMR program “Viscosity Solutions and Their Applications.”

where $T > 0$, $\mathcal{A}$ is a compact metric space, the control $\alpha(\cdot) \in L^\infty([0,T], \mathcal{A})$ and the vector field $b \in C(\overline{\Omega} \times [0,T] \times \mathcal{A}, I\!\!R^N)$ is Lipschitz continuous in the first variable uniformly with respect to the others.

From Lions and Sznitman [12] (see also Lions [11]), we know that, for any $(x,t) \in \overline{\Omega} \times [0,T]$, the system (1) admits a unique solution $(X^{x,t}, k^{x,t}) \in C([0,t], I\!\!R^N) \times BV([0,t], I\!\!R^N)$. The notation $|k^{x,t}|_s$ stands for the total variation of the bounded variation process $k^{x,t}$.

Remark 1.1 In our simple case, we have an explicit formula for $k^{x,t}$:

$$dk_s^{x,t} = 1\!\!I_{\partial\Omega}(X_s^{x,t}) n(X_s^{x,t}) \langle b(X_s^{x,t}, t-s, \alpha(s)), n(X_s^{x,t}) \rangle^+ \, ds. \tag{2}$$

We define the optimal control problem by introducing the value function

$$u[\psi](x,t) = \inf_{\alpha(\cdot) \in L^\infty([0,T],\mathcal{A})} \left\{ \int_0^t f(X_s^{x,t}, t-s, \alpha(s)) \, ds + \psi(X_t^{x,t}) \right\}, \tag{3}$$

where $f \in C(\overline{\Omega} \times [0,T] \times \mathcal{A})$ is uniformly continuous in the first variable uniformly with respect to the others, and the final cost $\psi : I\!\!R^N \to I\!\!R$, is locally bounded. The classical dynamical programming principle holds and provides the

Theorem 1.1 *For any locally bounded function ψ, the value function $u[\psi]$ is a viscosity solution of the Hamilton-jacobi equation*

$$\begin{cases} \dfrac{\partial u}{\partial t} + \sup\limits_{\alpha \in \mathcal{A}} \{ -\langle b(x,t,\alpha), Du \rangle - f(x,t,\alpha) \} = 0 \quad \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0,T), \qquad u(\cdot, 0) = \psi \quad \text{in } \overline{\Omega}. \end{cases} \tag{4}$$

For a proof of the theorem, see [11] and [10]. For the definition of viscosity solutions of this problem, we refer to [11] and [2]; notice that the boundary conditions has to be "relaxed" in the viscosity sense.

When ψ is continuous, Lions [11] characterized the value function using the Hamilton-Jacobi equation.

Theorem 1.2 *Under the previous assumptions, if, in addition, $\psi \in C(\overline{\Omega})$, then $u[\psi]$ is the unique viscosity solution of (4).*

We address the same problem but with a locally bounded final cost ψ. This case is of importance for applications (for instance when considering problems with punctual targets). It leads to a discontinuous value function $u[\psi]$ which is still a viscosity solution of (4) but its characterization appears to be more difficult since one does not have uniqueness for (4) anymore.

Many authors (see [8, 9], [3], [4, 5], [1], [7], [14], [6], etc.) have investigated the problem of characterizing the value function of such discontinuous control problems. Here we follow the discontinuous approach introduced by Barles and Perthame [3].

We need first to introduce the relaxed control problem associated to the control problem with reflection. For relaxed control problems, see for example [15] and [2]. We replace the first ordinary differential equation in (1) by

$$d\hat{X}_s^{x,t} = \int_{\mathcal{A}} b(\hat{X}_s^{x,t}, t-s, \alpha)\, d\mu_s(\alpha) ds - d\hat{k}_s^{x,t} \text{ in } [0,t],\ t \le T,$$

where the control $(\mu_s)_{s\in[0,T]} \in L^\infty([0,T], P(\mathcal{A}))$ and $P(\mathcal{A})$ is the space of the probability measures on $\mathcal{A}$. All the previous results (in particular the existence and uniqueness of a relaxed solution to the system (1)) apply; Therefore, defining the relaxed value function by

$$\hat{u}[\psi](x,t) = \inf_{\mu\in L^\infty([0,T],P(\mathcal{A}))} \left\{ \int_0^t \int_{\mathcal{A}} f(\hat{X}_s^{x,t}, t-s, \alpha)\, d\mu_s(\alpha) ds + \psi(\hat{X}_t^{x,t}) \right\},$$

this function turns out to be a viscosity solution of (4). Note that $\hat{u}[\psi] \le u[\psi]$ and, if ψ is continuous, then, by uniqueness, we have the equality.

Finally, we define the semicontinuous envelopes. For any locally bounded function $v : \overline{\Omega} \times [0,T] \to \mathbb{R}$, we define the upper-semicontinuous (USC in short) and lower-semicontinuous (LSC) envelopes by $v^*(x,t) = \limsup_{(y,s)\to(x,t)} v(y,s)$ and $v_*(x,t) = \liminf_{(y,s)\to(x,t)} v(y,s)$ respectively.

