On black hole uniqueness theorems

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Abstract

We obtain a classification of stationary, appropriately regular, non-degenerate and analytic electro-vacuum space-times in terms of Weinstein solutions. In particular, for connected horizons, we prove uniqueness of the Kerr-Newman black holes. This is done by means of a new and explicit definition of regularity (I^+–regularity) which allows us to overcome a considerable amount of technical gaps existing in the previous literature on the subject.

We also prove an upper bound for angular-momentum and charge in terms of the mass for electro-vacuum asymptotically flat axisymmetric initial data sets with simply connected orbit space.

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Contents

Chapter 1

Introduction

Wir müssen wissen.
Wir werden wissen.
David Hilbert

The main subject of this thesis is the *classification of stationary electro-vacuum "regular" black hole space-times*. Results providing such classifications are known in the relativity community as "Black Hole Uniqueness Theorems" or, in a wider community, as "No Hair Theorems" [1]. We will also present a detailed proof of a Dain inequality with charge, providing an upper bound for angular momentum and Maxwell charges in terms of the ADM mass. This last result will be exposed in a completely independent and self-contained manner although several links, some of them still mysterious, connect it to the main body of work. For instance, at a more technical level, both problems deal with axisymmetry and the harmonic maps that emanate from the Einstein-Maxwell equations in the presence of such symmetry. Moreover, another connection, at a more fundamental level, exists and justifies the chosen title for this thesis: this charged Dain inequality provides strong evidence that extreme Kerr-Newman initial data gives rise to the unique minimum of *mass* (4.1.18) for fixed *angular momentum* and *charges*, within a class of axisymmetric and asymptotically flat data.

[1] Strictly speaking these denominations correspond to inequivalent propositions.

1.1 Classification of stationary black holes

In Chapters 2 and 3 we address the following celebrated and long-standing conjecture:

Conjecture 1.1.1 *Let $(\mathscr{M}, \mathfrak{g}, F)$ be a stationary, asymptotically flat, electro-vacuum, four-dimensional regular space-time. Then the domain of outer communications $\langle\langle \mathscr{M}_{\mathrm{ext}} \rangle\rangle$ is either isometric to the domain of outer communications of a Kerr-Newman space-time or to the domain of outer communications of a (standard) Majumdar-Papapetrou space-time.*

This conjecture started being coined 42 years ago with the surprising and groundbreaking work of Israel where, modulo some explicit and other implicit technical assumptions, it was established that among "all static, asymptotically flat vacuum space-times with closed simply connected equipotential surfaces $g_{00} = constant$, the Schwarzschild solution is the only one which has a nonsingular infinite-red-shift surface $g_{00} = 0$." [87] [2]. This result immediately led to the belief that the "relevant Kerr subfamily might be the only stationary [vacuum] solutions that are well behaved all the way in to a regular black hole horizon" [21], which together with the generalization to electro-vacuum with Kerr replaced by Kerr-Newman became known as the "Israel-Carter conjecture". Far more ambitious generalizations were soon to appear within the framework of gravitational collapse where it was heuristically argued that "collapse leads to a black hole endowed with mass and charge and angular momentum but (...) no other adjustable parameters" [121] a suggestion that was summarized in the inspired *pop-culture* form: "a black hole has no hair" [121]; Penrose had gone even further by conjecturing that "if an absolute event horizon develops in an asymptotically flat space-time, then the solution exterior to this horizon approaches a Kerr-Newman solution asymptotically with time" [113].

In a series of papers culminating in [19] Brandon Carter addressed, with extraordinary successes of great relevance to the work presented here, the stationary and axisymmetric vacuum problem, which he later revisited and reconsidered

[2] The term "black hole" is neither used in this paper or in the subsequent work by the same author generalizing this result to electro-vacuum. In fact the exact origin of this icon generating term, sometime during the 1960s, seems to remain a matter of controversy.

in [20, 21]. One of the main difficulties and sources of confusion [3] resided in the fact that, contrary to what happens in Schwarzschild, the Kerr metrics with non-vanishing angular momentum parameter have non-empty ergoregions [4] (see Section 2.5.3). Nonetheless, Carter realized that the two-dimensional orbits of the subgroup of the isometry group generated by stationarity and axial symmetry are timelike throughout the domain of outer communications (compare Theorem 2.5.4); in here lay the key for constructing global Weyl coordinates and reducing the vacuum Einstein equations for asymptotically flat, stationary and axisymmetric space-times to a two-dimensional non-linear elliptic boundary valued problem. By following this program, in a mixture of semi-heuristic and rigorous steps, Carter was able to obtain a "'no hair'(i.e. no bifurcation) theorem to the effect that within a continuously differentiable family of solutions (such as the Kerr family) variation between neighbouring members is fully determined just by the corresponding variation of the pair of boundary value parameters" [21]. This, together with Israel's Theorem, provided strong evidence of uniqueness of Kerr but a proof of such fact had to wait for Robinson to discover his divergence identity [117].

Already in 1972, i.e., prior to Robinson's decisive work, Hawking had realized [70, 71] that, at the cost of assuming real-analyticity of all objects involved, a stationary rotating black hole had to be axisymmetric; this is the essence of Hawking's (strong) Rigidity Theorem. In fact, he went even further by arguing that in the non-rotating case the space-time had to be static. It should be noted that complete proofs of both the Rigidity and Staticity Theorems were not discovered until the 1990s. Consequently Israel's Theorem and the Carter-Robinson Theorem, which apparently only provided uniqueness results for two specific classes of stationary space-times, in fact exhausted all "regular", stationary, vacuum non-degenerate and connected black holes if one further assumed the infamous analyticity condition.

With Israel's extension of his result to electro-vacuum [88] and Robinson's generalization of Carter's no bifurcation result to the source free Einstein-Maxwell

[3] For instance in [87] the notions of *event horizon* and infinite-red-shift surface $g_{00} = 0$ seem indistinguishable.

[4] In fact it is not even obvious that for static space-times the ergoregion is a priori empty (see Section 2.1) and such proposition plays a key role in establishing uniqueness of Schwarzschild within this class.

setting [116] at the "end of the [1970s] decade the main gap in the uniqueness theorems appeared to be the lack of a proof of the uniqueness of a single charged stationary black hole" [119]. A more systematic approach was required for obtaining a generalization of Robinson's identity to electro-vacuum. Two formally distinct, although equivalent, approaches succeeded [20, 99]. Mazur obtained his divergence identity by algebraic methods within the framework of generalized σ-models, while Bunting's geometrical-analytical approach led him to his form of the identity via the study of more general harmonic maps, those whose target manifolds have non-positive sectional curvature.

Could one say that by the end of the 1980s there was a theorem establishing non-extremal Kerr-Newman as the unique "regular", analytic, stationary, and electro-vacuum space-times with non-degenerate and connected event horizon? The notion of what constitutes "proof" within the relativity community seems as wide and flexible as the notion of "regularity" within the extensive literature on black hole uniqueness; the first flexible enough to allow for heuristic arguments and educated guesses and the second wide enough to enclose undesirable "hairy" assumptions and technical difficulties. It is our opinion that this has made the state of the art concerning this problem difficult to assess; nonetheless an affirmative answer to the posed question, in accordance with the author's beliefs of what constitutes "proof", would at least require black hole "regularity" to correspond to a long list of stringent technical assumptions. [5] In fact, as it was demonstrated by "treating the shaky foundations" [21] of the underlying theory during the 1990s, such artificial solution is not only unsatisfying but also unnecessary, since most of these "extra" assumptions might be established from first principles. The Staticity Theorem was established in solid grounds by Sudarsky and Wald [127] by explicitly requiring a bifurcate structure for the event horizon as well as the existence of a maximal slicing of the domain of outer communications; the existence of such slicing was later proved by Chruściel and Wald [44] based on previous work of Bartnik [7], while the possibility of (isometrically) extending the closure of a domain of outer communications whose boundary is a

[5] An illustrative example of an "educated guess" is given by the product structure (3.2.15) that, although clear for Minkowski with the usual $\mathbb{R} \times U(1)$ action by isometries, seems far from obvious in the generality required. Other examples involve the regularity of the horizon, the necessary degree of differentiability is usually implicitly assumed a priori, and the asymptotic behavior of the relevant harmonic maps. Such problems are resolved in the present work.

non-degenerate horizon to a space-time containing a bifurcate horizon was proved by Rácz and Wald in [114] (see Section 2.7). One of the most stringent assumptions required by all classic results is simple connectedness of the domain of outer communications which Chruściel and Wald [45] have shown to be a consequence of *topological censorship* [57] for stationary and asymptotically flat space-times satisfying the dominant energy condition (see Section 2.2.3); in [45], Hawking's claim about spherical topology of compact cross sections of the event horizon in four-dimensional space-times [70] [6] was also clarified. During this decade Chruściel also clarified the relation between properties of Killing vectors and the existence of group actions by isometries [25], gave a complete proof of the Rigidity Theorem assuming analyticity [28] and highlighted some other insufficiencies of the existing theory, some of which he eventually overcame [26, 27, 29, 30].

Until now, this historical review has been confined to the class of non-degenerate and connected event horizons; we will finish it by briefly exposing some results where these undesirable restrictions are relaxed. Non-existence of stationary, vacuum, "regular" black holes with *all* components of the event horizon non-rotating and degenerate, follows immediately from the Komar identity and the Positive energy Theorem [84] (compare [29, Section 4]). For the static, *analytic* [7] and vacuum case non-existence of regular multi black hole configurations has been established in [29] without assuming any degeneracy conditions; such result is obtained by extending the ingenious strategy developed by Bunting and Masood-ul-Alam [17]. It should also be noted that the pioneering developments on this particular subject date back to 1973 with the results of Müller zum Hagen and Seifert [105] for static and axisymmetric space-times.

This time the situation changes considerably when passing from vacuum to electro-vacuum. As pointed out by Hartle and Hawking [69], the Majumdar-Papapetrou metrics provide regular many black holes solutions of the Einstein-Maxwell equations; such solutions are static, with all components of the horizon degenerate and it has been for long expected [26] that they exhaust all stationary, electro-vacuum black holes with disconnected horizons. Such a conjecture is far

[6] Until then the possibility of toroidal topologies was not completely excluded.

[7] The proof in [29] contains one mistake, and one gap, both of which are addressed and settled in Section 2.1. To overcome one of these problems we had to assume analyticity; this is done for reasons of different nature than the ones leading to the same requirement in Hawking's rigidity.

from established but when we restrict ourselves to the static and analytic [8] case a complete classification in terms of the Majumdar-Papapetrou and the Reissner-Norsdtröm families, with neither degeneracy or connectedness assumptions is already available by the work of Chruściel and Tod presented in [43]; this result builds upon previous work by Israel, Simon, Ruback, Heusler, Chruściel and Nadirashvili [40, 75, 88, 120, 124]. In fact more is known in the degenerate class, since it was established by Chruściel, Reall and Tod in [42] that appropriately regular, I^+–regular in particular, Israel-Wilson-Perjés Black holes belong to the Majumdar-Papapetrou family. Motivated by the many black hole equilibrium problem in general relativity Weinstein extensively studied the Dirichlet problem for harmonic maps with prescribed singularities and target manifolds with non-positive scalar curvature [131–136]. This work, besides providing important results concerning the foundation of the theory of stationary and axisymmetric black holes, for instance, by clarifying how in the presence of such symmetries the space-time metric of a solution of the source free Einstein-Maxwell equations is completely determined by a specific harmonic map, establishes the existence of multi-black hole, electro-vacuum, stationary and axisymmetric solutions which are regular, expect perhaps at the rotation axis. Such *Weinstein solutions* (see Sections 2.6.6 and 3.5) will play a fundamental role in the work presented here.

In every decade since the founding result of Israel the interest in this problem seems to prevail. This decade was no exception with important developments, most notably in the attempts to remove the analyticity condition [2,3] and, challenged by the celebrated discovery of Emparan-Reall's Black Ring solution [55], in the pursuit of classifications of higher dimensional black hole space-times [32,83]. [9] The work presented here revisits the foundations of the subject and overcomes a considerable amount of technical gaps, to be listed and described in a moment. We hope that such effort may help us in our *fundamental Hilbertian necessity*.

[8] See Section 2.1 and *Corrigendum* to [29].

[9] Some of these results and other recent developments will be discussed in the concluding remarks Section 3.7

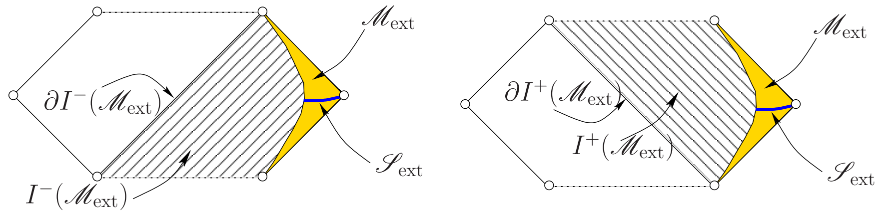

Figure 1.1.1: $\mathscr{S}_{\rm ext}$, $\mathscr{M}_{\rm ext}$, together with the future and the past of $\mathscr{M}_{\rm ext}$. One has $\mathscr{M}_{\rm ext} \subset I^\pm(\mathscr{M}_{\rm ext})$, even though this is not immediately apparent from the figure. The domain of outer communications is the intersection $I^+(\mathscr{M}_{\rm ext}) \cap I^-(\mathscr{M}_{\rm ext})$, compare Figure 1.1.2.

1.1.1 Domains of outer communications, event horizons and I^+–regularity

As usual, in mathematical Relativity, part of the challenge posed by a conjecture is to obtain a precise formulation. In the case of the "no-hair" conjectures this difficulty lies in the notion of *regularity* and as already stressed it is our opinion that such situation has obscured the status of the problem. So, we start exactly by collecting our technical assumptions in a new and explicit definition of *regularity*. To this end we need to establish some basic terminology:

A key notion in the theory of black holes is that of the *domain of outer communications*: A space-time $(\mathscr{M}, \mathfrak{g})$ will be called stationary if there exists on $\mathscr{M}$ a complete Killing vector field K which is *timelike* in the asymptotically flat region $\mathscr{S}_{\rm ext}$.[10] For $t \in \mathbb{R}$ let $\phi_t[K] : \mathscr{M} \to \mathscr{M}$ denote the one-parameter group of diffeomorphisms generated by K; we will write ϕ_t for $\phi_t[K]$ whenever ambiguities are unlikely to occur. The exterior region $\mathscr{M}_{\rm ext}$ and the *domain of outer communications* $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ are then defined as[11] (compare Figure 1.1.1)

$$\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle = I^+(\underbrace{\cup_t \phi_t(\mathscr{S}_{\rm ext})}_{=:\mathscr{M}_{\rm ext}}) \cap I^-(\cup_t \phi_t(\mathscr{S}_{\rm ext}))\ . \tag{1.1.1}$$

[10]In fact, in the literature it is always implicitly assumed that K is *uniformly timelike* in the asymptotic region $\mathscr{S}_{\rm ext}$, by this we mean that $\mathfrak{g}(K,K) < -\epsilon < 0$ for some ϵ and for all r large enough. This uniformity condition excludes the possibility of a timelike vector which asymptotes to a null one. This involves no loss of generality in well-behaved space-times: indeed, uniformity always holds for Killing vectors which are timelike for all large distances if the conditions of the positive energy theorem are met [10, 39].

[11]Recall that $I^-(\Omega)$, respectively $J^-(\Omega)$, is the set covered by past-directed timelike, respectively causal, curves originating from Ω, while $\dot I^-$ denotes the boundary of I^-, etc. The sets I^+, etc., are defined as I^-, etc., after changing time-orientation.

The *black hole region* $\mathscr{B}$ and the *black hole event horizon* $\mathscr{H}^+$ are defined as

$$\mathscr{B} = \mathscr{M} \setminus I^-(\mathscr{M}_{\rm ext}) , \quad \mathscr{H}^+ = \partial \mathscr{B} .$$

The *white hole region* $\mathscr{W}$ and the *white hole event horizon* $\mathscr{H}^-$ are defined as above after changing time orientation:

$$\mathscr{W} = \mathscr{M} \setminus I^+(\mathscr{M}_{\rm ext}) , \quad \mathscr{H}^- = \partial \mathscr{W} , \quad \mathscr{H} = \mathscr{H}^+ \cup \mathscr{H}^- .$$

It follows that the boundaries of $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ are included in the event horizons. We set

$$\mathscr{E}^\pm = \partial \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle \cap I^\pm(\mathscr{M}_{\rm ext}) , \qquad \mathscr{E} = \mathscr{E}^+ \cup \mathscr{E}^- . \tag{1.1.2}$$

There is considerable freedom in choosing the asymptotic region $\mathscr{S}_{\rm ext}$. However, it is not too difficult to show, using Lemma 2.3.6 below, that $I^\pm(\mathscr{M}_{\rm ext})$, and hence $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$, $\mathscr{H}^\pm$ and $\mathscr{E}^\pm$, are independent of the choice of $\mathscr{S}_{\rm ext}$ whenever the associated $\mathscr{M}_{\rm ext}$'s overlap.

We are now able to formulate the main new definition of this thesis:

Definition 1.1.2 *Let $(\mathscr{M}, \mathfrak{g})$ be a space-time containing an asymptotically flat end $\mathscr{S}_{\rm ext}$, and let K be a stationary Killing vector field on $\mathscr{M}$. We will say that $(\mathscr{M}, \mathfrak{g}, K)$ is* I^+–regular *if K is complete, if the domain of outer communications $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is globally hyperbolic, and if $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ contains a spacelike, connected, acausal hypersurface $\mathscr{S} \supset \mathscr{S}_{\rm ext}$, the closure $\overline{\mathscr{S}}$ of which is a topological manifold with boundary, consisting of the union of a compact set and of a finite number of asymptotic ends, such that the boundary $\partial\overline{\mathscr{S}} := \overline{\mathscr{S}} \setminus \mathscr{S}$ is a topological manifold satisfying*

$$\partial\overline{\mathscr{S}} \subset \mathscr{E}^+ := \partial \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle \cap I^+(\mathscr{M}_{\rm ext}) , \tag{1.1.3}$$

with $\partial\overline{\mathscr{S}}$ meeting every generator of $\mathscr{E}^+$ precisely once. (See Figure 1.1.2.)

In the previous definition, the hypothesis of asymptotic flatness (see Section 2.2.1) is made for definiteness, and is not needed for several of the results presented below. Thus, this definition is convenient in a wider context, e.g. if asymptotic flatness is replaced by Kaluza-Klein asymptotics, as in [32, 37, 83].

Some comments about the definition are in order. First we require completeness of the orbits of the stationary Killing vector because we need an action of

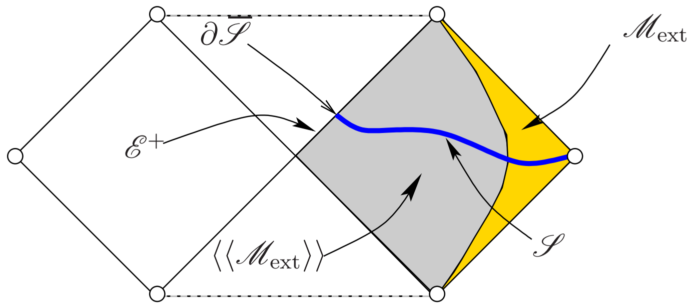

Figure 1.1.2: The hypersurface $\mathscr{S}$ from the definition of I^+–regularity.

$\mathbb{R}$ on $\mathscr{M}$ by isometries. Next, we require global hyperbolicity of the domain of outer communications to guarantee its simple connectedness, to make sure that the Area Theorem holds, and to avoid causality violations as well as certain kinds of naked singularities in $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$. Further, the existence of a well-behaved spacelike hypersurface gives us reasonable control of the geometry of $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$, and is a prerequisite to any elliptic PDEs analysis, as is extensively needed for the problem at hand. The existence of compact cross-sections of the future event horizon prevents singularities on the future part of the boundary of the domain of outer communications, and eventually guarantees the smoothness of that boundary. (Obviously I^+ could have been replaced by I^- throughout the definition, whence $\mathscr{E}^+$ would have become $\mathscr{E}^-$.) The main point of requirement (1.1.3) is to avoid certain zeros of the stationary Killing vector K at the boundary of $\mathscr{S}$, which otherwise create various difficulties; e.g., it is not clear how to guarantee then smoothness of $\mathscr{E}^+$, or the static-or-axisymmetric alternative.[12] Needless to say, all these conditions are satisfied by the Kerr-Newman and the Majumdar-Papapetrou solutions and, in particular, by Minkowski and Reissner-Nordström.

1.1.2 Main result

In this work we establish the following special case of Conjecture 1.1.1:

[12]In fact, this condition is not needed for *static* metrics if, e.g., one assumes at the outset that all horizons are non-degenerate, as we do in Theorem 1.1.3 below, see the discussion in the Corrigendum to [29].

Theorem 1.1.3 *Let* $(\mathscr{M}, \mathfrak{g}, F)$ *be a stationary, asymptotically flat,* I^+*–regular, electro-vacuum, four-dimensional analytic space-time, satisfying* (3.1.5) *and* (3.1.6). *If each component of the event horizon is mean non-degenerate, then* $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ *is isometric to the domain of outer communications of one of the Weinstein solutions of Section 3.5. In particular, if the event horizon is connected and mean non-degenerate, then* $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ *is isometric to the domain of outer communications of a Kerr-Newman space-time.*

It should be emphasized that the hypotheses of analyticity and non-degeneracy are highly unsatisfactory, and one believes that they are not needed for the conclusion. Note that by not allowing the existence of the "technically awkward" [21] degenerate horizons we eliminate extreme Kerr-Newman as well as the Majumdar-Papapetrou solutions from our classification. One also believes, in accordance with the statement of Conjecture 1.1.1, that all solutions with non-connected event horizon are in the Majumdar-Papapetrou family; consequently one expects all other (non-connected) Weinstein solutions, and in particular the ones referred to in the previous result, to be singular. We postpone further discussion of these issues to Section 3.7.

A critical remark comparing our work with the existing literature is in order. First, the event horizon in a smooth or analytic black hole space-time is a priori only a Lipschitz surface, which is way insufficient to prove the usual static-or-axisymmetric alternative. Here we use the results of [36] to show that event horizons in I^+–regular stationary black hole space-times are as differentiable as the differentiability of the metric allows. Next, the famous reduction of the Einstein-Maxwell source free equations to a singular harmonic map problem requires the use of Weyl coordinates. The local existence of such coordinates has been well known for some time now, but global existence has, to our knowledge, either been part of the ansatz, usually implicitly, or based on incorrect or incomplete analysis. The main reasons for this unsatisfactory situation resides in the existing proofs of non-negativity of the *area function* (3.2.10) in $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$, and existence of a global cross-section for the $\mathbb{R}\times U(1)$ action again in $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$; the first of these problems is due both to a potential lack of regularity of the intersection of the rotation axis with the zero-level-set of the area function, and to the fact that the gradient of the area function could vanish on its zero level set *regardless of whether or not the event horizon itself is degenerate*. We prove Theorem 2.5.4 which establishes this result. The difficulty here is to exclude *non-embedded Killing prehorizons* (for terminology, see Definition 2.5.7), and we have not been able to do it without assuming analyticity or axisymmetry, *even for static solutions*. The existence of such global coordinates, and in particular of the global cross-section for the action, also requires, in turn, the Structure Theorem 2.4.5 and the Ergoset Theorem 2.5.24, and relies heavily on the analysis in [31]. Also, no previous work known to us establishes the asymptotic behavior, as needed for the proof of uniqueness, of the relevant harmonic maps. More specifically: the necessity to control such behavior near points where the horizon meets the rotation axis, prior to [33], seems to have been neglected; at infinity, which requires special attention in the electro-vacuum case, part of the necessary estimates sometimes show up as extra conditions, beyond asymptotic flatness; [13] also, an apparent disregard for the singular character, at the axis, of the hyperbolic distance (3.3.14) between the maps, even at large distance, appears to be the norm. A detailed asymptotic analysis is carried out in Sections 2.6.5 and 3.4. Last but not least, we provide a coherent set of conditions under which all pieces of the proof can be combined to obtain the desired classification.

We note that various intermediate results are established under conditions weaker than previously cited, or are generalized to higher dimensions; this is of potential interest for further work on the subject.

1.2 Dain inequality

Gravitational collapse involving *suitable* matter is expected [49,113,130] to *generically* result in the formation of an event horizon whose exterior solution approaches a Kerr-Newman metric asymptotically with time, here we are assuming that the exterior region becomes electro-vacuum. Then, the characteristic inequality

$$m_\infty \geq \sqrt{\frac{|\vec{J}_\infty|^2}{m_\infty^2} + Q_{E,\infty}^2 + Q_{B,\infty}^2}\ , \tag{1.2.1}$$

relating the mass, angular momentum and the Maxwell charges of such black-holes should also be valid asymptotically with time. Now, mass is non-increasing

[13] See, for example, Theorem 2 in [100].

while the Maxwell charges are conserved quantities. If one further assumes axisymmetry we are able to define the angular momentum using a Komar integral [14] which is also conserved. So, letting m, $\vec{J}$, Q_E and Q_B denote the Poncaré and Maxwell charges of axisymmetric initial data for such a collapse we see that

$$m \geq m_\infty \geq \sqrt{\frac{|\vec{J}_\infty|^2}{m_\infty^2} + Q_{E,\infty}^2 + Q_{B,\infty}^2} \geq \sqrt{\frac{|\vec{J}|^2}{m^2} + Q_E^2 + Q_B^2} \,. \tag{1.2.2}$$

Besides their own intrinsic interest, results establishing such inequalities provide evidences in favor of this "current standard picture of gravitational collapse" [49], which is based upon *weak cosmic censorship* and a version of *black hole uniqueness* considerably stronger than the ones available (compare Theorem 1.1.3).

Dain [49, 50], besides providing the previous Penrose-like heuristic argument, proved an upper bound for angular-momentum in terms of the mass for a class of maximal, vacuum, axisymmetric initial data sets. The analysis of [50] has been extended in [38] to include all vacuum axisymmetric initial data, with simply connected orbit space, and manifolds which are asymptotically flat in the standard sense, allowing moreover several asymptotic ends. Recently a generalized Dain's inequality including electric and magnetic charges was obtained in [34]; there the proof of the main result, based on the methods of [38], was only sketched. The aim of this work is to provide a complete proof of this charged Dain inequality while simplifying the methods of [38].

1.3 Overview

In Chapter 2 we built the foundations of the black hole uniqueness theory for stationary I^+–regular space-times and provide a uniqueness theorem for vacuum solutions within such class. This is joint work with my supervisor Piotr Chruściel that was published in [33].

We revisit the static case in Section 2.1 and discuss the necessary adjustments to establish uniqueness of Schwarzschild by invoking [29]. In Section 2.2 we provide the basic definitions, discuss results concerning degenerate horizons and recall the information provided by topological censorship. The non-existence of zeros of linear combinations of the stationary and axial Killing vectors in a

[14] See the first equality in (4.1.12) and the paragraph presiding it, or equivalently take the integral (3.5.2) over the sphere at infinity.

chronological domain of outer communications is established in Section 2.3. In Section 2.4 we construct cross-sections of the horizon as differentiable as the metric allows, we prove the Structure Theorem, which provides a decomposition of $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ *natural* with respect to a $\mathbb{R} \times \mathbb{T}^{s-1}$ action by isometries, and by relying on this properties we obtain the aforementioned results concerning the regularity of the event horizon; we are then able to establish a Rigidity Theorem for I^+–regular space-times. The construction of global Weyl coordinates follows. The first step corresponds to the proof of non-negativity of the *area function* in the domain of outer communications. This is done in Section 2.5, where we also prove the Ergoset Theorem. The desired global representation then follows by an analysis of the $U(1)$ action on the Riemannian three-manifold provided by the Structure Theorem and by constructing isothermal coordinates for the orbit space, using the square root of the area function. This is carried out in Sections 2.6.1–2.6.3. The boundary conditions of the relevant harmonic maps are studied in Section 2.6.5 with most of the attention reserved to the behavior near points where the axis meets the horizon. (A more detailed asymptotic analysis is presented in Section 3.4). In Sections 2.6.6 and 2.6.7 the construction of *vacuum Weinstein solutions* is carried out and existence and uniqueness results for the relevant Dirichlet problem provided. In the final section we prove the main theorem of this chapter.

In Chapter 3 we generalize the results of the previous chapter to the electro-vacuum setting. Section 3.2 is devoted to the construction of global Weyl coordinates, with most of the hard work already carried out in the previous chapter. In this section we also provide a complete proof of the fundamental integrability conditions established by Proposition 3.2.1. The distance function for the 'upper half-space' model of $\mathbb{H}^2_{\mathbb{C}}$, in terms of which the criteria for the existence and uniqueness of the relevant harmonic maps is presented, is computed in Section 3.3.1. As already mentioned, a detailed asymptotic analysis of the boundary conditions is available in Section 3.4. Sections 3.5 and 3.6 are devoted to the construction of the electro-vacuum Weinstein solutions and to the proof of the main result of this thesis, Theorem 1.1.3. We finish with some remarks concerning the state of the art of black hole uniqueness: we discuss some new results on the subject and highlight what we consider to be the main weaknesses of the existing ones.

For the final chapter of this thesis a change of tone. We abandon the previous classification problem and prove our charged Dain inequality. This corresponds to an extended version of a joint paper with Piotr Chruściel [34], where a proof of the desired result was only sketched. Here we present a complete proof with some simplification of the argument suggested by [34] and provide a few suggestions for future research on the subject.

Chapter 2

On uniqueness of stationary vacuum black holes

In this chapter we establish the foundations of the black uniqueness theory for stationary and I^+–regular space-times and, by restricting ourselves to vacuum, prove the following special case of Conjecture 1.1.1:[1]

THEOREM 2.0.1 *Let* $(\mathscr{M}, \mathfrak{g})$ *be a stationary, asymptotically flat,* I^+*–regular, vacuum, four-dimensional analytic space-time. If each component of the event horizon is mean non-degenerate, then* $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ *is isometric to the domain of outer communications of one of the Weinstein solutions of Section 2.6.7. In particular, if* $\mathscr{E}^+$ *is connected and mean non-degenerate, then* $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ *is isometric to the domain of outer communications of a Kerr space-time.*

2.1 Static case

Assuming *staticity*, i.e., stationarity and hypersurface-orthogonality of the stationary Killing vector, a more satisfactory result is available in space dimensions less than or equal to seven, and in higher dimensions on manifolds on which the Riemannian rigid positive energy theorem holds: non-connected configurations are excluded, without any *a priori* restrictions on the gradient $\nabla(\mathfrak{g}(K, K))$ at event horizons.

[1] We refer to Section 2.2.2.1 for the definition of *mean non-degeneracy*. We also note that the usual definition of degeneracy (see Section 2.2.2) is insufficient since an equivalence between the notions of event horizon and Killing (pre)horizon (see Definition 2.5.7) is far from obvious (see Corollaries 2.5.17 and 2.5.22).

More precisely, we shall say that a manifold $\widehat{\mathscr{S}}$ is of *positive energy type* if there are no asymptotically flat complete Riemannian metrics on $\widehat{\mathscr{S}}$ with non-negative scalar curvature and vanishing mass except perhaps for a flat one. This property has been proved so far for all n–dimensional manifolds $\widehat{\mathscr{S}}$ obtained by removing a finite number of points from a compact manifold of dimension $3 \le n \le 7$ [122], or under the hypothesis that $\widehat{\mathscr{S}}$ is a spin manifold of any dimension $n \ge 3$, and is expected to be true in general [23, 98].

We have the following result, which finds its roots in the work of Israel [87], with further simplifications by Robinson [118], and with a significant strengthening by Bunting and Masood-ul-Alam [17]:

THEOREM 2.1.1 *Let $(\mathscr{M}, \mathfrak{g})$ be a stationary, vacuum, $(n+1)$-dimensional space-time, $n \ge 3$, containing a spacelike, connected, acausal hypersurface $\mathscr{S}$, such that $\overline{\mathscr{S}}$ is a topological manifold with boundary, consisting of the union of a compact set and of a finite number of asymptotically flat ends. Suppose that there exists on $\mathscr{M}$ a complete stationary Killing vector K, that $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ is globally hyperbolic, and that $\partial\overline{\mathscr{S}} \subset \mathscr{M} \setminus \langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$. Suppose moreover that $(\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle, \mathfrak{g})$ is analytic and K is hypersurface-orthogonal. Let $\widehat{\mathscr{S}}$ denote the manifold obtained by doubling $\mathscr{S}$ across the non-degenerate components of its boundary and compactifying, in the doubled manifold, all asymptotically flat regions but one to a point. If $\widehat{\mathscr{S}}$ is of positive energy type, then $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ is isometric to the domain of outer communications of a Schwarzschild space-time.*

REMARK 2.1.2 As a corollary of Theorem 2.1.1 one obtains non-existence of black holes as above with some components of the horizon degenerate. In space-time dimension four an elementary proof of this fact has been given in [42], but the simple argument there does not seem to generalize to higher dimensions in any obvious way.

REMARK 2.1.3 Analyticity is only needed to exclude non-embedded degenerate prehorizons (see Definition 2.5.7) within $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$. In space-time dimension four it can be replaced by the condition of axisymmetry and I^+–regularity, compare Theorem 2.5.2.

PROOF: We want to invoke [29], where $n = 3$ has been assumed; the argument given there generalizes immediately to those higher dimensional manifolds on which the positive energy theorem holds. However, the proof in [29] contains one mistake, and one gap, both of which need to be addressed.

First, in the case of degenerate horizons $\mathscr{H}$, the analysis of [29] assumes that the static Killing vector has no zeros on $\mathscr{H}$; this is used in the key Proposition 3.2 there, which could be wrong without this assumption. The non-vanishing of the static Killing vector is justified in [29] by an incorrectly quoted version of Boyer's theorem [14], see [29, Theorem 3.1]. Under a supplementary assumption of I^+–regularity, the zeros of a Killing vector which could arise in the closure of a degenerate Killing horizon can be excluded using Corollary 2.3.3. In general, the problem is dealt with in the addendum to the arXiv versions vN, $N \geq 3$, of [29] in space-dimension three, and in [32] in higher dimensions.

Next, neither the original proof, nor that given in [29], of the Vishveshwara-Carter Lemma, takes properly into account the possibility that the hypersurface $\mathscr{N}$ of [29, Lemma 4.1] could fail to be embedded.[2] This problem is taken care of by Theorem 2.5.4 below with $s = 1$, which shows that $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ cannot intersect the set where $W := -\mathfrak{g}(K, K)$ vanishes. This implies that K is timelike on $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle \supset \mathscr{S}$, and null on $\partial\overline{\mathscr{S}}$. The remaining details are as in [29]. □

2.2 Preliminaries

2.2.1 Asymptotically flat stationary metrics

A space-time $(\mathscr{M}, \mathfrak{g})$ will be said to possess an *asymptotically flat end* if $\mathscr{M}$ contains a spacelike hypersurface $\mathscr{S}_{\rm ext}$ diffeomorphic to $\mathbb{R}^n \setminus B(R)$, where $B(R)$ is an open coordinate ball of radius R, with the following properties: there exists a constant $\alpha > 0$ such that, in local coordinates on $\mathscr{S}_{\rm ext}$ obtained from $\mathbb{R}^n \setminus B(R)$, the metric γ induced by $\mathfrak{g}$ on $\mathscr{S}_{\rm ext}$, and the extrinsic curvature tensor K_{ij} of $\mathscr{S}_{\rm ext}$, satisfy the fall-off conditions

$$\gamma_{ij} - \delta_{ij} = O_k(r^{-\alpha}) \, , \qquad K_{ij} = O_{k-1}(r^{-1-\alpha}) \, , \tag{2.2.1}$$

for some $k > 1$, where we write $f = O_k(r^\alpha)$ if f satisfies

$$\partial_{k_1} \ldots \partial_{k_\ell} f = O(r^{\alpha-\ell}) \, , \quad 0 \leq \ell \leq k \, . \tag{2.2.2}$$

[2] This problem affects points 4c,d,e and f of [29, Theorem 1.3], which require the supplementary hypothesis of existence of an embedded closed hypersurface within $\mathscr{N}$; the remaining claims of [29, Theorem 1.3] are justified by the arguments described here.

A Killing vector K is said to be *complete* if for every $p \in \mathscr{M}$ the orbit $\phi_t[K](p)$ of K is defined for all $t \in \mathbb{R}$, i.e., if (the flow of) K generates an action of $\mathbb{R}$ by isometries; in an asymptotically flat context, K is called *stationary* if it is timelike at large distances.

For simplicity we assume that the space-time is vacuum, though similar results hold in general under appropriate conditions on matter fields, see [9, 39] and references therein. Along any spacelike hypersurface $\mathscr{S}$, a Killing vector field K of $(\mathscr{M}, \mathfrak{g})$ can be decomposed as

$$K = Nn + Y \ ,$$

where Y is tangent to $\mathscr{S}$, and n is the unit future-directed normal to $\mathscr{S}_{\text{ext}}$. The vacuum field equations, together with the Killing equations imply the following set of equations on $\mathscr{S}$, where $R_{ij}(\gamma)$ is the Ricci tensor of γ:

$$D_iY_j + D_jY_i = 2NK_{ij} \ , \tag{2.2.3}$$

$$R_{ij}(\gamma) + K^k{}_kK_{ij} - 2K_{ik}K^k{}_j - N^{-1}(\mathscr{L}_YK_{ij} + D_iD_jN) = 0 \ . \tag{2.2.4}$$

Under the boundary conditions (2.2.1) with $k \geq 2$, an analysis of (2.2.3)-(2.2.4) provides detailed information about the asymptotic behavior of (N, Y). In particular, one can prove that if the asymptotic region $\mathscr{S}_{\text{ext}}$ is contained in a hypersurface $\mathscr{S}$ satisfying the requirements of the positive energy theorem, and if K is timelike along $\mathscr{S}_{\text{ext}}$, then $(N, Y^i) \to_{r\to\infty} (A^0, A^i)$, where the A^μ's are constants satisfying $(A^0)^2 > \sum_i (A^i)^2$. One can then choose adapted coordinates so that the metric can, locally, be written as

$$\mathfrak{g} = -V^2(dt + \underbrace{\theta_idx^i}_{=\theta})^2 + \underbrace{\gamma_{ij}dx^idx^j}_{=\gamma} \ , \tag{2.2.5}$$

with

$$\partial_tV = \partial_t\theta = \partial_t\gamma = 0 \tag{2.2.6}$$

$$\gamma_{ij} - \delta_{ij} = O_k(r^{-\alpha}) \ , \quad \theta_i = O_k(r^{-\alpha}) \ , \quad V - 1 = O_k(r^{-\alpha}) \ , \tag{2.2.7}$$

for any $k \in \mathbb{N}$. As discussed in more detail in [12], in γ-harmonic coordinates, and in e.g. a maximal time-slicing, the vacuum equations for $\mathfrak{g}$ form a quasi-linear elliptic system with diagonal principal part, with principal symbol identical to that of the scalar Laplace operator. Methods known in principle show that, in this "gauge", all metric functions have a full asymptotic expansion[3] in terms of powers of $\ln r$ and inverse powers of r. In the new coordinates we can in fact take

$$\alpha = n - 2 \; . \tag{2.2.8}$$

By inspection of the equations one can further infer that the leading order corrections in the metric can be written in a Schwarzschild form, which in "isotropic" coordinates reads

$$\mathfrak{g}_m \quad = \quad -\left(\frac{1-\frac{m}{2|x|^{n-2}}}{1+\frac{m}{2|x|^{n-2}}}\right)^2 dt^2 + \left(1+\frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \left(\sum_{i=1}^n dx_i^2\right) \; ,$$

where $m \in \mathbb{R}$.

