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DISSERTATION / DOCTORAL THESIS

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Solutions of the Einstein equations with Kaluza-Klein asymptotics

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ABSTRACT

Kaluza-Klein theory constitutes an interesting extension of general relativity to five spacetime dimensions, originally emerged from a historic attempt to unify the known fundamental forces of nature at that time. The elegant feature of this theory lies in the fact, that the involved dimensional reduction leads naturally to electromagnetism coupled to four dimensional gravity without the need of the introduction of a source term in the five dimensional Einstein equations. This concept can also be extended to higher dimensions. Although not being a realistic theory of nature, Kaluza-Klein theory gives interesting insights to advanced theories, such as string theory, and provides a geometric link between higher- and lower dimensional theories. Within this theory, black hole spacetimes exist. An interesting class of black hole solutions in Kaluza-Klein theory has been derived independently by Rasheed and Larsen, describing a family of axisymmetric, rotating, dyonic (magnetically- and electrically charged) black holes.

We analyse the Rasheed-Larsen metrics in different limits of their parameter family, prove regularity at the outer Killing horizon, identify and analyse the singularities of the metrics and derive conditions, under which they are shielded by the outer Killing horizon, exclude the existence of regular metrics without horizons and derive a criterion for stable causality in the d.o.c. (domain of outer communications).

In Kaluza-Klein theory, as in any other physical theory, the notion of total energy, momentum and other global charges play a key role. Our analysis covers asymptotically anti–de Sitter spacetimes, asymptotically flat spacetimes, as well as Kaluza-Klein asymptotically flat spacetimes. We prove that the Komar mass equals the Arnowitt-Deser-Misner (ADM) mass in stationary asymptotically flat spacetimes in all dimensions, while this is no longer true with Kaluza-Klein asymptotics. Furthermore, we show that the Hamiltonian mass does not necessarily coincide with the ADM mass in Kaluza-Klein asymptotically flat spacetimes. A Witten-type argument is applied to derive global inequalities between the Hamiltonian energy-momentum and the Kaluza-Klein charges. Our formulae are applied to the five-dimensional Rasheed metrics, from which the corresponding global charges are computed. Furthermore, by a comparison of them with those of the Larsen metrics, we show that these classes of metrics are isometric.

We finish this thesis by a study of four-dimensional initial data with $\mathbb{R}^2 \times S^2$ topology in Kaluza-Klein theory, constructed by Brill and Pfister. The resulting spacetimes are particularly interesting because they have negative ADM mass. Those four-dimensional initial data contain a so-called bubble, causing this topology. We show that the initial data metric is non differentiable at the bubble, which leads to the question, how problematic the resulting singularity is. We show that the initial four-dimensional metric is at least twice weakly differentiable at this location, leading to a Riemann tensor without distributional components which could be responsible for the negativity of the ADM mass.

ZUSAMMENFASSUNG

Die Kaluza-Klein-Theorie stellt eine interessante Erweiterung der allgemeinen Relativitätstheorie auf fünf Raumzeitdimensionen dar, urprünglich hervorgegangen aus einem historischen Versuch, die damals bekannten Fundamentalkräfte der Natur zu vereinheitlichen. Die Eleganz dieser Theorie liegt darin, dass die damit verbundene Dimensionsreduzierung auf natürliche Weise zu Elektromagnetismus gekoppelt mit vierdimensionaler Gravitation führt, ohne dass es notwendig ist, einen entsprechenden Quellterm in den zugehörigen Einsteingleichungen einzuführen. Dieses Konzept kann auf höhere Dimensionen erweitert werden. Obwohl sie keine realistische Beschreibung der Natur darstellt, so ermöglicht die Kaluza-Klein-Theorie interessante Einsichten in fortgeschrittene Theorien, wie der String-Theorie, und stellt eine geometrische Verbindung zwischen höher- und niedrigerdimensionalen Theorien dar. Innerhalb dieser Theorie existierten Raumzeiten, die schwarze Löcher beschreiben. Eine interessante Klasse von schwarzen Löchern, innerhalb der Kaluza-Klein-Theorie, wurde unabhängig von Rasheed und Larsen gefunden, welche eine Familie axisymmetrischer, rotierender, dynonischer schwarzer Löcher beschreiben. Wir analysieren die von Rasheed gefundenen Lösungen in verschiedenen Grenzwerten ihrer Parameterfamilie, beweisen ihre Regularität an ihrem äußeren Killinghorizont, identifizieren und analysieren Singularitäten der Metrik und leiten Bedingungen her, unter denen diese durch den äußeren Killinghorizont abgeschirmt werden, schließen die Existenz von regulären Metriken ohne Killinghorizonte aus und leiten ein Kriterium für stabile Kausalität in der d.o.c. her. Im Rahmen der Kaluza-Klein-Theorie, wie in jeder anderen physikalischen Theorie, nehmen die Begriffe Gesamtenergie, Impuls und andere globalen Ladungen eine Schlüsselrolle ein. In unserer Analyse betrachten wir asymptotisch Anti-de Sitter-, asymptotisch flache- und Kaluza-Klein asymptotisch flache Raumzeiten. Wir beweisen, dass die Komarmasse und die ADM Masse in stationären, asymptotisch flachen Raumzeiten in beliebigen Dimensionen äquivalent sind. Weiters zeigen wir, dass die Hamiltonmasse nicht notwendigerweise äquivalent zur ADM Masse in Kaluza-Klein asymptotisch flachen Raumzeiten ist. Ein Argument nach Witten wird angewandt, um globale Ungleichungen zwischen dem Hamilton'schen Energie-Impuls und den Kaluza-Klein Ladungen herzuleiten. Wir wenden unsere Formeln auf die fünfdimensionale Rasheed-Metrik an, aus denen wir die entsprechenden globalen Ladungen berechnen. Durch einen Vergleich mit jenen der Larsen-Lösungen zeigen wird, dass die beiden Klassen von Metriken äquivalent sind. Zuletzt betrachten wir vierdimensionale Anfangsdaten mit $\mathbb{R}^2 \times S^2$ Topologie, konstruiert durch Brill und Pfister, in Kaluza-Klein-Theorie, welche speziell wegen ihrer negativen ADM Masse interessant sind. Diese vierdimensionalen Anfangsdaten enthalten eine sogenannte Bubble, die zu dieser Topologie führt. In einer sorgfältigen Analyse zeigen wir, dass die Metrik der Anfangsdaten nicht differenzierbar auf der Bubble ist, was uns zur Frage führt, wie problematisch diese Singularität ist. Wir zeigen, dass die Metrik der Anfangsdaten dort mindestens zweimal schwach differenzierbar ist.

AKNOWLEDGEMENTS

First of all, I would like to thank Piotr T. Chruściel for the supervision of this thesis, leading me to the topic of Kaluza-Klein theory, and most of all for his support and patience over the years. Furthermore, I would like to thank Hamed and Maciej for many helpful discussions and their collaboration, as well as Walter, David, Stefan, Natascha, Tobias, Thomas, Jerzy, Peter Aichelburg and Robert Beig for their time and the many pleasant conversations on and apart from the topic of general relativity, especially during lunch time, and our former group members Jérémie, Tim, Paul, Peter, Gernot and Maximilian.

Contents

1 Introduction

1.1 General relativity

General relativity is the geometric theory of gravitation, formulated by A. Einstein in 1915 [6]. In contrast to the Newtonian theory, where gravity is directly described as a force, it manifests in Einstein's theory as the consequence of the curvature of the spacetime, caused by the presence of matter. Freely falling test particles travel along geodesics through the curved spacetime, i.e. the shortest possible curve between two points of the space. This interplay is summarized in A. Wheeler's famous quote: "Space tells matter how to move, matter tells space how to curve.".
From the mathematical point of view, general relativity is formulated by the so-called Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}\,, \tag{1.1}$$

where $T_{\mu\nu}$ denotes the energy-momentum tensor, describing the distribution of energy and momentum in the spacetime, $R_{\mu\nu}$ is the Ricci tensor and R the Ricci scalar, which encode the curvature of the spacetime, defined over the metric tensor $g_{\mu\nu}$, which prescribes, how distances are measured locally in the spacetime. In four spacetime dimensions, when fully written out, the Einstein equations constitute a system of 10 nonlinear PDEs for the components of the metric tensor. General relativity predicts and describes effects, that cannot be understood by means of the Newtonian theory of gravity, among them the advance of the perihelion of Mercury, the deflection of light in a gravitational field and the emergence of gravitational waves, travelling through the spacetime.
The first class of metrics, providing a solution to the Einstein equations, was found by K. Schwarzschild in 1915, describing a spherically symmetric, asymptotically flat black hole in four dimensions. However, this metric and it's physical interpretation have been misunderstood for a long time. The prediction of general relativity, that light is deflected in a gravitational field, was confirmed, due to a observation by Eddington, by a shift of the observed positions of the stars during a solar ellipse in 1919. This confirmation has brought great interest to the field general relativity, as well as in the scientific community and in the public. After that phase, the interest in general relativity has been dormant for quite a long time, since the Newtonian theory provides an excellent approximation in the case of weak gravitational fields and general relativity had the reputation of being complicated and hard to understand. In the 30's J. Oppenheimer and H. Snyder [16] have been able to derive mathematically, based on the previous work of S. Chandrasekhar on neutron stars, that the ultimate fate of a dying star, running out of nuclear fuel, is a total gravitational collapse, resulting in a so-called black hole, provided the mass of the star is exceeding a certain limit. Later, in the 60's, the experimental discovery of pulsars and other compact X-ray sources with strong gravitational fields, requiring general relativity for an adequate description, and on the theoretical side the groundbreaking work of S. Hawking and R. Penrose and others on black holes, singularities and other important aspects of mathematical relativity caused a revival of Einstein's theory. Another source of interest emerged with theories, attempting to unifying gravity with the other three fundamental

forces of nature(electromagnetism, the weak- and the strong force), such as quantum gravity and string theory. The celebrated announcement of the detection of gravitational waves in 2016 [1] has put general relativity once more into the public spotlight, as a further impressive confirmation of it's predictions. Today, over 100 years after Einstein's fundamental work, general relativity is well established, explaining, or at least describing, every aspect of gravitational physics ever observed.

1.2 Black holes

Stars gain their energy from the process of nuclear fission. It theory, if the star is running out of nuclear fuel, a black hole forms under a total gravitational collapse, provided the mass M of the star is large enough, i.e. $M \gtrapprox 3 \times M_{\odot}$, where $M_{\odot}$ denotes the mass of our sun. The strongest evidence, that black holes do indeed exist as physical objects, has been provided recently by the Event Horizon Telescope(EHT)-cooperation [2]. Akiyama et al. have been able to obtain an image of plasma Figure 1.1, orbiting M87*, depicting the object itself therefore as a shadow. Further strong candidates for black holes are Cygnus X-1, the galactic

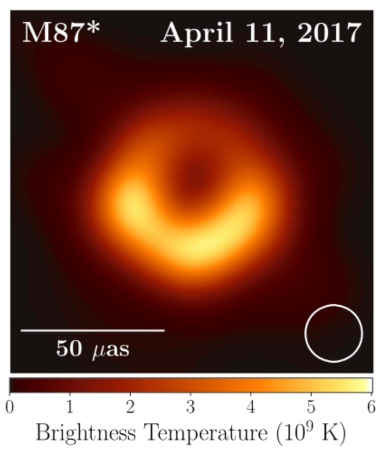

Figure 1.1: EHT image of M87* from observations on 2017 April 11

nuclei of NGC 4258 and the center of our own Milky Way.
Black hole spacetimes are predicted from the theory general relativity. As the main property of those spacetimes, an inner region, from which no matter or light can escape, called the black hole region, is separated from an outer region, called the domain of other communications(d.o.c.), through a so-called event horizon. The first black hole solution that has been discovered, as already mentioned, is the Schwarzschild metric. One year after, H. Reissner [18], G. Nordström and others extended this result to the electrovacuum case, nowadays called the Reissner-Nordström metric. In 1963 R. Kerr [12] derived a class of metrics, describing rotating black holes, later extended by Newman to the electrically charged case, finally describing black holes with mass, angular momentum and charge, called the Kerr-Newman family.

1.2.1 The four-dimensional case

In four dimensions, in the electrovacuum case, the picture, regarding the classification of asymptotically flat holes, is almost complete. The Hawking topology theorem asserts, that in four-dimensional black hole spacetimes the topology of the cross-sections of the event horizon is necessarily S^2. W. Israel [10] and later G. Bunting and A. Masood-ul Alam [4], under less restrictive requirements, have shown, that any four-dimensional static, asymptotically flat black hole is isometric to the Reissner-Nordström/Schwarzschild metrics. Later, in 2010, P. Chruściel and G. Galloway [5] have been able to remove the analyticity requirement of the metric, that was implicitly assumed in the previous proofs. It is widely believed that all stationary, asymptotically flat, sufficiently well-behaved, electrovacuum, four-dimensional black holes are isometric to the Kerr-Newman family. It should, however, be mentioned that the existing theorems still contain unsatisfactory assumptions on analyticity of the metric and connectedness of the event horizon.

1.2.2 The higher-dimensional case

In higher dimensions less is known regarding the topology and the uniqueness of black holes. After the discovery of the so-called black ring solutions, with $S^1 \times S^2$ horizon topology, by R. Emparan and H. Reall [7], the question on the restrictions on the topology of higher-dimensional black holes came up once more. This issue has been addressed by G. Galloway and R. Schoen [8], giving a generalization of Hawking's theorem to higher dimensions. In this work the authors show, that the cross-sections of the event horizon are of positive Yamabe type, i.e. admit metrics of positive scalar curvature. This is a much less restrictive statement in comparison to the four-dimensional case, and a full classification of the topology of higher-dimensional black holes is still open. Regarding the uniqueness of higher-dimensional black holes the territory is more or less open. In higher dimensions the no-hair theorem doesn't even hold in vacuum case, as in this setting, apart from the already mentioned black ring solutions, also the Myers-Perry [15] family additionally exists in five dimensions for example, for details see [9].

1.3 Kaluza-Klein theory

The Kaluza-Klein theory arose for a historic attempt by T. Kaluza [11] and O. Klein [13] to unify electromagnetism and gravity, starting from five-dimensional vacuum Einstein gravity, leading to a four-dimensional theory with a Maxwell- and a scalar field, called the dilaton field, therefore named Einstein-Maxwell-Dilaton theory, after a so-called dimensional reduction/compactification of the extradimension. This concept can also be extended to higher dimensions. Beside the relevance of Einstein-Maxwell-Dilaton theory in high energy physics, Kaluza-Klein theory gives further insights into the geometry of spacetimes, providing a geometric link between higher- and lower dimensional theories.

1.4 Kaluza-Klein black holes

Within Kaluza-Klein theory also black hole spacetimes exist, so-called Kaluza-Klein black holes. An interesting class of black hole solutions of five-dimensional Kaluza-Klein theory, being in the scope of this thesis, has been derived independently by D. Rasheed [17] and F. Larsen [14], via so-called solution generating methods, describing a family of axisymmetric, rotating, dyonic(magnetically- and electrically charged) black holes.

1.5 Energy in general relativity

The notion of total energy, momentum and other global charges play a key role in any physical theory. In general, the definition of the these quantities depends on the asymptotics of the corresponding spacetime. The most important applications are the no-hair theorem, i.e. that a black hole is fully determined via it's mass, charge and angular momentum, the positive energy theorem, and within the measurement of the energy transported by gravitational waves through the spacetime.

1.6 Outline

This thesis consists of three parts, of which we give an outline in the following:

- **Rasheed-Larsen black holes**

  We analyse the metrics presented by Rasheed and by Larsen in different limits of their parameter family, prove regularity at the outer Killing horizon, analyse and identify the singularities of the metrics and derive conditions, under which they are shielded by the outer Killing horizon, exclude the existence of regular metrics without horizons and derive a criterion for stable causality in the domain of outer communications. Furthermore, we analyse the asymptotic behaviour of the Rasheed-Larsen metrics and determine the corresponding global charges with our formulae developed in Section 4. Finally, we derive an isometric transformation, proving the equivalence of the metrics written by Rasheed and by Larsen. The fact that these metrics are isometric appears to be well known but, somewhat surprisingly, we have not been able to find this transformation in the literature.

  We list here some questions which remain to be answered in order to get a more complete understanding of the geometry of the Rasheed solutions:

Unanswered Questions	**Issue**
Do regular metrics without horizons exist in the $\lvert\mathscr{A}\rvert > 2$ or $\lvert\mathscr{B}\rvert > 2$ case?	The corresponding system of inequalities (2.62), (2.63) and (2.65) appears too hard to analyse in this case.
Do regular solutions with a double zero of B in the $a \neq 0$, $P \neq 0$ case in the d.o.c. exist?	It seems hard to show under which conditions the remainders of the polynomial divisions (2.145) are zero, which is necessary to answer this question.
Is stable causality also guaranteed in the $P \neq 0$ case?	This leads to the question if all zeros of the fourth-order polynomial (2.178) are located below the location r_+ of the outer Killing horizon. Due to the complexity of this problem in the general setting, it appears hard to derive a corresponding criterion. In the small $\lvert P\rvert$ case we have been able to answer this question positively for the equivalent Larsen metrics.

- **Energy in higher-dimensional spacetimes**

  We derive new expressions for the total Hamiltonian mass and the Komar mass in higher dimensions, in terms of the Riemann tensor, in asymptotically flat, Kaluza-Klein asymptotically flat, and asymptotically anti–de Sitter (AdS) spacetimes.
  Furthermore, we show that if the space-time is asymptotically flat, the Komar mass coincides with the ADM mass in all dimensions, generalizing the four-dimensional result of Beig. However, the quantities mentioned above differ from each other in the non-asymptotically flat setting in general. In line with our analysis, we derive formulae for the mass and momentum associated with asymptotically Anti-de Sitter spacetimes, generalising results by Ashtekar and Das, with stronger conditions required, in comparison to ours. Here it not only shows, that the ADM and Komar mass differ from each other in non-asymptotically flat space-times in general, but also from the Hamiltonian mass. Furthermore, a Witten-type argument is used to derive global inequalities between the Hamiltonian energy-momentum and the Kaluza-Klein charges. Finally, as a non-trivial example at hand to test our formulae, we apply our results to the metrics discovered by Rasheed, describing rotating, dyonic black holes in Kaluza-Klein theory.

We attach the following paper, the results of which are an integral part of this thesis:

”Energy in higher-dimensional spacetimes”,

Hamed Barzegar, Piotr T. Chruściel, and Michael Hörzinger

published in PHYSICAL REVIEW D **96**, 124002 (2017),

arXiv: 1708.03122

Abstract: We derive expressions for the total Hamiltonian energy of gravitating systems in higher-dimensional theories in terms of the Riemann tensor, allowing a cosmological constant $\Lambda \in \mathbb{R}$. Our analysis covers asymptotically anti–de Sitter spacetimes, asymptotically flat spacetimes, as well as Kaluza-Klein asymptotically flat spacetimes. We show that the Komar mass equals the Arnowitt-Deser-Misner (ADM) mass in stationary asymptotically flat spacetimes in all dimensions, generalizing the four-dimensional result of Beig, and that this is no longer true with Kaluza-Klein asymptotics.We show that the Hamiltonian mass does not necessarily coincide with the ADM mass in Kaluza-Klein asymptotically flat spacetimes, and that the Witten positivity argument provides a lower bound for the Hamiltonian mass and not for the ADM mass in terms of the electric charge. We illustrate our results on the five-dimensional Rasheed metrics, which we study in some detail, pointing out restrictions that arise from the requirement of regularity, which have gone seemingly unnoticed so far in the literature.

- **The Brill-Pfister initial data**

Brill and Pfister [3] have constructed initial data with $\mathbb{R}^2 \times S^2$ topology in Kaluza-Klein theory, which are particularly interesting because they have negative ADM mass. Those four-dimensional initial data contain a so-called bubble, causing this topology. A careful analysis shows, that the initial data metric is non differentiable at the bubble, arising the question how problematic the resulting singularity is. We show that the initial four-dimensional metric is at least twice weakly differentiable at this location, leading to a non-singular Riemann tensor. This fills a gap in the original paper, excluding the possibility that the negativity of the total energy could be caused by distributional negative energy density. We apply our formulae of Section 4 to this initial data metric, obtaining a negative ADM- and Hamiltonian mass as a upper bound for the energy, associated with this class of initial data, in accordance with the work of Brill and Pfister.

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Univ. Prof. Dr. Piotr T. Chruściel
Gravitationsphysik

May 30, 2021

PhD Committee
Faculty of Physics
University of Vienna

Dear Colleagues,

One of the chapters of the PhD thesis of Michael Hörzinger is a paper written jointly with me and Hamed Barzegar. Michael made substantial contributions to the writing of the paper and to the underlying calculations. In fact, his studies of the Rasheed-Larsen metrics prompted us to carry-out the analysis there, in order to provide a solid theoretical foundation to the calculation of global charges associated with the Rasheed-Larsen metrics.

Sincerely yours,

Piotr T. Chruściel
Professor of Gravitational Physics

2 The Rasheed solutions

The Rasheed-Larsen metrics are particularly noteworthy by providing an example of five dimensional solutions of vacuum Einstein equations with a non-product structure in the Kaluza-Klein directions. They have been discovered by Rasheed [17] and, independently by Larsen [14]. We will use the name "Rasheed metric" for the Rasheed-Larsen metric written in the original coordinates used by Rasheed, and "Larsen metric" when the coordinates in [14] are used.

The line element of the metrics in [17] is given by

$$ds^2_{(5)} = \frac{B}{A}\left(dx^4 + 2A_\mu dx^\mu\right)^2 + \sqrt{\frac{A}{B}}\, ds^2_{(4)}\,, \tag{2.1}$$

where we assume that

$$\begin{gathered} M^2+\Sigma^2-P^2-Q^2 \neq 0\,, \quad \left(M+\Sigma/\sqrt{3}\right)^2 - Q^2 \neq 0\,, \quad \left(M-\Sigma/\sqrt{3}\right)^2 - P^2 \neq 0\,, \\ M \pm \frac{\Sigma}{\sqrt{3}} \neq 0\,, \quad F^2 := \frac{\left[\left(M+\Sigma/\sqrt{3}\right)^2-Q^2\right]\left[\left(M-\Sigma/\sqrt{3}\right)^2-P^2\right]}{M^2+\Sigma^2-P^2-Q^2} > 0\,, \end{gathered} \tag{2.2}$$

and where

$$ds^2_{(4)} = -\frac{\Delta_\theta}{\sqrt{AB}}\left(dt + \omega^0{}_\phi d\phi\right)^2 + \frac{\sqrt{AB}}{\Delta}dr^2 + \sqrt{AB}\,d\theta^2 + \frac{\Delta\sqrt{AB}}{\Delta_\theta}\sin^2\theta\, d\phi^2\,, \tag{2.3}$$

with

$$\begin{aligned} A &= \left(r-\Sigma/\sqrt{3}\right)^2 - \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} + a^2\cos^2\theta + \frac{2JPQ\cos\theta}{\left(M+\Sigma/\sqrt{3}\right)^2 - Q^2}\,, \\ B &= \left(r+\Sigma/\sqrt{3}\right)^2 - \frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} + a^2\cos^2\theta - \frac{2JPQ\cos\theta}{\left(M-\Sigma/\sqrt{3}\right)^2 - P^2}\,, \\ \Delta_\theta &= r^2 - 2Mr + P^2 + Q^2 - \Sigma^2 + a^2\cos^2\theta\,, \\ \Delta &= r^2 - 2Mr + P^2 + Q^2 - \Sigma^2 + a^2\,, \\ \omega^0{}_\phi &= \frac{2J\sin^2\theta}{\Delta_\theta}(r+E)\,, \\ J^2 &= a^2F^2\,, \end{aligned} \tag{2.4}$$

whereas E is given by

$$E = -M + \frac{\left(M^2+\Sigma^2-P^2-Q^2\right)\left(M+\Sigma/\sqrt{3}\right)}{\left(M+\Sigma/\sqrt{3}\right)^2 - Q^2}\,. \tag{2.5}$$

The Maxwell field in (2.1) is given by

$$2A_\mu dx^\mu = \frac{C}{B}dt + \left(\omega^5{}_\phi + \frac{C}{B}\omega^0{}_\phi\right)d\phi\,, \tag{2.6}$$

where

$$C = 2Q\left(r-\Sigma/\sqrt{3}\right)-\frac{2PJ\cos\theta\left(M+\Sigma/\sqrt{3}\right)}{\left(M-\Sigma/\sqrt{3}\right)^2-P^2}\,, \tag{2.7}$$

$$\omega^5{}_\phi = \frac{H}{\Delta_\theta}\,, \tag{2.8}$$

and

$$H = 2P\Delta\cos\theta-\frac{2QJ\sin^2\theta\left[r\left(M-\Sigma/\sqrt{3}\right)+M\Sigma/\sqrt{3}+\Sigma^2-P^2-Q^2\right]}{\left[\left(M+\Sigma/\sqrt{3}\right)^2-Q^2\right]}\,. \tag{2.9}$$

In addition, the parameters (M,Σ,P,Q) have to fulfil the constraint

$$\frac{Q^2}{\Sigma+M\sqrt{3}}+\frac{P^2}{\Sigma-M\sqrt{3}}=\frac{2\Sigma}{3}\,, \tag{2.10}$$

otherwise the metric does not satisfy the five-dimensional vacuum field equations.

The inverse metric of (2.1) reads

$$\begin{aligned} g^{-1} = {} & \left(\frac{A}{B}-4\frac{B}{\Delta_\theta}A_t^2+\frac{4\Delta_\theta(A_\phi-A_t\omega^0{}_\phi)^2}{A\Delta}\frac{1}{\sin^2\theta}\right)(\partial_{x^4})^2 \\ & +4\left(\frac{B}{\Delta_\theta}A_t+\frac{\Delta_\theta\omega^0{}_\phi(A_\phi-A_t\omega^0{}_\phi)}{A\Delta}\frac{1}{\sin^2\theta}\right)\partial_t\partial_{x^4} \\ & -4\frac{\Delta_\theta(A_\phi-A_t\omega^0{}_\phi)}{A\Delta}\frac{1}{\sin^2\theta}\partial_\phi\partial_{x^4}+g_4^{-1}\,, \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} g_4^{-1} = {} & -\left(\frac{B}{\Delta_\theta}+\frac{(\omega^0{}_\phi)^2\Delta_\theta}{A\Delta}\frac{1}{\sin^2\theta}\right)(\partial_t)^2+\frac{\Delta}{A}(\partial_r)^2+\frac{1}{A}(\partial_\theta)^2 \\ & +\frac{\Delta_\theta}{A\Delta}\frac{1}{\sin^2\theta}(\partial_\phi)^2-2\frac{\Delta_\theta\omega^0{}_\phi}{A\Delta\sin^2\theta}\partial_t\partial_\phi\,. \end{aligned} \tag{2.12}$$

We have verified with SAGE, that the metric fulfils the five-dimensional vacuum field equations in the $P=0$ case, whereas in the $P\neq 0$ case, due to the complexity of the metric, we have have only been able to obtain a corresponding verification for a sample of parameters, fulfilling (2.10), with MATHEMATICA.

2.1 The "Kerr" case ($\Sigma = Q = P = 0$)

If we set $\Sigma = Q = P = 0$ in (2.4)-(2.2), we obtain for (2.3)

$$ds^2_{(4)} = -\frac{\Delta_\theta}{\sqrt{AB}}\left(dt + \omega^0{}_\phi d\phi\right)^2 + \frac{\sqrt{AB}}{\Delta}dr^2 + \sqrt{AB}d\theta^2 + \frac{\Delta\sqrt{AB}}{\Delta_\theta}\sin^2\theta d\phi^2\,, \tag{2.13}$$

where

$$\begin{aligned} A &= r^2 + a^2\cos^2\theta\,, \\ B &= r^2 + a^2\cos^2\theta\,, \\ \omega^0{}_\phi &= \frac{2J\sin^2\theta}{\Delta_\theta}r\,, \\ \Delta &= r^2 - 2Mr + a^2\,, \end{aligned} \tag{2.14}$$

with

$$\begin{aligned} \Delta_\theta &= r^2 - 2Mr + a^2\cos^2\theta\,, \\ J &= Ma\,. \end{aligned} \tag{2.15}$$

By introducing

$$\rho = r^2 + a^2cos^2\theta\,, \quad Z = \frac{2Mr}{\rho}\,, \tag{2.16}$$

we obtain

$$A = B = \rho\,, \quad \Delta_\theta = \rho - 2Mr\,. \tag{2.17}$$

With this conventions we obtain for the terms, appearing in the metric (2.13),

$$\frac{\Delta\sqrt{AB}}{\Delta_\theta} = \frac{\Delta\rho}{\rho - 2Mr} = \frac{\Delta}{\rho - \frac{2Mr}{\rho}} = \frac{\Delta}{1-Z}\,, \tag{2.18}$$

$$\sqrt{AB} = \rho\,, \tag{2.19}$$

$$\frac{\sqrt{AB}}{\Delta} = \frac{\rho}{\Delta}\,, \tag{2.20}$$

$$-\frac{\Delta_\theta}{\sqrt{AB}} = \frac{\rho - 2Mr}{\rho} = \frac{1}{1 - \frac{2Mr}{\rho}} = -\frac{1}{1-Z}\,, \tag{2.21}$$

$$\begin{aligned} \omega^0{}_\phi &= \frac{2Mra\sin^2\theta}{\Delta_\theta} \\ &= \frac{2Mar\sin^2\theta}{\rho - 2Mr} \\ &= \frac{\frac{2Mar}{\rho}\sin^2\theta r}{1 - \frac{2Mar}{\rho}} = \frac{Zsin^2\theta}{1-Z}\,. \end{aligned} \tag{2.22}$$

The insertion of (2.18)-(2.22) in (2.13) yields

$$ds^2_{(4)} = -(1-Z)\left(dt + \frac{aZ\sin^2\theta}{1-Z}d\phi\right)^2 + \frac{\rho}{\Delta}dr^2 + \rho d\theta^2 + \frac{\Delta}{1-Z}\sin^2\theta d\phi^2\,. \tag{2.23}$$

To show that this metric coincides with the standard Kerr metric in Boyer-Lindquist coordinates, we expand (2.23) and insert (2.16)

$$
\begin{aligned}
ds^2_{(4)} &= -(1-Z)\left(dt+\frac{aZ\sin^2\theta}{1-Z}d\phi\right)^2+\frac{\rho}{\Delta}dr^2+\rho d\theta^2+\frac{\Delta}{1-Z}\sin^2\theta d\phi^2 \\
&= -(1-Z)\left(dt^2+2\frac{aZ\sin^2\theta}{1-Z}dtd\phi+\left(\frac{aZ\sin^2\theta}{1-Z}\right)^2d\phi^2\right)^2+\frac{\rho}{\Delta}dr^2+\rho d\theta^2+\frac{\Delta}{1-Z}\sin^2\theta d\phi^2 \\
&= -(1-Z)\,dt^2-2aZ\sin^2\theta dtd\phi-\frac{\left(aZ\sin^2\theta\right)^2}{1-Z}d\phi^2+\frac{\rho}{\Delta}dr^2+\rho d\theta^2+\frac{\Delta}{1-Z}\sin^2\theta d\phi^2 \\
&= (-1+Z)\,dt^2-2aZ\sin^2\theta dtd\phi+\frac{\Delta\sin^2\theta-\left(aZ\sin^2\theta\right)^2}{1-Z}d\phi^2+\frac{\rho}{\Delta}dr^2+\rho d\theta^2 \\
&= \left(-1+\frac{2Mr}{\rho}\right)dt^2-2a\frac{2Mr}{\rho}\sin^2\theta dtd\phi+\frac{\Delta\sin^2\theta-\left(aZ\sin^2\theta\right)^2}{1-Z}d\phi^2 \\
&\quad +\frac{\rho}{\Delta}dr^2+\rho d\theta^2\,. \qquad (2.24)
\end{aligned}
$$

In the next step we write the $\phi\phi$-component of (2.24) in the form

$$
\begin{aligned}
\frac{\Delta\sin^2\theta-\left(aZ\sin^2\theta\right)^2}{1-Z} &= \sin^2\theta\frac{\Delta-a^2Z^2\sin^2\theta}{1-Z} \\
&= \sin^2\theta\frac{(\rho-2Mr+a^2\sin^2\theta)-a^2Z^2\sin^2\theta}{1-Z} \\
&= \sin^2\theta\frac{\rho-2Mr+a^2\sin^2\theta(1-Z^2)}{1-Z} \\
&= \sin^2\theta\left(\frac{\rho-2Mr}{(1-Z)}+\frac{a^2\sin^2\theta(1-Z^2)}{1-Z}\right) \\
&= \sin^2\theta\left(\frac{\rho-2Mr}{(1-\frac{2Mr}{\rho})}+a^2\sin^2\theta(1+Z)\right) \\
&= \sin^2\theta\left(\rho+a^2\sin^2\theta(1+Z)\right) \\
&= \sin^2\theta\left(\rho+a^2\sin^2\theta+a^2\sin^2\theta Z\right) \\
&= \sin^2\theta\left((r^2+a^2)+a^2\sin^2\theta\frac{2Mr}{\rho}\right). \qquad (2.25)
\end{aligned}
$$

Finally, by the insertion of (2.25) in (2.24), we obtain

$$
\begin{aligned}
ds^2_{(4)} &= \left(-1+\frac{2Mr}{\rho}\right)dt^2-2a\frac{2Mr}{\rho}\sin^2\theta dtd\phi+\sin^2\theta\left((r^2+a^2)+a^2sin^2\theta\frac{2Mr}{\rho}\right)d\phi^2+\frac{\rho}{\Delta}dr^2+\rho d\theta^2 \\
&= -dt^2+\frac{2Mr}{\rho}dt^2-2a\frac{2Mr}{\rho}\sin^2\theta dtd\phi+a^2\frac{2Mr}{\rho}sin^4\theta d\phi^2+\sin^2\theta(r^2+a^2)d\phi^2+\frac{\rho}{\Delta}dr^2+\rho d\theta^2 \\
&= -dt^2+\frac{2Mr}{\rho}dt^2-2a\frac{2Mr}{\rho}\sin^2\theta dtd\phi+a^2\frac{2Mr}{\rho}sin^4\theta d\phi^2+\sin^2\theta(r^2+a^2)d\phi^2+\frac{\rho}{\Delta}dr^2+\rho d\theta^2 \\
&= -dt^2+\frac{2Mr}{\rho}\left(dt-a\sin^2\theta d\phi\right)^2+\sin^2\theta(r^2+a^2)d\phi^2+\frac{\rho}{\Delta}dr^2+\rho d\theta^2\,, \qquad (2.26)
\end{aligned}
$$

where the last line represents the standard Kerr metric in Boyer-Lindquist coordinates. (2.6)-(2.9) with $P=0$ and $Q=0$ imply $A_t=0$, $A_\phi=0$. Thus in the

$\Sigma = 0$, $P = 0$, $Q = 0$ case we obtain from (2.1), together with (2.26), for the corresponding five-dimensional line element

$$\begin{aligned} ds^2_{(5)} &= -dt^2 + \frac{2Mr}{\rho}\left(dt - a\sin^2\theta d\phi\right)^2 \\ &\quad + \sin^2\theta(r^2+a^2)d\phi^2 + \frac{\rho}{\Delta}dr^2 + \rho d\theta^2 + (dx^4)^2 . \end{aligned} \tag{2.27}$$

2.2 The $a = 0$ case

In the $a = 0$ case (2.4) reduces to

$$\begin{aligned} A &= \left(r - \Sigma/\sqrt{3}\right)^2 - \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}}, \quad B = \left(r + \Sigma/\sqrt{3}\right)^2 - \frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}}, \\ \Delta &= \Delta_\theta = r^2 - 2Mr + P^2 + Q^2 - \Sigma^2, \quad \omega^0{}_\phi = 0 . \end{aligned} \tag{2.28}$$

The Maxwell field (2.6) is then given by

$$2A_\mu dx^\mu = \frac{C}{B}dt + \omega^5{}_\phi d\phi , \tag{2.29}$$

where now (2.8) and (2.9) take the form

$$C = 2Q\left(r - \Sigma/\sqrt{3}\right), \quad \omega^5{}_\phi = 2P\cos\theta . \tag{2.30}$$

In addition the parameters (M, Σ, P, Q) have to fulfil (2.10), i.e.

$$\frac{Q^2}{\Sigma + M\sqrt{3}} + \frac{P^2}{\Sigma - M\sqrt{3}} = \frac{2\Sigma}{3} . \tag{2.31}$$

Inserting (2.28) -(2.30) in (2.1) yields

$$\begin{aligned} ds^2 &= -\frac{\frac{4Q^2}{\frac{18P^2\Sigma}{\left(\Sigma-\sqrt{3}M\right)\left(\sqrt{3}\Sigma-3r\right)^2}-1} + \Delta}{\left(r+\frac{\Sigma}{\sqrt{3}}\right)^2 - \frac{2Q^2\Sigma}{\sqrt{3}M+\Sigma}} dt^2 \\ &\quad + \frac{\frac{2P^2\Sigma}{\sqrt{3}M-\Sigma} + \left(r - \frac{\Sigma}{\sqrt{3}}\right)^2}{\Delta} dr^2 + \left(\frac{2P^2\Sigma}{\sqrt{3}M-\Sigma} + \left(r-\frac{\Sigma}{\sqrt{3}}\right)^2\right) d\theta^2 \\ &\quad + \left(\frac{4P^2\cos^2(\theta)\left(\left(r+\frac{\Sigma}{\sqrt{3}}\right)^2 - \frac{2Q^2\Sigma}{\sqrt{3}M+\Sigma}\right)}{\frac{2P^2\Sigma}{\sqrt{3}M-\Sigma} + \left(r-\frac{\Sigma}{\sqrt{3}}\right)^2} + \sin^2(\theta)\left(\frac{2P^2\Sigma}{\sqrt{3}M-\Sigma} + \left(r-\frac{\Sigma}{\sqrt{3}}\right)^2\right)\right) d\phi^2 \\ &\quad + \frac{\left(r+\frac{\Sigma}{\sqrt{3}}\right)^2 - \frac{2Q^2\Sigma}{\sqrt{3}M+\Sigma}}{\frac{2P^2\Sigma}{\sqrt{3}M-\Sigma} + \left(r-\frac{\Sigma}{\sqrt{3}}\right)^2}(dx^4)^2 + \frac{8PQ\cos(\theta)\left(r-\frac{\Sigma}{\sqrt{3}}\right)}{\frac{2P^2\Sigma}{\sqrt{3}M-\Sigma} + \left(r-\frac{\Sigma}{\sqrt{3}}\right)^2} dt d\phi \\ &\quad + \frac{4Q\left(r-\frac{\Sigma}{\sqrt{3}}\right)}{\frac{2P^2\Sigma}{\sqrt{3}M-\Sigma} + \left(r-\frac{\Sigma}{\sqrt{3}}\right)^2} dt dx^4 + \frac{4P\cos(\theta)\left(\left(r+\frac{\Sigma}{\sqrt{3}}\right)^2 - \frac{2Q^2\Sigma}{\sqrt{3}M+\Sigma}\right)}{\frac{2P^2\Sigma}{\sqrt{3}M-\Sigma} + \left(r-\frac{\Sigma}{\sqrt{3}}\right)^2} d\phi dx^4 . \end{aligned} \tag{2.32}$$

All results of the geometric analysis of the generic case apply directly to the $a = 0$ case. In this special case it holds as well, that the d.o.c. is non-singular if and only if the zeros of A and B are located below the outer Killing horizon, cf. Lemma 2.5. It also follows directly from Lemma 2.6 that (2.32) is stably causal.

2.3 The metric in the $M \to \pm \frac{\Sigma}{\sqrt{3}}$ limit

In the $M \to \pm \frac{\Sigma}{\sqrt{3}}$ limit the line element (2.1) becomes singular. This arises from $M \pm \frac{\Sigma}{\sqrt{3}}$ expressions, appearing in the denominators of terms of the metric components, in particular in A and B, given by (2.4). We parametrize either P^2 by the curve

$$P^2 = \lambda \left(M - \frac{\Sigma}{\sqrt{3}} \right), \tag{2.33}$$

or Q^2 by the curve

$$Q^2 = \lambda \left(M + \frac{\Sigma}{\sqrt{3}} \right), \tag{2.34}$$

where λ is the affine parameter. Under those parametrizations the expressions, leading to a singularity in the $M \to \frac{\Sigma}{\sqrt{3}}$ or $M \to -\frac{\Sigma}{\sqrt{3}}$ limit respectively, in the second term of A or B are cancelled out. By inserting F and J, defined by (2.2) and (2.4), in (2.4), (2.5), (2.7) and (2.9), we obtain the terms

$$\begin{aligned}
A &= \left(r - \frac{\Sigma}{\sqrt{3}} \right)^2 - \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} a^2 \cos^2\theta \pm 2aPQ\cos\theta \sqrt{\frac{\left(M - \frac{\Sigma}{\sqrt{3}} \right)^2 - P^2}{(M^2+\Sigma^2-P^2-Q^2)\left(\left(M + \frac{\Sigma}{\sqrt{3}} \right)^2 - Q^2 \right)}}, \\
B &= \left(r + \frac{\Sigma}{\sqrt{3}} \right)^2 - \frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} + a^2 \cos^2\theta \mp 2aPQ\cos\theta \sqrt{\frac{\left(M + \frac{\Sigma}{\sqrt{3}} \right)^2 - Q^2}{(M^2+\Sigma^2-P^2-Q^2)\left(\left(M - \frac{\Sigma}{\sqrt{3}} \right)^2 - P^2 \right)}}, \\
C &= 2Q\left(r - \frac{\Sigma}{\sqrt{3}} \right) \mp 2aP\cos\theta \sqrt{\frac{\left(\left(M + \frac{\Sigma}{\sqrt{3}} \right)^2 - Q^2 \right)\left(M + \frac{\Sigma}{\sqrt{3}} \right)^2}{(M^2+\Sigma^2-P^2-Q^2)\left(\left(M - \frac{\Sigma}{\sqrt{3}} \right)^2 - P^2 \right)}}, \\
H &= 2P\Delta\cos\theta \mp 2a\sin^2\theta \left[r\left(M - \frac{\Sigma}{\sqrt{3}} \right) + M\frac{\Sigma}{\sqrt{3}} + \Sigma^2 - P^2 - Q^2 \right] \times \\
&\quad Q\sqrt{\frac{\left(\left(M - \frac{\Sigma}{\sqrt{3}} \right)^2 - P^2 \right)}{(M^2+\Sigma^2-P^2-Q^2)\left(\left(M + \frac{\Sigma}{\sqrt{3}} \right)^2 - Q^2 \right)}}, \\
F^2 &= \frac{\left[\left(M + \frac{\Sigma}{\sqrt{3}} \right)^2 - Q^2 \right]\left[\left(M - \frac{\Sigma}{\sqrt{3}} \right)^2 - P^2 \right]}{M^2+\Sigma^2-P^2-Q^2}, \\
E &= -M + \frac{(M^2+\Sigma^2-P^2-Q^2)\left(M + \frac{\Sigma}{\sqrt{3}} \right)}{\left(M + \frac{\Sigma}{\sqrt{3}} \right)^2 - Q^2},
\end{aligned} \tag{2.35}$$

which are crucial for the analysis of the metric in the limits outlined above.