We have

Theorem 1.3 *Under the previous assumption, for any locally bounded final cost ψ, let v be a viscosity solution of (4). Then $\hat{u}[\psi_*] \le v_*$ and $v^* \le u[\psi^*]$ in $\overline{\Omega} \times [0,T]$. The value function $u[\psi^*]$ is the maximal USC subsolution and $\hat{u}[\psi_*]$ is the minimal LSC supersolution.*

This result was first proved in Barles and Perthame [3] in the case of a optimal stopping time problem with discontinuous stopping cost which corresponds to a time-independent Hamilton-Jacobi equation set in the whole space $\mathbb{R}^N$. We refer to [10] for a proof in the Neumann case.

From this result, Barles and Perthame obtain the following uniqueness result for their problem in $\mathbb{R}^N$: if the final cost ψ satisfies a "regularity" condition, namely

$$(\psi^*)_* = \psi_*, \tag{5}$$

then all the discontinuous viscosity solutions have the same LSC envelope. It means that the LSC envelope of the value function $u[\psi]$ is the unique LSC viscosity solution of the Hamilton-jacobi equation.

The question we address: is it possible to prove such a characterization for the Neumann problem? In the next section, we provide a counter-example answering the question in a negative way. To our knowledge, the problem of uniqueness for discontinuous solutions to (4) is still open. We learn recently that this problem is investigated by Serea [13] who obtained some uniqueness results defining a new notion of solution which is related to the other main discontinuous approach of Barron and Jensen [4, 5].

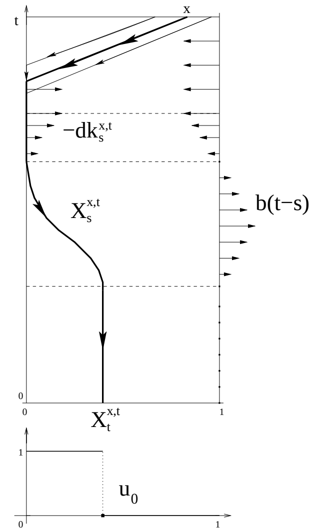

Figure 1: *Reflected trajectories of the system (1)*

2 The counter-example

We construct a control problem with reflection for which $\hat{u}[\psi_*] < u_*[\psi^*]$.

Set $\Omega = (0,1)$ and take a space and control-independent vector field b in (1) such that

$$b(\tau) = \begin{cases} 0 & \text{for } \tau \in [0,1], \\ \frac{\pi}{4}\sin[\pi(2-\tau)] & \text{for } \tau \in [1,2], \\ 2-\tau & \text{for } \tau \in [2,3], \\ -1 & \text{for } \tau \in [3,+\infty). \end{cases}$$

From (2), we have an explicit formula for the reflected process, $dk_s^{x,t} = \mathbb{I}_{\{0\}}(X_s^{x,t})\min\{0, b(t-s)\}ds + \mathbb{I}_{\{1\}}(X_s^{x,t})\max\{0, b(t-s)\}ds$, and we can compute explicitely the reflected trajectories of (1). We claim that we chose the vector field b such that,

$$X_t^{x,t} = 1/2 \quad \text{for any } (x,t) \in [0,1] \times [3,+\infty).$$

Indeed, let $x \in [0,1]$ and $t \geq 3$. For $s \in [0, t-3]$, $dX_s^{x,t} = -ds$ if $X_s^{x,t} \in (0,1]$ and $dX_s^{x,t} = 0$ if $X_s^{x,t} = 0$. In any case, $X_{t-3}^{x,t} = 0$. For $s \in [t-3, t-2]$, $dX_s^{x,t} = 0$ and $X_{t-2}^{x,t} = 0$. For $s \in [t-2, t-1]$, we have to integrate $dX_s^{x,t} = \pi \sin[\pi(2-t+s)]/4$ with the initial data $X_{t-2}^{x,t} = 0$, which gives $X_{t-1}^{x,t} = 1/2$. And for $s \in [t-1, t]$, $dX_s^{x,t} = 0$. It proves the claim. Such trajectories are drawn on Figure 1.

We then consider the control problem governed by (1) with the running cost $f \equiv 0$ and the final cost ψ such that $\psi(y) = 1$ if $y \in [0, 1/2)$ and $\psi(y) = 0$ if $y \in [1/2, 1]$. The function ψ is *LSC* in $[0,1]$ and satisfies (5). Since (1) is independent of the control, the value function is $u[\psi](x,t) = \psi(X_t^{x,t})$.

On the one hand, $\hat{u}[\psi_*](x,t) = u[\psi_*](x,t) = \psi_*(X_t^{x,t}) = \psi_*(1/2) = 0$.

On the second hand, $u[\psi^*](x,t) = \psi^*(X_t^{x,t})$. For any sequence (x_n, t_n) which converges to (x,t), there exists n_0 such that $t_n \geq 3$ for $n \geq n_0$. It follows $u[\psi^*](x_n,t_n) = \psi^*(X_{t_n}^{x_n,t_n}) = \psi^*(1/2) = 1$; Taking the infimum over all such sequences, we get $u_*[\psi^*](x,t) = 1$.

Remark 2.1 Note that we recover the classical continuous dependence of the trajectory $X^{x,t}$ with respect to the data (x,t) for the system (1). But we point out that, contrary to the system without the term "$dk_s^{x,t}$," when $X_{t_n}^{x_n,t_n} \to X_t^{x,t}$ as $n \to +\infty$, we do not have anymore $(x_n,t_n) \to (x,t)$ (see Figure 1 for an illustration).

Acknowledgment. I am grateful to Guy Barles, who brought this problem to my attention. I would like to thank Élisabeth Rouy for helpful discussions and valuable suggestions.

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