2.2.2 Killing horizons, bifurcate horizons

A null hypersurface, invariant under the flow of a Killing vector K, which coincides with a connected component of the set

$$\mathscr{N}(K) := \{\mathfrak{g}(K,K) = 0 \; , \; K \neq 0\} \; ,$$

is called a *Killing horizon* associated to K.

A set will be called a *bifurcate Killing horizon* if it is the union of four Killing horizons, the intersection of the closure of which forms a smooth submanifold S of co-dimension two, called the *bifurcation surface*. The four Killing horizons consist then of the four null hypersurfaces obtained by shooting null geodesics in the four distinct null directions normal to S. For example, the Killing vector $x\partial_t + t\partial_x$ in Minkowski space-time has a bifurcate Killing horizon, with the bifurcation surface $\{t = x = 0\}$.

The *surface gravity* κ of a Killing horizon $\mathscr{N}$ is defined by the formula

$$d\left(\mathfrak{g}(K,K)\right)|_{\mathscr{N}} = -2\kappa K^\flat \; , \tag{2.2.9}$$

where $K^\flat = \mathfrak{g}_{\mu\nu} K^\nu dx^\mu$. A fundamental property is that the surface gravity κ is constant over each horizon in vacuum, or in electro-vacuum, see e.g. [74, Theorem 7.1]. The proof given in [129] generalizes to all space-time dimensions $n+1 \geq 4$; the result also follows in all dimensions from the analysis in [80] when

[3] One can use the results in, e.g., [24] together with a simple iterative argument to obtain the expansion. This analysis holds in any dimension.

the horizon has compact spacelike sections. (The constancy of κ can also be established without assuming any field equations in some cases, see [90, 114].) A Killing horizon is called *degenerate* if κ vanishes, and *non-degenerate* otherwise.

2.2.2.1 Near-horizon geometry

Following [103], near a smooth event horizon [4] one can introduce *Gaussian null coordinates*, in which the metric takes the form

$$\mathfrak{g} = r\varphi dv^2 + 2dvdr + 2rh_a dx^a dv + h_{ab} dx^a dx^b \ . \tag{2.2.10}$$

(These coordinates can be introduced for any null hypersurface, not necessarily an event horizon, in any number of dimensions). The horizon is given by the equation $\{r = 0\}$, replacing r by $-r$ if necessary we can without loss of generality assume that $r > 0$ in the domain of outer communications. Assuming that the horizon admits a smooth compact cross-section S, the *average surface gravity* $\langle\kappa\rangle_S$ is defined as

$$\langle\kappa\rangle_S = -\frac{1}{|S|}\int_S \varphi d\mu_h \ , \tag{2.2.11}$$

where $d\mu_h$ is the measure induced by the metric h on S, and $|S|$ is the volume of S. We emphasize that this is defined regardless of whether or not some Killing vector K is tangent to the horizon generators; but if $K = \partial_v$ is, and if the surface gravity κ of K is constant on S, then $\langle\kappa\rangle_S$ equals κ.

On a degenerate Killing horizon the surface gravity vanishes by definition, so that the function φ in (2.2.10) can itself be written as rA, for some smooth function A. The vacuum Einstein equations imply (see [103, eq. (2.9)] in dimension four and [95, eq. (5.9)] in higher dimensions)

$$\mathring{R}_{ab} = \frac{1}{2}\mathring{h}_a\mathring{h}_b - \mathring{D}_{(a}\mathring{h}_{b)} \ , \tag{2.2.12}$$

where $\mathring{R}_{ab}$ is the Ricci tensor of $\mathring{h}_{ab} := h_{ab}|_{r=0}$, and $\mathring{D}$ is the covariant derivative thereof, while $\mathring{h}_a := h_a|_{r=0}$. The Einstein equations also determine $\mathring{A} := A|_{r=0}$ uniquely in terms of $\mathring{h}_a$ and $\mathring{h}_{ab}$:

$$\mathring{A} = \frac{1}{2}\mathring{h}^{ab}\left(\mathring{h}_a\mathring{h}_b - \mathring{D}_a\mathring{h}_b\right) \tag{2.2.13}$$

[4] In Section 2.4.3 it will be established that event horizons in smooth I^+–regular space-times are smooth.

(this equation follows again e.g. from [103, eq. (2.9)] in dimension four, and can be checked by a calculation in all higher dimensions). We have the following:[5]

THEOREM 2.2.1 ([41]) *Let the space-time dimension be* $n+1$, $n \geq 3$, *suppose that a degenerate Killing horizon* $\mathscr{N}$ *has a compact cross-section, and that* $\mathring{h}_a = \partial_a \lambda$ *for some function* λ *(which is necessarily the case in vacuum static space-times). Then* (2.2.12) *implies* $\mathring{h}_a \equiv 0$, *so that* $\mathring{h}_{ab}$ *is Ricci-flat.*

THEOREM 2.2.2 ([67, 95]) *In space-time dimension four and in vacuum, suppose that a degenerate Killing horizon* $\mathscr{N}$ *has a spherical cross-section, and that* $(\mathscr{M}, \mathfrak{g})$ *admits a second Killing vector field with periodic orbits. For every connected component* $\mathscr{N}_0$ *of* $\mathscr{N}$ *there exists an embedding of* $\mathscr{N}_0$ *into a Kerr space-time which preserves* $\mathring{h}_a$, $\mathring{h}_{ab}$ *and* $\mathring{A}$.

It would be of interest to understand fully (2.2.12), in all dimensions, without restrictive conditions.

In the four-dimensional static case, Theorem 2.2.1 enforces toroidal topology of cross-sections of $\mathscr{N}$, with a flat $\mathring{h}_{ab}$. On the other hand, in the four-dimensional axisymmetric case, Theorem 2.2.2 guarantees that the geometry tends to a Kerr one, at a rate made clear in the statement of the theorem, when the horizon is approached. (Somewhat more detailed information can be found in [67].) So, in the degenerate case, the vacuum equations impose strong restrictions on the near-horizon geometry.

It seems that this is not the case any more for non-degenerate horizons, at least in the analytic setting. Indeed, we claim that for any triple $(N, \mathring{h}_a, \mathring{h}_{ab})$, where N is a two-dimensional analytic manifold (compact or not), $\mathring{h}_a$ is an analytic one-form on N, and $\mathring{h}_{ab}$ is an analytic Riemannian metric on N, there exists a vacuum space-time $(\mathscr{M}, \mathfrak{g})$ with a bifurcate (and thus non-degenerate) Killing horizon, so that the metric $\mathfrak{g}$ takes the form (2.2.10) near each Killing horizon branching out of the bifurcation surface $S \approx N$, with $\mathring{h}_{ab} = h_{ab}|_{r=0}$ and $\mathring{h}_a = h_a|_{r=0}$; in fact $\mathring{h}_{ab}$ is the metric induced by $\mathfrak{g}$ on S. When N is the two-dimensional torus $\mathbb{T}^2$ this can be inferred from [102] as follows: using [102, Theorem (2)] with $(\phi, \beta_a, g_{ab})|_{t=0} = (0, 2\mathring{h}_a, \mathring{h}_{ab})$ one obtains a vacuum space-time $(\mathscr{M}' = S^1 \times \mathbb{T}^2 \times (-\epsilon, \epsilon), \mathfrak{g}')$ with a compact Cauchy horizon $S^1 \times \mathbb{T}^2$ and Killing vector K tangent to the S^1 factor

[5] Some partial results with a non-zero cosmological constant have also been proved in [41].

of $\mathscr{M}'$. One can then pass to a covering space where S^1 is replaced by $\mathbb{R}$, and use a construction of Rácz and Wald [114, Theorem 4.2] to obtain the desired $\mathscr{M}$ containing the bifurcate horizon. This argument generalizes to any analytic $(N, \mathring{h}_a, \mathring{h}_{ab})$ without difficulties.

2.2.3 Globally hyperbolic asymptotically flat domains of outer communications are simply connected

Simple connectedness of the domain of outer communication is an essential ingredient in several steps of the uniqueness argument below. It was first noted in [45] that this stringent topological restriction is a consequence of the "topological censorship theorem" of Friedman, Schleich and Witt [57] for asymptotically flat, stationary and globally hyperbolic domains of outer communications satisfying the null energy condition:

$$R_{\mu\nu} Y^\mu Y^\nu \geq 0 \quad \text{for null} \quad Y^\mu \, . \tag{2.2.14}$$

In fact, stationarity is not needed. To make things precise, consider a space-time $(\mathscr{M}, \mathfrak{g})$ with several asymptotically flat regions $\mathscr{M}^i_{\rm ext}$, $i = 1, \ldots, N$, each generating its own domain of outer communications. It turns out [61] (compare [62]) that the null energy condition prohibits causal interactions between distinct such ends:

Theorem 2.2.3 *If $(\mathscr{M}, \mathfrak{g})$ is a globally hyperbolic and asymptotically flat space-time satisfying the null energy condition* (2.2.14), *then*

$$\langle\langle \mathscr{M}^i_{\rm ext} \rangle\rangle \cap J^\pm(\langle\langle \mathscr{M}^j_{\rm ext} \rangle\rangle) = \emptyset \textit{ for } i \neq j \, . \tag{2.2.15}$$

A clever covering/connectedness argument[6] [61] shows then:[7]

Corollary 2.2.4 *A globally hyperbolic and asymptotically flat domain of outer communications satisfying the null energy condition is simply connected.*

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[6]Under more general asymptotic conditions it was proved in [64] that inclusion induces a surjective homeomorphism between the fundamental groups of the exterior region and the domain of outer communications. In particular, $\pi_1(\mathscr{M}_{\rm ext}) = 0 \Rightarrow \pi_1(\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle) = 0$.

[7]Strictly speaking, our applications below of [61] require checking that the conditions of asymptotic flatness in [61] coincide with ours; this, however, can be avoided by invoking directly [45].

In space-time dimension four this, together with standard topological results [72, Lemma 4.9], leads to a spherical topology of horizons (see [45] together with Proposition 2.4.4 below):

COROLLARY 2.2.5 *In I^+–regular, stationary, asymptotically flat space-times satisfying the null energy condition, cross-sections of $\mathscr{E}^+$ have spherical topology.*

2.3 Zeros of Killing vectors

Let $\mathscr{S}$ be a spacelike hypersurface in $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$; in the proof of Theorem 2.0.1 it will be essential to have no zeros of the stationary Killing vector K on $\overline{\mathscr{S}}$. Furthermore, in the axisymmetric scenario, we need to exclude zeros of Killing vectors of the form $K_{(0)} + \alpha K_{(1)}$ on $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$, where $K_{(0)} = K$ and $K_{(1)}$ is a generator of the axial symmetry. The aim of this section is to present conditions which guarantee that; for future reference, this is done in arbitrary space-time dimension.

We start with the following:

LEMMA 2.3.1 *Let $\mathscr{S}_{\rm ext} \subset \mathscr{S} \subset \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$, and suppose that $\mathscr{S}$ is achronal in $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$. Then for any $p \in \mathscr{M}_{\rm ext}$ there exists $t_0 \in \mathbb{R}$ such that*

$$\overline{\mathscr{S}} \cap I^+(\phi_{t_0}(p)) = \emptyset \ .$$

PROOF: Let $p \in \mathscr{M}_{\rm ext}$. There exists t_0 such that $r := \phi_{t_0}(p) \in \mathscr{S}_{\rm ext}$. Suppose that $\overline{\mathscr{S}} \cap I^+(\phi_{t_0}(p)) \neq \emptyset$. Then there exists a timelike future directed curve γ from r to $q \in \overline{\mathscr{S}}$. Let $q_i \in \mathscr{S}$ converge to q; then $q_i \in I^+(r)$ for i large enough, which contradicts achronality of $\mathscr{S}$ within $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$.

LEMMA 2.3.2 *Let $S \subset I^+(\mathscr{M}_{\rm ext})$ be compact.*

1. *There exists $p \in \mathscr{M}_{\rm ext}$ such that S is contained in $I^+(p)$.*
2. *If $S \subset \partial\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle \cap I^+(\mathscr{M}_{\rm ext})$ and if $(\overline{\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle}, \mathfrak{g})$ is strongly causal at S,[8] then for any $p \in \mathscr{M}_{\rm ext}$ there exists $t_0 \in \mathbb{R}$ such that $S \cap I^+(\phi_{t_0}(p)) = \emptyset$.*

[8] In a sense made clear in the last sentence of the proof below.

Proof: 1: Let $q \in S$; there exists $p_q \in \mathscr{M}_{\rm ext}$ such that $q \in I^+(p_q)$, and since $I^+(p_q)$ is open there exists an open neighborhood $\mathscr{O}_q \subset S$ of q such that $\mathscr{O}_q \subset I^+(p_q)$. By compactness there exists a finite collection $\mathscr{O}_{q_i}$, $i = 1, \ldots, I$, covering S, thus $S \subset \cup_i I^+(p_{q_i})$. Letting $p \in \mathscr{M}_{\rm ext}$ be any point such that $p_{q_i} \in I^+(p)$ for $i = 1, \ldots, I$, the result follows.

2: Suppose not. Then $\phi_i(p) \in I^-(S)$ for all $i \in \mathbb{N}$, hence there exists $q_i \in S$ such that $q_i \in I^+(\phi_i(p))$. By compactness there exits $q \in S$ such that $q_i \to q$. Let $\mathscr{O}$ be an arbitrary neighborhood of q; since $q \in \mathscr{E}^+$, there exists $r \in \mathscr{O} \cap \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$, $p_+ \in \mathscr{M}_{\rm ext}$, and a future directed causal curve γ from r to p_+. For all i large, this can be continued by a future directed causal curve from p_+ to $\phi_i(p)$, which can then be continued by a future directed causal curve to q_i. But $q_i \in \mathscr{O}$ for i large enough. This implies that every small neighborhood of q meets a future directed causal curve entirely contained within $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ which leaves the neighborhood and returns, contradicting strong causality of $\overline{\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle}$. □

It follows from Lemma 2.3.1, together with point 1 of Lemma 2.3.2 with $S = \{r\}$, that

Corollary 2.3.3 *If $r \in \overline{\mathscr{S}} \cap I^+(\mathscr{M}_{\rm ext})$, then the stationary Killing vector K does not vanish at r. In particular if $(\mathscr{M}, \mathfrak{g})$ is I^+–regular, then K has no zeros on $\overline{\mathscr{S}}$.* □

To continue, we assume the existence of a commutative group of isometries $\mathbb{R} \times \mathbb{T}^{s-1}$, $s \geq 1$. We denote by $K_{(0)}$ the Killing vector tangent to the orbits of the $\mathbb{R}$ factor, and we assume that $K_{(0)}$ is timelike in $\mathscr{M}_{\rm ext}$. We denote by $K_{(i)}$, $i = 1, \ldots, s-1$ the Killing vector tangent to the orbits of the i'th S^1 factor of $\mathbb{T}^{s-1}$. We assume that each $K_{(i)}$ is spacelike in $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ wherever non-vanishing, which will necessarily be the case if $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is chronological[9]. Note that asymptotic flatness imposes $s-1 \leq n/2$, though most of the results of this section remain true without this hypothesis, when properly formulated.

We say that a Killing orbit $\gamma : \mathbb{R} \to \mathscr{M}$ is future-oriented if there exist numbers $\tau_1 > \tau_0$ such that $\gamma(\tau_1) \in I^+(\gamma(\tau_0))$. Clearly all orbits of a Killing vector K are future-oriented in the region where K is timelike. A less-trivial example is given by orbits of the Killing vector $\partial_t + \Omega\partial_\varphi$ in Minkowski space-time. Similarly,

[9]No closed timelike curves allowed.

in stationary axisymmetric space-times, those orbits of this last Killing vector on which ∂_t is timelike are future-oriented (let $\tau_0 = 0$ and $\tau_1 = 2\pi/\Omega$).

We have:

LEMMA 2.3.4 *Orbits through $\mathscr{M}_{\rm ext}$ of Killing vector fields K of the form $K_{(0)} + \sum \alpha_{(i)} K_{(i)}$ are future-oriented.*

PROOF: Recall that for any Killing vector field Z we denote by $\phi_t[Z]$ the flow of Z. Let

$$Y := \sum \alpha_{(i)} K_{(i)} .$$

Suppose, first, that there exists $\tau > 0$ such that $\phi_\tau[Y]$ is the identity. Since $K_{(0)}$ and Y commute we have

$$\phi_\tau[K] = \phi_\tau[K_{(0)} + Y] = \phi_\tau[K_{(0)}] \circ \phi_\tau[Y] = \phi_\tau[K_{(0)}] .$$

Setting $\tau_0 = 0$ and $\tau_1 = \tau$, the result follows.

Otherwise, there exists a sequence $t_i \to \infty$ such that $\phi_{t_i}[Y](p)$ converges to p. Since $I^+(p)$ is open there exists a neighborhood $\mathscr{U}^+ \subset I^+(p)$ of $\phi_1[K_{(0)}](p)$. Let $\mathscr{V}^+ = \phi_{-1}[K_{(0)}](\mathscr{U}^+)$, then every point in $\mathscr{U}^+$ lies on a future directed timelike path starting in $\mathscr{V}^+$, namely an integral curve of $K_{(0)}$. There exists $i_0 \geq 1$ so that $t_i \geq 1$ and $\phi_{t_i}[Y](p) \in \mathscr{V}^+$ for $i \geq i_0$. We then have

$$\phi_{t_i}[K](p) = \phi_{t_i}[K_{(0)} + Y](p) = \phi_{t_i - 1}[K_{(0)}]\big(\underbrace{\phi_1[K_{(0)}](\underbrace{\phi_{t_i}[Y](p)}_{\in \mathscr{V}^+})}_{\in \mathscr{U}^+ \subset I^+(p)} \big) \in I^+(p) .$$

The numbers $\tau_0 = 0$ and $\tau_1 = t_{i_0}$ satisfy then the requirements of the definition.

For future reference we note the following:

LEMMA 2.3.5 *The orbits through $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ of any Killing vector K of the form $K_{(0)} + \sum \alpha_{(i)} K_{(i)}$ are future-oriented.*

PROOF: Let $p \in \langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$, thus there exist points $p_\pm \in \mathscr{M}_{\rm ext}$ such that $p_\pm \in I^\pm(p)$, with associated future directed timelike curves $\gamma_\pm$. It follows from Lemma 2.3.4 together with asymptotic flatness that there exists τ such that $\phi_\tau[K](p_-) \in I^+(p_+)$ for some τ, as well as an associated future directed curve γ from p_+ to $\phi_\tau[K](p_-)$. Then the curve $\gamma_+ \cdot \gamma \cdot \phi_\tau[K](\gamma_-)$, where $\cdot$ denotes concatenation of curves, is a timelike curve from p to $\phi_\tau(p)$. □

The following result, essentially due to [44], turns out to be very useful:

Lemma 2.3.6 *Let $\alpha_i \in \mathbb{R}$. For any set C invariant under the flow of $K = K_{(0)} + \sum_i \alpha_i K_i$, the set $I^{\pm}(C) \cap \mathscr{M}_{\rm ext}$ coincides with $\mathscr{M}_{\rm ext}$, if non-empty.*

Proof: The null achronal boundaries $\dot{I}^{\mp}(C) \cap \mathscr{M}_{\rm ext}$ are invariant under the flow of K. This is compatible with Lemma 2.3.4 if and only if $\dot{I}^{\mp}(C) \cap \mathscr{M}_{\rm ext} = \emptyset$. If C intersects $I^+(\mathscr{M}_{\rm ext})$ then $I^-(C) \cap \mathscr{M}_{\rm ext}$ is non-empty, hence $I^-(C) \supset \mathscr{M}_{\rm ext}$ since $\mathscr{M}_{\rm ext}$ is connected. A similar argument applies if C intersects $I^-(\mathscr{M}_{\rm ext})$.

We have the following strengthening of Lemma 2.3.2:

Lemma 2.3.7 *Let $\alpha_i \in \mathbb{R}$. If $(\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle, \mathfrak{g})$ is chronological, then there exists no nonempty set N which is invariant under the flow of $K_{(0)} + \sum_i \alpha_i K_i$ and which is included in a compact set $C \subset \langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$.*

Proof: Assume that $N \subset \langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ is not empty. From Lemma 2.3.6 we obtain $\mathscr{M}_{\rm ext} \subset I^+(N)$, hence $I^+(\mathscr{M}_{\rm ext}) \subset I^+(N)$. Arguing similarly with I^- we infer that

$$\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle \subset I^+(N) \cap I^-(N) \ .$$

Hence every point q in $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ is in $I^+(p)$ for some $p \in N$. We conclude that $\{I^+(p) \cap C\}_{p \in N}$ is an open cover of C. Assuming compactness, we may then choose a finite subcover $\{I^+(p_i) \cap C\}_{i=1}^I$. This implies that each p_i must be in the future of at least one p_j, and since there is a finite number of them one eventually gets a closed timelike curve, which is not possible in chronological space-times.

Since each zero of a Killing vector provides a compact invariant set, from Lemma 2.3.7 we conclude

Corollary 2.3.8 *Let $\alpha_i \in \mathbb{R}$. If $(\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle, \mathfrak{g})$ is chronological, then Killing vectors of the form $K_{(0)} + \sum_i \alpha_i K_i$ have no zeros in $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$.*

2.4 Horizons and domains of outer communications in regular space-times

In this section we analyze the structure of a class of horizons, and of domains of outer communications.

2.4.1 Sections of horizons

The aim of this section is to establish the existence of cross-sections of the event horizon with good properties.

By standard causality theory the future event horizon $\mathscr{H}^+ = \dot I^-(\mathscr{M}_{\rm ext})$ (recall that $\dot I^\pm$ denotes the boundary of $I^\pm$) is the union of Lipschitz topological hypersurfaces. Furthermore, through every point $p \in \mathscr{H}^+$ there is a future inextendible null geodesic entirely contained in $\mathscr{H}^+$ (though it may leave $\mathscr{H}^+$ when followed to the past of p). Such geodesics are called *generators*. A topological submanifold S of $\mathscr{H}^+$ will be called a *local section*, or simply *section*, if S meets the generators of $\mathscr{H}^+$ transversally; it will be called a *cross-section* if it meets all the generators precisely once. Similar definitions apply to any null achronal hypersurfaces, such as $\mathscr{H}^-$ or $\mathscr{E}^\pm$.

We start with the proof of existence of sections of the event horizon which are moved to their future by the isometry group. The existence of such sections has been claimed in Lemma 5.2 of [27]; here we give the proof of a somewhat more general result:

Proposition 2.4.1 *Let $\mathscr{H}_0 \subset \mathscr{H} := \mathscr{H}^+ \cup \mathscr{H}^- \equiv \dot I^-(\mathscr{M}_{\rm ext}) \cup \dot I^+(\mathscr{M}_{\rm ext})$ be a connected component of the event horizon $\mathscr{H}$ in a space-time $(\mathscr{M}, \mathfrak{g})$ with stationary Killing vector $K_{(0)}$, and suppose that there exists a compact cross-section S of $\mathscr{H}_0$ satisfying*

$$S \subset \mathscr{E}_0 := \mathscr{H}_0 \cap I^+(\mathscr{M}_{\rm ext}) \, .$$

Assume that

1. *either*
$$\overline{\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle} \cap I^+(\mathscr{M}_{\rm ext}) \text{ is strongly causal,}$$
2. *or there exists in $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ a spacelike hypersurface $\mathscr{S} \supset \mathscr{S}_{\rm ext}$, achronal in $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$, so that S above coincides with the boundary of $\overline{\mathscr{S}}$:*
$$S = \partial \overline{\mathscr{S}} \subset \mathscr{E}^+ \, .$$

Then there exists a compact Lipschitz hypersurface S_0 of $\mathscr{E}_0$ which is transverse to both the stationary Killing vector field $K_{(0)}$ and to the generators of $\mathscr{E}_0$, and which meets every generator of $\mathscr{E}_0$ precisely once; in particular

$$\mathscr{E}_0 = \cup_t \phi_t(S_0) \, .$$

PROOF: Changing time orientation if necessary, and replacing $\mathscr{M}$ by $I^+(\mathscr{M}_{\mathrm{ext}}) \setminus (\mathscr{H} \setminus \mathscr{H}_0)$, we can without loss of generality assume that $\mathscr{E} = \mathscr{E}_0 = \mathscr{H}_0 = \mathscr{H} = \mathscr{H}^+$. Choose a point $p \in \mathscr{M}_{\mathrm{ext}}$, where the Killing vector $K_{(0)}$ is timelike, and let

$$\gamma_p = \cup_{t\in\mathbb{R}}\phi_t(p)$$

be the orbit of $K_{(0)}$ through p. Then $I^-(S)$ must intersect γ_p (since $\mathscr{E}_0$ is contained in the future of $\mathscr{M}_{\mathrm{ext}}$). Further, $I^-(S)$ cannot contain all of γ_p, by Lemma 2.3.1 or by part 2 of Lemma 2.3.2. Let $q \in \gamma_p$ lie on the boundary of $I^-(S)$, then $I^+(q)$ cannot contain any point of S, so it does not contain any complete null generator of $\mathscr{E}_0$. On the other hand, if $I^+(q)$ failed to intersect some generator of $\mathscr{E}_0$, then (by invariance under the flow of $K_{(0)}$) the future of each point of γ_p would also fail to intersect some generator. By considering a sequence, $\{q_n = \phi_{t_n}(q)\}$, along γ_p with $t_n \to -\infty$, one would obtain a corresponding sequence of horizon generators lying entirely outside the future of $\{q_n\}$. Using compactness, one would get an "accumulation generator" that lies outside the future of all $\{q_n\}$ and thus lies outside of $I^+(\gamma_p) = I^+(\mathscr{M}_{\mathrm{ext}})$, contradicting the fact that S lies to the future of $\mathscr{M}_{\mathrm{ext}}$.

Set

$$S_0 := \dot{I}^+(q) \cap \mathscr{E}_0 \ ,$$

and we have just proved that every generator of $\mathscr{E}_0$ intersects S_0 at least once.

The fact that the only null geodesics tangent to $\mathscr{E}_0$ are the generators of $\mathscr{E}_0$ shows that the generators of $\dot{I}^+(q)$ intersect $\mathscr{E}_0$ transversally. (Otherwise a generator of $\dot{I}^+(q)$ would become a generator, say Γ, of $\mathscr{E}_0$. Thus Γ would leave $\mathscr{E}_0$ when followed to the past at the intersection point of $\dot{I}^+(q)$ and $\mathscr{E}_0$, reaching q, which contradicts the fact that $\mathscr{E}_0$ lies at the boundary of $I^-(\mathscr{M}_{\mathrm{ext}})$.) As in [36], Clarke's Lipschitz implicit function theorem [46] shows now that S_0 is a Lipschitz submanifold intersecting each horizon generator; while the argument just given shows that it intersects each generator at most one point. Thus, S_0 is a cross-section with respect to the null generators. However, S_0 also is a cross-section with respect to the flow of $K_{(0)}$, because for all t we have

$$\phi_t(S_0) = \dot{I}^+(\phi_t(q)) \cap \mathscr{E} \ ,$$

and for $t > 0$ the boundary of $I^+(\phi_t(q))$ is contained within $I^+(q)$. In other words, $\phi_t(S_0)$ cannot intersect S_0, which is equivalent to saying that each orbit of the flow of $K_{(0)}$ on the horizon cannot intersect S_0 at more than one point. On the other hand, each orbit must intersect S_0 at least once by the type of argument already given — one will run into a contradiction if complete Killing orbits on the horizon are either contained within $I^+(q)$ or lie entirely outside of $I^+(q)$. $\Box$

Now, both S and S_0 are compact cross-sections of $\mathscr{E}_0$. Flowing along the generators of the horizon, one obtains:

PROPOSITION 2.4.2 *S is homeomorphic to S_0.*

We note that so far we only have a $C^{0,1}$ cross-section of the horizon, and in fact this is the best one can get at this stage, since this is the natural differentiability of $\mathscr{E}_0$. However, if $\mathscr{E}_0$ is smooth, we claim:

PROPOSITION 2.4.3 *Under the hypotheses of Proposition 2.4.1, assume moreover that $\mathscr{E}_0$ is smooth, and that $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is globally hyperbolic. Then S_0 can be chosen to be smooth.*

PROOF: The result is obtained with the following regularization argument: Choose a point $p \in \mathscr{M}_{\rm ext}$, such that the section S of Proposition 2.4.1 does *not* intersect the future of p. Let the function u be the retarded time associated with the orbit γ_p through p parameterized by the Killing time from p; this is defined as follows: For any $q \in \mathscr{M}$ we consider the intersection $J^-(q) \cap \gamma_p$. If that intersection is empty we set $u(q) = \infty$. If $J^-(q)$ contains γ_p we set $u(q) = -\infty$. Otherwise, as $\dot J^-(q)$ is achronal, the set $\dot J^-(q) \cap \gamma_p$ contains precisely one point $\phi_\tau(p)$ for some τ. We then set $u(q) = \tau$. Note that, with appropriate conventions, this is the same as setting

$$u(q) = \inf\{t : \phi_t(p) \in J^-(q)\} \ . \tag{2.4.1}$$

It follows from the definition of u that we have, for all r,

$$u(\phi_t(r)) = u(r) + t \ . \tag{2.4.2}$$

In particular, u is differentiable in the direction tangent to the orbits of $K_{(0)}$, with

$$K_{(0)}(u) = \mathfrak{g}(K_{(0)}, \nabla u) = 1 \ , \tag{2.4.3}$$

everywhere.

The proof of Proposition 2.4.1 shows that u is finite in a neighborhood of $\mathscr{E}_0$; let

$$S_0 = u^{-1}(0) \cap \mathscr{E}_0 \ ,$$

and let $\mathscr{O}$ denote a conditionally compact neighborhood of S_0 on which u is finite; note that S_0 here is a $\phi_t[K_{(0)}]$–translate of the section S_0 of Proposition 2.4.1.

Let n be the field of future directed tangents to the generators of $\mathscr{E}_0$, normalized to unit length with some auxiliary smooth Riemannian metric on $\mathscr{M}$. For $q \in S_0$ let $\mathscr{N}_q \subset T_q\mathscr{M}$ denote the collection of all similarly normalized null vectors that are tangent to an achronal past directed null geodesic γ from q to $\phi_{u(q)}(p)$, with γ contained in $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ except for its initial point. (If u is differentiable at q then $\mathscr{N}_q$ contains one single element, proportional to ∇u, but $\mathscr{N}_q$ can contain more than one null vector in general.) We claim that there exists $c > 0$ such that

$$\inf_{q\in S_0, l_q \in \mathscr{N}_q} \mathfrak{g}(l_q, n_q) \geq c > 0 \ . \tag{2.4.4}$$

Indeed, suppose that this is not the case; then there exists a sequence $q_i \in S_0$ and a sequence of past directed null achronal geodesic segments γ_i from q_i to p, with tangents l_i at q_i, such that $\mathfrak{g}(l_i, n) \to 0$. Compactness of S_0 implies that there exists $q \in S_0$ such that $q_i \to q$.

Let γ be an accumulation curve of the γ_i's passing through q. By hypothesis, $\mathscr{E}_0$ is a smooth null hypersurface contained in the boundary of $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$, with $q \in \mathscr{E}_0$. This implies that either γ immediately enters $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$, or γ is a subsegment of a generator of $\mathscr{E}_0$ through q. In the latter case γ intersects S when followed from q towards the past, and therefore the γ_i's intersect $\dot J^-(S) \cap \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ for all i large enough. But this is not possible since $S \cap J^+(p) = \emptyset$. We conclude that there exists $s_0 > 0$ such that $\gamma(s_0) \in \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$. Thus a subsequence, still denoted by $\gamma_i(s_0)$, converges to $\gamma(s_0)$, and global hyperbolicity of $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ implies that the γ_i's converge to an achronal null geodesic segment γ through p, with tangent l at S_0 satisfying $\mathfrak{g}(l, n) = 0$. Since both l and n are null we conclude that l is proportional to n, which is not possible as the intersection must be transverse, providing a contradiction, and establishing (2.4.4).

Let $\mathscr{O}_i$, $i = 1, \dots, N$, be a family of coordinate balls of radii $3r_i$ such that the balls of radius r_i cover $\overline{\mathscr{O}}$, and let φ_i be an associated partition of unity; by this we mean that the φ_i's are supported in $\mathscr{O}_i$, and they sum to one on $\mathscr{O}$. For $\epsilon \leq r := \min r_i$ let $\varphi_\epsilon(x) = \epsilon^{-n-1}\varphi(x/\epsilon)$ (recall that the dimension of $\mathscr{M}$ is

$n+1$), where φ is a positive smooth function supported in the ball of radius ϵ, with integral one. Set

$$u_\epsilon := \sum_{i=1}^N \varphi_i \, \varphi_\epsilon * u \,, \tag{2.4.5}$$

where $*$ denotes a convolution in local coordinates. Strictly speaking, φ_ϵ should be denoted by $\varphi_{\epsilon,i}$, as it depends explicitly on the local coordinates on $\mathscr{O}_i$, but we will not overburden the notation with yet another index.[10] Then u_ϵ tends uniformly to u. Further, using the Stokes theorem for Lipschitz functions [104],

$$\begin{aligned} du_\epsilon &= \sum_{i=1}^N \Big\{ \varphi_\epsilon * u \, d\varphi_i + \varphi_i \, \varphi_\epsilon * du \Big\} \\ &= \sum_{i=1}^N \Big\{ \underbrace{(\varphi_\epsilon * u - u)}_{I} \, d\varphi_i + \varphi_i \, \underbrace{\varphi_\epsilon * du}_{II} \Big\} \,, \end{aligned} \tag{2.4.6}$$

where we have also used $\sum_i d\varphi_i = d\sum_i \varphi_i = d1 = 0$. It immediately follows that the term I uniformly tends to zero as ϵ goes to zero. Now, the term II, when contracted with $K_{(0)}$, gives a contribution

$$\begin{aligned} i_{K_{(0)}}(\varphi_\epsilon * du)(x) &= \int_{|y-x|\le\epsilon} K^i_{(0)}(x) \, \partial_i u(y) \varphi_\epsilon(x-y) d^{n+1}y \\ &= \int_{|y-x|\le\epsilon} \Big[\underbrace{(K^i_{(0)}(x) - K^i_{(0)}(y))}_{=O(\epsilon)} \partial_i u(y) \\ &\qquad + \underbrace{K^i_{(0)}(y) \partial_i u(y)}_{=1 \text{ by (2.4.3)}} \Big] \varphi_\epsilon(x-y) d^{n+1}y \\ &= 1 + O(\epsilon) \,. \end{aligned} \tag{2.4.7}$$

It follows that, for all ϵ small enough, the differential du_ϵ is nowhere vanishing, and that $K_{(0)}$ is transverse to the level sets of u_ϵ.

To conclude, let n denote any future directed causal smooth vector field on $\mathscr{O}$ which coincides with the field of tangents to the null generators of $\mathscr{E}_0$ as defined above. By (2.4.4) the terms II in the formula for du_ϵ, when contracted with n,

[10] This is admittedly somewhat confusing since, e.g., $\sum_{i=1}^N \varphi_i \, \varphi_\epsilon * u \neq (\sum_{i=1}^N \varphi_i) \, \varphi_\epsilon * u$.

will give a contribution

$$
\begin{aligned}
i_n(\varphi_\epsilon * du)(x) &= \int_{|y-x|\le\epsilon} [\underbrace{(n^i(x)-n^i(y))}_{=O(\epsilon)}\partial_i u(y) + \underbrace{n^i(y)\partial_i u(y)}_{\ge c}]\varphi_\epsilon(x-y)d^{n+1}y \\
&\ge c + O(\epsilon)\ ,
\end{aligned}
\tag{2.4.8}
$$

and transversality of the generators of $\mathscr{E}_0$ to the level sets of u_ϵ, for ϵ small enough, follows. $\Box$

2.4.2 The structure of the domain of outer communications

The aim of this section is to establish the product structure of I^+–regular domains of outer communication, Theorem 2.4.5 below. The analysis here is closely related to that of [44].

As in Section 2.3, we assume the existence of a commutative group of isometries $\mathbb{R}\times\mathbb{T}^{s-1}$ with $s\ge 1$. We use the notation there, with $K_{(0)}$ timelike in $\mathscr{M}_{\rm ext}$, and each $K_{(i)}$ spacelike in $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$.

Let $r=\sqrt{\sum_i (x^i)^2}$ be the radius function in $\mathscr{M}_{\rm ext}$. By the asymptotic analysis of [39] there exists R so that for $r\ge R$ the orbits of the $K_{(i)}$'s are entirely contained in $\mathscr{M}_{\rm ext}$, so that the function

$$\hat r(p) = \int_{g\in\mathbb{T}^{s-1}} r(g(p))d\mu_g\ ,$$

is well defined, and invariant under $\mathbb{T}^{s-1}$. Here $d\mu_g$ is the translation invariant measure on $\mathbb{T}^{s-1}$ normalized to total volume one, and $g(p)$ denotes the action on $\mathscr{M}$ of the isometry group generated by the $K_{(i)}$'s. Similarly, let t be any time function on $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$, the level sets of which are asymptotically flat Cauchy surfaces. Averaging over $\mathbb{T}^{s-1}$ as above, we obtain a new time function $\hat t$, with asymptotically flat level sets, which is invariant under $\mathbb{T}^{s-1}$. (The interesting question, whether or not the level sets of $\hat t$ are Cauchy, is irrelevant for our further considerations here.) It is then easily seen that, for σ large enough, the level sets

$$\hat S_{\tau,\sigma} := \{\hat t=\tau, \hat r=\sigma\}$$

are smooth embedded spheres included in $\mathscr{M}_{\rm ext}$.

Throughout this section we assume that $(\mathscr{M},\mathfrak{g})$ is I^+–regular. Let $\mathscr{S}$ be as in the definition of regularity, thus $\mathscr{S}$ is an asymptotically flat spacelike acausal hypersurface in $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$ with compact boundary, the latter coinciding with a compact cross-section of $\mathscr{E}^+$. Deforming $\mathscr{S}$ if necessary, without loss of generality we may assume that $\mathscr{S}\cap\mathscr{M}_{\rm ext}$ is a level set of $\hat t$. We choose R large enough so that $\hat S_{0,R}$ is a smooth sphere, and so that the slopes of light cones on the $\hat S_{\tau,\sigma}$'s, for $\sigma\ge R$, are bounded from above by two, and from below by one half, and redefine $\mathscr{S}_{\rm ext}$ so that $\partial\mathscr{S}_{\rm ext}=\hat S_{0,R}$.