2.3.1 The $P^2=\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)$, $\Sigma\to\sqrt{3}M$ case

Under the parametrization $P^2=\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)$ (2.35) takes the form

$$
\begin{aligned}
A &= \left(r-\frac{\Sigma}{\sqrt{3}}\right)^2+\frac{2\Sigma\lambda}{\sqrt{3}}+a^2\cos^2\theta\,, \\
&\quad \pm 2a\sqrt{\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)}Q\cos\theta\sqrt{\frac{\left(M-\frac{\Sigma}{\sqrt{3}}\right)^2-\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)}{\left(M^2+\Sigma^2-\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)-Q^2\right)\left(\left(M+\frac{\Sigma}{\sqrt{3}}\right)^2-Q^2\right)}}\,, \\
B &= \left(r+\frac{\Sigma}{\sqrt{3}}\right)^2-\frac{2Q^2\Sigma}{\Sigma+M\sqrt{3}}+a^2\cos^2\theta \\
&\quad \mp 2a\sqrt{\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)}Q\cos\theta\sqrt{\frac{\left(M+\frac{\Sigma}{\sqrt{3}}\right)^2-Q^2}{\left(M^2+\Sigma^2-\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)-Q^2\right)\left(\left(M-\frac{\Sigma}{\sqrt{3}}\right)^2-\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)\right)}} \\
&= \left(r+\frac{\Sigma}{\sqrt{3}}\right)^2-\frac{2Q^2\Sigma}{\Sigma+M\sqrt{3}}+a^2\cos^2\theta\,, \\
&\quad \mp 2aQ\cos\theta\sqrt{\frac{\left(\left(M+\frac{\Sigma}{\sqrt{3}}\right)^2-Q^2\right)\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)}{\left(M^2+\Sigma^2-\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)-Q^2\right)\left(\left(M-\frac{\Sigma}{\sqrt{3}}\right)^2-\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)\right)}} \\
&= \left(r+\frac{\Sigma}{\sqrt{3}}\right)^2-\frac{2Q^2\Sigma}{\Sigma+M\sqrt{3}}+a^2\cos^2\theta \\
&\quad \mp 2aQ\cos\theta\sqrt{\frac{\left(\left(M+\frac{\Sigma}{\sqrt{3}}\right)^2-Q^2\right)\lambda}{\left(M^2+\Sigma^2-\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)-Q^2\right)\left(\left(M-\frac{\Sigma}{\sqrt{3}}\right)-\lambda\right)}}\,, \\
C &= 2Q\left(r-\frac{\Sigma}{\sqrt{3}}\right)\mp 2a\sqrt{\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)}\cos\theta\sqrt{\frac{\left(\left(M+\frac{\Sigma}{\sqrt{3}}\right)^2-Q^2\right)\left(M+\frac{\Sigma}{\sqrt{3}}\right)^2}{\left(M^2+\Sigma^2-\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)-Q^2\right)\left(\left(M-\frac{\Sigma}{\sqrt{3}}\right)^2-\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)\right)}} \\
&= 2Q\left(r-\frac{\Sigma}{\sqrt{3}}\right)\mp 2a\cos\theta\sqrt{\frac{\left(\left(M+\frac{\Sigma}{\sqrt{3}}\right)^2-Q^2\right)\left(M+\frac{\Sigma}{\sqrt{3}}\right)^2\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)}{\left(M^2+\Sigma^2-\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)-Q^2\right)\left(\left(M-\frac{\Sigma}{\sqrt{3}}\right)^2-\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)\right)}} \\
&= 2Q\left(r-\frac{\Sigma}{\sqrt{3}}\right)\mp 2a\cos\theta\sqrt{\frac{\left(\left(M+\frac{\Sigma}{\sqrt{3}}\right)^2-Q^2\right)\left(M+\frac{\Sigma}{\sqrt{3}}\right)^2\lambda}{\left(M^2+\Sigma^2-\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)-Q^2\right)\left(\left(M-\frac{\Sigma}{\sqrt{3}}\right)-\lambda\right)}}\,, \\
H &= 2\sqrt{\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)}\Delta\cos\theta\mp 2a\sin^2\theta\left[r\left(M-\frac{\Sigma}{\sqrt{3}}\right)+M\frac{\Sigma}{\sqrt{3}}+\Sigma^2-\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)-Q^2\right]\times
\end{aligned}
$$

$$
\begin{aligned}
& Q\sqrt{\frac{\left(\left(M-\frac{\Sigma}{\sqrt{3}}\right)^2-\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)\right)}{\left(M^2+\Sigma^2-\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)-Q^2\right)\left(\left(M+\frac{\Sigma}{\sqrt{3}}\right)^2-Q^2\right)}}\,,\\
F^2 &= \frac{\left[\left(M+\frac{\Sigma}{\sqrt{3}}\right)^2-Q^2\right]\left[\left(M-\frac{\Sigma}{\sqrt{3}}\right)^2-\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)\right]}{M^2+\Sigma^2-\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)-Q^2}\,,\\
E &= -M+\frac{\left(M^2+\Sigma^2-\lambda\left(M-\frac{\Sigma}{\sqrt{3}}\right)-Q^2\right)\left(M+\frac{\Sigma}{\sqrt{3}}\right)}{\left(M+\frac{\Sigma}{\sqrt{3}}\right)^2-Q^2}\,.
\end{aligned}
\tag{2.36}
$$

In the limit $\Sigma\to\sqrt{3}M$ (2.36) gives

$$
\begin{aligned}
\lim_{\Sigma\to\sqrt{3}M} A &= (r-M)^2+2\lambda M+a^2\cos^2\theta\,,\\
\lim_{\Sigma\to\sqrt{3}M} B &= (r+M)^2-Q^2+a^2\cos^2\theta\mp 2aQ\cos\theta\sqrt{-\frac{4M^2-Q^2}{4M^2-Q^2}}\\
&= (r+M)^2-Q^2+a^2\cos^2\theta\mp 2iaQ\cos\theta\,,\\
\lim_{\Sigma\to\sqrt{3}M} C &= 2Q\,(r-M)\mp 2a\cos\theta\sqrt{-\frac{4M^2(4M^2-Q^2)}{4M^2-Q^2}}\\
&= 2Q\,(r-M)\mp 4iaM\cos\theta\,,\\
\lim_{\Sigma\to\sqrt{3}M} H &= 0\,,\\
\lim_{\Sigma\to\sqrt{3}M} F^2 &= 0\,,\\
\lim_{\Sigma\to\sqrt{3}M} E &= -M+\frac{\left(4M^2-Q^2\right)2M}{4M^2-Q^2}=M\,.
\end{aligned}
\tag{2.37}
$$

Furthermore, the constraint (2.10) reduces in this limit to

$$
\frac{Q^2}{2M}-\lambda=2M\,. \tag{2.38}
$$

Together with (2.1) we conclude that metric is real, apart from the massless Kerr metric case ($Q=0$ and $M=0$), if and only if $a=0$. In this case (2.37) yields

$$
\begin{aligned}
A &= (r-M)^2+2\lambda M\,,\qquad B=(r+M)^2-Q^2\,,\\
&= r^2-2Mr+Q^2-3M^2\,,\\
\Delta &= \Delta_\theta=r^2-2Mr+Q^2-3M^2\,,\quad \omega^0{}_\phi=0\,,
\end{aligned}
\tag{2.39}
$$

and for the Maxwell field

$$
2A_\mu dx^\mu=\frac{C}{B}dt+\omega^5{}_\phi d\phi\,, \tag{2.40}
$$

with

$$
C \;=\; 2Q\,(r-M)\,,\quad \omega^5{}_\phi=0\,. \tag{2.41}
$$

The insertion of the expressions above in (2.1) yields

$$\begin{aligned} ds^2 &= -\frac{(r-3M)^2-Q^2}{\Delta}dt^2+dr^2+\Delta d\theta^2 \\ &\quad +\Delta\sin^2\theta d\phi^2+\frac{(r+M)^2-Q^2}{\Delta}(dx^4)^2+\frac{4Q(r-M)}{\Delta}dx^4dt. \end{aligned} \tag{2.42}$$

By calculating corresponding resultants, it follows that the numerators of g_{tt} and g_{44}, representing the norms of the Killing fields ∂_t and ∂_{x^4} respectively, factorize in Δ, necessary to avoid singularities, requires $Q=\pm 2M$. With this choice for Q (2.42) reads as

$$\begin{aligned} ds^2 &= \left(-1+\frac{4M}{r-M}\right)dt^2+dr^2+(r-M)^2d\theta^2+(r-M)^2\sin^2\theta d\phi^2 \\ &\quad +\left(1+\frac{4M}{r-M}\right)(dx^4)^2\pm\frac{4M}{r-M}dx^4dt, \end{aligned} \tag{2.43}$$

from which it follows, that the metric is singular at $r=M$.

2.3.2 The $Q^2=\lambda\left(M+\frac{\Sigma}{\sqrt{3}}\right)$, $\Sigma\to\sqrt{3}M$ case

Under the parametrization $Q^2=\lambda\left(M+\frac{\Sigma}{\sqrt{3}}\right)$ (2.35) takes the form

$$\begin{aligned} A &= \left(r-\frac{\Sigma}{\sqrt{3}}\right)^2-\frac{2P^2\Sigma}{\Sigma-M\sqrt{3}}+a^2\cos^2\theta \\ &\quad +2a\sqrt{\lambda\left(M+\frac{\Sigma}{\sqrt{3}}\right)}P\cos\theta\sqrt{\frac{\left(M-\frac{\Sigma}{\sqrt{3}}\right)^2-P^2}{\left(M^2+\Sigma^2-P^2-\lambda\left(M+\frac{\Sigma}{\sqrt{3}}\right)\right)\left(\left(M+\frac{\Sigma}{\sqrt{3}}\right)^2-\lambda\left(M+\frac{\Sigma}{\sqrt{3}}\right)\right)}} \\ &= \left(r-\frac{\Sigma}{\sqrt{3}}\right)^2-\frac{2P^2\Sigma}{\Sigma-M\sqrt{3}}+a^2\cos^2\theta \\ &\quad \pm 2aP\cos\theta\sqrt{\frac{\left(\left(M-\frac{\Sigma}{\sqrt{3}}\right)^2-P^2\right)\lambda\left(M+\frac{\Sigma}{\sqrt{3}}\right)}{\left(M^2+\Sigma^2-P^2-\lambda\left(M+\frac{\Sigma}{\sqrt{3}}\right)\right)\left(\left(M+\frac{\Sigma}{\sqrt{3}}\right)^2-\lambda\left(M+\frac{\Sigma}{\sqrt{3}}\right)\right)}} \\ &= \left(r-\frac{\Sigma}{\sqrt{3}}\right)^2-\frac{2P^2\Sigma}{\Sigma-M\sqrt{3}}+a^2\cos^2\theta \\ &\quad \pm 2aP\cos\theta\sqrt{\frac{\left(\left(M-\frac{\Sigma}{\sqrt{3}}\right)^2-P^2\right)\lambda}{\left(M^2+\Sigma^2-P^2-\lambda\left(M+\frac{\Sigma}{\sqrt{3}}\right)\right)\left(\left(M+\frac{\Sigma}{\sqrt{3}}\right)-\lambda\right)}}, \\ B &= \left(r+\frac{\Sigma}{\sqrt{3}}\right)^2-\frac{2\lambda\Sigma}{\sqrt{3}}+a^2\cos^2\theta \\ &\quad \mp 2a\sqrt{\lambda\left(M+\frac{\Sigma}{\sqrt{3}}\right)}P\cos\theta\sqrt{\frac{\left(M+\frac{\Sigma}{\sqrt{3}}\right)^2-\lambda\left(M+\frac{\Sigma}{\sqrt{3}}\right)}{\left(M^2+\Sigma^2-P^2-\lambda\left(M+\frac{\Sigma}{\sqrt{3}}\right)\right)\left(\left(M-\frac{\Sigma}{\sqrt{3}}\right)^2-P^2\right)}}, \end{aligned}$$

$$
\begin{aligned}
C &= 2\sqrt{\lambda\left(M+\frac{\Sigma}{\sqrt{3}}\right)}\left(r-\frac{\Sigma}{\sqrt{3}}\right)\mp 2aP\cos\theta\sqrt{\frac{\left(\left(M+\frac{\Sigma}{\sqrt{3}}\right)^2-\lambda\left(M+\frac{\Sigma}{\sqrt{3}}\right)\right)\left(M+\frac{\Sigma}{\sqrt{3}}\right)^2}{\left(M^2+\Sigma^2-P^2-\lambda\left(M+\frac{\Sigma}{\sqrt{3}}\right)\right)\left(\left(M-\frac{\Sigma}{\sqrt{3}}\right)^2-P^2\right)}}\,,\\
H &= 2P\Delta\cos\theta\mp 2a\sin^2\theta\left[r\left(M-\frac{\Sigma}{\sqrt{3}}\right)+M\frac{\Sigma}{\sqrt{3}}+\Sigma^2-P^2-\lambda\left(M+\frac{\Sigma}{\sqrt{3}}\right)\right]\times\\
&\quad\sqrt{\frac{\left(\left(M-\frac{\Sigma}{\sqrt{3}}\right)^2-P^2\right)\lambda\left(M+\frac{\Sigma}{\sqrt{3}}\right)}{\left(M^2+\Sigma^2-P^2-\lambda\left(M+\frac{\Sigma}{\sqrt{3}}\right)\right)\left(\left(M+\frac{\Sigma}{\sqrt{3}}\right)^2-\lambda\left(M+\frac{\Sigma}{\sqrt{3}}\right)\right)}}\\
&= 2P\Delta\cos\theta\mp 2a\sin^2\theta\left[r\left(M-\frac{\Sigma}{\sqrt{3}}\right)+M\frac{\Sigma}{\sqrt{3}}+\Sigma^2-P^2-\lambda\left(M+\frac{\Sigma}{\sqrt{3}}\right)\right]\times\\
&\quad\sqrt{\frac{\left(\left(M-\frac{\Sigma}{\sqrt{3}}\right)^2-P^2\right)\lambda}{\left(M^2+\Sigma^2-P^2-\lambda\left(M+\frac{\Sigma}{\sqrt{3}}\right)\right)\left(\left(M+\frac{\Sigma}{\sqrt{3}}\right)-\lambda\right)}}\,,\\
F^2 &= \frac{\left[\left(M+\frac{\Sigma}{\sqrt{3}}\right)^2-\lambda\left(M+\frac{\Sigma}{\sqrt{3}}\right)\right]\left[\left(M-\frac{\Sigma}{\sqrt{3}}\right)^2-P^2\right]}{M^2+\Sigma^2-P^2-\lambda\left(M+\frac{\Sigma}{\sqrt{3}}\right)}\,,\\
E &= -M+\frac{\left(M^2+\Sigma^2-P^2-\lambda\left(M+\frac{\Sigma}{\sqrt{3}}\right)\right)\left(M+\frac{\Sigma}{\sqrt{3}}\right)}{\left(M+\frac{\Sigma}{\sqrt{3}}\right)^2-\lambda\left(M+\frac{\Sigma}{\sqrt{3}}\right)}\,. \qquad (2.44)
\end{aligned}
$$

In the limit $\Sigma\to-\sqrt{3}M$ we obtain

$$
\begin{aligned}
\lim_{\Sigma\to-\sqrt{3}M} A &= (r+M)^2-P^2+a^2\cos^2\theta\pm 2aP\cos\theta\sqrt{-\frac{4M^2-P^2}{4M^2-P^2}}\\
&= (r+M)^2-P^2+a^2\cos^2\theta\pm 2iaP\cos\theta\,,\\
\lim_{\Sigma\to-\sqrt{3}M} B &= (r-M)^2+2\lambda M+a^2\cos^2\theta\,,\\
\lim_{\Sigma\to-\sqrt{3}M} C &= 0\,,\\
\lim_{\Sigma\to-\sqrt{3}M} H &= 2P\Delta\cos\theta\mp 2ia\sin^2\theta\big(2Mr-P^2\big)\,,\\
\lim_{\Sigma\to-\sqrt{3}M} F^2 &= 0\,,\\
\lim_{\Sigma\to-\sqrt{3}M} E &= -M-\frac{4M^2-P^2}{\lambda}\,. \qquad (2.45)
\end{aligned}
$$

Furthermore, the constraint (2.10) reduces in this limit to

$$
\lambda-\frac{P^2}{2M}=-2M\,. \qquad (2.46)
$$

Together with (2.1) we conclude that metric is real, apart from the massless Kerr metric case($P=0$ and $M=0$), if and only if $a=0$. In this case we obtain from (2.45)

$$\begin{aligned} A &= (r+M)^2 - P^2\,, \qquad B = (r-M)^2 + 2\lambda M\,, \\ & \qquad\qquad\qquad\qquad = r^2 - 2Mr + P^2 - 3M^2\,, \\ \Delta &= \Delta_\theta = r^2 - 2Mr + P^2 - 3M^2\,, \quad {\omega^0}_\phi = 0\,, \end{aligned} \tag{2.47}$$

and that the Maxwell field is then given by

$$2A_\mu dx^\mu = \frac{C}{B} dt + {\omega^5}_\phi d\phi, \tag{2.48}$$

where

$$C = 0\,, \quad {\omega^5}_\phi = 2P\cos\theta\,. \tag{2.49}$$

The insertion of the previous expressions in (2.1) yields

$$\begin{aligned} ds^2 &= -dt^2 + \frac{\Xi}{\Delta} dr^2 + \Xi d\theta^2 + \left(\frac{4P^2 \Delta \cos^2\theta}{\Xi} + \Xi \sin^2\theta \right) d\phi^2 \\ & \quad + \frac{\Delta}{\Xi} (dx^4)^2 + \frac{4P\cos\theta\Delta}{\Xi} d\phi dx^4\,, \end{aligned} \tag{2.50}$$

where $\Xi := (r+M)^2 - P^2$. By calculating a corresponding resultant, it follows that the numerator Δ of g_{44} factorizes in Ξ, necessary to avoid singularities, if and only if $P = \pm 2M$. With this choice for P (2.50) takes the form

$$\begin{aligned} ds^2 &= -dt^2 + \left(1 + \frac{4M}{r-M}\right) dr^2 + \left((r+M)^2 - 4M^2\right) d\theta^2 \\ & \quad + \frac{(r-M)\left(16M^2\cos^2(\theta) + \sin^2(\theta)(r+3M)^2\right)}{r+3M} d\phi^2 \\ & \quad + \left(1 - \frac{4M}{r+3M}\right)(dx^4)^2 \pm \frac{8M\cos(\theta)(r-M)}{r+3M} d\phi dx^4\,, \end{aligned} \tag{2.51}$$

from which it follows, that the metric is singular at $r = -3M$.

2.3.3 The $P = 0$, $\Sigma \to \sqrt{3}M$ case

In the $P = 0$, $\Sigma \to \sqrt{3}M$ case the line element (2.1) reduces to

$$\begin{aligned} ds^2_{(5)} &= \left(-1 + \frac{4M(r-M)}{\Delta_{P=0}}\right) dt^2 + \frac{\Delta_{P=0}}{(r-M)^2 + a^2} dr^2 \\ & \quad + \Delta_{P=0} d\theta^2 + \left((r-M)^2 + a^2\right)\sin^2(\theta) d\phi^2 \\ & \quad + \left(1 + \frac{4M(r-M)}{\Delta_{P=0}}\right)(dx^4)^2 \pm \frac{8M(r-M)}{\Delta_{P=0}} dt dx^4\,, \end{aligned} \tag{2.52}$$

where

$$\Delta_{P=0} := (r-M)^2 + a^2\cos^2(\theta)\,, \tag{2.53}$$

and all previous constraints are satisfied, so that both M and a are unconstrained parameters. A computation with MATHEMATICA confirms, that (2.52) is a solution of the five-dimensional vacuum field equations.

By rewriting (2.52) in the form

$$
\begin{aligned}
ds^2_{(5)} &= \left(-1+\frac{4M(r-M)}{\Delta_{P=0}}\right)dt^2+\frac{\Delta_{P=0}}{(r-M)^2+a^2}dr^2 \\
&\quad +\left(1+\frac{4M(r-M)}{\Delta_{P=0}}\right)(dx^4)^2 \pm \frac{8M(r-M)}{\Delta_{P=0}}dt dx^4 \\
&\quad +\Delta_{P=0}(d\theta^2+\sin^2(\theta)d\phi^2) \\
&\quad +\left((r-M)^2+a^2-\Delta_{P=0}\right)\sin^2(\theta)d\phi^2 \\
&= \left(-1+\frac{4M(r-M)}{\Delta_{P=0}}\right)dt^2+\frac{\Delta_{P=0}}{(r-M)^2+a^2}dr^2 \\
&\quad +\left(1+\frac{4M(r-M)}{\Delta_{P=0}}\right)(dx^4)^2 \pm \frac{8M(r-M)}{\Delta_{P=0}}dt dx^4 \\
&\quad +\underbrace{\Delta_{P=0}d\Omega^2+a^2\sin^4(\theta)d\phi^2}_{\text{smooth if } r\neq 0 \text{ and } \Delta_{P=0}\neq 0}\,,
\end{aligned}
\tag{2.54}
$$

we conclude that the metric is smooth at the rotation axes $\sin\theta = 0$ away from the set $\Delta_{P=0} = 0$. From (2.53)

$$
\Delta_{P=0} = 0 \Longleftrightarrow r = M \quad \text{and} \quad \cos\theta = 0\,, \tag{2.55}
$$

follows. On the hyperplane $\cos\theta = 0$ the norm g_{tt} of the Killing vector ∂_t equals

$$
g_{tt} = -1+\frac{4M(r-M)}{\Delta_{P=0}} = -1+\frac{4M}{r-M}\,,
$$

which blows up as $r \to M$, which implies, by the usual arguments, that the singularities of the metrics are represented by $\{r = M, \cos\theta = 0\}$.
In the $a = 0$ case (2.55) reduces to

$$
\Delta_{P=0} = 0 \Longleftrightarrow r = M\,. \tag{2.56}
$$

The asymptotic expansion of (2.52), using the usual asymptotically Minkowskian space-time coordinates $\{x^0,\cdots,x^3\}$, gives

$$
ds^2_{(5)} = \eta_{\alpha\beta}dx^\alpha dx^\beta + \frac{4M}{r}\left(dt^2+(dx^4)^2+2dx^4dt\right)+O(r^{-2})\,, \tag{2.57}
$$

which yields that the metric is asymptotically flat.

2.3.4 The $Q = 0$, $\Sigma \to -\sqrt{3}M$ case

In the $Q = 0$, $\Sigma \to -\sqrt{3}M$ case the line element (2.1) reduces to

$$
\begin{aligned}
ds^2_{(5)} &= -dt^2+\frac{\Delta_{Q=0}}{(r-M)^2+a^2}dr^2+\Delta_{Q=0}d\theta^2 \\
&\quad -\frac{\left((r-M)^2+a^2\right)\left(a^2\cos(4\theta)-a^2-4\left(25M^2+6Mr+r^2\right)-4\cos(2\theta)(M-r)(7M+r)\right)}{8\Delta_{Q=0}}d\phi^2 \\
&\quad +\left(1+\frac{4M(M-r)}{\Delta_{Q=0}}\right)(dx^4)^2 \pm \frac{8M\cos(\theta)\left(a^2+(M-r)^2\right)}{\Delta_{Q=0}}d\phi dx^4\,,
\end{aligned}
\tag{2.58}
$$

where

$$\Delta_{Q=0} =: (r-M)(r+3M) + a^2\cos^2(\theta)\,, \tag{2.59}$$

and M and a are unconstrained parameters again .
A computation with MATHEMATICA confirms, that (2.58) is a solution of the five-dimensional vacuum Einstein equations.
From (2.58) it follows directly, that the norm of ∂_4, i.e. the geometric invariant g_{44} of the metric, is singular in the $a \neq 0$ case at the locations of the zeros of $\Delta_{Q=0}$, bounded by $-3M \leq r < M$. In the $a = 0$ case the singularities are attained at $r \in \{-3M, M\}$. Furthermore, from the asymptotic expansion

$$\begin{aligned} ds^2_{(5)} &= -dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2 \\ &\quad + \left(dx^4 - 4M\cos(\theta)d\phi\right)^2 + O(r^{-1})\,, \end{aligned} \tag{2.60}$$

of (2.58) we obtain, that the metric is not asymptotically flat.

In the following we give a summary of the results of this subsection:

	$P^2 = \lambda\left(M - \frac{\Sigma}{\sqrt{3}}\right),\ M \to \frac{\Sigma}{\sqrt{3}}$	$Q^2 = \lambda\left(M + \frac{\Sigma}{\sqrt{3}}\right),\ M \to -\frac{\Sigma}{\sqrt{3}}$
$\lambda \neq 0$	"Complex metric"	"Complex metric"
$\lambda \neq 0,\ a = 0$	Naked singularity at $r = M$	Naked singularity at $r = -3M$
$\lambda = 0$	Naked singularity at $r = M$, $\cos\theta = 0$	Naked singularities in the bound $-3M \leq r < M$
$\lambda = 0,\ a = 0$	Naked singularity at $r = M$	Naked singularities at $r \in \{-3M, M\}$

2.4 On the existence of regular metrics with no horizons

The location of the Killing horizons of the metric is determined by the zeros of Δ. From (2.88) it follows that Δ attains no real zeros if and only if

$$a^2 > M^2 + \Sigma^2 - P^2 - Q^2\,. \tag{2.61}$$

The singularities of the metric are determined by the zeros of A and B. The conditions (2.124) and (2.125), necessary to avoid singularities, are in this case given by

$$\begin{aligned} &|\mathscr{A}| > 2 \ \text{ and } \ \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} - a^2(1 - |\mathscr{A}|) < 0 \\ \text{or} \\ &|\mathscr{A}| \leq 2 \ \text{ and } \ \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} + \frac{a^2\mathscr{A}^2}{4} < 0\,, \end{aligned} \tag{2.62}$$

and

$$|\mathscr{B}| > 2 \text{ and } \tfrac{2Q^2\Sigma}{\Sigma+M\sqrt{3}} - a^2(1-|\mathscr{B}|) < 0$$

or

$$|\mathscr{B}| \leq 2 \text{ and } \tfrac{2Q^2\Sigma}{\Sigma+M\sqrt{3}} + \tfrac{a^2\mathscr{B}^2}{4} < 0\,, \tag{2.63}$$

where $\mathscr{A}$ and $\mathscr{B}$ are given by (2.125) and (2.127), i.e.

$$\mathscr{A} := \frac{2FPQ}{a\left(\left(M+\Sigma/\sqrt{3}\right)^2 - Q^2\right)}\,, \quad \mathscr{B} := -\frac{2FPQ}{a\left(\left(M-\Sigma/\sqrt{3}\right)^2 - P^2\right)}\,. \tag{2.64}$$

Furthermore, if $a \neq 0$, from (2.2) the condition

$$\frac{\left[\left(M+\Sigma/\sqrt{3}\right)^2 - Q^2\right]\left[\left(M-\Sigma/\sqrt{3}\right)^2 - P^2\right]}{M^2+\Sigma^2-P^2-Q^2} \geq 0\,, \tag{2.65}$$

as a requirement to obtain a real-valued metric, follows.

2.4.1 $P = 0$ case

$P = 0$ in (2.64) directly implies $\mathscr{A} = 0$ and $\mathscr{B} = 0$. Then (2.61) and (2.63) reduce to

$$a^2 > M^2 + \Sigma^2 - Q^2\,, \quad \Sigma < M\sqrt{3}\,. \tag{2.66}$$

Furthermore, it follows immediately that (2.62) is violated. Thus in this setting a metric with no horizons possesses naked singularities, determined by the zeros of A, for any choice of the parameters.

2.4.2 The "large" $|a|$ case

If $|a|$ is chosen large enough, it follows that (2.61) holds and from (2.64) that $|\mathscr{A}| \leq 2$ and $|\mathscr{B}| \leq 2$ holds. In this case (2.62), (2.63) and (2.65) reduce to

$$\begin{aligned}
&\frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} + \frac{a^2\mathscr{A}^2}{4} < 0\,,\\
&\frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} + \frac{a^2\mathscr{B}^2}{4} < 0\,,\\
&\frac{\left[\left(M+\Sigma/\sqrt{3}\right)^2 - Q^2\right]\left[\left(M-\Sigma/\sqrt{3}\right)^2 - P^2\right]}{M^2+\Sigma^2-P^2-Q^2} \geq 0\,.
\end{aligned} \tag{2.67}$$

$F > 0$ case:
The insertion of the constraint (2.10) in (2.67) yields

$$\begin{aligned}
&\frac{3Q^2\left(\sqrt{3}M-\Sigma\right)}{3\sqrt{3}M^3+9M^2\Sigma+3\sqrt{3}M\left(\Sigma^2-2Q^2\right)+\Sigma^3} - \frac{2\Sigma}{\sqrt{3}M-\Sigma} < 0\,,\\
&\frac{3Q^2\left(3M^2-2\sqrt{3}M\Sigma+\Sigma^2\right)}{\left(\sqrt{3}M+\Sigma\right)\left(3\sqrt{3}M^3+9M^2\Sigma+3\sqrt{3}M\left(\Sigma^2-2Q^2\right)+\Sigma^3\right)} < 0\,,\\
&\frac{\left(\left(M+\frac{\Sigma}{\sqrt{3}}\right)^2 - Q^2\right)\left(\left(M-\frac{\Sigma}{\sqrt{3}}\right)^2 - \left(\Sigma-\sqrt{3}M\right)\left(\frac{2\Sigma}{3} - \frac{Q^2}{\sqrt{3}M+\Sigma}\right)\right)}{M^2 - \left(\Sigma-\sqrt{3}M\right)\left(\frac{2\Sigma}{3} - \frac{Q^2}{\sqrt{3}M+\Sigma}\right) - Q^2 + \Sigma^2} > 0\,.
\end{aligned} \tag{2.68}$$

From an analysis with MATHEMATICA we obtain, that the inequalities (2.68) are not fulfilled simultaneously.

$F = 0$ case:
In the $F = 0$ case (2.67) reduces to

$$\frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} < 0\,, \quad \frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} < 0\,. \tag{2.69}$$

By the insertion of the constraint (2.10) in (2.69), we obtain

$$\frac{4\Sigma^2}{3} - \frac{2Q^2\Sigma}{\sqrt{3}M + \Sigma} < 0\,, \quad \frac{2Q^2\Sigma}{\sqrt{3}M + \Sigma} < 0\,. \tag{2.70}$$

If the second inequality in (2.70) is fulfilled, it follows immediately, that the first inequality is violated. In conclusion for any choice of the parameters (M, P, Q, Σ) the Rasheed metrics have naked singularities if $|a|$ exceeds a certain threshold.

In the $|\mathscr{A}| > 2$ and/or $|\mathscr{B}| > 2$ case, corresponding to the "small" $|a|$ setting, we have not been able to analyse the resulting inequalities, determining if (2.61)-(2.63) and (2.65) can be fulfilled simultaneously, appropriately due to their complexity.

2.4.3 The $M^2 + \Sigma^2 - P^2 - Q^2 \to 0$ limit

The critical term in this limit is given by (2.2), i.e.

$$F^2 = \frac{\left[\left(M + \Sigma/\sqrt{3}\right)^2 - Q^2\right]\left[\left(M - \Sigma/\sqrt{3}\right)^2 - P^2\right]}{M^2 + \Sigma^2 - P^2 - Q^2}\,. \tag{2.71}$$

By defining $q := M^2 + \Sigma^2 - P^2 - Q^2$, we can rewrite (2.71) as

$$F^2 = \frac{\left(3M^2 + 2\sqrt{3}M\Sigma - 3Q^2 + 7\Sigma^2\right)\left(-2\Sigma\left(\sqrt{3}M + \Sigma\right) + 3q + 3Q^2\right)}{9q}\,. \tag{2.72}$$

The insertion of the solution of the constraint (2.10) for Q^2 in (2.72) yields

$$F^2 = \frac{\left(3\sqrt{3}M^3 - 3M^2\Sigma + \sqrt{3}M\left(5\Sigma^2 - 3P^2\right) - 3P^2\Sigma - 5\Sigma^3\right)\left(\sqrt{3}M(P^2 + q) + \Sigma(P^2 - q)\right)}{3q\left(\Sigma - \sqrt{3}M\right)^2}\,. \tag{2.73}$$

Thus a regular metric is obtained in the $q \to 0$ limit if and only if the numerator of (2.73) fulfils

$$\left(3\sqrt{3}M^3 - 3M^2\Sigma + \sqrt{3}M\left(5\Sigma^2 - 3P^2\right) - 3P^2\Sigma - 5\Sigma^3\right)\left(\sqrt{3}M(P^2 + q) + \Sigma(P^2 - q)\right) = f \cdot q^n\,, \tag{2.74}$$

where f is a smooth function and $n \in \mathbb{N}$. (2.74) is solved by

$$f = \frac{f_1(M, P, \Sigma)}{q^{n-1}} + \frac{9M^4P^2 - 9M^2P^4 + 12M^2P^2\Sigma^2 - 6\sqrt{3}MP^4\Sigma - 3P^4\Sigma^2 - 5P^2\Sigma^4}{q^n}\,, \tag{2.75}$$

where $f_1(M, P, \Sigma)$ is a smooth function of the parameters, or

$$P = \frac{9M^3 - 3\sqrt{3}M^2\Sigma + 15M\Sigma^2 - 5\sqrt{3}\Sigma^3}{3\left(3M + \sqrt{3}\Sigma\right)} . \tag{2.76}$$

It follows that only in the $n = 1$ case it is potentially possible to obtain a suitable function if and only if the remainder of the polynomial division

$$(9M^4P^2 - 9M^2P^4 + 12M^2P^2\Sigma^2 - 6\sqrt{3}MP^4\Sigma - 3P^4\Sigma^2 - 5P^2\Sigma^4) : q , \tag{2.77}$$

is zero, which turns out not to hold. The insertion of (2.76) in (2.71) yields $F = 0$, which is just a special case of the metric, which has been already covered in the analysis above. Summarizing, there exist no regular metrics in this limit.

2.5 Asymptotic form and global charges

With the expansion of the metric coefficients of (2.1)

$$\begin{aligned}
g_{tt} &= -1+\frac{2M}{r}+\frac{2\Sigma}{\sqrt{3}r}+O\left(r^{-2}\right), \\
g_{rr} &= 1+\frac{2M}{r}-\frac{2\Sigma}{\sqrt{3}r}+O\left(r^{-2}\right), \\
g_{\theta\theta} &= r^2-\frac{2r\Sigma}{\sqrt{3}}+O(1), \\
g_{\phi\phi} &= r^2\sin^2(\theta)-\frac{2r\Sigma\sin^2(\theta)}{\sqrt{3}}+4P^2\cos^2(\theta)+O(1), \\
g_{44} &= 1+\frac{4\Sigma}{\sqrt{3}r}+O\left(r^{-2}\right), \\
g_{t\phi} &= O\left(r^{-2}\right), \\
g_{t4} &= \frac{2Q}{r}+O\left(r^{-2}\right), \\
g_{\phi 4} &= 2P\cos(\theta)+O\left(r^{-1}\right),
\end{aligned} \tag{2.78}$$

the line element can be taken into the asymptotic form

$$\begin{aligned}
ds^2 &= \left(-1+\frac{2M}{r}+\frac{2\Sigma}{\sqrt{3}r}\right)dt^2+\left(1+\frac{2M}{r}-\frac{2\Sigma}{\sqrt{3}r}\right)dr^2+\left(r^2-\frac{2r\Sigma}{\sqrt{3}}\right)d\theta^2 \\
&\quad+\left(r^2\sin^2(\theta)-\frac{2r\Sigma\sin^2(\theta)}{\sqrt{3}}+4P^2\cos^2(\theta)\right)d\phi^2 \\
&\quad+\left(1+\frac{4\Sigma}{\sqrt{3}r}\right)(dx^4)^2+4P\cos(\theta)d\phi dx^4+\frac{4Q}{r}dt dx^4+O\left(r^{-2}\right) \\
&= -dt^2+dr^2+r^2d\theta^2+r^2\sin^2(\theta)d\phi^2+(dx^4)^2+4P\cos(\theta)d\phi dx^4+4P^2\cos^2(\theta)d\phi^2 \\
&\quad-\frac{2\Sigma}{\sqrt{3}r}\Big(\underbrace{-dt^2+dr^2+r^2d\theta^2+r^2\sin^2(\theta)d\phi^2}_{=\eta_{ab}dx^a dx^b}\Big)+\frac{2M}{r}dt^2+\frac{2M}{r}dr^2+\frac{4Q}{r}dt dx^4+O\left(r^{-2}\right) \\
&= -dt^2+dr^2+r^2d\theta^2+r^2\sin^2(\theta)d\phi^2+\left(dx^4+2P\cos(\theta)d\phi\right)^2-\frac{2\Sigma}{\sqrt{3}r}\eta_{ab}dx^a dx^b \\
&\quad+\frac{2M}{r}dr^2+\frac{4Q}{r}dt dx^4+O\left(r^{-2}\right) \\
&= \hat{g}_{\mu\nu}dx^\mu dx^\nu-\frac{2\Sigma}{\sqrt{3}r}\eta_{ab}dx^a dx^b+\frac{2M}{r}dt^2+\frac{2M}{r}dr^2+\frac{4Q}{r}dt dx^4+O\left(r^{-2}\right),
\end{aligned} \tag{2.79}$$

where we have defined

$$\hat{g} =: -dt^2+dr^2+r^2d\theta^2+r^2\sin^2(\theta)d\phi^2+\left(dx^4+2P\cos(\theta)d\phi\right)^2, \tag{2.80}$$

as the background metric. From (2.80) it follows, that the metric is not asymptotically flat for $P \neq 0$ and that x^4 has to be $8\pi P$-periodic(for a detailed analysis see Section 4). In the following we compute the global charges of the Rasheed solution. For this sake it is convenient to switch to Cartesian coordinates.

In the $P=0$ case in a Cartesian-type basis (t,x,y,z,x^4) (2.79) takes the form

$$g=\begin{pmatrix} -1+\frac{2M}{r}+\frac{2\Sigma}{\sqrt{3}r} & 0 & 0 & 0 & \frac{2Q}{r} \\ 0 & 1+\frac{2Mx^2}{r^3}-\frac{2\Sigma}{\sqrt{3}r} & \frac{2Mxy}{r^3} & \frac{2Mxz}{r^3} & 0 \\ 0 & \frac{2Mxy}{r^3} & 1+\frac{2My^2}{r^3}-\frac{2\Sigma}{\sqrt{3}r} & \frac{2Myz}{r^3} & 0 \\ 0 & \frac{2Mxz}{r^3} & 1+\frac{2Myz}{r^3} & \frac{2Mz^2}{r^3}-\frac{2\Sigma}{\sqrt{3}r} & 0 \\ \frac{2Q}{r} & 0 & 0 & 0 & 1+\frac{4\Sigma}{\sqrt{3}r} \end{pmatrix}+O\big(r^{-2}\big)\,. \tag{2.81}$$

When $P\neq 0$ the expansions are considerably more complicated and not very enlightening, therefore we do not include them here. From (2.79), with the formulae derived in Section 4, we obtain for the Hamiltonian momentum $p_\mu$of the level sets of t, and the ADM four-momentum $p_{\mu,ADM}$ of the space-metric $g_{ij}dx^i dx^j$ the following results:

$$p_{i,ADM}=p_i=0\,,\quad p_{0,ADM}=M-\frac{\Sigma}{\sqrt{3}}\,,\quad p_0=\begin{cases} 2\pi M, & P=0, \\ 4\pi PM, & P\neq 0,\end{cases}\quad p_4=\begin{cases} 2\pi Q, & P=0, \\ 8\pi PQ, & P\neq 0.\end{cases} \tag{2.82}$$

The Komar integrals associated with $X=\partial_t$ are

$$\frac{1}{8\pi}\lim_{R\to\infty}\int_{S(R)}\int_{S^1}X^{\alpha;\beta}dS_{\alpha\beta}=\begin{cases} 2\pi\big(M+\frac{\Sigma}{\sqrt{3}}\big), & P=0, \\ 8\pi P\big(M+\frac{\Sigma}{\sqrt{3}}\big), & P\neq 0,\end{cases} \tag{2.83}$$

whereas those associated with $X=\partial_4$ are given by

$$\frac{1}{8\pi}\lim_{R\to\infty}\int_{S(R)}\int_{S^1}X^{\alpha;\beta}dS_{\alpha\beta}=\begin{cases} 4\pi Q, & P=0, \\ 16\pi PQ, & P\neq 0.\end{cases} \tag{2.84}$$

2.6 Regularity at the $\sin\theta=0$ axis

To show that the metric is regular at the $\sin\theta=0$ axis, we write (2.3) in the following form

$$\begin{aligned} ds^2_{(4)} &= -\frac{\Delta_\theta}{\sqrt{AB}}\big(dt+\omega^0{}_\phi d\phi\big)^2+\frac{\sqrt{AB}}{\Delta}dr^2+\sqrt{AB}d\theta^2+\frac{\Delta\sqrt{AB}}{\Delta_\theta}\sin^2\theta d\phi^2 \\ &= -\frac{\Delta_\theta}{\sqrt{AB}}\big(dt+\omega^0{}_\phi d\phi\big)^2+\underbrace{dr^2+r^2\big(d\theta^2+\sin^2\theta d\phi^2\big)}_{\delta_{ij}dx^idx^j} \\ &\quad +\big(\sqrt{AB}-r^2\big)\underbrace{\big(d\theta^2+\sin^2\theta d\phi^2\big)}_{r^{-2}(\delta_{ij}dx^idx^j-dr^2)}+\left(\frac{\sqrt{AB}}{\Delta}-1\right)dr^2+\sqrt{AB}\left(\frac{\Delta}{\Delta_\theta}-1\right)\sin^2\theta d\phi^2\,. \end{aligned} \tag{2.85}$$

From the last line in (2.85) it follows, that the metric is regular at the $\sin\theta=0$ axis if and only if $\left(\frac{\Delta}{\Delta_\theta}-1\right)$ factorizes in $\sin^2\theta$. This is true since

$$\left(\frac{\Delta}{\Delta_\theta}-1\right)=\frac{a^2\sin^2\theta}{a^2\cos^2(\theta)-2Mr+P^2+Q^2+r^2-\Sigma^2}\,. \tag{2.86}$$

In the $P=0$ case A_ϕ (2.6) factorizes in $\sin^2\theta$ also, together with (2.1) this yields, that the five-dimensional metric is also regular at the $\sin\theta=0$ axis.