Consider

$$\mathscr{C}^+ := (\dot J^+(\hat S_{0,R})\setminus\mathscr{M}_{\rm ext})\cap\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle\ .$$

Then $\mathscr{C}^+$ is a null, achronal, Lipschitz hypersurface generated by null geodesics initially orthogonal to $\hat S_{0,R}$. Let us write ϕ_t for $\phi_t[K_{(0)}]$, and set

$$\mathscr{C}^+_t := \phi_t(\mathscr{C}^+)\ ;$$

we then have

$$\mathscr{C}^+_t := (\dot J^+(\hat S_{t,R})\setminus\mathscr{M}_{\rm ext})\cap\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle\ ,$$

(recall that the flow of $K_{(0)}$ consists of translations in t in $\mathscr{M}_{\rm ext}$) which implies that every orbit of $K_{(0)}$ intersects $\mathscr{C}^+$ at most once.

Since $\mathscr{S}$ is achronal it partitions $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$ as

$$\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle=\mathscr{S}\cup I^+(\mathscr{S};\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle)\cup I^-(\mathscr{S};\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle)\ \text{(disjoint union)}\ . \tag{2.4.9}$$

Indeed, as $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$ is globally hyperbolic, the boundaries $(\dot I^\pm(\mathscr{S})\setminus\mathscr{S})\cap\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$ are generated by null geodesics with end points on $\mathrm{edge}(\mathscr{S})\cap\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle=\emptyset$.

We claim that every orbit of $K_{(0)}$ intersects $\mathscr{S}$. For this, recall that for any q in $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$ there exist points $p_\pm\in\mathscr{M}_{\rm ext}$ such that $q\in I^\mp(p_\pm)$. Since the flow of $K_{(0)}$ in $\mathscr{M}_{\rm ext}$ is by time translations there exist $t_\pm\in\mathbb{R}$ so that $\phi_{t_\pm}(p_\pm)\in\mathscr{S}_{\rm ext}$. Hence $\phi_{t_\pm}(q)\in I^\mp(\mathscr{S}_{\rm ext})$, which shows that every orbit of $K_{(0)}$ meets both the future and the past of $\mathscr{S}$. By continuity and (2.4.9) every orbit meets $\mathscr{S}$ (perhaps more than once). Hence

$$\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle=\cup_t\phi_t(\mathscr{S})\ ,\quad \overline{\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle}\cap I^+(\mathscr{M}_{\rm ext})=\cup_t\phi_t(\overline{\mathscr{S}}) \tag{2.4.10}$$

(for the second equality Proposition 2.4.1 has been used). Setting $\mathscr{M}_{\rm int} = \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle\setminus \mathscr{M}_{\rm ext}$, one similarly obtains

$$\mathscr{M}_{\rm int} = \mathscr{C}^+ \cup I^+(\mathscr{C}^+;\mathscr{M}_{\rm int}) \cup I^-(\mathscr{C}^+;\mathscr{M}_{\rm int}) \text{ (disjoint union) }, \quad (2.4.11)$$

$$\mathscr{M}_{\rm int} = \cup_t \phi_t(\mathscr{C}^+) \; . \quad (2.4.12)$$

By hypothesis $\overline{\mathscr{S}\setminus\mathscr{S}_{\rm ext}}$ is compact and so, by the first part of Lemma 2.3.2, there exists $p_-\in\mathscr{M}_{\rm ext}$ such that

$$\overline{\mathscr{S}\setminus\mathscr{S}_{\rm ext}} \subset I^+(p_-) \; . \quad (2.4.13)$$

Choose $t_- < 0$ so that $p_-\in I^+(\hat S_{t_-,R})$; we obtain that $\overline{\mathscr{S}\setminus\mathscr{S}_{\rm ext}} \subset I^+(\hat S_{t_-,R})$, hence

$$\overline{\mathscr{S}\setminus\mathscr{S}_{\rm ext}} \subset I^+(\mathscr{C}^+_{t_-}) \; .$$

Since $\hat S_{0,R}\subset \mathscr{S}$ we have $\mathscr{C}^+\subset I^+(\mathscr{S})$. By acausality of $\mathscr{S}$ and (2.4.9) we infer that $\overline{\mathscr{S}\setminus\mathscr{S}_{\rm ext}} \subset I^-(\mathscr{C}^+)$, and hence $\phi_{t_-}(\overline{\mathscr{S}\setminus\mathscr{S}_{\rm ext}}) \subset I^-(\mathscr{C}^+_{t_-})$.

So, for $p\in\overline{\mathscr{S}\setminus\mathscr{S}_{\rm ext}}$ the orbit segment

$$[t_-,0]\ni t\mapsto \phi_t(p)$$

starts in the past of $\mathscr{C}^+_{t_-}$ and finishes to its future. From (2.4.10) we conclude that

$$\overline{\mathscr{C}^+_{t_-}} \subset \cup_{t\in[t_-,0]}\phi_t(\overline{\mathscr{S}\setminus\mathscr{S}_{\rm ext}}) \; ; \quad (2.4.14)$$

equivalently,

$$\overline{\mathscr{C}^+} \subset \cup_{t\in[0,-t_-]}\phi_t(\overline{\mathscr{S}\setminus\mathscr{S}_{\rm ext}}) \; .$$

As the set at the right-hand-side is compact, we have established:

PROPOSITION 2.4.4 *Suppose that $(\mathscr{M},\mathfrak{g})$ is I^+–regular, then $\overline{\mathscr{C}^+}$ is compact.*

We are ready to prove now the following version of point 2 of Lemma 5.1 of [27]:

THEOREM 2.4.5 (Structure theorem) *Suppose that $(\mathscr{M},\mathfrak{g})$ is an I^+–regular stationary space-time invariant under a commutative group of isometries $\mathbb{R}\times\mathbb{T}^{s-1}$, $s\geq 1$, with the stationary Killing vector $K_{(0)}$ tangent to the orbits of the $\mathbb{R}$ factor.*

There exists on $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ a smooth time function t, invariant under $\mathbb{T}^{s-1}$, which together with the flow of $K_{(0)}$ induces the diffeomorphisms

$$\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle \approx \mathbb{R}\times \mathring{\mathscr{S}}\;, \qquad \overline{\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle}\cap I^+(\mathscr{M}_{\rm ext}) \approx \mathbb{R}\times \overline{\mathring{\mathscr{S}}}\;, \tag{2.4.15}$$

where $\mathring{\mathscr{S}} := t^{-1}(0)$ is asymptotically flat, (invariant under $\mathbb{T}^{s-1}$), with the boundary $\partial\overline{\mathring{\mathscr{S}}}$ being a compact cross-section of $\mathscr{E}^+$. The smooth hypersurface with boundary $\overline{\mathring{\mathscr{S}}}$ is acausal, spacelike up-to-boundary, and the flow of $K_{(0)}$ is a translation along the $\mathbb{R}$ factor in (2.4.15).

PROOF: From what has been said, every orbit of $K_{(0)}$ through $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle \setminus \mathscr{M}_{\rm ext}$ intersects $\mathscr{C}^+$ precisely once. For $p \in \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle \setminus \mathscr{M}_{\rm ext}$ we let $u(p)$ be the unique real number such that $\phi_{u(p)}(p) \in \mathscr{C}^+$, while for $p\in \mathscr{M}_{\rm ext}$ we let $u(p)$ be the unique real number such that $\phi_{u(p)}(p)\in \mathscr{S}_{\rm ext}$. The function $u:\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle \to \mathbb{R}$ is Lipschitz, smooth in $\mathscr{M}_{\rm ext}$, with achronal level sets transverse to the flow of $K_{(0)}$, and provides a homeomorphism

$$\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle \setminus \mathscr{M}_{\rm ext} \approx \mathbb{R}\times \mathscr{C}^+\;, \qquad \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle \approx \mathbb{R}\times (\mathscr{C}^+\cup \mathscr{S}_{\rm ext})\;.$$

The desired hypersurface $\mathring{\mathscr{S}}$ will be a small spacelike smoothing of $u^{-1}(0)$, obtained by first deforming the metric $\mathfrak{g}$ to a metric $\mathfrak{g}_\epsilon$, the null vectors of which are spacelike for $\mathfrak{g}$. The associated corresponding function u_ϵ will have Lipschitz level sets which are uniformly spacelike for $\mathfrak{g}$. A smoothing of u_ϵ will provide the desired function t. The details are as follows:

We start by finding a smooth hypersurface, not necessarily spacelike, transverse to the flow of $K_{(0)}$. We shall use the following general result, pointed out to us by R. Wald (private communication):

PROPOSITION 2.4.6 *Let S_0 be a two-sided, smooth, hypersurface in a manifold M with an open neighborhood $\mathscr{O}$ such that $M\setminus\mathscr{O}$ consists of two disconnected components M_- and M_+. Let X be a complete vector field on M and suppose that there exists $T>0$ such that for every orbit $\phi_t(p)$ of X, $t\in\mathbb{R}$, $p\in M$, there is an interval $[t_0,t_1]$ with $(t_1-t_0)<T$ such that $\phi_t(p)$ lies in M_- for all $t<t_0$, and $\phi_t(p)$ lies in M_+ for all $t>t_1$. If M has a boundary, assume moreover that $\partial S_0\subset\partial M$, and that X is tangent to ∂M. Then there exists a smooth hypersurface $S_1\subset M$ such that every orbit of X intersects S_1 once and only once.*

Proof: Let f be a smooth function with the property that $f = 0$ in M_-, $0 \leq f \leq 1$ in $\mathscr{O}$, and $f = 1$ in M_+; such a function is easily constructed by introducing Gauss coordinates, with respect to some auxiliary Riemannian metric, near S_0. For $t \in \mathbb{R}$ and $p \in M$ let $\phi_t(p)$ denote the flow generated by X. Define $F : M \to \mathbb{R}$ by

$$F(p) = \int_{-\infty}^{0} f \circ \phi_s(p) ds \ .$$

Then F is a smooth function on M increasing monotonically from zero to infinity along every orbit of X. Furthermore F is strictly increasing along the orbits at points at which $F \geq T$ (since such points must lie in M_+, where $f = 1$). In particular, the gradient of F is non-vanishing at all points where $F \geq T$. Setting $S_1 = \{F = T\}$, the result follows. $\square$

Returning to the proof of Theorem 2.4.5, we use Proposition 2.4.6 with $X = K_{(0)}$,

$$M = \overline{\overline{\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle} \cap I^+(\mathscr{M}_{\rm ext}) \setminus \mathscr{M}_{\rm ext}} \ ,$$

and $S_0 = \mathscr{S} \cap M$. Letting t_- be as in (2.4.14) we set

$$\mathscr{O} := \cup_{t \in (t_-, -t_-)} \phi_t(\mathscr{S}) \ ;$$

by what has been said, $\mathscr{O}$ is an open neighborhood of $\mathscr{S}$. Finally

$$M_- := \cup_{t \in (-\infty, t_-]} \phi_t(\mathscr{S}) \ , \quad M_+ := \cup_{t \in [-t_-, \infty)} \phi_t(\mathscr{S}) \ .$$

It follows now from Proposition 2.4.6 that there exists a hypersurface $S_1 \subset M$ which is transverse to the flow of $K_{(0)}$.

Let $\hat{T}$ be any smooth, timelike vector field defined along S_1, and define the smooth timelike vector field T on M as the unique solution of the Cauchy problem

$$\mathscr{L}_{K_{(0)}} T = 0 \ , \quad T = \hat{T} \ \text{ on } S_1 \ . \tag{2.4.16}$$

Since the flow of $K_{(0)}$ acts by time translations on $\mathscr{M}_{\rm ext}$, it is straightforward to extend T to a smooth vector field defined on $\mathscr{M}$, timelike wherever non vanishing, still denoted by T, which is invariant under the flow of $K_{(0)}$, the support of which on $\mathscr{S}$ is compact. Replacing T by its average over $\mathbb{T}^{s-1}$, we can assume that T is invariant under the action of $\mathbb{T}^{s-1}$.

For all $\epsilon \geq 0$ sufficiently small, the formula

$$\mathfrak{g}_\epsilon(Z_1, Z_2) = \mathfrak{g}(Z_1, Z_2) - \epsilon \mathfrak{g}(T, Z_1)\mathfrak{g}(T, Z_2) \; . \tag{2.4.17}$$

defines a Lorentzian, $\mathbb{R}\times\mathbb{T}^{s-1}$ invariant metric on the manifold with ($\mathfrak{g}_\epsilon$–timelike) boundary $\overline{\langle\langle \mathscr{M}_{\mathrm{ext}}\rangle\rangle}\cap I^+(\mathscr{M}_{\mathrm{ext}})$. By definition of $\mathfrak{g}_\epsilon$, vectors which are causal for $\mathfrak{g}$ are timelike for $\mathfrak{g}_\epsilon$. Wherever $T \neq 0$ the light cones of $\mathfrak{g}_\epsilon$ are spacelike for $\mathfrak{g}$, provided $\epsilon \neq 0$.

Since $\mathfrak{g}$-causal curves are also $\mathfrak{g}_\epsilon$-causal, $(\langle\langle \mathscr{M}_{\mathrm{ext}}\rangle\rangle, \mathfrak{g}_\epsilon)$ is also a domain of outer communications with respect to $\mathfrak{g}_\epsilon$.

Set

$$\mathscr{C}^+_\epsilon = (\dot J^+_\epsilon(\hat S_{0,R}) \setminus \mathscr{M}_{\mathrm{ext}}) \cap \langle\langle \mathscr{M}_{\mathrm{ext}}\rangle\rangle \; ,$$

where we denote by $J^+_\epsilon(\Omega)$ the future of a set Ω with respect to the metric $\mathfrak{g}_\epsilon$. Then the $\mathscr{C}^+_\epsilon$'s are Lipschitz, $\mathfrak{g}$-spacelike wherever differentiable, $\mathbb{T}^{s-1}$ invariant, hypersurfaces. Continuous dependence of geodesics upon the metric together with Proposition 2.4.4 shows that the $\mathscr{C}^+_\epsilon$'s accumulate at $\mathscr{C}^+$ as ϵ tends to zero.

Let $u_\epsilon : M \to \mathbb{R}$ be defined as in (2.4.1) using the metric $\mathfrak{g}_\epsilon$ instead of $\mathfrak{g}$. As before we have

$$u_\epsilon(\phi_t(p)) = u_\epsilon(p) + t \; , \text{ so that } K_{(0)}(u_\epsilon) = 1 \; . \tag{2.4.18}$$

We perform a smoothing procedure as in the proof of Proposition 2.4.3, with $\mathscr{O}$ there replaced by a conditionally compact neighborhood of $\mathscr{C}^+$. The vector field $\hat T$ in (2.4.16) is chosen to be timelike on $\overline{\mathscr{O}}$; the same will then be true of T. Analogously to (2.4.5) we set

$$u_{\epsilon,\eta} := \sum_{i=1}^N \varphi_i \; \varphi_\eta * u_\epsilon \; , \tag{2.4.19}$$

so that the $u_{\epsilon,\eta}$'s converge uniformly on $\mathscr{O}$ to u_ϵ as η tends to zero. The calculation in (2.4.7) shows that

$$K_{(0)}(u_{\epsilon,\eta}) \geq \frac{1}{2}$$

for η small enough, so that the level sets of $u_{\epsilon,\eta}$ near $\mathscr{C}^+$ are transverse to the flow of $K_{(0)}$.

It remains to show that the level sets of $u_{\epsilon,\eta}$ are spacelike. For this we start with some lemmata:

Lemma 2.4.7 *Let $\mathfrak{g}$ be a Lipschitz-continuous metric on a coordinate ball $B(p,3r_i)\equiv \mathscr{O}_i$ of coordinate radius $3r_i$. There exists a constant C such that for any $q\in B(p,r_i)$ and for any timelike, respectively causal, vector $N_q = N^\mu_q\partial_\mu \in T_q\mathscr{M}$ satisfying*

$$\sum_\mu (N^\mu_q)^2 = 1 \tag{2.4.20}$$

there exists a timelike, respectively causal, vector field $N=N^\mu\partial_\mu$ on $B(p,2r_i)$ such that for all points $y,z\in B(p,2r_i)$ we have

$$|N^\mu_y - N^\mu_z| \le C|y-z|\ , \quad C^{-1}\le \sum_\mu (N^\mu_y)^2 \le C\ . \tag{2.4.21}$$

Proof: We will write both N^μ_q and $N^\mu(q)$ for the coordinate components of a vector field at q. For $\nu=0,\ldots,n$, let $e_{(\nu)} = e^\mu_{(\nu)}\partial_\mu$ be any Lipschitz-continuous ON basis for $\mathfrak{g}$ on $\mathscr{O}_i$. there exists a constant c such that on $B(p,2r_i)$ we have

$$|e^\mu_{(\nu)}(y) - e^\mu_{(\nu)}(z)| \le c|y-z|\ .$$

Decompose N_q as $N_q = N^{(\nu)}_q e_{(\nu)}(q)$, and for $y\in\mathscr{O}_i$ set $N_y = N^{(\nu)}_q e_{(\nu)}(y)$; (2.4.21) easily follows.

Lemma 2.4.8 *Under the hypotheses of Lemma 2.4.7, let f be differentiable on $\mathscr{O}_i$. Then ∇f is timelike past directed on $B(p,2r_i)$ if and only if $N^\mu\partial_\mu f<0$ on $\mathscr{O}_i$ for all causal past directed vector fields satisfying* (2.4.20) *and* (2.4.21).

Proof: The condition is clearly necessary. For sufficiency, suppose that there exists $q\in B(p,2r_i)$ such that ∇f is null, let $N_q=\lambda\nabla f(q)$, where λ is chosen so that (2.4.20) holds, and let N be as in Lemma 2.4.7; then $N^\mu\partial_\mu f$ vanishes at q. If ∇f is spacelike at q the argument is similar, with N_q chosen to be any timelike vector orthogonal to $\nabla f(q)$ satisfying (2.4.20).

Let N be any $\mathfrak{g}$–timelike past directed vector field satisfying (2.4.20) and (2.4.21). Returning to (2.4.6) we find,

$$i_N du_{\epsilon,\eta} \;=\; \sum_{i=1}^N \Big\{ \underbrace{(\varphi_\eta * u_\epsilon - u_\epsilon)}_{I}\, i_N d\varphi_i + \varphi_i\ \underbrace{i_N(\varphi_\eta * du_\epsilon)}_{II} \Big\}\ . \tag{2.4.22}$$

For any fixed ϵ, and for any $\delta>0$ we can choose η_δ so that the term I is smaller than δ for all $0<\eta<\eta_\delta$.

To obtain control of II, we need uniform spacelikeness of du_ϵ:

LEMMA 2.4.9 *There exists a constant c such that, for N as in Lemma 2.4.7,*

$$N^\mu \partial_\mu u_\epsilon < -c\epsilon \tag{2.4.23}$$

almost everywhere, for all $\epsilon > 0$ sufficiently small.

PROOF: Let $\{e_{(\nu)}\}$ be an $\mathfrak{g}$–ON frame in which the vector field T of (2.4.17) equals $T^{(0)}e_{(0)}$. Let $\alpha_{(\nu)}$ denote the components of du_ϵ in a frame dual to $\{e_{(\nu)}\}$. In this frame we have

$$\mathfrak{g} = \mathrm{diag}(-1, 1, \ldots, 1) \ , \qquad \mathfrak{g}_\epsilon = \mathrm{diag}(-(1 + (T^{(0)})^2\epsilon), 1, \ldots, 1) \ .$$

Since du_ϵ is $\mathfrak{g}_\epsilon$–null and past pointing we have

$$\alpha_{(0)} = \sqrt{1 + (T^{(0)})^2\epsilon}\sqrt{\sum \alpha^2_{(i)}} \ .$$

The last part of (2.4.18) reads

$$K^{(0)}_{(0)}\alpha_{(0)} + K^{(i)}_{(0)}\alpha_{(i)} = 1 \ .$$

It is straightforward to show from these two equations that there exists a constant c_1 such that, for all ϵ sufficiently small,

$$\alpha_{(0)} > c_1^{-1} \ , \quad \sqrt{\sum \alpha^2_{(i)}} > c_1^{-1} \ , \quad \sum |\alpha_{(\mu)}| \le c_1 \ .$$

Since N is $\mathfrak{g}_\epsilon$ causal past directed, (2.4.20) and (2.4.21) together with the construction of N show that there exists a constant c_2 such that

$$N^{(0)} < -c_2 \ .$$

We then have

$$\begin{aligned}
N^\mu \partial_\mu u_\epsilon &= N^{(0)}\alpha_{(0)} + N^{(i)}\alpha_{(i)} \\
&= N^{(0)}\sqrt{1 + (T^{(0)})^2\epsilon}\sqrt{\sum \alpha^2_{(i)}} + N^{(i)}\alpha_{(i)} \\
&= N^{(0)}(\sqrt{1 + (T^{(0)})^2\epsilon} - 1)\sqrt{\sum \alpha^2_{(i)}} + \underbrace{N^{(0)}\sqrt{\sum \alpha^2_{(i)}} + N^{(i)}\alpha_{(i)}}_{<0 \text{ as both } N \text{ and } \nabla u_\epsilon \text{ are } \mathfrak{g}\text{–past pointing}} \\
&< -\frac{c_2}{4c_1} \inf_{\overline{\mathscr{O}}} (T^{(0)})^2 \, \epsilon =: -c\epsilon \ ,
\end{aligned}$$

for ϵ small enough.

Now, calculating as in (2.4.8), using (2.4.23),

$$\begin{aligned} i_N(\varphi_\eta * du_\epsilon)(x) &= \int_{|y-x|\le\eta} [\underbrace{(N^\mu(x)-N^\mu(y))}_{\le C\eta}\partial_\mu u_\epsilon(y) + \underbrace{N^\mu(y)\partial_\mu u_\epsilon(y)}_{\le -c\epsilon}]\varphi_\eta(x-y)d^{n+1}y \\ &\le -c\epsilon + O(\eta)\ , \end{aligned}$$

so that for η small enough each such term will give a contribution to (2.4.22) smaller than $-c\epsilon/2$. Timelikeness of $\nabla u_{\epsilon,\eta}$ on $\overline{\mathscr{O}}$ follows now from Lemma 2.4.8.

Summarizing, we have shown that we can choose ϵ and η small enough so that the function $u_{\epsilon,\eta} : M \to \mathbb{R}$ is a time function near its zero level set. It is rather straightforward to extend $u_{\epsilon,\eta}$ to a function on $\langle\langle \mathscr{M}_{\mathrm{ext}}\rangle\rangle \to \mathbb{R}$, with smooth spacelike zero-level-set, which coincides with $\mathscr{S}$ at large distances. Letting $\mathring{\mathscr{S}}$ be this zero level set, the function $t(p)$ is defined now as the unique value of parameter t so that $\phi_t(p) \in \mathring{\mathscr{S}}$; since the level sets of t are smooth spacelike hypersurface, t is a smooth time function. This completes the proof of Theorem 2.4.5. $\square$

2.4.3 Smoothness of event horizons

The starting point to any study of event horizons in stationary space-times is a corollary to the area theorem, essentially due to [36], which shows that event horizons in well-behaved stationary space-times are as smooth as the metric allows. In order to proceed, some terminology from that last reference is needed; we restrict ourselves to asymptotically flat space-times; the reader is referred to [36, Section 4] for the general case. Let $(\widetilde{\mathscr{M}}, \widetilde{\mathfrak{g}})$ be a C^3 completion of $(\mathscr{M}, \mathfrak{g})$ obtained by adding a null conformal boundary at infinity, denoted by $\mathscr{I}^+$, to $\mathscr{M}$, such that $\mathfrak{g} = \Omega^{-2}\widetilde{\mathfrak{g}}$ for a non-negative function Ω defined on $\widetilde{\mathscr{M}}$, vanishing precisely on $\mathscr{I}^+$, and $d\Omega$ without zeros on $\mathscr{I}^+$. Let $\mathscr{E}^+$ be the future event horizon in $\mathscr{M}$. We say that $(\widetilde{\mathscr{M}}, \widetilde{\mathfrak{g}})$ is $\mathscr{E}^+$–regular if there exists a neighborhood $\mathscr{O}$ of $\mathscr{E}^+$ such that for every compact set $C \subset \mathscr{O}$ for which $I^+(C; \widetilde{\mathscr{M}}) \neq \emptyset$ there exists a generator of $\mathscr{I}^+$ intersecting $\overline{I^+(C; \widetilde{\mathscr{M}})}$ which leaves this last set when followed to the past. (Compare Remark 4.4 and Definition 4.3 in [36]).

We note the following:

Proposition 2.4.10 *Consider an asymptotically flat stationary space-time which is vacuum at large distances, recall that* $\mathscr{E}^+ = \dot{I}^-(\mathscr{M}_{\mathrm{ext}}) \cap I^+(\mathscr{M}_{\mathrm{ext}})$. *If* $\langle\langle \mathscr{M}_{\mathrm{ext}}\rangle\rangle$ *is globally hyperbolic, then* $(\mathscr{M}, \mathfrak{g})$ *admits an* $\mathscr{E}^+$*–regular conformal completion.*

Proof: Let $\widetilde{\mathscr{M}}$ be obtained by adding to $\mathscr{M}_{\rm ext}$ the surface $\tilde r = 0$ in the coordinate system $(u, \tilde r, \theta, \varphi)$ of [53, Appendix A] (see also [48], where the construction of [53] is corrected; those results generalize without difficulty to higher dimensions). Let t be any time function on $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ which tends to infinity when $\mathscr{E}^+$ is approached, which tends to $-\infty$ when $\dot I^+(\mathscr{M}_{\rm ext})$ is approached, and which coincides with the coordinate t in $\mathscr{M}_{\rm ext}$ as in [53, Appendix A]. Let

$$\mathscr{O} = \{p \mid t(p) > 0\} \cup I^+(\mathscr{E}^+) \cup \mathscr{E}^+ \;;$$

then $\mathscr{O}$ forms an open neighborhood of $\mathscr{E}^+$. Let C be any compact subset of $\mathscr{O}$ such that $I^+(C; \widetilde{\mathscr{M}}) \cap \mathscr{I}^+ \neq \emptyset$; then $\emptyset \neq C \cap \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle \subset \{t > 0\}$. Let γ be any future directed causal curve from C to $\mathscr{I}^+$, then γ is entirely contained in $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$, with $t \circ \gamma > 0$. In particular any intersection of γ with $\partial \mathscr{M}_{\rm ext}$ belongs to the set $\{t > 0\}$, so that at each intersection point

$$u \circ \gamma > \inf u|_{\{t=0\}\cap \partial \mathscr{M}_{\rm ext}} =: c > -\infty \;.$$

The coordinate u of [53, Appendix A] is null, hence non-increasing along causal curves, so $u \circ \gamma > c$, which implies the regularity condition. $\square$

We are ready to prove now:

Theorem 2.4.11 *Let $(\mathscr{M}, \mathfrak{g})$ be a smooth, asymptotically flat, $(n+1)$–dimensional space-time with stationary Killing vector $K_{(0)}$, the orbits of which are complete. Suppose that $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is globally hyperbolic, vacuum at large distances in the asymptotic region, and assume that the* null energy condition (2.2.14) *holds. Assume that a connected component $\mathscr{H}_0$ of*

$$\mathscr{H} := \mathscr{H}^- \cup \mathscr{H}^+$$

admits a compact cross-section satisfying $S \subset I^+(\mathscr{M}_{\rm ext})$. If

1. *either*
$$\overline{\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle} \cap I^+(\mathscr{M}_{\rm ext}) \textit{ is strongly causal,}$$

2. *or there exists in $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ a spacelike hypersurface $\mathscr{S} \supset \mathscr{S}_{\rm ext}$, achronal in $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$, so that S as above coincides with the boundary of $\overline{\mathscr{S}}$:*
$$S = \partial \overline{\mathscr{S}} \subset \mathscr{E}^+ \;,$$

then

$$\cup_t \phi_t[K_{(0)}](S) \subset \mathscr{H}_0$$

is a smooth null hypersurface, which is analytic if the metric is.

Remark 2.4.12 The condition that the space-time is vacuum at large distances can be replaced by the requirement of existence of an $\mathscr{E}^+$–regular conformal completion at null infinity.

Proof: Let Σ be a Cauchy surface for $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$, and let $\widetilde{\mathscr{M}}$ be the conformal completion of $\mathscr{M}$ provided by Proposition 2.4.10. By [36, Proposition 4.8] the hypotheses of [36, Proposition 4.1] are satisfied, so that the Aleksandrov divergence $\theta_{\mathscr{A}l}$ of $\mathscr{E}^+$, as defined in [36], is nonnegative. Let S_1 be given by Proposition 2.4.1. Since isometries preserve area we have $\theta_{\mathscr{A}l} = 0$ almost everywhere on $\cup_t\phi_t(S_1) = \cup_t\phi_t(S)$. The result follows now from [36, Theorem 6.18]. □

2.4.4 Event horizons vs Killing horizons in analytic vacuum space-times

We have the following result, first proved by Hawking for $n = 3$ [71] (compare [58] or [27, Theorem 5.1]), while the result for $n \geq 4$ in the mean-non-degenerate case is due to Hollands, Ishibashi and Wald [80], see also [79, 86, 96]:

Theorem 2.4.13 *Let $(\mathscr{M}, \mathfrak{g})$ be an analytic, $(n+1)$–dimensional, vacuum space-time with complete Killing vector $K_{(0)}$. Assume that $\mathscr{M}$ contains an analytic null hypersurface $\mathscr{E}$ with a compact cross-section S transverse both to $K_{(0)}$ and to the generators of $\mathscr{E}$. Suppose that*

1. *either $\langle\kappa\rangle_S \neq 0$, where $\langle\kappa\rangle_S$ is defined in* (2.2.11)*,*

2. *or $n = 3$.*

Then there exists a neighborhood $\mathscr{U}$ of $\mathscr{E}$ and a Killing vector defined on $\mathscr{U}$ which is null on $\mathscr{E}$.

In fact, if $K_{(0)}$ is not tangent to the generators of $\mathscr{E}$, then there exist, near $\mathscr{E}$, N commuting linearly independent Killing vector fields $K_{(1)}, \ldots, K_{(N)}$, $N \geq$

1, *(not necessarily complete but) with* 2π*–periodic orbits near* $\mathscr{E}$*, and numbers* $\Omega_{(1)}, \ldots, \Omega_{(N)}$*, such that*

$$K_{(0)} + \Omega_{(1)} K_{(1)} + \ldots + \Omega_{(N)} K_{(N)}$$

is null on $\mathscr{E}$*.*

In the black hole context, Theorem 2.4.13 implies:

THEOREM 2.4.14 *Let* $(\mathscr{M}, \mathfrak{g})$ *be an analytic, asymptotically flat, strongly causal, vacuum,* $(n+1)$*–dimensional space-time with stationary Killing vector* $K_{(0)}$*, the orbits of which are complete. Assume that* $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ *is globally hyperbolic, that a connected component* $\mathscr{H}_0^+$ *of* $\mathscr{H}^+$ *contains a compact cross-section* S *satisfying*

$$S \subset I^+(\mathscr{M}_{\rm ext}) \;,$$

and that

1. *either* $\langle \kappa \rangle_S \neq 0$*,*
2. *or the flow defined by* $K_{(0)}$ *on the space of the generators of* $\mathscr{H}_0^+$ *is periodic.*

Suppose moreover that

a) *either*
$$\overline{\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle} \cap I^+(\mathscr{M}_{\rm ext}) \text{ is strongly causal,}$$

b) *or there exists in* $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ *an asymptotically flat spacelike hypersurface* $\mathscr{S}$*, achronal in* $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$*, so that* S *as above coincides with the boundary of* $\overline{\mathscr{S}}$*:*
$$S = \partial \overline{\mathscr{S}} \subset \mathscr{E}^+ \;.$$

If $K_{(0)}$ *is* not *tangent to the generators of* $\mathscr{H}$*, then there exist, on* $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle \cup \mathscr{H}_0^+$*,* N *complete, commuting, linearly independent Killing vector fields* $K_{(1)}, \ldots, K_{(N)}$*,* $N \geq 1$*, with* 2π*–periodic orbits, and numbers* $\Omega_{(1)}, \ldots, \Omega_{(N)}$*, such that the Killing vector field*

$$K_{(0)} + \Omega_{(1)} K_{(1)} + \ldots + \Omega_{(N)} K_{(N)}$$

is null on $\mathscr{H}_0$*.*

Remark 2.4.15 For I^+–regular four-dimensional black holes S is a two-dimensional sphere (see Corollary 2.2.5), and then every Killing vector field acts periodically on the generators of $\mathscr{H}_0^+$.

Proof: Theorem 2.4.11 shows that $\mathscr{E}_0^+ := \cup_t \phi_t[K_{(0)}](S)$ is an analytic null hypersurface. By Proposition 2.4.3 there exists a smooth compact section of $\mathscr{E}_0^+$ which is transverse both to its generators and to the stationary Killing vector.[11] We can thus invoke Theorem 2.4.13 to conclude existence of Killing vector fields $K_{(i)}$, $i = 1, \ldots, N$, defined near $\mathscr{E}_0^+$. By Corollary 2.2.4 and a theorem of Nomizu [109] we infer that the $K_{(i)}$'s extend globally to $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$. It remains to prove that the orbits of all Killing vector fields are complete. In order to see that, we note that by the asymptotic analysis of Killing vectors of [10, 39] there exists R large enough so that the flows of all $K_{(i)}$'s through points in the asymptotically flat region with $r \geq R$ are defined for all parameter values $t \in [0, 2\pi]$. The arguments in the proof of Theorem 1.2 of [28] then show that the flows $\phi_t[K_{(i)}]$'s are defined for $t \in [0, 2\pi]$ throughout $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$. But $\phi_{2\pi}[K_{(i)}]$ is an isometry which is the identity on an open set near $\mathscr{E}_0^+$, hence everywhere, and completeness of the orbits follows.

2.5 Stationary axisymmetric black hole space-times: the area function

As will be explained in detail below, it follows from Theorem 2.4.14 together with the results on Killing vectors in [11, 28], that I^+–regular, $3+1$ dimensional, asymptotically flat, rotating black holes have to be axisymmetric. The next step of the analysis of such space-times is the study of the area function

$$W := -\det\Big(\mathfrak{g}(K_{(\mu)}, K_{(\nu)})\Big)_{\mu,\nu=0,1}\,, \qquad (2.5.1)$$

with $K_{(0)}$ being the asymptotically timelike Killing vector, and $K_{(1)}$ the axial one. Whenever $\sqrt{W}$ can be used as a coordinate, one obtains a dramatic simplification of the field equations, whence the interest thereof.

The function W is clearly positive in a region where $K_{(0)}$ is timelike and $K_{(1)}$ is spacelike, in particular it is non-negative on $\mathscr{M}_{\rm ext}$. As a starting point for further considerations, one then wants to show that W is non-negative on $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$:

[11] The hypothesis of existence of such a section needs to be added to those of [80, Theorem 2.1].

THEOREM 2.5.1 *Let $(\mathscr{M}, \mathfrak{g})$ be a four-dimensional, analytic, asymptotically flat, vacuum space-time with stationary Killing vector $K_{(0)}$ and periodic Killing vector $K_{(1)}$, jointly generating an $\mathbb{R} \times \mathrm{U}(1)$ subgroup of the isometry group of $(\mathscr{M}, \mathfrak{g})$. If $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is globally hyperbolic, then the area function* (2.5.1) *is non-negative on $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$, vanishing precisely on the union of its boundary with the (non-empty) set $\{\mathfrak{g}(K_{(1)}, K_{(1)}) = 0\}$.*

We also have a version of Theorem 2.5.1, where the hypothesis of analyticity is replaced by that of I^+–regularity:

THEOREM 2.5.2 *Under the remaining hypotheses of Theorem 2.5.1, instead of analyticity assume that $(\mathscr{M}, \mathfrak{g})$ is I^+–regular. Then the conclusion of Theorem 2.5.1 holds.*

Keeping in mind our discussion above, Theorem 2.5.1 follows from Proposition 2.5.3 and Theorem 2.5.4 below. Similarly, Theorem 2.5.2 is a corollary of Theorem 2.5.6.

2.5.1 Integrability

The first key fact underlying the analysis of the area function W is the following purely local fact, observed independently by Kundt and Trümper [93] and by Papapetrou [111] in dimension four (for a modern derivation see [74, 135]). The result, which does neither require $K_{(0)}$ to be stationary, nor the $K_{(i)}$'s to generate S^1 actions, generalizes to higher dimensions as follows (compare [18, 54]):

PROPOSITION 2.5.3 *Let $(\mathscr{M}, \mathfrak{g})$ be a vacuum, possibly with a cosmological constant, $(n+1)$–dimensional pseudo-Riemannian manifold with $n-1$ linearly independent commuting Killing vector fields $K_{(\mu)}$, $\mu = 0, \ldots, n-2$. If*

$$\mathscr{Z}_{dgt} := \{p \in \mathscr{M} \mid K_{(0)} \wedge ... \wedge K_{(n-2)}|_p = 0\} \neq \emptyset \;, \tag{2.5.2}$$

then [12]

$$dK_{(\mu)} \wedge K_{(0)} \wedge \ldots \wedge K_{(n-2)} = 0 \;. \tag{2.5.3}$$

A proof of a generalization of the previous result to the electro-vacuum setting, Proposition 3.2.1, will be presented in the next chapter.

[12] By an abuse of notation, we use the same symbols for vector fields and for the associated 1-forms.

2.5.2 The area function for a class of space-times with a commutative group of isometries

The simplest non-trivial reduction of the Einstein equations by isometries, which does *not* reduce the equations to ODEs, arises when orbits have co-dimension two, and the isometry group is abelian. It is useful to formulate the problem in a general setting, with $1 \le s \le n-1$ commuting Killing vector fields $K_{(\mu)}$, $\mu = 0, \ldots, s-1$, satisfying the following orthogonal integrability condition:

$$\forall \mu = 0, \ldots, s-1 \qquad dK_{(\mu)} \wedge K_{(0)} \wedge \ldots \wedge K_{(s-1)} = 0 \; . \tag{2.5.4}$$

For the problem at hand, (2.5.4) will hold when $s = n-1$ by Proposition 2.5.3. Note further that (2.5.4) with $s = 1$ is the definition of staticity. So, the analysis that follows covers simultaneously static analytic domains of dependence in all dimensions $n \ge 3$ (filling a gap in previous proofs), or stationary axisymmetric analytic four-dimensional space-times, or five dimensional stationary analytic space-times with two further periodic Killing vectors as in [81]. It further covers stationary axisymmetric I^+–regular black holes in $n = 3$, in which case analyticity is not needed.

Similarly to (2.5.2) we set

$$\mathscr{Z}_{dgt} := \{K_{(0)} \wedge ... \wedge K_{(s-1)} = 0\} \; , \tag{2.5.5}$$

$$\widetilde{\mathscr{Z}} := \{p \in \mathscr{M} : \det\Big(\mathfrak{g}(K_{(i)}, K_{(j)})\Big)_{i,j=1,\ldots s-1} = 0\}. \tag{2.5.6}$$

In the following result, the proof of which builds on key ideas of Carter [18,19], we let $K_{(0)}$ denote the Killing vector associated to the $\mathbb{R}$ factor of $\mathbb{R} \times \mathbb{T}^{s-1}$, and we let $K_{(i)}$ denote the Killing vector field associated with the $i - th$ S^1 factor of $\mathbb{T}^{s-1}$:

THEOREM 2.5.4 *Let $(\mathscr{M}, \mathfrak{g})$ be an $(n+1)$–dimensional, asymptotically flat, analytic space-time with a metric invariant under an action of the abelian group $G = \mathbb{R} \times \mathbb{T}^{s-1}$ with s–dimensional principal orbits, $1 \le s \le n-1$, and assume that (2.5.4) holds. If $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ is globally hyperbolic, then the function*

$$W := -\det\Big(\mathfrak{g}(K_{(\mu)}, K_{(\nu)})\Big)_{\mu,\nu=0,\ldots,s-1} \tag{2.5.7}$$

is non-negative on $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$, vanishing on $\partial\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle \cup \widetilde{\mathscr{Z}}$.

REMARK 2.5.5 Here analyticity could be avoided if, in the proof below, one could show that one can extract out of the degenerate $\hat{S}_p$'s (if any) a closed embedded hypersurface. Alternatively, the hypothesis of analyticity can be replaced by that of non-existence of non-embedded degenerate prehorizons within $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$. Moreover, one also has:

THEOREM 2.5.6 *Let $n = 3$, $s = 2$ and, under the remaining conditions of Theorem 2.5.4, instead of analyticity assume that $(\mathscr{M}, \mathfrak{g})$ is I^+–regular. Then the conclusion of Theorem 2.5.4 holds.*

Before passing to the proof, some preliminary remarks are in order. The fact that $\mathscr{M} \setminus \mathscr{Z}_{dgt}$ is open, where $\mathscr{Z}_{dgt}$ is as in (2.5.5), together with (2.5.4), establishes the conditions of the Frobenius theorem (see, e.g., [76]). Therefore, for every $p \notin \mathscr{Z}_{dgt}$ there exists a unique, maximal submanifold (not necessarily embedded), passing through p and orthogonal to $\mathrm{Span}\{K_{(0)}, ..., K_{(s-1)}\}$, that we denote by $\mathscr{O}_p$. Carter builds his further analysis of stationary axisymmetric black holes on the sets $\mathscr{O}_p$. This leads to severe difficulties at the set $\widetilde{\mathscr{Z}}$ of (2.5.6), which we were not able to resolve using either Carter's ideas, or those in [131]. There is, fortunately, an alternative which we provide below. In order to continue, some terminology is needed:

DEFINITION 2.5.7 *Let K be a Killing vector and set*

$$\mathscr{N}[K] := \{\mathfrak{g}(K, K) = 0\,,\ K \neq 0\}\,. \tag{2.5.8}$$

Every connected, not necessarily embedded, null hypersurface $\mathscr{N}_0 \subset \mathscr{N}[K]$ to which K is tangent will be called a Killing prehorizon*.*

In this terminology, a Killing horizon is a Killing prehorizon which forms an *embedded* hypersurface which *coincides* with a connected component of $\mathscr{N}[K]$.

The Minkowskian Killing vector $\partial_t - \partial_x$ provides an example where $\mathscr{N}$ is not a hypersurface, with every hyperplane $t + x = \mathrm{const}$ being a prehorizon. The Killing vector $K = \partial_t + Y$ on $\mathbb{R} \times \mathbb{T}^n$, equipped with the flat metric, where $\mathbb{T}^n$ is an n-dimensional torus, and where Y is a unit Killing vector on $\mathbb{T}^n$ with dense orbits, admits prehorizons which are not embedded. This last example is globally hyperbolic, which shows that causality conditions are not sufficient to eliminate this kind of behavior.

Our first step towards the proof of Theorem 2.5.4 will be Theorem 2.5.8, inspired again by some key ideas of Carter, together with their variations by Heusler. We will assume that the $K_{(i)}$'s, $i = 1, \ldots, s-1$, are spacelike (by this we mean that they are spacelike away from their zero sets), but no periodicity or completeness assumptions are made concerning their orbits. This can always be arranged locally, and therefore does not involve any loss of generality for the local aspects of our claim; but we emphasize that our claims are global when the $K_{(i)}$'s are spacelike everywhere.

In our analysis below we will be mainly interested in what happens in $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ where, by Corollary 2.3.8, we have

$$\widetilde{\mathscr{Z}} \cap \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle = \mathscr{Z}_{dgt} \cap \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle \ ,$$

in a chronological domain of outer communications.

We note that $\mathscr{Z}_{dgt} \subset \{W = 0\}$, but equality does not need to hold for Lorentzian metrics. For example, consider in $\mathbb{R}^{1,2}$, $K_{(0)} = \partial_x + \partial_t$ and $K_{(1)} = \partial_y$; then $K_{(0)} \wedge K_{(1)} = dx \wedge dy - dt \wedge dy \not\equiv 0$ and $W \equiv 0$.

If the $K_{(i)}$'s generate a torus action on a stably causal manifold,[13] it is well known that $\widetilde{\mathscr{Z}}$ is a closed, totally geodesic, timelike, stratified, embedded submanifold of $\mathscr{M}$ with codimension of each stratum at least two (this follows from [91] or [5, Appendix C]). So, under those hypotheses, within $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$, we will have

$$\begin{aligned}&\text{the intersection of } \mathscr{Z}_{dgt} \text{ with any null hypersurface } \mathscr{N} \text{ is a}\\ &\text{stratified submanifold of } \mathscr{N}, \text{ with } \mathscr{N}\text{–codimension at least two.}\end{aligned} \tag{2.5.9}$$

This condition will be used in our subsequent analysis. We expect this property not to be needed, but we have not investigated this question any further.

THEOREM 2.5.8 *Let* $(\mathscr{M}, \mathfrak{g})$ *be an* $(n+1)$*–dimensional Lorentzian manifold with* $s \geq 1$ *linearly independent commuting Killing vectors* $K_{(\mu)}$*,* $\mu = 0, \ldots, s-1$*, satisfying the integrability conditions* (2.5.4)*, as well as* (2.5.9)*, with the* $K_{(i)}$*'s,*

[13] Let t be a time-function on $(\mathscr{M}, \mathfrak{g})$; averaging t over the orbits of the torus generated by the $K_{(i)}$'s we obtain a new time function such that the $K_{(i)}$'s are tangent to its level sets. This reduces the problem to the analysis of zeros of Riemannian Killing vectors.

$i = 1, \ldots, s-1$, spacelike. Suppose that $\{W = 0\} \setminus \mathscr{Z}_{dgt}$ is not empty, and for each p in this set consider the Killing vector field l_p defined as[14]

$$l_p = K_{(0)} - (h^{(i)(j)}\mathfrak{g}(K_{(0)}, K_{(i)}))|_p K_{(j)} \,, \tag{2.5.10}$$

where $h^{(i)(j)}$ is the matrix inverse to

$$h_{(i)(j)} := \mathfrak{g}(K_{(i)}, K_{(j)}) \;, \quad i, j \in \{1, ..., s-1\} \;. \tag{2.5.11}$$

Then the distribution $l_p^\perp \subset T\mathscr{M}$ of vectors orthogonal to l_p is integrable over the non-empty set

$$\overline{\{q \in \mathscr{M} \setminus \mathscr{Z}_{dgt} \mid \mathfrak{g}(l_p, l_p)|_q = 0 \;,\; W(q) = 0\}} \setminus \{q \in \mathscr{M} \mid l_p(q) = 0\} \;. \tag{2.5.12}$$

If we define $\hat{S}_p$ to be the maximally extended over $\{W = 0\}$, connected, integral leaf of this distribution[15] *passing through p, then all $\hat{S}_p$'s are Killing prehorizons, totally geodesic in $\mathscr{M} \setminus \{l_p = 0\}$.*

In several situations of interest the $\hat{S}_p$'s form embedded hypersurfaces which coincide with connected components of the set defined in (2.5.12), but this is certainly not known at this stage of the argument:

Remark 2.5.9 Null translations in Minkowski space-time, or in *pp*-wave space-times, show that the $\hat{S}_p$'s might be different from connected components of $\mathscr{N}[l_p]$.

Remark 2.5.10 It follows from our analysis here that for $q \in \hat{S}_p \setminus \mathscr{Z}_{dgt}$ we have $l_q = l_p$. For $q \in \hat{S}_p \cap \mathscr{Z}_{dgt}$ we can define l_q by setting $l_q := l_p$. We then have $l_p = l_q$ for all $q \in \hat{S}_p$.

Proof: Let

$$w := K_{(0)} \wedge \ldots \wedge K_{(s-1)} \;. \tag{2.5.13}$$

We need an equation of Carter [18]:

Lemma 2.5.11 ([18]) *We have*

$$w \wedge dW = (-1)^s W dw \;. \tag{2.5.14}$$

[14] If $s = 1$ then $\widetilde{\mathscr{Z}} = \emptyset$ and $l_p = K_{(0)}$.

[15] To avoid ambiguities, we emphasize that points at which l_p vanishes do not belong to $\hat{S}_p$.

Proof: Let $F = \{W = 0\}$. The result is trivial on the interior $\mathring{F}$ of F, if non-empty. By continuity, it then suffices to prove (2.5.14) on $\mathscr{M} \setminus F$. Let $\mathscr{O}$ be the set of points in $\mathscr{M} \setminus F$ at which the Killing vectors are linearly independent. Consider any point $p \in \mathscr{O}$, and let (x^a, x^A), $a = 0, \ldots, s-1$, be local coordinates near p chosen so that $K_{(a)} = \partial_a$ and $\mathrm{Span}\{\partial_a\} \perp \mathrm{Span}\{\partial_A\}$; this is possible by (2.5.4). Then

$$w = -W dx^0 \wedge \ldots \wedge dx^{s-1} \ ,$$

and (2.5.14) follows near p. Since $\mathscr{O}$ is open and dense, the lemma is proved. $\square$

Returning to the proof of Theorem 2.5.8, as already said, (2.5.4) implies that for every $p \notin \mathscr{Z}_{dgt}$ there exists a unique, maximal, $(n+1-s)$–dimensional submanifold (not necessarily embedded), passing through p and orthogonal to $\mathrm{Span}\{K_{(0)}, \ldots, K_{(s-1)}\}$, that we denote by $\mathscr{O}_p$. By definition,

$$\mathscr{O}_p \cap \mathscr{Z}_{dgt} = \emptyset \ , \tag{2.5.15}$$

and clearly

$$\mathscr{O}_p \cap \mathscr{O}_q \neq \emptyset \quad \Longleftrightarrow \quad \mathscr{O}_p = \mathscr{O}_q \ . \tag{2.5.16}$$

Recall that $p \in \{W = 0\} \setminus \mathscr{Z}_{dgt}$; then $K_{(0)} \wedge \ldots \wedge K_{(s-1)} \neq 0$ in $\mathscr{O}_p$ and we may choose vector fields $u_{(\mu)} \in TM$, $\mu = 0, \ldots, s-1$, such that

$$K_{(0)} \wedge \ldots \wedge K_{(s-1)}(u_{(0)}, \ldots, u_{(s-1)}) = 1$$

in some neighborhood of p. Let γ be a C^k curve, $k \geq 1$, passing through p and contained in $\mathscr{O}_p$. Since $\dot\gamma(s) \in T_{\gamma(s)}\mathscr{O}_p = \mathrm{Span}\{K_{(0)}, \ldots, K_{(s-1)}\}^\perp|_{\gamma(s)}$, after contracting (2.5.14) with $(u_0, \ldots, u_{s-1}, \dot\gamma)$ we obtain the following Cauchy problem

$$\begin{cases} \frac{d}{ds}(W \circ \gamma)(s) \sim W \circ \gamma(s) \ , \\ W|_p = 0 \ . \end{cases} \tag{2.5.17}$$

Uniqueness of solutions of this problem guarantees that $W \circ \gamma(s) \equiv 0$ and therefore W vanishes along the $(n+1-s)$-dimensional submanifold $\mathscr{O}_p$. Since G preserves W, W must vanish on the sets

$$S_p := G_s \cdot \mathscr{O}_p \ . \tag{2.5.18}$$

Here $G_s\cdot$ denotes the motion of a set using the group generated by the $K_{(i)}$'s, $i = 1, \ldots, s-1$; if the orbits of some of the $K_{(i)}$'s are not complete, by this we mean "the motion along the orbits of all linear combinations of the $K_{(i)}$'s starting in the given set, as far as those orbits exist". Since $T_q\mathscr{O}_p$ is orthogonal to all Killing vectors by definition, and the $K_{(i)}$'s are spacelike, the $K_{(i)}$'s are transverse to $\mathscr{O}_p$, so that the S_p's are smooth (not necessarily embedded) submanifolds of codimension one.

On $\{W=0\}\setminus\mathscr{Z}_{dgt}$ the metric $\mathfrak{g}$ restricted to $\mathrm{Span}\{K_{(0)},\ldots,K_{(s-1)}\}$ is degenerate, so that $\mathrm{Span}\{K_{(0)},\ldots,K_{(s-1)}\}$ is a null subspace of $T\mathscr{M}$. It follows that for $q\in\{W=0\}\setminus\mathscr{Z}_{dgt}$ some linear combination of Killing vectors is null and orthogonal to $\mathrm{Span}\{K_{(0)},\ldots,K_{(s-1)}\}$, thus in $T_q\mathscr{O}_p$. So for $q\in\{W=0\}\setminus\mathscr{Z}_{dgt}$ the tangent spaces T_qS_p are orthogonal sums of the null spaces $T_q\mathscr{O}_p$ and the spacelike ones $\mathrm{Span}\{K_{(1)},\ldots,K_{(s-1)}\}$. We conclude that the S_p's form smooth, null, not necessarily embedded, hypersurfaces, with

$$S_p = G\cdot\mathscr{O}_p \subset \{W=0\}\setminus\mathscr{Z}_{dgt}\ , \tag{2.5.19}$$

where the action of G is understood as explained after (2.5.18).

Let the vector $\ell=\Omega^{(\mu)}K_{(\mu)}$, $\Omega^{(\mu)}\in\mathbb{R}$ be tangent to the null generators of S_p, thus

$$\Omega^{(\mu)}\mathfrak{g}(K_{(\mu)},K_{(\nu)})\Omega^{(\nu)}=0\ . \tag{2.5.20}$$

Since $\det(\mathfrak{g}(K_{(\mu)},K_{(\nu)}))=0$ with one-dimensional null space on $\{W=0\}\setminus\mathscr{Z}_{dgt}$, (2.5.20) is equivalent there to

$$\mathfrak{g}(K_{(\mu)},K_{(\nu)})\Omega^{(\nu)}=0\ . \tag{2.5.21}$$

Since the $K_{(i)}$'s are spacelike we must have $\Omega^{(0)}\neq 0$, and it is convenient to normalize ℓ so that $\Omega^{(0)}=1$. Assuming $p\notin\widetilde{\mathscr{Z}}$, from (2.5.21) one then immediately finds

$$\ell = K_{(0)}+\Omega^{(i)}K_{(i)} = K_{(0)} - h^{(i)(j)}\mathfrak{g}(K_{(0)},K_{(j)})K_{(i)}\ , \tag{2.5.22}$$

where $h^{(i)(j)}$ is the matrix inverse to

$$h_{(i)(j)}=\mathfrak{g}(K_{(i)},K_{(j)})\ ,\ i,j\in\{1,\ldots,s-1\}\ . \tag{2.5.23}$$

To continue, we show that:

Proposition 2.5.12 *For each $j=1,\ldots,n$, the function*

$$S_p\ni q\mapsto \Omega^{(j)}(q) := -h^{(i)(j)}(q)\mathfrak{g}(K_{(0)},K_{(i)})(q)$$

is constant over S_p.

PROOF: The calculations here are inspired by, and generalize those of [74, pp. 93-94]. As is well known,

$$dh^{(i)(j)} = -h^{(i)(m)}h^{(j)(s)}dh_{(m)(s)} \, . \tag{2.5.24}$$

From (3.2.2)-(3.2.3), in the next chapter, together with $\mathscr{L}_{K_{(i)}}K_{(j)} = 0$ we have

$$\begin{aligned} dh_{(i)(j)} &= d[\mathfrak{g}(K_{(i)}, K_{(j)})] = di_{K_{(i)}}K_{(j)} = -i_{K_{(i)}}dK_{(j)} \\ &= -i_{K_{(i)}}(-1)^{2(n+1-2)-1} * *dK_{(j)} = (-1)^n * (K_{(i)} \wedge *dK_{(j)}) \, , \end{aligned}$$

with a similar formula for $d[\mathfrak{g}(K_{(0)}, K_{(j)})]$. Next,

$$\begin{aligned} d\Omega^{(i)} &= d(-h^{(i)(j)}\mathfrak{g}(K_{(0)}, K_{(j)})) \\ &= -[\mathfrak{g}(K_{(0)}, K_{(j)})dh^{(i)(j)} + h^{(i)(j)}d[\mathfrak{g}(K_{(0)}, K_{(j)})]] \\ &= -[-\mathfrak{g}(K_{(0)}, K_{(j)})h^{(i)(m)}h^{(j)(s)}dh_{(s)(m)} + h^{(i)(m)}d[\mathfrak{g}(K_{(0)}, K_{(m)})]] \\ &= -h^{(i)(m)}[-(-1)^n\mathfrak{g}(K_{(0)}, K_{(j)})h^{(j)(s)} * (K_{(s)} \wedge *dK_{(m)}) \\ &+ (-1)^n * (K_{(0)} \wedge *dK_{(m)})] \\ &= (-1)^{n+1}h^{(i)(m)} * [(\Omega^{(s)}K_{(s)} + K_{(0)}) \wedge *dK_{(m)}] \\ &= (-1)^{n+1}h^{(i)(m)} * (\ell \wedge *dK_{(m)}) \, , \end{aligned}$$

and

$$\begin{aligned} i_{K_{(0)}} \dots i_{K_{(s-1)}} * d\Omega^{(i)} &= (-1)^{n+1}i_{K_{(0)}} \dots i_{K_{(s-1)}}h^{(i)(m)} * *(\ell \wedge *dK_{(m)}) \\ &= h^{(i)(m)}i_{K_{(0)}} \dots i_{K_{(s-1)}}(\ell \wedge *dK_{(m)}) \, . \end{aligned}$$

Since $i_{K_{(i)}}\ell|_{S_p} = \mathfrak{g}(\ell, K_{(i)})|_{S_p} = 0$, we obtain

$$\begin{aligned} i_{K_{(0)}} \dots i_{K_{(s-1)}}(\ell \wedge *dK_{(m)})|_{S_p} &= i_{K_{(0)}} \dots i_{K_{(s-2)}}[i_{K_{(s-1)}}\ell \wedge *dK_{(m)} \\ &+ (-1)^1\ell \wedge i_{K_{(s-1)}} * dK_{(m)}]|_{S_p} \\ &= -i_{K_{(0)}} \dots i_{K_{(s-2)}}(\ell \wedge i_{K_{(s-1)}} * dK_{(m)})|_{S_p} = \dots \\ &= (-1)^s\ell \wedge i_{K_{(0)}} \dots i_{K_{(s-1)}} * dK_{(m)} \,|_{S_p} \\ &= (-1)^s\ell \wedge *(dK_{(m)} \wedge K_{(s-1)} \wedge \dots \wedge K_{(0)})|_{S_p} \\ &\overset{(2.5.3)}{=} 0 \, , \end{aligned}$$

and therefore

$$i_{K_{(0)}} \dots i_{K_{(s-1)}} * d\Omega^{(i)}|_{S_p} = 0 \, . \tag{2.5.25}$$

This last result says that $d\Omega^{(i)}|_{S_p}$ is a linear combination of the $K_{(\mu)}$'s, so for each i there exist numbers $\alpha^{(\mu)} \in \mathbb{R}$ such that

$$d\Omega^{(i)}|_{S_p} = \alpha^{(\mu)} K_{(\mu)}. \tag{2.5.26}$$

Now, the $\Omega^{(i)}$'s are clearly invariant under the action of the group generated by the $K_{(\mu)}$'s, which implies

$$0 = i_{K_{(\mu)}} d\Omega^{(i)} = \mathfrak{g}(K_{(\mu)}, \alpha^{(\nu)} K_{(\nu)}) \ .$$

This shows that $\alpha^{(\mu)} K_{(\mu)}$ is orthogonal to all Killing vectors, so it must be proportional to ℓ. Since $T_q S_p = \ell^\perp$, we are done. □

Returning to the proof of Theorem 2.5.8, we have shown so far that S_p is a null hypersurface in $\{W = 0\} \setminus \mathscr{Z}_{dgt}$, with the Killing vector $l_p := \ell$ as in (2.5.10) tangent to the generators of S_p. In other words, S_p is a prehorizon. Furthermore,

$$\begin{aligned} T_q\mathscr{M} \ni Y \in T_q S_p \ \text{ for some } p \quad &\Longleftrightarrow \\ W(q) = 0 \ , \ K_{(0)} \wedge \ldots \wedge K_{(s-1)}|_q \neq 0 \ , &\quad Y \perp l_p \ . \end{aligned} \tag{2.5.27}$$

For further purposes it is necessary to extend this result to the hypersurface $\hat S_p$ defined in the statement of Theorem 2.5.8. This proceeds as follows:

It is well known [63] that Killing horizons are *locally totally geodesic*, by which we mean that geodesics initially tangent to the horizon remain on the horizon for some open interval of parameters. This remains true for prehorizons:

COROLLARY 2.5.13 *S_p is locally totally geodesic. Furthermore, if $\gamma : [0,1) \to S_p$ is a geodesic such that $\gamma(1) \notin S_p$, then $\gamma(1) \in \mathscr{Z}_{dgt}$.*

PROOF: Let $\gamma : I \to \mathscr{M}$ be an affinely-parameterized geodesic satisfying $\gamma(0) = q \in S_p$ and $\dot\gamma(0) \in T_q S_p \Longleftrightarrow \mathfrak{g}(\dot\gamma(0), l_p) = 0$. Then

$$\frac{d}{dt}\mathfrak{g}(\dot\gamma(t), l_p) = \mathfrak{g}(\nabla_{\dot\gamma(t)}\dot\gamma(t), l_p) + \mathfrak{g}(\dot\gamma(t), \nabla_{\dot\gamma(t)} l_p) = 0 \ , \tag{2.5.28}$$

where the first term vanishes because γ is an affinely parameterized geodesic, while the second is zero by the Killing equation. Since $\mathfrak{g}(\dot\gamma(0), l_p) = 0$, we get

$$\mathfrak{g}(\dot\gamma(t), l_p) = 0 \ , \ \ \forall t \in I \ . \tag{2.5.29}$$

We conclude that $\dot\gamma$ remains perpendicular to l_p, hence remains within S_p as long as a zero of $K_{(0)} \wedge \ldots \wedge K_{(s-1)}$ is not reached, compare (2.5.27).

Consider, now, the following set of points which can be reached by geodesics initially tangent to S_p:

$$\begin{aligned} \tilde S_p \;\; := \;\; & \{q : \; \exists \text{ a geodesic segment } \gamma : [0,1] \to \mathscr{M} \text{ such} \\ & \text{that } \gamma(1) = q \text{ and } \gamma(s) \in S_p \text{ for } s \in [0,1)\} \setminus \{q : l_p(q) = 0\} \; . \end{aligned} \tag{2.5.30}$$

Then $S_p \subset \tilde S_p$, and if $q \in \tilde S_p \setminus S_p$ then $q \in \mathscr{Z}_{dgt}$ by Corollary 2.5.13. We wish to show that $\tilde S_p$ is a smooth hypersurface, included and maximally extended in the set (2.5.12); equivalently

$$\tilde S_p = \hat S_p \; . \tag{2.5.31}$$

For this, let $q \in \tilde S_p$, let $\mathscr{O}$ be a geodesically convex neighborhood of q not containing zeros of l_p, and for $r \in \mathscr{O}$ define

$$R_r = \exp_{\mathscr{O},r}(l_q(r)^\perp) \; , \tag{2.5.32}$$

here $\exp_{\mathscr{O},r}$ is the exponential map at the point $r \in \mathscr{O}$ in the space-time $(\mathscr{O}, \mathfrak{g}|_{\mathscr{O}})$. It is convenient to require that $\mathscr{O}$ is included within the radius of injectivity of all its points (see [92, Theorem 8.7]). Let γ be as in the definition of $\tilde S_p$. Without loss of generality we can assume that $\gamma(0) \in \mathscr{O}$. We have $\dot\gamma(s) \perp l_p$ for all $s \in [0,1)$, and by continuity also at $s = 1$. This shows that $\gamma([0,1]) \subset R_q$.

Now, $R_{\gamma(0)}$ is a smooth hypersurface in $\mathscr{O}$. It coincides with S_p near $\gamma(0)$, and every null geodesic starting at $\gamma(0)$ and normal to l_p there belongs both to $R_{\gamma(0)}$ and S_p until a point in $\mathscr{Z}_{dgt}$ is reached. This shows that $R_{\gamma(0)}$ is null near every such geodesic until, and including, the first point on that geodesic at which $\mathscr{Z}_{dgt}$ is reached (if any). By (2.5.9) $R_{\gamma(0)} \cap S_p$ is open and dense in $R_{\gamma(0)}$. Thus the tangent space to $R_{\gamma(0)}$ coincides with $l_p^\perp$ at the open dense set of points $R_{\gamma(0)} \cap S_p$, with that intersection being a null, locally totally geodesic (not necessarily embedded) hypersurface. By continuity $R_{\gamma(0)}$ is a subset of (2.5.12), with $TR_{\gamma(0)} = l_p^\perp$ everywhere. Since $R_{\gamma(0)} \subset \tilde S_p$, Equation (2.5.31) follows.

The construction of the $\tilde S_p$'s shows that every integral manifold of the distribution $l_p^\perp$ over the set

$$\Omega := \{q \in \mathscr{M} \setminus \mathscr{Z}_{dgt} \mid \mathfrak{g}(l_p, l_p)|_q = 0 \; , \; W(q) = 0\} \; , \tag{2.5.33}$$

can be extended to a maximal leaf contained in $\overline{\Omega} \setminus \{q \mid l_p(q) = 0\}$, compare (2.5.12). To finish the proof of Theorem 2.5.8 it thus remains to show that there exists a leaf through every point in $\overline{\Omega} \setminus \{q \mid l_p(q) = 0\}$. Since this last set is contained in the closure of Ω, we need to analyze what happens when a sequence of null leaves $\hat{S}_{p_n}$, all normal to a fixed Killing vector field l_q, has an accumulation point. We show in Lemma 2.5.14 below that such sequences accumulate to an integral leaf through the limit point, which completes the proof of the theorem. $\Box$

We shall say that S is an *accumulation set* of a sequence of sets S_n if S is the collection of limits, as i tends to infinity, of sequences $q_{n_i} \in S_{n_i}$.

LEMMA 2.5.14 *Let $\hat{S}_{p_n}$ be a sequence of leaves such that $l_{p_n} = l_q$, for some fixed q, and suppose that $p_n \to p$. If $l_q(p) \neq 0$, then p belongs to a leaf $\hat{S}_p$ with $l_p = l_q$. Furthermore there exists a neighborhood $\mathscr{U}$ of p such that $exp_{\mathscr{U},p}(l_q(p)^\perp) \subseteq \hat{S}_p \cap \mathscr{U}$ is the accumulation set of the sequence $exp_{\mathscr{U},p_n}(l_q(p_n)^\perp) \subseteq \hat{S}_{p_n} \cap \mathscr{U}$, $n \in \mathbb{N}$.*

PROOF: Let $\mathscr{U}$ be a small, open, conditionally compact, geodesically convex neighborhood of p which does not contain zeros of l_q. Let $\hat{S}_{p_n}$ be that leaf, within $\mathscr{U}$, of the distribution $l_q^\perp$ which contains p_n. The $\hat{S}_{p_n}$'s are totally geodesic submanifolds of $\mathscr{U}$ by Corollary 2.5.15, and therefore are uniquely determined by prescribing $T_{p_n}\hat{S}_{p_n}$. Now, the subspaces $T_{p_n}\hat{S}_{p_n} = l_q(p_n)^\perp$ obviously converge to $l_q(p)^\perp$ in the sense of accumulation sets. Smooth dependence of geodesics upon initial values implies that $\exp_{\mathscr{U},p_n}(l_q(p_n)^\perp)$ converges in C^k, for any k, to $\exp_{\mathscr{U},p}(l_q(p)^\perp)$. Since W vanishes on $\exp_{\mathscr{U},p_n}(l_q(p_n)^\perp)$, we obtain that W vanishes on $\exp_{\mathscr{U},p}(l_q(p)^\perp)$. Since $T_{q_n}\exp_{\mathscr{U},p_n}(l_q(p_n)^\perp) = l_p^\perp(q_n)$ for any $q_n \in \exp_{\mathscr{U},p_n}(l_q(p_n)^\perp)$ we conclude that $T_r \exp_{\mathscr{U},p}(l_q(p)^\perp) = l_p^\perp(r)$ for any $r \in \exp_{\mathscr{U},p}(l_q(p)^\perp)$. So $\exp_{\mathscr{U},p}(l_q(p)^\perp)$ is a leaf, within $\mathscr{U}$, through p of the distribution $l_q^\perp$ over the set (2.5.12), and $\exp_{\mathscr{U},p}(l_q(p)^\perp) \subseteq \hat{S}_p \cap \mathscr{U}$ is the accumulation set of the totally geodesic submanifolds $\hat{S}_{p_n} \cap \mathscr{U}$'s.

The remainder of the proof of Theorem 2.5.4 consists in showing that the $\hat{S}_p$'s cannot intersect $\langle\langle \mathscr{M}_{\mathrm{ext}} \rangle\rangle$. We start with an equivalent of Corollary 2.5.13, with identical proof:

COROLLARY 2.5.15 *$\hat{S}_p$ is locally totally geodesic. Furthermore, if $\gamma : [0, 1) \to \hat{S}_p$ is a geodesic segment such that $\gamma(1) \notin \hat{S}_p$, then l_p vanishes at $\gamma(1)$.* $\square$

Corollary 2.3.8 shows that Killing vectors as described there have no zeros in $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$, and Corollary 2.5.15 implies now:

COROLLARY 2.5.16 *$\hat{S}_p \cap \langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ is totally geodesic in $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ (possibly empty).* $\square$

To continue, we want to extract, out of the $\hat{S}_p$'s, a closed, embedded, Killing horizon S_0^+. Now, e.g. the analysis in [80] shows that the gradient of $\mathfrak{g}(l_p, l_p)$ is either everywhere zero on $\hat{S}_p$ (we then say that $\hat{S}_p$ is degenerate), or nowhere vanishing there. One immediately concludes that non-degenerate $\hat{S}_p$'s, if non-empty, are embedded, closed hypersurfaces in $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$. Then, if there exists non-empty non-degenerate $\hat{S}_p$'s, we choose one and we set

$$S_0^+ = \hat{S}_p \ . \tag{2.5.34}$$

Otherwise, all non-empty $\hat{S}_p$'s are degenerate; to show that such prehorizons, if non-empty, are embedded, we will invoke analyticity (which has not been used so far). So, consider a degenerate component $\hat{S}_p$, and note that $\hat{S}_p$ does not self-intersect, being a subset of the union of integral manifolds of a smooth distribution of hyperplanes. Suppose that $\hat{S}_p$ is not embedded. Then there exists a point $q \in \hat{S}_p$, a conditionally compact neighborhood $\mathscr{O}$ of q, and a sequence of points $p_n \in \hat{S}_p$ lying on pairwise disjoint components of $\mathscr{O} \cap \hat{S}_p$, with p_n converging to q. Now, Killing vectors are solutions of the overdetermined set of PDEs

$$\nabla_\mu \nabla_\nu X_\rho = R^\alpha{}_{\mu\nu\rho} X_\alpha \ ,$$

which imply that they are analytic if the metric is. So $\mathfrak{g}(l_p, l_p)$ is an analytic function that vanishes on an accumulating family of hypersurfaces. Consequently $\mathfrak{g}(l_p, l_p)$ vanishes everywhere, which is not compatible with asymptotic flatness. Hence the $\hat{S}_p$'s are embedded, coinciding with connected components of the set $\{\mathfrak{g}(l_p, l_p) = 0 = W\} \setminus \{l_p = 0\}$; it should be clear now that they are closed in $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$. We define S_0^+ again using (2.5.34), choosing one non-empty $\hat{S}_p$,

We can finish the proof of Theorem 2.5.4. Suppose that W changes sign within $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$. Then S_0^+ is a non-empty, closed, connected, embedded null hypersurface within $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$. Now, any embedded null hypersurface S_0^+ is locally two-sided, and we can assign an intersection number one to every intersection point of S_0^+ with a curve that crosses S_0^+ from its local past to its local future, and minus one for the remaining ones (this coincides with the oriented intersection number as in [66, Chapter 3]). Let $p \in S_0^+$, there exists a smooth timelike future directed curve γ_1 from some point $q \in \mathscr{M}_{\rm ext}$ to p. By definition there also exists a future directed null geodesic segment γ_2 from p to some point $r \in \mathscr{M}_{\rm ext}$. Since $\mathscr{M}_{\rm ext}$ is connected there exists a curve $\gamma_3 \subset \mathscr{M}_{\rm ext}$ (which, in fact, cannot be causal future directed, but this is irrelevant for our purposes) from r to q. Then the path γ obtained by following γ_1, then γ_2, and then γ_3 is closed. Since S_0^+ does not extend into $\mathscr{M}_{\rm ext}$, γ intersects S_0^+ only along its timelike future directed part, where every intersection has intersection number one, and γ intersects S_0^+ at least once at p, hence the intersection number of γ with S_0^+ is strictly positive. Now, Corollary 2.2.4 shows that $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is simply connected. But, by standard intersection theory [66, Chapter 3], the intersection number of a closed curve with a closed, externally orientable, embedded hypersurface in a simply connected manifold vanishes, which gives a contradiction and proves that W cannot change sign on $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$.

It remains to show that W vanishes at the boundary of $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$. For this, note that, by definition of W, in the region $\{W > 0\}$ the subspace of $T\mathscr{M}$ spanned by the Killing vectors $K_{(\mu)}$ is timelike. Hence at every p such that $W(p) > 0$ there exist vectors of the form $K_{(0)} + \sum \alpha_i K_{(i)}$ which are timelike. But $\partial\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle \subset \dot I^-(\mathscr{M}_{\rm ext}) \cup \dot I^+(\mathscr{M}_{\rm ext})$, and each of the boundaries $\dot I^-(\mathscr{M}_{\rm ext})$ and $\dot I^+(\mathscr{M}_{\rm ext})$ is invariant under the flow of any linear combination of $K_{(\mu)}$'s, and each is achronal, hence $W \le 0$ on $\partial\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$, whence the result. □

In view of what has been said, the reader will conclude:

Corollary 2.5.17 (Killing horizon theorem) *Under the conditions of Theorem 2.5.4, away from the set $\mathscr{Z}_{dgt}$ as defined in* (2.5.5)*, the boundary $\overline{\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle} \setminus \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is a union of embedded Killing horizons.* □

Let us pass now to the

Proof of Theorem 2.5.6: Let

$$\pi : \langle\langle \mathscr{M}_{\rm ext} \rangle\rangle \cup \mathscr{E}^+ \to \underbrace{\Big(\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle \cup \mathscr{E}^+\Big)/\Big(\mathbb{R}\times \mathrm{U}(1)\Big)}_{=:\mathscr{Q}}$$

denote the quotient map. As discussed in more detail in Sections 2.6.1 and 2.6.2 (keeping in mind that, by topological censorship, $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ has only one asymptotically flat end), the orbit space $\mathscr{Q}$ is diffeomorphic to the half-plane $\{(x,y) \mid x \geq 0\}$ from which a finite number $\mathring{n} \geq 0$ of open half-discs, centred at the axis $\{x=0\}$, have been removed. As explained at the beginning of Section 2.7, the case $\mathring{n}=0$ leads to Minkowski space-time, in which case the result is clear, so from now on we assume $\mathring{n} \geq 1$.

Suppose that $\{W=0\} \cap \langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ is non-empty. Let p_0 be an element of this set, with corresponding Killing vector field $l_0 := l_{p_0}$. Let W_0 be the norm squared of l_0:

$$W_0 := \mathfrak{g}(l_0, l_0)\ .$$

In the remainder of the proof of Theorem 2.5.2 we consider only those $\hat{S}_p$'s for which $l_p = l_0$:

$$\hat{S}_p \subset \{W=0\} \cap \{W_0 = 0\}\ .$$

We denote by $C_{\pi(p)}$ the image in $\mathscr{Q}$, under the projection map π, of $\hat{S}_p \cap (\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle \cup \mathscr{E}^+)$. Define

$$\mathring{\mathscr{Q}} = \langle\langle \mathscr{M}_{\rm ext} \rangle\rangle/\Big(\mathbb{R}\times \mathrm{U}(1)\Big)\ ,$$

$$\mathscr{W}_0^\flat := \Big(\{W_0=0\} \cap \{W=0\} \cap (\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle \cup \mathscr{E}^+)\Big)/\Big(\mathbb{R}\times \mathrm{U}(1)\Big)\ ,$$

Then $\mathscr{W}_0^\flat$ is a closed subset of $\mathscr{Q}$, with the following property: through every point q of $\mathscr{W}_0^\flat$ there exists a smooth maximally extended curve C_q, which will be called *orbit*, entirely contained in $\mathscr{W}_0^\flat$. The C_q's are pairwise disjoint, or coincide. Their union forms a closed set, and locally they look like a subcollection of leaves of a foliation. (Such structures are called laminations; see, e.g., [59].)

An orbit will be called a *Jordan orbit* if C_q forms a Jordan curve.

We need to consider several possibilities; we start with the simplest one:

Case I: If an orbit C_q forms a Jordan curve entirely contained in $\mathring{\mathscr{Q}}$, then the corresponding $\hat{S}_p = \pi^{-1}(C_q)$ forms a closed embedded hypersurface in $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$, and a contradiction arises as at the end of the proof of Theorem 2.5.4.

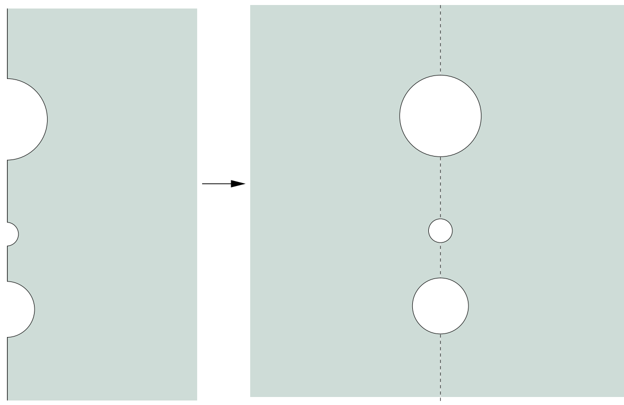

Figure 2.5.1: The quotient space $\mathcal{Q}$ and its double $\widehat{\mathcal{Q}}$.

Case II: Consider, next, an orbit C_q which meets the boundary of $\mathcal{Q}$ at two or more points which belong to $\pi(\mathscr{A})$, and only at such points. Let $I_q \subset C_q$ denote that part of C_q which connects any two subsequent such points, in the sense that I_q meets $\partial\mathcal{Q}$ at its end points only. Now, every $\hat{S}_p$ is a smooth hypersurface in $\mathscr{M}$ invariant under $\mathbb{R} \times \mathrm{U}(1)$, and therefore meets the rotation axis $\mathscr{A}$ orthogonally. This implies that $\pi^{-1}(I_q)$ is a closed, smooth, embedded hypersurface in $\langle\langle \mathscr{M}_{\mathrm{ext}} \rangle\rangle$, providing again a contradiction.