2.7 Killing horizons

The location of the Killing horizons of the metric (2.3) is determined by the zeros of the determinant

$$\begin{vmatrix} g_{tt} & g_{t\phi} & g_{t4} \\ g_{\phi t} & g_{\phi\phi} & g_{\phi 4} \\ g_{4t} & g_{4\phi} & g_{44} \end{vmatrix} = -\Delta \sin^2\theta\,, \tag{2.87}$$

and thus by the roots $r_+ \geq r_-$ of Δ, which, therefore, give the locations of the outer- and inner Killing horizon respectively. From (2.4), by denoting the zeros of Δ_θ by $R_+ \geq R_-$, we obtain

$$\begin{aligned} r_\pm &= M \pm \sqrt{M^2 + \Sigma^2 - P^2 - Q^2 - a^2}\,, \\ R_\pm &= M \pm \sqrt{M^2 + \Sigma^2 - P^2 - Q^2 - a^2\cos^2\theta}\,. \end{aligned} \tag{2.88}$$

from which

$$r_+ \leq R_+\,, \tag{2.89}$$

follows for $\theta \in [0,\pi]$. Thus we have to pay special attention to the zero-set of Δ_θ, since potential singularities arising from this set are not shielded by the outer Killing horizon. This problem, among others, is addressed in the next subsection, resulting in that there are no singularities associated with the zeros R_+ and R_- of Δ_θ, corresponding to the ergosurfaces of the metric.

2.8 Singularities related with the zeros of A and B

To write some terms in a more compact form, we define $s := \sin\theta$.

The metric (2.1) is a solution to the five-dimensional field equations if (2.10) holds. Solving this constraint for P yields

$$P = \pm\sqrt{\left(\Sigma - \sqrt{3}M\right)\left(\frac{2\Sigma}{3} - \frac{Q^2}{\sqrt{3}M + \Sigma}\right)}\,, \tag{2.90}$$

then $P \in \mathbb{R}$ implies

$$(\Sigma - \sqrt{3}M)\left(\frac{2\Sigma}{3} - \frac{Q^2}{\sqrt{3}M + \Sigma}\right) \geq 0\,, \tag{2.91}$$

which is used throughout this subsection.

LEMMA 2.1 *The invariant*

$$g^{(5)}_{\phi\phi}(r,\theta) = \frac{B}{A} 4A_\phi^2 + \sqrt{\frac{A}{B}} g_{\phi\phi}\,, \tag{2.92}$$

is real and $C^\infty(S)$, where $S = (r_+,\infty) \times [0,\pi]$, with r_+ given by (2.88), if and only if

$$\frac{\left[\left(M + \Sigma/\sqrt{3}\right)^2 - Q^2\right]\left[\left(M - \Sigma/\sqrt{3}\right)^2 - P^2\right]}{M^2 + \Sigma^2 - P^2 - Q^2} \geq 0\,, \quad \text{if } a, P, Q, \neq 0\,, \tag{2.93}$$

and

$$|\mathscr{A}| > 2 \;\; and \;\; M + \sqrt{M^2 + \Sigma^2 - P^2 - Q^2 - a^2} > \frac{\Sigma}{3} + \sqrt{\frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} - a^2(1 - |\mathscr{A}|)}\,,$$

or

$$|\mathscr{A}| \leq 2 \;\; and \;\; M + \sqrt{M^2 + \Sigma^2 - P^2 - Q^2 - a^2} > \frac{\Sigma}{3} + \sqrt{\frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} + \frac{a^2\mathscr{A}^2}{4}}\,, \tag{2.94}$$

where

$$\mathscr{A} = \frac{2JPQ}{a^2\left(\left(M + \Sigma/\sqrt{3}\right)^2 - Q^2\right)}\,, \tag{2.95}$$

and $P = P(M, \Sigma, Q)$, given by (2.90), with (2.91) holds.

PROOF:
The strategy of the proof is to write (2.92) as a quotient of two polynomial expressions in the variables (r, θ), in order to factorize out as many potential problematic terms as possible. We write the $\phi\phi$-component of the four-dimensional line element in the following form

$$\begin{aligned} g_{\phi\phi} &= -\frac{\Delta_\theta}{\sqrt{AB}}\left(\omega^0{}_\phi\right)^2 + \frac{\Delta\sqrt{AB}}{\Delta_\theta}s^2 \\ &= \frac{s^2 AB\Delta - \Delta_\theta^2(\omega^0{}_\phi)^2}{\sqrt{AB}\Delta_\theta} \\ &= \frac{ABs^2\Delta - 4J^2s^4(r+E)^2}{\sqrt{AB}\Delta_\theta}\,. \end{aligned} \tag{2.96}$$

The insertion of (2.6) in (2.92) yields

$$g^{(5)}_{\phi\phi} = \frac{B}{A}\underbrace{\left(\omega^5{}_\phi + \frac{C}{B}\omega^0{}_\phi\right)^2}_{:=g_{\phi\phi A}} + \sqrt{\frac{A}{B}}g_{\phi\phi}\,. \tag{2.97}$$

The evaluation of $g_{\phi\phi A}$ yields

$$\begin{aligned} g_{\phi\phi A} &= \left(\omega^5{}_\phi + \frac{C}{B}\omega^0{}_\phi\right)^2 \\ &= \left((\omega^5{}_\phi)^2 + 2\frac{C}{B}\omega^5{}_\phi\omega^0{}_\phi + \frac{C^2}{B^2}(\omega^0{}_\phi)^2\right) \\ &= \left(\frac{H^2 + 4Js^2(r+E)\frac{C}{B}H + \frac{C^2}{B^2}4J^2s^4(r+E)^2}{\Delta_\theta^2}\right) \\ &= \frac{1}{B^2}\left(\frac{B^2H^2 + 4Js^2(r+E)CBH + 4C^2J^2s^4(r+E)^2}{\Delta_\theta^2}\right). \end{aligned} \tag{2.98}$$

Finally, (2.96) and (2.98) inserted in (2.97) gives the desired polynomial expression

$$\begin{aligned}
g_{\phi\phi}^{(5)} &= \frac{ABs^2\Delta - 4J^2s^4(r+E)^2}{B\Delta_\theta} + \frac{1}{AB}\left(\frac{B^2H^2 + 4Js^2(r+E)CBH + 4C^2J^2s^4(r+E)^2}{\Delta_\theta^2}\right) \\
&= \frac{A\Delta_\theta(ABs^2\Delta - 4J^2s^4(r+E)^2)}{AB\Delta_\theta^2} + \frac{1}{AB}\left(\frac{B^2H^2 + 4Js^2(r+E)CBH + 4C^2J^2s^4(r+E)^2}{\Delta_\theta^2}\right) \\
&= \frac{1}{AB}\left(\frac{A\Delta_\theta(ABs^2\Delta - 4J^2s^4(r+E)^2) + B^2H^2 + 4CBHJ(r+E)s^2 + 4C^2J^2(r+E)^2s^4}{\Delta_\theta^2}\right).
\end{aligned} \tag{2.99}$$

A MATHEMATICA computation shows, that the numerator of (2.99) factorizes in Δ_θ^2 and B, but not in A. Therefore, $g_{\phi\phi}^{(5)} \in C^\infty(S)$ if A attains no zeros on S. From Lemma 2.2 it follows, that this is the case if and only if (2.94) holds. From (2.4) and (2.2) one observes directly, that J is real if and only if the inequality in (2.93) holds, then it follows, that (2.99) is also real. From (2.2), (2.4) and (2.7) it follows, that (2.99) is real.

□

LEMMA 2.2 *The function*

$$A = \left(r - \Sigma/\sqrt{3}\right)^2 - \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} + a^2\cos^2\theta + \frac{2JPQ\cos\theta}{\left(M + \Sigma/\sqrt{3}\right)^2 - Q^2}, \tag{2.100}$$

i) is real if and only if

$$\frac{\left[\left(M + \Sigma/\sqrt{3}\right)^2 - Q^2\right]\left[\left(M - \Sigma/\sqrt{3}\right)^2 - P^2\right]}{M^2 + \Sigma^2 - P^2 - Q^2} \geq 0, \quad \text{if } a, P, Q, \neq 0, \tag{2.101}$$

ii) has real zeros in the variable r, for some $\theta \in [0,\pi]$, if and only if

$$0 \leq \begin{cases} \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} - a^2(1 - |\mathscr{A}|), & \text{if } |\mathscr{A}| > 2, \\ \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} + \frac{a^2\mathscr{A}^2}{4}, & \text{if } |\mathscr{A}| \leq 2, \end{cases} \tag{2.102}$$

iii) The largest zero of A, in the variable r, for all $\theta \in [0,\pi]$, is given by

$$r_{max,+}^A = \begin{cases} \frac{\Sigma}{\sqrt{3}} + \sqrt{\frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} - a^2(1 - |\mathscr{A}|)}, & \text{if } |\mathscr{A}| > 2, \\ \frac{\Sigma}{\sqrt{3}} + \sqrt{\frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} + \frac{a^2\mathscr{A}^2}{4}}, & \text{if } |\mathscr{A}| \leq 2, \end{cases} \tag{2.103}$$

where $\mathscr{A}$ is given by (2.95), $P = P(M,\Sigma,Q)$ by (2.90), and (2.91) holds.

PROOF:

i): The parameter J is given via (2.4) and (2.2) by

$$J = \pm a\sqrt{\frac{\left[\left(M+\Sigma/\sqrt{3}\right)^2 - Q^2\right]\left[\left(M-\Sigma/\sqrt{3}\right)^2 - P^2\right]}{M^2+\Sigma^2-P^2-Q^2}}\,. \tag{2.104}$$

With (2.100) and $M, \Sigma, a, P, Q \in \mathbb{R}$ it follows, that $A \in \mathbb{R}$ if and only if $J \in \mathbb{R}$ or $P = 0$, or $Q = 0$ or $a = 0$, which implies (2.101).

ii): We write the equation $A = 0$ in the form

$$\begin{aligned}\left(r-\Sigma/\sqrt{3}\right)^2 &= \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} - a^2\cos^2\theta - \frac{2JPQ\cos\theta}{\left(M+\Sigma/\sqrt{3}\right)^2 - Q^2} \\ &= \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} - a^2\underbrace{\left(\cos^2\theta + \frac{2JPQ}{a^2\left(\left(M+\Sigma/\sqrt{3}\right)^2 - Q^2\right)}\cos\theta\right)}_{=:h(\theta)}.\end{aligned} \tag{2.105}$$

A will have real zeros in the variable r for some $\theta \in [0,\pi]$ if and only if the maximum of the right-hand side of (2.105), in the variable θ, is larger than or equal to zero. The right side is maximal if h is minimal. By using the definition (2.95), we write

$$h(\theta) = \cos^2\theta + \mathscr{A}\cos\theta\,. \tag{2.106}$$

Since h is a periodic function, the global minimum coincides with a local minimum. The first and second derivative of h are given by

$$\begin{aligned} h'(\theta) &= -\sin\theta(\mathscr{A} + 2\cos\theta)\,, \\ h''(\theta) &= 2 - \mathscr{A}\cos\theta - 4\cos^2\theta\,. \end{aligned} \tag{2.107}$$

For the analysis of the local minima it is sufficient to restrict to the $h' = 0$ and $h'' > 0$ requirement, since it follows easily from (2.107) that $h' = h'' = 0$ is attained if and only if $|\mathscr{A}| = 2 \wedge \theta \in \{0,\pi\}$. This special case is covered within the analysis. From $h' = 0$ we obtain

$$\sin\theta(\mathscr{A} + 2\cos\theta) = 0 \Longleftrightarrow \theta \in \left\{0,\pi\right\} \text{ or } \cos\theta = -\frac{\mathscr{A}}{2}\,. \tag{2.108}$$

If $\theta \in \{0,\pi\}$, then the condition $h'' > 0$ and (2.107) imply

$$-2 \pm \mathscr{A} > 0 \Longleftrightarrow |\mathscr{A}| > 2\,. \tag{2.109}$$

In the $\cos\theta = -\frac{\mathscr{A}}{2}$ case, $h'' > 0$ and (2.107) imply

$$2 - \mathscr{A}^2/2 > 0 \Longleftrightarrow |\mathscr{A}| < 2\,. \tag{2.110}$$

In the $|\mathscr{A}| = 2$ case if follows easily, that the minimum of h is located where $\cos\theta = -\frac{\mathscr{A}}{2} = -1$. Therefore, we can finally conclude, by inserting $\theta \in \left\{0, \pi\right\}$ (in the $|\mathscr{A}| > 2$ case) and $\cos\theta = -\frac{\mathscr{A}}{2}$ (in the $|\mathscr{A}| \leq 2$ case), that the minimum h_{min} of h is given by

$$h_{min} = \begin{cases} (1 - |\mathscr{A}|), & \text{if } |\mathscr{A}| > 2, \\ \\ -\frac{\mathscr{A}^2}{4}, & \text{if } |\mathscr{A}| \leq 2, \end{cases} \quad . \tag{2.111}$$

From (2.111) it follows, that the right side of (2.105) is non-negative if and only if

$$0 \leq \begin{cases} \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} - a^2(1 - |\mathscr{A}|), & \text{if } |\mathscr{A}| > 2, \\ \\ \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} + \frac{a^2\mathscr{A}^2}{4}, & \text{if } |\mathscr{A}| \leq 2, \end{cases} \quad . \tag{2.112}$$

iii): From ii) and (2.105) the expression for the largest zero of A

$$r^A_{max,+} = \begin{cases} \frac{\Sigma}{\sqrt{3}} + \sqrt{\frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} - a^2(1 - |\mathscr{A}|)}, & \text{if } |\mathscr{A}| > 2, \\ \\ \frac{\Sigma}{\sqrt{3}} + \sqrt{\frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} + \frac{a^2\mathscr{A}^2}{4}}, & \text{if } |\mathscr{A}| \leq 2, \end{cases} \quad , \tag{2.113}$$

follows.

□

LEMMA 2.3 *The function*

$$B \quad = \quad \left(r + \Sigma/\sqrt{3}\right)^2 - \frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} + a^2\cos^2\theta - \frac{2JPQ\cos\theta}{\left(M - \Sigma/\sqrt{3}\right)^2 - P^2}\,, \tag{2.114}$$

i) is real if and only if

$$\frac{\left[\left(M + \Sigma/\sqrt{3}\right)^2 - Q^2\right]\left[\left(M - \Sigma/\sqrt{3}\right)^2 - P^2\right]}{M^2 + \Sigma^2 - P^2 - Q^2} \geq 0\,, \quad \textit{if } a, P, Q, \neq 0\,, \tag{2.115}$$

ii) has real zeros in the variable r, for some $\theta \in [0, \pi]$, if and only if

$$0 \leq \begin{cases} \frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} - a^2(1 - |\mathscr{B}|)\,, & \textit{if } |\mathscr{B}| > 2\,, \\ \\ \frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} + \frac{a^2\mathscr{B}^2}{4}\,, & \textit{if } |\mathscr{B}| \leq 2\,, \end{cases} \quad , \tag{2.116}$$

iii) The largest zero of B, in the variable r, for all $\theta \in [0, \pi]$, is given by

$$r^B_{max,+} = \begin{cases} -\frac{\Sigma}{\sqrt{3}} + \sqrt{\frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} - a^2(1 - |\mathscr{B}|)}\,, & \textit{if } |\mathscr{B}| > 2\,, \\ \\ -\frac{\Sigma}{\sqrt{3}} + \sqrt{\frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} + \frac{a^2\mathscr{B}^2}{4}}\,, & \textit{if } |\mathscr{B}| \leq 2\,, \end{cases} \quad , \tag{2.117}$$

where $P = P(M,\Sigma,Q)$ is given by (2.90), (2.91) holds, and $\mathscr{B}$ is given by

$$\mathscr{B} = -\frac{2JPQ}{a^2\left(\left(M+\Sigma/\sqrt{3}\right)^2 - P^2\right)}. \tag{2.118}$$

PROOF:

i): Analogous to i) of Lemma 2.2.

ii): We write the equation $B = 0$ in the following form

$$\begin{aligned}\left(r+\Sigma/\sqrt{3}\right)^2 &= \frac{2Q^2\Sigma}{\Sigma+M\sqrt{3}} - a^2\cos^2\theta + \frac{2JPQ\cos\theta}{\left(M-\Sigma/\sqrt{3}\right)^2 - P^2} \\ &= \frac{2Q^2\Sigma}{\Sigma+M\sqrt{3}} - a^2\underbrace{\left(\cos^2\theta - \frac{2JPQ}{a^2\left(\left(M+\Sigma/\sqrt{3}\right)^2-P^2\right)}\cos\theta\right)}_{=:h_B(\theta)}.\end{aligned} \tag{2.119}$$

B will have real zeros in the variable r for some $\theta \in [0,\pi]$ if and only if the maximum of the right-hand side of (2.119), in the variable θ, is larger than or equal to zero. The right side is maximal if h_B is minimal. By using the definition (2.118), we write

$$h_B(\theta) = \cos^2\theta + \mathscr{B}\cos\theta. \tag{2.120}$$

Since (2.106) of Lemma 2.2 is of the exact same form as (2.120), we obtain the minimum $h_{B,min}$ of h_B simply by replacing $\mathscr{A}$ by $\mathscr{B}$ in (2.111), i.e.

$$h_{B,min} = \begin{cases} (1-|\mathscr{B}|), & \text{if } |\mathscr{B}| > 2, \\ -\frac{\mathscr{B}^2}{4}, & \text{if } |\mathscr{B}| \le 2, \end{cases}. \tag{2.121}$$

Thus we can write the positivity condition for the maximum of the left side of (2.119) in the form

$$0 \le \begin{cases} \frac{2Q^2\Sigma}{\Sigma+M\sqrt{3}} - a^2(1-|\mathscr{B}|), & \text{if } |\mathscr{B}| > 2, \\ \frac{2Q^2\Sigma}{\Sigma+M\sqrt{3}} + \frac{a^2\mathscr{B}^2}{4}, & \text{if } |\mathscr{B}| \le 2, \end{cases}. \tag{2.122}$$

iii): From ii) and (2.119)

$$r^B_{max,+} = \begin{cases} -\frac{\Sigma}{\sqrt{3}} + \sqrt{\frac{2Q^2\Sigma}{\Sigma+M\sqrt{3}} - a^2(1-|\mathscr{B}|)}, & \text{if } |\mathscr{B}| > 2, \\ -\frac{\Sigma}{\sqrt{3}} + \sqrt{\frac{2Q^2\Sigma}{\Sigma+M\sqrt{3}} + \frac{a^2\mathscr{B}^2}{4}}, & \text{if } |\mathscr{B}| \le 2, \end{cases}, \tag{2.123}$$

for the largest zero of B, follows. □

Summarising, a necessary and sufficient condition, so that all A has no zeros in the d.o.c., is given by

$$|\mathscr{A}| > 2 \ \text{ and } \left\{ \begin{array}{l} \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} - a^2(1-|\mathscr{A}|) < 0\,, \\ M + \sqrt{M^2+\Sigma^2-P^2-Q^2-a^2} > \frac{\Sigma}{3} + \sqrt{\frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} - a^2(1-|\mathscr{A}|)}\,, \end{array} \right. \quad \text{or}$$

or

$$|\mathscr{A}| \le 2 \ \text{ and } \left\{ \begin{array}{l} \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} + \frac{a^2\mathscr{A}^2}{4} < 0\,, \\ M + \sqrt{M^2+\Sigma^2-P^2-Q^2-a^2} > \frac{\Sigma}{3} + \sqrt{\frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} + \frac{a^2\mathscr{A}^2}{4}}\,, \end{array} \right. \quad \text{or} \tag{2.124}$$

where

$$\mathscr{A} := \frac{2JPQ}{a^2\left(\left(M+\Sigma/\sqrt{3}\right)^2 - Q^2\right)}\,, \tag{2.125}$$

and the same condition for B is given by

$$|\mathscr{B}| > 2 \ \text{ and } \left\{ \begin{array}{l} \frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} - a^2(1-|\mathscr{B}|) < 0\,, \\ M + \sqrt{M^2+\Sigma^2-P^2-Q^2-a^2} > -\frac{\Sigma}{3} + \sqrt{\frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} - a^2(1-|\mathscr{B}|)}\,, \end{array} \right. \quad \text{or}$$

or

$$|\mathscr{B}| \le 2 \ \text{ and } \left\{ \begin{array}{l} \frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} + \frac{a^2\mathscr{B}^2}{4} < 0\,, \\ M + \sqrt{M^2+\Sigma^2-P^2-Q^2-a^2} > -\frac{\Sigma}{3} + \sqrt{\frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} + \frac{a^2\mathscr{B}^2}{4}}\,, \end{array} \right. \quad \text{or} \tag{2.126}$$

where

$$\mathscr{B} := -\frac{2JPQ}{a^2\left(\left(M-\Sigma/\sqrt{3}\right)^2 - P^2\right)}\,. \tag{2.127}$$

LEMMA 2.4 *For $M = 0$ there exist no $a, P, Q, \Sigma \in \mathbb{R}$, such that $r_+ > r^A_{max,+}$ and $r_+ > r^B_{max,+}$ and (2.10) holds.*

PROOF:
Solving (2.10) for Σ yields

$$\Sigma = \pm\sqrt{\frac{3}{2}}\sqrt{P^2+Q^2}\,. \tag{2.128}$$

- $a = 0$ case:
  (2.103) and (2.123) reduce to

$$r^A_{max,+} = \frac{\Sigma}{\sqrt{3}} + \sqrt{2}|P|\,, \quad r^B_{max,+} = -\frac{\Sigma}{\sqrt{3}} + \sqrt{2}|Q|\,. \tag{2.129}$$

Inserting the negative case of (2.128) in (2.129) and (2.88) yields for the $r_+ > r^A_{max,+}$ and $r_+ > r^B_{max,+}$ condition

$$\sqrt{P^2+Q^2} > |P|\,, \quad |Q| < 0\,. \tag{2.130}$$

while insertion of the positive case of (2.128) yields

$$|P| < 0\,, \quad \sqrt{P^2+Q^2} > |Q|\,. \tag{2.131}$$

Thus those both special cases of (2.128) yield a contradiction.

- $a \neq 0$ case:
  By inserting (2.95) and (2.4) in the second line of (2.94), we obtain

$$|\mathscr{A}| \leq 2 \ \text{ and } \ \sqrt{\Sigma^2 - P^2 - Q^2 - a^2} > \frac{\Sigma}{3} + \sqrt{2P^2 + \frac{2FPQ}{4\left(\left(\Sigma/\sqrt{3}\right)^2 - Q^2\right)}}\,, \tag{2.132}$$

  for the $r_+ > r^A_{max,+}$ constraint. The left side(r_+) of (2.132) decreases with increasing $|a|$, while the right side ($r^A_{max,+}$) remains constant. Next, we write the first line of (2.94) in the form

$$|\mathscr{A}| > 2 \ \text{ and } \ \sqrt{\Sigma^2 - P^2 - Q^2 - a^2} > \frac{\Sigma}{3} + \sqrt{2P^2 + a^2 \underbrace{(|\mathscr{A}| - 1)}_{>1}}\,. \tag{2.133}$$

  The left side(r_+) of (2.133) decreases with increasing $|a|$, while the right side($r^A_{max,+}$) increases. Together with analogous considerations for the $r_+ > r^B_{max,+}$ condition and the analysis for the $a = 0$ case, Lemma 2.4 follows.

□

LEMMA 2.5 *The norm of the Killing fields ∂_t and ∂_4, i.e.*

$$g_{tt} = \frac{W}{AB}\,, \quad g_{44} = \frac{B}{A}\,, \tag{2.134}$$

where $W := -GA + C^2$, is non-singular in the closure of the d.o.c, i.e. $\{r \geq r_+\}$, if and only if A and B have no zeros in the d.o.c., except perhaps in a special setting, where B attains an isolated double zero in the d.o.c., if $a \neq 0$ and $P \neq 0$.

PROOF:
We write A and B in the following form

$$A = (r - r^A_+)(r - r^A_-)\,, \quad B = (r - r^B_+)(r - r^B_-)\,, \tag{2.135}$$

where

$$\begin{aligned} r^A_\pm &= \frac{\Sigma}{\sqrt{3}} \pm \sqrt{-a^2\cos^2(\theta) - \frac{2aFPQ\cos(\theta)}{\left(M + \frac{\Sigma}{\sqrt{3}}\right)^2 - Q^2} + \frac{2P^2\Sigma}{\sqrt{3}M - \Sigma}}\,, \\ r^B_\pm &= -\frac{\Sigma}{\sqrt{3}} \pm \sqrt{-a^2\cos^2(\theta) + \frac{2aFPQ\cos(\theta)}{\left(M - \frac{\Sigma}{\sqrt{3}}\right)^2 - P^2} + \frac{2Q^2\Sigma}{\sqrt{3}M + \Sigma}}\,. \end{aligned} \tag{2.136}$$

- **a ≠ 0 case**:

  i) $r^A_{max,+} \neq \frac{\Sigma}{\sqrt{3}}$, $r^B_{max,+} \neq -\frac{\Sigma}{\sqrt{3}}$ case:

  A and B have zeros in the d.o.c. if and only if $r^A_{max,+} > r_+$ and $r^B_{max,+} > r_+$, where $r^A_{max,+}$ and $r^B_{max,+}$ are given by (2.103) and (2.117) respectively. In those cases the continuity of (2.136) in the variable θ implies that $\exists I_A \subset [0,\pi)$, $\forall\theta \in I_A$, $r_+ < r^A_+(\theta) < r^A_{max,+}$, and $\exists I_B \subset [0,\pi)$, $\forall\theta \in I_B$, $r_+ < r^B_+(\theta) < r^B_{max,+}$.
  Therefore, in the case of the occurrence of zeros of A and/or B in the d.o.c., it is necessary to obtain a non-singular term $\frac{W}{AB}$ in this domain, which requires the remainder of the polynomial divisions

$$W : (r - r^A_+)\,, \quad \text{and/or} \quad W : (r - r^B_+)\,, \tag{2.137}$$

  to vanish $\forall\theta \in I_A$ and $\forall\theta \in I_B$ respectively. A computation of the corresponding remainder polynomials with MATHEMATICA yields the conditions

$$\begin{aligned} p_1(z)\sqrt{-z^2 - za\mathscr{A} + \frac{2P^2\Sigma}{\Sigma - \sqrt{3}M}} + p_2(z) = 0\,, \\ p_3(z)\sqrt{-z^2 - za\mathscr{B} + \frac{2Q^2\Sigma}{\sqrt{3}M + \Sigma}} + p_4(z) = 0\,, \end{aligned} \tag{2.138}$$

  where we have applied the replacement $a\cos\theta \to z$, and p_1 and p_3 are first order and p_2 and p_4 are second order polynomials in the variable z. From (2.138) it follows, that the solution-set of those equations is discrete. Thus W will potentially only factorize for discrete elements of I_A and I_B and not on the full intervals.

  $P = 0$ case: In this case (2.138) reduces to

$$-4Q^2z^2 = 0\,, \tag{2.139}$$

$$\begin{aligned} &-\frac{4Q^4\Sigma^2}{\left(\sqrt{3}M+\Sigma\right)^2} + \frac{6Q^4\Sigma}{\sqrt{3}M+\Sigma} - \frac{20Q^2\Sigma^3}{3\left(\sqrt{3}M+\Sigma\right)} - \frac{4\sqrt{3}MQ^2\Sigma^2}{\sqrt{3}M+\Sigma} - \frac{8MQ^2\Sigma^2}{\sqrt{3}\left(\sqrt{3}M+\Sigma\right)} \\ &-\frac{8M\Sigma^3}{3\sqrt{3}} + 2\sqrt{3}M\Sigma z^2 + \frac{2M\Sigma z^2}{\sqrt{3}} + 4Q^2\Sigma^2 - 4Q^2z^2 + \frac{8\Sigma^4}{9} + \frac{8\Sigma^2z^2}{3} \\ &\left(\frac{4\sqrt{3}Q^2\Sigma^2}{\sqrt{3}M+\Sigma} + \frac{4MQ^2\Sigma}{\sqrt{3}M+\Sigma} + \frac{16M\Sigma^2}{3} - 4\sqrt{3}Q^2\Sigma\right)\sqrt{\frac{2Q^2\Sigma}{\sqrt{3}M+\Sigma} - z^2} = 0\,. \end{aligned} \tag{2.140}$$

  The first equation corresponds to a double zero case discussed below, the second one cannot be fulfilled for the whole range of values of I_B, by the same arguments as in the $P \neq 0$ case.

ii) $r^A_{max,+} = \frac{\Sigma}{\sqrt{3}}$, $r^B_{max,+} = -\frac{\Sigma}{\sqrt{3}}$ case(double zero case):

From (2.124) and (2.126) it follows that A and/or B attain a double zero, located at $r = \frac{\Sigma}{\sqrt{3}}$ and $r = -\frac{\Sigma}{\sqrt{3}}$ respectively, if

$$\begin{cases} \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} - a^2(1-|\mathscr{A}|) = 0, \text{ if } |\mathscr{A}| > 2 \quad \text{or} \\ \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} + \frac{a^2\mathscr{A}^2}{4} = 0, \text{ if } |\mathscr{A}| \leq 2, \end{cases} \tag{2.141}$$

and/or

$$\begin{cases} \frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} - a^2(1-|\mathscr{B}|) = 0, \text{ if } |\mathscr{B}| > 2 \quad \text{or} \\ \frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} + \frac{a^2\mathscr{B}^2}{4} = 0, \text{ if } |\mathscr{B}| \leq 2. \end{cases} \tag{2.142}$$

We rewrite (2.136) in the form

$$r^A_{\pm} = \frac{\Sigma}{\sqrt{3}} \pm \sqrt{p_5(z)}, \quad r^B_{\pm} = -\frac{\Sigma}{\sqrt{3}} \pm \sqrt{p_6(z)}, \tag{2.143}$$

where

$$\begin{aligned} p_5(z) &= -z^2 - az\mathscr{A} + \frac{2P^2\Sigma}{\sqrt{3}M - \Sigma}, \\ p_6(z) &= -z^2 - az\mathscr{B} + \frac{2Q^2\Sigma}{\sqrt{3}M + \Sigma}, \end{aligned} \tag{2.144}$$

and $\mathscr{A}$ and $\mathscr{B}$ are given by (2.94) and (2.118) respectively. The leading monomial of p_5 and p_6 is negative, thus those polynomials only attain positive real values between their zeros. Thus a necessary criterion for A or B to have an isolated double zero $r^A_{max,+} = \frac{\Sigma}{\sqrt{3}}$, $r^B_{max,+} = -\frac{\Sigma}{\sqrt{3}}$ is, that p_5 or p_6 have a double zero $z^*_A = -\frac{a\mathscr{A}}{2} \in [-a, a]$ or $z^*_B = -\frac{a\mathscr{B}}{2} \in [-a, a]$. In this case of a double zero of A or B the remainder polynomial of the polynomial division

$$W : \left(r - \frac{\Sigma}{\sqrt{3}}\right)^2, \quad \text{or} \quad W : \left(r + \frac{\Sigma}{\sqrt{3}}\right)^2, \tag{2.145}$$

has to vanish. An analysis with MATHEMATICA yields the conditions

$$\begin{aligned} r\,p_7(z) + p_8(z) = 0, \\ r\,p_9(z) + p_{10}(z) = 0, \end{aligned} \tag{2.146}$$

where p_7, p_8, p_9 and p_{10} are polynomials in the variable z. This finally yields the following conditions to obtain a non-singular function $\frac{W}{AB}$ in the case of a double zero of A or B in the d.o.c.

$$p_7(z^*_A) = p_8(z^*_A) = 0, \quad z^*_A \in [-a, a], \text{ or} \tag{2.147}$$

$$p_9(z^*_B) = p_{10}(z^*_B) = 0, \quad z^*_B \in [-a, a]. \tag{2.148}$$

• Double zeros of A in the d.o.c.

From $p_7(z_A^*) = 0$ we obtain the equation

$$\mathscr{A} JPQ\left(3P^2\left(3M+\sqrt{3}\Sigma\right)-\left(3M-\sqrt{3}\Sigma\right)\left(3M^2-\Sigma^2\right)\right)=0\,. \qquad (2.149)$$

By taking the constraint (2.10) for P into account, (2.149) reads as

$$27\sqrt{3}M^4+54M^3\Sigma-27\sqrt{3}M^2Q^2-18M\Sigma^3+9\sqrt{3}Q^2\Sigma^2-3\sqrt{3}\Sigma^4=0\,. \qquad (2.150)$$

The set of allowed solutions of this equation for Σ is given by $\left\{-\sqrt{3}(M+Q), \sqrt{3}(-M+Q)\right\}$. Both solutions inserted in (2.3) yield $F=0$ and thus (2.95) implies $\mathscr{A}=0$ and $z_A^*=0$. With those values (2.103) yields

$$r^A_{max,+} \quad = \quad \frac{\Sigma}{\sqrt{3}}+\sqrt{\frac{2P^2\Sigma}{\Sigma-M\sqrt{3}}}\,, \qquad (2.151)$$

and therefore the condition $P=0$ for a double zero. Thus in the generic case a double zero of A is forbidden in the d.o.c.

$P=0$ case: In this case (2.95) and (2.118) yield $\mathscr{A}=0$, $\mathscr{B}=0$ and $z_A^*=0$. Thus from (2.103) we obtain

$$r^A_{max,+} \quad = \quad \frac{\Sigma}{\sqrt{3}}\,. \qquad (2.152)$$

Inserting those values and the solution of (2.10) for Q in (2.117) yields

$$r^B_{max,+} \quad = \quad \frac{\Sigma}{\sqrt{3}}\,. \qquad (2.153)$$

As a result a double zero of B is excluded and a curve γ, parametrized by θ of zeros of B is located in the d.o.c., which is forbidden.

• Double zeros of B in the d.o.c.

In the generic case $p_9(z_B^*) = 0$ and $p_{10}(z_B^*) = 0$ yield complicated non-polynomial equations, which are hard to analyse.

In the $P=0$ case the conditions $p_9(z_B^*) = 0$ and $p_{10}(z_B^*) = 0$ reduce to

$$\sqrt{3}M+\Sigma=0 \;\text{ and }\; M+\sqrt{3}\Sigma=0\,, \qquad (2.154)$$

with only the trivial solution $M=\Sigma=0$ allowed, which is not of interest.

- **The a = 0 case**:

  In this case (2.138) reduces to

$$
\begin{aligned}
&\frac{8P^2Q^2\Sigma}{\Sigma-\sqrt{3}M}=0\,,\\
&36M^4\Sigma\left(\Sigma-2\sqrt{6}\sqrt{\frac{Q^2\Sigma}{\sqrt{3}M+\Sigma}}\right)\\
&-36M^3\left(3\sqrt{2}P^2\sqrt{\frac{Q^2\Sigma}{\sqrt{3}M+\Sigma}}+2\sqrt{2}\Sigma^2\sqrt{\frac{Q^2\Sigma}{\sqrt{3}M+\Sigma}}-3\sqrt{2}Q^2\sqrt{\frac{Q^2\Sigma}{\sqrt{3}M+\Sigma}}+\left(-\sqrt{3}\right)\Sigma\left(P^2+Q^2\right)\right)\\
&+3M^2\left(6P^2\left(2\Sigma\left(\Sigma-2\sqrt{6}\sqrt{\frac{Q^2\Sigma}{\sqrt{3}M+\Sigma}}\right)+3Q^2\right)+8\sqrt{6}\Sigma^3\sqrt{\frac{Q^2\Sigma}{\sqrt{3}M+\Sigma}}+9P^4-27Q^4+12Q^2\Sigma^2-8\Sigma^4\right)\\
&+6M\Sigma\left(3\left(P^2+Q^2\right)-2\Sigma^2\right)\left(-2\sqrt{2}\Sigma\sqrt{\frac{Q^2\Sigma}{\sqrt{3}M+\Sigma}}+\sqrt{3}P^2+\sqrt{3}Q^2\right)\\
&+\left(3\Sigma\left(P^2+Q^2\right)-2\Sigma^3\right)^2=0\,.
\end{aligned}
\tag{2.155}
$$

  The first equation corresponds to a double zero case, discussed above, the second one is fulfilled under the constraint (2.10). From (2.134) the requirement follows additionally, that B has to factorize in the larger zero of A, i.e.

$$B:(r-r_+^A) \tag{2.156}$$

  has to vanish. This condition yields in the $a=0$ case $P=\pm\sqrt{\frac{2}{3}}\sqrt{\Sigma^2-\sqrt{3}M\Sigma}$. By the insertion of this expression in (2.134) we obtain $r^B_{max,+}=r^B_{max,-}=-\frac{\Sigma}{\sqrt{3}}$ and thus the double zero case, which has been discussed above.

  Summarising, we can exclude the possibility of zeros in the d.o.c., except perhaps if $P\neq 0, a\neq 0$ and (2.142) holds.

□

2.9 Regularity at the outer Killing horizon $\mathscr{H}_+$

The outer Killing horizon $\mathscr{H}_+$ of the Killing field

$$k=\partial_t+\Omega_\phi\partial_\phi+\Omega_4\partial_{x^4}\,, \tag{2.157}$$

is given by the larger root r_+ (2.88) of Δ. The condition that $\mathscr{H}_+$ is a Killing horizon for k is that the pullback of

$$g_{\mu\nu}k^\nu\,, \tag{2.158}$$

to $\mathscr{H}_+$ vanishes. This, together with

$$\Delta\big|_{\mathscr{H}_+}=0\,,\quad \Delta_\theta\big|_{\mathscr{H}_+}=-a^2\sin^2(\theta)\,, \tag{2.159}$$

yields

$$
\begin{aligned}
\Omega_\phi &= -\frac{1}{\omega^0{}_\phi}\bigg|_{\mathcal{H}_+} \\
&= \frac{a^2}{2J}(r_+ + E)^{-1}\,, \\
\Omega_4 &= -\frac{2(A_t\omega^0{}_\phi - A_\phi)}{\omega^0{}_\phi}\bigg|_{\mathcal{H}_+} \\
&= \frac{Q\left(-3Mr_+ - \sqrt{3}M\Sigma + 3P^2 + 3Q^2 + \sqrt{3}r\Sigma - 3\Sigma^2\right)}{(E + r_+)\left(3M^2 + 2\sqrt{3}M\Sigma - 3Q^2 + \Sigma^2\right)}\,.
\end{aligned}
\tag{2.160}
$$

Under the coordinate transformation

$$
\bar\phi = \phi - \Omega_\phi\, dt\,, \quad \bar x^4 = x^4 - \Omega_4\, dt\,, \tag{2.161}
$$

the metric (2.1) takes the following form

$$
g = g_S + \frac{dr^2}{\Delta} + \Delta U dt^2\,, \tag{2.162}
$$

where g_S is a smooth $(0,2)$-tensor and $U = \frac{g_{tt}}{\Delta}$. Introducing a new time coordinate by

$$
\tau = t - \sigma \ln(r - r_+) \quad \Longrightarrow \quad d\tau = dt - \frac{\sigma}{r - r_+}dr\,, \tag{2.163}
$$

where σ is a constant to be chosen, in (2.162) yields

$$
\begin{aligned}
g &= g_S + \Delta U\left(d\tau + \frac{\sigma}{r - r_+}dr\right)^2 + \frac{dr^2}{\Delta} \\
&= g_S + \Delta U d\tau^2 + \frac{2\Delta U\sigma}{r - r_+}d\tau dr + \left(\frac{1}{\Delta} + \frac{\Delta U\sigma^2}{(r - r_+)^2}\right)dr^2 \\
&= g_S + \Delta U d\tau^2 + \frac{2\Delta U\sigma}{r - r_+}d\tau dr + \underbrace{\frac{(r - r_+)^2 + \Delta^2 U\sigma^2}{\Delta(r - r_+)^2}}_{V}dr^2\,.
\end{aligned}
\tag{2.164}
$$

In order to obtain a smooth metric in the d.o.c., σ has to be chosen in a way that the numerator of V attains a triple-zero at $r = r_+$. A computation, using MATHEMATICA, gives a lengthy algebraic expression(therefore, not given in explicit form her) for σ, fulfilling this requirement.

2.9.1 Kerr case: $\Sigma = 0$, $Q = 0$ and $P = 0$

In the Kerr case (2.160) reduces to

$$
\begin{aligned}
\Omega_\phi &= \frac{a}{Mr_+} \\
&= \frac{a}{r_+^2 + a^2}\,, \\
\Omega_4 &= 0\,.
\end{aligned}
\tag{2.165}
$$

The coordinate transformation

$$v = t + \int \frac{r^2 + a^2}{\Delta} dr\,, \quad u = \phi + \int \frac{a}{\Delta} dr\,, \tag{2.166}$$

resolves the $\Delta = 0$ coordinate singularity in the Kerr case and thus provides an analytic extension of the metric.

2.10 Stable causality

LEMMA 2.6 *If (2.124) and (2.126) hold,*

i) g^{00} *has no zeros in the d.o.c. if*

$$r_+ \geq \mathscr{C}\,, \quad r_+ > -E\,, \tag{2.167}$$

or $a = 0$ *holds, where* $\mathscr{C} := \frac{EM+q}{M+E}$ *and* $q := P^2 + Q^2 - \Sigma^2 + a^2$.

ii) The metric is stably causal for small values of $|P|$ *if* $M > \frac{\Sigma}{\sqrt{3}}$.

PROOF:

i): With (2.11) and the insertion of the expression (2.4) for ${\omega^0}_\phi$ we obtain

$$\begin{aligned} g^{00} &= \left(-\frac{B}{\Delta_\theta} + \frac{({\omega^0}_\phi)^2 \Delta_\theta}{A\Delta} \frac{1}{\sin^2\theta} \right) \\ &= \left(-\frac{B}{\Delta_\theta} + \frac{4J^2[r+E]^2 \sin^2\theta}{A\Delta\Delta_\theta} \right) \\ &= \frac{1}{\Delta_\theta} \underbrace{\left(-B + \frac{4J^2[r+E]^2 \sin^2\theta}{A\Delta} \right)}_{:= w(r,\theta)}. \end{aligned} \tag{2.168}$$

We list the following properties of important functions involved in the proof:

- A, B(if (2.124) and (2.126) hold) and Δ are strictly monotonically increasing(s.m.i.) for $r \in [r_+, \infty)$ for all values of θ,
- $\Delta_\theta > 0$ if $r > R_+$, $\Delta_\theta < 0$ for $r_+ < r < R_+$, for all values of θ,
- $g^{00} \to -1$ as $r \to \infty$,
- $w(r = R_+(\theta), \theta) = 0$ (MATHEMATICA result: The numerator of w factorizes in Δ_θ),

where r_+ and R_+ are given by (2.88).