To handle the remaining cases, some preliminary work is needed. It is convenient to double $\mathcal{Q}$ across $\{x = 0\}$ to obtain a manifold $\widehat{\mathcal{Q}}$ diffeomorphic to $\mathbb{R}^2$ from which a finite number of open discs, centered at the axis $\{x = 0\}$, have been removed, see Figure 2.5.1. Connected components of the event horizon $\mathscr{E}^+$ correspond to smooth circles forming the boundary of $\widehat{\mathcal{Q}}$, regardless of whether or not they are degenerate. From what has been said, every C_q which has an end point at $\pi(\mathscr{A})$ is smoothly extended in $\widehat{\mathcal{Q}}$ across $\{x = 0\}$ by its image under the map $(x, y) \mapsto (-x, y)$. We will continue to denote by C_q the orbits so extended in $\widehat{\mathcal{Q}}$.

The analysis of Cases I and II also shows:

Lemma 2.5.18 *An orbit C_q which does not meet $\partial\widehat{\mathcal{Q}}$ can cross the axis $\{x = 0\}$ at most once.* □

An orbit C_q will be called an *accumulation orbit* of an orbit C_r if there exists a sequence $q_n \in C_r$ such that $q_n \to q$. Every orbit is its own accumulation orbit. It is a simple consequence of the accumulation Lemma 2.5.14 that:

Lemma 2.5.19 *Let C_q be an accumulation orbit of C_r. Then for every $p \in C_q$ there exists a sequence $p_n \in C_r$ such that $p_n \to p$.* $\square$

We will need the following:

Lemma 2.5.20 *Let $r_n \in C_r$ be a sequence accumulating at $p \in \pi(\mathscr{A}) \setminus \partial\widehat{\mathcal{Q}}$. Then $p \in C_r$, and C_r continues smoothly across $\{x = 0\}$ at p.*

Proof: By Lemma 2.5.14 there exists an orbit C_p crossing the axis $\{x = 0\}$ transversally at p. Lemma 2.5.19 shows that C_r crosses the axis. But, by Lemma 2.5.18, C_r can cross the axis only once. It follows that $C_r = C_p$ and that $p \in C_r$. $\square$

Abusing notation, we still denote by W and W_0 the functions $W \circ \pi$ and $W_0 \circ \pi$. If W and W_0 vanish at a point lying at the boundary $\partial\widehat{\mathcal{Q}}$, then the corresponding circle forms a Jordan orbit. We have:

Lemma 2.5.21 *The only orbits accumulating at $\partial\widehat{\mathcal{Q}}$ are the boundary circles.*

Proof: Suppose that $r_n \in C_q$ accumulates at $p \in \partial\widehat{\mathcal{Q}}$. Then, by continuity, $W(p) = W_0(p) = 0$, which implies that the boundary component through p is a Jordan orbit. But it follows from Lemma 2.5.19 that any orbit accumulating at $\partial\widehat{\mathcal{Q}}$ has to cross the axis more than once, and the result follows from Lemma 2.5.18. $\square$

The remaining possibilities will be excluded by a lamination version of the Poincaré-Bendixson theorem. We will make use of a smooth transverse orientation of all the $\hat{S}_p$'s; such a structure is not available for a general lamination, but exists in the problem at hand. More precisely, we will endow $\langle\langle \mathscr{M}_{\mathrm{ext}} \rangle\rangle \cup \mathscr{E}^+$ with a smooth vector field Z transverse to all $\hat{S}_p$'s. The construction proceeds as follows: Choose any decomposition of $\langle\langle \mathscr{M}_{\mathrm{ext}} \rangle\rangle \cup \mathscr{E}^+$ as $\mathbb{R} \times \overline{\mathring{\mathscr{S}}}$, as in Theorem 2.4.5: thus each level set $\overline{\mathring{\mathscr{S}}_t}$ of the time function t is transverse to the stationary Killing vector field K_0, with the periodic Killing vector K_1 tangent to $\overline{\mathring{\mathscr{S}}_t}$. Let $q \in \hat{S}_p \cap \mathring{\mathscr{S}}_0$; as the null leaf $\hat{S}_p$ is transversal to $\mathring{\mathscr{S}}_0$, the intersection $\mathring{\mathscr{S}}_0 \cap \hat{S}_p$ is a hypersurface in $\mathscr{M}$ of co-dimension two. There exist precisely two null directions at q which are normal to $\mathring{\mathscr{S}}_0 \cap \hat{S}_p$, one of them is spanned by $l_0(q)$; we denote by $\mathring{Z}_q$ the unique future directed null vector spanning the other direction and satisfying $\mathring{Z}_q = T_q + \tilde{Z}_q$, where T_q is the unit timelike future directed normal to $\mathring{\mathscr{S}}_0$ at q, and $\tilde{Z}_q$ is tangent to $\mathring{\mathscr{S}}$.

The above definition of $\tilde{Z}_q$ extends by continuity to $q \in \hat{S}_p \cap \overline{\mathring{\mathscr{S}}_0}$.

Transversality and smoothness of l_0 imply that there exists a neighborhood $\mathscr{O}_q$ of q and an extension $\hat{Z}_q$ of $\tilde{Z}_q$ to $\mathscr{O}_q$ with the property that $\hat{Z}_q(r)$ is transverse to $\hat{S}_r$ for every $r \in \mathscr{O}_q$ satisfying $W_0(r) = W(r) = 0$. The neighborhood $\mathscr{O}_q$ can, and will, be chosen to be invariant under $\mathbb{R} \times \mathrm{U}(1)$; similarly for $\hat{Z}_q(r)$.

Consider the covering of $\overline{\mathring{\mathscr{S}}_0} \cap \{W_0 = 0\} \cap \{W = 0\}$ by sets of the form $\mathscr{O}_q \cap \overline{\mathring{\mathscr{S}}_0}$. Asymptotic flatness implies that $\overline{\mathring{\mathscr{S}}_0} \cap \{W_0 = 0\} \cap \{W = 0\}$ is compact, which in turn implies that a finite subcovering $\mathscr{O}_i := \mathscr{O}_{q_i}$ can be chosen. Let φ_i be a partition of unity subordinated to the covering of $\langle\langle \mathscr{M}_{\mathrm{ext}} \rangle\rangle \cup \mathscr{E}^+$ by the $\mathscr{O}_i$'s together with

$$\mathscr{O}_0 := \Big(\langle\langle \mathscr{M}_{\mathrm{ext}} \rangle\rangle \cup \mathscr{E}^+ \Big) \setminus \Big(\{W = 0\} \cap \{W_0 = 0\} \Big) \,.$$

The φ_i's can, and will, be chosen to be $\mathbb{R} \times \mathrm{U}(1)$–invariant. Set

$$Z := \sum_{i \geq 1} \varphi_i \hat{Z}_{q_i} \,.$$

Then Z is smooth, tangent to $\mathring{\mathscr{S}}_0$, and transverse to all $\hat{S}_p$'s.

Choose an orientation of $\widehat{\mathcal{Q}}$. The vector field Z projects under π to a vector field $Z^\flat$ on $\widehat{\mathcal{Q}}$ transverse to each C_q. For each $r \in C_q$ we define a vector $V_q(r)$ by requiring $V_q(r)$ to be tangent to C_q at r, with $\{V_q, Z^\flat\}$ positively oriented, and with V_q having length one with respect to some auxiliary Riemannian metric on $\widehat{\mathcal{Q}}$. Then V_q varies smoothly along C_q, and each C_q is in fact a complete integral curve of its own V_q. The vector field V_p along C_p defines an order, and diverging sequences, on C_p in the obvious way: we say that a point $r' \in C_p$ is subsequent to $r \in C_p$ if one flows from r to r' along V_p in the forward direction; a sequence $r_n \in V_p$ is diverging if $r_n = \phi(s_n)(p)$, where $\phi(s)$ is the flow of V_p along C_p, with $s_n \nearrow \infty$ or $s_n \searrow -\infty$.

By Lemma 2.5.14, if a sequence $r_n \in C_{q_n}$ tends to $r \in C_q$, then the tangent spaces TC_{q_n} accumulate on TC_q. This implies that there exist numbers $\epsilon_n \in \{\pm 1\}$ such that $\epsilon_n V_{q_n}(r_n) \to V_q(r)$, and this is the best one can say in general. However, the existence of Z guarantees that $V_{q_n}(r_n) \to V_q(r)$.

We are ready now to pass to the analysis of

Case III: In view of Lemmata 2.5.18 and 2.5.21, it remains to exclude the existence of orbits C_q which are entirely contained within $\widehat{\mathcal{Q}} \setminus \partial\widehat{\mathcal{Q}}$, and which do not intersect $\pi(\mathscr{A})$, or which intersect $\pi(\mathscr{A})$ only once, and which do *not* form Jordan curves in $\mathring{\mathcal{Q}}$. Since $\{W = 0\} \cap \overline{\mathring{\mathscr{S}}_0}$ is compact, there exists $p \in \widehat{\mathcal{Q}}$ and a diverging sequence $q_n \in C_q$ such that $q_n \to p$. Again by Lemmata 2.5.18 and 2.5.21, $p \notin \partial\widehat{\mathcal{Q}}$. The fact that C_p is a closed embedded curve follows now by the standard arguments of the proof of the Poincaré–Bendixson theorem, as e.g. in [78]. The orbit C_p does not meet $\partial\widehat{\mathcal{Q}}$ by Lemma 2.5.21. If C_p met $\pi(\mathscr{A})$, it would have an intersection number with $\{x = 0\}$ equal to one by Lemma 2.5.18, which is impossible for a Jordan curve in the plane. Thus C_p is entirely contained in $\mathring{\mathcal{Q}}$, which has already been shown to be impossible in Case I, and the result is established. □

Similarly to Corollary 2.5.17, we have the following Corollary of Theorem 2.5.6, which is essentially a rewording of Lemma 2.5.21:

Corollary 2.5.22 (Embedded prehorizons theorem) *Under the conditions of Theorem 2.5.2, away from the set $\mathscr{Z}_{dgt}$ as defined in* (2.5.5)*, the boundary $\overline{\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle} \setminus \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is a union of embedded Killing prehorizons.* □

2.5.3 The ergoset in space-time dimension four

The *ergoset* E is defined as the set where the stationary Killing vector field $K_{(0)}$ is spacelike or null:

$$E := \{p \mid \mathfrak{g}(K_{(0)}, K_{(0)})|_p \geq 0\} . \tag{2.5.35}$$

In this section we wish to show that, in vacuum, the ergoset cannot intersect the rotation axis within $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$, if we assume the latter to be chronological.

The first part of the argument is purely local. For this we will assume that the space-time dimension is four, that $K_{(0)} \equiv X$ has no zeros near a point p, that $K_{(1)} \equiv Y$ has 2π–periodic orbits and vanishes at p, and that X and Y commute.

Let $\hat{T}$ be any timelike vector at p, set

$$T := \int_0^{2\pi} \phi_t[Y]_* \hat{T} dt , \tag{2.5.36}$$

then T is invariant under the flow of Y. Hence $T^\perp$ is also invariant under Y. Let $\mathscr{S}_\mathscr{O}$ denote $\exp_{p,\mathscr{O}}(T^\perp)$ (recall (2.5.32)), where $\mathscr{O}$ is any neighborhood of p lying within the injectivity radius of $\exp_p$, sufficiently small so that $\mathscr{S}_{\mathscr{O}}$ is spacelike; note that $\mathscr{S}_{\mathscr{O}}$ is invariant under the flow of Y. A standard argument (see, e.g., [5] Appendix C) shows that Y vanishes on

$$\mathscr{A}_p := \exp_p(\mathrm{Ker}\, \nabla Y) ,$$

and that $\mathscr{A}_p$ is totally geodesic. Note that $T \in \mathrm{Ker}\, \nabla Y$, which implies that $\mathscr{A}_p$ is timelike.

We are interested in the behavior of the area function W near $\mathscr{A}$, the set of points where Y vanishes. We have $\nabla W|_{\mathscr{A}} = 0$ and

$$\begin{aligned} \nabla_\mu \nabla_\nu W|_{\mathscr{A}} &= -\nabla_\mu \nabla_\nu \left(\mathfrak{g}(X,X)\mathfrak{g}(Y,Y) - \mathfrak{g}(X,Y)^2 \right) \\ &= -2\left(\mathfrak{g}(X,X)\mathfrak{g}(\nabla_\mu Y, \nabla_\nu Y) - \mathfrak{g}(X,\nabla_\mu Y)\mathfrak{g}(X,\nabla_\nu Y)\right) . \end{aligned} \tag{2.5.37}$$

The second term vanishes because $[X,Y]=0$, with Y vanishing on $\mathscr{A}$:

$$X^\alpha \nabla_\nu Y_\alpha|_{\mathscr{A}} = -X^\alpha \nabla_\alpha Y_\nu = -X^\alpha \nabla_\alpha Y_\nu + \underbrace{Y^\alpha}_{=0} \nabla_\alpha X_\nu = -[X,Y]_\nu = 0 .$$

Now, the axis $\mathscr{A}$ is timelike, and the only non-vanishing components of the tensor $\nabla_\mu Y_\nu$ have a spacelike character on $\mathscr{A}$. This implies that the quadratic form $\nabla_\mu Y^\alpha \nabla_\nu Y_\alpha$ is semi-positive definite. We have therefore shown

LEMMA 2.5.23 *If X is spacelike at $p \in \mathscr{A}$, then $W < 0$ in a neighborhood of p away from $\mathscr{A}$.*

Under the conditions of Theorem 2.5.1, we conclude that X cannot be spacelike on $\mathscr{A} \cap \langle\langle \mathscr{M}_{\mathrm{ext}} \rangle\rangle$. To exclude the possibility that $g(X,X)=0$ there,[16] let w be defined as in (2.5.13),

$$w = X^\flat \wedge Y^\flat ;$$

here, and throughout this section, we explicitly distinguish between a vector Z and its dual $Z^\flat := \mathfrak{g}(Z,\cdot)$. We will further assume that X is causal at p, and that the conclusion of Lemma 2.5.11 holds:

$$dW \wedge w = W dw . \tag{2.5.38}$$

[16]The analysis in Section 2.6 shows that X cannot become null on $\mathscr{A} \cap \langle\langle \mathscr{M}_{\mathrm{ext}} \rangle\rangle$ when the vacuum equations hold *and* the axis can be identified with a smooth boundary for the metric q; this can be traced to the "boundary point Lemma", which guarantees that the gradient of the harmonic function ρ has no zeros at the boundary $\{\rho = 0\}$. But the behavior of q at those axis points which are not on a non-degenerate horizon and on which X is null is not clear.

Let T denote the field of vectors normal to $\mathscr{S}_{\mathscr{O}}$ normalized so that $\mathfrak{g}(T,X) = 1$; note that T_p is, up to a multiplicative factor, as in (2.5.36). Let γ be any affinely parameterized geodesic such that $\gamma(0) = p$, $\dot\gamma(0) \perp T_p$ and $\dot\gamma(0) \perp X_p$; a calculation as in (2.5.28) shows that

$$\mathfrak{g}(Y,\dot\gamma) = \mathfrak{g}(X,\dot\gamma) = 0$$

along γ. As Y is tangent to $\mathscr{S}_{\mathscr{O}}$, from (2.5.38) we obtain

$$\underbrace{\frac{dW}{ds}\mathfrak{g}(Y,Y)}_{=dW\wedge X^\flat\wedge Y^\flat(\dot\gamma,T,Y)} = W dw(\dot\gamma,T,Y)\ . \tag{2.5.39}$$

Now, $i_Y dw = \mathscr{L}_Y w - d(i_Y w) = -d(i_Y w)$, so that

$$\begin{aligned} dw(\dot\gamma,T,Y) &= -d\Big(i_Y(X^\flat\wedge Y^\flat)\Big)(\dot\gamma,T) \\ &= d\Big(-\mathfrak{g}(Y,X)Y^\flat + \mathfrak{g}(Y,Y)X^\flat\Big)(\dot\gamma,T) \\ &= \Big(-\mathfrak{g}(Y,X)dY^\flat + \mathfrak{g}(Y,Y)dX^\flat\Big)(\dot\gamma,T) + \frac{d(\mathfrak{g}(Y,Y))}{ds}\ . \end{aligned}$$

Inserting this in (2.5.39), we conclude that

$$\frac{d}{ds}\left(\frac{W}{\mathfrak{g}(Y,Y)}\right) = \underbrace{\Big(-\frac{\mathfrak{g}(Y,X)}{\mathfrak{g}(Y,Y)}dY^\flat + dX^\flat\Big)(\dot\gamma,T)}_{=:f}\times\frac{W}{\mathfrak{g}(Y,Y)}\ . \tag{2.5.40}$$

Let h be the metric induced on $\mathscr{S}_{\mathscr{O}}$ by $\mathfrak{g}$. Then h is a Riemannian metric invariant under the flow of Y. As is well known (compare [31]) we have $c^{-1}s^2 \le \mathfrak{g}(Y,Y) = h(Y,Y) \le cs^2$. Since $T \in \mathrm{Ker}\nabla Y$ we have $dY^\flat(T,\cdot) = 0$ at p. It follows that the function f defined in (2.5.40) is bounded along γ near p. If $\mathfrak{g}(X,X) = 0$ at p, then the limit at p of $W/\mathfrak{g}(Y,Y)$ along γ vanishes by (2.5.37). Using uniqueness of solutions of ODE's, it follows from (2.5.40) that W vanishes along γ. But this is not possible in $\langle\langle \mathscr{M}_{\mathrm{ext}}\rangle\rangle$ away from $\mathscr{A}$ by Theorem 2.5.1. We have therefore proved that the ergoset does *not* intersect the axis within $\langle\langle \mathscr{M}_{\mathrm{ext}}\rangle\rangle$:

Theorem 2.5.24 (Ergoset theorem) *In space-time dimension four, and under the conditions of Theorem 2.5.1, $K_{(0)}$ is timelike on $\langle\langle \mathscr{M}_{\mathrm{ext}}\rangle\rangle \cap \mathscr{A}$.* □

A higher dimensional version of Theorem 2.5.24 can be found in [32].

A corollary of Theorem 2.5.24 is that, under the conditions there, the existence of an ergoset intersecting the axis implies that of an event horizon. Here one should keep in mind a similar result of Hajiček [68], under conditions that include the hypothesis of smoothness of ∂E (which does not hold e.g. in Kerr [112]), and affine completeness of those Killing orbits which are geodesics, and non-existence of degenerate Killing horizons. On the other hand, Hajiček assumes the existence of only one Killing vector, while in our work two Killing vectors are required.

2.6 The reduction to a harmonic map problem

2.6.1 The orbit space in space-time dimension four

Let $(\mathscr{M}, \mathfrak{g})$ be a chronological, four-dimensional, asymptotically flat space-time invariant under a $\mathbb{R} \times \mathrm{U}(1)$ action, with stationary Killing vector field $K_{(0)} \equiv X$ and 2π–periodic Killing vector field $K_{(1)} \equiv Y$. Throughout this section we shall assume that

$\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle = \mathbb{R} \times M$, where M is a three dimensional, simply connected manifold with boundary, invariant under the flow of Y, with the flow of X consisting of translations along the $\mathbb{R}$ factor. Moreover the closure $\bar{M}$ of M is the union of a compact set and of a finite number of asymptotically flat ends. (2.6.1)

Recall that (2.6.1) follows from Corollary 2.2.4 and Theorem 2.4.5 under appropriate conditions.

Because X and Y commute, the periodic flow of Y on $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ defines naturally a periodic flow on M; in our context this flow consists of rotations around an axis in the asymptotically flat regions. Now, every asymptotic end can be compactified by adding a point, with the action of $\mathrm{U}(1)$ extending to the compactified manifold by fixing the point at infinity. Similarly every boundary component has to be a sphere [72, Lemma 4.9], which can be filled in by a ball, with the (unique) action of $\mathrm{U}(1)$ on S^2 extending to the interior as the associated rotation of a ball in $\mathbb{R}^3$, reducing the analysis of the group action to the boundaryless case. Existence of asymptotically flat regions, or of boundary spheres, implies that the set of fixed points of the action is non-empty (see, e.g., [11, Proposition 2.4]). Assuming, for notational simplicity, that there is only one asymptotically flat end, it then follows from [115] (see the italicized paragraph on p.52 there) that, after the addition of a ball B_i to every boundary component, and after the addition of a point i_0 at infinity to the asymptotic region, the new manifold $M \cup B_i \cup \{i_0\}$ is homeomorphic to S^3, with the action of U(1) conjugate, by a homeomorphism, to the usual rotations of S^3. On the other hand, it is shown in [110, Theorem 1.10] that the actions are classified, up to smooth conjugation, by topological invariants, so that the action of U(1) is smoothly conjugate to the usual rotations of S^3. It follows that the manifold $M \cup B_i$ is diffeomorphic to $\mathbb{R}^3$, with the U(1) action smoothly conjugate to the usual rotations of $\mathbb{R}^3$. In particular: a) there exists a global cross-section $\mathring{M}^2$ for the action of U(1) on $M \cup B_i$ away from the set of fixed points $\mathscr{A}$,[17] with $\mathring{M}^2$ diffeomorphic to an open half-plane; b) all isotropy groups are trivial or equal to U(1); c) $\mathscr{A}$ is diffeomorphic to $\mathbb{R}$.[18]

Somewhat more generally, the above analysis applies whenever M can be compactified by adding a finite number of points or balls. A nontrivial example is provided by manifolds with a finite number of asymptotically flat and asymptotically cylindrical ends, as is the case for the Cauchy surfaces for the domain of outer communication of the extreme Kerr solution.

Summarizing, under (2.6.1) there exists in $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ an embedded two-dimensional manifold $\bar{M}^2$, diffeomorphic to $\hat{M}^2 \approx [0, \infty) \times \mathbb{R}$ minus a finite number of points (corresponding to the remaining asymptotic ends), and minus a finite number of open half-discs (the boundary of each corresponding to a connected component of the horizon). We denote by M^2 the manifold obtained by removing from $\bar{M}^2$ all its boundaries.

2.6.2 Global coordinates on the orbit space

We turn our attention now to the construction of a convenient coordinate system on a four-dimensional, globally hyperbolic, $\mathbb{R} \times \mathrm{U}(1)$ invariant, simply connected domain of outer communications $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$. Let $\bar{M}^2$ and $\mathring{M}^2$ be as in Section 2.6.1. We will invoke the uniformization theorem to understand the geometry of $\bar{M}^2$; however, some preparatory work is useful, which will allow us to control both the asymptotic behavior of the fields involved, as well as the boundary conditions at various boundaries.

[17] We will use the symbol $\mathscr{A}$ to denote the set of fixed points of the Killing vector Y in M *or* in $\mathscr{M}$, as should be clear from the context.

[18] We are grateful to Allen Hatcher for clarifying comments on the classification of U(1) actions.

For simplicity we assume that $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ contains only one asymptotically flat region, which is necessarily the case under the hypotheses of Theorem 2.2.3. On M^2 there is a naturally defined orbit space-metric which, away from the rotation axis $\{Y=0\}$, is defined as follows. Let us denote by $\mathfrak{g}$ the metric on space-time, let $X_1 = X$, $X_2 = Y$, set $h_{ij} = \mathfrak{g}(X_i, X_j)$, let h^{ij} denote the matrix inverse to h_{ij} wherever defined, and on that last set for $Z_1, Z_2 \in T_p\mathring{M}^2$ set

$$q(Z_1, Z_2) = \mathfrak{g}(Z_1, Z_2) - h^{ij}\mathfrak{g}(Z_1, X_i)\mathfrak{g}(Z_2, X_j) \ . \tag{2.6.2}$$

Note that if Z_1 and Z_2 are orthogonal to the Killing vectors, then $q(Z_1, Z_2) = \mathfrak{g}(Z_1.Z_2)$. This implies that if the linear span of the Killing vectors is timelike (which, under our hypotheses below, is the case away from the axis $\{Y=0\}$ in the domain of outer communications), then q is positive definite on the space orthogonal to the Killing vectors. Also note that q is independent of the choice of the basis of the space of Killing vectors.

To take advantage of the asymptotic analysis in [31], a straightforward calculation shows that q equals

$$q(Z_1, Z_2) = \gamma(Z_1, Z_2) - \frac{\gamma(Y, Z_1)\gamma(Y, Z_2)}{\gamma(Y, Y)} \ , \tag{2.6.3}$$

where γ is the (obviously U(1)–invariant) metric on the level sets of t (where t is any time function as in Section 2.6.1) obtained from the space-time metric by a formula similar to (2.6.2):

$$\gamma(Z_1, Z_2) = \mathfrak{g}(Z_1, Z_2) - \frac{\mathfrak{g}(Z_1, X)\mathfrak{g}(Z_2, X)}{\mathfrak{g}(X, X)} \ . \tag{2.6.4}$$

(So γ is *not* the metric induced on the level sets of t by $\mathfrak{g}$.) The right-hand-side is manifestly well-behaved in the region where X is timelike; this is the case in the asymptotic region, and near the axis on $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ under the conditions of Theorem 2.5.24.

In any case, the asymptotic analysis of [31] can be invoked directly to obtain information about the metric q at large distances. Recall that if the asymptotic flatness conditions (2.2.1) hold with $k \geq 1$, then by the field equations (2.2.1) holds with k arbitrarily large. We can thus use [31] to conclude that there exist coordinates x^A, covering the complement of a compact set in $\mathbb{R}^2$ after the quotient space has been doubled across the rotation axis, in which q is manifestly asymptotically flat as well (see Proposition 2.2 and Remark 2.8 in [31]):

$$q_{AB} - \delta_{AB} = o_{k-3}(r^{-1}) \ . \tag{2.6.5}$$

To gain insight into the geometry of q near the horizons, one can use (2.6.4) with X being instead the Killing vector which is null on the horizon. It is then shown in [29] that each non-degenerate component of the horizon corresponds to a smooth totally geodesic boundary for γ. (It is also shown there that every degenerate component corresponds to a metrically complete end of infinite extent *provided* that the Killing vector tangent to the generators of the horizon is time-like on $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ near the horizon, but it is not clear that this property holds.) Some information on the asymptotic geometry of γ in the degenerate case can be obtained from [67, 94]; whether or not the information there suffices to extend our analysis below to the non-degenerate case remains to be seen.

2.6.3 All horizons non-degenerate

Assuming that all horizons are non-degenerate, we proceed as follows: Every non-degenerate component of the boundary ∂M is a smooth sphere S^2 invariant under U(1). As is well known, every isometry of S^2 is smoothly conjugate to the action of rotations around the z axis in a flat $\mathbb{R}^3$, with the rotation axis meeting S^2 at exactly two points. Thus, as already mentioned in Section 2.6.1, we can fill each component of the boundary ∂M by a smooth ball B^3, with a rotation-invariant metric there. We denote by γ any rotation-invariant smooth Riemannian metric on $\mathbb{R}^3$ which extends the original metric γ, and by q the associated two-dimensional metric as in (2.6.3). From what has been said we conclude that every non-degenerate component of the horizon corresponds to a smooth boundary $\partial M/\mathrm{U}(1)$ for the metric q, consisting of a segment which meets the rotation axis at precisely two points. The filling-in just described is equivalent to filling in a half-disc in the quotient manifold. Since the boundary ∂M is a smooth U(1) invariant surface for γ, it meets the rotation axis orthogonally. This implies that each one-dimensional boundary segment of $\partial M/\mathrm{U}(1)$ meets the rotation axis orthogonally in the metric q.

Consider, then, a black hole space-time which contains one asymptotically flat end and N non-degenerate spherical horizons. After adding N half-discs as described above, the quotient space, denoted by $\hat{M}^2$, is then a two-dimensional non-compact asymptotically flat manifold diffeomorphic to a half-plane. Recall that we are assuming (2.6.1), and that there is only one asymptotically flat region. We will also suppose that

$$W > 0 \text{ on } \langle\langle \mathscr{M}_{\rm ext} \rangle\rangle \setminus \mathscr{A}, \text{ and} \tag{2.6.6}$$

$$\text{on } \langle\langle \mathscr{M}_{\rm ext} \rangle\rangle \cap \mathscr{A} \text{ the stationary Killing vector field } X \text{ is timelike.} \tag{2.6.7}$$

Note that those conditions necessarily hold under the hypotheses of Theorem 2.5.1, compare Theorem 2.5.24.

By (2.6.6) the metric q is positive definite away from $\mathscr{A}$. Near $\mathscr{A}$ the metric γ defined in (2.6.4) is Riemannian and smooth by (2.6.7), and the analysis in [31] shows that $\mathscr{A}$ is a smooth boundary for q. After doubling across the boundary, one obtains an asymptotically flat metric on $\mathbb{R}^2$. By [31, Proposition 2.3], for $k \geq 5$ in (2.2.1) there exist global isothermal coordinates for q:

$$q = e^{2u}(dx^2 + dy^2) \ , \quad \text{with } u \longrightarrow_{\sqrt{x^2+y^2}\to\infty} 0 \ . \tag{2.6.8}$$

In fact, $u = o_{k-4}(r^{-1})$. The existence of such coordinates also follows from the uniformization theorem (see, e.g., [1]), but this theorem does not seem to provide the information about the asymptotic behavior in various regimes, needed here, in any obvious way. As explained in the proof of [31, Theorem 2.7], the coordinates (x, y) can be chosen so that the rotation axis corresponds to $x = 0$, with $\hat{M}^2 = \{x \geq 0\}$.

The next step of the construction is to modify the coordinates (x, y) of (2.6.8) to a coordinate system (ρ, z) on the quotient manifold $\bar{M}^2$, covering $[0, \infty) \times \mathbb{R}$, so that ρ vanishes on the rotation axis *and* the event horizons. This is done by first solving the equation

$$\Delta_q \rho_R = 0 \ ,$$

on $\Omega_R := \bar{M}^2 \cap \{x^2 + y^2 \leq R^2\}$, with zero boundary value on $\partial\bar{M}^2$, and with $\rho_R = x$ on $\{x^2 + y^2 = R^2\}$. Note that

$$C = \sup_{\partial\Omega_R \setminus \mathscr{A}} x - \rho_R \ ,$$

is independent of R, for R large, since x and ρ_R differ only on the event horizons. Since $\Delta_q x = 0$, the maximum principle implies

$$x - C \leq \rho_R \leq x \ \text{ on } \Omega_R \ .$$

By usual arguments there exists a subsequence ρ_{R_i} which converges, as i tends to infinity, to a q–harmonic function ρ on $\bar{M}^2$, satisfying the desired boundary values. By standard asymptotic expansions (see, e.g., [24]) we find that $\nabla\rho$ approaches ∇x as $\sqrt{x^2+y^2}\to\infty$. In fact, for any $j\in\mathbb{N}$ we have

$$\rho - x = \sum_{i=0}^{j} \frac{\alpha_i(\varphi)}{(x^2+y^2)^{i/2}} + O((x^2+y^2)^{-(j+1)/2}) \ , \tag{2.6.9}$$

where φ denotes an angular coordinate in the (x,y) plane, with α_i being linear combinations of $\cos(i\varphi)$ and $\sin(i\varphi)$, with the expansion being preserved under differentiation in the obvious way. In particular $\nabla\rho$ does not vanish for large x, so that for R sufficiently large the level sets $\{\rho = R\}$ are smooth submanifolds. The strips $0<\rho<R$ are simply connected so, by the uniformization theorem, there exists a holomorphic diffeomorphism

$$(x,y)\mapsto(\alpha(x,y),\beta(x,y))$$

from that strip to the set $\{0<\alpha<R \ , \ \beta\in\mathbb{R}\}$. By composing with a Möbius map we can further arrange so that the point at infinity of the (x,y)–variables is mapped to the point at infinity of the (α,β)–variables. As the map is holomorphic, the function $\alpha(x,y)$ is harmonic, with the same boundary values and asymptotic conditions as ρ, hence $\alpha(x,y)=\rho(x,y)$ wherever both are defined. If we denote by z a harmonic conjugate to ρ, we similarly obtain that $z-\beta$ is a constant, so that the map

$$(x,y)\mapsto(\rho,z) \tag{2.6.10}$$

is a holomorphic diffeomorphism between the strips described above. Since the constant R was arbitrarily large, we conclude that the map (2.6.10) provides a holomorphic diffeomorphism from the interior of $\bar{M}^2$ to $\{\rho>0 \ , \ z\in\mathbb{R}\}$, and provides the desired coordinate system in which q takes the form

$$q = e^{2\hat{u}}(d\rho^2+dz^2) \ . \tag{2.6.11}$$

From (2.6.9) and its equivalent for z (which is immediately obtained from the defining equations $\partial_x\rho=\partial_y z$, $\partial_y\rho=-\partial_x z$) we infer that $\hat{u}\to 0$ as $\sqrt{\rho^2+z^2}$ goes to infinity, with the decay rate $\hat{u}=o_{k-4}(r^{-1})$ remaining valid in the new coordinates.

In vacuum the area function W satisfies $\Delta_q\sqrt{W} = 0$ (see, e.g., [131]). If we assume that W vanishes on $\partial\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle\cup\mathscr{A}$ (which is the case under the hypotheses of Theorem 2.5.1), then $W = \rho$ on $\partial\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle \cup \mathscr{A}$. Since $\Delta_q\rho = 0$ as well, we have $\Delta_q(\sqrt{W}-\rho) = 0$, with $\sqrt{W}-\rho$ going to zero as one tends to infinity by [31], and the maximum principle gives

$$\sqrt{W} = \rho \,. \tag{2.6.12}$$

2.6.4 Global coordinates on $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$

According to Section 2.6.1 we have

$$\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle \setminus \mathscr{A} \approx \mathbb{R}\times S^1\times\mathbb{R}^*_+\times\mathbb{R} \,,$$

and this diffeomorphism defines a global coordinate system (t,φ,ρ,z) on $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle\setminus\mathscr{A}$, with $X = \partial_t$ and $Y = \partial_\varphi$. Letting $(x^A) = (\rho,z)$ and $(x^a) = (t,\varphi)$, we can write the metric in the form

$$\mathfrak{g} = \mathfrak{g}_{ab}(dx^a + \underbrace{\theta^a{}_A dx^A}_{=:\theta^a})(dx^b + \theta^b{}_B dx^B) + q_{AB}dx^A dx^B \,,$$

with all functions independent of t and φ. The orthogonal integrability condition of Proposition 2.5.3 gives

$$d\theta^a = 0 \,,$$

so that, by simple connectedness of $\mathbb{R}^*_+\times\mathbb{R}$, there exist functions f^a such that $\theta^a = df^a$. Redefining the x^a's to $x^a + f^a$, and keeping the same symbols for the new coordinates, we conclude that the metric on $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle\setminus\mathscr{A}$ has a global coordinate representation as

$$\mathfrak{g} = -\rho^2 e^{2\lambda}dt^2 + e^{-2\lambda}(d\varphi - v dt)^2 + e^{2\hat u}(d\rho^2 + dz^2) \tag{2.6.13}$$

for some functions $v(\rho,z)$, $\lambda(\rho,z)$, with ρ, z and $\hat u$ as in Section 2.6.3, see in particular (2.6.12). We set

$$U = \lambda + \ln\rho \,, \quad \text{so that} \quad \mathfrak{g}(\partial_\varphi,\partial_\varphi) = \rho^2 e^{-2U} = e^{-2\lambda} \,. \tag{2.6.14}$$

Let ω be the *twist potential* defined by the equation

$$d\omega = *(dY\wedge Y) \,, \tag{2.6.15}$$

its existence follows from simple-connectedness of $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ and from $d*(dY\wedge Y)=0$ (see, e.g., [131]). As discussed in more detail in Section 2.6.7 below (compare [131, Proposition 2]), the space-time metric is uniquely determined by the axisymmetric map

$$\Phi = (\lambda,\omega) : \mathbb{R}^3 \setminus \mathscr{A} \to \mathbb{H}^2 \;, \tag{2.6.16}$$

where $\mathbb{H}^2$ is the hyperbolic space with metric

$$b := d\lambda^2 + e^{4\lambda} d\omega^2 \;, \tag{2.6.17}$$

and $\mathscr{A}$ is the rotation axis $\mathscr{A} := \{(0,0,z)\;, z\in\mathbb{R}\}\subset\mathbb{R}^3$. The metric coefficients can be determined from Φ by solving equations (2.6.45)-(2.6.47) below. The map Φ solves the harmonic map equations [56, 126]:

$$|T|_b^2 := (\Delta\lambda - 2e^{4\lambda}|D\omega|^2)^2 + e^{4\lambda}(\Delta\omega + 4D\lambda\cdot D\omega)^2 = 0 \;, \tag{2.6.18}$$

where both D and Δ refer to the flat metric on $\mathbb{R}^3$, together with a set of asymptotic conditions depending upon the configuration at hand.

We continue with the derivation of those boundary conditions.

2.6.5 Boundary conditions at non-degenerate horizons

Near the points at which the boundary is analytic (so, e.g., at those points of the axis at which X is timelike), the map defined by (2.6.10) extends to a holomorphic map across the boundary (see, e.g., [47]). This implies that $\hat{u}$ extends across the axis as a smooth function of ρ^2 and z *away* from the set of points $\{\mathfrak{g}(X,X)=0\}$.

Let us now analyze the behavior of $\hat{u}$ near the points $z_i\in\mathscr{A}$ where non-degenerate horizons meet the axis. As described above, after performing a constant shift in the y coordinate, any component of a non-degenerate horizon can locally be described by a smooth curve in the $\zeta := x+iy$ plane of the form

$$y=\gamma(x)\;,\;\; \gamma(0)=0\;,\;\; \gamma(x)=\gamma(-x)\;. \tag{2.6.19}$$

Near the origin, the points lying in the domain of outer communications correspond then to the values of $x+iy$ lying in a region, say Ω, bounded by the half-axis $\{x=0, y\geq 0\}$ and by the curve $x+i\gamma(x)$, with $x\geq 0$.

To get rid of the right-angle-corner where the curve $x + i\gamma(x)$ meets the axis, the obvious first attempt is to introduce a new complex coordinate

$$w := \alpha + i\beta = -i\zeta^2 . \tag{2.6.20}$$

If we write $\gamma(x) = a_2 x^2 + O(x^4)$, then the image of $\{x + i\gamma(x) \ , \ x \geq 0\}$ under (2.6.20) becomes

$$\begin{aligned} f_1(x + i\gamma(x)) &= 2a_2x^3 + O(x^5) - i\,\underbrace{(x^2 - a_2^2x^4 + O(x^6))}_{=:-t} \qquad (2.6.21)\\ &= it + 2a_2|t|^{3/2} + O(|t|^{5/2}) . \end{aligned}$$

The remaining part $\{iy,\ y \in \mathbb{R}^+\}$, of the boundary of Ω, is mapped to itself. It follows that the boundary of the image of Ω by the map (2.6.20) is a $C^{1,1/2}$ curve. Here $C^{k,\lambda}$ denotes the space of k-times differentiable functions, the k'th derivatives of which satisfy a Hölder condition with index λ.

To improve the regularity we replace $-i\zeta^2$ by $f_2(\zeta) = -i\zeta^2 + \sigma_3\zeta^3$ for some constant σ_3. Then (2.6.21) becomes

$$\begin{aligned} f_2(x + i\gamma(x)) &= (2a_2 + \Re\sigma_3)x^3 + O(x^5) - i\,\underbrace{(x^2 + O(x^4))}_{=:-t} - \Im(\sigma_3)O(x^4)\\ &= it + (2a_2 + \Re\sigma_3)|t|^{3/2} + O(|t|^{5/2}) . \qquad (2.6.22) \end{aligned}$$

The remaining part of the boundary of Ω is mapped to the curve $f_2(iy)$, with $y \geq 0$:

$$\begin{aligned} f_2(iy) &= \Im\sigma_3 y^3 + i\,\underbrace{(y^2 - \Re\sigma_3 y^3)}_{=:t} \qquad (2.6.23)\\ &= it + \Im\sigma_3(|t|^{3/2} + O(|t|^2)) . \end{aligned}$$

and is thus mapped to itself if σ_3 is real. Choosing $\sigma_3 = -2a_2 \in \mathbb{R}$ one gets rid of the offending $|t|^{3/2}$ terms in (2.6.22)-(2.6.23), resulting in the boundary of $f_2(\Omega)$ of $C^{2,1/2}$ differentiability class.

More generally, suppose that the image of $x + i\gamma(x)$ by the polynomial map $\zeta \mapsto w = f_{k-1}(\zeta) = -i\zeta^2 + \ldots$ has a real part equal to $\beta_{2k-1}x^{2k-1} + O(x^{2k+1})$; then the subtraction from f_{k-1} of a term $\beta_{2k-1}\zeta^{2k-1}$ leads to a new polynomial map $\zeta \to w = f_k(\zeta)$ which has real part $\beta_{2k+1}x^{2k+1} + O(x^{2k+3})$, and the differentiability of the image has been improved by one. Since all the coefficients β_{2k+1} are real, the maps f_k map the imaginary axis to itself. One should note that this argument wouldn't work if γ had odd powers of x in its Taylor expansion.

Summarizing, for any k we can choose a finite polynomial $f_k(\zeta)$, with lowest order term $-i\zeta^2$, and with the remaining coefficients real and involving only odd powers of ζ, which maps the boundary of Ω to a curve

$$(-\epsilon,\epsilon) \ni t \mapsto (\mu(t),\nu(t)) := \begin{cases} (0,t), & t \geq 0; \\ (O(t^{k+1/2}),t), & t \leq 0, \end{cases} \tag{2.6.24}$$

which is $C^{k,1/2}$.

Note that

$$\psi_k(\zeta) := \sqrt{if_k(\zeta)} = \zeta\Big(1+O(|\zeta|)\Big) \,, \tag{2.6.25}$$

where $\sqrt{\cdot}$ denotes the principal branch of the square root, is a holomorphic diffeomorphism near the origin. So

$$w = f_k(\zeta) = -i\psi_k^2(\zeta) \tag{2.6.26}$$

and we have

$$dw\,d\bar{w} = 4|\psi_k\psi_k'|^2 d\zeta\,d\bar{\zeta} = 4|w||\psi_k'|^2 d\zeta\,d\bar{\zeta} \,. \tag{2.6.27}$$

We claim that the map

$$w \mapsto \eta := \rho + iz$$

extends across $\rho = 0$ to a C^k diffeomorphism near the origin. To see this, note that we have again $\Delta\rho = 0$ with respect to the metric $dwd\bar{w}$, with ρ vanishing on a $C^{k,1/2}$ boundary. We can straighten the boundary using the transformation

$$w = (\alpha,\beta) \mapsto (\alpha-\mu(\beta),\beta) = w + (O(|\beta|^{k+1/2}),0) = w + O(|w|^{k+1/2}) \,, \tag{2.6.28}$$

where μ is as (2.6.24), and $O(\cdot)$ is understood for small $|w|$. Extending ρ with $-\rho$ across the new boundary, one can use the standard interior Schauder estimates on the extended function to conclude that $w \mapsto \rho(w)$ is $C^{k,1/2}$ up-to-boundary. Now, the condition $dz = \star d\rho$, where $\star$ is the Hodge dual of the metric q, is conformally invariant and therefore holds in the metric $dw\,d\bar{w}$, so z is a $C^{k,1/2}$ function of w. By the boundary version of the maximum principle we have $d\rho \neq 0$ at the boundary (when understood as a function of w), and hence near the boundary, so dz is non-vanishing near the boundary and orthogonal to $d\rho$. The implicit function theorem allows us to conclude that the map $w \mapsto \eta$ is a $C^{k,1/2}$ diffeomorphism near $w = 0$.

Comparing (2.6.8) and (2.6.11) we have

$$e^{2\hat{u}} d\eta\, d\bar{\eta} = q = e^{2u} d\zeta\, d\bar{\zeta} = \frac{e^{2u}}{4|w||\psi_k'|^2} dw\, d\bar{w}\ , \tag{2.6.29}$$

in particular $dw\, d\bar{w} = e^{2\tilde{u}_k} d\eta\, d\bar{\eta}$, and from what has been said the function $\tilde{u}_k$ is $C^{k-1,1/2}$ up to boundary. Hence

$$e^{2\hat{u}} = \frac{e^{2u+2\tilde{u}_k}}{4|w||\psi_k'|^2} \tag{2.6.30}$$

where u is a smooth function of (x^2, y), while ψ_k' is a non-vanishing holomorphic function of $\zeta = x + iy$, $\tilde{u}_k$ is a C^{k-1} function of $\eta = \rho + iz$, and $\eta \mapsto w$ is a C^k diffeomorphism, with w having a zero of order one where the horizon meets the axis. Finally $x + iy$ is a holomorphic function of $\sqrt{iw}$, compare (2.6.26).

Choosing $k = 2$ we obtain

$$\hat{u} = -\frac{1}{2} \ln |w| + \hat{u}_1 + \hat{u}_2\ , \tag{2.6.31}$$

where w is a smooth complex coordinate which vanishes where the horizon meets the axis, $\hat{u}_2 = -\ln |\psi_2'|^2/2$ is a smooth function of (x, y), and $\hat{u}_1$ is a C^1 function of (ρ, z).

Taylor expanding at the origin, from what has been said (recall that $\eta \mapsto w$ is conformal and that, near the origin, $\{\rho = 0\}$ coincides with $\{\alpha - \mu(\beta) = 0\}$) it follows that there exists a real number $a > 0$ such that

$$(\rho, z) = \big(a^{-2}(\alpha - \mu(\beta)), a^{-2}\beta\big) + O((\alpha - \mu(\beta))^2 + \beta^2)\ ,$$

which implies

$$(\alpha, \beta) = (a^2\rho, a^2 z) + O(\rho^2 + z^2)\ . \tag{2.6.32}$$

Here we have assumed that z has been shifted by a constant so that it vanishes at the chosen intersection point of the axis and of the event horizon.

We conclude that there exists a constant C such that

$$|\hat{u} + \frac{1}{2} \ln \sqrt{\rho^2 + z^2}| \le C \quad \text{near } (0,0)\ . \tag{2.6.33}$$

This is the desired equation describing the leading order behavior of $\hat{u}$ near the meeting point of the axis and a non-degenerate horizon.

2.6.5.1 The Ernst potential

We continue by deriving the boundary conditions satisfied by the Ernst potential (U,ω) near the point where the horizon meets the axis. Here U is as in (2.6.13)-(2.6.14), and ω is obtained from the function v appearing in the metric by solving (2.6.45) below.

Our analysis so far can be summarized as:

$$x+iy=\zeta \;\mapsto\; \psi_k(\zeta)=\sqrt{if_k(\zeta)} \;\mapsto\; -i(\psi_k(\zeta))^2=w \;\mapsto\; \rho+iz\,. \tag{2.6.34}$$

Each map is invertible on the sets under consideration; and each is a C^k diffeomorphism up-to-boundary except for the middle one, which involves the squaring of a complex number.

Using $\zeta=\psi_k^{-1}(\sqrt{iw})$, the expansion

$$\psi_k^{-1}(c+id)=(c+id)\Big(1+O(\sqrt{c^2+d^2})\Big)\,,$$

which follows from (2.6.25), together with (2.6.32), we obtain

$$x+iy=a\sqrt{-z+i\rho}+O(\rho^2+z^2)\,.$$

Equivalently,

$$x=\frac{a\rho}{\sqrt{2(z+\sqrt{z^2+\rho^2})}}+O(\rho^2+z^2)\,,\quad y=a\sqrt{\frac{z+\sqrt{z^2+\rho^2}}{2}}+O(\rho^2+z^2)\,. \tag{2.6.35}$$

To continue, in addition to (2.6.1), (2.6.6) and (2.6.7) we assume that

> The level sets of the function t, defined as the projection on the $\mathbb{R}$ factor in (2.6.1), are spacelike, with $\partial_\varphi t=0$; (2.6.36)

this is justified for our purposes by Theorem 2.4.5. Thus, the Killing vector ∂_φ is tangent to the level sets of t, so that

$$\mathfrak{g}(\partial_\varphi,\partial_\varphi)=h(\partial_\varphi,\partial_\varphi)\,,$$

where h is the Riemannian metric induced on the level sets of t. As shown in [31], we have

$$h(\partial_\varphi,\partial_\varphi)=f(x,y)x^2\,, \tag{2.6.37}$$

where the function $f(x,y)$ is uniformly bounded above and below on compact sets.

Recall that U has been defined as $-\frac{1}{2}\ln(\mathfrak{g}_{\varphi\varphi}\rho^{-2})$, and that (ρ,z) have been normalized so that $(0,0)$ corresponds to a point where a non-degenerate horizon meets the axis. We want to show that

$$U = \ln\sqrt{z+\sqrt{z^2+\rho^2}} + O(1) \quad \text{near } (0,0)\ . \tag{2.6.38}$$

(This formula can be checked for the Kerr metrics by a direct calculation, but we emphasize that we are considering a general non-degenerate horizon.) To see that, we use (2.6.37) to obtain

$$\ln(\mathfrak{g}_{\varphi\varphi}\rho^{-2}) = \ln(x^2\rho^{-2}) + \ln(\mathfrak{g}_{\varphi\varphi}x^{-2}) = 2\ln(x\rho^{-1}) + O(1)\ .$$

We assume that ρ^2+z^2 is sufficiently small, as required by the calculations that follow. In the region $0\le|z|\le 2\rho$ we use (2.6.35) as follows:

$$\begin{aligned}
\ln(x\rho^{-1}) &= \ln\left(\frac{a+\sqrt{2\left(\frac{z}{\rho}+\sqrt{\frac{z^2}{\rho^2}+1}\right)O(\rho^{3/2}+\frac{z^2}{\rho^{1/2}})}}{\sqrt{2(z+\sqrt{z^2+\rho^2})}}\right)\\
&= -\ln\left(\sqrt{2(z+\sqrt{z^2+\rho^2})}\right) + O(1)\ .
\end{aligned}$$

In the region $z\le 0$ we note that

$$\begin{aligned}
\frac{1}{\rho}\sqrt{2(z+\sqrt{z^2+\rho^2})} &= \frac{\sqrt{2(z+\sqrt{z^2+\rho^2})}\sqrt{2(-z+\sqrt{z^2+\rho^2})}}{\rho\sqrt{2(-z+\sqrt{z^2+\rho^2})}}\\
&= \frac{2}{\sqrt{2(-z+\sqrt{z^2+\rho^2})}} \le \frac{\sqrt{2}}{(z^2+\rho^2)^{1/4}}\ .
\end{aligned}$$

Hence, again by (2.6.35),

$$\begin{aligned}
\ln(x\rho^{-1}) &= \ln\left(\frac{a+\frac{1}{\rho}\sqrt{2(z+\sqrt{z^2+\rho^2})}O(\rho^2+z^2)}{\sqrt{2(z+\sqrt{z^2+\rho^2})}}\right)\\
&= \ln\left(\frac{a+O\big((\rho^2+z^2)^{3/4}\big)}{\sqrt{2(z+\sqrt{z^2+\rho^2})}}\right) = -\ln\left(\sqrt{2(z+\sqrt{z^2+\rho^2})}\right) + O(1)\ .
\end{aligned}$$

In the region $0 \leq \rho \leq z/2$ some more work is needed. Instead of (2.6.35), we want to use a Taylor expansion of ρ around the axis $\alpha = 0$, where α is as in (2.6.20). To simplify the calculations, note that there is no loss of generality in assuming that the map ψ_k of (2.6.25) is the identity, by redefining the original (x, y) coordinates to the new ones obtained from ψ_k. Since in the region $0 \leq \rho \leq z/2$ we have $\beta \geq 0$, the function $\mu(\beta)$ in (2.6.28) vanishes, so

$$\alpha(\rho, z) = \underbrace{\alpha(0, z)}_{=\mu\left(\beta(0,z)\right)=0} + \partial_\rho \alpha(0, z)\rho + O(\rho^2) = \partial_\rho \alpha(0, z)\rho + O(\rho^2) \, .$$

Note that $\partial_\rho \alpha(0, z)$ tends to a^2 as z tends to zero, so is strictly positive for z small enough. Instead of (2.6.35) we now have directly

$$x = \frac{\alpha}{\sqrt{2(\beta + \sqrt{\beta^2 + \alpha^2})}} \quad \Longrightarrow \quad \frac{x}{\rho} = \frac{\partial_\rho \alpha(0, z) + O(\rho)}{\sqrt{2(\beta + \sqrt{\beta^2 + \alpha^2})}} \, .$$

In the current region α is equivalent to ρ, β is equivalent to z, $\sqrt{\beta^2 + \alpha^2}$ is equivalent to z, and z is equivalent to $2(z + \sqrt{z^2 + \rho^2})$, which leads to the desired formula:

$$\begin{aligned}
\ln(x\rho^{-1}) &= -\ln\left(\sqrt{2(\beta + \sqrt{\beta^2 + \alpha^2})}\right) + O(1) \\
&= -\ln\left(\sqrt{2(z + \sqrt{z^2 + \rho^2})}\frac{\sqrt{2(\beta + \sqrt{\beta^2 + \alpha^2})}}{\sqrt{2(z + \sqrt{z^2 + \rho^2})}}\right) + O(1) \\
&= -\ln\left(\sqrt{2(z + \sqrt{z^2 + \rho^2})}\right) + O(1) \, .
\end{aligned}$$

This finishes the proof of (2.6.38).

Let us turn our attention now to the twist potential ω: as is well known, or from [38, Equation (2.6)] together with the analysis in [31], ω is a smooth function of (x, y), constant on the axis $\{x = 0\}$, with odd x–derivatives vanishing there. So, Taylor expanding in x, there exists a constant ω_0 and a bounded function $\mathring{\omega}$ such that

$$\begin{aligned}
\omega &= \omega_0 + \mathring{\omega}(x, y)x^2 \\
&= \omega_0 + \frac{\mathring{\omega}(x, y)\left(a\rho + \sqrt{2(z + \sqrt{z^2 + \rho^2})}O(\rho^2 + z^2)\right)^2}{2(z + \sqrt{z^2 + \rho^2})} \, .
\end{aligned} \qquad (2.6.39)$$

In our approach below, the proof of black hole uniqueness requires a uniform bound on the distance between the relevant harmonic maps. Now, using the coordinates (λ, ω) on hyperbolic space as in (2.6.17), the distance d_b between two points (x_1, ω_1) and (x_2, ω_2) is implicitly defined by the formula [8, Theorem 7.2.1]:

$$\cosh(d_b) - 1 \quad = \quad \frac{(e^{-2x_1} - e^{-2x_2})^2 + 4(\omega_1 - \omega_2)^2}{2e^{-2x_1 - 2x_2}} \; .$$

Using the (U, ω) parameterization of the maps, with U as in (2.6.14), the distance measured in the hyperbolic plane between two such maps is the supremum of the function d_b:

$$\begin{aligned} \cosh(d_b) - 1 \quad &= \quad \frac{\rho^4(e^{-2U_1} - e^{-2U_2})^2 + 4(\omega_1 - \omega_2)^2}{2\rho^4 e^{-2U_1 - 2U_2}} \\ &= \quad \frac{1}{2}\underbrace{(e^{2(U_1-U_2)} + e^{2(U_2-U_1)} - 2)}_{(a)} + 2\underbrace{\rho^{-4}e^{2(U_1+U_2)}(\omega_1 - \omega_2)^2}_{(b)} \; . \end{aligned}$$

Inserting (2.6.38) and the analogous expansion for the Ernst potential of a second metric into (a) above we obviously obtain a bounded contribution. Finally, assuming $\omega_1(0,0) = \omega_2(0,0)$, up to a multiplicative factor which is uniformly bounded above and bounded away from zero, (b) can be rewritten as a square of the difference of two terms of the form

$$f_i := \mathring{\omega}_i\Big(a_i + \rho^{-1}\sqrt{2(z + \sqrt{z^2+\rho^2})}O(\rho^2 + z^2)\Big)^2 \; , \tag{2.6.40}$$

with $i = 1, 2$. We have the following, for all $z^2 + \rho^2 \le 1$:

1. The functions f_i in (2.6.40) are uniformly bounded in the sector $|z| \le \rho$:

   $$|f_i| \le C\Big(a_i + \sqrt{2(z + \sqrt{z^2+\rho^2})}O(\rho + z^2/\rho)\Big)^2 \le C' \; .$$

2. For $0 \le \rho \le -z$ we write

   $$0 \le z + \sqrt{z^2+\rho^2} = |z|(\sqrt{1+\frac{\rho^2}{z^2}} - 1) \le C\frac{\rho^2}{|z|} \; ,$$

   so that

   $$|f_i| \le C\Big(a_i + \frac{1}{|z|^{1/2}}O(\rho^2 + z^2)\Big)^2 = C(a_i + O(|z|^{3/2}))^2 \le C' \; .$$

3. For $0 \le \rho \le z$ one can proceed as follows: by (2.6.37), together with the analysis of ω in [31], there exists a constant C such that near the axis we have

$$C^{-1}x^2 \le \mathfrak{g}(\partial_\varphi, \partial_\varphi) = h(\partial_\varphi, \partial_\varphi) \le Cx^2 \ , \quad \Big|\omega - \underbrace{\omega|_{x=0}}_{=:\omega_0}\Big| \le Cx^2 \tag{2.6.41}$$

(recall that h denotes the metric induced by $\mathfrak{g}$ on the slices $t = \text{const}$, where t is a time function invariant under the flow of ∂_φ). But

$$\begin{aligned}
\frac{(\omega_1-\omega_2)^2}{\rho^4 e^{-2U_1-2U_2}} &= \frac{(\omega_1-\omega_2)^2}{\mathfrak{g}_1(\partial_\varphi,\partial_\varphi)\mathfrak{g}_2(\partial_\varphi,\partial_\varphi)} \le 2\frac{(\omega_1-\omega_0)^2+(\omega_2-\omega_0)^2}{\mathfrak{g}_1(\partial_\varphi,\partial_\varphi)\mathfrak{g}_2(\partial_\varphi,\partial_\varphi)} \\
&= 2\underbrace{\left(\frac{\omega_1-\omega_0}{\mathfrak{g}_1(\partial_\varphi,\partial_\varphi)}\right)^2}_{\le C^2}\underbrace{\frac{\mathfrak{g}_1(\partial_\varphi,\partial_\varphi)}{\mathfrak{g}_2(\partial_\varphi,\partial_\varphi)}}_{=e^{2(U_2-U_1)}} + 2\underbrace{\left(\frac{\omega_2-\omega_0}{\mathfrak{g}_2(\partial_\varphi,\partial_\varphi)}\right)^2}_{\le C^2}\underbrace{\frac{\mathfrak{g}_2(\partial_\varphi,\partial_\varphi)}{\mathfrak{g}_1(\partial_\varphi,\partial_\varphi)}}_{=e^{2(U_1-U_2)}} ,
\end{aligned} \tag{2.6.42}$$

where $\mathfrak{g}_i$ denotes the respective space-time metric, while x_i denotes the respective x coordinate. Uniform boundedness of this expression, in a neighborhood of the intersection point, follows now from (2.6.38).

We are ready now to prove one of the significant missing elements of all previous uniqueness claims for the Kerr metric:

Theorem 2.6.1 *Suppose that* (2.6.1)*,* (2.6.6)*-*(2.6.7) *and* (2.6.36) *hold. Let* (U_i, ω_i)*,* $i = 1, 2$*, be the Ernst potentials associated with two vacuum, stationary, asymptotically flat axisymmetric metrics with smooth non-degenerate event horizons. If* $\omega_1 = \omega_2$ *on the rotation axis, then the hyperbolic-space distance between* (U_1, ω_1) *and* (U_2, ω_2) *is bounded, going to zero as* r *tends to infinity in the asymptotic region.*

Proof: We have just proved that the distance between two different Ernst potentials is bounded near the intersection points of the horizon and of the axis. In view of (2.6.7), the distance is bounded on bounded subsets of the axis away from the horizon intersection points by the analysis in [31]. Next, both ω_a's are bounded on the horizon, and both functions $\rho^2 e^{-2U_a}$'s are bounded on the horizon away from its end points. Finally, both ω_a's approach the Kerr twist potential at infinity by the results in [125] (the asymptotic Poincaré Lemma 8.7 in [35] is useful in this context), so the distance approaches zero as one recedes to infinity by a calculation as in (2.6.42), together with the asymptotic analysis of [31]; a more detailed exposition can be found in Section 3.4. □

2.6.6 The harmonic map problem: existence and uniqueness

In this section we consider Ernst maps satisfying the following conditions, modeled on the local behavior of the Kerr solutions:

1. There exists $N_{\rm dh} \geq 0$ degenerate event horizons, which are represented by punctures $(\rho = 0, z = b_i)$, together with a mass parameter $m_i > 0$ and angular momentum parameter $a_i = \pm m_i$, with the following behavior for small $r_i := \sqrt{\rho^2 + (z - b_i)^2}$,
$$U = \ln\left(\frac{r_i}{2m_i}\right) + \frac{1}{2}\ln\left(1 + \frac{(z-b_i)^2}{r_i^2}\right) + O(r_i). \tag{2.6.43}$$
The twist potential ω is a bounded, angle-dependent function which jumps by $-4J_i = -4a_i m_i$ when crossing b_i from $z < b_i$ to $z > b_i$, where J_i is the "angular momentum of the puncture".

2. There exists $N_{\rm ndh} \geq 0$ non-degenerate horizons, which are represented by bounded open intervals $(c_i^-, c_i^+) = I_i \subset \mathscr{A}$, with none of the previous b_j's belonging to the union of the closures of the I_i. The functions $U - 2\ln\rho$ and ω extend smoothly across each interval I_i, with the following behavior near the end points, for some constant C, as derived in (2.6.38):
$$|U - \frac{1}{2}\ln(\sqrt{\rho^2 + (z - c_i^\pm)^2} + z - c_i^\pm)| \leq C \quad \text{near } (0, c_i^\pm)\ . \tag{2.6.44}$$
The function ω is assumed to be locally constant on $\mathscr{A} \setminus (\cup_i\{b_i\} \cup_j I_j)$, with expansions as in (2.6.39) nearby.

3. The functions U and ω are smooth across $\mathscr{A} \setminus (\cup_i\{b_i\} \cup_j I_j)$.

A collection $\{b_i, m_i\}_{i=1}^{N_{\rm dh}}$, I_j, $j = 1, \ldots, N_{\rm ndh}$, and $\{\omega_k\}$, where the ω_k's are the values of ω_i on the connected components of $\mathscr{A} \setminus (\cup_i\{b_i\} \cup_j I_j)$, will be called *"axis data"*.

We have the following [38, Appendix C] (compare [50, 136] and references therein for previous related results):

THEOREM 2.6.2 *For any set of axis data there exists a unique harmonic map* $\Phi : \mathbb{R}^3 \setminus \mathscr{A} \to \mathbb{H}^2$ *which lies a finite distance from an axisymmetric map (not necessarily harmonic) with the properties 1.–3. above, and such that* $\omega = 0$ *on* $\mathscr{A}$ *for large positive* z. □

Here the distance between two maps Φ_1 and Φ_2 is defined as

$$d(\Phi_1, \Phi_2) = \sup_{p \in \mathbb{R}^3 \setminus \mathscr{A}} d_b(\Phi_1(p), \Phi_2(p)) ,$$

where the distance d_b is taken with respect to the hyperbolic metric (2.6.17).

We emphasize the following corollary, first established by Robinson [117] using different methods (and assuming $|a| < m$ [19] which Weinstein [131] does not); the approach presented here is due to Weinstein [131]:[20]

COROLLARY 2.6.3 *For each mass parameter* m *and angular momentum parameter* $a \in (-m, m)$ *there exists only one map* Φ *with the behavior at the axis corresponding to an* I^+*–regular axisymmetric vacuum black hole with a connected non-degenerate horizon centered at the origin and with* ω *vanishing on* $\mathscr{A}$ *for large positive* z*. Furthermore, no* I^+*–regular non-degenerate axisymmetric vacuum black holes with* $|a| \geq m$ *exist.*

PROOF: Theorem 2.4.5 shows that (2.6.1) and (2.6.36) hold, (2.6.6) follows from Theorem 2.5.1, while (2.6.7) holds by the Ergoset Theorem 2.5.24. One can thus introduce (ρ, z) coordinates on the orbit space as in Section 2.6.2, then the event horizon corresponds to a connected interval of the axis of length ℓ, for some $\ell > 0$. Let (U, ω) be the Ernst potential corresponding to the black hole under consideration, with ω normalized to vanish on $\mathscr{A}$ for large positive z. Let J be the total angular momentum of the black hole, there exists a Kerr solution (U_K, ω_K), with ω_K normalized to vanish on $\mathscr{A}$ for large positive z, and such that the corresponding "horizon interval" has the same length ℓ. We can adjust

[19] Where $a := J/m$, with J the total angular momentum.

[20] Yet another approach can be found in [108]; compare [101, Section 2.4]. In order to become complete, the proof there needs to be complemented by a justification of the assumed behavior of their potential Φ (not to be confused with the map Φ here) on the set $\{\rho = 0\}$. More precisely, one needs to justify differentiability of Φ on $\{\rho = 0\}$ away from the horizons, continuity of Φ and Φ' at the points where the horizon meets the rotation axis, as well as the detailed differentiability properties of Φ near degenerate horizons as implicitly assumed in [101, Section 2.4].

the z coordinate so that the horizon intervals coincide. The value of ω on the axis for large negative z equals $4J$, similarly for ω_K, hence $\omega = \omega_K$ on the axis except possibly on the horizon interval. Theorem 2.6.1 shows that (U, ω) lies at a finite distance from (U_K, ω_K). By the uniqueness part of Theorem 2.6.2 we find $(U, \omega) = (U_K, \omega_K)$, thus the ADM mass of the black hole equals the mass of the comparison Kerr solution, and $|a| < m$ follows.

2.6.7 Candidate solutions

Each harmonic map (λ, ω) of Theorem 2.6.2 with $N_{\mathrm{dh}} + N_{\mathrm{ndh}} \geq 1$ provides a candidate for a solution with $N_{\mathrm{dh}} + N_{\mathrm{ndh}}$ components of the event horizon, as follows: let the functions v and $\hat{u}$ be the unique solutions of the set of equations

$$\partial_\rho v = -e^{4\lambda}\rho\, \partial_z \omega \ , \qquad \partial_z v = e^{4\lambda}\rho\, \partial_\rho \omega \ , \tag{2.6.45}$$

$$\partial_\rho \hat{u} = \rho \left[(\partial_\rho \lambda)^2 - (\partial_z \lambda)^2 + \tfrac{1}{4} e^{4\lambda}((\partial_\rho \omega)^2 - (\partial_z \omega)^2)\right] + \partial_\rho \lambda \ , \tag{2.6.46}$$

$$\partial_z \hat{u} = 2\,\rho \left[\partial_\rho \lambda\, \partial_z \lambda + \tfrac{1}{4} e^{4\lambda} \partial_\rho \omega\, \partial_z \omega\right] + \partial_z \lambda \ , \tag{2.6.47}$$

which go to zero at infinity. (Those equations are compatible whenever (λ, ω) satisfy the harmonic map equations.) Then the metric (2.6.13) satisfies the vacuum Einstein equations (see, e.g., [135, Eqs. (2.19)-(2.22)]). Every such solution provides a candidate for a regular, vacuum, stationary, axisymmetric black hole with several components of the event horizon. If $N_{\mathrm{dh}} + N_{\mathrm{ndh}} = 1$ the resulting metrics are of course the Kerr ones.

At the time of writing of this work, it is not known whether any such candidate solution other than Kerr itself describes an I^+–regular black hole. It should be emphasized that there are two separate issues here: The first is that of uniqueness, which is settled by the uniqueness part of Theorem 2.6.2 together with the remaining analysis in this section: *if* there exist stationary axisymmetric multi-black hole solutions, with all components of the horizon non-degenerate, *then* they belong to the family described by the harmonic maps of Theorem 2.6.2. Note that Theorem 2.6.2 extends to those solutions with degenerate horizons with the behavior described in (2.6.43). Conceivably this covers all degenerate horizons, but this remains to be established.

Another question is that of the global properties of the candidate solutions: for this one needs, first, to study the behavior of the harmonic maps of Theorem 2.6.2 near the singular set in much more detail in order to establish e.g. existence of a smooth event horizon; an analysis of this issue has only been done so far [97,131] if $N_{\rm dh} = 0$ away from the points where the axis meets the horizon, and the question of space-time regularity at those points is wide open. Regardless of this, one expects that for all such solutions the integration of the remaining equations (2.6.45)-(2.6.47) will lead to singular "struts" [6,131,132] in the space-time metric (2.6.13) somewhere on $\mathscr{A}$.

2.7 Proof of Theorem 2.0.1

If $\mathscr{E}^+$ is empty, the conclusion follows from the Komar identity and the rigid positive energy theorem (see, e.g. [29, Section 4]). Otherwise the proof splits into two cases, according to whether or not X is tangent to the generators of $\mathscr{E}^+$, to be covered separately in Sections 2.7.1 and 2.7.2.

2.7.1 Rotating horizons

Suppose, first, that the Killing vector is not tangent to the generators of some connected component $\mathscr{E}^+_0$ of $\mathscr{E}^+ = \mathscr{H}^+ \cap I^+(\mathscr{M}_{\rm ext})$. Theorem 2.4.14 shows that the isometry group of $(\mathscr{M}, \mathfrak{g})$ contains $\mathbb{R} \times \mathrm{U}(1)$. By Corollary 2.2.4 $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is simply connected so that, in view of Theorem 2.4.5, the analysis of Section 2.6 applies, leading to the global representation (2.6.13) of the metric. The analysis of the behavior near the symmetry axis of the harmonic map Φ of Section 2.6.5 shows that Φ lies a finite distance from one of the solutions of Theorem 2.6.2, and the uniqueness part of that last theorem allows us to conclude; compare Corollary 2.6.3 in the connected case.

2.7.2 Non-rotating case

The case where the stationary Killing vector X is tangent to the generators of every component of $\mathscr{H}^+$ will be referred to as the *non-rotating one*. By hypothesis $\nabla(\mathfrak{g}(X,X))$ has no zeros on $\mathscr{E}^+$, so all components of the future event horizon are non-degenerate.

Deforming $\mathscr{S}$ near $\partial\mathscr{S}$ if necessary, we may without loss of generality assume that $\mathscr{S}$ can be extended across $\mathscr{E}^+$ to a smooth spacelike hypersurface there.

For the proof we need a new hypersurface $\mathscr{S}''$ which is maximal, Cauchy for $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$, with X vanishing on $\partial\mathscr{S}''$. Under our hypotheses such a hypersurface will not exist in general, so we start by replacing $(\mathscr{M}, \mathfrak{g})$ by a new space-time $(\mathscr{M}', \mathfrak{g}')$ with the following properties:

1. $(\mathscr{M}', \mathfrak{g}')$ contains a region $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle'$ isometric to $(\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle, \mathfrak{g})$;
2. $(\mathscr{M}', \mathfrak{g}')$ is invariant under the flow of a Killing vector X' which coincides with X on $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$;
3. Each connected component of the horizon $\mathscr{E}_0^{+\prime}$ is contained in a bifurcate Killing horizon, which contains a "bifurcation surface" where X' vanishes. We will denote by S the union of these bifurcation surfaces.

This is done by attaching to $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ a bifurcate horizon near each connected component of $\mathscr{E}^+$ as in [114].

We wish, now to construct a Cauchy surface $\mathscr{S}'$ for $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle'$ such that $\partial\mathscr{S}' = S$. To do that, for $\epsilon > 0$ let $\mathfrak{g}_\epsilon$ denote a family of metrics such that $\mathfrak{g}_\epsilon$ tends to $\mathfrak{g}$, as ϵ goes to zero, uniformly on compact sets, with the property that null directions for $\mathfrak{g}_\epsilon$ are spacelike for $\mathfrak{g}$. Consider the family of $\mathfrak{g}_\epsilon$–null Lipschitz hypersurfaces

$$\mathscr{N}_\epsilon := \dot J^+_\epsilon(S) \cap \mathscr{M} \ ,$$

where $\dot J^+_\epsilon$ denotes the boundary of the causal future with respect to the metric $\mathfrak{g}_\epsilon$. The $\mathscr{N}_\epsilon$'s are threaded with $\mathfrak{g}_\epsilon$–null geodesics, with initial points on S, which converge uniformly to $\mathfrak{g}$-null geodesics starting from S, hence to the generators of $\mathscr{E}^+$ (within $\mathscr{M}'$). It follows that, for all ϵ small enough, $\mathscr{N}_\epsilon$ intersects $\mathscr{S}$ transversally. Furthermore, since $\mathscr{E}^+$ is smooth, decreasing ϵ if necessary, continuity of Jacobi fields with respect to ϵ implies that the $\mathscr{N}_\epsilon$'s remain smooth in the portion between S and their intersection with $\mathscr{S}$. Choosing ϵ small enough, one obtains a smooth $\mathfrak{g}$-spacelike hypersurface $\mathscr{S}'$, with boundary at S, by taking the union of the portion of $\mathscr{N}_\epsilon$ between S and where it meets $\mathscr{S}$, with that portion of $\mathscr{S}$ which extends to infinity and which is bounded by the intersection with $\mathscr{N}_\epsilon$, and smoothing out the intersection. The hypersurface $\mathscr{S}'$ can be shown to be Cauchy by the usual arguments [16, 60].

By [44] there exists an asymptotically flat Cauchy hypersurface $\mathscr{S}''$ for $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$, with boundary on S, which is maximal.

We wish to show, now, that $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle'$, and hence $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$, are static; this has been first proved in [127], but a rather simple proof proceeds as follows: Let us decompose X' as $Nn+Z$, where n is the future-directed normal to $\mathscr{S}''$, while Z is tangent. The space-time Killing equations imply

$$D_iZ_j + D_jZ_i = -2NK_{ij} \ , \tag{2.7.1}$$

where g_{ij} is the metric induced on $\mathscr{S}''$, K_{ij} is its extrinsic curvature tensor, and D is the covariant derivative operator of g_{ij}. Since $\mathscr{S}''$ is maximal, the (vacuum) momentum constraint reads

$$D_iK^{ij} = 0 \ . \tag{2.7.2}$$

From (2.7.1)-(2.7.2) one obtains

$$D_i(K^{ij}Z_j) = -NK^{ij}K_{ij} \ . \tag{2.7.3}$$

Integrating (2.7.3) over $\mathscr{S}''$, the boundary integral in the asymptotically flat regions gives no contribution because K_{ij} approaches zero there as $O(1/r^{n-1})$, while Z approaches zero there as $O(1/r^{n-2})$ [39]. The boundary integral at the horizons vanishes since Z and N vanish on $S = \partial\mathscr{S}''$ by construction. Hence

$$\int_{\mathscr{S}''} NK^{ij}K_{ij} = 0 \ . \tag{2.7.4}$$

On a maximal hypersurface the normal component N of a Killing vector satisfies the equation

$$\Delta N = K^{ij}K_{ij}N \ , \tag{2.7.5}$$

and the maximum principle shows that N is strictly positive except at $\partial\mathscr{S}''$. Staticity of $\langle\langle\mathscr{M}_{\mathrm{ext}}\rangle\rangle'$ along $\mathscr{S}''$ follows now from (2.7.4). Moving the $\mathscr{S}''$'s with the isometry group one covers $\langle\langle\mathscr{M}_{\mathrm{ext}}\rangle\rangle'$ [44], and staticity of $\langle\langle\mathscr{M}_{\mathrm{ext}}\rangle\rangle'$ follows. Hence $\langle\langle\mathscr{M}_{\mathrm{ext}}\rangle\rangle$ is static as well, and Theorem 2.1.1 allows us to conclude that $\langle\langle\mathscr{M}_{\mathrm{ext}}\rangle\rangle$ is Schwarzschildian. This achieves the proof of Theorem 2.0.1. □

Chapter 3

On the classification of stationary electro-vacuum black holes

We will now consider I^+–regular electro-vacuum space-times and prove the following generalization of Theorem 2.0.1:

THEOREM 3.0.1 *Let $(\mathscr{M}, \mathfrak{g}, F)$ be a stationary, asymptotically flat, I^+–regular, electro-vacuum, four-dimensional analytic space-time, satisfying* (3.1.5) *and* (3.1.6). *If each component of the event horizon is mean non-degenerate, then $\langle\langle\mathscr{M}_{\mathrm{ext}}\rangle\rangle$ is isometric to the domain of outer communications of one of the Weinstein solutions of Section 3.5. In particular, if the event horizon is connected and mean non-degenerate, then $\langle\langle\mathscr{M}_{\mathrm{ext}}\rangle\rangle$ is isometric to the domain of outer communications of a Kerr-Newman space-time.*

3.1 Preliminaries

An *electro-vacuum* space-time is a triple $(\mathscr{M}, \mathfrak{g}, F)$, assembled by a $(n+1)$–dimensional Lorentzian manifold $(\mathscr{M}, \mathfrak{g})$ endowed with a 2-form F, that satisfies the *source free Einstein-Maxwell* field equations *with global electromagnetic potential* A[1]

$$\mathrm{Ric} - \frac{1}{2}\mathrm{R}\mathfrak{g} = 2\,\mathrm{T}_F\ , \tag{3.1.1}$$

$$F = dA\ , \tag{3.1.2}$$

[1]In this chapter, when refereing to *electro-vacuum* or to the *source free Einstein-Maxwell* equations the existence of a global electromagnetic potential will always be assumed.

The existence of the global potential (3.1.2) as well as the decay rate (3.1.5) imply non-existence of magnetic charges a priori (compare (3.4.44) and (4.1.23)). This is a highly unsatisfactory situation.

$$d * F = 0 \ , \tag{3.1.3}$$

where Ric is the Ricci curvature tensor of the metric $\mathfrak{g}$, R its scalar curvature and T_F is the energy-momentum tensor of the *electromagnetic* 2-form F,

$$\mathrm{T}_F(u,v) := \mathfrak{g}(i_u F, i_v F) - \frac{1}{2}|F|^2 \mathfrak{g}(u,v) \ . \tag{3.1.4}$$

We will be interested in asymptotically flat space-times as defined in section 2.2 and in addition we will impose the following decay rate for the electromagnetic potential

$$A_\mu = O_k(r^{-\alpha}) \ . \tag{3.1.5}$$

We will also require the electromagnetic field to be invariant under the flow of the relevant Killing vectors

$$\mathscr{L}_{K_{(\mu)}} F = 0 \ . \tag{3.1.6}$$

As mentioned before, in the stationary and asymptotically flat scenario one is able to choose adapted coordinates so that the metric can, in a neighborhood of infinity, be written as

$$\mathfrak{g} = -V^2(dt + \underbrace{\theta_i dx^i}_{=\theta})^2 + \underbrace{\gamma_{ij} dx^i dx^j}_{=\gamma} \ . \tag{3.1.7}$$

Since we are also assuming electro-vacuum we get the following improvement of the original decay rates [26, Section 1.3],

$$\gamma_{ij} - \delta_{ij} = O_\infty(r^{-1}) \ , \quad \theta_i = O_\infty(r^{-1}) \ , \quad V - 1 = O_\infty(r^{-1}) \ , \tag{3.1.8}$$

and

$$A_\mu = O_\infty(r^{-1}) \ , \tag{3.1.9}$$

where the infinity symbol means that (2.2.2) holds for arbitrary k.