Since w factorizes in Δ_θ, g^{00} has no poles in the doc and R_+ is a zero of w. From the properties above it follows, if w is strictly monotonically decreasing for $r \in [r_+, \infty)$, for all values of θ, we have $\Delta_\theta > 0$, $w < 0 \Rightarrow g^{00} < 0$ if $r > R_+$ and $\Delta_\theta < 0$, $w > 0 \Rightarrow g^{00} < 0$ for $r_+ < r < R_+$, for all values of θ, and thus by the continuity of the functions involved no zeros of g^{00} in the d.o.c. In order to derive the conditions, so that w is strictly monotonically decreasing , we write (2.168) in the form

$$w(r,\theta) = \underbrace{-B}_{\text{strictly monotonically decreasing}} + \underbrace{\frac{4J^2\sin^2\Theta}{A}}_{\text{monotonically decreasing}} \cdot \underbrace{\frac{(r+E)^2}{\Delta}}_{:=u(r)} . \quad (2.169)$$

Thus it remains to derive the restrictions on the parameters, so that $u(r)$ is a monotonically decreasing function. We define $q := P^2 + Q^2 - \Sigma^2 + a^2$ and write $u(r)$ in the form

$$\begin{aligned} u(r) &= \frac{(r+E)^2}{\Delta} = \frac{(r+E)^2}{r^2 - 2Mr + q} = \frac{(r+E)^2}{(r+E)^2 - 2Mr - 2rE - E^2 + q} \\ &= \frac{(r+E)^2}{(r+E)^2 - 2r(E+M) - E^2 + q} \\ &= \frac{(r+E)^2}{(r+E)^2 - 2(r+E)(E+M) + 2E(E+M) - E^2 + q} \\ &= \frac{1}{1 - \frac{2(E+M)}{(r+E)} + \frac{2E(E+M)-E^2+q}{(r+E)^2}} . \end{aligned} \quad (2.170)$$

We define the denominator function of (2.170) by

$$D(r) := 1 - \frac{2(E+M)}{(r+E)} + \frac{2E(E+M) - E^2 + q}{(r+E)^2} . \quad (2.171)$$

If $r > -E$, (2.170) is monotonically decreasing, if

$$\begin{aligned} D'(r) &= \frac{2(E+M)}{(r+E)^2} - 2\frac{2E(E+M) - E^2 + q}{(r+E)^3} \geq 0 \\ &\Rightarrow \quad (E+M)(r+E) \geq 2E(E+M) - E^2 + q \\ &\Rightarrow \quad r > -E + \frac{2E(E+M) - E^2 + q}{M+E} , \end{aligned} \quad (2.172)$$

Thus if

$$r_+ \geq -E + \frac{2E(E+M) - E^2 + q}{M+E} = \underbrace{\frac{EM+q}{M+E}}_{:=\mathscr{C}} , \quad (2.173)$$

and $r_+ > -E$ it follows, that u is monotonically decreasing and thus w is strictly monotonically decreasing on $r \in [r_+, \infty)$ for all values of θ, where $r_+ = M + \sqrt{M^2 - q}$, given by (2.88).

From (2.173) we can derive the following inequality

$$\begin{aligned}
M+\sqrt{M^2-q} &\geq \frac{EM+q}{M+E} \\
(M+E)(M+\sqrt{M^2-q}) &\geq EM+q \\
M(M+\sqrt{M^2-q})+E\sqrt{M^2-q} &\geq q \\
(E+M)\sqrt{M^2-q} &\geq q-M^2=-\left(\sqrt{M^2-q}\right)^2 \\
E+M &\geq -\sqrt{M^2-q}\,. \qquad (2.174)
\end{aligned}$$

In the $E+M\geq 0$ case (2.174) is fulfilled trivially, for the $E+M<0$ case we obtain

$$\begin{aligned}
E^2+2ME+M^2 &\leq M^2-q \\
-E(E+2M) &\geq q\,. \qquad (2.175)
\end{aligned}$$

Thus finally

$$r_+\geq \mathscr{C} \Longleftrightarrow E+M\geq 0 \vee \left(E+M<0 \wedge -E(E+2M)\geq q\right). \qquad (2.176)$$

$a=0$: In this case (2.168) yields $g^{00}=-\frac{B}{\Delta}$. Since B and Δ are positive functions in the d.o.c. if (2.126) holds, $g^{00}<0$ in the d.o.c. holds as well.

Remarks:

- For $M=8$, $a=\frac{33}{10}$, $Q=\frac{8}{5}$, $\Sigma=-\frac{23}{5}$, $P=-\frac{1}{5}\sqrt{\frac{2\left(4105960\sqrt{3}+2770943\right)}{12813}}\approx -7.86$ (see Fig. 1 in Section 4) for example one obtains $r_+\approx 11.16>\mathscr{C}\approx 5.67$ and $r_+>E\approx -3.70$ and thus a stably causal d.o.c.

- The if and only if statement:

  (2.169) can alternatively be written in the form

  $$w=\frac{-AB\Delta+4J^2(E+r)^2\sin^2\theta}{A\Delta}\,. \qquad (2.177)$$

  We define the numerator function of (2.177) by

  $$N_w:=-AB\Delta+4J^2(E+r)^2\sin^2\theta\,. \qquad (2.178)$$

  Since w factorizes in Δ_θ, it follows from (2.168) that the exclusion of zeros of g^{00} in the d.o.c. leads to the following question:

  Are all real zeros of $N_w:G$ located inside the outer Killing horizon($r=r_+$), for all values of θ, if the constraint and $a, M, Q, \Sigma, P\in\mathbb{R}$ hold?

The question leads to the general localisation of the zeros of the fourth-order polynomial $N_w : G$ in relation to the location outer Killing horizon r_+ in the variable r. We have neither been able to pursue a reasonable strategy, nor to construct a counter example. For small values of $|P|$ the problem has been solved in the section on the equivalent Larsen metrics, see Lemma 3.3, where the relevant terms take an easier form due to the more favourable parametrization.

ii):

- $P = 0$ case:

  With (2.88), we can write $r_+ < \mathscr{C}$ in the form

$$\sqrt{-a^2 + M^2 - Q^2 + \Sigma^2} < \frac{\left(3M^2 + 2\sqrt{3}M\Sigma - 3Q^2 + \Sigma^2\right)\left(a^2 - M^2 + Q^2 - \Sigma^2\right)}{\left(3M + \sqrt{3}\Sigma\right)\left(M^2 - Q^2 + \Sigma^2\right)} . \tag{2.179}$$

  Solving (2.10) for Q and insertion in (2.179) yields

$$M + \sqrt{\left(M - \frac{\Sigma}{\sqrt{3}}\right)^2 - a^2} < \frac{\Sigma}{\sqrt{3}} + \frac{a^2}{M - \frac{\Sigma}{\sqrt{3}}} . \tag{2.180}$$

  Simplifying (2.180) yields

$$1 < -\frac{\sqrt{\left(M - \frac{\Sigma}{\sqrt{3}}\right)^2 - a^2}}{\left(M - \frac{\Sigma}{\sqrt{3}}\right)} , \tag{2.181}$$

  and thus a contradiction.

- Small $|P|$ case:

  From iii) of Lemma 3.3, requiring $m > 0$, together with $m = M - \frac{\Sigma}{\sqrt{3}}$ from (3.20), the statement follows.

□

3 The Larsen solutions

The line element of the Larsen solutions [14] is given by

$$ds_5^2 = \frac{H_2}{H_1}(dx^4 + \mathbf{A}_L)^2 - \frac{H_3}{H_2}(dt + \mathbf{B}_L)^2 + H_1\left(\frac{dr^2}{\Delta_L} + d\theta^2 + \frac{\Delta_L}{H_3}\sin^2\theta d\phi^2\right), \tag{3.1}$$

where

$$\begin{aligned} H_1 &= r^2 + a_L^2\cos^2\theta + r(p-2m) + \frac{p}{p+q}\frac{(p-2m)(q-2m)}{2} \\ &\quad - \frac{p}{2m(p+q)}\sqrt{(q^2-4m^2)(p^2-4m^2)}\; a_L\cos\theta, \end{aligned} \tag{3.2}$$

$$\begin{aligned} H_2 &= r^2 + a_L^2\cos^2\theta + r(q-2m) + \frac{q}{p+q}\frac{(p-2m)(q-2m)}{2} \\ &\quad + \frac{q}{2m(p+q)}\sqrt{(q^2-4m^2)(p^2-4m^2)}\; a_L\cos\theta, \end{aligned} \tag{3.3}$$

$$H_3 = r^2 + a_L^2\cos^2\theta - 2mr, \tag{3.4}$$

$$\Delta_L = r^2 + a_L^2 - 2mr, \tag{3.5}$$

the 1-forms in (3.1) are given by

$$\mathbf{A}_L = A_t dt + A_\phi d\phi, \quad \mathbf{B}_L = B_\phi d\phi, \tag{3.6}$$

where

$$\begin{aligned} A_t &= -\left[2Q_L\left(r + \frac{p-2m}{2}\right) + \sqrt{\frac{q^3(p^2-4m^2)}{4m^2(p+q)}}\, a_L\cos\theta\right] H_2^{-1} \\ A_\phi &= -\Bigg[2P_L(H_2 + a_L^2\sin^2\theta)\cos\theta + \sqrt{\frac{p(q^2-4m^2)}{4m^2(p+q)^3}} \times \\ &\qquad \times \left[(p+q)(pr - m(p-2m)) + q(p^2-4m^2)\right] a_L\sin^2\theta\Bigg] H_2^{-1}, \\ B_\phi &= -\sqrt{pq}\,\frac{(pq+4m^2)r - m(p-2m)(q-2m)}{2m(p+q)H_3}\; a_L\sin^2\theta. \end{aligned} \tag{3.7}$$

The parameters (m, a, q, p) are related to the physical mass M, angular momentum J, electric charge Q, and magnetic charge P by

$$G_4 M_L = \frac{p+q}{4}, \tag{3.8}$$

$$G_4 J_L = a\,\frac{\sqrt{pq}(pq+4m^2)}{4m(p+q)}, \tag{3.9}$$

$$Q_L^2 = \frac{q(q^2-4m^2)}{4(p+q)}, \tag{3.10}$$

$$P_L^2 = \frac{p(p^2-4m^2)}{4(p+q)}, \tag{3.11}$$

where G_4 is the gravitational constant of four-dimensional gravity. Furthermore, the requirement

$$q, p \geq 2m\,,$$

is imposed. Note that the equality case corresponds to the absence of electric or magnetic charge, respectively.
Here we have corrected a typographic error in [14], where the overall sign of B_ϕ was opposite to the one in (3.7). I am grateful to Maciej Maliborski for pointing this out. We have not been able to check by a direct MATHEMATICA calculation that the metric (3.1) satisfies the Einstein equation for all parameters, but have checked that it does so for a sample of random values of parameters. We note that the opposite sign in (3.7) does not lead to a vacuum metric
In order to make explicit the correspondence between the parameters of the Larsen and the Rasheed solutions, we will calculate the global charges of the Larsen metric, compare them to the corresponding parameters of the Rasheed solutions, and use this correspondence to derive an isometric transformation between the metrics.

3.1 Asymptotic expansion and global charges

With the expansion of the metric coefficients

$$\begin{aligned}
g_{tt} &= -1+\frac{q}{r}+O\left(r^{-2}\right),\\
g_{rr} &= 1+\frac{p}{r}+O\left(r^{-2}\right),\\
g_{\theta\theta} &= r^2+r(p-2m)+O(1)\,,\\
g_{\phi\phi} &= r^2\sin^2\theta+r\sin^2\theta(p-2m)+4P_L^2\cos^2\theta+O(1)\,,\\
g_{44} &= 1+\frac{q-p}{r}+O\left(r^{-2}\right),\\
g_{t\phi} &= O\left(r^{-1}\right),\\
g_{t4} &= -\frac{2Q_L}{r}+O\left(r^{-2}\right),\\
g_{\phi 4} &= -2P_L\cos\theta+O\left(r^{-1}\right),
\end{aligned} \tag{3.12}$$

we can take the line element (3.1) of the Larsen solutions in the asymptotic form

$$\begin{aligned}
ds^2 &= \left(-1+\frac{q}{r}\right)dt^2+\left(1+\frac{p}{r}\right)dr^2+\left(r^2+r(p-2m)\right)d\theta^2\\
&\quad+\left(r^2\sin^2\theta+r\sin^2\theta(p-2m)+4P_L^2\cos^2\theta\right)d\phi^2\\
&\quad+\left(1+\frac{q-p}{r}\right)(dx^4)^2-4P_L\cos\theta\, d\phi\, dx^4-\frac{4Q_L}{r}dt\,dx^4+O\left(r^{-2}\right)\\
&= -dt^2+dr^2+r^2d\theta^2+r^2\sin^2\theta\, d\phi^2+(dx^4)^2-4P_L\cos\theta\, d\phi\, dx^4+4P_L^2\cos^2\theta\, d\phi^2\\
&\quad+\frac{q}{r}dt^2+\frac{p}{r}dr^2+r(p-2m)d\theta^2-\frac{4Q_L}{r}dt\,dx^4+O\left(r^{-2}\right)\\
&= -dt^2+dr^2+r^2d\theta^2+r^2\sin^2\theta\, d\phi^2+\left(dx^4-2P_L\cos\theta\, d\phi\right)^2\\
&\quad+\frac{q}{r}dt^2+\frac{p}{r}dr^2+r(p-2m)d\theta^2-\frac{4Q_L}{r}dt\,dx^4+O\left(r^{-2}\right)
\end{aligned}$$

$$= (\hat{g}_L)_{\mu\nu} dx^\mu dx^\nu + \frac{q}{r} dt^2 + \frac{p}{r} dr^2 + r(p-2m) d\theta^2 - \frac{4Q_L}{r} dt dx^4 + O\left(r^{-2}\right), \quad (3.13)$$

where the decay order of the error terms is indicated with respect to the obvious asymptotically Cartesian coordinates (t, x, y, z, x^4). Here we have defined

$$\hat{g}_L := -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 + \left(dx^4 - 2P_L \cos\theta d\phi\right)^2, \quad (3.14)$$

as the (asymptotic) background metric of the Larsen solutions. By comparing (3.14) with the background metric (2.80) of the Rasheed solutions, we obtain that the magnetic charge parameter of the Rasheed- and Larsen solutions are related by $P = -P_L$ and that x^4 has to be $8\pi P$-periodic (for details see Section 4). In the $P_L = 0$ case in a Cartesian-type basis (t, x, y, z, x^4) (3.13) takes the form

$$g = \begin{pmatrix} -1+\frac{q}{r} & 0 & 0 & 0 & -\frac{2Q_L}{r} \\ 0 & 1+\frac{2mx^2}{r^3} & \frac{2mxy}{r^3} & \frac{2mxz}{r^3} & 0 \\ 0 & \frac{2mxy}{r^3} & 1+\frac{2my^2}{r^3} & \frac{2myz}{r^3} & 0 \\ 0 & \frac{2mxz}{r^3} & \frac{2myz}{r^3} & 1+\frac{2mz^2}{r^3} & 0 \\ -\frac{2Q_L}{r} & 0 & 0 & 0 & 1-\frac{2m}{r}+\frac{q}{r} \end{pmatrix} + O\left(r^{-2}\right) \quad (3.15)$$

When $P_L \neq 0$ the expansions are considerably more complicated and not very enlightening, therefore we do not include them here.
From (3.15) and for the analogous expansion for $P \neq 0$, with the formulae derived in Section 4, we obtain for the Hamiltonian momentum p_μ of the level sets of t, the ADM four-momentum $p_{\mu,ADM}$ of the space-metric $g_{ij} dx^i dx^j$:

$$p_{i,ADM} = p_i = 0, \quad p_{0,ADM} = \begin{cases} m, & P=0, \\ \frac{p}{2}, & P\neq 0, \end{cases}, \quad p_0 = \begin{cases} 2\pi\left(\frac{m}{2}+\frac{q}{4}\right), & P=0, \\ 4\pi P\left(\frac{p}{4}+\frac{q}{4}\right) & P\neq 0, \end{cases} \quad p_4 = \begin{cases} -2\pi Q_L, & P=0, \\ -8\pi P Q_L, & P\neq 0. \end{cases} \quad (3.16)$$

The Komar integrals associated with $X = \partial_t$ are

$$\frac{1}{8\pi} \lim_{R\to\infty} \int_{S(R)} \int_{S^1} X^{\alpha;\beta} dS_{\alpha\beta} = \begin{cases} 2\pi \frac{q}{2}, & P=0, \\ 8\pi P \frac{q}{2}, & P\neq 0, \end{cases} \quad (3.17)$$

wheras those associated with $X = \partial_4$ are given by

$$\frac{1}{8\pi} \lim_{R\to\infty} \int_{S(R)} \int_{S^1} X^{\alpha;\beta} dS_{\alpha\beta} = \begin{cases} -4\pi Q_L, & P=0, \\ -16\pi P Q_L, & P\neq 0. \end{cases} \quad (3.18)$$

Furthermore, we note that the inequality (6.9) of Section 4, resulting from a Witten-type positive energy argument, in terms of the parameters of the Larsen solutions reads as

$$(p+q)^2 \geq 4q(q^2-4m^2). \quad (3.19)$$

3.1.1 A comparison with the global charges of the Rasheed metrics

By comparing the ADM mass and the Komar integrals (3.16)- (3.18) with those of the Rasheed solutions (2.82)-(2.84), we obtain

$$M - \frac{\Sigma}{\sqrt{3}} = \frac{p}{2}, \quad M + \frac{\Sigma}{\sqrt{3}} = \frac{q}{2}, \quad Q = -Q_L. \quad (3.20)$$

Solving (3.20) for M, P, Q, Σ yields

$$M = \frac{p+q}{4}\,, \quad \Sigma = \frac{\sqrt{3}(q-p)}{4}\,, \quad Q = -Q_L\,, \quad P = -P_L\,. \tag{3.21}$$

By insertion of (3.21) in (2.10) it verifies, that the Rasheed constraint holds. The insertion of (3.21) in (2.2) and (2.4) yields

$$J = \pm a_L \, \frac{\sqrt{pq}(pq+4m^2)}{4m(p+q)}\,. \tag{3.22}$$

A comparison of (3.21) and (3.22) with (3.8), (3.9), setting $G_4 = 1$ in those expressions, finally yields the following relations

$$M = M_L\,, \quad Q = -Q_L\,, \quad P = -P_L\,, \quad |J| = |J_L|\,, \quad |a| = |a_L|\,, \tag{3.23}$$

between the physical parameters of the Rasheed and Larsen solutions.

3.1.2 An isometric transformation

LEMMA 3.1

i) The Rasheed and Larsen metrics are isometric.

ii) The isometric transformation is given by the parameter transformation

$$M = \frac{p+q}{4}\,, \quad \Sigma = \frac{\sqrt{3}(q-p)}{4}\,, \quad Q = -Q_L\,, \quad P = -P_L\,, \quad a = -a_L\,, \tag{3.24}$$

and the coordinate transformation

$$r = \hat{r} + M_L - m\,. \tag{3.25}$$

PROOF:

We rewrite the Rasheed metrics (2.1) in the form

$$ds^2_{(5)} = \frac{B}{A}\Big(dx^4 + \underbrace{2A_\mu dx^\mu}_{:=\mathbf{A}}\Big)^2 - \frac{\Delta_\theta}{B}\Big(dt + \underbrace{\omega^0{}_\phi d\phi}_{:=\mathbf{B}}\Big)^2 + A\left(\frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta}{\Delta_\theta}\sin^2\theta d\phi^2\right). \tag{3.26}$$

A comparison of (3.26) with the Larsen metrics (3.1), given by

$$ds^2_5 = \frac{H_2}{H_1}(dx^4 + \mathbf{A}_L)^2 - \frac{H_3}{H_2}(dt + \mathbf{B}_L)^2 + H_1\left(\frac{d\hat{r}^2}{\Delta_L} + d\theta^2 + \frac{\Delta_L}{H_3}\sin^2\theta d\phi^2\right), \tag{3.27}$$

where we have applied the replacement $r \to \hat{r}$ to distinguish the radial coordinate from that of the Rasheed solutions, yields that both metrics are isometric if

$$A = H_1\,, \quad B = H_2\,, \quad \Delta = \Delta_L\,, \quad \Delta_\theta = H_3\,, \quad \mathbf{A} = \mathbf{A_L}\,, \quad \mathbf{B} = \mathbf{B_L}\,, \tag{3.28}$$

provided that the coordinates r and $\hat{r}$ differ by an additive constant. A computation with MATHEMATICA yields, that if the reparametrizations

$$M = \frac{p+q}{4}, \quad \Sigma = \frac{\sqrt{3}(q-p)}{4}, \quad Q = -Q_L, \quad P = -P_L, \quad a = -a_L, \tag{3.29}$$

and the coordinate transformation

$$r = \hat{r} + M_L - m, \tag{3.30}$$

where M_L, Q_L and P_L are given by (3.8)-(3.11), are applied to (2.4)-(2.8), then (3.28) holds. □

3.2 Killing horizons, the ergosurface and the zeros of H_1 and H_2

From (3.1), by the same arguments as in the case of the equivalent Rasheed metrics, it follows, that the Killing horizons of the Larsen solutions are given by the zeros of (3.5), i.e.

$$r_\pm = m \pm \sqrt{m^2 - a_L^2}, \tag{3.31}$$

and the ergosurface is given by the lager zero of (3.4), i.e.

$$R_+ = m + \sqrt{m^2 - a_L^2 \cos^2\theta}. \tag{3.32}$$

In the following, for completeness we analyse directly the zeros of H_1 (3.2) and H_2 (3.3), which determine the singularities in the Larsen solutions.

LEMMA 3.2

i) H_1 (3.2) has no zeros in the d.o.c., if and only if

$$|\mathscr{A}_L| > 2 \;\; and \;\; \begin{cases} \mathscr{A}_L^0 - a_L^2(1-|\mathscr{A}_L|) < 0, & or \\ m + \sqrt{m^2 - a_L^2} > -\frac{p-2m}{2} + \sqrt{\mathscr{A}_L^0 - a_L^2(1-|\mathscr{A}_L|)}, & \end{cases}$$

or

$$|\mathscr{A}_L| \le 2 \;\; and \;\; \begin{cases} \mathscr{A}_L^0 + \frac{a_L^2\mathscr{A}_L^2}{4} < 0, & or \\ m + \sqrt{m^2 - a_L^2} > -\frac{p-2m}{2} + \sqrt{\mathscr{A}_L^0 + \frac{a_L^2\mathscr{A}_L^2}{4}}, & \end{cases}$$

where

$$\mathscr{A}_L^0 = \frac{(p-2m)^2}{4} - \frac{p}{p+q}\frac{(p-2m)(q-2m)}{2}, \quad \mathscr{A}_L = \frac{p\sqrt{(q^2-4m^2)(p^2-4m^2)}}{2ma_L(p+q)}.$$

ii) H_2 (3.3) has no zeros in the d.o.c., if and only if

$$|\mathscr{B}_L| > 2 \;\; and \;\; \begin{cases} \mathscr{B}_L^0 - a_L^2(1-|\mathscr{B}_L|) < 0, & or \\ m + \sqrt{m^2 - a_L^2} > -\frac{q-2m}{2} + \sqrt{\mathscr{B}_L^0 - a_L^2(1-|\mathscr{B}_L|)}, & \end{cases}$$

or

$$|\mathscr{B}_L| \le 2 \;\; and \;\; \begin{cases} \mathscr{B}_L^0 + \frac{a_L^2\mathscr{B}_L^2}{4} < 0, & or \\ m + \sqrt{m^2 - a_L^2} > -\frac{q-2m}{2} + \sqrt{\mathscr{B}_L^0 + \frac{a_L^2\mathscr{B}_L^2}{4}}, & \end{cases}$$

where

$$\mathscr{B}_L^0 = \frac{(q-2m)^2}{4} - \frac{q}{p+q}\frac{(p-2m)(q-2m)}{2}\,, \quad \mathscr{B}_L = -\frac{q\sqrt{(q^2-4m^2)(p^2-4m^2)}}{2ma_L(p+q)}\,.$$

iii) i) and ii) are equivalent to the corresponding conditions (2.124) and (2.126) for A and B of the Rasheed metrics.

PROOF:
We rewrite (3.2) and (3.3) in the form

$$\begin{aligned} H_1 &= \left(r+\frac{p-2m}{2}\right)^2 + a_L^2\cos^2\theta + \frac{p}{p+q}\frac{(p-2m)(q-2m)}{2} - \frac{(p-2m)^2}{4} \\ &\quad -\frac{p}{2m(p+q)}\sqrt{(q^2-4m^2)(p^2-4m^2)}\,a\cos\theta\,, \qquad (3.33)\\ H_2 &= \left(r+\frac{q-2m}{2}\right)^2 + a_L^2\cos^2\theta + \frac{q}{p+q}\frac{(p-2m)(q-2m)}{2} - \frac{(q-2m)^2}{4} \\ &\quad +\frac{q}{2m(p+q)}\sqrt{(q^2-4m^2)(p^2-4m^2)}\,a_L\cos\theta\,. \qquad (3.34)\end{aligned}$$

i): From (3.33) the set of zeros of H_1 is given by

$$\begin{aligned}\left(r+\frac{p-2m}{2}\right)^2 &= \frac{(p-2m)^2}{4} - \frac{p}{p+q}\frac{(p-2m)(q-2m)}{2} \\ &\quad -a^2\left(\cos^2\theta + \frac{p\sqrt{(q^2-4m^2)(p^2-4m^2)}}{2ma_L(p+q)}\cos\theta\right).\end{aligned} \tag{3.35}$$

We define

$$h_A(\theta) := \cos^2\theta + \mathscr{A}_L\cos\theta\,, \tag{3.36}$$

where

$$\mathscr{A}_L = \frac{p\sqrt{(q^2-4m^2)(p^2-4m^2)}}{2ma_L(p+q)}\,, \tag{3.37}$$

and

$$\mathscr{A}_L^0 = \frac{(p-2m)^2}{4} - \frac{p}{p+q}\frac{(p-2m)(q-2m)}{2}\,. \tag{3.38}$$

With these notations (3.35) reads as

$$\left(r+\frac{p-2m}{2}\right)^2 = \mathscr{A}_L^0 - a_L^2 h_A(\theta)\,. \tag{3.39}$$

By a comparison of the analogous function in the case of the Rasheed solutions (2.100) it follows, that the minimum of (3.36) is given by

$$h_{A,min} := \begin{cases} 1-|\mathscr{A}_L|)\,, & \text{if } |\mathscr{A}_L|>2\,, \\ -\frac{\mathscr{A}_L^2}{4}\,, & \text{if } |\mathscr{A}_L|\le 2 \end{cases}\,. \tag{3.40}$$

From (3.40) it follows, that maximum of the right side of (3.39) is given by

$$\begin{cases} \mathscr{A}_L^0 - a_L^2(1-|\mathscr{A}_L|)\,, & \text{if } |\mathscr{A}_L| > 2\,, \\ \mathscr{A}_L^0 + \frac{a_L^2\mathscr{A}_L^2}{4}\,, & \text{if } |\mathscr{A}_L| \le 2 \end{cases} . \tag{3.41}$$

(3.41) together with (3.31) yields, that H_1 has no zeros in the d.o.c., if and only if

$$|\mathscr{A}_L| > 2 \text{ and } \begin{cases} \mathscr{A}_L^0 - a_L^2(1-|\mathscr{A}_L|) < 0\,, \\ m + \sqrt{m^2 - a_L^2} > -\frac{p-2m}{2} + \sqrt{\mathscr{A}_L^0 - a_L^2(1-|\mathscr{A}_L|)}\,, \end{cases} \text{or}$$

or

$$|\mathscr{A}_L| \le 2 \text{ and } \begin{cases} \mathscr{A}_L^0 + \frac{a_L^2\mathscr{A}_L^2}{4} < 0\,, \\ m + \sqrt{m^2 - a_L^2} > -\frac{p-2m}{2} + \sqrt{\mathscr{A}_L^0 + \frac{a_L^2\mathscr{A}_L^2}{4}}\,. \end{cases} \text{or} \tag{3.42}$$

ii): From (3.34) the set of zeros of H_2 is given by

$$\begin{aligned} \left(r + \frac{q-2m}{2}\right)^2 \quad = \quad & \frac{(q-2m)^2}{4} - \frac{q}{p+q}\frac{(p-2m)(q-2m)}{2} \\ & - a_L^2\left(\cos^2\theta - \frac{q\sqrt{(q^2-4m^2)(p^2-4m^2)}}{2ma_L(p+q)}\cos\theta\right). \end{aligned} \tag{3.43}$$

We define

$$h_B(\theta) := \cos^2\theta + \mathscr{B}_L\cos\theta\,, \tag{3.44}$$

where

$$\mathscr{B}_L = -\frac{q\sqrt{(q^2-4m^2)(p^2-4m^2)}}{2ma_L(p+q)}\,, \tag{3.45}$$

and

$$\mathscr{B}_L^0 = \frac{(q-2m)^2}{4} - \frac{q}{p+q}\frac{(p-2m)(q-2m)}{2}\,. \tag{3.46}$$

With these notations (3.43) reads as

$$\left(r + \frac{q-2m}{2}\right)^2 \quad = \quad \mathscr{B}_L^0 - a_L^2 h_B(\theta)\,. \tag{3.47}$$

Then it follows from a completely analogous analysis as in the case for H_1, that H_2 has no zeros in the d.o.c., if and only if

$$|\mathscr{B}_L| > 2 \text{ and } \begin{cases} \mathscr{B}_L^0 - a_L^2(1-|\mathscr{B}_L|) < 0\,, \\ m + \sqrt{m^2 - a_L^2} > -\frac{q-2m}{2} + \sqrt{\mathscr{B}_L^0 - a_L^2(1-|\mathscr{B}_L|)}\,, \end{cases} \text{or}$$

or

$$|\mathscr{B}_L| \le 2 \text{ and } \begin{cases} \mathscr{B}_L^0 + \frac{a_L^2\mathscr{B}_L^2}{4} < 0\,, \\ m + \sqrt{m^2 - a_L^2} > -\frac{q-2m}{2} + \sqrt{\mathscr{B}_L^0 + \frac{a_L^2\mathscr{B}_L^2}{4}}\,. \end{cases} \text{or} \tag{3.48}$$

iii): By applying the isometric transformation, given in Lemma 3.1, we obtain

$$\Delta = \Delta_L\,, \quad A = H_1\,, \quad B = H_2\,. \tag{3.49}$$

From (3.49) iii) follows immediately. □

3.3 Stable causality

In this section we revisit the issue of stable causality of the Rasheed-Larsen metrics using the Larsen coordinates. Recall that well behaved black holes should be globally hyperbolic, and stable causality is a necessary condition for global hyperbolicity.
The analysis here allows us to find new regions of parameters where stable causality holds, namely small values of $|P_L|$. This was not apparent in an analysis in Rasheed coordinates.

LEMMA 3.3
If i) and ii) of Lemma 3.2 hold, the Larsen solutions (3.1) are stably causal

i) *if (but not if and only if)*

$$r_+ \geq \frac{\mathscr{E} m + a_L^2}{m + \mathscr{E}}\,, \quad \textit{and} \quad r_+ > -\mathscr{E}\,, \tag{3.50}$$

where $\mathscr{E} := -\frac{m(p-2m)(q-2m)}{(pq+4m^2)}$,

ii) *if* $P_L = 0$ *or* $Q_L = 0$,

iii) *for small values of* $|P_L|$ *if* $m > 0$.

iv) *i) is equivalent to Lemma 2.6.*

PROOF:

i): We write g^{00} in the form

$$\begin{aligned} g^{00} &= -\frac{H_2}{H_3} + \frac{B_\phi^2 H_3}{\sin^2\theta \Delta_L H_1} \\ &= \frac{1}{H_3}\left(-H_2 + \frac{B_\phi^2 H_3^2}{\sin^2\theta \Delta_L H_1}\right). \end{aligned} \tag{3.51}$$

By rewriting (3.7)

$$\begin{aligned} B_\phi &= -\sqrt{pq}\frac{(pq+4m^2)r - m(p-2m)(q-2m)}{2m(p+q)H_3}\, a\sin^2\theta \\ &= -\sqrt{pq}\frac{(pq+4m^2)}{2m(p+q)H_3}\left(r - \frac{m(p-2m)(q-2m)}{(pq+4m^2)}\right) a\sin^2\theta \\ &= -\sqrt{pq}\frac{(pq+4m^2)}{2m(p+q)H_3}\left(r - \frac{m(p-2m)(q-2m)}{(pq+4m^2)}\right) a\sin^2\theta \\ &= -\frac{2J_L}{H_3}\left(r - \frac{m(p-2m)(q-2m)}{(pq+4m^2)}\right)\sin^2\theta\,, \end{aligned} \tag{3.52}$$

where we have used (3.9), and the insertion of this expression in (3.51), we obtain

$$
\begin{aligned}
g^{00} &= -\frac{H_2}{H_3} + \frac{B_\phi^2 H_3}{\sin^2\theta \Delta_L H_1} \\
&= \frac{1}{H_3}\left(\underbrace{-H_2 + \frac{4J_L^2 (r+\mathcal{E})^2 \sin^2\theta}{\Delta_L H_1}}_{=:w(r,\theta)}\right), \qquad (3.53)
\end{aligned}
$$

where we have defined $\mathcal{E} := -\frac{m(p-2m)(q-2m)}{(pq+4m^2)}$.

We list the following properties of important functions involved, where we have defined r_+ and R_+ as the largest zero of Δ and H_3 respectively:

- H_1, H_2 (if they have no zeros in the d.o.c.(see Lemma 3.2)) and Δ are strictly increasing for $r \in [r_+, \infty)$, for all values of θ,

- $H_3 < 0$ if $r_+ < r < R_+$, $H_3 > 0$ if $r > R_+$, for all values of θ,

- $g^{00} \to -1$ as $r \to \infty$,

- $w(r = R_+(\theta), \theta) = 0$ (MATHEMATICA result: The numerator of w factorizes in H_3),

where r_+ and R_+ are given by (3.31) and (3.32) respectively.
Since the numerator of w factorizes in H_3, g^{00} has no poles in the d.o.c. and R_+ is a zero of w. From the properties above it follows, that if w is strictly monotonically decreasing for $r \in [r_+, \infty)$, for all values of θ, we obtain $H_3 > 0$, $w < 0 \Rightarrow g^{00} < 0$ if $r > R_+$ and $H_3 < 0$, $w > 0 \Rightarrow g^{00} < 0$ if $r_+ < r < R_+$, for all values of θ, and it follows, together with the properties of the functions involved, g^{00} has no zeros the d.o.c. To derive the corresponding condition, we write w in the form

$$
w(r,\theta) = \underbrace{-H_2}_{\text{strictly monotonically decreasing}} + \underbrace{\frac{4J_L^2 \sin^2\theta}{H_1}}_{\text{monotonically decreasing}} \cdot \underbrace{\frac{(r+\mathcal{E})^2}{\Delta}}_{:=u(r)}. \qquad (3.54)
$$

Thus it remains to derive the restrictions on the parameters so that $u(r)$ is a monotonically decreasing function. Assuming $r > -\mathcal{E}$, we write $u(r)$ in the form

$$
\begin{aligned}
u(r) &= \frac{(r+\mathcal{E})^2}{\Delta} = \frac{(r+\mathcal{E})^2}{r^2 - 2mr + a_L^2} = \frac{(r+\mathcal{E})^2}{(r+\mathcal{E})^2 - 2mr - 2r\mathcal{E} - \mathcal{E}^2 + a_L^2} \\
&= \frac{(r+\mathcal{E})^2}{(r+\mathcal{E})^2 - 2r(\mathcal{E}+m) - \mathcal{E}^2 + a_L^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(r+\mathscr{E})^2}{(r+\mathscr{E})^2 - 2(r+\mathscr{E})(\mathscr{E}+m) + 2\mathscr{E}(\mathscr{E}+m) - \mathscr{E}^2 + a_L^2} \\
&= \frac{1}{1 - \frac{2(\mathscr{E}+m)}{(r+\mathscr{E})} + \frac{2\mathscr{E}(\mathscr{E}+m)-\mathscr{E}^2+a_L^2}{(r+\mathscr{E})^2}}, \qquad (3.55)
\end{aligned}
$$

and define the denominator function of (3.55) by

$$
D(r) = 1 - \frac{2(\mathscr{E}+m)}{(r+\mathscr{E})} + \frac{2\mathscr{E}(\mathscr{E}+m) - \mathscr{E}^2 + a_L^2}{(r+\mathscr{E})^2}. \qquad (3.56)
$$

If $r > -\mathscr{E}$, (3.56) is monotonically decreasing if

$$
\begin{aligned}
D'(r) &= \frac{2(\mathscr{E}+m)}{(r+\mathscr{E})^2} - 2\frac{2\mathscr{E}(\mathscr{E}+m) - \mathscr{E}^2 + a_L^2}{(r+\mathscr{E})^3} \geq 0 \\
&\Rightarrow \quad (\mathscr{E}+m)(r+\mathscr{E}) \geq 2\mathscr{E}(\mathscr{E}+m) - \mathscr{E}^2 + a_L^2 \\
&\Rightarrow \quad r \geq \underbrace{-\mathscr{E} + \frac{2\mathscr{E}(\mathscr{E}+m) - \mathscr{E}^2 + a_L^2}{m+\mathscr{E}}}_{:=\mathscr{C}_L}. \qquad (3.57)
\end{aligned}
$$

Thus if

$$
r_+ \geq -\mathscr{E} + \frac{2\mathscr{E}(\mathscr{E}+m) - \mathscr{E}^2 + a_L^2}{m+\mathscr{E}} = \underbrace{\frac{\mathscr{E}m + a_L^2}{m+\mathscr{E}}}_{\mathscr{C}_L}, \qquad (3.58)
$$

and $r_+ > -\mathscr{E}$, then u is monotonically decreasing and therefore w is strictly monotonically decreasing on $r \in [r_+, \infty)$ for all values of θ. Then from (3.54) we finally obtain, that g^{00} has no zeros in the d.o.c.

An if and only if statement would require to derive conditions, so that the polynomial $h := N(w) : H_3$, where $N(w)$ denotes the numerator of w, has no zeros in the d.o.c. In the $P_L = 0 (p = 2m)$ case the polynomial is given by

$$
\begin{aligned}
h \quad &= \left(4m^2(2m+q)^2\right) r^4 + \left(4m^2(q-2m)(2m+q)^2\right) r^3 \\
&+ \left(2a_L^2 m^2(\cos(2\theta)+3)(2m+q)^2\right) r^2 \\
&+ \left(-4a_L^2 m^2(2m+q)^2(m\cos(2\theta)+m-q)\right) r \\
&+ 4a^4 m^2 \cos^2(\theta)(2m+q)^2. \qquad (3.59)
\end{aligned}
$$

Even for this reduced problem it seems hard to derive a compact system of inequalities, imposed on the parameters, guaranteeing that all real zeros of h are smaller than r_+ for all values of θ. At least in this setting the question can be answered in the following.

ii): In the $P_L = 0 (p = 2m)$ and/or $Q_L = 0 (q = 2m)$ case (3.58) reduces to

$$
m + \sqrt{m^2 - a_L^2} \geq \frac{a_L^2}{m} \Longleftrightarrow |m| \geq |a_L|, \qquad (3.60)
$$

which is imposed anyway from (3.31) to avoid naked singularities.

iii): We write the criterion for stable causality (3.58) in the form

$$\underbrace{r_+ - \mathscr{C}_L}_{:=\mathscr{W}_L} \geq 0\,. \tag{3.61}$$

With the expansion of $\mathscr{C}_L$ at $P_L = 0 \Longleftrightarrow p = 2m$

$$\mathscr{C}_L = \frac{a_L^2}{m} + \frac{\left(m^2 - a_L^2\right)(2m-q)}{\left(2m^2\right)(2m+q)}(p-2m) + O\left((p-2m)^2\right) \tag{3.62}$$

$\mathscr{W}_L$ can be written near $P_L = 0$ in the non-extremal case($|m| > |a_L|$) in the form

$$\begin{aligned}
\mathscr{W}_L \quad &= m + \sqrt{m^2 - a_L^2} - \frac{a_L^2}{m} - \frac{\left(m^2 - a_L^2\right)(2m-q)}{\left(2m^2\right)(2m+q)}(p-2m) + O\left((p-2m)^2\right) \\
&= \underbrace{m\left(1 - \left(\frac{a_L}{m}\right)^2\right) + \sqrt{m^2\left(1 - \left(\frac{a_L}{m}\right)^2\right)}}_{:=\epsilon>0} - \frac{\left(m^2 - a_L^2\right)(2m-q)}{\left(2m^2\right)(2m+q)}(p-2m) + O\left((p-2m)^2\right) \\
&= \epsilon - \frac{\left(m^2 - a_L^2\right)(2m-q)}{\left(2m^2\right)(2m+q)}(p-2m) + O\left((p-2m)^2\right)\,,
\end{aligned} \tag{3.63}$$

(3.61) together with (3.63) yields stable causality for small values of $|P_L|$ if $m > 0$ holds.

iv): By applying the transformation, given by Lemma 3.1, to the expressions in i) of Lemma 2.6, a computation with MATHEMATICA yields i).

□

4 Energy in higher-dimensional spacetimes

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Energy in higher-dimensional spacetimes

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We derive expressions for the total Hamiltonian energy of gravitating systems in higher-dimensional theories in terms of the Riemann tensor, allowing a cosmological constant $\Lambda \in \mathbb{R}$. Our analysis covers asymptotically anti–de Sitter spacetimes, asymptotically flat spacetimes, as well as Kaluza-Klein asymptotically flat spacetimes. We show that the Komar mass equals the Arnowitt-Deser-Misner (ADM) mass in stationary asymptotically flat spacetimes in all dimensions, generalizing the four-dimensional result of Beig, and that this is no longer true with Kaluza-Klein asymptotics. We show that the Hamiltonian mass does not necessarily coincide with the ADM mass in Kaluza-Klein asymptotically flat spacetimes, and that the Witten positivity argument provides a lower bound for the Hamiltonian mass—and not for the ADM mass—in terms of the electric charge. We illustrate our results on the five-dimensional Rasheed metrics, which we study in some detail, pointing out restrictions that arise from the requirement of regularity, which have gone seemingly unnoticed so far in the literature.