3.2 Weyl coordinates

On a region charted by Weyl coordinates the source free Einstein-Maxwell equations simplify considerably. It has been for long expected and showed in Theorem 2.5.4 that such global chart is available away from the axis of a stationary and axisymmetric vacuum domain of outer communications. In fact the role of the vacuum field equations in the referred analysis – they imply the *orthogonal integrability conditions* (3.2.1) and allow us to show that, whenever defined, the squared root of the *area function* (3.2.10) is harmonic with respect to the orbit space metric – is fulfilled by the electro-vacuum field equations.

The first of these well known results, which neither requires $K_{(0)}$ to be stationary, nor $K_{(1)}$ to be a generator of axisymmetry, generalizes to higher dimensions as follows (compare [18]):

PROPOSITION 3.2.1 *Let $(\mathscr{M},\mathfrak{g},F)$ be an $(n{+}1)$–dimensional electro-vacuum space-time, possibly with a cosmological constant, with $n-1$ commuting Killing vector fields satisfying*

$$\mathscr{L}_{K_{(\mu)}}F=0 \quad , \ \mu=0,\ldots,n-2\ .$$

If $n-2$ of the zero sets $\mathscr{A}_\mu:=\{p\in\mathscr{M}\ |\ K_{(\mu)}|_p=0\}$ are non-empty then [2]

$$dK_{(\mu)}\wedge K_{(0)}\wedge\ldots\wedge K_{(n-2)}=0 \quad , \ \forall\mu=0,\ldots,n-2\ . \tag{3.2.1}$$

PROOF: To fix conventions, we use a Hodge star defined through the formula

$$\alpha\wedge\beta=\pm\langle *\alpha,\beta\rangle\mathrm{Vol}\ ,$$

where the plus sign is taken in the Riemannian case, minus in our Lorentzian one, while Vol is the volume form. The following (well known) identities are useful [74]:

$$**\theta=(-1)^{s(n+1-s)-1}\theta\ , \qquad \forall\theta\in\Lambda^s\ , \tag{3.2.2}$$

$$i_X*\theta=*(\theta\wedge X)\ , \qquad \forall\theta\in\Lambda^s\ , \quad X\in\Lambda^1\ . \tag{3.2.3}$$

Further, for any Killing vector K,

$$[\mathscr{L}_K,*]=0\ . \tag{3.2.4}$$

The Leibniz rule for the divergence $\delta:=*d*$ reads, for $\theta\in\Lambda^s$,

$$\begin{aligned}\delta(\theta\wedge K)&=*d*(\theta\wedge K)\overset{(3.2.3)}{=}*d(i_K*\theta)=*(\mathscr{L}_K*\theta-i_Kd*\theta)\\&\overset{(3.2.2),(3.2.4)}{=}**\mathscr{L}_K\theta-*i_K(-1)^{(n+1-s+1)(n+1-(n+1-s+1))-1}**d*\theta\\&=(-1)^{s(n+1-s)-1}\mathscr{L}_K\theta-(-1)^{s(n+1-s)-n+1}**(\delta\theta\wedge K)\\&=(-1)^{s(n+1-s)-1}\mathscr{L}_K\theta+(-1)^{n+1}\delta\theta\wedge K\ .\end{aligned}$$

[2]By an abuse of notation, we use the same symbols for vector fields and for the associated 1-forms.

Applying this to $\theta = dK$ one obtains

$$\begin{aligned}
d(dK \wedge K) &= -\mathscr{L}_K dK + (-1)^{n+1}\delta dK \wedge K \\
&= (-1)^{n+1}\delta dK \wedge K \; .
\end{aligned}$$

As any Killing vector is divergence free, we see that

$$\delta dK = (-1)^n \Delta K = (-1)^n 2\,\mathrm{tr}\nabla^2 K = (-1)^{n+1} 2\, i_K \,\mathrm{Ric}\;,$$

where Δ is the Laplace-Beltrami operator. The assumed field equations (with cosmological constant Λ) imply

$$\mathrm{Ric} = 2\,\mathrm{T}_F + \frac{2}{n-1}\Lambda \mathfrak{g}\;,$$

from which

$$\begin{aligned}
d(dK \wedge K) &= (-1)^{n+1}(-1)^{n+1} 2\, i_K (2\,\mathrm{T}_F + \frac{2}{n-1}\Lambda\mathfrak{g}) \wedge K \\
&= 2\left(2\, i_K \mathrm{T}_F \wedge K + \frac{2}{n-1}\Lambda\, K\right) \wedge K = 4\, i_K \mathrm{T}_F \wedge K \; .
\end{aligned}$$

Letting $\alpha := i_K F$ for any vector field X we have

$$\begin{aligned}
\alpha \cdot i_X F &= - * (\alpha \wedge * i_X F) = -(-1)^n * (* i_X F \wedge \alpha) \\
&= (-1)^{n+1} i_\alpha * * F = (-1)^{n+1}(-1)^{n+1} i_\alpha i_X F \\
&= -F(\alpha, X)\;,
\end{aligned}$$

which inserted into (3.1.4) gives

$$i_K T_F = -i_\alpha F - \frac{1}{2}|F|^2 K\;,$$

and consequently

$$*d*(dK \wedge K) = -4(i_\alpha F + \frac{1}{2}|F|^2 K) \wedge K = -4\, i_\alpha F \wedge K = 4\, K \wedge i_\alpha F\; .$$

Meanwhile, since (modulo sign)

$$\begin{aligned}
i_\alpha K &= \pm * (K \wedge * \alpha) = \pm * (K \wedge * i_K F) = \pm * (K \wedge * i_K * * F) \\
&= \pm * (K \wedge * * (* F \wedge K)) = \pm * (K \wedge * F \wedge K) = 0\;,
\end{aligned}$$

for $\beta := i_K * F \in \Lambda^{n-2}$, we have

$$\begin{aligned} *(\alpha \wedge \beta) &= (-1)^{1\times(n-2)} * (\beta \wedge \alpha) = (-1)^{n-2} i_\alpha * \beta \\ &= (-1)^{n-2} i_\alpha * i_K * F = (-1)^{n-2} i_\alpha * *(F \wedge K) \\ &= (-1)^{n-2} i_\alpha (-1)^{3(n+1-3)-1} F \wedge K \\ &= -(i_\alpha F \wedge K + (-1)^2 F \wedge i_\alpha K) = -(i_\alpha F \wedge K + 0) \\ &= K \wedge i_\alpha F \end{aligned}$$

which leads to the significant

$$d * (dK \wedge K) = 4\,\alpha \wedge \beta = 4\, i_K F \wedge i_K * F\ . \tag{3.2.5}$$

Now, for any two commuting Killing vectors and an arbitrary differential form we have

$$\begin{aligned} [\mathscr{L}_{K_{(\mu)}}, i_{K_{(\nu)}}]\theta &= \mathscr{L}_{K_{(\mu)}}(i_{K_{(\nu)}}\theta) - i_{K_{(\nu)}}(\mathscr{L}_{K_{(\mu)}}\theta) \\ &= \mathscr{L}_{K_{(\mu)}}[\theta(K_{(\nu)}\,, \ldots)] - (\mathscr{L}_{K_{(\mu)}}\theta)(K_{(\nu)}\,, \ldots) \\ &= (\mathscr{L}_{K_{(\mu)}}\theta)(K_{(\nu)}\,, \ldots) + \theta(\mathscr{L}_{K_{(\mu)}}K_{(\nu)}\,, \ldots) - (\mathscr{L}_{K_{(\mu)}}\theta)(K_{(\nu)}\,, \ldots) = 0\ , \end{aligned}$$

giving us the commutation relation

$$[K_{(\mu)}, K_{(\nu)}] = 0 \Longrightarrow [\mathscr{L}_{K_{(\mu)}}, i_{K_{(\nu)}}] = 0\ , \tag{3.2.6}$$

from which it follows that

$$\begin{aligned} dF(K_{(\nu)}, K_{(\mu)}) &= d i_{K_{(\mu)}}\alpha_{(\nu)} = -i_{K_{(\mu)}} d\alpha_{(\nu)} + \mathscr{L}_{K_{(\mu)}}\alpha_{(\mu)} \\ &= -i_{K_{(\mu)}}(-i_{K_{(\nu)}} dF + \mathscr{L}_{K_{(\nu)}}F) + i_{K_{(\mu)}}\mathscr{L}_{K_{(\mu)}}F = 0\ , \end{aligned}$$

where we used the fact that F is exact and invariant under the flow of these Killing vectors. By the hypothesis on the zero sets, for any pair $\mu \neq \nu$, we may take $\mathscr{A}_{(\mu)} \neq \emptyset$. We then have $F(K_{(\mu)}, K_{(\nu)})|_{\mathscr{A}_{(\mu)}} \equiv 0$ and consequently

$$F(K_{(\mu)}, K_{(\nu)}) \equiv 0\ ,\ \forall\, \mu, \nu \in \{0, ..., n-2\}\ . \tag{3.2.7}$$

A similar computation leads to

$$i_{K_{(\mu)}} i_{K_{(\nu)}} * F = 0\ ,\ \forall\, \mu, \nu \in \{0, ..., n-2\}\ . \tag{3.2.8}$$

Now, let $\omega_{(\mu)}$ be the μ'th twist form,

$$\omega_{(\mu)} := *(dK_{(\mu)} \wedge K_{(\mu)}) \ .$$

The identity

$$\begin{aligned}\mathscr{L}_{K_{(\mu)}}\omega_{(\nu)} &= \mathscr{L}_{K_{(\mu)}} * (dK_{(\mu)} \wedge K_{(\nu)}) \\ &= *(\mathscr{L}_{K_{(\mu)}}dK_{(\nu)} + dK_{(\nu)} \wedge \mathscr{L}_{K_{(\mu)}}K_{(\nu)}) = 0 \ ,\end{aligned}$$

together with

$$\mathscr{L}_{K_{(\mu_1)}}(i_{K_{(\mu_2)}} \dots i_{K_{(\mu_\ell)}}\omega_{(\mu_{\ell+1})}) = i_{K_{(\mu_2)}} \dots i_{K_{(\mu_{n-1})}}\mathscr{L}_{K_{(\mu_\ell)}}\omega_{(\mu_{\ell+1})} = 0 \ ,$$

and Cartan's formula for the Lie derivative, gives

$$d(i_{K_{(\mu_1)}} \dots i_{K_{(\mu_\ell)}}\omega_{(\mu_{\ell+1})}) = (-1)^\ell i_{K_{(\mu_1)}} \dots i_{K_{(\mu_{n-1})}}d\omega_{(\mu_{\ell+1})} \ . \qquad (3.2.9)$$

We thus have

$$\begin{aligned}d * (dK_{(\mu_0)} \wedge K_{(\mu_0)} \wedge \dots \wedge K_{(\mu_{n-2})}) &= d(i_{K_{(\mu_{n-2})}} \dots i_{K_{(\mu_1)}} * (dK_{(\mu_0)} \wedge K_{(\mu_0)})) \\ &= (-1)^{n-2} i_{K_{(\mu_{n-2})}} \dots i_{K_{(\mu_1)}} d\omega_{(\mu_0)} \\ &\overset{(3.2.5)}{=} (-1)^n i_{K_{(\mu_{n-2})}} \dots i_{K_{(\mu_1)}} 4\, \alpha_{(\mu_0)} \wedge \beta_{(\mu_0)} \\ &= 4\,(-1)^n i_{K_{(\mu_{n-2})}} \dots i_{K_{(\mu_2)}} \\ &\qquad (i_{K_{(\mu_1)}}\alpha_{(\mu_0)} \wedge \beta_{(\mu_0)} - \alpha_{(\mu_0)} \wedge i_{K_{(\mu_1)}}\beta_{(\mu_0)}) \\ &= 4\,(-1)^n i_{K_{(\mu_{n-2})}} \dots i_{K_{(\mu_2)}} \\ &\qquad (F(K_{(\mu_0)}, K_{(\mu_1)})\beta_{(\mu_0)} - \alpha_{(\mu_0)} \wedge i_{K_{(\mu_1)}} i_{K_{(\mu_0)}} * F) \\ &\overset{(3.2.7,3.2.8)}{=} 0 \ .\end{aligned}$$

So the function $*(dK_{(\mu_0)} \wedge K_{(\mu_0)} \wedge K_{(\mu_1)} \wedge \dots \wedge K_{(\mu_{n-2})})$ is constant, and, as before, the result follows from the hypothesis on the zero sets.

Noting that a globally hyperbolic, stationary and asymptotically flat domain of outer communications satisfying the null energy condition is necessarily simply-connected [45, 61, 62], in view of the previous result Theorem 2.5.6 translates to the electro-vacuum setting as:

THEOREM 3.2.2 *Let $(\mathscr{M}, \mathfrak{g}, F)$ be a four-dimensional, I^+–regular, asymptotically flat, electro-vacuum space-time with stationary Killing vector $K_{(0)}$ and periodic*

Killing vector $K_{(1)}$, jointly generating an $\mathbb{R}\times \mathrm{U}(1)$ subgroup of the isometry group of $(\mathscr{M},\mathfrak{g})$. If $\langle\langle \mathscr{M}_{\mathrm{ext}}\rangle\rangle$ is globally hyperbolic, then the area function

$$W := -\det \Big(\mathfrak{g}(K_{(\mu)},K_{(\nu)})\Big)_{\mu,\nu=0,1} , \tag{3.2.10}$$

is non-negative on $\langle\langle \mathscr{M}_{\mathrm{ext}}\rangle\rangle$, vanishing precisely on the union of its boundary with the (non-empty) set $\{\mathfrak{g}(K_{(1)},K_{(1)})=0\}$.

Away from points where $K_{(0)}\wedge K_{(1)}$ vanishes, which according to Corollary 2.3.8 correspond, in a chronological $\langle\langle \mathscr{M}_{\mathrm{ext}}\rangle\rangle$, exactly to axis points

$$\mathscr{A} := \{q\in \mathscr{M} \mid K_{(1)}|_q = 0\} , \tag{3.2.11}$$

there is a well defined and differentiable local cross-section for the $\mathbb{R}\times U(1)$ action. We can endow this cross-section with the orbit space metric

$$q(Z_1,Z_2) = \mathfrak{g}(Z_1,Z_2) - h^{\mu\nu}\mathfrak{g}(Z_1,K_{(\mu)})\mathfrak{g}(Z_2,K_{(\nu)}) , \tag{3.2.12}$$

whenever $h_{\mu\nu} := \mathfrak{g}(K_{(\mu)},K_{(\nu)})$ is non-singular. The established orthogonality conditions allow us to identify, at least locally, the previous orbit space structure with a 2-surface orthogonal to the Killing vectors, provided by (3.2.1), endowed with the induced metric. From this and Theorem (3.2.2) we see that q is well defined and Riemannian throughout $\langle\langle \mathscr{M}_{\mathrm{ext}}\rangle\rangle\setminus\mathscr{A}$; it is then well known [135] that

$$\Delta_q\sqrt{W} = 0 , \tag{3.2.13}$$

whenever W is non-negative and q is Riemannian, which again is the case within $\langle\langle \mathscr{M}_{\mathrm{ext}}\rangle\rangle\setminus\mathscr{A}$.

According to the Structure Theorem 2.4.5, I^+–regularity allows for the decomposition

$$\overline{\langle\langle \mathscr{M}_{\mathrm{ext}}\rangle\rangle}\cap I^+(\mathscr{M}_{\mathrm{ext}}) = \mathbb{R}\times\overline{\mathscr{S}} , \tag{3.2.14}$$

with $K_{(1)}$ tangent to $\overline{\mathscr{S}}$, a simply-connected spacelike hypersurface with boundary which is an asymptotically flat global cross-section for the action generated by the stationary vector. We are now allowed to use the classification of circle actions on simply-connected 3-manifolds of Orlik and Raymond [110, 115] to obtain a global cross-section for the $\mathbb{R}\times\mathbf{U}(1)$ action in $\langle\langle \mathscr{M}_{\mathrm{ext}}\rangle\rangle\setminus\mathscr{A}$. Then, by (3.2.13) and relying on the results of [31], while disallowing the existence of degenerate horizons, we are able to undertake an analysis leading to

$$\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle \setminus \mathscr{A} \approx \mathbb{R} \times S^1 \times \mathbb{R}^+ \times \mathbb{R} \; , \tag{3.2.15}$$

while showing that this diffeomorphism defines a global coordinate system (t, φ, ρ, z) with

$$K_{(0)} = \partial_t \; , \quad K_{(1)} = \partial_\varphi \quad \text{and} \quad \rho = \sqrt{W} \; . \tag{3.2.16}$$

After invoking (3.2.1) once more, the desired global expression for the space-time metric in terms of Weyl coordinates [3]

$$\mathfrak{g} = -\rho^2 e^{2\lambda} dt^2 + e^{-2\lambda}(d\varphi - w dt)^2 + e^{2u}(d\rho^2 + dz^2) \; , \tag{3.2.17}$$

follows, with

$$u = O_{k-4}(r^{-1}) \;\; , \; r = \sqrt{\rho^2 + z^2} \to \infty \; . \tag{3.2.18}$$

3.3 Reduction to a harmonic map problem

The electro-vacuum field equations (3.1.1)-(3.1.3) and simple-connectedness of $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ guarantee the global existence of the following potentials:

$$d\chi = i_{K_{(1)}} F \;\; , \quad d\psi = i_{K_{(1)}} * F \quad \text{and} \quad dv = \omega - 2(\chi d\psi - \psi d\chi) \; , \tag{3.3.1}$$

where

$$\omega := *(dK^\flat_{(1)} \wedge K^\flat_{(1)}) \; , \tag{3.3.2}$$

is the axial twist form. As discussed in detail in [135], when a global representation in terms of Weyl coordinates like (3.2.17) is allowed, the space-time metric is uniquely determined by an axisymmetric harmonic map

$$\Phi = (\lambda, v, \chi, \psi) : \mathbb{R}^3 \setminus \mathscr{A} \longrightarrow \mathbb{H}^2_{\mathbb{C}} \; , \tag{3.3.3}$$

here $\mathscr{A} = \{(0,0,z) \mid z \in \mathbb{R}\}$ and $\mathbb{H}^2_{\mathbb{C}}$ is the 'upper half-space model' of the 2-dimensional complex hyperbolic space, i.e., $\mathbb{R}^4$ with metric given by

$$ds^2 = d\lambda^2 + e^{4\lambda}(dv + \chi d\psi - \psi d\chi)^2 + e^{2\lambda}(d\chi^2 + d\psi^2) \; . \tag{3.3.4}$$

[3]For convenience we will change the notation used in Chapter 2 for the metric components in Weyl coordinates (compare (2.6.13)).

The metric coefficient λ is part of the harmonic map and the remaining unknowns of the metric can be determined from Φ by considering the unique solution (w, u) of the set of equations

$$\partial_\rho w = -e^{4\lambda}\rho\,\omega_z\,, \qquad \partial_z w = e^{4\lambda}\rho\,\omega_\rho\,, \tag{3.3.5}$$

$$\partial_\rho u - \partial_\rho\lambda = \rho\left[(\partial_\rho\lambda)^2 - (\partial_z\lambda)^2 + \frac{1}{4}e^{4\lambda}(\omega_\rho^2 - \omega_z^2) + e^{2\lambda}\left((\partial_\rho\chi)^2 - (\partial_z\chi)^2 + (\partial_\rho\psi)^2 - (\partial_z\psi)^2\right)\right] \tag{3.3.6}$$

$$\partial_z u - \partial_z\lambda = 2\,\rho\left[\partial_\rho\lambda\,\partial_z\lambda + \frac{1}{4}e^{4\lambda}\omega_\rho\,\omega_z + e^{2\lambda}(\partial_\rho\chi\,\partial_z\chi + \partial_\rho\psi\,\partial_z\psi)\right]\,, \tag{3.3.7}$$

that go to zero at infinity, and where we write $\omega_a := \omega(\partial_a)$ for $a \in \{\rho, z\}$.

3.3.1 Distance function on the target manifold

The criteria for uniqueness of harmonic maps used in this thesis (see Theorem 3.5.1 and compare [38, Appendix C]), is stated in terms of the pointwise distance between the maps. For the 'disk model' of $\mathbb{H}^2_{\mathbb{C}}$ the distance between two points $z = (z_1, z_2)$ and $w = (w_1, w_2)$ is given by [133, eq 55, pg 26]

$$\cosh(d) = \frac{|1 - \bar{z}_1\,w_1 - \bar{z}_2\,w_2|}{\sqrt{1 - |z|^2}\sqrt{1 - |w|^2}}\,. \tag{3.3.8}$$

To obtain the distance function for the 'upper half-space model' we will use the isometry between the two referred models presented in [133, Appendix]: first we perform the coordinate transformation

$$z_1 = \frac{1 - x_1}{1 + x_1} \quad , \quad z_2 = \frac{2x_2}{1 + x_1}\,,$$

with analogous expressions for $w_i = w_i(y_1, y_2)$ to obtain

$$|1 - \bar{z}_1\,w_1 - \bar{z}_2\,w_2| = \frac{2\,|\bar{x}_1 + y_1 - 2\,\bar{x}_2\,y_2|}{|1 + x_1|\,|1 + y_1|}\,;$$

then we take

$$e^{\lambda_1} = \frac{|1 + z_1|}{\sqrt{1 - |z|^2}} \quad \text{and} \quad e^{\lambda_2} = \frac{|1 + h_1|}{\sqrt{1 - |h|^2}}$$

so that

$$\cosh(d) = \frac{1}{2}\,|\bar{x}_1 + y_1 - 2\,\bar{x}_2\,y_2|\,e^{\lambda_1 + \lambda_2}\,; \tag{3.3.9}$$

and finally, by writing

$$x_1 = e^{-2\lambda_1} + \chi_1^2 + \psi_1^2 + 2\,iv_1 \quad \text{and} \quad x_2 = \chi_1 + i\psi_1 \;, \tag{3.3.10}$$

with similar expressions for $y_i = y_i(\lambda_2, v_2, \chi_2, \psi_2)$, we see that the distance function satisfies [4]

$$\begin{aligned} \cosh^2(d) = &\ \frac{1}{4}\, e^{2(\lambda_1+\lambda_2)}(e^{-2\lambda_1} + e^{-2\lambda_2} + (\chi_1 - \chi_2)^2 + (\psi_1 - \psi_2)^2)^2 \\ &+ e^{2(\lambda_1+\lambda_2)}(v_2 - v_1 - \chi_1\psi_2 + \chi_2\psi_1)^2 \\ = &\ \frac{1}{4}\left\{e^{-\lambda_1+\lambda_2} + e^{\lambda_1-\lambda_2} + e^{\lambda_1+\lambda_2}(\chi_1 - \chi_2)^2 + e^{\lambda_1+\lambda_2}(\psi_1 - \psi_2)^2\right\}^2 \\ &+ e^{2(\lambda_1+\lambda_2)}\left\{(v_2 - v_1) + (\chi_2\psi_1 - \chi_1\psi_2)\right\}^2 \;, \end{aligned} \tag{3.3.11}$$

or in an apparently more intrinsic way

$$\begin{aligned} \cosh^2(d) = &\ \frac{1}{4}\left\{\sqrt{\frac{\mathfrak{g}_2(\partial_\varphi, \partial_\varphi)}{\mathfrak{g}_1(\partial_\varphi, \partial_\varphi)}} + \sqrt{\frac{\mathfrak{g}_1(\partial_\varphi, \partial_\varphi)}{\mathfrak{g}_2(\partial_\varphi, \partial_\varphi)}} + \frac{(\chi_1 - \chi_2)^2 + (\psi_1 - \psi_2)^2}{\sqrt{\mathfrak{g}_1(\partial_\varphi, \partial_\varphi)}\sqrt{\mathfrak{g}_2(\partial_\varphi, \partial_\varphi)}}\right\}^2 \\ &+ \left\{\frac{(v_2 - v_1) + (\chi_2\psi_1 - \chi_1\psi_2)}{\sqrt{\mathfrak{g}_1(\partial_\varphi, \partial_\varphi)}\sqrt{\mathfrak{g}_2(\partial_\varphi, \partial_\varphi)}}\right\}^2 \;. \end{aligned} \tag{3.3.12}$$

It will also be helpful to use the usual rescaling

$$U_i = \lambda_i + \ln\rho \;, \quad \text{so that} \quad \mathfrak{g}_i(\partial_\varphi, \partial_\varphi) = \rho^2 e^{-2U_i} = e^{-2\lambda_i} \;, \tag{3.3.13}$$

from which we get our final expression for the distance in the 'upper half-space':

$$\begin{aligned} \cosh^2(d) = &\frac{1}{4}\left\{e^{U_1-U_2} + e^{-U_1+U_2} + \rho^{-2}e^{U_1+U_2}(\chi_1 - \chi_2)^2 + \rho^{-2}e^{U_1+U_2}(\psi_1 - \psi_2)^2\right\}^2 \\ &+ \left\{\rho^{-2}e^{U_1+U_2}(v_2 - v_1) - \rho^{-2}e^{U_1+U_2}(\chi_1\psi_2 - \chi_2\psi_1)\right\}^2 \;. \end{aligned} \tag{3.3.14}$$

3.4 Boundary conditions

3.4.1 The Axis

From now on we will be controlling the distance, as given by any of the formulae in the previous section, between the harmonic maps arising from two I^+–regular,

[4] By taking $\chi_i = \psi_i \equiv 0$ we see that this distance function is related to the one used in the vacuum case in Section 2.6.5.1 by $d = 2d_b$. This discrepancy has its genesis in an analogous relation between the line elements of the different disk models used.

stationary-axisymmetric and electro-vacuum space-times $(\mathscr{M}_i, \mathfrak{g}_i)$, $i = 1, 2$. We will start by showing that

$$d(\Phi_1, \Phi_2) \text{ is bounded near } \overline{\mathscr{A} \cap \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle} \, . \tag{3.4.1}$$

In this section we will be working with the following coordinate systems: isothermal coordinates $(\hat{x}_i, \hat{z}_i)$ globally defined in the doubling across the axis of the orbit space of an appropriate extension of the $\mathbf{U}(1)$ action to the manifold obtained by the addition of 3-discs to every connected component of $\partial\mathscr{S}_i$;[5] "canonical coordinates" (ρ, z) of the half plane $\mathbb{R}_0^+ \times \mathbb{R}$, which is the image of each (physical) orbit space by the map Ψ_i defined by $(\hat{x}_i, \hat{z}_i) \mapsto (\rho_i(\hat{x}_i, \hat{z}_i), z_i(\hat{x}_i, \hat{z}_i))$.

Let ϕ_s be the flow generated by the axial Killing vector $K_{(1)}$. In the doubling of the orbit space the isothermal coordinates satisfy

$$\hat{x} \circ \phi_\pi = -\hat{x} \quad \text{ and } \quad \hat{z} \circ \phi_\pi = \hat{z} \, .$$

Then, invariance of a function $(\hat{x}, \hat{z}) \mapsto f(\hat{x}, \hat{z})$ under the axial flow, which is the case for the fields v, χ and ψ, implies that the function $\hat{x} \mapsto f(\hat{x}, \hat{z})$ is even for all $\hat{z}$. In this case, if f is C^2, Taylor expanding on $\hat{x}$, from the axis, gives

$$f(\hat{x}, \hat{z}) = f(0, \hat{z}) + \frac{1}{2}\frac{\partial^2 f}{\partial \hat{x}^2}(c(\hat{x}), \hat{z})\, \hat{x}^{\,2} \ \ , \ |c(\hat{x})| \leq |\hat{x}| \ . \tag{3.4.2}$$

Now fix a point in $\overline{\mathscr{A} \cap \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle}$ and by rescaling $\hat{z}$ assume it lies at the origin. Suppose also that $f \equiv f_0 := f(0, 0)$ along the connected component of $\overline{\mathscr{A} \cap \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle}$, in $\overline{\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle}$, containing $(0, 0)$; this is clearly the case for all the functions appearing in 3.3.11 and it also implies that we can realize the aforementioned extension of the doubling of the orbit space to $\mathbb{R}^2$ while preserving the constancy of f along the extended axis near the poles, i.e., near the points where the axis meets the event horizon. Then (3.4.2) implies

$$|f(\hat{x}, \hat{z}) - f_0| \leq C\hat{x}^{\,2} \quad \text{near } (0, 0) \in \overline{\mathscr{A} \cap \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle} \, . \tag{3.4.3}$$

We will need better control over the functions $e^{-2\lambda} = \mathfrak{g}_{\varphi\varphi}$. To this end let $\{x, y, z\}$ be Gaussian coordinates along the axis, in the extension of $\mathscr{S}$, with

[5] The resulting space is diffeomorphic to $\mathbb{R}^2$, see Figure 2.5.1, and for more details concerning this construction see Section 2.6; also, the fact that I^+-regularity, stationarity and the null energy condition imply spherical topology for the connected components of the cross-section of the event horizon $\partial\mathscr{S}_i$ follows from [45].

$\mathscr{A} = \{x = y = 0\}$ and for which $K_{(1)} = x\partial_y - y\partial_x$ (see [31, pg 5] and compare with (3.4.23)). For any path with initial velocity transverse to $\mathscr{A}$ we have

$$\nabla_{\dot\gamma(0)} K_{(1)}|_{x=y=0} = \nabla_{\gamma^i\partial_i}(x\partial_y - y\partial_x)|_{x=y=0} = \gamma^x\partial_y - \gamma^y\partial_x \ , \tag{3.4.4}$$

and consequently $\mathfrak{g}(\nabla_{\dot\gamma(0)}K_{(1)}, \nabla_{\dot\gamma(0)}K_{(1)}) = (\gamma^x)^2 + (\gamma^y)^2 \neq 0$. Since $\nabla_\mu \mathfrak{g}(K_{(1)}, K_{(1)}) = 2\,\mathfrak{g}(\nabla_\mu K_{(1)}, K_{(1)})$ we see that the gradient of $\mathfrak{g}_{\varphi\varphi}$ vanishes at the axis and

$$\nabla_\mu\nabla_\nu\, \mathfrak{g}(K_{(1)}, K_{(1)})|_{\mathscr{A}} = 2\,\mathfrak{g}(\nabla_\mu K_{(1)}, \nabla_\nu K_{(1)})|_{\mathscr{A}} \ .$$

Taylor expanding along γ yields

$$\mathfrak{g}_{\varphi\varphi}\circ\gamma(s) = (\underbrace{\mathfrak{g}(\nabla_{\dot\gamma(0)}K_{(1)}, \nabla_{\dot\gamma(0)}K_{(1)})}_{\neq 0} + O(s))s^2 \ ,$$

from which it follows that *for any path transverse to* $\mathscr{A}$ *and small* s

$$C^{-1}\, s^2 \leq \mathfrak{g}_{\varphi\varphi}\circ\gamma(s) \leq C\, s^2 \ . \tag{3.4.5}$$

We will need to consider two separate cases. First, fix, in each space-time, a point belonging to $\mathscr{A}_i \cap \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ and rescale all the previous coordinate systems so that each of the fixed points corresponds to its respective origin and $\Psi_i(0,0) = (0,0)$. At these points, since there the boundary of the orbit space is analytic, the function $\rho_i = \rho_i(\hat x_i, \hat z_i)$ may be extended analytically across the origin, therefore, as an immediate consequence of (3.4.5) we get control over the first terms appearing in (3.3.14)

$$e^{U_j - U_i} = \sqrt{\frac{\mathfrak{g}_i(\partial_\varphi, \partial_\varphi)}{\mathfrak{g}_j(\partial_\varphi, \partial_\varphi)}} \leq \sqrt{\frac{C_i\rho^2}{C_j^{-1}\rho^2}} \leq C \quad \text{near } (0,0) \in \mathscr{A}\cap\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle \ . \tag{3.4.6}$$

Since the χ_i's and the ψ_i's are all bounded near the origin our goal gets reduced to showing that

$$\rho^{-2} e^{U_1+U_2}(f_1 - f_2) = O(1) \text{ near } (0,0) \in \mathscr{A}\cap\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle \ , \tag{3.4.7}$$

when $f_1 = \chi_1, \psi_1, v_1, \chi_1\psi_2$ and $f_2 = \chi_2, \psi_2, v_2, \chi_2\psi_1$, where by this we mean that if, for example, we set $f_1 = \chi_1$ then $f_2 = \chi_2$.

Let us start with $f_1 = \chi_1, \psi_1, v_1$ and $f_2 = \chi_2, \psi_2, v_2$. Each f_i is invariant under the respective axial flow and constant along each connected component of

$\mathscr{A}_i \cap \langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$, so if we impose $f_1(0,0) = f_2(0,0) = f_0$, which is always achievable if the space-times $(\mathscr{M}_i, \mathfrak{g}_i)$ have the same set of masses, angular momenta and charges (see Section 3.5 and [135, Section 2.3]), we see that (3.4.3) holds and using (3.4.5) and (3.4.6) we get

$$
\begin{aligned}
\left|\rho^{-2} e^{U_1+U_2}(f_1 - f_2)\right| &= \left|\frac{f_1 - f_2}{\sqrt{\mathfrak{g}_1(\partial_\varphi, \partial_\varphi)}\sqrt{\mathfrak{g}_2(\partial_\varphi, \partial_\varphi)}}\right| \leq \frac{|f_1 - f_0| + |f_2 - f_0|}{\sqrt{\mathfrak{g}_1(\partial_\varphi, \partial_\varphi)}\sqrt{\mathfrak{g}_2(\partial_\varphi, \partial_\varphi)}} \\
&= \frac{|f_1 - f_0|}{\mathfrak{g}_1(\partial_\varphi, \partial_\varphi)}\sqrt{\frac{\mathfrak{g}_1(\partial_\varphi, \partial_\varphi)}{\mathfrak{g}_2(\partial_\varphi, \partial_\varphi)}} + \frac{|f_2 - f_0|}{\mathfrak{g}_2(\partial_\varphi, \partial_\varphi)}\sqrt{\frac{\mathfrak{g}_2(\partial_\varphi, \partial_\varphi)}{\mathfrak{g}_1(\partial_\varphi, \partial_\varphi)}} \\
&\leq \frac{C_1 \hat{x}_1^2}{C_2^{-1}\hat{x}_1^2} C_3 + \frac{C_4 \hat{x}_2^2}{C_5^{-1}\hat{x}_2^2} C_6 \\
&\leq C \quad \text{near } (0,0) \in \mathscr{A} \cap \langle\langle \mathscr{M}_{\rm ext} \rangle\rangle \ .
\end{aligned}
\tag{3.4.8}
$$

We take the chance to stress the fact that the previous argument does not apply to the fields $\chi_1\psi_2$ and $\chi_2\psi_1$ since these products involve functions originating from different space-times and therefore only make sense as functions of (ρ, z) for which estimates like 3.4.3 are not available a priori.[6]

To bypass this problem we write

$$
\chi_1\psi_2 - \chi_2\psi_1 = (\chi_1 + \chi_2)(\psi_2 - \psi_1) + \chi_1\psi_1 - \chi_2\psi_2 \ .
$$

Since $\chi_1 + \chi_2$ is bounded, to control the first term we just need to take $f_i = \psi_i$ as before. Setting $f_1 = \chi_1\psi_1$ and $f_2 = \chi_2\psi_2$ we see that the previous argument still applies as these are also axially symmetric functions which are constant along the axis components. The desired result follows.

To finish the proof of boundedness of (3.3.14) near the singular set $\mathscr{A} \cap \langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ we still have to analyze what happens near points where the axis meets the horizon. Choose such a point in each space-time and, without loss of generality, assume that these are 'north poles' which, as before, lie at the origin of the coordinate systems $(\hat{x}_i, \hat{z}_i)$, and satisfy $(\rho, z) = \Psi_i(0,0) = (0,0)$.

[6] In fact, extending ρ_i and z_i near these axis points by $\rho_i(-\hat{x}_i, \hat{z}_i) = -\rho_i(\hat{x}_i, \hat{z}_i)$ and $z_i(-\hat{x}_i, \hat{z}_i) = z_i(\hat{x}_i, \hat{z}_i)$ shows that invariance under the axial flow implies that $f(\rho, z) := f \circ \Psi_i^{-1}(\rho, z)$ is an even function of ρ. Then, direct estimates in terms of ρ analogous to (3.4.3) may be obtained for all the fields and the presented procedure including (3.4.8) may be bypassed. Unfortunately this is no longer possible near points where the axis meets the horizon as the ρ_i are no longer differentiable.

As already mentioned, a careful extension of the doubling of the orbits spaces validates (3.4.3) in a neighborhood of these 'north poles', but, on the other hand, the ρ_i's are now non-differentiable at such points and (3.4.6) no longer holds. Nonetheless, if we are able to control $e^{U_i - U_j}$ by other means, then the inequalities established in (3.4.8) extend to the case under consideration and boundedness of the distance near the axis follows. This problem, which is in fact the major difficulty that arises in the analysis of the boundary conditions of these axisymmetric harmonic maps, has been recently solved for the vacuum case in Section 2.6.5.1 by obtaining the following uniform estimate

$$U = \ln \sqrt{z + \sqrt{z^2 + \rho^2}} + O(1) \quad \text{near } (0,0) \in \mathscr{A} \cap \mathscr{E}^+ \ , \tag{3.4.9}$$

from which the desired consequence immediately follows. This result, which requires this component of the horizon to be non-degenerate, extends to the electro-vacuum case immediately.

3.4.2 Spatial infinity

In this section we want to show that

$$\lim_{\sqrt{\rho^2+z^2} \to +\infty} d(\Phi_1, \Phi_2) = 0 \ , \tag{3.4.10}$$

with d implicitly defined by (3.3.14). For this we will assume stationarity and asymptotic flatness as given by the system of equations (3.1.7)–(3.1.9). It turns out that the estimates provided by asymptotic flatness, even in the way just defined, seem insufficient to control the relevant fields; even in an adapted frame provided by the results of Section 3.2 integration of the defining equations (3.3.1) yields divergent logarithmic terms. Fortunately, in the stationary and electro-vacuum setting, the asymptotic analysis of Beig and Simon [13,123] provides relevant improvements of the initial decay rates by means of the expansion (3.4.15).