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I. INTRODUCTION

A key notion in any physical theory is that of total energy, momentum, and similar global charges. The corresponding definitions, and their properties, depend very much upon the asymptotic conditions satisfied by the fields. There are various possibilities here, dictated by the physical problem at hand. For instance, the vanishing and the sign of the cosmological constant play a crucial role. Next, one may find it convenient to use direct coordinate methods [1–3] or conformal methods [4,5], or else [6], to define the asymptotic conditions and the objects at hand. Finally, one may want to use definitions arising from Hamiltonian techniques [7,8], or appeal to the Noether theorem [9], or use *ad hoc* conserved currents [10–14]. See also Ref. [15] for an excellent review of early work on the subject.

A natural class of asymptotic conditions arises when considering isolated systems in Kaluza-Klein-type theories; see Sec. II below. Much to our surprise, no systematic study of the notion of energy in this context appears to exist in the literature, and one of the aims of this work is to fill this gap. For this, we derive new expressions for the total Hamiltonian energy in higher dimensions in terms of the Riemann tensor, in asymptotically flat, asymptotically Kaluza-Klein (KK), or asymptotically anti–de Sitter (AdS) spacetimes. Our definitions arise from a Hamiltonian analysis of the fields and invoke direct coordinate- or tetrad-based asymptotic conditions. We relate these integrals to Komar-type integrals. We use Witten's argument to derive global inequalities between the Hamiltonian energy-momentum and the Kaluza-Klein charges. We test our energy expressions on the Rasheed family of five-dimensional vacuum metrics, clarifying furthermore some aspects of the global structure of these solutions.

This paper is organized as follows. In Sec. II we make precise our notion of Kaluza-Klein asymptotic flatness. At the beginning of Sec. III we review the definition of energy within the Hamiltonian framework of Refs. [16,17]. In Sec. III A we apply the framework to spacetimes which are asymptotically flat in a Kaluza-Klein sense. In Sec. III B we derive general formulas which apply for a large class of asymptotic conditions. In Sec. IV we show how to rewrite the formulas derived so far in terms of the curvature tensor. This is done in Sec. IV A for KK-asymptotically flat solutions, and in Sec. IV B for general backgrounds. The formulas are then specialized in Sec. IV B 1 to asymptotically anti–de Sitter solutions, and in Sec. IV B 2 to a class of Kaluza-Klein solutions with vanishing cosmological constant which are not KK-asymptotically flat. In Sec. IV C we rewrite some of our Riemann-integral energy expressions in terms of a space-and-time decomposition of the metric. In Sec. V we show how to establish Komar-type expressions for energy in spacetimes with Killing vectors. In Sec. VI we show how a Witten-type positivity argument applies to obtaining global inequalities for KK-asymptotically flat metrics. Appendix A is devoted to a study of the geometry of Rasheed's Kaluza-Klein black holes, which provide a nontrivial family of examples for which our energy expressions can be explicitly calculated.

II. KALUZA-KLEIN ASYMPTOTICS

The starting point for our notion of Kaluza-Klein asymptotics is initial data surfaces in an $(n + K + 1)$-dimensional spacetime containing asymptotic ends of the form

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$$\mathcal{S}_{\rm ext} := (\mathbb{R}^n \backslash B(0,R)) \times \underbrace{S^1 \times \cdots \times S^1}_{K \text{ factors}} =: (\mathbb{R}^n \backslash B(0,R)) \times \mathbb{T}^K, \tag{2.1}$$

where S^1 is the unit circle. We will say that the metric is KK-asymptotically flat if g has the following asymptotic form along $\mathcal{S}_{\rm ext} \equiv \{x^0 = 0\}$:

$$g = \underbrace{\eta_{ab}dx^a dx^b + \delta_{AB}dx^A dx^B}_{=:\eta_{\mu\nu}dx^\mu dx^\nu} + o(r^{-\alpha}),$$
$$\partial_\mu g_{\nu\rho} = o(r^{-\alpha-1}), \tag{2.2}$$

where Greek indices run from 0 to $n+K$, uppercase Latin indices from the beginning of the alphabet run from $n+1$ to $n+K$, lowercase Latin indices from the beginning of alphabet run from 0 to n, and lowercase Latin indices from the middle of alphabet run from 1 to n. Finally, uppercase latin indices from the middle of the alphabet run from 1 to $n+K$. Summarizing:

$$(x^\mu) \equiv (x^0, x^i, x^A) \equiv (x^a, x^A) \equiv (x^0, x^I). \tag{2.3}$$

Last but not least,

$$r := \sqrt{(x^1)^2 + \cdots + (x^n)^2}.$$

The exponent α will be chosen to be the optimal one for the purpose of a well-posed definition of the total energy, namely,

$$\alpha = \frac{n-2}{2}, \tag{2.4}$$

where, as in Eq. (2.1), n is the space dimension *without counting the Kaluza-Klein directions*.

In Kaluza-Klein theories it is often assumed that the vector fields ∂_A are Killing vectors, but we will not make this assumption unless explicitly indicated otherwise.

III. HAMILTONIAN CHARGES

In this section we adapt the Hamiltonian analysis of Ref. [17] (based on Ref. [16], cf. Ref. [18]) to the asymptotically KK setting, which also provides convenient alternative expressions for the formulas for the Hamiltonians derived there. We use a background metric $\bar{g}_{\mu\nu}$, which is assumed to be asymptotically KK as defined in Sec. II, to determine the asymptotic conditions. The metric $\bar{g}_{\mu\nu}$ should be thought of as being the metric $\eta_{\mu\nu}$ of Sec. II at large distances, but it might be convenient in some situations to use coordinate systems where $\bar{g}_{\mu\nu}$ does not take an explicitly flat form.

Every such metric $\bar{g}_{\mu\nu}$ determines a family of metrics $g_{\mu\nu}$ which asymptote to it in the sense of Eq. (2.2). We will denote by $\bar{\Gamma}^\alpha{}_{\beta\gamma}$ the Christoffel symbols of the Levi-Civita connection of $\bar{g}_{\mu\nu}$.

Given a vector field X, the calculations in Ref. [17] showed that the flow of X in the spacetime obtained by evolving the initial data on $\mathcal{S}$ is Hamiltonian with respect to a suitable symplectic structure, with a Hamiltonian $H(X,\mathcal{S})$ which, in vacuum, is given by the formula

$$H(X,\mathcal{S}) = \int_{\mathcal{S}} (p^\mu_{\alpha\beta}\mathcal{L}_X \mathfrak{g}^{\alpha\beta} - X^\mu L)d\Sigma_\mu, \tag{3.1}$$

where

$$\begin{aligned} L := \mathfrak{g}^{\mu\nu}\Big[&(\Gamma^\alpha{}_{\sigma\mu} - \bar{\Gamma}^\alpha{}_{\sigma\mu})(\Gamma^\sigma{}_{\alpha\nu} - \bar{\Gamma}^\sigma{}_{\alpha\nu}) \\ &- (\Gamma^\alpha{}_{\mu\nu} - \bar{\Gamma}^\alpha{}_{\mu\nu})(\Gamma^\sigma{}_{\alpha\sigma} - \bar{\Gamma}^\sigma{}_{\alpha\sigma}) + \bar{\mathbf{R}}_{\mu\nu} - \frac{2}{d+K}\Lambda g_{\mu\nu}\Big] \\ &- \frac{1}{16\pi}\sqrt{-\det \bar{g}}\bar{g}^{\mu\nu}\left(\bar{\mathbf{R}}_{\mu\nu} - \frac{2}{d+K}\Lambda \bar{g}_{\mu\nu}\right), \end{aligned} \tag{3.2}$$

with $\bar{\mathbf{R}}_{\mu\nu}$ being the Ricci tensor of the background metric $\bar{g}_{\mu\nu}$, Λ the cosmological constant, d the dimension of the physical spacetime, K the number of Kaluza-Klein dimensions (possibly zero), and

$$\mathfrak{g}^{\mu\nu}\frac{1}{16\pi}\sqrt{-\det g}g^{\mu\nu},$$
$$p^\lambda_{\mu\nu} := \frac{\partial L}{\partial \mathfrak{g}^{\mu\nu}{}_{,\lambda}} = (\bar{\Gamma}^\lambda_{\mu\nu} - \delta^\lambda_{(\mu}\bar{\Gamma}^\kappa{}_{\nu)\kappa}) - (\Gamma^\lambda_{\mu\nu} - \delta^\lambda_{(\mu}\Gamma^\kappa{}_{\nu)\kappa}). \tag{3.3}$$

Finally, the volume forms $d\Sigma_\alpha$ and $d\Sigma_{\alpha\beta}$ are defined as

$$d\Sigma_\alpha = \partial_\alpha \rfloor (dx^0 \wedge \cdots \wedge dx^{n+K}), \quad d\Sigma_{\alpha\beta} = \partial_\beta \rfloor d\Sigma_\alpha, \tag{3.4}$$

where $\rfloor$ denotes the contraction: for any vector field X and skew-form α we have $X\rfloor\alpha(\cdot,\ldots) := \alpha(X,\ldots)$.

We note that the last two, g-independent "renormalization" terms in Eq. (3.2) have been added for convergence of the integrals at hand.

We will write $\det g \equiv \det(g_{\mu\nu})$ for the determinant of the full metric tensor, and explicitly write $\det(g_{IJ})$ for the determinant of the metric $g_{IJ}dx^I dx^J$ induced on the level sets of x^0, etc., when the need arises.

We emphasize that the formal considerations in Ref. [17] were quite general, and they apply regardless of the asymptotic conditions and of dimensions. However, the question of the convergence and well posedness of the resulting formulas appears to require a case-by-case analysis, once a set of asymptotic conditions has been imposed.

If X is a Killing vector field of $\bar{g}_{\mu\nu}$ and if the Einstein equations with sources and with a cosmological constant Λ are satisfied,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{3.5}$$

the integrand (3.1) can be rewritten as the divergence of a "Freud-type superpotential," up to source and renormalization terms:

$$\begin{aligned} H^\mu &\equiv p^\mu_{\alpha\beta}\mathcal{L}_X \mathfrak{g}^{\alpha\beta} - X^\mu L \\ &= \partial_\alpha \mathbb{U}^{\mu\alpha} - \sqrt{-\det g}T^\mu{}_\alpha X^\alpha + \frac{1}{16\pi}\sqrt{-\det \bar g}\bar g^{\alpha\beta} \\ &\quad \times \left(\bar{\mathbf{R}}_{\alpha\beta} - \frac{2}{d+K}\Lambda \bar g_{\alpha\beta}\right) X^\mu, \end{aligned} \tag{3.6}$$

with

$$\mathbb{U}^{\nu\lambda} = \mathbb{U}^{\nu\lambda}{}_\beta X^\beta - \frac{1}{8\pi}\sqrt{|\det g|}g^{\alpha[\nu}\delta^{\lambda]}_\beta \bar\nabla_\alpha X^\beta, \tag{3.7}$$

$$\mathbb{U}^{\nu\lambda}{}_\beta = \frac{2|\det \bar g|}{16\pi\sqrt{|\det g|}} g_{\beta\gamma}\bar\nabla_\kappa (e^2 g^{\gamma[\lambda}g^{\nu]\kappa}), \tag{3.8}$$

where $\bar\nabla$ denotes the covariant derivative of the background metric $\bar g_{\mu\nu}$ and

$$e^2 \equiv \frac{\det g}{\det \bar g}. \tag{3.9}$$

In vacuum this leads to the formula

$$H(X,\mathcal{S}) = H_b(X,\mathcal{S}) \coloneqq \frac{1}{2}\int_{\partial\mathcal{S}} (\mathbb{U}^{\nu\lambda} - \mathbb{U}^{\nu\lambda}|_{g=\bar g}) d\Sigma_{\nu\lambda}, \tag{3.10}$$

where the subscript "b" on H_b stands for "boundary." For vector fields X which are not necessarily Killing vector fields of the background, the Hamiltonian might have some supplementary volume terms, cf. Refs. [18,19]. In nonvacuum Lagrangian diffeomorphism-invariant field theories, this formula for the total Hamiltonian of the coupled system of fields remains true after adding to H^μ a contribution from the matter fields; cf., e.g., Refs. [16,19,20].

A. Kaluza-Klein asymptotics

For Kaluza-Klein asymptotically flat field configurations we have

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + o(r^{-\alpha}), \qquad \partial_\sigma g_{\mu\nu} = o(r^{-\alpha-1}), \\ \bar g_{\mu\nu} &= \eta_{\mu\nu} + o(r^{-\alpha}), \qquad \partial_\sigma \bar g_{\mu\nu} = o(r^{-\alpha-1}). \end{aligned} \tag{3.11}$$

In particular, this implies

$$\bar\Gamma^\alpha{}_{\beta\gamma} = o(r^{-\alpha-1}).$$

First, let us assume that X is $\bar g$-covariantly constant (and hence also a Killing vector of the background metric $\bar g_{\mu\nu}$). One then checks that in the coordinates of Eq. (3.11) the vector field X has to be of the form

$$X^\mu = X^\mu_\infty + o(r^{-\alpha}), \qquad \partial_\nu X^\mu_\infty = 0. \tag{3.12}$$

As $\Lambda = 0$ in the current case, the convergence of the boundary integrals in vacuum will be guaranteed if one assumes, e.g.,

$$\sum_{\mu\alpha\beta}\int_{\mathcal{S}\cap\{r\ge R\}} |\partial_\mu g_{\alpha\beta}|^2 d^{n+K}x < \infty. \tag{3.13}$$

This follows immediately from Stokes' theorem together with Eqs. (3.1)–(3.3) and (3.6), keeping in mind that $\Lambda = 0 = \bar{\mathbf{R}}_{\mu\nu}$ in the current context.

We note that Eq. (3.13) will hold if Eq. (2.4) is replaced by $\alpha > (n-2)/2$, which provides a sufficient but not a necessary condition.

While we are mostly interested in vacuum solutions, the analysis below applies to nonvacuum ones, provided that one also has

$$T_{\mu\nu} = o(r^{-n}) \text{ and } \sum_{\alpha\beta}\int_{\mathcal{S}\cap\{r\ge R\}} |T_{\alpha\beta}| d^{n+K}x < \infty. \tag{3.14}$$

Equations (3.13)–(3.14) will be assumed in the calculations that follow.

Since the last term in Eq. (3.7) drops out when $\bar\nabla_\beta X_\alpha = 0$, we obtain

$$\begin{aligned} \mathbb{U}^{\nu\lambda} &= \mathbb{U}^{\nu\lambda}{}_\beta X^\beta \\ &= -\frac{1}{16\pi}(1+o(r^{-\alpha}))(\eta_{\beta\gamma}+o(r^{-\alpha}))X^\beta[(\eta^{\gamma\nu}\eta^{\lambda\kappa}\eta_{\rho\sigma} - \eta^{\gamma\lambda}\eta^{\nu\kappa}\eta_{\rho\sigma})g^{\rho\sigma}{}_{,\kappa} + g^{\gamma\lambda}{}_{,\kappa}\eta^{\nu\kappa} - g^{\gamma\nu}{}_{,\kappa}\eta^{\lambda\kappa} + \eta^{\gamma\lambda}g^{\nu\kappa}{}_{,\kappa} - \eta^{\gamma\nu}g^{\lambda\kappa}{}_{,\kappa} + o(r^{-2\alpha-1})] \\ &= -\frac{1}{16\pi}[(\eta^{\lambda\kappa}X^\nu - \eta^{\nu\kappa}X^\lambda)\eta_{\rho\sigma}g^{\rho\sigma}{}_{,\kappa} + \eta^{\nu\kappa}\eta_{\beta\gamma}g^{\gamma\lambda}{}_{,\kappa}X^\beta - \eta^{\lambda\kappa}\eta_{\beta\gamma}g^{\gamma\nu}{}_{,\kappa}X^\beta + g^{\nu\kappa}{}_{,\kappa}X^\lambda - g^{\lambda\kappa}{}_{,\kappa}X^\nu] + o(r^{-2\alpha-1}) \\ &= -\frac{1}{16\pi}\eta^{\delta\kappa}\eta_{\beta\gamma}g^{\gamma\tau}{}_{,\kappa}X^\xi(\delta^\lambda_\delta\delta^\nu_\xi\delta^\beta_\tau - \delta^\lambda_\xi\delta^\nu_\delta\delta^\beta_\tau + \delta^\lambda_\tau\delta^\nu_\delta\delta^\beta_\xi - \delta^\lambda_\delta\delta^\nu_\tau\delta^\beta_\xi + \delta^\lambda_\xi\delta^\nu_\tau\delta^\beta_\delta - \delta^\lambda_\tau\delta^\nu_\xi\delta^\beta_\delta) + o(r^{-2\alpha-1}) \\ &= \frac{3}{8\pi}\eta^{\delta\kappa}\eta_{\beta\gamma}g^{\gamma\tau}{}_{,\kappa}X^\xi\delta^{\nu\lambda\beta}_{\tau\delta\xi} + o(r^{-2\alpha-1}). \end{aligned} \tag{3.15}$$

Plugging the result into Eq. (3.7) and renaming indices, in the limit $r \to \infty$, we obtain the following form of Eq. (3.10), which will be seen to be convenient in our further considerations:

$$H_b(X,\mathcal{S}) = \frac{3}{16\pi}\lim_{R\to\infty}\int_{S(R)\times\mathbb{T}^{\mathbb{K}}} \delta^{\alpha\beta\gamma}_{\lambda\mu\nu} X^\nu \eta^{\lambda\rho}\eta_{\gamma\sigma}\partial_\rho g^{\sigma\mu} dS_{\alpha\beta}, \tag{3.16}$$

where $S(R)$ denotes a sphere of radius R in the $\mathbb{R}^n$ factor of $\mathcal{S}_{\rm ext}$, and

$$\delta^{\alpha\beta\gamma}_{\lambda\mu\nu} := \delta^\alpha_{[\lambda}\delta^\beta_\mu\delta^\gamma_{\nu]}. \tag{3.17}$$

We see from Eq. (3.12) that $H_b(X,\mathcal{S})$ can be written as

$$H_b(X,\mathcal{S}) =: p_\mu X^\mu_\infty. \tag{3.18}$$

When $K = 0$, the coefficients p_μ are called the Arnowitt-Deser-Misner (ADM) four-momentum of $\mathcal{S}$ [1].

If $X = \partial_0$, we find a formula somewhat resembling the usual one:

$$\begin{aligned} p_0 &:= H_b(\partial_t,\mathcal{S}) \\ &= \frac{1}{16\pi}\lim_{R\to\infty}\int_{S(R)}\int_{\mathbb{T}^K}\sum_{I=1}^{n+K}(\partial_I g_{iI} - \partial_i g_{II})\frac{x^i}{R} d^{n+K-1}\mu \\ &= |\mathbb{T}^K| p_{0,\rm ADM} \\ &\quad + \frac{1}{16\pi}\lim_{R\to\infty}\int_{S(R)}\int_{\mathbb{T}^K}\sum_{A=n+1}^{n+K}(\partial_A g_{iA} - \partial_i g_{AA})\frac{x^i}{R} d^{n+K-1}\mu. \end{aligned} \tag{3.19}$$

Here $d^{n+K-1}\mu$ is the measure induced on $S(R)\times\mathbb{T}^K$ by the flat metric, $|\mathbb{T}^K|$ denotes the volume of $\mathbb{T}^K$, and $p_{0,\rm ADM}$ is the usual (total) ADM energy of the physical-space metric $g_{ij}dx^i dx^j$. Perhaps not unexpectedly, the ADM energy $p_{0,\rm ADM}$ does *not* coincide with the Hamiltonian generating time translations in general.

Next, when $X^0 = 0$, after using Stokes' theorem in the integral

$$\int_{S(R)\times\mathbb{T}^K} \partial_J(g_{J0}\delta^L_I - g_{L0}\delta^J_I)\partial_L\rfloor(dx^1\wedge\cdots\wedge dx^{K+n}) = 0, \tag{3.20}$$

we obtain the formula

$$p_I := H_b(\partial_I,\mathcal{S}) = \frac{1}{8\pi R}\lim_{R\to\infty}\int_{S(R)}\int_{\mathbb{T}^K} P_{Ii}x^i d^{n+K-1}\mu. \tag{3.21}$$

Here, P_{IJ} is the usual canonical ADM momentum

$$\begin{aligned} P_{IJ} &:= g^{LM}k_{LM}g_{IJ} - k_{IJ}, \\ k_{IJ} &:= \frac{1}{2}\mathcal{L}_T g_{IJ} = \frac{1}{2}(\partial_0 g_{IJ} - \partial_I g_{0J} - \partial_J g_{0I}) + o(r^{-2\alpha-1}), \end{aligned} \tag{3.22}$$

while $\mathcal{L}_T$ denotes the Lie derivative in the direction of the unit-timelike future-directed field T of normals to the level sets of x^0.

As an example, we compute the above integrals for the Rasheed metrics, described in Appendix A, with $P = 0$:

$$p_0 = 2\pi M, \qquad p_i = 0, \qquad p_4 = 2\pi Q. \tag{3.23}$$

Equation (3.23) includes a 2π factor arising from a normalization in which the Kaluza-Klein coordinate x^4 in the Rasheed solutions runs over a circle of length 2π.

This should be compared with the ADM four-momentum $p_{\mu,\rm ADM}$ of the n-dimensional space metric $g_{ij}dx^i dx^j$, which reads

$$p_{0,\rm ADM} = M - \frac{\Sigma}{\sqrt{3}}, \qquad p_{i,\rm ADM} = 0. \tag{3.24}$$

B. General backgrounds

As discussed in detail in Appendix A 3, the Rasheed solutions with $P \neq 0$ are not KK-asymptotically flat in the sense set forth above. To cover this case we need to generalize the calculations so far to the case where the background metric is not flat, with an asymptotic region $\mathcal{S}_{\rm ext} \subset \mathcal{S}$ diffeomorphic to

$$\mathcal{S}_{\rm ext} \approx E(R_0), \quad \text{where } E(R) := (\mathbb{R}^n \backslash B(R)) \times {}^K\mathcal{N}, \tag{3.25}$$

with some K-dimensional compact manifold ${}^K\mathcal{N}$, for some $R_0 \geq 0$. We therefore have an associated global coordinate system x^i on $\mathbb{R}^n\backslash B(R_0)$, as well as the dilation vector field $Z = x^i\partial_i \equiv r\partial_r$ which will play a key role in some calculations below.

Somewhat more generally, in order to be able to include general "Birmingham-Kottler-Schwarzschild anti–de Sitter" metrics, we will consider ends $E(R)$ equipped with a radial function r so that

$$\mathcal{S}_{\rm ext} \approx E(R_0), \quad \text{with} \quad E(R) := \{r \geq R\} \equiv [R,\infty)\times\mathcal{K}, \tag{3.26}$$

where $\mathcal{K}$ is a compact manifold. Here r is a coordinate running along the $[R_0,\infty)$ factor of $\mathcal{S}_{\rm ext}$, and the dilation vector Z is defined as $Z := r\partial_r$.

For the usual $(n+1)$-dimensional Schwarzschild–anti de Sitter metric the manifold $\mathcal{K}$ will be an $(n-1)$-dimensional sphere, but it can be an arbitrary compact manifold admitting Einstein metrics in the case of metrics (B1)–(B3) below.

Along $\mathcal{S}_{\rm ext}$ we are given two Lorentzian metrics g and $\bar g$, with g asymptotic to the background $\bar g$ in a sense which we make precise now. Denoting by $\bar\nabla$ the Levi-Civita connection associated with $\bar g$, we assume the existence of a $\bar g$-orthonormal frame $\{\bar e_{\hat\mu}\}$ defined along $\mathcal{S}_{\rm ext}$ such that (decorating frame indices with hats)

$$g_{\hat\mu\hat\nu} \coloneqq g(\bar e_{\hat\mu}, \bar e_{\hat\nu}) = \bar g_{\hat\mu\hat\nu} + o(r^{-\alpha}),\qquad \bar\nabla_{\hat\lambda} g_{\hat\mu\hat\nu} = o(r^{-\beta}). \tag{3.27}$$

It seems that the specific values of α and β as needed for our mass formulas can only be chosen after a case-by-case study of the background metric $\bar g$; cf. Eqs. (3.31)–(3.32) below.

In what follows we will use the following convention: given two tensor fields u and v, we will write

$$u = v + o(r^{-\alpha}) \tag{3.28}$$

if the frame components of $u - v$, within the class of $\bar g$-ON frames chosen, decay as $o(r^{-\alpha})$. If $\bar e_{\hat 0}$ is orthogonal to $\mathcal{S}_{\rm ext}$ (which will often be assumed) then, if we denote by $\bar g_{\mathcal{S}} \coloneqq \bar g_{IJ} dx^I dx^J$ the Riemannian metric induced by $\bar g$ on $\mathcal{S}_{\rm ext}$, and by $|\cdot|_{\bar g_{\mathcal{S}}}$ the associated norm, we have, e.g.,

$$u_{\mu\nu} = o(r^{-\alpha}) \Leftrightarrow |u_{\hat 0\hat 0}| + |u_{\hat 0 I} dx^I|_{\bar g_{\mathcal{S}}} + |u_{IJ} dx^I dx^J|_{\bar g_{\mathcal{S}}} = o(r^{-\alpha}).$$

Assuming again that X is $\bar g$-covariantly constant, the second term of Eq. (3.7) vanishes and for the first term we have the same expression as in the KK-asymptotically flat case, with the difference that instead of $\eta_{\mu\nu}$ we have $\bar g_{\mu\nu}$ and instead of partial derivatives we have covariant derivatives of the background metric, i.e.,

$$\mathbb{U}^{\nu\lambda} = \mathbb{U}^{\nu\lambda}{}_\xi X^\xi = \left(\frac{3}{8\pi}\delta^{\nu\lambda\sigma}_{\tau\delta\xi}\bar g^{\delta\kappa}\bar g_{\sigma\gamma}X^\xi\bar\nabla_\kappa g^{\gamma\tau} + o(|X| r^{-\alpha-\beta})\right)\sqrt{|\det g|}, \tag{3.29}$$

where

$$|X|^2 \coloneqq \sum_\mu (X^{\hat\mu})^2. \tag{3.30}$$

In order to control the error terms appearing in Eq. (3.29) we will assume that

$$\alpha \text{ and } \beta \text{ are such that the subleading terms } o(|X| r^{-\alpha-\beta}) \text{ in Eq. (III.29) give a vanishing contribution to the boundary integrals after passing to the limit.} \tag{3.31}$$

This will be the case, e.g., for all Rasheed metrics when $\alpha = (n-2)/2$ as in Eq. (2.4), $\beta = \alpha + 1$, with X asymptotic to ∂_μ in coordinates as in Eq. (A33).

Similarly, Eq. (3.31) will be satisfied for asymptotically anti–de Sitter metrics with

$$\alpha = \beta = n/2, \tag{3.32}$$

where r is the area coordinate for the anti–de Sitter metric. Note that in this case we have $|X| = O(r)$.

Instead of Eq. (3.16) we now obtain

$$H_b(X, \mathcal{S}) = \frac{3}{16\pi}\lim_{R\to\infty}\int_{\partial E(R)} \delta^{\alpha\beta\gamma}_{\lambda\mu\nu} X^\nu \bar g^{\lambda\rho}\bar g_{\gamma\sigma}\bar\nabla_\rho g^{\sigma\mu} dS_{\alpha\beta}, \tag{3.33}$$

where the two-forms $dS_{\alpha\beta}$ in $d + K \equiv n + 1 + K$ spacetime dimensions take the form

$$dS_{\alpha\beta} = \frac{1}{(n+K-1)!}\epsilon_{\alpha\beta\xi_1\cdots\xi_{n+K-1}} dx^{\xi_1}\wedge\cdots\wedge dx^{\xi_{n+K-1}} \equiv \partial_\beta\rfloor\partial_\alpha\rfloor\underbrace{\sqrt{|\det g|}dx^0\wedge\cdots\wedge dx^{n+K}}_{=: d\mu_g}. \tag{3.34}$$

We can now compute the Hamiltonian charges for this general case. We have

$$\begin{aligned}
&\frac{3}{16\pi}\delta^{\alpha\beta\gamma}_{\lambda\mu\nu}X^\nu\bar g^{\lambda\rho}\bar g_{\gamma\sigma}\bar\nabla_\rho g^{\sigma\mu}dS_{\alpha\beta}\\
&= \frac{1}{16\pi}(\delta^{\alpha\beta}_{\lambda\mu}\delta^\gamma_\nu + \delta^{\alpha\beta}_{\mu\nu}\delta^\gamma_\lambda + \delta^{\alpha\beta}_{\nu\lambda}\delta^\gamma_\mu)X^\nu\bar g^{\lambda\rho}\bar g_{\gamma\sigma}\bar\nabla_\rho g^{\sigma\mu}dS_{\alpha\beta}\\
&= \frac{1}{16\pi}(X^\gamma\bar g^{\lambda\rho}\bar g_{\gamma\sigma}\bar\nabla_\rho g^{\sigma\mu}dS_{\lambda\mu} + X^\nu\bar\nabla_\sigma g^{\sigma\mu}dS_{\mu\nu}\\
&\quad + X^\nu\bar g^{\lambda\rho}\bar g_{\gamma\sigma}\bar\nabla_\rho g^{\sigma\gamma}dS_{\nu\lambda}).
\end{aligned} \tag{3.35}$$

To continue, it is best to use a $\bar g$-orthonormal frame $\bar e_{\hat i}$ with $\bar e_{\hat 0}$ orthogonal to $\mathcal{S}$ and $\bar e_{\hat A}$ tangent to $\partial E(R)$. Then only the forms $dS_{\hat 0\hat i}$ give a nonvanishing contribution to the boundary integral. In the calculations that follow we will write "n.c." for the sum of those terms which do not contribute to the integral either because of the integration domain, or by Stokes' theorem, or by passage to the limit.

If $X = \partial_0$, and assuming that

$$\partial_0 = X^{\hat 0}\bar e_{\hat 0}, \tag{3.36}$$

one finds, using frame indices throughout the calculation,

$$\frac{3}{16\pi}\delta^{\alpha\beta\gamma}_{\lambda\mu\nu}X^\nu\bar g^{\lambda\rho}\bar g_{\gamma\sigma}\bar\nabla_\rho g^{\sigma\mu}dS_{\alpha\beta} = \frac{3}{8\pi}X^{\hat 0}\delta^{\hat 0\hat k\hat\jmath}_{\hat\lambda\hat\mu\hat 0}\bar g^{\hat\lambda\hat\rho}\bar g_{\hat\jmath\hat\nu}\bar\nabla_{\hat\rho}g^{\hat\nu\hat\mu}dS_{\hat 0\hat k} = \frac{1}{16\pi}X^{\hat 0}[\bar\nabla^{\hat k}(\bar g_{\hat J\hat L}g^{\hat J\hat L}) - \bar\nabla_{\hat\jmath}g^{\hat\jmath\hat k}]d\sigma_{\hat k} + \text{n.c.}, \quad (3.37)$$

where

$$d\sigma_{\hat\imath} := dS_{\hat 0\hat\imath} \equiv \bar e_{\hat\imath}\rfloor\bar e_{\hat 0}\rfloor(\sqrt{|\det g|}dx^0\wedge\cdots\wedge dx^{n+K}) = \bar e_{\hat\imath}\rfloor(\sqrt{|\det g_{IJ}|}dx^1\wedge\cdots\wedge dx^{n+K}) + \text{n.c.} \quad (3.38)$$

Hence, we obtain the following generalization of the ADM energy:

$$p_0 = H_b(\partial_t,\mathcal{S}) = \frac{1}{16\pi}\lim_{R\to\infty}\int_{\partial E(R)}X^{\hat 0}[\bar\nabla^i(\bar g_{JK}g^{JK}) - \bar\nabla_J g^{Ji}]d\sigma_i. \quad (3.39)$$

The existence of the limit in Eq. (3.39) will be guaranteed if, instead of Eqs. (3.13)–(3.14), one now assumes, e.g.,

$$\int_{\mathcal{S}\cap\{r\geq R\}}|X|\left(\sum_{\hat\mu\hat\alpha\hat\beta}|\bar\nabla_{\hat\mu}g_{\hat\alpha\hat\beta}|^2+\sum_{\hat\alpha\hat\beta}|T_{\hat\alpha\hat\beta}|+|\Lambda||g^{\mu\nu}(g_{\mu\nu}-\bar g_{\mu\nu})|\right)\times d\mu_{\bar g_{\mathcal{S}}}<\infty, \quad (3.40)$$

where $d\mu_{\bar g_{\mathcal{S}}}$ is the $(n+K)$-dimensional Riemannian measure induced on $\mathcal{S}$ by $\bar g$. A condition on the metric and the energy-momentum tensor of matter fields naturally associated with Eq. (3.40) is

$$\lim_{R\to\infty}|X||\partial E(R)|T_{\hat\mu\hat\nu} = 0,$$
$$\lim_{R\to\infty}|X||\partial E(R)||\Lambda||g^{\mu\nu}(g_{\mu\nu}-\bar g_{\mu\nu})| = 0, \quad (3.41)$$

where $|\partial E(R)|$ denotes the area of $\partial E(R)$; cf. Eq. (3.14). This will be assumed whenever relevant.

As an example, we consider the Rasheed metrics of Appendix A with $P\neq 0$, which are vacuum. The $\bar g$-Killing vector $X=\partial_t$ is $\bar g$-covariantly constant so that Eq. (3.39) applies. The asymptotic behavior of the metric coefficients in the frame (A39) coincides with the asymptotic behavior of the metric coefficients in manifestly asymptotically Minkowskian coordinates we have seen in the case $P=0$, and is given by Eq. (A33). One obtains

$$p_0 = 4\pi PM, \quad (3.42)$$

where the extra factor $4P$, as compared to Eq. (3.23), is due to the $8P\pi$ periodicity of the coordinate x^4 [cf. Eq. (A37)], as enforced by the requirement of the smoothness of the metric. Note that the formulas (3.24) for the ADM four-momentum remain unchanged.

We emphasize that the calculations above are done at fixed P, since every P defines its own class of asymptotic backgrounds. As a result, the phase space of all configurations considered above splits into sectors parametrized by P. It would be interesting to investigate the question of the existence of a Hamiltonian in a phase space where P is allowed to vary. We leave this question to future work.

If X is not $\bar g$-covariantly constant, the second term of Eq. (3.7) does not vanish. Thus, disregarding those terms which do *not* involve the forms $d\Sigma_{0i}$, we obtain (keeping in mind that X is a Killing vector field of $\bar g$)

$$\frac{1}{2}(U^{\alpha\beta}-U^{\alpha\beta}|_{g=\bar g})d\Sigma_{\alpha\beta} = \frac{1}{2}\left(\mathbb{U}^{\alpha\beta}{}_\lambda X^\lambda - \frac{1}{8\pi}\sqrt{|\det g|}g^{\mu[\alpha}\delta^{\beta]}_\nu\bar\nabla_\mu X^\nu - U^{\alpha\beta}|_{g=\bar g}\right)d\Sigma_{\alpha\beta}$$
$$= \frac{3}{16\pi}\delta^{\alpha\beta\gamma}_{\lambda\mu\nu}X^\nu\bar g^{\lambda\rho}\bar g_{\gamma\sigma}\bar\nabla_\rho g^{\sigma\mu}dS_{\alpha\beta} - \frac{1}{16\pi}\left(\sqrt{|\det g|}g^{\mu[\alpha}-\sqrt{|\det\bar g|}\bar g^{\mu[\alpha}\right)\bar\nabla_\mu X^{\beta]}d\Sigma_{\alpha\beta} + \text{n.c.}$$
$$= \frac{1}{8\pi}(3\delta^{0i\gamma}_{\lambda\mu\nu}X^\nu\bar g^{\lambda\rho}\bar g_{\gamma\sigma}\bar\nabla_\rho g^{\sigma\mu} - (g^{\mu[0}-e^{-1}\bar g^{\mu[0})\bar\nabla_\mu X^{i]})Nd\sigma_i + \text{n.c.}$$

Here we have used $dS_{0i}=Nd\sigma_i$, where N is the lapse function of the foliation by the level sets of t, defined by writing the metric as $g=-N^2dt^2+g_{IJ}(dx^I+N^Idt)(dx^J+N^Jdt)$. We conclude that

$$H_b(X,\mathcal{S}) = \frac{1}{8\pi}\lim_{R\to\infty}\int_{\partial E(R)}(3\delta^{0i\gamma}_{\lambda\mu\nu}X^\nu\bar g^{\lambda\rho}\bar g_{\gamma\sigma}\bar\nabla_\rho g^{\sigma\mu} - (g^{\mu[0}-e^{-1}\bar g^{\mu[0})\bar\nabla_\mu X^{i]})Nd\sigma_i. \quad (3.43)$$

We can apply the last formula to the background Killing vectors ∂_i and ∂_4 for Rasheed metrics with $P\neq 0$. A calculation gives

$$p_i = 0, \qquad p_4 = 4\pi PQ. \quad (3.44)$$

Here one can note that ∂_z is $\bar g$-covariantly constant so that the last term in Eq. (3.43) certainly does not contribute, while $p_x=p_y=0$ follows from the axisymmetry of the Rasheed metrics. [In fact, $\bar\nabla X = O(r^{-2})$ or better for these Killing vectors so that the last term never contributes in the current case.]

Equation (3.43) applies for completely general background metrics $\bar g$, assuming that Eqs. (3.40) and (3.31) hold, for a large class of field equations. In particular, it applies to asymptotically Kottler ("anti–de Sitter") metrics, cf. Refs. [19,21–23].

IV. ENERGY-MOMENTUM AND THE CURVATURE TENSOR

For our further purposes it is convenient to rewrite Eq. (3.10) in terms of the Christoffel symbols. As a first step towards this we note the following consequence of Eq. (3.34):

$$\delta^{\alpha\beta\gamma}_{\lambda\mu\nu} dS_{\alpha\beta} = \frac{1}{3} \cdot \frac{1}{(n+K-2)!} \epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n+K-2}} dx^\gamma \wedge dx^{\xi_1} \wedge \cdots \wedge dx^{\xi_{n+K-2}}. \tag{4.1}$$

A. KK-asymptotic flatness

We assume again that X is $\bar{g}$-covariantly constant; of course, it would suffice to assume that $\bar{\nabla}X$ falls off fast enough to provide a vanishing contribution to the integral defining the Hamiltonian in the limit.

In the KK-asymptotically flat case, Eq. (3.16) can be rewritten as

$$p_\mu X^\mu_\infty = \frac{(-1)^{n+K-1}}{16\pi(n+K-2)!} \lim_{R\to\infty} \int_{S(R)\times\mathbb{T}^n} \epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n+K-2}} \times X^\nu g^{\lambda\rho} \Gamma^\mu{}_{\gamma\rho} dx^{\xi_1} \wedge \cdots \wedge dx^{\xi_{n+K-2}} \wedge dx^\gamma. \tag{4.2}$$

In the standard asymptotically flat case, without Kaluza-Klein directions, Eq. (4.2) can be used to obtain an expression for the ADM energy-momentum in terms of the Riemann tensor, generalizing a similar formula derived by Ashtekar and Hansen in spacetime dimension four [5] (cf. Refs. [4,24]), as follows. We can write

$$\begin{aligned}
&\epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n-2}} X^\nu g^{\lambda\rho} \Gamma^\mu{}_{\gamma\rho} dx^{\xi_1} \wedge \cdots \wedge dx^{\xi_{n-2}} \wedge dx^\gamma \\
&= d(\epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n-2}} X^\nu x^{\xi_1} g^{\lambda\rho} \Gamma^\mu{}_{\gamma\rho} dx^{\xi_2} \wedge \cdots \wedge dx^{\xi_{n-2}} \wedge dx^\gamma) \\
&\quad - (-1)^{n-3} \epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n-2}} X^\nu g^{\lambda\rho} x^{\xi_1} dx^{\xi_2} \wedge \cdots \wedge dx^{\xi_{n-2}} \\
&\quad \wedge \underbrace{(\partial_\sigma \Gamma\gamma\mu\rho dx^\sigma \wedge dx^\gamma)}_{=\frac{1}{2}R^\mu\rho\sigma\gamma dx^\sigma\wedge dx^\gamma} + \text{n.c.}
\end{aligned} \tag{4.3}$$

Inserting this into Eq. (4.2) and applying Stokes' theorem, one obtains

$$\begin{aligned}
p_\mu X^\mu_\infty &= \frac{1}{32\pi(n-2)!} \lim_{R\to\infty} \int_{S(R)} \epsilon_{\nu\xi_1\cdots\xi_{n-2}\lambda\mu} X^\nu x^{\xi_1} R^{\lambda\mu}{}_{\delta\gamma} \\
&\quad \times \underbrace{dx^{\xi_2} \wedge \cdots \wedge dx^{\xi_{n-2}} \wedge dx^\delta \wedge dx^\gamma}_{-\frac{1}{2}\epsilon^{\xi_2\cdots\xi_{n-2}\delta\gamma\mu\nu} dS_{\mu\nu}} \\
&= \frac{1}{16(n-2)\pi} \lim_{R\to\infty} \int_{S(R)} X^\mu x^\nu R_{\mu\nu\rho\sigma} dS^{\rho\sigma},
\end{aligned} \tag{4.4}$$

which is the desired new formula.

Let us now pass to a derivation of a version of Eq. (4.4) relevant for Kaluza-Klein asymptotically flat spacetimes. In this case we will be integrating the integrand of Eq. (4.2) over

$$S^{n-1} \times \mathbb{T}^K = S^{d-2} \times \mathbb{T}^K.$$

So only those forms in the sum which contain a $dx^d \wedge \cdots \wedge dx^{d+K-1}$ factor will survive integration. We will use the following symbols:

1. $\mathbf{R}^\alpha{}_{\beta\gamma\delta}$ denotes the Riemann tensor of the $(d+K)$-dimensional metric $g_{\mu\nu}dx^\mu dx^\nu$.
2. $R^a{}_{bcd}$ denotes the Riemann tensor of the d-dimensional metric $g_{ab}dx^a dx^b$.
3. $\mathsf{R}^I{}_{JKL}$ denotes the Riemann tensor of the $(n+K)$-dimensional metric $g_{IJ}dx^I dx^J$.
4. $\mathcal{R}^i{}_{jk\ell}$ denotes the Riemann tensor of the n-dimensional metric $g_{ij}dx^i dx^j$.

No distinction between $g_{ab}dx^a dx^b$ and $g_{\mu\nu}dx^\mu dx^\nu$ will be made when $K=0$. Keeping in mind that n.c. denotes the sum of those terms which do not contribute to the integral either because of the integration domain, or by Stokes' theorem, or by passage to the limit, we find

$$\begin{aligned}
&\epsilon_{\lambda\mu\nu\xi_1\ldots\xi_{d+K-3}} X^\nu g^{\lambda\rho} \Gamma^\mu{}_{\gamma\rho} dx^{\xi_1} \wedge \cdots \wedge dx^{\xi_{d+K-3}} \wedge dx^\gamma \\
&= \frac{(d+K-3)!}{(d-3)!N!} \epsilon_{bcfa_1\ldots a_{d-3}A_1\ldots A_K} X^f g^{be} \Gamma^c{}_{ae} dx^{a_1} \wedge \cdots \wedge dx^{A_K} \wedge dx^a \\
&\quad + \frac{(d+K-3)!}{(d-2)!(N-1)!} \epsilon_{\lambda\mu\nu a_1\ldots a_{d-2}A_1\ldots A_{K-1}} X^\nu g^{\lambda\rho} \Gamma^\mu{}_{A\rho} dx^{a_1} \wedge \cdots \wedge dx^{A_{K-1}} \wedge dx^A \\
&= \frac{(d+K-3)!}{(d-3)!N!} d(\epsilon_{bcfa_1\ldots a_{d-3}A_1\ldots A_K} X^f \eta^{be} \Gamma^c{}_{ae} x^{a_1} dx^{a_2} \wedge \cdots \wedge dx^{A_K} \wedge dx^a) \\
&\quad - \frac{(d+K-3)!}{(d-3)!N!} \epsilon_{bcfa_1\ldots a_{d-3}A_1\ldots A_K} X^f \eta^{be} x^{a_1} d\Gamma^c{}_{ae} \wedge \cdots \wedge dx^{A_K} \wedge dx^a \\
&\quad + \frac{(d+K-3)!}{(d-2)!(N-1)!} d(\epsilon_{\lambda\mu\nu a_1\ldots a_{d-2}A_1\ldots A_{K-1}} X^\nu \eta^{\lambda\rho} \Gamma^\mu{}_{A\rho} x^{a_1} dx^{a_2} \wedge \cdots \wedge dx^{A_{K-1}} \wedge dx^A) \\
&\quad - \frac{(d+K-3)!}{(d-2)!(N-1)!} \epsilon_{\lambda\mu\nu a_1\ldots a_{d-2}A_1\ldots A_{K-1}} X^\nu \eta^{\lambda\rho} x^{a_1} d\Gamma^\mu{}_{A\rho} \wedge \cdots \wedge dx^{A_{K-1}} \wedge dx^A + \text{n.c.}
\end{aligned}$$

$$= \frac{(d+K-3)!}{2(d-3)!N!} \epsilon_{bcfa_1\ldots a_{d-3}A_1\ldots A_K} X^f x^{a_1} \mathbf{R}^{bc}{}_{a_{d-1}a_{d-2}} dx^{a_{d-1}} \wedge dx^{a_2} \wedge \cdots \wedge dx^{A_K} \wedge dx^{a_{d-2}}$$
$$+ \frac{(d+K-3)!}{(d-2)!(N-1)!} \epsilon_{\lambda\mu\nu a_1\ldots a_{d-2}A_1\ldots A_{K-1}} X^\nu x^{a_1} \mathbf{R}^{\lambda\mu}{}_{a_{d-1}A} dx^{a_{d-1}} \wedge dx^{a_2} \wedge \cdots \wedge dx^{A_{K-1}} \wedge dx^A + \text{n.c.} \quad (4.5)$$

Using

$$\epsilon_{\lambda\mu\nu a_1\ldots a_{d-2}A_1\ldots A_{K-1}} dx^{a_{d-1}} \wedge dx^{a_2} \wedge \cdots \wedge dx^{A_{K-1}} \wedge dx^A = 3(N-1)!(-1)^{d+K-1} \delta^{Aab}_{\lambda\mu\nu} \epsilon_{aba_1\ldots a_{d-2}} dx^{a_{d-1}} \wedge dx^{a_2} \wedge \cdots \wedge dx^{a_{d-2}} \wedge dx^{d+1} \wedge \cdots \wedge dx^{d+K},$$

after some reordering of indices one obtains

$$p_\mu X^\mu_\infty = \frac{(-1)^n}{32\pi(n-1)!} \lim_{R\to\infty} \int_{S(R)} \int_{\mathbb{T}^K} x^{a_1} [(n-1)\epsilon_{a_1a_2\cdots a_{n-2}abc} X^a \mathbf{R}^{bc}{}_{a_{n-1}a_n} - \underbrace{\epsilon_{a_1a_2\cdots a_{n-1}ab}(4X^a\mathbf{R}^{bA}{}_{a_nA} + 2X^A\mathbf{R}^{ab}{}_{a_nA})}_{(*)}] dx^{a_2} \wedge \cdots \wedge dx^{a_n} \wedge dx^{d+1} \wedge \cdots \wedge dx^{d+K}. \quad (4.6)$$

Using

$$dx^{a_2} \wedge \cdots \wedge dx^{a_n} \wedge dx^{d+1} \wedge \cdots \wedge dx^{d+K} = -\frac{1}{2} \epsilon^{a_2\cdots a_n ef} dS_{ef} \quad (4.7)$$

and

$$\mathbf{R}^{ad}{}_{bd} = \mathbf{R}^a{}_b - \mathbf{R}^{aA}{}_{bA},$$

one obtains for the first term of the Hamiltonian integral [where in the fourth line below we use Eq. (C3)]

$$\begin{aligned}
&\epsilon_{a_1a_2\cdots a_{n-2}abc} x^{a_1} X^a \mathbf{R}^{bc}{}_{a_{n-1}a_n} dx^{a_2} \wedge \cdots \wedge dx^{a_n} \wedge dx^{d+1} \wedge \cdots \wedge dx^{d+K} \\
&= -\frac{1}{2} \epsilon_{a_1a_2\cdots a_{n-2}abc} \epsilon^{a_2\cdots a_n ef} x^{a_1} X^a \mathbf{R}^{bc}{}_{a_{n-1}a_n} dS_{ef} \\
&= \frac{1}{2}(-1)^{n-1}(n-3)!4!\delta^{a_{n-1}a_n ef}_{a_1 a\, b\, c} x^{a_1} X^a \mathbf{R}^{bc}{}_{a_{n-1}a_n} dS_{ef} \\
&= 2(-1)^{n-1}(n-3)! x^{a_1} X^a (\mathbf{R}^{ef}{}_{a_1a} + \delta^{ef}_{a_1a} \mathbf{R}^{bc}{}_{bc} - 4\delta^{[e}_{[a_1} \mathbf{R}^{f]c}{}_{a]c}) dS_{ef} \\
&= 2(-1)^{n-1}(n-3)! x^{a_1} X^a [\mathbf{R}^{ef}{}_{a_1a} + \delta^{ef}_{a_1a}(\mathbf{R}^c{}_c - \mathbf{R}^{cA}{}_{cA}) - 4(\delta^{[e}_{[a_1} \mathbf{R}^{f]}{}_{a]} - \delta^{[e}_{[a_1} \mathbf{R}^{f]A}{}_{a]A})] dS_{ef}. \quad (4.8)
\end{aligned}$$

Now, recall that finiteness of the total energy of matter fields together with the dominant energy condition requires, essentially, that

$$T_{\mu\nu} = o(r^{-n}); \quad (4.9)$$

cf. Eq. (3.14). This, together with the Einstein equations, implies that the Ricci-tensor contribution to the integrals will vanish in the limit $R\to\infty$. Nevertheless, we will keep the Ricci tensor terms for future reference.

Using

$$-\frac{1}{2}\epsilon_{a_1a_2\cdots a_{n-1}ab}\epsilon^{a_2\cdots a_n ef} dS_{ef} = 3(-1)^n(n-2)!\delta^{a_n ef}_{a_1ab} dS_{ef},$$

the terms involving $(*)$ in Eq. (4.6) can be manipulated as

$$\begin{aligned}
&6(-1)^n(n-2)!\delta^{a_n ef}_{a_1ab} x^{a_1}(2X^a\mathbf{R}^{bA}{}_{a_nA} + X^A\mathbf{R}^{ab}{}_{a_nA}) dS_{ef} \\
&= 2(-1)^n(n-2)! x^{a_1}(\delta^{a_n e}_{a_1a}\delta^f_b + \delta^{ef}_{a_1a}\delta^{a_n}_b + \delta^{fa_n}_{a_1a}\delta^e_b)(2X^a\mathbf{R}^{bA}{}_{a_nA} + X^A\mathbf{R}^{ab}{}_{a_nA}) dS_{ef} \\
&= 2(-1)^n(n-2)![2(x^{[a_n}X^{e]}\mathbf{R}^{fA}{}_{a_nA} + x^{[e}X^{f]}\mathbf{R}^{a_nA}{}_{a_nA} + x^{[f}X^{a_n]}\mathbf{R}^{eA}{}_{a_nA}) \\
&\quad + X^A(x^{[a_n}\mathbf{R}^{e]f}{}_{a_nA} + x^{[e}\mathbf{R}^{f]a_n}{}_{a_nA} + x^{[f}\mathbf{R}^{a_n]e}{}_{a_nA})] dS_{ef}.
\end{aligned}$$

Renaming the indices, rearranging terms, and plugging the results into the integral, one obtains our final expression

$$
\begin{aligned}
p_\mu X^\mu_\infty = & \frac{1}{32\pi(n-2)} \lim_{R\to\infty} \int_{S(R)} \int_{\mathbb{T}^K} \Big\{ -2x^b X^a [\mathbf{R}^{ef}{}_{ba} + \delta^{ef}_{ba}(\mathbf{R}^c{}_c - \mathbf{R}^{cA}{}_{cA}) - 4(\delta^{[e}_{[b}\mathbf{R}^{f]}{}_{a]} - \delta^{[e}_{[b}\mathbf{R}^{f]A}{}_{a]A})] \\
& - 2\frac{n-2}{n-1}[2(2x^{[b}X^{e]}\mathbf{R}^{fA}{}_{bA} + x^e X^f \mathbf{R}^{bA}{}_{bA}) + 3X^A x^e \mathbf{R}^{fb}{}_{bA}] \Big\} dS_{ef} \\
= & \frac{1}{16(n-2)\pi} \lim_{R\to\infty} \int_{S(R)} \int_{\mathbb{T}^K} \Big(X^a x^b \mathbf{R}_{ab}{}^{ef} + 4x^{[e}X^{a]}\mathbf{R}^f{}_a - x^e X^f \mathbf{R}^c{}_c \\
& - \frac{1}{n-1}[(n-3)x^e X^f \mathbf{R}^{bA}{}_{bA} + 4x^{[e}X^{a]}\mathbf{R}^{fA}{}_{aA} + 3(n-2)X^A x^e \mathbf{R}^{fb}{}_{bA}] \Big) dS_{ef}. \qquad (4.10)
\end{aligned}
$$

Some special cases are of interest:

(1) Suppose that $X^\mu = \delta^\mu_0$; thus, X has only a time component. At $x^0 = 0$ we have

$$
\begin{aligned}
p_0 = & \frac{1}{8(n-2)\pi} \\
& \times \lim_{R\to\infty} \int_{S(R)} \int_{\mathbb{T}^K} \Big[x^j \mathbf{R}_{0j}{}^{0i} + \frac{1}{2}x^i(\mathbf{R}^j{}_j - \mathbf{R}^0{}_0) \\
& - x^j \mathbf{R}^i{}_j - \frac{1}{n-1}\Big(\frac{1}{2}x^i \mathbf{R}^{0A}{}_{0A} - x^j \mathbf{R}^{iA}{}_{jA}\Big)\Big] dS_{0i},
\end{aligned}
\qquad (4.11)
$$

where the terms involving the Ricci tensor give a vanishing contribution in view of Eq. (4.9) [and similarly for Eq. (4.12) below].

(2) Suppose that $X^A = 0$; thus, X has only spacetime components. Then

$$
\begin{aligned}
p_a X^a_\infty = & \frac{1}{16(n-2)\pi} \\
& \times \lim_{R\to\infty} \int_{S(R)} \int_{\mathbb{T}^K} \Big(X^a x^b \mathbf{R}_{ab}{}^{ef} + 4x^{[e}X^{a]}\mathbf{R}^f{}_a \\
& - x^e X^f \mathbf{R}^c{}_c - \frac{1}{n-1}[(n-3)x^e X^f \mathbf{R}^{bA}{}_{bA} \\
& + 4x^{[e}X^{a]}\mathbf{R}^{fA}{}_{aA}] \Big) dS_{ef}. \qquad (4.12)
\end{aligned}
$$

We will see below that the first term on the right-hand side is related to the Komar integral. It is not clear whether or not the remaining terms vanish in general. However, when $X^0 = 0$, at $t = 0$ the third term in the integrand gives a vanishing contribution, so that the generators of space translations read

$$
\begin{aligned}
p_i X^i_\infty = & \frac{1}{8(n-2)\pi} \lim_{R\to\infty} \int_{S(R)} \int_{\mathbb{T}^K} \Big[X^i x^k \mathbf{R}_{ik}{}^{0j} \\
& + 2x^{[i}X^{j]}\Big(\frac{2}{n-1}\mathbf{R}^{0A}{}_{iA} + \mathbf{R}^0{}_i\Big)\Big] dS_{0j}.
\end{aligned}
\qquad (4.13)
$$

We also note that when $K = 1$ the contribution of the fourth term in the integrand in Eq. (4.12) always vanishes because then, denoting by x^4 the Kaluza-Klein coordinate,

$$
\mathbf{R}^{bA}{}_{bA} = \mathbf{R}^{b4}{}_{b4} = \mathbf{R}^{\mu 4}{}_{\mu 4} = \mathbf{R}^4{}_4 = o(r^{-n}),
$$

which gives a zero contribution in the limit.

(iii) Suppose instead that $X^a = 0$; thus, X has only components tangential to the Kaluza-Klein fibers. Then, again at $x^0 = 0$,

$$
\begin{aligned}
p_A X^A_\infty = & \frac{3}{16(n-1)\pi} \lim_{R\to\infty} \int_{S(R)} \int_{\mathbb{T}^K} X^A x^e \mathbf{R}_{Ab}{}^{fb} dS_{ef} \\
= & \frac{3}{16(n-1)\pi} \lim_{R\to\infty} \int_{S(R)} \int_{\mathbb{T}^K} X^A x^i \mathbf{R}_{AB}{}^{0B} dS_{0i},
\end{aligned}
\qquad (4.14)
$$

where the decay $o(r^{-n})$ of the Ricci tensor of the $(n+K+1)$-dimensional metric has been used.

B. General case

For general background metrics, still assuming a covariantly-constant $\bar{g}$-Killing vector, we start by rewriting Eq. (3.33) as

$$
\begin{aligned}
H_b(X, \mathcal{S}) = & \frac{(-1)^{n+K-1}}{16\pi(n+K-2)!} \lim_{R\to\infty} \int_{\partial E(R)} \epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n+K-2}} \\
& \times X^\nu g^{\lambda\rho} \delta\Gamma^\mu{}_{\gamma\rho} dx^{\xi_1} \wedge \cdots \wedge dx^{\xi_{n+K-2}} \wedge dx^\gamma,
\end{aligned}
\qquad (4.15)
$$

where

$$
\Gamma\delta^\alpha{}_{\beta\gamma} := \Gamma^\alpha{}_{\beta\gamma} - \bar{\Gamma}^\alpha{}_{\beta\gamma} = o(r^{-\beta}), \qquad (4.16)
$$

with the last equality following from Eq. (3.27).

In order to obtain a version of Eq. (4.3) suitable to the current setting. we will assume that there exists a vector field Z with $Z^A = 0$ and a real number $\gamma > 0$ such that

$$\bar{\nabla}_a Z^b = \delta^b_a + O(r^{-\gamma}) \quad \text{mod } (\delta^r_\mu, \delta^t_\mu). \tag{4.17}$$

Here we write "mod $(\delta^r_\mu, \delta^t_\mu)$" for a tensor which has the form $\delta^r_\mu \mathring{\alpha}_{\dots} + \delta^t_\mu \mathring{\beta}_{\dots}$ for some tensor fields $\mathring{\alpha}$ and $\mathring{\beta}$. That is to say, if X is a vector field tangent to the submanifolds of constant t, r, and if "$u_{\mu\dots} = 0$ mod $(\delta^r_\mu, \delta^t_\mu)$," then $X^\mu u_{\mu\dots} = 0$.

We show in Appendix B that the vector field defined in appropriate coordinates as

$$Z = r\partial_r \tag{4.18}$$

satisfies Eq. (4.17) for a) asymptotically anti–de Sitter metrics and b) general Rasheed metrics, in both cases without the error term $O(r^{-\gamma})$; equivalently, the exponent γ can be taken as large as desired. We have introduced the $O(r^{-\gamma})$ term for possible future generalizations.

We further assume that

$$\bar{\nabla}_\mu X^\nu = O(|X| r^{-\beta}) \quad \text{mod } (\delta^r_\mu, \delta^t_\mu), \tag{4.19}$$

which will certainly be the case if X is $\bar{g}$-covariantly constant. Last but not least, we replace Eq. (3.31) by the requirement that

$$\text{terms } o(|X| r^{-\alpha-\beta}),\ o(|Z||X| r^{-2\beta}), \text{ and } o(|X| r^{-\beta-\gamma}) \text{ give a vanishing contribution to boundary integrals at fixed } r \text{ and } t, \text{ after passing to the limit } r \to \infty. \tag{4.20}$$

Now, the identity that we are about to derive will be integrated on submanifolds of fixed r and t, so that any forms containing a factor dr or dt will give zero integral. Assuming that there are no Kaluza-Klein directions ($K = 0$), we find

$$\begin{aligned}
&d(\epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n-2}} X^\nu Z^{\xi_1} g^{\lambda\rho} \delta\Gamma^\mu{}_{\gamma\rho} dx^{\xi_2} \wedge \cdots \wedge dx^{\xi_{n-2}} \wedge dx^\gamma) \\
&= \bar{\nabla}_\sigma (\sqrt{|\det g|} \mathring{\epsilon}_{\lambda\mu\nu\xi_1\cdots\xi_{n-2}} X^\nu Z^{\xi_1} g^{\lambda\rho} \delta\Gamma^\mu{}_{\gamma\rho}) dx^\sigma \wedge dx^{\xi_2} \wedge \cdots \wedge dx^{\xi_{n-2}} \wedge dx^\gamma \\
&= Z^{\xi_1} \epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n-2}} g^{\lambda\rho} \delta\Gamma^\mu{}_{\gamma\rho} \underbrace{\bar{\nabla}_\sigma X^\nu dx^\sigma}_{\text{n.c.}} \wedge dx^{\xi_2} \wedge \cdots \wedge dx^{\xi_{n-2}} \wedge dx^\gamma \\
&\quad + \epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n-2}} X^\nu g^{\lambda\rho} \delta\Gamma^\mu{}_{\gamma\rho} \underbrace{\bar{\nabla}_\sigma Z^{\xi_1} dx^\sigma}_{dx^{\xi_1} + \text{n.c.}} \wedge dx^{\xi_2} \wedge \cdots \wedge dx^{\xi_{n-2}} \wedge dx^\gamma \\
&\quad + (-1)^{n-3} \epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n-2}} X^\nu g^{\lambda\rho} Z^{\xi_1} dx^{\xi_2} \wedge \cdots \wedge dx^{\xi_{n-2}} \wedge \underbrace{(\bar{\nabla}_\sigma \delta\Gamma^\mu_\gamma dx^\sigma \wedge dx^\gamma)}_{=(\frac{1}{2}\delta R^\mu_\rho \sigma\gamma + o(r^{-2\beta})) dx^\sigma \wedge dx^\gamma} + \text{n.c.} \\
&= \epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n-2}} X^\nu g^{\lambda\rho} \delta\Gamma^\mu{}_{\gamma\rho} dx^{\xi_1} \wedge \cdots \wedge dx^{\xi_{n-2}} \wedge dx^\gamma \\
&\quad + (-1)^{n-3} \frac{1}{2} \epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n-2}} X^\nu g^{\lambda\rho} \delta R^\mu{}_{\rho\sigma\gamma} Z^{\xi_1} dx^{\xi_2} \wedge \cdots \wedge dx^{\xi_{n-2}} \wedge dx^\sigma \wedge dx^\gamma + \text{n.c.}
\end{aligned} \tag{4.21}$$

This identity replaces Eq. (4.3) in the current setting. One can now repeat the remaining calculations of Sec. IV A by replacing every occurrence of the Christoffel symbols by the difference of those of g and $\bar{g}$, every occurrence of the Riemann tensor by the difference of the Riemann tensors of g and $\bar{g}$, and every occurrence of an undifferentiated x^α by Z^α. Some care must be taken when generalizing Eq. (4.10) when passing from the background Riemann tensor to the background Ricci tensor, because in Eq. (C1) all indices are lowered and raised with g. Thus, Eq. (C1) is now replaced by

$$\begin{aligned}
&3!\delta^{\sigma\gamma\alpha\beta}_{\lambda\mu\nu\xi} (\mathbf{R}^\mu{}_{\rho\sigma\gamma} - \bar{\mathbf{R}}^\mu{}_{\rho\sigma\gamma}) g^{\lambda\rho} \\
&\quad = (\mathbf{R}^{\alpha\beta}{}_{\xi\nu} - \bar{\mathbf{R}}^{[\alpha}{}_{\rho\xi\nu} g^{\beta]\rho}) + (\mathbf{R} - \bar{\mathbf{R}}_{\rho\lambda} g^{\rho\lambda}) \delta^{\alpha\beta}_{\xi\nu} \\
&\qquad - 4\delta^{[\alpha}_{[\xi} \mathbf{R}^{\beta]}{}_{\nu]} + 2\delta^{[\alpha}_{[\xi} g^{\beta]\rho} \bar{\mathbf{R}}_{\nu]\rho} - 2\bar{\mathbf{R}}^{[\alpha}{}_{\rho\lambda[\xi} \delta^{\beta]}_{\nu]} g^{\rho\lambda}.
\end{aligned} \tag{4.22}$$

The simplest situation is obtained when $K = 0$ so that ${}^K\mathcal{N}$ is reduced to a point, and Eq. (3.43) becomes

$$\begin{aligned}
H_b(X, \mathcal{S}) = \frac{1}{8(n-2)\pi} \lim_{R\to\infty} \int_{S(R)} &\{X^\nu Z^\xi (R^{0i}{}_{\nu\xi} - \bar{R}^{[0}{}_{\rho\nu\xi} g^{i]\rho}) \\
&+ X^{[0} Z^{i]} (R - \bar{R}_{\rho\lambda} g^{\rho\lambda}) + 2X^\nu Z^{[0} R^{i]}{}_\nu \\
&- 2Z^\nu X^{[0} R^{i]}{}_\nu + (Z^\nu X^{[0} g^{i]\rho} - X^\nu Z^{[0} g^{i]\rho}) \bar{R}_{\nu\rho} \\
&- X^{[\nu} Z^{i]} \bar{R}^0{}_{\rho\lambda\nu} g^{\rho\lambda} + X^{[\nu} Z^{0]} \bar{R}^i{}_{\rho\lambda\nu} g^{\rho\lambda} \\
&- (n-2)(g^{\mu[0} - e^{-1} \bar{g}^{\mu[0}) \bar{\nabla}_\mu X^{i]})\} N d\sigma_i.
\end{aligned} \tag{4.23}$$

1. Λ ≠ 0

We wish to analyze Eq. (4.23) for metrics g which asymptote a maximally symmetric background $\bar{g}$ with

$\Lambda \neq 0$. This case requires separate attention as then the background curvature tensor does not approach zero as we recede to infinity. We note that the calculations in this section are formally correct independently of the sign of Λ, but to the best of our knowledge they are only relevant in the case $\Lambda < 0$.

It is useful to decompose the Riemann tensor into its irreducible components,

$$R_{\alpha\beta\gamma\delta} = W_{\alpha\beta\gamma\delta} + \frac{1}{d-2}(R_{\alpha\gamma}g_{\beta\delta} - R_{\alpha\delta}g_{\beta\gamma} + R_{\beta\delta}g_{\alpha\gamma} - R_{\beta\gamma}g_{\alpha\delta}) - \frac{R}{(d-1)(d-2)}(g_{\beta\delta}g_{\alpha\gamma} - g_{\beta\gamma}g_{\alpha\delta})$$
$$= W_{\alpha\beta\gamma\delta} + \frac{1}{d-2}(P_{\alpha\gamma}g_{\beta\delta} - P_{\alpha\delta}g_{\beta\gamma} + P_{\beta\delta}g_{\alpha\gamma} - P_{\beta\gamma}g_{\alpha\delta}) + \frac{R}{d(d-1)}(g_{\beta\delta}g_{\alpha\gamma} - g_{\beta\gamma}g_{\alpha\delta}),$$

where $W_{\alpha\beta\gamma\delta}$ is the Weyl tensor and $P_{\alpha\beta}$ is the trace-free part of the Ricci tensor,

$$P_{\mu\nu} = R_{\mu\nu} - \frac{R}{d}g_{\mu\nu}.$$

This leads to the following rewriting of Eq. (4.23):

$$H_b(X,\mathcal{S}) = \frac{1}{8(n-2)\pi}\lim_{R\to\infty}\int_{\partial E(R)}\Big\{X^\nu Z^\xi\Big(W^{0i}{}_{\nu\xi} - \bar{W}^{[0}{}_{\rho\nu\xi}g^{i]\rho} + \frac{2R}{n(n+1)}\delta^{[0}_{[\nu}\delta^{i]}_{\xi]} - \frac{2\bar{R}}{n(n+1)}\delta^{[0}_{[\nu}\bar{g}_{\xi]\rho}g^{i]\rho}\Big)$$
$$+ X^{[0}Z^{i]}(R - \bar{R}_{\rho\lambda}g^{\rho\lambda}) + 2X^\nu Z^{[0}R^{i]}{}_\nu - 2Z^\nu X^{[0}R^{i]}{}_\nu + (Z^\nu X^{[0}g^{i]\rho} - X^\nu Z^{[0}g^{i]\rho})\bar{R}_{\nu\rho}$$
$$- \frac{2\bar{R}}{n(n+1)}(X^{[\nu}Z^{i]}\delta^0_{[\lambda}\bar{g}_{\nu]\rho} - X^{[\nu}Z^{0]}\delta^i_{[\lambda}\bar{g}_{\nu]\rho})g^{\rho\lambda} - (n-2)(g^{\mu[0} - e^{-1}\bar{g}^{\mu[0]})\bar{\nabla}_\mu X^{i]}\Big\}Nd\sigma_i. \quad (4.24)$$

Assuming that the background Weyl tensor falls off sufficiently fast so that it does not contribute to the integrals (e.g., vanishes, when the background is a space-form such as the anti–de Sitter metric), that both the energy-momentum tensor of matter and $e-1$ decay fast enough [cf. Eq. (3.41)], and setting

$$\Delta^{\mu\nu} := g^{\mu\nu} - \bar{g}^{\mu\nu},$$

we obtain

$$H_b(X,\mathcal{S}) = \frac{1}{8(n-2)\pi}\lim_{R\to\infty}\int_{\partial E(R)}\Big\{X^\nu Z^\xi\Big(W^{0i}{}_{\nu\xi} - \frac{2\bar{R}}{n(n+1)}\delta^{[0}_{[\nu}\bar{g}_{\xi]\rho}\Delta^{i]\rho}\Big) - X^{[0}Z^{i]}\bar{R}_{\rho\lambda}\Delta^{\rho\lambda} + (Z^\nu X^{[0}\Delta^{i]\rho} - X^\nu Z^{[0}\Delta^{i]\rho})\bar{R}_{\nu\rho}$$
$$- \frac{2\bar{R}}{n(n+1)}(X^{[\nu}Z^{i]}\delta^0_{[\lambda}\bar{g}_{\nu]\rho} - X^{[\nu}Z^{0]}\delta^i_{[\lambda}\bar{g}_{\nu]\rho})\Delta^{\rho\lambda} - (n-2)\Delta^{\mu[0}\bar{\nabla}_\mu X^{i]}\Big\}Nd\sigma_i, \quad (4.25)$$

where we have also used the hypothesis (3.31) that terms such as $|X||Z|\Delta^{\mu\nu}\Delta_{\rho\sigma}$ and $|X||Z|g_{\mu\nu}\Delta^{\mu\nu}$ fall off fast enough so that they give no contribution to the integral in the limit. With some further work, one gets

$$H_b(X,\mathcal{S}) = \frac{1}{8(n-2)\pi}\lim_{R\to\infty}\int_{\partial E(R)}\Big\{X^\nu Z^\xi W^{0i}{}_{\nu\xi} + (n-2)\Delta^{\mu[0}\Big[\frac{\bar{R}}{n(n+1)}(X_\mu Z^{i]} - Z_\mu X^{i]}) - \bar{\nabla}_\mu X^{i]}\Big]\Big\}Nd\sigma_i. \quad (4.26)$$

To continue, we assume the Birmingham-Kottler form (B1)–(B3) of the background metric $\bar{g}$. If X is the $\bar{g}$-Killing vector field ∂_t then, writing momentarily X_ν for $\bar{g}_{\nu\mu}X^\nu$,

$$\bar{\nabla}_\sigma X_\nu dx^\sigma \otimes dx^\nu = \bar{\nabla}_{[\sigma}X_{\nu]}dx^\sigma \otimes dx^\nu = \partial_{[\sigma}X_{\nu]}dx^\sigma \otimes dx^\nu$$
$$= \partial_{[\sigma}\bar{g}_{\nu]0}dx^\sigma \otimes dx^\nu = \frac{1}{2}\partial_r\bar{g}_{00}dx^r \wedge dx^0$$
$$= \frac{1}{2}\partial_r\bar{g}_{00}\bar{\Theta}^1 \wedge \bar{\Theta}^0.$$

Using this, one checks that all terms linear in Δ in Eq. (4.26) cancel out, leading to the elegant formulas

$$H_b(X,\mathcal{S}) = \frac{1}{8(n-2)\pi}\lim_{R\to\infty}\int_{\partial E(R)}X^\nu Z^\xi W^{0i}{}_{\nu\xi}Nd\sigma_i$$
$$= \frac{1}{16(n-2)\pi}\lim_{R\to\infty}\int_{\partial E(R)}X^\nu Z^\xi W^{\alpha\beta}{}_{\nu\xi}dS_{\alpha\beta}, \quad (4.27)$$

which, at this stage, hold for all X belonging to the $(n+1)$-dimensional family of Killing vectors of the anti–de Sitter background which are normal to $\{t=0\}$.

If $X = \partial_\varphi$, then we have

$$\bar{\nabla}_\sigma X_\nu dx^\sigma \otimes dx^\nu = \bar{\nabla}_{[\sigma} X_{\nu]} dx^\sigma \otimes dx^\nu = \frac{1}{2}\partial_\sigma \bar{g}_{\nu\varphi} dx^\sigma \wedge dx^\nu = \frac{1}{2}\partial_r \bar{g}_{\varphi\varphi} dr \wedge d\varphi + \frac{1}{2}\partial_\theta \bar{g}_{\varphi\varphi} d\theta \wedge d\varphi$$
$$= \sqrt{V}\sin\theta \bar{\Theta}^1 \wedge \bar{\Theta}^3 + \cos\theta \bar{\Theta}^2 \wedge \bar{\Theta}^3,$$

where we used the coframe of the background metric (B1) with the following cobasis:

$$\bar{\Theta}^0 = \sqrt{V}dt, \qquad \bar{\Theta}^1 = \frac{1}{\sqrt{V}}dr, \qquad \bar{\Theta}^2 = rd\theta, \qquad \bar{\Theta}^3 = r\sin\theta d\varphi. \tag{4.28}$$

Hence, in this coframe one obtains

$$\bar{\nabla}_{\hat{1}} X_{\hat{3}} = \sqrt{V}\sin\theta = -\bar{\nabla}_{\hat{3}} X_{\hat{1}}, \qquad \bar{\nabla}_{\hat{2}} X_{\hat{3}} = \cos\theta = -\bar{\nabla}_{\hat{3}} X_{\hat{2}}.$$

Therefore, the second term of the integrand in Eq. (4.26) vanishes for $r \to \infty$, since (keeping in mind that $dS_{\hat{0}\hat{i}}$ for $i \neq 1$ gives zero contribution to the integrals)

$$\frac{\bar{R}}{n(n+1)}\Delta^{\mu[0}(X_\mu Z^{1]} - Z_\mu X^{1]}) - \Delta^{\mu[0}\bar{\nabla}_\mu X^{1]} = -\frac{\lambda}{2}\Delta^{\hat{\mu}\hat{0}} X_{\hat{\mu}} Z^{\hat{1}} - \frac{1}{2}\Delta^{\hat{\mu}\hat{0}}\bar{\nabla}_{\hat{\mu}} X^{\hat{1}} = -\frac{\lambda}{2}\Delta^{\hat{3}\hat{0}} \underbrace{X_{\hat{3}}}_{=(\bar{g}_{\hat{3}\hat{3}}+\Delta_{\hat{3}\hat{3}})X^{\hat{3}}} Z^{\hat{1}} - \frac{1}{2}\Delta^{\hat{3}\hat{0}}\bar{\nabla}_{\hat{3}} X_{\hat{1}}$$
$$= \left(-\frac{\lambda}{2}\cdot\frac{r^2}{\sqrt{V}} + \frac{1}{2}\sqrt{V}\right)\sin\theta\Delta^{\hat{3}\hat{0}} \overset{r\to\infty}{\to} 0.$$

Hence, Eq. (4.27) also holds for $X = \partial_\varphi$. Since all Killing vectors of AdS spacetime can be obtained as linear combinations of images of these two vectors by isometries preserving $\{t = 0\}$, we conclude that Eq. (4.27) holds for all Killing vectors of the AdS metric.

Once this work was completed, we were informed that Eq. (4.27) had already been observed in Ref. [25], following up on the pioneering definitions in Refs. [26,27]. We note that our conditions for the equality in Eq. (4.27) are quite weaker than those in Ref. [25].

2. Λ = 0

We pass to the case $\Lambda = 0$. We will impose conditions which guarantee that all terms which are quadratic or higher in $g_{\mu\nu} - \bar{g}_{\mu\nu}$ give zero contribution to the integrals in the limit $R \to \infty$. Without these assumptions the final formulas become unreasonably long. Hence, we assume Eqs. (4.16), (4.17), (4.19), and (4.20).

In the current context, the calculation (4.5) is replaced by

$$\begin{aligned}
&\epsilon_{\lambda\mu\nu\xi_1\ldots\xi_{d+K-3}} X^\nu g^{\lambda\rho}\delta\Gamma^\mu{}_{\gamma\rho} dx^{\xi_1}\wedge\ldots\wedge dx^{\xi_{d+K-3}}\wedge dx^\gamma \\
&= \frac{(d+K-3)!}{(d-3)!N!}\epsilon_{bcfa_1\ldots a_{d-3}A_1\ldots A_K} X^f g^{be}\delta\Gamma^c{}_{ae} \underbrace{dx^{a_1}}_{\bar{\nabla}_h Z^{a_1}dx^h+\text{n.c.}=\delta^{a_1}_h dx^h+\text{n.c.}} \wedge\ldots\wedge dx^{A_K}\wedge dx^a \\
&\quad + \frac{(d+K-3)!}{(d-2)!(N-1)!}\epsilon_{\lambda\mu\nu a_1\ldots a_{d-2}A_1\ldots A_{K-1}} X^\nu g^{\lambda\rho}\delta\Gamma^\mu{}_{A\rho} dx^{a_1}\wedge\ldots\wedge dx^{A_{K-1}}\wedge dx^A \\
&= \frac{(d+K-3)!}{(d-3)!N!}[\bar{\nabla}_h(\epsilon_{bcfa_1\ldots a_{d-3}A_1\ldots A_K} X^f g^{be}\delta\Gamma^c{}_{ae} Z^{a_1}) dx^h\wedge dx^{a_2}\wedge\ldots\wedge dx^{A_K}\wedge dx^a \\
&\quad - \epsilon_{bcfa_1\ldots a_{d-3}A_1\ldots A_K} \underbrace{\bar{\nabla}_h X^f}_{\text{n.c.}} g^{be}\delta\Gamma^x{}_{ae} Z^{a_1} dx^h\wedge dx^{a_2}\wedge\ldots\wedge dx^{A_K}\wedge dx^a \\
&\quad - \epsilon_{bcfa_1\ldots a_{d-3}A_1\ldots A_K} X^f g^{be} Z^{a_1}\bar{\nabla}_h\delta\Gamma^c{}_{ae} dx^h\wedge dx^{a_2}\wedge\ldots\wedge dx^{A_K}\wedge dx^a + \text{n.c.}] \\
&\quad + \frac{(d+K-3)!}{(d-2)!(N-1)!}[\bar{\nabla}_h(\epsilon_{\lambda\mu\nu a_1\ldots a_{d-2}A_1\ldots A_{K-1}} X^\nu g^{\lambda\rho}\delta\Gamma^\mu{}_{A\rho} Z^{a_1}) dx^h\wedge dx^{a_2}\wedge\ldots\wedge dx^{A_{K-1}}\wedge dx^A \\
&\quad - \epsilon_{\lambda\mu\nu a_1\ldots a_{d-2}A_1\ldots A_{K-1}} \underbrace{\bar{\nabla}_h X^\nu}_{\text{n.c.}} g^{\lambda\rho}\delta\Gamma^\mu{}_{A\rho} Z^{a_1} dx^h\wedge dx^{a_2}\wedge\ldots\wedge dx^{A_{K-1}}\wedge dx^A \\
&\quad - \epsilon_{\lambda\mu\nu a_1\ldots a_{d-2}A_1\ldots A_{K-1}} X^\nu g^{\lambda\rho} Z^{a_1}\bar{\nabla}_h\delta\Gamma^\mu{}_{A\rho} dx^h\wedge dx^{a_2}\wedge\ldots\wedge dx^{A_{K-1}}\wedge dx^A + \text{n.c.}]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(d+K-3)!}{(d-3)!N!} d(\epsilon_{bcfa_1\ldots a_{d-3}A_1\ldots A_K} X^f g^{be} \delta\Gamma^c{}_{ae} Z^{a_1} dx^{a_2} \wedge \ldots \wedge dx^{A_K} \wedge dx^a) \\
&\quad + \frac{(d+K-3)!}{(d-2)!(N-1)!} d(\epsilon_{\lambda\mu\nu a_1\ldots a_{d-2}A_1\ldots A_{K-1}} X^\nu g^{\lambda\rho} \delta\Gamma^\mu{}_{A\rho} Z^{a_1} dx^h \wedge dx^{a_2} \wedge \ldots \wedge dx^{A_{K-1}} \wedge dx^A) \\
&\quad - \frac{(d+K-3)!}{(d-3)!N!} \epsilon_{bcfa_1\ldots a_{d-3}A_1\ldots A_K} X^f x^{a_1} g^{be} \delta\mathbf{R}^c{}_{ea_{d-1}a_{d-2}} dx^{a_{d-1}} \wedge dx^{a_2} \wedge \cdots \wedge dx^{A_K} \wedge dx^{a_{d-2}} \\
&\quad - \frac{(d+K-3)!}{2(d-3)!N!} \epsilon_{\lambda\mu\nu a_1\ldots a_{d-2}A_1\ldots A_{K-1}} X^\nu x^{a_1} g^{\lambda\rho} \delta\mathbf{R}^\mu{}_{\rho_{a_{d-1}A}} dx^{a_{d-1}} \wedge dx^{a_2} \wedge \ldots \wedge dx^{A_{K-1}} \wedge dx^A + \text{n.c.}
\end{aligned}
$$

As before, in the last equality we have used the fact that the first $\bar{\nabla}_h$ terms in the first expression in each of the square brackets can be replaced by $\bar{\nabla}_\mu$, because each form appearing in the first line above must already contain K factors of the KK differentials dx^A; otherwise, it will give zero contribution to the integral.

In addition to all of the hypotheses so far, we will also assume that the Riemann tensor decays at a rate $o(r^{-\beta_R})$:

$$\mathbf{R}^\alpha{}_{\beta\gamma\delta} = o(r^{-\beta_R}), \qquad \bar{\mathbf{R}}^\alpha{}_{\beta\gamma\delta} = o(r^{-\beta_R}), \tag{4.29}$$

with β_R chosen so that

$$\text{terms}|X||Z|o(r^{-\alpha-\beta_R}) \text{ give no contribution to the integral in the limit } R \to \infty. \tag{4.30}$$

All of these conditions are satisfied by the five-dimensional Rasheed metrics, with $\alpha > 0$ as close to one as one wishes, $\beta = 1 + \alpha$, $\beta_R = 3$, and with γ as large as desired.

In line with our previous notation, we will write $\mathbf{R}^\alpha{}_{\beta\gamma\delta} - \bar{\mathbf{R}}^\alpha{}_{\beta\gamma\delta}$ for the difference of Riemann tensors of the $(d+K)$-dimensional metrics $g_{\mu\nu}dx^\mu dx^\nu$ and $\bar{g}_{\mu\nu}dx^\mu dx^\nu$, $R^a{}_{bcd} - \bar{R}^a{}_{bcd}$ for that of the d-dimensional metrics $g_{ab}dx^a dx^b$ and $\bar{g}_{ab}dx^a dx^b$, $\mathsf{R}^I{}_{JKL} - \bar{\mathsf{R}}^I{}_{JKL}$ for that of the $(n+K)$-dimensional metrics $g_{IJ}dx^I dx^J$ and $\bar{g}_{IJ}dx^I dx^J$, and $\mathcal{R}^i{}_{jk\ell} - \bar{\mathcal{R}}^i{}_{jk\ell}$ for that of the n-dimensional metrics $g_{ij}dx^i dx^j$ and $\bar{g}_{ij}dx^i dx^j$.

With the above hypotheses, the derivation of the key formula (4.10) follows closely the remaining calculations in Sec. IV A, and leads to

$$
\begin{aligned}
H_b(X,\mathcal{S}) = \frac{1}{16(n-2)\pi} \lim_{R\to\infty} \Bigg\{ \int_{\partial E(R)} \Bigg(& X^a Z^b (\mathbf{R}_{ab}{}^{ef} - \bar{\mathbf{R}}_{ab}{}^{ef}) + 4Z^{[e}X^{a]}(\mathbf{R}^f{}_a - \bar{\mathbf{R}}^f{}_a) \\
& - Z^e X^f (\mathbf{R}^c{}_c - \bar{\mathbf{R}}^c{}_c) - \frac{1}{n-1}[(n-3)Z^e X^f (\mathbf{R}^{bA}{}_{bA} - \bar{\mathbf{R}}^{bA}{}_{bA}) + 4Z^{[e}X^{a]}(\mathbf{R}^{fA}{}_{aA} - \bar{\mathbf{R}}^{fA}{}_{aA}) \\
& + 3(n-2)X^A Z^e (\mathbf{R}^{fb}{}_{bA} - \bar{\mathbf{R}}^{fb}{}_{bA})]\Bigg) dS_{ef} - (n-2)\int_{\partial E(R)} (g^{\mu[a} - e^{-1}\bar{g}^{\mu[a})\bar{\nabla}_\mu X^{b]} dS_{ab} \Bigg\}.
\end{aligned} \tag{4.31}
$$

For Rasheed solutions, or more generally for solutions which asymptote to the Rasheed backgrounds $\bar{g}$ given by Eq. (A34) with the usual decay $o(r^{-(n-2)/2})$, with $T_{\mu\nu} = o(r^{-3})$, one has [cf. Eqs. (A43)–(A44)] $\bar{\mathbf{R}}^0{}_{\lambda\mu\nu} = 0$, $\bar{\mathbf{R}}^{\hat{\mu}}{}_{\hat{4}\hat{\alpha}\hat{4}} = O(r^{-4})$, and $\bar{\mathbf{R}}_{\mu\nu} = O(r^{-4})$. Thus, for $X = \partial_\mu$ and after passing to the limit $R \to \infty$, we obtain an integrand which is formally identical to that for metrics which are KK-asymptotically flat:

$$
\begin{aligned}
H_b(X,\mathcal{S}) &= \frac{1}{16(n-2)\pi} \lim_{R\to\infty} \int_{S(R)\times S^1} X^\nu Z^\mu \mathbf{R}^{\alpha\beta}{}_{\nu\mu} dS_{\alpha\beta} \\
&= \frac{1}{8(n-2)\pi} \lim_{R\to\infty} \int_{S(R)\times S^1} X^\nu x^j \mathbf{R}^{0i}{}_{\nu j} N d\sigma_i.
\end{aligned} \tag{4.32}
$$

Some special cases, without necessarily assuming that g asymptotes to the Rasheed background, are of interest:

(1) Suppose that $X^\mu = \delta^\mu_0$; thus, X has only a time component. Keeping in mind that $Z^0 = 0$ and $\partial E(R) \subset \{x^0 = 0\}$, we have

$$\begin{aligned} H_b(\partial_0, \mathcal{S}) = &\frac{1}{8(n-2)\pi} \lim_{R\to\infty} \left\{ \int_{\partial E(R)} \left(Z^j (\mathbf{R}_{0j}{}^{0i} - \bar{\mathbf{R}}_{0j}{}^{0i}) + \frac{1}{2} Z^i (\mathbf{R}^j{}_j - \mathbf{R}^0{}_0 + \bar{\mathbf{R}}^j{}_j - \bar{\mathbf{R}}^0{}_0) - Z^j (\mathbf{R}^i{}_j - \bar{\mathbf{R}}^i{}_j) \right.\right. \\ &\left. - \frac{1}{2(n-1)} [Z^i (\mathbf{R}^{0A}{}_{0A} - \bar{\mathbf{R}}^{0A}{}_{0A}) - 2Z^j (\mathbf{R}^{iA}{}_{jA} - \bar{\mathbf{R}}^{iA}{}_{jA})] \right) dS_{0i} \\ &\left. - (n-2) \int_{\partial E(R)} (g^{\mu[0} - e^{-1} \bar{g}^{\mu[0}) \bar{\nabla}_\mu X^{i]} dS_{0i} \right\} \\ = &\frac{1}{8(n-2)\pi} \lim_{R\to\infty} \left\{ \int_{\partial E(R)} \left(\frac{1}{2} Z^i (\mathbf{R}^j{}_j - \bar{\mathbf{R}}^j{}_j) - Z^j (\mathbf{R}_{Ij}{}^{Ii} - \bar{\mathbf{R}}_{Ij}{}^{Ii}) \right.\right. \\ &\left. - \frac{1}{2(n-1)} [Z^i (\mathbf{R}^{0A}{}_{0A} - \bar{\mathbf{R}}^{0A}{}_{0A}) - 2Z^j (\mathbf{R}^{iA}{}_{jA} - \bar{\mathbf{R}}^{iA}{}_{jA})] \right) dS_{0i} \\ &\left. - (n-2) \int_{\partial E(R)} (g^{\mu[0} - e^{-1} \bar{g}^{\mu[0}) \bar{\nabla}_\mu X^{i]} dS_{0i} \right\}. \end{aligned} \tag{4.33}$$

(2) Suppose that $X^A = 0$; thus, X has only spacetime components. Then

$$\begin{aligned} H_b(X, \mathcal{S}) = &\frac{1}{16(n-2)\pi} \lim_{R\to\infty} \left\{ \int_{\partial E(R)} \left(X^a Z^b (\mathbf{R}_{ab}{}^{ef} - \bar{\mathbf{R}}_{ab}{}^{ef}) + 4 Z^{[e} X^{a]} (\mathbf{R}^f{}_a - \bar{\mathbf{R}}^f{}_a) \right.\right. \\ &\left. - Z^e X^f (\mathbf{R}^c{}_c - \bar{\mathbf{R}}^c{}_c) - \frac{1}{n-1} [(n-3) Z^e X^f (\mathbf{R}^{bA}{}_{bA} - \bar{\mathbf{R}}^{bA}{}_{bA}) + 4 Z^{[e} X^{a]} (\mathbf{R}^{fA}{}_{aA} - \bar{\mathbf{R}}^{fA}{}_{aA})] \right) dS_{ef} \\ &\left. - 2(n-2) \int_{\partial E(R)} (g^{\mu[0} - e^{-1} \bar{g}^{\mu[0}) \bar{\nabla}_\mu X^{i]} dS_{0i} \right\}. \end{aligned} \tag{4.34}$$

We will see below that the first term on the right-hand side is related to the Komar integral. It is not clear whether or not the remaining terms vanish in general. However, when $X^0 = 0$, at $t = 0$ the third and fourth terms in the integrand in Eq. (4.34) give a vanishing contribution so that the generators of space translations read

$$\begin{aligned} H_b(X, \mathcal{S}) = &\frac{1}{8(n-2)\pi} \lim_{R\to\infty} \left\{ \int_{\partial E(R)} \left(X^i Z^k (\mathbf{R}_{ik}{}^{0j} - \bar{\mathbf{R}}_{ik}{}^{0j}) + Z^{[i} X^{j]} \left[\frac{2}{n-1} (\mathbf{R}^{0A}{}_{iA} - \bar{\mathbf{R}}^{0A}{}_{iA}) + \mathbf{R}^0{}_i - \bar{\mathbf{R}}^0{}_i \right] \right) dS_{0j} \right. \\ &\left. - (n-2) \int_{\partial E(R)} (g^{\mu[0} - e^{-1} \bar{g}^{\mu[0}) \bar{\nabla}_\mu X^{i]} dS_{0i} \right\}. \end{aligned} \tag{4.35}$$

(3) Suppose instead that $X^a = 0$; thus, X has only components tangential to the Kaluza-Klein fibers. Then, again at $x^0 = 0$,

$$H_b(X, \mathcal{S}) = \lim_{R\to\infty} \left\{ \frac{3}{16(n-1)\pi} \int_{\partial E(R)} X^A Z^e (\mathbf{R}_{AB}{}^{fb} - \bar{\mathbf{R}}_{Ab}{}^{fb}) dS_{ef} - \frac{1}{8\pi} \int_{\partial E(R)} (g^{\mu[0} - e^{-1} \bar{g}^{\mu[0}) \bar{\nabla}_\mu X^{i]} dS_{0i} \right\}. \tag{4.36}$$

C. $(n+K)+1$–decomposition

In a Cauchy-data context it is convenient to express the global charges explicitly in terms of Cauchy data. Here one can use the Gauss-Codazzi-Mainardi embedding equations to reexpress our spacetime-Riemann-tensor integrals in terms of the Riemann tensor of the initial-data metric and of the extrinsic curvature tensor. For this we consider $X^\mu = \delta^\mu_0$ and $x^0 = 0$, i.e., we consider Eq. (4.33).

We start with the case of KK-asymptotically flat initial data sets. Keeping in mind our convention that $(x^I) = (x^i, x^A)$, we can replace $\mathbf{R}^i{}_{JKL}$ with the $(n+K)$-dimensional

Riemann tensor, which we denote by $\mathsf{R}^I{}_{JKL}$, by means of the Gauss-Codazzi relation

$$\mathbf{R}^I{}_{JKL} = \mathsf{R}^I{}_{JKL} + o(r^{-2\alpha-2}). \tag{4.37}$$

Hence, from Eq. (4.11) we obtain

$$p_0 = -\frac{1}{8(n-2)\pi}\lim_{R\to\infty}\int_{S(R)}\int_{\mathbb{T}^N} x^j\left(\mathsf{R}^i{}_j + \frac{1}{2(n-1)}\mathsf{R}^k{}_k\delta^i_j + \frac{1}{n-1}\mathsf{R}^{iA}{}_{jA}\right)Nd\sigma_i. \tag{4.38}$$

We note that in the usual asymptotically flat case, $K=0$, the last integral is not present. Further, $\mathsf{R}^k{}_k$ then becomes the Ricci scalar of the initial data metric, with $\mathsf{R}^k{}_k = o(r^{-2\alpha-2})$ because of the scalar constraint equation, and hence does not contribute to the integral. Thus, the above reproduces the well-known-by-now formula for the ADM mass in terms of the Ricci tensor of the initial data metric [28–31] when the Ricci scalar decays fast enough, as we assumed here.

We pass now to the case covered in Sec. IV B 1, namely, $K=0$ but $\Lambda<0$, with the background metric $\bar g$ as in Eqs. (B1)–(B3). Let k_{IJ} be the extrinsic curvature tensor of the slices $\{x^0 = \text{const}\}$. If we assume that k_{IJ} satisfies

$$|k| \coloneqq \sqrt{g^{IJ}g^{LM}k_{IL}k_{JM}} = o(r^{-n/2}), \tag{4.39}$$

from Eq. (4.27) we obtain a formula first observed in Ref. [30]:

$$H_b(X,\mathcal{S}) = -\frac{1}{16(n-2)\pi}\lim_{R\to\infty}\int_{\partial E(R)} X^0 Z^j\left(\mathsf{R}^i{}_j - \frac{\mathsf{R}}{n}\delta^i_j\right) \times Nd\sigma_i, \tag{4.40}$$

where in Eq. (4.40) we have assumed that X is a Killing vector of the anti–de Sitter background which is normal to the hypersurface $\{t=0\}$.

Finally, consider general configurations as in Sec. IV B 2. Under the hypothesis that

$$|k|^2|Z||\partial E(R)| \to_{R\to\infty} 0, \tag{4.41}$$

from Eq. (4.33) we find

$$\begin{aligned}H_b(\partial_0,\mathcal{S}) = \frac{1}{8(n-2)\pi}\lim_{R\to\infty}\Bigg\{\int_{\partial E(R)}\Bigg(\left[\frac{1}{2}Z^i(\mathbf{R}^j{}_j - \bar{\mathbf{R}}^j{}_j) - Z^j(\mathsf{R}^i{}_j - \bar{\mathsf{R}}^i{}_j)\right] \\ - \frac{1}{2(n-1)}[Z^i((\mathbf{R}^A{}_A - \bar{\mathbf{R}}^A{}_A) - (\mathsf{R}^A{}_A - \bar{\mathsf{R}}^A{}_A)) - 2Z^j((\mathsf{R}^i{}_j - \bar{\mathsf{R}}^i{}_j) - (\mathsf{R}^{ik}{}_{jk} - \bar{\mathsf{R}}^{ik}{}_{jk}))]\Bigg)Nd\sigma_i \\ -(n-2)\lim_{R\to\infty}\int_{\partial E(R)}(g^{\mu[0} - e^{-1}\bar g^{\mu[0})\bar\nabla_\mu X^{i]}Nd\sigma_i\Bigg\}.\end{aligned} \tag{4.42}$$

V. KOMAR INTEGRALS

If X^α is a Killing vector field of both g and $\bar g$, we have

$$X^\mu\mathbf{R}_{\mu bcd} = \nabla_b\nabla_c X_d, \quad \text{and} \quad X^\mu\bar{\mathbf{R}}_{\mu bcd} = \bar\nabla_b\bar\nabla_c X_d. \tag{5.1}$$

This allows us to express some of the integrals above as Komar-type integrals.

We start with the setup of Sec. IV B 2; the KK-asymptotically flat case can be obtained directly from the calculations here by setting $\bar{\mathbf{R}}_{\alpha\beta\gamma\delta} = 0$. To make things clear, we assume Eqs. (4.16)–(4.17) and (4.19)–(4.20), together with Eqs. (4.29)–(4.30), and recall that all these hypotheses are satisfied under the corresponding hypotheses made in the KK-asymptotically flat case.

The contribution from the first integrand in Eq. (4.31) can be manipulated as [32]

$$\begin{aligned}\lim_{R\to\infty}\int_{\partial E(R)} X^a Z^b(\mathbf{R}_{ab}{}^{ef} - \bar{\mathbf{R}}_{ab}{}^{ef})dS_{ef} &= \lim_{R\to\infty}\int_{\partial E(R)}[(X^{[f;e]}{}_{;b} - X^{[f\|e]}{}_{\|b})Z^b - X^A Z^b(\mathbf{R}_{Ab}{}^{ef} - \bar{\mathbf{R}}_{Ab}{}^{ef})]dS_{ef} \\ &= \lim_{R\to\infty}\int_{\partial E(R)}\{(n-1)(X^{[e;f]} - X^{[e\|f]}) - 3(X^{[e;f}Z^{b]})_{;b} + 3(X^{[e\|f}Z^{b]})_{\|b} \\ &\quad + 2(\mathbf{R}_{\mu b}{}^{b[f}Z^{e]} - \bar{\mathbf{R}}_{\mu b}{}^{b[f}Z^{e]})X^\mu dS_{ef} - X^A Z^b(\mathbf{R}_{Ab}{}^{ef} - \bar{\mathbf{R}}_{Ab}{}^{ef})\}dS_{ef} \\ &= \lim_{R\to\infty}\Bigg\{(n-1)\int_{\partial E(R)}(X^{[\alpha;\beta]} - X^{[\alpha\|\beta]})dS_{\alpha\beta} + \int_{\partial E(R)}[2X^\mu Z^e(\mathbf{R}^{fb}{}_{b\mu} - \bar{\mathbf{R}}^{fb}{}_{b\mu}) \\ &\quad - X^A Z^b(\mathbf{R}_{Ab}{}^{ef} - \bar{\mathbf{R}}_{Ab}{}^{ef})]dS_{ef}\Bigg\},\end{aligned} \tag{5.2}$$

where the semicolon (;) denotes the covariant derivative of the metric g and the double bar (‖) denotes the covariant derivative of the background metric $\bar{g}$. Moreover, we used Gauss' theorem, e.g.,

$$\lim_{R\to\infty}\int_{\partial E(R)}(X^{[e\|f}Z^{b]})_{\|b}\sqrt{|\det g|}d\Sigma_{ef} = \lim_{R\to\infty}\int_{\partial E(R)}(X^{[e\|f}Z^{b]})_{\|b}\sqrt{|\det \bar{g}|}d\Sigma_{ef} = 0. \quad (5.3)$$

Hence, under the hypotheses used in the derivation of Eq. (4.31), we can rewrite Eq. (4.31) as

$$\begin{aligned} H_b(X,\mathcal{S}) = \frac{1}{16(n-2)\pi}\lim_{R\to\infty}\Bigg\{&\int_{\partial E(R)}(n-1)(X^{[\alpha;\beta]} - X^{[\alpha\|\beta]})dS_{\alpha\beta} \\ &+\int_{\partial E(R)}\Bigg(2X^\mu Z^e(\mathbf{R}^{fb}{}_{b\mu} - \bar{\mathbf{R}}^{fb}{}_{b\mu}) - X^A Z^b(\mathbf{R}_{Ab}{}^{ef} - \bar{\mathbf{R}}_{Ab}{}^{ef}) + 4Z^{[e}X^{a]}(\mathbf{R}^f{}_a - \bar{\mathbf{R}}^f{}_a) \\ &- Z^e X^f(\mathbf{R}^c{}_c - \bar{\mathbf{R}}^c{}_c) - \frac{1}{n-1}[(n-3)Z^e X^f(\mathbf{R}^{bA}{}_{bA} - \bar{\mathbf{R}}^{bA}{}_{bA}) + 4Z^{[e}X^{a]}(\mathbf{R}^{fA}{}_{aA} - \bar{\mathbf{R}}^{fA}{}_{aA}) \\ &+ 3(n-2)X^A Z^e(\mathbf{R}^{fb}{}_{bA} - \bar{\mathbf{R}}^{fb}{}_{bA})]\Bigg)dS_{ef} - (n-2)\lim_{R\to\infty}\int_{\partial E(R)}(g^{\mu[a} - e^{-1}\bar{g}^{\mu[a})\bar{\nabla}_\mu X^{b]}dS_{ab}\Bigg\}. \end{aligned} \quad (5.4)$$

The first integrand is the difference of the Komar integrands of g and $\bar{g}$.

Specializing to the KK-asymptotically flat case for background-covariantly constant Killing vectors, this reads

$$\begin{aligned} p_\mu X^\mu_\infty = \frac{1}{16(n-2)\pi}\lim_{R\to\infty}\Bigg\{&(n-1)\int_{S(R)}\int_{\mathbb{T}^N}X^{\alpha;\beta}dS_{\alpha\beta} \\ &+\int_{S(R)}\int_{\mathbb{T}^N}\Bigg(2X^\mu x^e\mathbf{R}^{fb}{}_{b\mu} - X^A x^b\mathbf{R}_{Ab}{}^{ef} - \frac{1}{n-1}[(n-3)x^e X^f\mathbf{R}^{bA}{}_{bA} + 4x^{[e}X^{a]}\mathbf{R}^{fA}{}_{aA} \\ &+ 3(n-2)X^A x^e\mathbf{R}^{fb}{}_{bA}]\Bigg)dS_{ef}\Bigg\}. \end{aligned} \quad (5.5)$$

Thus, it appears that in general Komar-type integrals do *not* coincide with the Hamiltonian generators. This is really the case, as can be seen for the Rasheed solutions. Using Eq. (A33), one readily finds for $X = \partial_t$ (keeping in mind that $n = 3$)

$$\frac{1}{8\pi}\lim_{R\to\infty}\int_{S(R)}\int_{S^1}X^{\alpha;\beta}dS_{\alpha\beta} = \begin{cases} 2\pi(M + \frac{\Sigma}{\sqrt{3}}), & P = 0, \\ 8\pi P(M + \frac{\Sigma}{\sqrt{3}}), & P \neq 0, \end{cases} \quad (5.6)$$

which does *neither* coincide with p_0 [cf. Eq. (3.23)] *nor* with the ADM mass of the space metric $g_{ij}dx^i dx^j$. Note that the Komar integral of the spacetime metric $g_{ab}dx^a dx^b$ will equal $M + \frac{\Sigma}{\sqrt{3}}$ regardless of the value of P.

Next, for $X = \partial_4$ we obtain

$$\frac{1}{8\pi}\lim_{R\to\infty}\int_{S(R)}\int_{S^1}X^{\alpha;\beta}dS_{\alpha\beta} = \begin{cases} 4\pi Q, & P = 0, \\ 16\pi PQ, & P \neq 0, \end{cases} \quad (5.7)$$

which is twice the Hamiltonian charge p_4.

As a simple application of Eq. (5.6), suppose that there exists a Rasheed metric without a black-hole region. Since the divergence of the Komar integrand is zero, we obtain $M = -\Sigma/\sqrt{3}$. But this is precisely one of the parameter values excluded in the Rasheed metrics, cf. Eq. (A4) below. We conclude that the regular metrics in the Rasheed family must be black-hole solutions.

For the case of metrics which asymptote to a maximally symmetric background $\bar{g}$ with $\Lambda \neq 0$, as in Sec. IV B 1, the Komar integral resulting from Eq. (4.27) reads

$$\begin{aligned} H_b(X,\mathcal{S}) &= \frac{1}{16(n-2)\pi}\lim_{R\to\infty}\int_{\partial E(R)}X^\nu Z^\xi W^{\alpha\beta}{}_{\nu\xi}dS_{\alpha\beta} \\ &= \lim_{R\to\infty}\Bigg\{\frac{n-1}{16(n-2)\pi}\int_{\partial E(R)}X^{[\alpha;\beta]}dS_{\alpha\beta} \\ &\quad - \frac{\Lambda}{4(n-2)(n-1)n\pi}\int_{\partial E(R)}X^\alpha Z^\beta dS_{\alpha\beta}\Bigg\}. \end{aligned} \quad (5.8)$$

VI. WITTEN'S POSITIVITY ARGUMENT

The Witten positive-energy argument [33,34] (cf. Ref. [35]) generalizes in an obvious manner to KK-asymptotically flat metrics. Assuming that the initial data hypersurface $\mathcal{S}$ is spin, we consider the Witten boundary integral $\mathcal{W}$ defined as

$$\mathcal{W}(\phi_\infty) \coloneqq \lim_{R\to\infty}\int_{S(R)\times\mathbb{T}^K}\mathcal{U}^i d\sigma_i, \tag{6.1}$$

$$\mathcal{U}^I = \langle\phi, D^I\phi + \gamma^I \not{D}\phi\rangle, \tag{6.2}$$

where ϕ is a spinor field which asymptotes to a constant spinor ϕ_∞ at an appropriate rate as one recedes to infinity in the asymptotic end, and $\not{D} \coloneqq \gamma^J D_J$ is the Dirac operator on $\mathcal{S}$. (Note that the asymptotic spinors ϕ_∞ might be incompatible with the spin structure of $\mathcal{S}$, in which case the argument below of course does not apply; cf. Refs. [36–38].) It is standard to show that in the natural spin frame we have

$$\mathcal{U}^I = \frac{1}{4}\sum_{L=1}^{n+K}(\partial_L g_{IL} - \partial_I g_{LL})|\phi_\infty|^2 + o(r^{-2\alpha-1}). \tag{6.3}$$

Assuming a positive and suitably decaying energy density on a maximal (i.e., $g^{IJ}K_{IJ} = 0$) initial data hypersurface, such that

$\mathcal{S}$ is metrically complete and either is boundaryless

or has a trapped compact boundary, (6.4)

the proof of the existence of the desired solutions of the Witten equation $\not{D}\phi = 0$ can be carried out along lines identical to the usual asymptotically flat case, cf., e.g., Refs. [39,40]. Comparing with Eq. (3.19), we conclude that the positivity of $\mathcal{W}$ is equivalent to positivity of the Hamiltonian mass:

$$p_0 \geq 0.$$

It should be emphasized that p_0 does *not* necessarily coincide with the ADM mass of $g_{IJ}dx^I dx^J$.

The above argument required the positivity of the scalar curvature of $g_{IJ}dx^I dx^J$. This is not needed if one replaces the usual spinor covariant derivative in Eq. (6.2) by

$$D_I \to D_I + \frac{1}{2}K_I{}^J\gamma_J\gamma_0. \tag{6.5}$$

The Witten quadratic form $\mathcal{W}$ instead becomes

$$\lim_{R\to\infty}\oint_{S(R)\times\mathbb{T}^K}\mathcal{U}^i d\sigma_i = 4\pi p_\mu\langle\phi_\infty, \gamma^\mu\gamma^0\phi_\infty\rangle, \tag{6.6}$$

and is non-negative for all ϕ_∞ when the dominant energy condition is assumed on initial data hypersurfaces as in Eq. (6.4). The positivity of $\mathcal{W}$ is equivalent to the time-likeness of the $(n+K+1)$-vector p_μ. Equivalently,

$$p_0^2 - \sum_{i=1}^n p_i^2 \geq \sum_{A=n+1}^{n+K} p_A^2 \geq 0. \tag{6.7}$$

The first inequality is saturated if and only if the initial data set can be isometrically embedded in $\mathbb{R}\times\mathbb{R}^n\times\mathbb{T}^K$ equipped with the flat Lorentzian metric (cf. Ref. [41]).

As an example, consider the Rasheed metrics with $P = 0$. The corresponding domains of outer communications have the topology $\mathbb{R}\times S^1\times(\mathbb{R}^3\backslash B(R))$, where the $\mathbb{R}$ factor corresponds to the time variable, S^1 is the Kaluza-Klein factor, and the $\mathbb{R}^3\backslash B(R)$ factor describes the space topology of the black hole. It thus has the obvious spin structure inherited from a flat $\mathbb{R}\times S^1\times\mathbb{R}^3$, together with the obvious associated parallel spinors. Therefore the Witten-type argument just described applies, leading to

$$M^2 \geq Q^2, \tag{6.8}$$

where the inequality is strict for black-hole solutions. If we denote by $M_{\rm ADM}$ the ADM mass of the three-dimensional-space part of the Rasheed metric, this can be equivalently rewritten as

$$\left(M_{\rm ADM} + \frac{\Sigma}{\sqrt{3}}\right)^2 \geq Q^2, \tag{6.9}$$

cf. Ref. [42].

Note that Eq. (6.9) does not exclude the possibility of a negative or vanishing $M_{\rm ADM}$ (cf. Refs. [36,43,44]). We have not attempted a systematic analysis of this issue, and only checked that all Rasheed solutions with $a = 0$ and $M = 0$ have naked singularities outside of the horizon.

VII. SUMMARY

In this work we have considered families of metrics asymptotic to various background metrics, and studied the Hamiltonians associated with the flow of Killing vectors of the background. We have derived several new formulas for these Hamiltonians, generalizing previous work by allowing a cosmological constant, or nonstandard backgrounds, and allowing higher dimensions. In particular:

We have derived an ADM-type formula for Hamiltonians generating time translations for a wide class of background metrics, cf. Eq. (3.39).

We have provided a formula for Hamiltonians generating translations for KK-asymptotically flat metrics in terms of the spacetime curvature tensor [Eq. (4.10)].

We have derived a formula for Hamiltonians associated with generators of all background Killing fields for asymptotically anti–de Sitter spacetimes in terms of the spacetime curvature tensor [Eq. (4.27)].

Equation (4.31) provides a similar formula for a wide class of backgrounds with $\Lambda = 0$.

Equations (4.40) and (4.42) provide space-and-time decomposed versions of the last two Hamiltonians.

In Sec. V we have derived several Komar-type formulas for the Hamiltonians above for vector fields X which are

Killing vectors for both the background and the physical metric.

In Sec. VI we have pointed out the consequences of a Witten-type positivity argument for KK-asymptotically flat spacetimes: instead of proving the positivity of the ADM energy, the argument provides an inequality involving the Kaluza-Klein charges and the energy. An explicit version of the inequality has been established for KK-asymptotically flat Rasheed metrics.

In addition to the above, we have carried out a careful study of Rasheed metrics (see Appendix A) to obtain a nontrivial family of metrics with singularity-free domains of outer communications to which our formulas apply. We have pointed out the restrictions (A20) and (A22) on the parameters needed to guarantee the absence of naked singularities in the metric. We have shown that all metrics satisfying these conditions together with $P = 0$ have a stably causal domain of outer communications, and we have given sufficient conditions for stable causality when $P \neq 0$ in Eq. (A24). In Appendix A 3 we point out that the Rasheed metrics with $P \neq 0$ are not KK-asymptotically flat, and describe their asymptotics. We have determined their global charges, which are significantly different according to whether or not P vanishes.

Last but perhaps not least, Eq. (C3) provides a useful identity—which we have not seen in the literature—that is satisfied by the Riemann tensor in any dimensions and generalizing the usual double-dual identity valid in four dimensions.

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APPENDIX A: AN EXAMPLE: RASHEED'S SOLUTIONS

Rasheed [45] has constructed a family of stationary axisymmetric solutions of the five-dimensional vacuum Einstein equations which take the form

$$ds^2_{(5)} = \frac{B}{A}(dx^4 + 2A_\mu dx^\mu)^2 + \sqrt{\frac{A}{B}} ds^2_{(4)}, \qquad \text{(A1)}$$

where a, M, P, Q, and Σ are real numbers satisfying

$$\frac{Q^2}{\Sigma + M\sqrt{3}} + \frac{P^2}{\Sigma - M\sqrt{3}} = \frac{2\Sigma}{3}, \qquad \text{(A2)}$$

$$M^2 + \Sigma^2 - P^2 - Q^2 \neq 0, \qquad (M + \Sigma/\sqrt{3})^2 - Q^2 \neq 0,$$
$$(M - \Sigma/\sqrt{3})^2 - P^2 \neq 0, \qquad \text{(A3)}$$

$$M \pm \frac{\Sigma}{\sqrt{3}} \neq 0,$$
$$F^2 := \frac{[(M + \Sigma/\sqrt{3})^2 - Q^2][(M - \Sigma/\sqrt{3})^2 - P^2]}{M^2 + \Sigma^2 - P^2 - Q^2} > 0, \qquad \text{(A4)}$$

and where

$$ds^2_{(4)} = -\frac{G}{\sqrt{AB}}(dt + \omega^0{}_\phi d\phi)^2 + \frac{\sqrt{AB}}{\Delta} dr^2 + \sqrt{AB} d\theta^2 + \frac{\Delta\sqrt{AB}}{G} \sin^2(\theta) d\phi^2, \qquad \text{(A5)}$$

with

$$A = (r - \Sigma/\sqrt{3})^2 - \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} + a^2\cos^2(\theta) + \frac{2JPQ\cos(\theta)}{(M + \Sigma/\sqrt{3})^2 - Q^2},$$
$$B = (r + \Sigma/\sqrt{3})^2 - \frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} + a^2\cos^2(\theta) - \frac{2JPQ\cos(\theta)}{(M - \Sigma/\sqrt{3})^2 - P^2},$$
$$G = r^2 - 2Mr + P^2 + Q^2 - \Sigma^2 + a^2\cos^2(\theta),$$
$$\Delta = r^2 - 2Mr + P^2 + Q^2 - \Sigma^2 + a^2,$$
$$\omega^0{}_\phi = \frac{2J\sin^2(\theta)}{G}[r + E],$$
$$J^2 = a^2F^2, \qquad \text{(A6)}$$

whereas E is given by

$$E = -M + \frac{(M^2 + \Sigma^2 - P^2 - Q^2)(M + \Sigma/\sqrt{3})}{(M + \Sigma/\sqrt{3})^2 - Q^2}. \qquad \text{(A7)}$$

The physical-space Maxwell potential is given by

$$2A_\mu dx^\mu = \frac{C}{B} dt + \left(\omega^5{}_\phi + \frac{C}{B}\omega^0{}_\phi\right) d\phi, \qquad \text{(A8)}$$

where

$$C = 2Q(r - \Sigma/\sqrt{3}) - \frac{2PJ\cos(\theta)(M + \Sigma/\sqrt{3})}{(M - \Sigma/\sqrt{3})^2 - P^2}, \qquad \text{(A9)}$$

$$\omega^5{}_\phi = \frac{H}{G}, \qquad \text{(A10)}$$

and

$$H := 2P\Delta\cos(\theta) - \frac{2QJ\sin^2(\theta)[r(M-\Sigma/\sqrt{3})+M\Sigma/\sqrt{3}+\Sigma^2-P^2-Q^2]}{[(M+\Sigma/\sqrt{3})^2-Q^2]}. \tag{A11}$$

The Rasheed metrics (A1) have been obtained by applying a solution-generating technique [45] (cf. Ref. [46]) to the Kerr metrics. This guarantees that these metrics solve the five-dimensional vacuum Einstein equations when the constraint (A3) is satisfied. As the procedure is somewhat involved, it appears useful to cross-check the vanishing of the Ricci tensor using computer algebra. We have been able to verify this in the $P = 0$ case with SAGE (which required a week-long computation on a personal computer), as well as for a set of samples for the parameters (M, a, P, Q, Σ) in the $P \neq 0$ case with MATHEMATICA. We have, however, not been able to do it for the full set of parameters.

Let us address the question of the global structure of the metrics above. We have

$$\det g = -A^2 \sin^2(\theta),$$

which shows that the metrics are smooth and Lorentzian except possibly at the zeros of A, B, G, Δ, and $\sin(\theta)$.

After a suitable periodicity of ϕ (as in Sec. A 3 below) has been imposed, regularity at the axes of rotation away from the zeros of denominators follows from the factorizations

$$\left(\frac{\Delta}{G} - 1\right) = \frac{a^2\sin^2(\theta)}{a^2\cos^2(\theta) - 2Mr + P^2 + Q^2 + r^2 - \Sigma^2}, \tag{A12}$$

$$2A_\phi - 2P\frac{\Delta}{G}\cos(\theta) = \frac{\sin^2(\theta)}{G}\left(\mathcal{H} + \frac{2JC}{B}[r+E]\right), \tag{A13}$$

where

$$\mathcal{H} := -\frac{2QJ[r(M-\Sigma/\sqrt{3}) + M\Sigma/\sqrt{3} + \Sigma^2 - P^2 - Q^2]}{[(M+\Sigma/\sqrt{3})^2 - Q^2]}. \tag{A14}$$

It will be seen below that, after restricting the parameter ranges as in Eqs. (A20) and (A22), the location of Killing horizons is determined by the zeros of

$$\begin{vmatrix} g_{tt} & g_{t\phi} & g_{t4} \\ g_{\phi t} & g_{\phi\phi} & g_{\phi 4} \\ g_{4t} & g_{4\phi} & g_{44} \end{vmatrix} = -\Delta\sin^2(\theta), \tag{A15}$$

and thus by the real roots $r_+ \geq r_-$ of Δ, if any:

$$r_\pm = M \pm \sqrt{M^2 + \Sigma^2 - P^2 - Q^2 - a^2}. \tag{A16}$$

1. Zeros of the denominators

The norms

$$g_{tt} = \frac{W}{AB} \quad \text{and} \quad g_{44} = \frac{B}{A}$$

of the Killing vectors ∂_t and ∂_4 are geometric invariants, where $W = -GA + C^2$. So zeros of A and AB correspond to singularities in the five-dimensional geometry except if

(1) a zero of A is a joint zero of A, B, and W, or if
(2) a zero of B which is not a zero of A is also a zero of W.

Setting

$$\mathcal{A} := \frac{2JPQ}{a^2((M+\Sigma/\sqrt{3})^2 - Q^2)}, \tag{A17}$$

one checks that if

$$\begin{cases} \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} - a^2(1-|\mathcal{A}|) = 0, & \text{when } |\mathcal{A}| > 2 \quad \text{or} \\ \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} + \frac{a^2\mathcal{A}^2}{4} = 0, & \text{when } |\mathcal{A}| \leq 2, \end{cases} \tag{A18}$$

then A vanishes exactly at one point. Otherwise the set of zeros of A forms a curve in the (r,θ) plane. Let $\theta \mapsto r_A^+(\theta)$ denote the curve (say, γ) corresponding to the set of largest zeros of A.

Note that W and A are polynomials in r, with A of second order. If W/A is smooth, the remainder of the polynomial division of W by $r - r_A^+$ must vanish on the part of γ that lies outside the horizon. One can calculate this remainder with MATHEMATICA, obtaining a function of θ which vanishes at most at isolated points, if at all. It follows that the division of W by A is singular on the closure of the domain of outer communications (d.o.c.), i.e., the region $\{r \geq r_+\}$, if A has zeros there, except perhaps when Eq. (A18) holds.

One can likewise exclude a joint zero of W and B in the closure of the d.o.c. without a zero of A, except possibly for the case where this zero is isolated for B as well, which happens if

$$\begin{cases} \frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} - a^2(1-|\mathcal{B}|) = 0, & \text{if } |\mathcal{B}| > 2 \quad \text{or} \\ \frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} + \frac{a^2\mathcal{B}^2}{4} = 0, & \text{if } |\mathcal{B}| \leq 2. \end{cases} \tag{A19}$$

See Ref. [47] for a more detailed analysis of the borderline cases.

To summarize, a necessary condition for a black hole without obvious singularities in the closure of the domain of outer communications is that all zeros of A lie under the outermost Killing horizon $r = r_+$. One finds that this will be the case if and only if

$$|\mathcal{A}| > 2 \quad \text{and} \quad \begin{cases} \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} - a^2(1 - |\mathcal{A}|) < 0, \quad \text{or} \\ M + \sqrt{M^2 + \Sigma^2 - P^2 - Q^2 - a^2} > \frac{\Sigma}{3} + \sqrt{\frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} - a^2(1 - |\mathcal{A}|)}, \end{cases}$$

or

$$|\mathcal{A}| \leq 2 \quad \text{and} \quad \begin{cases} \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} + \frac{a^2\mathcal{A}^2}{4} < 0, \quad \text{or} \\ M + \sqrt{M^2 + \Sigma^2 - P^2 - Q^2 - a^2} > \frac{\Sigma}{3} + \sqrt{\frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} + \frac{a^2\mathcal{A}^2}{4}}, \end{cases} \tag{A20}$$

except perhaps when Eq. (A18) holds.

An identical argument applies to the zeros of B, with the zeros of B lying on a curve unless Eq. (A19) holds. Ignoring this last case, the zeros of B need to be similarly hidden behind the outermost Killing horizon. Setting

$$\mathcal{B} := -\frac{2JPQ}{a^2((M - \Sigma/\sqrt{3})^2 - P^2)}, \tag{A21}$$

one finds that this will be the case if and only if

$$|\mathcal{B}| > 2 \quad \text{and} \quad \begin{cases} \frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} - a^2(1 - |\mathcal{B}|) < 0, \quad \text{or} \\ M + \sqrt{M^2 + \Sigma^2 - P^2 - Q^2 - a^2} > -\frac{\Sigma}{3} + \sqrt{\frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} - a^2(1 - |\mathcal{B}|)}, \end{cases}$$

or

$$|\mathcal{B}| \leq 2 \quad \text{and} \quad \begin{cases} \frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} + \frac{a^2\mathcal{B}^2}{4} < 0, \quad \text{or} \\ M + \sqrt{M^2 + \Sigma^2 - P^2 - Q^2 - a^2} > -\frac{\Sigma}{3} + \sqrt{\frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} + \frac{a^2\mathcal{B}^2}{4}}, \end{cases} \tag{A22}$$

except perhaps when Eq. (A19) holds.

While the above guarantees the lack of obvious singularities in the d.o.c. $\{r > r_+\}$, there could still be causality violations there. Ideally, the d.o.c. should be globally hyperbolic, a question which we have not attempted to address. Barring global hyperbolicity, a decent d.o.c. should at least admit a time function, and the function t provides an obvious candidate. In order to study the issue we note the identity

$$g^{00} = \frac{4J^2[r + E]^2 \sin^2(\theta) - AB\Delta}{A\Delta G}. \tag{A23}$$

A Mathematica calculation shows that the numerator factorizes through G, so that g^{00} extends smoothly through the ergosphere. When $P = 0$, one can verify that g^{00} is negative on the d.o.c. For $P \neq 0$ one can find open sets of parameters which guarantee that g^{00} is strictly negative for $r > r_+$ when A and B have no zeros there. An example is given by the condition

$$r_+ \geq \frac{EM + q}{M + E}, \tag{A24}$$

which is sufficient but not necessary, where $q := P^2 + Q^2 - \Sigma^2 + a^2$. We hope to return to the question of causality violations in the future.

In Fig. 1 we show the locations of the zeros of A and B for some specific sets of parameters satisfying, or violating, the conditions above.

Another potential source of singularities of the metric (A1) could be the zeros of G. It turns out that they are irrelevant, which can be seen as follows. The relevant metric coefficient is $g_{\phi\phi}$, which reads

$$g_{\phi\phi} = \frac{B}{A}\left(\omega^5{}_\phi + \frac{C}{B}\omega^0{}_\phi\right)^2 + \sqrt{\frac{A}{B}}\left(-\frac{G}{\sqrt{AB}}(\omega^0{}_\phi)^2 + \frac{\Delta\sqrt{AB}}{G}\sin^2(\theta)\right). \tag{A25}$$

Taking into account a G^{-1} factor in $\omega^0{}_\phi$, it follows that $g_{\phi\phi}$ can be written as a fraction $(\ldots)/ABG^2$. A Mathematica

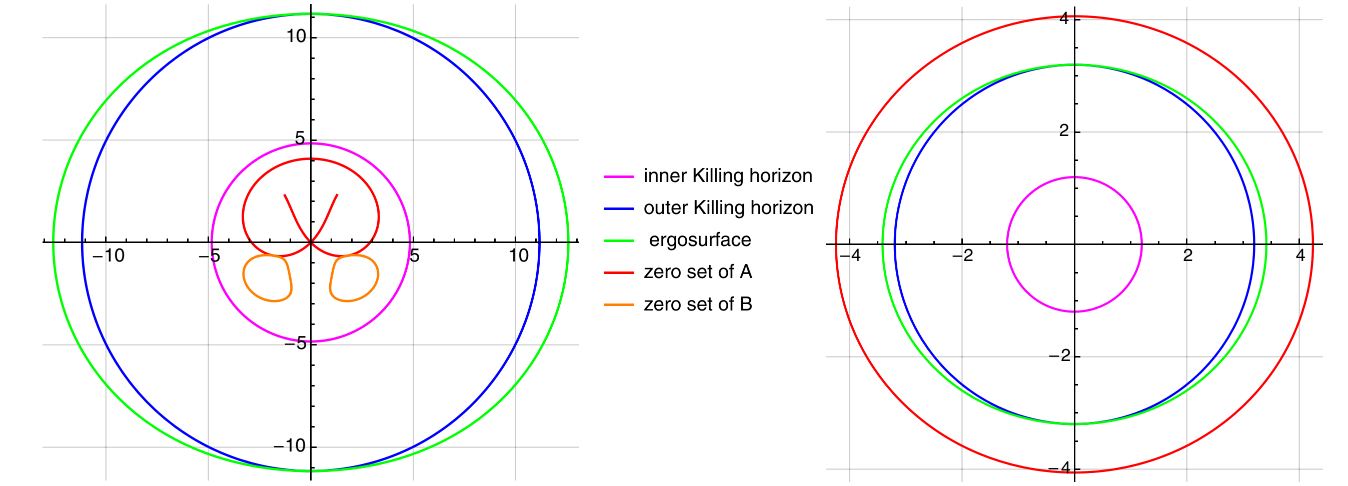

FIG. 1. Two sample plots for the location of the ergosurface (zeros of G), the outer and inner Killing horizons (zeros of Δ), and the zeros of A, B. Left plot: $M = 8, a = \frac{33}{10}, Q = \frac{8}{5}, \Sigma = -\frac{23}{5}, P = -\frac{1}{5}\sqrt{\frac{2(4105960\sqrt{3}+2770943)}{12813}} \approx -7.86$, with zeros of A and B under both horizons, consistently with Eq. (A20) and (A22). Right plot: $M = 1, a = 1, Q = 0, \Sigma = \sqrt{6}, P = \sqrt{4-2\sqrt{2}} \approx 1.08$; here, Eq. (A20) is violated, while the zeros of B occur at negative r.

calculation shows that the denominator $(\ldots)$ factorizes through AG^2, which shows indeed that the zeros of G are innocuous for the problem at hand.

Let us write $ds^2_{(4)}$ as ${}^{(4)}g_{ab}dx^a dx^b$. The factorization just described works for $g_{\phi\phi}$ but does *not* work for ${}^{(4)}g_{\phi\phi}$. From what has been said we see that the quotient metric ${}^{(4)}g_{ab}dx^a dx^b$ is always singular in the d.o.c., a fact which seems to have been ignored, and unnoticed, in the literature so far.

2. Regularity at the outer Killing horizon $\mathcal{H}_+$

The location of the outer Killing horizon $\mathcal{H}_+$ of the Killing field

$$k = \partial_t + \Omega_\phi \partial_\phi + \Omega_4 \partial_{x^4} \tag{A26}$$

is given by the larger root r_+ of Δ, cf. Eq. (A16). The condition that $\mathcal{H}_+$ is a Killing horizon for k is that the pullback of $g_{\mu\nu}k^\nu$ to $\mathcal{H}_+$ vanishes. This, together with

$$\Delta|_{\mathcal{H}_+} = 0, \qquad G|_{\mathcal{H}_+} = -a^2\sin^2(\theta), \tag{A27}$$

yields

$$\begin{aligned}\Omega_\phi &= -\frac{1}{\omega^0{}_\phi}\Big|_{\mathcal{H}_+} = \frac{a^2}{2J}(r_+ + E)^{-1},\\ \Omega_4 &= -\frac{2(A_t\omega^0{}_\phi - A_\phi)}{\omega^0{}_\phi}\Big|_{\mathcal{H}_+}\\ &= \frac{Q(-3Mr_+ - \sqrt{3}M\Sigma + 3P^2 + 3Q^2 + \sqrt{3}r\Sigma - 3\Sigma^2)}{(E + r_+)(3M^2 + 2\sqrt{3}M\Sigma - 3Q^2 + \Sigma^2)}.\end{aligned} \tag{A28}$$

After the coordinate transformation

$$\bar{\phi} = \phi - \Omega_\phi dt, \qquad \bar{x}^4 = x^4 - \Omega_4 dt, \tag{A29}$$

the metric (A1) becomes

$$g = g_S + \frac{dr^2}{\Delta} + \Delta U dt^2, \tag{A30}$$

where g_S is a smooth (0, 2) tensor, with $U \coloneqq g_{tt}/\Delta$ extending smoothly across $\Delta = 0$. Introducing a new time coordinate by

$$\tau = t - \sigma \ln(r - r_+) \Rightarrow d\tau = dt - \frac{\sigma}{r - r_+}dr, \tag{A31}$$

where σ is a constant to be determined, Eq. (A30) takes the form

$$\begin{aligned}g &= g_S + \Delta U\left(d\tau + \frac{\sigma}{r - r_+}dr\right)^2 + \frac{dr^2}{\Delta}\\ &= g_S + \Delta U d\tau^2 + \frac{2\Delta U\sigma}{r - r_+}d\tau dr + \left(\frac{1}{\Delta} + \frac{\Delta U\sigma^2}{(r - r_+)^2}\right)dr^2\\ &= g_S + \Delta U d\tau^2 + \frac{2\Delta U\sigma}{r - r_+}d\tau dr + \underbrace{\frac{(r - r_+)^2 + \Delta^2\sigma^2 U}{\Delta(r - r_+)^2}}_{V}dr^2.\end{aligned} \tag{A32}$$

In order to obtain a smooth metric in the domain of outer communications the constant σ has to be chosen so that the numerator of V has a triple zero at $r = r_+$. A MATHEMATICA computation gives an explicit formula for the desired constant σ, which is too lengthy to be explicitly presented here. This establishes the smooth extendibility of the metric in suitable coordinates across $r = r_+$.

3. Asymptotic behavior

When $P = 0$ the Rasheed metrics satisfy the KK-asymptotic flatness conditions. This can be seen by introducing manifestly asymptotically-flat coordinates (t, x, y, z) in the usual way. With some work one finds that the metric takes the form

$$\begin{pmatrix} \frac{2M}{r} + \frac{2\Sigma}{\sqrt{3}r} - 1 & 0 & 0 & 0 & \frac{2Q}{r} \\ 0 & \frac{2Mx^2}{r^3} - \frac{2\Sigma}{\sqrt{3}r} + 1 & \frac{2Mxy}{r^3} & \frac{2Mxz}{r^3} & 0 \\ 0 & \frac{2Mxy}{r^3} & \frac{2My^2}{r^3} - \frac{2\Sigma}{\sqrt{3}r} + 1 & \frac{2Myz}{r^3} & 0 \\ 0 & \frac{2Mxz}{r^3} & \frac{2Myz}{r^3} & \frac{2Mz^2}{r^3} - \frac{2\Sigma}{\sqrt{3}r} + 1 & 0 \\ \frac{2Q}{r} & 0 & 0 & 0 & \frac{4\Sigma}{\sqrt{3}r} + 1 \end{pmatrix} + O(r^{-2}). \qquad \text{(A33)}$$

It turns out that when $P \neq 0$, the Rasheed metrics do *not* satisfy the KK-asymptotic flatness requirements anymore; indeed, the phase space decomposes into sectors, labeled by $P \in \mathbb{R}$, in which the metrics g asymptote to the background metric

$$\bar{g} \coloneqq (dx^4 + 2P\cos(\theta)d\varphi)^2 - dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta)d\varphi^2. \qquad \text{(A34)}$$

The metrics (A1) and (A34) are singular at $\sin(\theta) = 0$. This can be resolved by replacing x^4 by $\bar{x}^4$ (respectively, by $\tilde{x}^4$) on the following coordinate patches:

$$\begin{cases} \bar{x}^4 \coloneqq x^4 + 2P\varphi, & \theta \in [0, \pi), \\ \tilde{x}^4 \coloneqq x^4 - 2P\varphi, & \theta \in (0, \pi]. \end{cases} \qquad \text{(A35)}$$

Indeed, the one-form

$$dx^4 + 2P\cos(\theta)d\varphi = d\bar{x}^4 + 2P(\cos(\theta) - 1)d\varphi = d\bar{x}^4 - \frac{2P}{r(r+z)}(xdy - ydx)$$

is smooth for $r > 0$ on $\{\theta \in [0, \pi)\}$. Similarly the one-form

$$dx^4 + 2P\cos(\theta)d\varphi = d\tilde{x}^4 + 2P(\cos(\theta) + 1)d\varphi = d\tilde{x}^4 + \frac{2P}{r(r-z)}(xdy - ydx)$$

is smooth on $\{\theta \in (0, \pi], r > 0\}$. The smoothness of both g and $\bar{g}$ in the d.o.c., under the constraints discussed above, readily follows.

We note the relation

$$\bar{x}^4 = \tilde{x}^4 + 4P\varphi, \qquad \text{(A36)}$$

which implies a smooth geometry with periodic coordinates $\bar{x}^4$ and $\tilde{x}^4$ if and only if

$$\text{both } \bar{x}^4 \text{ and } \tilde{x}^4 \text{ are periodic with period } 8P\pi. \qquad \text{(A37)}$$

From this perspective x^4 is *not a coordinate* anymore; instead, the basic coordinates are $\bar{x}^4$ for $\theta \in [0, \pi)$ and $\tilde{x}^4$ for $\theta \in (0, \pi]$, with dx^4 (but *not* x^4) well defined away from the axes of rotation $\{\sin(\theta) = 0\}$ as

$$dx^4 = \begin{cases} d\bar{x}^4 - 2Pd\varphi, & \theta \in [0, \pi), \\ d\tilde{x}^4 + 2Pd\varphi, & \theta \in (0, \pi]. \end{cases} \qquad \text{(A38)}$$

a. Curvature of the asymptotic background

We continue with a calculation of the curvature tensor of the asymptotic background. It is convenient to work in the coframe

$$\bar{\Theta}^{\hat{0}} = dt, \qquad \bar{\Theta}^{\hat{1}} = dx, \qquad \bar{\Theta}^{\hat{2}} = dy, \qquad \bar{\Theta}^{\hat{3}} = dz, \qquad \bar{\Theta}^{\hat{4}} = dx^4 + 2P\cos(\theta)d\varphi, \qquad \text{(A39)}$$

which is manifestly smooth after replacing dx^4 as in Eq. (A38). Using

$$d\bar{\Theta}^{\hat{4}} = -2P\sin(\theta)d\theta \wedge d\varphi = -2P\frac{x^i}{r^3}\partial_i \rfloor (dx \wedge dy \wedge dz) = -\frac{P}{r^3}\mathring{\epsilon}_{\hat{i}\hat{j}\hat{k}}x^i dx^{\hat{j}} \wedge dx^{\hat{k}}, \qquad \text{(A40)}$$

where $\mathring{\epsilon}_{\hat{i}\hat{j}\hat{k}} \in \{0, \pm 1\}$ denotes the usual epsilon symbol, one finds the following nonvanishing connection coefficients:

$$\bar{\omega}^{\hat{4}}{}_{\hat{i}} = \frac{P}{r^3}\overset{\circ}{\epsilon}_{\hat{i}\hat{j}\hat{k}} x^{\hat{j}} \bar{\Theta}^{\hat{k}}, \qquad \bar{\omega}^{\hat{i}}{}_{\hat{j}} = \frac{P}{r^3}\overset{\circ}{\epsilon}_{\hat{i}\hat{j}\hat{k}} x^{\hat{k}} \bar{\Theta}^{\hat{4}}, \tag{A41}$$

where $x^{\hat{i}} \equiv x^i$. This leads to the curvature forms

$$\begin{aligned} \bar{\Omega}^{\hat{i}}{}_{\hat{j}} &= \frac{P}{r^3}\overset{\circ}{\epsilon}_{\hat{i}\hat{j}\hat{k}}\left(-\frac{3}{r^2}x^{\hat{k}}x^{\hat{\ell}} + \delta^{\hat{k}}_{\hat{\ell}}\right)\bar{\Theta}^{\hat{\ell}} \wedge \bar{\Theta}^{\hat{4}} \\ &\quad - \frac{2P^2}{r^6}\overset{\circ}{\epsilon}_{\hat{i}\hat{m}(\hat{k}}\overset{\circ}{\epsilon}_{\hat{j})\hat{n}\hat{\ell}} x^{\hat{m}} x^{\hat{n}} \bar{\Theta}^{\hat{k}} \wedge \bar{\Theta}^{\hat{\ell}}, \\ \bar{\Omega}^{\hat{4}}{}_{\hat{i}} &= \frac{P}{r^3}\overset{\circ}{\epsilon}_{\hat{i}\hat{j}\hat{k}}\left(-\frac{3}{r^2}x^{\hat{j}}x^{\hat{\ell}} + \delta^{\hat{j}}_{\hat{\ell}}\right)\bar{\Theta}^{\hat{\ell}} \wedge \bar{\Theta}^{\hat{k}} \\ &\quad + \frac{P^2}{r^6}\overset{\circ}{\epsilon}_{\hat{k}\hat{m}\hat{j}}\overset{\circ}{\epsilon}_{\hat{k}\hat{i}\hat{\ell}} x^{\hat{m}} x^{\hat{\ell}} \bar{\Theta}^{\hat{j}} \wedge \bar{\Theta}^{\hat{4}}, \end{aligned} \tag{A42}$$

and hence the following nonvanishing curvature tensor components:

$$\begin{aligned} \bar{\mathbf{R}}^{\hat{i}}{}_{\hat{j}\hat{k}\hat{4}} &= \frac{P}{r^3}\overset{\circ}{\epsilon}_{\hat{i}\hat{j}\hat{\ell}}\left(-\frac{3}{r^2}x^{\hat{\ell}}x^{\hat{k}} + \delta^{\hat{\ell}}_{\hat{k}}\right), \\ \bar{\mathbf{R}}^{\hat{4}}{}_{\hat{i}\hat{j}\hat{4}} &= \frac{P^2}{r^6}\overset{\circ}{\epsilon}_{\hat{k}\hat{m}\hat{j}}\overset{\circ}{\epsilon}_{\hat{k}\hat{i}\hat{\ell}} x^{\hat{m}} x^{\hat{\ell}}, \\ \bar{\mathbf{R}}_{\hat{i}\hat{j}\hat{k}\hat{\ell}} &= -\frac{2P^2}{r^6}(\overset{\circ}{\epsilon}_{\hat{i}\hat{j}\hat{n}}\overset{\circ}{\epsilon}_{\hat{k}\hat{\ell}\hat{m}} + \overset{\circ}{\epsilon}_{\hat{i}\hat{m}[\hat{k}}\overset{\circ}{\epsilon}_{\hat{\ell}]\hat{j}\hat{n}}) x^{\hat{m}} x^{\hat{n}}. \end{aligned} \tag{A43}$$

The nonvanishing components of the Ricci tensor read

$$\begin{aligned} \bar{\mathbf{R}}_{\hat{i}\hat{j}} &= -\frac{2P^2}{r^6}\overset{\circ}{\epsilon}_{\hat{k}\hat{m}\hat{i}}\overset{\circ}{\epsilon}_{\hat{k}\hat{n}\hat{j}} x^{\hat{m}} x^{\hat{n}}, \\ \bar{\mathbf{R}}_{\hat{4}\hat{4}} &= -\frac{P^2}{r^6}\overset{\circ}{\epsilon}_{\hat{k}\hat{m}\hat{i}}\overset{\circ}{\epsilon}_{\hat{k}\hat{i}\hat{\ell}} x^{\hat{m}} x^{\hat{\ell}}. \end{aligned} \tag{A44}$$

Subsequently, the Ricci scalar is $\bar{\mathbf{R}} = -2P^2/r^4$.

4. Global charges: A summary

For ease of future reference, we summarize the global charges of the Rasheed metrics. Let p_μ be the Hamiltonian momentum of the level sets of t, and let $p_{\mu,\mathrm{ADM}}$ be the ADM four-momentum of the space metric $g_{ij}dx^i dx^j$. Then,

$$p_{i,\mathrm{ADM}} = p_i = 0, \qquad p_{0,\mathrm{ADM}} = M - \frac{\Sigma}{\sqrt{3}},$$
$$p_0 = \begin{cases} 2\pi M, & P = 0, \\ 4\pi PM, & P \neq 0, \end{cases} \qquad p_4 = \begin{cases} 2\pi Q, & P = 0, \\ 8\pi PQ, & P \neq 0. \end{cases} \tag{A45}$$

The Komar integrals associated with $X = \partial_t$ are

$$\frac{1}{8\pi}\lim_{R\to\infty}\int_{S(R)}\int_{S^1} X^{\alpha;\beta} dS_{\alpha\beta} = \begin{cases} 2\pi(M + \frac{\Sigma}{\sqrt{3}}), & P = 0, \\ 8\pi P(M + \frac{\Sigma}{\sqrt{3}}), & P \neq 0, \end{cases} \tag{A46}$$

The Komar integrals associated with $X = \partial_4$ are

$$\frac{1}{8\pi}\lim_{R\to\infty}\int_{S(R)}\int_{S^1} X^{\alpha;\beta} dS_{\alpha\beta} = \begin{cases} 4\pi Q, & P = 0, \\ 16\pi PQ, & P \neq 0. \end{cases} \tag{A47}$$

APPENDIX B: THE VECTOR FIELD Z

Let

$$Z = r\partial_r.$$

We wish to calculate $\bar{\nabla}_\mu Z_\nu$ for the Kottler metrics and the Rasheed metrics.

First, let $\bar{g}$ be the $(n+1)$-dimensional anti–de Sitter (Kottler) metric,

$$\bar{g} = -V dt^2 + V^{-1} dr^2 + r^2 \bar{h}, \tag{B1}$$

with

$$V = \lambda r^2 + \kappa, \tag{B2}$$

where $\kappa \in \{0, \pm 1\}$ is a constant,

$$\lambda = -\frac{2\Lambda}{n(n-1)}, \tag{B3}$$

and where $\bar{h}$ is an (r-independent) Einstein metric on an $(n-1)$-dimensional compact manifold $\mathcal{K}$, with scalar curvature $(n-1)(n-2)\kappa$. It holds that (cf., e.g., Ref. [48])

$$\bar{R} = -n(n+1)\lambda. \tag{B4}$$

Further,

$$\begin{aligned}\bar{\nabla}_{(\mu}Z_{\nu)}dx^\mu\otimes dx^\nu &= \frac{1}{2}\mathcal{L}_Z\bar{g} = \frac{1}{2}(Z^\alpha\partial_\alpha\bar{g}_{\mu\nu}+\partial_\mu Z^\alpha\bar{g}_{\alpha\nu}+\partial_\nu Z^\alpha\bar{g}_{\alpha\mu})dx^\mu dx^\nu\\ &= \frac{1}{2}(r(\partial_r(-V)dt^2+\partial_r(V^{-1})dr^2+\partial_r(r^2)d\Omega)+2V^{-1}dr^2)\\ &= \frac{1}{2}\left(\frac{r\partial_r V}{V}(-Vdt^2)+(2-rV^{-1}\partial_r V)V^{-1}dr^2+2r^2d\Omega^2\right), \end{aligned} \tag{B5}$$

$$\bar{\nabla}_{[\mu}Z_{\nu]}dx^\mu\otimes dx^\nu = \partial_{[\mu}Z_{\nu]}dx^\mu\otimes dx^\nu = 0. \tag{B6}$$

Adding, we find

$$\bar{\nabla}_\mu Z_\nu dx^\mu\otimes dx^\nu = \bar{g} \quad \mathrm{mod}\ (\delta^t_\mu,\delta^r_\mu), \tag{B7}$$

which gives Eq. (4.17).

Next, for the Rasheed background metrics (A34) one finds

$$\begin{aligned}\mathcal{L}_Z\bar{g} &= 2(dr^2+r^2d\Omega^2),\\ d(\bar{g}_{\alpha\beta}Z^\alpha dx^\beta) &= d(rdr)=0,\end{aligned} \tag{B8}$$

and Eq. (4.17) without the $o(r^{-\gamma})$ term readily follows.

APPENDIX C: AN IDENTITY FOR THE RIEMANN TENSOR

We write $\delta^{\alpha\beta}_{\gamma\delta}$ for $\delta^{[\alpha}_\gamma\delta^{\beta]}_\delta \equiv \frac{1}{2}(\delta^\alpha_\gamma\delta^\beta_\delta-\delta^\beta_\gamma\delta^\alpha_\delta)$, etc.

For completeness, we prove the following identity satisfied by the Riemann tensor, which is valid in any dimension, is clear in dimensions two and three, implies the double-dual identity for the Weyl tensor in dimension four, and is probably well known in higher dimensions as well:

$$\delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}R^{\rho\sigma}{}_{\gamma\delta} = \frac{1}{3!}(R^{\alpha\beta}{}_{\mu\nu}+\delta^{\alpha\beta}_{\mu\nu}R-4\delta^{[\alpha}_{[\mu}R^{\beta]}{}_{\nu]}). \tag{C1}$$

The above holds for any tensor field satisfying

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = R_{\beta\alpha\delta\gamma}. \tag{C2}$$

To prove Eq. (C1) one can calculate as follows:

$$\begin{aligned}4!\delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}R^{\rho\sigma}{}_{\gamma\delta} &= 2[\delta^\alpha_\mu(\delta^\beta_\nu\delta^\gamma_\rho\delta^\delta_\sigma-\delta^\beta_\rho\delta^\gamma_\nu\delta^\delta_\sigma+\delta^\beta_\sigma\delta^\gamma_\nu\delta^\delta_\rho)-\delta^\alpha_\nu(\delta^\beta_\mu\delta^\gamma_\rho\delta^\delta_\sigma-\delta^\beta_\rho\delta^\gamma_\mu\delta^\delta_\sigma+\delta^\beta_\sigma\delta^\gamma_\mu\delta^\delta_\rho)\\ &\quad+\delta^\alpha_\rho(\delta^\beta_\mu\delta^\gamma_\nu\delta^\delta_\sigma-\delta^\beta_\nu\delta^\gamma_\mu\delta^\delta_\sigma+\delta^\beta_\sigma\delta^\gamma_\mu\delta^\delta_\nu)-\delta^\alpha_\sigma(\delta^\beta_\mu\delta^\gamma_\nu\delta^\delta_\rho-\delta^\beta_\nu\delta^\gamma_\mu\delta^\delta_\rho+\delta^\beta_\rho\delta^\gamma_\mu\delta^\delta_\nu)]R^{\rho\sigma}{}_{\gamma\delta}\\ &= 2(2\delta^{\alpha\beta}_{\mu\nu}\delta^\gamma_\rho\delta^\delta_\sigma-4\delta^{\alpha\gamma}_{\mu\nu}\delta^{\beta\delta}_{\rho\sigma}+4\delta^{\beta\gamma}_{\mu\nu}\delta^{\alpha\delta}_{\rho\sigma}+2\delta^\alpha_\rho\delta^\beta_\sigma\delta^\gamma_\mu\delta^\delta_\nu)R^{\rho\sigma}{}_{\gamma\delta}\\ &= 4(\delta^{\alpha\beta}_{\mu\nu}R^{\gamma\delta}{}_{\gamma\delta}-2\delta^{\alpha\gamma}_{\mu\nu}R^{\beta\sigma}{}_{\gamma\sigma}+2\delta^{\beta\gamma}_{\mu\nu}R^{\alpha\sigma}{}_{\gamma\sigma}+R^{\alpha\beta}{}_{\mu\nu})\\ &= 4(R^{\alpha\beta}{}_{\mu\nu}+\delta^{\alpha\beta}_{\mu\nu}R^{\gamma\delta}{}_{\gamma\delta}-4\delta^{[\alpha}_{[\mu}R^{\beta]\gamma}{}_{\nu]\gamma}).\end{aligned} \tag{C3}$$

If the sums are over all indices we obtain Eq. (C1). The reader is warned, however, that in some of our calculations the sums will be only over a subset of all possible indices, in which case the last equation remains valid but the last two terms in Eq. (C3) *cannot* be replaced by the Ricci scalar and the Ricci tensor.

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5 The Brill and Pfister solution

Another family of interesting solutions with Kaluza-Klein asymptotics has been constructed by Brill and Pfister. More precisely, in [3] the authors provide time-symmetric initial data(with vanishing extrinsic curvature of the initial data surface) in Kaluza-Klein theory with a negative lower limit for the ADM- as well for the Hamiltonian mass, as defined in Section 4. It should be emphasised, that they do not provide the full spacetime metric, but only the initial data, which satisfy the general relativistic vacuum constraint equations(with vanishing cosmological constant). But this is good enough to obtain an associated maximal globally hyperbolic solution, by evolving the data using the vacuum Einstein equations.

The presentation in [3] does not provide convincing justification that the metrics there are singularity-free. In particular it is not completely clear whether or not the Riemann tensor has distributional components which could be responsible for the negativity of mass. The aim of what follows is to fill this gap.

The four-dimensional initial data, which we refer to as the Brill-Pfister solution in the following, are given by

$$ds^2 = \psi^4 d\sigma^2 + V^2 (dx^4)^2 \,, \tag{5.1}$$

where $d\sigma^2 = dr^2 + r^2 d\Omega^2$, with ψ, V being C^2-functions of r, and x^4 is the fifth coordinate of Kaluza-Klein theory, being $2\pi R$-periodic, where R denotes the compactification radius.
Asymptotic flatness is guaranteed if ψ and V take asymptotically the form

$$\psi = 1 + \frac{m}{2r} + O\left(r^{-2}\right), \quad V = 1 + \frac{\mu}{2r} + O\left(r^{-2}\right), \tag{5.2}$$

with the obvious associated decay conditions on the derivatives. The corresponding ADM- and Hamiltonian mass are discussed in section 5.2. The inner boundary of a space with topology $\mathbb{R}^2 \times S^2$ is called a bubble. The location of the bubble is determined by the zeros of the Killing vector $\frac{\partial}{\partial x^4}$ and thus from (5.1) by the zeros of V. We denote this location by $r = B > 0$. The five-dimensional Einstein equations ${}^5G_{\mu\nu} = 0$, imply $R = 0$ on the Hamiltonian constraint, where denotes R the four-dimensional scalar curvature. The momentum-constraint is fulfilled automatically, due to time-symmetry. By introducing

$$W = V\psi \,, \tag{5.3}$$

the four-dimensional scalar curvature of (5.1) in terms of W and ψ is given by

$$R = -2\psi^{-4}\Big(W^{-1}\Delta W + 3\psi^{-1}\Delta\psi\Big), \tag{5.4}$$

where Δ denotes the flat-space Laplacian. The explicit solution, constructed by Brill and Pfister, is given by

$$\begin{aligned} W &= \begin{cases} \frac{D}{r}\sin\Big(k(r-B)\Big), & B \le r \le A, \\ 1 + \frac{m}{2r}, & r > A, \end{cases} \\ \psi &= \begin{cases} \frac{E}{r}\cosh\Big(\frac{k}{\sqrt{3}}(r-C)\Big), & B \le r \le A, \\ 1 + \frac{m}{2r}, & r > A, \end{cases} \end{aligned} \tag{5.5}$$

where $k, C, D, E, m \in \mathbb{R}$ are real constants, constrained by (13)-(15) of [3], i.e.

$$\begin{aligned} 2b &= \sqrt{3}\coth\left(\frac{c}{\sqrt{3}}\right), \quad (5.6)\\ \tan(a) &= \sqrt{3}\coth\left(\frac{a+c}{\sqrt{3}}\right), \quad (5.7)\\ \frac{m}{2B} &= \frac{1}{b}\Big(\tan(a)-a\Big)-1, \quad (5.8) \end{aligned}$$

where $a = k(A-B)$, $b = kB$ and $c = k(B-C)$, obtained by the boundary conditions at $r = B$ and continuity conditions, imposed on V, W and ψ. By inserting (5.5) in (5.3), we obtain

$$V = \begin{cases} \frac{D}{E}\frac{\sin\left(k(r-B)\right)}{\cosh\left(\frac{k}{\sqrt{3}}(r-C)\right)}, & B \le r \le A,\\ 1, & r > A. \end{cases} \quad (5.9)$$

We point out a misprint in the original paper [3], i.e. in (5.5) the function cos is printed instead of cosh. By computing the flat-space Laplacian for (5.5), we obtain

$$\begin{aligned} W^{-1}\Delta W &= \begin{cases} -k^2, & B \le r \le A,\\ 0, & r > A, \end{cases}\\ \psi^{-1}\Delta\psi &= \begin{cases} \frac{k^2}{3}, & B \le r \le A,\\ 0, & r > A. \end{cases} \end{aligned} \quad (5.10)$$

After this correction, we see that the scalar constraint equation (5.4) is indeed solved by the Brill-Pfister solution.

5.1 Smoothness at $r = B$

In the following we show that the metric is not differentiable at $r = B$. The expansions of (5.5) and (5.9) at $r = B$ take the form

$$\begin{aligned} \psi^4 &= \psi_0 + \psi_1(r-B) + O((r-B)^2),\\ V^2 &= (r-B)^2\big(\alpha + \beta(r-B)\big) + O((r-B)^4), \end{aligned} \quad (5.11)$$

where $\psi_0, \psi_1, \alpha, \beta \in \mathbb{R}$. The insertion of (5.11) in (5.1) yields

$$\begin{aligned} ds^2 &= \Big(\psi_0 + \psi_1(r-B) + O((r-B)^2)\Big)(dr^2 + r^2 d\Omega^2) + (r-B)^2\Big(\alpha + \beta(r-B) + O((r-B)^2)\Big)(dx^4)^2\\ &= \Big(\psi_0 + \psi_1(r-B) + O((r-B)^2)\Big)dr^2 + (r-B)^2\Big(\alpha + \beta(r-B) + O((r-B)^2)\Big)(dx^4)^2\\ &\quad + r^2\Big(\psi_0 + \psi_1(r-B) + O((r-B)^2)\Big)d\Omega^2. \end{aligned} \quad (5.12)$$

Differentiability at $r = B$ would require the expansions of even functions to appear in the variable $r - B$ in the parentheses of (5.12), thus it follows immediately, that the metric is not differentiable at $r = B$, unless perhaps a better coordinate system can be found. To better investigate the behaviour of the curvature tensor as r tends to B, we introduce the orthonormal co-frame

$$\Theta^{\hat{1}} = \psi^2 dr, \quad \Theta^{\hat{2}} = \psi^2 r d\theta, \quad \Theta^{\hat{3}} = \psi^2 r\sin(\theta) d\theta, \quad \Theta^{\hat{4}} = V dx^4, \quad (5.13)$$

adapted to the metric, leading to the following non-vanishing connection one-forms

$$\begin{aligned}
\omega^{\hat{1}}{}_{\hat{2}} &= -\frac{2r\psi'+\psi}{\psi}d\theta\,, \quad \omega^{\hat{1}}{}_{\hat{3}} = -\frac{2r\psi'+\psi}{\psi}\sin\theta d\phi\,, \\
\omega^{\hat{1}}{}_{\hat{4}} &= -\frac{V'}{\psi^2}dx^4\,, \quad \omega^{\hat{2}}{}_{\hat{3}} = -\cos\theta d\phi\,. \qquad (5.14)
\end{aligned}$$

The corresponding non-vanishing curvature-forms read as

$$\begin{aligned}
\Omega^{\hat{1}}{}_{\hat{2}} &= -\frac{2\left(\psi\left(r\psi''+\psi'\right)-r\psi'^2\right)}{r\psi^6}\Theta^1\wedge\Theta^2\,, \\
\Omega^{\hat{1}}{}_{\hat{3}} &= -\frac{2\left(\psi\left(r\psi''+\psi'\right)-r\psi'^2\right)}{r\psi^6}\Theta^1\wedge\Theta^3\,, \\
\Omega^{\hat{1}}{}_{\hat{4}} &= \frac{2V'\psi'-\psi V''}{V\psi^5}\Theta^1\wedge\Theta^4\,, \\
\Omega^{\hat{2}}{}_{\hat{3}} &= -\frac{4\psi'\left(r\psi'+\psi\right)}{r\psi^6}\Theta^2\wedge\Theta^3\,, \\
\Omega^{\hat{2}}{}_{\hat{4}} &= -\frac{V'\left(2r\psi'+\psi\right)}{rV\psi^5}\Theta^2\wedge\Theta^4\,, \\
\Omega^{\hat{3}}{}_{\hat{4}} &= -\frac{V'\left(2r\psi'+\psi\right)}{rV\psi^5}\Theta^3\wedge\Theta^4\,. \qquad (5.15)
\end{aligned}$$

From these curvature forms we can read off the relevant components of the Riemann tensor, with potential critical behaviour where V vanishes, which are given by

$$\begin{aligned}
R_{\hat{4}\hat{1}\hat{4}\hat{1}} &= \frac{2V'\psi'-\psi V''}{2V\psi^5} \\
&= \frac{\frac{2\psi'}{\psi}V'-V''}{2V\psi^4}\,, \\
R_{\hat{4}\hat{2}\hat{4}\hat{2}} &= -\frac{V'\left(2r\psi'+\psi\right)}{2rV\psi^5}\,, \\
R_{\hat{4}\hat{3}\hat{4}\hat{3}} &= -\frac{V'\left(2r\psi'+\psi\right)}{2rV\psi^5}\,. \qquad (5.16)
\end{aligned}$$

From (5.16) we obtain the boundary conditions

$$\begin{aligned}
V''+\frac{V'}{r} &= 0 \quad \text{at} \quad r=B\,, \\
\frac{2\psi'}{\psi} &= -\frac{1}{r} \quad \text{at} \quad r=B\,, \qquad (5.17)
\end{aligned}$$

necessary to avoid singularities, at $r = B$. By implementing those boundary conditions in the coefficients of the series expansions for ψ and V, we obtain

$$\begin{aligned}
\psi &= \psi(B)\left[1-\frac{2}{B}(r-B)+\frac{1}{2B^2}(r-B)^2\right]+O((r-B)^3)\,, \\
V &= V'(B)\left[(r-B)-\frac{1}{2B}(r-B)^2\right]+O((r-B)^3)\,, \qquad (5.18)
\end{aligned}$$

and in conclusion for their powers, appearing in the metric (5.1),

$$
\begin{aligned}
\psi^4 &= \psi(B)^4\left[1-\frac{2}{B}(r-B)+\frac{3}{2B^2}(r-B)^2-\frac{1}{2B}(r-B)^3\right]+O((r-B)^4)\,,\\
V^2 &= V'(B)^2\left[(r-B)^2-\frac{1}{B}(r-B)^3\right]+O((r-B)^4)\,. \qquad (5.19)
\end{aligned}
$$

By introducing the coordinates $\rho=r-B$, $\phi=\frac{x^4}{\lambda}$, where $\lambda\in\mathbb{R}\setminus\{0\}$, chosen to a fixed value later, we rewrite the metric in the form

$$
\begin{aligned}
ds^2 &= \psi^4 d\rho^2+(\rho+B)^2\psi^4 d\Omega^2+V^2\lambda^2 d\phi^2\\
&= \psi^4 d\rho^2+(V^2\lambda^2-\psi^4\rho^2)d\phi^2+(\rho+B)^2\psi^4 d\Omega^2+\psi^4\rho^2 d\phi^2\,. \qquad (5.20)
\end{aligned}
$$

By ignoring and simplifying already smooth terms in (5.20) respectively, we obtain the metric

$$
ds^2 = \psi^4 d\rho^2+(V^2\lambda^2-\psi^4\rho^2)d\phi^2+(\rho+B)^2\psi^4 d\Theta^2\,, \qquad (5.21)
$$

conserving the potential critical behaviour of (5.20) at $\rho=0$. By introducing polar coordinates

$$
x=\rho\cos(\phi)\,,\quad y=\rho\sin(\phi)\,, \qquad (5.22)
$$

we write (5.21) in the from

$$
ds^2=\psi^4\left(dx^2+dy^2\right)+\Phi\left(xdy-ydx\right)^2+\psi^4\left(\rho+B\right)^2 d\Theta^2\,, \qquad (5.23)
$$

where

$$
\Phi=\frac{\lambda^2\frac{V^2}{\rho^2}-\psi^4}{\rho^2}\,. \qquad (5.24)
$$

We require

$$
\lim_{\rho\to 0}\left(\lambda^2\frac{V^2}{\rho^2}-\psi^4\right)=0\,, \qquad (5.25)
$$

i.e. that the numerator of Φ is vanishing in this limit. (5.25) is solved by $\lambda=\frac{\psi(B)^2}{V'(B)}$. We compute for this choice for λ the first-order expansions of the non-vanishing components of the Riemann tensor of (5.23) at $\rho=0$ with MATHEMATICA, which are given by

$$
\begin{aligned}
R^x{}_{xxy} &= -\frac{5\sin(\phi)\cos(\phi)}{4B^3}\rho+O(\rho^2)\,,\\
R^x{}_{yxy} &= \frac{5}{4B^2}+\frac{-6B^2+5\cos(2\phi)+27}{8B^3}\rho+O(\rho^2)\,,\\
R^x{}_{AxA} &= \frac{3}{2}+\frac{3\left(B^2+1\right)(\cos(2\phi)+3)}{8B}\rho+O(\rho^2)\,,\\
R^x{}_{AyA} &= \frac{3\left(B^2+1\right)\sin(2\phi)}{8B}\rho+O(\rho^2)\,,
\end{aligned}
$$

$$
\begin{aligned}
R^y{}_{xxy} &= -\frac{5}{4B^2}+\frac{\rho\left(6B^2+5\cos(2\phi)-27\right)}{8B^3}\rho+O(\rho^2)\,,\\
R^y{}_{yxy} &= \frac{5\rho\sin(2\phi)}{8B^3}+O(\rho^2)\,,\\
R^y{}_{AxA} &= \frac{3\left(B^2+1\right)\sin(2\phi)}{8B}\rho+O(\rho^2)\,,\\
R^y{}_{AyA} &= \frac{3}{2}-\frac{3\left(\left(B^2+1\right)(\cos(2\phi)-3)\right)}{8B}\rho+O(\rho^2)\,,\\
R^A{}_{xxA} &= -\frac{3}{2B^2}-\frac{3\rho\left(\left(B^2-1\right)(\cos(2\phi)+3)\right)}{8B^3}\rho+O(\rho^2)\,,\\
R^A{}_{xyA} &= -\frac{3\left(B^2-1\right)\sin(2\phi)}{8B^3}\rho+O(\rho^2)\,,\\
R^A{}_{yxA} &= -\frac{3\left(B^2-1\right)\sin(2\phi)}{8B^3}\rho+O(\rho^2)\,,\\
R^A{}_{yyA} &= -\frac{3}{2B^2}+\frac{3\left(B^2-1\right)(\cos(2\phi)-3)}{8B^3}\rho+O(\rho^2)\,. && (5.26)
\end{aligned}
$$

Now, with the expressions above, some care has to be taken because of second derivatives of ψ and W which could give a distributional contribution at $r=B$. In order to address this problem, we introduce local Cartesian coordinates centred at $r=B$:

$$
(x^i)\equiv(x,y):=(\rho\cos\phi,\rho\sin\phi)=((r-B)\cos\phi,(r-B)\sin\phi)\,.
$$

For r close to B we have

$$
\begin{aligned}
\psi &= \frac{E}{B+\rho}\cosh\frac{k}{\sqrt{3}}(\rho+B-C)\,,\\
\partial_i\psi &= \frac{E\left(\sqrt{3}k(B+\rho)\sinh\left(\frac{k(B-C+\rho)}{\sqrt{3}}\right)-3\cosh\left(\frac{k(B-C+\rho)}{\sqrt{3}}\right)\right)}{3(B+\rho)^2}\frac{x^i}{\rho}=O(1)\,, && (5.27)\\
\partial_i\partial_j\psi &= \frac{E\left(\sqrt{3}k(B+\rho)\sinh\left(\frac{k(B-C+\rho)}{\sqrt{3}}\right)-3\cosh\left(\frac{k(B-C+\rho)}{\sqrt{3}}\right)\right)}{3(B+\rho)^2}\frac{\rho^2\delta^i_j-x^ix^j}{\rho^3}\\
&\quad+\frac{E\left(k^2(B+\rho)^2+6\right)\cosh\left(\frac{k(B-C+\rho)}{\sqrt{3}}\right)-2\sqrt{3}Ek(B+\rho)\sinh\left(\frac{k(B-C+\rho)}{\sqrt{3}}\right)}{3(B+\rho)^3}\frac{x^ix^j}{\rho^2}\\
&= O(\rho^{-1})\,. && (5.28)
\end{aligned}
$$

From the definition of the distributional derivative, given a smooth compactly supported function f we have

$$
\begin{aligned}
\langle\partial_i\psi,f\rangle &:= -\int_{\mathbb{R}^2}\psi\partial_i f\,d^2\mu=-\lim_{\epsilon\to0}\int_{\mathbb{R}^2\setminus D(\epsilon)}\psi\partial_i f\,d^2\mu\\
&= -\lim_{\epsilon\to0}\int_{\mathbb{R}^2\setminus D(\epsilon)}(\partial_i(\psi f)-f\partial_i\psi)\,d^2\mu\\
&= \underbrace{-\lim_{\epsilon\to0}\int_{\partial D(\epsilon)}\psi f n^i\epsilon\,d\phi}_{=0}+\int_{\mathbb{R}^2\setminus D(\epsilon)}f\partial_i\psi\,d^2\mu\,,
\end{aligned}
$$

and we see that the first distributional derivatives of ψ do not give any contribution at the origin. One similarly finds

$$\begin{aligned}
\langle \partial_j \partial_i \psi, f \rangle &:= -\langle \partial_i \psi, \partial_j f \rangle \\
&= -\int_{\mathbb{R}^2} \partial_j \psi \partial_i f d^2\mu = -\lim_{\epsilon \to 0} \int_{\mathbb{R}^2 \setminus D(\epsilon)} \partial_j \psi \partial_i f d^2\mu \\
&= -\lim_{\epsilon \to 0} \int_{\mathbb{R}^2 \setminus D(\epsilon)} (\partial_i(\partial_j \psi f) - f \partial_j \partial_i \psi) d^2\mu \\
&= \underbrace{-\lim_{\epsilon \to 0} \int_{\partial D(\epsilon)} \partial_j \psi f n^i \epsilon d\phi}_{=0} + \int_{\mathbb{R}^2} f \partial_i \partial_j \psi d^2\mu ,
\end{aligned}$$

and there is no distributional contribution from second derivatives either.

5.2 The Hamiltonian and the ADM mass

Evaluating our formulae in Section 4 for the Hamiltonian mass m_0 and the ADM mass m_{ADM} for the Brill-Pfister initial data, under the decay assumptions (5.2), gives

$$m_0 = 2\pi(m + \mu), \quad m_{ADM} = m. \tag{5.29}$$

From (5.2) and (5.9) it follows that $\mu = 0$. Therefore, we obtain

$$m_0 = 2\pi m, \quad m_{ADM} = m. \tag{5.30}$$

By combining (5.6) and (5.8) we obtain

$$\frac{m}{2B} = \frac{2(\tan(a) - a)\left(\sqrt{3}\coth\left(\frac{a}{\sqrt{3}}\right) - \tan(a)\right)}{\sqrt{3}\coth\left(\frac{a}{\sqrt{3}}\right)\tan(a) - 3} - 1, \tag{5.31}$$

which attains a negative maximum -0.2092 at $a = \frac{\pi}{3}$. Together with $B > 0$ it follows that m, and thus m_0 and m_{ADM}, are negative.

6 Conclusions

In the framework of this thesis we have investigated the Rasheed-Larsen black hole solutions, i.e. we have proved their regularity at the outer Killing horizon, have analysed and identified the singularities of the metrics and have derived conditions, under which they are shielded by the outer Killing horizon. Furthermore, we have excluded the existence of regular metrics without horizons and have derived a criterion for stable causality in the domain of outer communications. In our analysis of global quantities we have covered asymptotically anti–de Sitter spacetimes, asymptotically flat spacetimes, as well as Kaluza-Klein asymptotically flat spacetimes. We have shown that the Komar mass equals the Arnowitt-Deser-Misner (ADM) mass in stationary asymptotically flat spacetimes in all dimensions. It has been shown that the Hamiltonian mass does not necessarily coincide with the ADM mass in Kaluza-Klein asymptotically flat spacetimes. Furthermore, we have applied a Witten-type argument to derive global inequalities between the Hamiltonian energy-momentum and the Kaluza-Klein charges. We have applied our formulae to the five-dimensional Rasheed metrics, from which we have computed the corresponding global charges. Finally, by a comparison of them to those of the Larsen metrics, we have shown, that those classes of metrics are isometric.
In our analysis of the four-dimensional initial data in Kaluza-Klein theory, constructed by Brill and Pfister, we have pointed out and analysed, although the corresponding initial data metric is not differentiable at the bubble, it is at least twice weakly differentiable at this location, leading to a Riemann tensor without distributional components which could be responsible for the negativity of the ADM mass.

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