Let D and ϵ_{ijk} denote the covariant derivative and volume element of γ, the induced metric in $\mathscr{S}_{ext}$ as in Section 3.1. A well known consequence of the source free Einstein-Maxwell equations (3.1.1)-(3.1.3) and simple-connectedness of $\langle\langle \mathscr{M}_{\text{ext}} \rangle\rangle$ is the global existence of functions τ and σ satisfying [89,123,135]

$$D_i \tau = V^2 \epsilon_i{}^{jk} (D_j A_k + \theta_j D_k A_0) \ , \tag{3.4.11}$$

and

$$D_i\sigma = -V^4\epsilon_i{}^{jk}D_j\theta_k + i(\bar{\Psi}\partial_i\Psi - \Psi\partial_i\bar{\Psi}) \ , \tag{3.4.12}$$

where $\Psi := A_0 + i\tau$. If we introduce the Ernst potential

$$\mathscr{C} = V^2 - \Psi\bar{\Psi} + i\sigma \ , \tag{3.4.13}$$

and consider the complex valued fields ζ and ϑ, implicitly defined by

$$\mathscr{C} = \frac{1-\vartheta}{1+\vartheta} \quad , \quad \Psi = \frac{\zeta}{1+\vartheta} \ , \tag{3.4.14}$$

then [123, eq 3.11] provides the following expansion for the vector $\mathscr{E}^A := (\vartheta, \zeta) \in \mathbb{C}^2$ in terms of an arbitrary asymptoticaly flat coordinate system

$$\mathscr{E}^A = \frac{M^A}{r} + \frac{M^A_k x^k}{r^3} + O_\infty(\log r/r^3) \ . \tag{3.4.15}$$

We note that the apparent discrepancy between the error term here with the one in the original paper comes from the fact that the result there is presented in adapted coordinates obtained from arbitrary asymptotically flat coordinates by a transformation of the form $x^i \mapsto x^i + O_\infty(\log r)$.

Using the identity $\frac{A}{B+C} = \frac{A}{B} - \frac{AC}{B(B+C)}$ we get

$$\begin{aligned}\mathscr{C} = \frac{1-\vartheta}{1+\vartheta} = \frac{1+\vartheta-2\,\vartheta}{1+\vartheta} = 1 - \frac{2\,\vartheta}{1+\vartheta} = 1 - 2\left(\vartheta - \frac{\vartheta^2}{1+\vartheta}\right) \\ = 1 - 2\left(\vartheta - \vartheta^2 + \frac{\vartheta^3}{1+\vartheta}\right) \ .\end{aligned}$$

Inserting the ϑ-component of (3.4.15) into the last expression yields

$$\begin{aligned}\mathscr{C} &= 1 - 2\left(\frac{M^\vartheta}{r} + \frac{M^\vartheta_k x^k}{r^3} + O_\infty(\log r/r^3)\right) + 2\left(\frac{M^\vartheta}{r} + \frac{M^\vartheta_k x^k}{r^3} + O_\infty(\log r/r^3)\right)^2 + O_\infty(r^{-3}) \\ &= 1 - 2\frac{M^\vartheta}{r} - 2\frac{M^\vartheta_k x^k}{r^3} + 2\frac{(M^\vartheta)^2}{r^2} + O_\infty(r^{-4}) + O_\infty(\log r/r^3) \ .\end{aligned}$$

Noting that the topological restrictions imposed by asymptotic flatness imply that the imaginary part of M^ϑ vanishes, $\Im M^\vartheta = 0$ [123, Section IV], we write $M^\vartheta = M$ and by setting $M^\vartheta_k = M_k + iS_k$ we get

$$\mathscr{C} = 1 - 2\frac{M}{r} + 2\frac{M^2}{r^2} - 2\frac{M_k x^k}{r^3} - 2\,i\frac{S_k x^k}{r^3} + O_\infty(\log r/r^3) \ . \tag{3.4.16}$$

Consequently

$$\sigma = \Im \mathscr{C} = -2\frac{S_k x^k}{r^3} + O_\infty(\log r/r^3) \ . \tag{3.4.17}$$

Similarly for Ψ we get

$$\Psi = \frac{\zeta}{1+\vartheta} = \zeta - \frac{\zeta\vartheta}{1+\vartheta} = \zeta - \zeta\vartheta + \frac{\zeta\vartheta^2}{1+\vartheta} \ .$$

Inserting (3.4.15) into the last expression yields

$$\begin{aligned}
\Psi &= \zeta(1-\vartheta) + O_\infty(r^{-3}) \\
&= \left(\frac{M^\zeta}{r} + \frac{M^\zeta_k x^k}{r^3} + O_\infty(\log r/r^3)\right)\left(1 - \frac{M^\vartheta}{r} - \frac{M^\vartheta_k x^k}{r^3} + O_\infty(\log r/r^3)\right) + O_\infty(r^{-3}) \\
&= \frac{M^\zeta}{r} - \frac{M^\zeta M^\vartheta}{r^2} + \frac{M^\zeta_k x^k}{r^3} + O_\infty(\log r/r^3) \ .
\end{aligned}$$

As before $\Im M^v = 0$. So now, by setting $M^\zeta = \frac{Q}{2}$ and $M^\zeta_k = Q_k + iB_k$, we see that

$$A_0 = \Re\Psi = \frac{Q}{2\,r} - \frac{MQ}{2\,r^2} + \frac{Q_k x^k}{r^3} + O_\infty(\log r/r^3) \ , \tag{3.4.18}$$

$$\tau = \Im\Psi = \frac{B_k x^k}{r^3} + O_\infty(\log r/r^3) \ . \tag{3.4.19}$$

We have $\bar\Psi\partial_i\Psi - \Psi\partial_i\bar\Psi = 2i(A_0\partial_i\tau - \tau\partial_i A_0) = O_\infty(r^{-4})$ and using (3.1.8), (3.4.12) and (3.4.17) we get

$$\epsilon_i{}^{jk} D_j\theta_k = -V^{-4}\left(D_i\sigma - i(\bar\Psi D_i\Psi - \Psi D_i\bar\Psi)\right) = D_i\left(2\frac{S_k x^k}{r^3} + O_\infty(\log r/r^3)\right) \ . \tag{3.4.20}$$

With the exception of the already noted $\log r$ discrepancy in the error term, this is [13, eq 4.1, pg 1010] and so we get

$$\theta_i := \frac{\mathfrak{g}_{it}}{\mathfrak{g}_{tt}} = 2e_{ijk}\frac{S^j x^k}{r^3} + O_k(\log r/r^3) \ , \tag{3.4.21}$$

where $e_{[ijk]} = e_{ijk}$ with $e_{123} = 1$.

3.4.2.1 The electromagnetic twist potential and the norm of the axial Killing vector

Until now we have been working with a generic asymptotically flat coordinate system, but to estimate the electromagnetic twist potential v via the Ernst equations (3.3.5) and the results of the previous section we will need to use adapted coordinates. So, let $\{t, \varphi, \rho, z\}$ be the Weyl coordinates as constructed in Section 3.2 and define the cylindrical type coordinates

$$\begin{cases} x = \rho \cos \varphi \\ y = \rho \sin \varphi \end{cases} . \tag{3.4.22}$$

A simple but noteworthy fact is that in this coordinate system we have

$$K_{(1)} = \partial_\varphi = x\partial_y - y\partial_x . \tag{3.4.23}$$

The estimates of the previous section will only be available to us in these coordinates if $\{t, x^i\} = \{t, x, y, z\}$ is an asymptotically flat coordinate system. This is in fact the case. To see it note that in the orbit space $\{t = \varphi = 0\}$ the identity (2.6.9) yields

$$\partial_\rho = (1 + O_\infty(\hat{r}^{-1}))\partial_{\hat{x}} + O_\infty(\hat{r}^{-1})\partial_{\hat{z}} \quad (\varphi = 0) , \tag{3.4.24}$$

$$\partial_z = O_\infty(\hat{r}^{-1})\partial_{\hat{x}} + (1 + O_\infty(\hat{r}^{-1}))\partial_{\hat{z}} \quad (\varphi = 0) . \tag{3.4.25}$$

Recall that $\{\hat{x}, \hat{z}\}$ are asymptotically flat isothermal coordinates (for the orbit space metric). Direct computations yield $\mathfrak{g}_{xx}|_{\varphi=0} = 1 + O_\infty(r^{-1})$, a similar expression for $\mathfrak{g}_{yy}$ and, using $\mathfrak{g}_{\rho\varphi} \equiv 0$, also $\mathfrak{g}_{xy}|_{\varphi=0} = 0$. The defining decay rates are now obtained by flowing the previous estimates along the integral lines of the axial Killing vector. We illustrate this with an explicit calculation:

$$\begin{aligned} \mathfrak{g}_{xy}|_{\varphi=-\varphi_0} &= \mathfrak{g}((\phi_{\varphi_0})_*\partial_x, (\phi_{\varphi_0})_*\partial_y)|_{\varphi=0} \\ &= \mathfrak{g}(\cos\varphi_0 \partial x + \sin\varphi_0 \partial y, -\sin\varphi_0 \partial x + \cos\varphi_0 \partial y)|_{\varphi=0} \\ &= -\sin\varphi_0 \cos\varphi_0 \underbrace{\mathfrak{g}_{xx}|_{\varphi=0}}_{=1+O_\infty(r^{-1})} (\cos^2\varphi_0 - \sin^2\varphi_0) \underbrace{\mathfrak{g}_{xy}|_{\varphi=0}}_{=0} + \sin\varphi_0 \cos\varphi_0 \underbrace{\mathfrak{g}_{yy}|_{\varphi=0}}_{=1+O_\infty(r^{-1})} \\ &= O_\infty(r^{-1}) . \end{aligned}$$

So we have constructed asymptotically flat coordinates for which the following uniform estimate holds

$$\mathfrak{g}_{\varphi\varphi}|_{\varphi=0} = \rho^2 \mathfrak{g}_{yy}|_{\varphi=0} = \rho^2(1 + O_\infty(r^{-1})) . \tag{3.4.26}$$

As a nice consequence we get

$$e^{-2U} := \frac{\mathfrak{g}_{\varphi\varphi}}{\rho^2} = 1 + O_\infty(r^{-1}) , \tag{3.4.27}$$

from which we see that

$$e^{U_i \pm U_j} := (1 + O_\infty(r^{-1}))(1 + O_\infty(r^{-1}))^{\pm 1} = 1 + O_\infty(r^{-1}) \to_{r\to+\infty} 1 \ , \quad (3.4.28)$$

and our goal (3.4.10) gets reduced to showing that

$$\begin{aligned}\lim_{\sqrt{\rho^2+z^2}\to+\infty} \frac{(\psi_1-\psi_2)^2}{\rho^2} &= \lim_{\sqrt{\rho^2+z^2}\to+\infty} \frac{(\chi_1-\chi_2)^2}{\rho^2} \\ &= \lim_{\sqrt{\rho^2+z^2}\to+\infty} \frac{v_1-v_2}{\rho^2} \\ &= \lim_{\sqrt{\rho^2+z^2}\to+\infty} \frac{\chi_1\psi_2-\chi_2\psi_1}{\rho^2} = 0 \ .\end{aligned} \quad (3.4.29)$$

It follows from (3.4.21) and (3.2.17) that

$$\mathfrak{g}_{zt} \equiv 0 \Rightarrow \theta_z \equiv 0 \Rightarrow S_x = S_y = 0 \ .$$

So we set $J := -S_z$ and by using (3.4.21) with (3.1.8) we get

$$\mathfrak{g}_{yt}|_{\varphi=0} = 2J\frac{\rho}{r^3} + O_\infty(\log r/r^3) \ , \quad (3.4.30)$$

from which

$$\mathfrak{g}_{\varphi t}|_{\varphi=0} = \mathfrak{g}(\rho\partial_y, \partial_t)|_{\varphi=0} = 2J\,\frac{\rho^2}{r^3} + \rho\, O_\infty(\log r/r^3) \ , \quad (3.4.31)$$

and therefore

$$\frac{\mathfrak{g}_{\varphi t}}{\mathfrak{g}_{\varphi\varphi}}|_{\varphi=0} = \frac{2J}{r^3} + \frac{1}{\rho}O_\infty(\log r/r^3) \ . \quad (3.4.32)$$

The Ernst equations (3.3.5) together with the estimates (3.4.42), (3.4.44), (3.4.47) and (3.4.48), that will be established in the next section, provide

$$\begin{cases} \partial_z v = -6J\rho^4/r^5 + \rho\, O_\infty(\log r/r^3) \\ \partial_\rho v = 6J\rho^3 z/r^5 + \rho^2 O_\infty(\log r/r^4) \end{cases} . \quad (3.4.33)$$

Integrating this system by using the polar coordinates $\rho = r\cos\theta$, $z = r\sin\theta$, while imposing the standard condition

$$v(0,z) \equiv 0, \ \text{for } z \gg 0 \ , \quad (3.4.34)$$

yields

$$v = 4J - \frac{J}{2}\,\frac{z}{r}\left(\frac{3\rho^2 - z^2}{r^2} + 9\right) + \rho\, O_\infty(\log r/r^2) \ . \quad (3.4.35)$$

We note the following relevant relation with the total angular momentum as given by the Komar integral formula

$$
\begin{aligned}
\lim_{R\to+\infty}\frac{1}{16\pi}\int_{\{r=R\}} *dK^\flat_{(1)} &= \lim_{R\to+\infty} -\frac{1}{16\pi}2\pi\int_{\{r=R\}\cap\{\varphi=0\}} i_{K_{(1)}} * dK^\flat_{(1)} \\
&= -\frac{1}{8}\lim_{R\to+\infty}\int_{\{r=R\}\cap\{\varphi=0\}} *(dK^\flat_{(1)}\wedge K^\flat_{(1)}) \\
&= -\frac{1}{8}\lim_{R\to+\infty}\int_{\{r=R\}\cap\{\varphi=0\}} \big(dv + 2\underbrace{(\chi d\psi - \psi d\chi)}_{=O(r^{-2})}\big) \\
&= -\frac{1}{8}\lim_{R\to+\infty}\big(v(0,R) - v(0,-R)\big) \\
&= -\frac{1}{8}(0-8J) \\
&= J\ .
\end{aligned}
$$

We are now able to establish the electromagnetic twist potential part of (3.4.29). For two twist potentials satisfying (3.4.34) we have

$$
\lim_{\sqrt{\rho^2+z^2}\to+\infty\ ,\rho\not\to 0}\frac{v_1-v_2}{\rho^2} = 0\ . \tag{3.4.36}
$$

To take care of the asymptotic behavior of v near the axis we Taylor expand on ρ around a point $(0,z)$, away from the poles, to get

$$
v(\rho,z) = v(0,z) + \partial_\rho v(c(\rho),z)\rho \quad , \quad |c(\rho)|\leq\rho\ . \tag{3.4.37}
$$

Then, using (3.4.33) to obtain

$$
\partial_\rho v = \rho^2 O_\infty(r^{-3})\ , \tag{3.4.38}
$$

we conclude that for $|z|\gg 0$ and $\rho\leq|z|$

$$
v = v(0,z) + \rho^2 O(r^{-3})\ . \tag{3.4.39}
$$

Finally, for two twist potentials that agree along the axis for both large positive and negative z we have, in the region $\rho\leq|z|$,

$$
\rho^{-2}(v_1-v_2) = O(r^{-3})\ , \tag{3.4.40}
$$

and the desired result follows.

3.4.2.2 The electromagnetic potentials

Asymptotic flatness (3.1.5) together with (3.4.18) and (3.4.19) yield the desired improvement of the initial decay rates

$$\partial_{[i}A_{j]} = O_\infty(r^{-3}) \ . \tag{3.4.41}$$

Now in the $\{x^\mu\} = \{t, x, y, z\}$ coordinates of the previous section we have

$$\begin{aligned}
d\chi &:= i_{K_{(1)}}F \\
&= F_{\mu\nu}dx^\mu \wedge dx^\nu(K_{(1)}, \cdot) \\
&= F_{\mu\nu}\left(dx^\mu(K_{(1)})\, dx^\nu - dx^\nu(K_{(1)})\, dx^\mu\right) = 2F_{\mu\nu}dx^\mu(K_{(1)})\, dx^\nu \\
&= 2F_{\mu\nu}dx^\mu(x\partial_y - y\partial_x)\, dx^\nu = 2F_{\mu\nu}(x\delta^\mu_y - y\delta^\mu_x)\, dx^\nu \\
&= 2\left(xF_{y\nu} - yF_{x\nu}\right)\, dx^\nu = 4\left(x\partial_{[y}A_{\nu]} - y\partial_{[y}A_{\nu]}\right)\, dx^\nu \ .
\end{aligned}$$

With (3.4.41) we see that, in the orbit space $\{\varphi = 0\}$ (where $y = 0$, $x = \rho$ and $\partial_x = \partial_\rho$), we have

$$\left\{ \begin{array}{l} \partial_\rho\chi|_{\varphi=0} = 4\rho\,\partial_{[y}A_{\rho]} = \rho\, O_\infty(r^{-3}) \\ \partial_z\chi|_{\varphi=0} = 4\rho\,\partial_{[y}A_{z]} = \rho\, O_\infty(r^{-3}) \end{array} \right. \ . \tag{3.4.42}$$

Imposing the boundary condition

$$\chi(0, z) \equiv 0, \text{ for } z \gg 0 \ , \tag{3.4.43}$$

integration yields

$$\chi|_{\varphi=0} = \rho\, O_\infty(r^{-2}) \ . \tag{3.4.44}$$

Arguing as in the end of section 3.4.2.1 the equation in (3.4.29) corresponding to the potentials χ_i follows.

To obtain a coordinate expression for $d\psi$ it will be helpful to rearrange our preferred coordinate system and consider $\{x^\mu\} = \{t, y, x, z\}$, then

$$\begin{aligned}
d\psi &:= i_{K_{(1)}} * F \\
&= \frac{1}{2}F^{\mu\nu}\epsilon_{\mu\nu\lambda\sigma}dx^\lambda \wedge dx^\sigma(K_{(1)}, \cdot) = F^{\mu\nu}\epsilon_{\mu\nu\lambda\sigma}dx^\lambda(x\partial_y - y\partial_x)\, dx^\sigma \\
&= F^{\mu\nu}\epsilon_{\mu\nu\lambda\sigma}(x\delta^\lambda_y - y\delta^\lambda_x)\, dx^\sigma = F^{\mu\nu}(x\,\epsilon_{\mu\nu y\sigma} - y\,\epsilon_{\mu\nu\, x\sigma})\, dx^\sigma \ .
\end{aligned}$$

Now, in the orbit space and away from the axis, we have (compare with (3.2.17))

$$\mathfrak{g}_{\mu\nu}|_{\varphi=0} = \begin{pmatrix} \mathfrak{g}_{tt} & \rho^{-1}\mathfrak{g}_{t\varphi} & 0 & 0 \\ \rho^{-1}\mathfrak{g}_{t\varphi} & \rho^{-2}\mathfrak{g}_{\varphi\varphi} & 0 & 0 \\ 0 & 0 & e^{2u} & 0 \\ 0 & 0 & 0 & e^{2u} \end{pmatrix} , \tag{3.4.45}$$

therefore

$$\det(\mathfrak{g}_{\mu\nu}|_{\varphi=0}) = \left(\mathfrak{g}_{tt}\frac{\mathfrak{g}_{\varphi\varphi}}{\rho^2} - \frac{\mathfrak{g}_{t\varphi}^2}{\rho^2}\right) e^{4u} = \frac{1}{\rho^2}(-\rho^2)e^{4u} = -e^{4u} \; ; \tag{3.4.46}$$

so

$$\begin{aligned} \partial_\rho\psi|_{\varphi=0} &= \rho\,\epsilon_{\mu\nu yx}F^{\mu\nu} = \rho(\epsilon_{tzyx}F^{tz} + \epsilon_{ztyx}F^{zt}) = 2\rho\,\epsilon_{tzyx}F^{tz} \\ &= 2\rho\sqrt{|\det(\mathfrak{g}_{\mu\nu})|}F^{tz} = 2\rho\,e^{2u}\mathfrak{g}^{\mu t}\mathfrak{g}^{\nu z}F_{\mu\nu} \\ &= 2\rho\,e^{2u}\mathfrak{g}^{\mu t}\mathfrak{g}^{zz}F_{\mu z} = 2\rho\,e^{2u}e^{-2u}(\mathfrak{g}^{tt}F_{tz} + \mathfrak{g}^{yt}F_{yz}) \\ &= 2\rho\left\{-\frac{\mathfrak{g}_{\varphi\varphi}}{\rho^2}(\partial_t A_z - \partial_z A_t) + 2\frac{\mathfrak{g}_{t\varphi}}{\rho}\partial_{[y}A_{z]}\right\} \\ &= 2\frac{\mathfrak{g}_{\varphi\varphi}}{\rho}\partial_z A_t + 4\mathfrak{g}_{t\varphi}\partial_{[y}A_{z]} = 2\rho(1 + O_\infty(r^{-1}))\partial_z A_0 + \rho\,O_\infty(r^{-5}) , \end{aligned}$$

where in the last equality we used (3.4.26), (3.4.31) and (3.4.41); also

$$\begin{aligned} \partial_z\psi|_{\varphi=0} &= \rho\,\epsilon_{\mu\nu yz}F^{\mu\nu} = 2\rho\,\epsilon_{txyz}F^{tx} \\ &= -2\rho\sqrt{|\det(\mathfrak{g}_{\mu\nu})|}F^{tx} = -2\rho\,e^{2u}\mathfrak{g}^{\mu t}\mathfrak{g}^{\nu x}F_{\mu\nu} \\ &= -2\rho\,e^{2u}\mathfrak{g}^{\mu t}\mathfrak{g}^{xx}F_{\mu x} = -2\rho\,(\mathfrak{g}^{tt}F_{tx} + \mathfrak{g}^{yt}F_{yx}) \\ &= -2\rho\left\{-\frac{\mathfrak{g}_{\varphi\varphi}}{\rho^2}(\partial_t A_x - \partial_x A_t) + 2\frac{\mathfrak{g}_{t\varphi}}{\rho}\partial_{[y}A_{x]}\right\} \\ &= -2\rho(1 + O_\infty(r^{-1}))\partial_\rho A_0 + \rho\,O_\infty(r^{-5}) . \end{aligned}$$

From (3.4.18) we get

$$\left\{ \begin{array}{l} \partial_\rho\psi|_{\varphi=0} = -Q\frac{\rho\,z}{r^3} + \rho\,O_\infty(r^{-3}) \\ \partial_z\psi|_{\varphi=0} = Q\frac{\rho^2}{r^3} + \rho\,O_\infty(r^{-3}) \end{array} \right. . \tag{3.4.47}$$

Integrating as before while using a standard boundary condition provides

$$\psi = Q\left(-1 + \frac{z}{r}\right) + \rho O_\infty(r^{-2}) . \tag{3.4.48}$$

We note the following relevant and expected relation with the *total electric charge* given by the Komar integral

$$\lim_{R\to+\infty} -\frac{1}{4\pi}\int_{\{r=R\}} *F = Q\ . \tag{3.4.49}$$

It should be now clear that (3.4.29) follows.

The results of these last two sections establish one of the significant missing elements of all previous uniqueness claims for the Kerr-Newman metric:

PROPOSITION 3.4.1 *Let $\Psi_i = (U_i, v_i, \chi_i, \psi_i)$, $i = 1, 2$, be the Ernst potentials associated with two I^+–regular, electro-vacuum, stationary, asymptotically flat axisymmetric metrics with non-degenerate event horizons. If $v_1 = v_2$ and $\psi_1 = \psi_2$ on the rotation axis, then the hyperbolic-space distance between Ψ_1 and Ψ_2 is bounded, going to zero as r tends to infinity in the asymptotic region.*

3.5 Weinstein Solutions: existence and uniqueness

In this section we construct axisymmetric Ernst maps

$$\Phi = (U, v, \chi, \psi) : \mathbb{R}^3 \setminus \mathscr{A} \to \mathbb{H}^2_{\mathbb{C}}\ ,$$

which are "close" to some reference maps, not necessarily harmonic, satisfying conditions modeled on the local behavior of the Kerr-Newman solutions. First recall the definitions of mass, angular momentum and electric charge of the k-th black hole as given by the Komar integrals

$$m_k := -\frac{1}{8\pi}\int_{S_k} *dK^\flat_{(0)}\ , \tag{3.5.1}$$

$$J_k := \frac{1}{16\pi}\int_{S_k} *dK^\flat_{(1)}\ , \tag{3.5.2}$$

$$q_k := -\frac{1}{4\pi}\int_{S_k} *F\ . \tag{3.5.3}$$

for some 2–sphere S_k whose interior intersects the event horizon exactly at its k-th component.

We are now able to characterize the reference maps $\tilde\Phi = (\tilde U, \tilde v, \tilde\chi, \tilde\psi)$:

1. The components $\tilde{f} = \tilde{v}, \tilde{\chi}$ and $\tilde{\psi}$ are locally bounded, constant along each connected component of $\mathscr{A} \setminus \mathscr{E}^+ = \cup_{k=0}^N \mathscr{A}_k$ and we write $\tilde{f}|_{\mathscr{A}_k} \equiv \tilde{f}_k$. These functions are normalized to satisfy $\tilde{f}_N = 0$. Moreover, we will require $\tilde{\chi}_k = 0$, $\forall k$, which imposes vanishing magnetic charges for all horizons (compare (4.1.23)).

2. There exist $N_{\mathrm{dh}} \geq 0$ degenerate event horizons, which are represented by punctures $(\varphi = 0, \rho = 0, z = b_i)$, together with a mass parameter $m_i > 0$. In a neighborhood of such puncture, containing only this component of the horizon, the map $\tilde{\Phi}$ corresponds to the harmonic map of the (extreme) Kerr-Newman solution parameterized by

$$(m_i, q_i) = (m_i, \frac{\tilde{\psi}_{i+1} - \tilde{\psi}_i}{2}) \ .$$

3. There exist $N_{\mathrm{ndh}} \geq 0$ non-degenerate horizons, which are represented by bounded open intervals $(c_i^-, c_i^+) = I_i \subset \mathscr{A}$, with none of the previous b_j's belonging to the union of the closures of the I_i. In a neighborhood of such interval, containing only this component of the horizon, the map $\tilde{\Phi}$ corresponds to the harmonic map of the Kerr-Newman solution parameterized by

$$(\mu_j, \lambda_j, q_j) = (2 \int_{I_j} dz, \tilde{v}_{i+1} - \tilde{v}_i, \frac{\tilde{\psi}_{i+1} - \tilde{\psi}_i}{2}) \ .$$

   To retrieve the usual parametrization using mass, angular momentum and charge one uses the known explicit formulas for Kerr-Newman (e.g., equations 2.31. of [135]) together with the following relations [135, section 2.3.]

$$J_j = \frac{\lambda_j + l_j}{4} \ , \tag{3.5.4}$$

$$m_j = \mu_j + 2 w_j J_j \ , \tag{3.5.5}$$

   where the auxiliary parameters are defined by $\lambda_j := \int_{I_j} dv$, $l_j = \int_{I_j} \chi d\psi - \psi d\chi$, and $w|_{I_j} \equiv w_j$, with w defined by (3.2.17).

4. In a neighborhood of infinity the functions $\tilde{U}$, $\tilde{v}$, $\tilde{\chi}$ and $\tilde{\psi}$ coincide with the components of the harmonic map associated with the Kerr-Newman solution with mass $M := \sum_k m_k$ angular momentum $J := \sum_k J_k = v_0/8$ and electric charge $Q := \sum_k q_k = -\psi_0/2$, where the sums are taken over all the components of the event horizon.

5. The functions $\tilde{U}$, $\tilde{v}$, $\tilde{\chi}$ and $\tilde{\psi}$ are smooth across $\mathscr{A} \setminus (\cup_i \{b_i\} \cup_j I_j)$.

A collection $\{b_i, m_i\}_{i=1}^{N_{\rm dh}}$, $\{I_j, v(c_j^-), v(c_j^+)\}_{j=1}^{N_{\rm dh}}$, and $\{\psi_k\}_{k=0}^{N-1}$ will be called *"electro-vacuum axis data"*.

A map $\tilde{\Phi}$ satisfying condition 1.–5. above defines *singular Dirichlet data* [136, Definition 2] (compare [135, Section 2.4.]) with a target manifold with constant negative sectional curvature. We then have the following version of [136, Theorem 2] (compare [38, Appendix C] where the uniqueness claim is clarified, and [135] for a similar result stated purely in terms of axis data):

Theorem 3.5.1 *For any set of electro-vacuum axis data there exists a unique harmonic map $\Phi : \mathbb{R}^3 \setminus \mathscr{A} \to \mathbb{H}^2_{\mathbb{C}}$ whose distance, as given by* (3.3.14)*, from an axisymmetric map $\tilde{\Phi} : \mathbb{R}^3 \setminus \mathscr{A} \to \mathbb{H}^2_{\mathbb{C}}$, not necessarily harmonic but with the properties 1.–5. above, satisfies:*

$$d(\Phi, \tilde{\Phi}) \in L^\infty(\mathbb{R}^3 \setminus \mathscr{A}) \ , \tag{3.5.6}$$

and

$$d(\Phi, \tilde{\Phi}) \to 0 \text{ as } r \to +\infty \ . \tag{3.5.7}$$

□

From an harmonic map $\Phi : \mathbb{R}^3 \setminus \mathscr{A} \to \mathbb{H}^2_{\mathbb{C}}$ one can construct a stationary and axisymmetric solution of the source free Einstein-Maxwell field equations [135, Section 4.1.]. Such (not necessarily I^+–regular) space-times, arising from the harmonic maps of the previous theorem will be referred to as *Weinstein solutions*.

3.6 Proof of Theorem 3.0.1

If $\mathscr{E}^+$ is empty we obtain Minkowski [26, Theorem 2.7]. Otherwise the proof splits into two cases, according to whether or not X is tangent to the generators of $\mathscr{E}^+$.

Rotating horizons:

Suppose, first, that the Killing vector is not tangent to the generators of some connected component $\mathscr{E}^+_0$. Proposition 1.9 of [26] allows us to generalize Proposition 2.4.10 to electro-vacuum and then Theorem 2.4.11 together with

Remark 2.4.12 show that the event horizon is analytic if the metric is; also, by (3.4.41) and Einstein's equations, $G_{\mu\nu} = 2\,T_{\mu\nu} = O(r^{-5})$. So the Rigidity Theorem, as presented in [27, Theorem 5.1], applies and establishes the existence of a $\mathbb{R}\times \mathrm{U}(1)$ subgroup of the isometry group of $(\mathscr{M}, \mathfrak{g})$. The analysis of Section 3.2, leading to the global representation (3.2.17) of the metric, is now available. As stressed throughout this thesis, in this gauge, the field equations (3.1.1)-(3.1.3) reduce to a harmonic map Φ (3.3.3). The analysis of the asymptotic behavior of such map, whose results are compiled in Proposition 3.4.1, shows that Φ lies a finite distance from one of the harmonic maps associated to the Weinstein solutions of Theorem 3.5.1 and the uniqueness part of such theorem allows us to conclude; note that in the connected and non-degenerate setting the Weinstein solutions correspond to the non-extreme Kerr-Newman metrics.

Non-rotating case:

Now let us consider the case when the stationary Killing vector $K_{(0)}$ is tangent to the generators of every component of $\mathscr{E}^+$. Following the procedure in Section 2.7.2, based on [114], we extend $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ to a space-time where each connected component of the event horizon is contained in a bifurcate horizon. Then, by [44] there exists an asymptotically flat Cauchy hypersurface for the domain of outer communications, with boundary on the union of the bifurcate spheres, which is maximal. We are now able to conclude from Theorem 3.4 of [127] that $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is static. By taking in account the discussion in Section 2.1 we can invoke, after relying on analyticity once more, the non-degenerate part of the conclusion of Theorem 1.3 of [30], yielding non-extreme Reissner-Nordström as the only non-rotating solution satisfying the remaining conditions of the desired result.

3.7 Concluding remarks

To obtain a satisfactory classification in four dimensions, the following issues remain to be addressed:

1. **Analyticity.** The previous versions of the uniqueness theorem required analyticity of *both* the metric *and* the horizon. As shown in Section 2.4.3 (see also the proof of Theorem 3.0.1), the latter follows from the former.

This is a worthwhile improvement, as even C^1-*differentiability of the horizon* is not clear a priori. But the hypothesis of analyticity of the metric remains to be removed. In this context one should keep in mind the Curzon solution, where analyticity of the metric fails precisely at the horizon [128].

We further note that a new approach to Hawking's rigidity without analyticity [2, 3] has yielded significant breakthroughs in the vacuum case. According to A.D. Ionescu (private communication) the generalization of the results to electro-vacuum should follow by similar techniques. However, some problems still need to be settled, even for vacuum: the local claim requires a non-expanding horizon which we expect to be a consequence of I^+–regularity and the results and techniques of Section 2.4.3 and [36], but such claim require checking; also, as it stands, the global result is restricted to near Kerr geometries.

The hypothesis of analyticity is particularly annoying in the static context, being needed there only to exclude non-embedded Killing prehorizons (recall Definition 2.5.7). The nature of that problem seems to be rather different from Hawking's rigidity, with presumably a simpler solution, yet to be found.

2. **Degeneracy.** The classification of black holes with degenerate components of the event horizon requires further investigations. We believe that the results here go a long way to obtain a classification, in terms of Weinstein solutions, of stationary, axisymmetric, rotating configurations allowing both degenerate and non-degenerate components of the horizon: the foundations are settled but we are still missing an equivalent of Proposition 3.4.1. We expect that Theorem 2.2.2 can be useful for solving this problem.

   In the pure vacuum case one could exclude degenerate non-rotating components solutions by proving existence of maximal hypersurfaces within $\langle\langle \mathscr{M}_{\text{ext}} \rangle\rangle$ with an appropriate asymptotic behavior at the cylindrical ends. The argument presented in Section 2.7.2 would then apply to give staticity, and non-existence would then follow from [42], or from Theorem 2.1.1.

   It has been announced [4] that the question of uniqueness of degenerate black holes (with connected event horizon) has been settled. Unfortunately, that reference does not contain any new results, as compared to what had already been published in [33], or is contained in this work, and so, it is our belief that this problem remains open. Indeed, the existence of global Weyl coordinates *with controlled behavior at the singular set* is assumed. In the non-degenerate case this issue was first settled for vacuum in [33] (see Sections 2.6 and 3.4), but the degenerate case appears to present serious technical difficulties, and requires further study.

3. **Multi Component Solutions.** In agreement with the statement of Conjecture 1.1.1, one believes that all solutions with non-connected $\mathscr{E}^+$ are in the Majumdar-Papapetrou family. It remains to show that non-static Weinstein solutions with non-connected horizons are singular; besides the already quoted result dealing with the Israel-Wilson-Perjés family [42], this has been established for slowly rotating black holes in vacuum by a regularity analysis of the relevant harmonic maps [97,132] and recent and promising results seem to have settled the problem for two-body configurations, also in vacuum [107].

4. **Magnetic Charges.** Both by assuming the existence of a global electromagnetic potential (3.1.2) and by prescribing the decay rate (3.1.5) we eliminate, a priori, the existence of magnetic charges; this can be seem from either (3.4.44) or (4.1.23). We expect such restrictions to be unnecessary. In fact, they play no role in either the construction of Weyl coordinates or the asymptotic analysis of the relevant maps near the axis. We also note that the construction of Weinstein solutions including magnetic charges is a straightforward generalization of the work presented here.

The question of classification of higher dimensional stationary black holes is largely unchartered territory. Nonetheless there have been significant progresses in the asymptotically Kaluza-Klein case [32,37,83], with the work presented here playing a role by providing the general framework.

Chapter 4

A Dain Inequality with charge

Consider a three dimensional electro-vacuum smooth initial data set (M, g, K, E, B), where M is the union of a compact set and of two asymptotically flat regions M_1 and M_2. Here g is a Riemannian metric on M, K is the extrinsic curvature tensor, E is the electric field and B the magnetic one; electro-vacuum means that the constraints (4.1.4) are satisfied with E and B both divergence-free. We suppose that the initial data set is *axisymmetric*, by which we mean that it is invariant under an action of U(1), and maximal: $\mathrm{tr}_g K = 0$. It is further assumed that $M/\mathrm{U}(1)$ is simply connected, so that the results of [31] can be used. The notion of asymptotic flatness is made precise in (4.1.1) and (4.1.3), where moreover $k \geq 6$ needs to be assumed when invoking [31]. We will prove the following result:

THEOREM 4.0.1 *Under the conditions just described, let m, $\vec{J}$, Q_E and Q_B denote respectively the ADM mass* (4.1.18)*, the ADM angular momentum* (4.1.12) *and the total electric and magnetic charges* (4.1.10) *of M_1. Then*

$$m \geq \sqrt{\frac{|\vec{J}|^2}{m^2} + Q_E^2 + Q_B^2}\,. \tag{4.0.1}$$

REMARK 4.0.2 *We expect the equality to be attained only for the magnetically and electrically charged extreme Kerr-Newman space-times, which do not satisfy the hypotheses of Theorem 4.0.1. Indeed, any spacelike manifold in an extreme Kerr-Newman space-time is either incomplete, or contains a boundary, or a singularity, or an asymptotically cylindrical end.*

Remark 4.0.3 *If M contains only one asymptotic flat end and $\partial M = \emptyset$ we have $Q_E = Q_B = 0$ and, under the supplementary hypothesis* (4.1.11), *we also have $\vec{J} = 0$ (see* (4.1.12) *below). Whence our interest in initial data sets containing two ends. In fact one expects our main result to generalize to several ends along the lines of [38], but a proof of this lies beyond the scope of this work.*

Remark 4.0.4 *The proof applies to Einstein–Abelian Yang-Mills fields configurations, giving in this case*

$$m \geq \sqrt{\frac{|\vec{J}|^2}{m^2} + (\sum_i Q_{E_i})^2 + (\sum_i Q_{B_i})^2}\ , \tag{4.0.2}$$

where the Q_{E_i}'s and the Q_{B_i}'s are the electric and magnetic charges associated with the i'th Maxwell field.

A slightly more general version of Theorem 4.0.1 can be found in Theorem 4.1.3 below. The reader should also note an inequality relating area, angular momentum, and charge, proved for *stationary* Einstein-Maxwell black holes in [73], as well as the discussion of the Penrose inequality in electrovacuum of [137].

4.1 Mass, angular momentum and charge inequalities

Recall that an *asymptotically flat end* is a region $\mathscr{M}_{\rm ext} \subset M$ diffeomorphic to $\mathbb{R}^3 \setminus B(R)$, where $B(R)$ is a coordinate ball of radius R, such that in coordinates on $\mathscr{M}_{\rm ext}$ obtained from $\mathbb{R}^3 \setminus B(R)$ we have, for some $k \geq 1$,[1]

$$g_{ij} = \delta_{ij} + o_k(r^{-1/2})\ ,\ \partial_k g_{ij} \in L^2(\mathscr{M}_{\rm ext})\ , \quad K_{ij} = O_{k-1}(r^{-\beta})\ ,\ \beta > \frac{5}{2}\ . \tag{4.1.1}$$

(The asymptotic conditions on g arise from the requirement of well defined ADM mass, with the integrability condition satisfied if, e.g., $\partial_k g_{ij} = O(r^{-\alpha-1})$, for some $\alpha > 1/2$ [31]. The restriction on the decay rate of K above arose already in the vacuum case, and can be traced back to the unnumbered equation after (2.37) in [38].)

[1]We write $f = o_k(r^{-\alpha})$ if the limits $\lim_{r\to\infty} r^{\alpha+\ell}\partial_{k_1}\ldots\partial_{k_\ell} f$ vanish for all $0 \leq \ell \leq k$, and $f = O_k(r^{-\alpha})$ if there exists a constant C such that $|r^{\alpha+\ell}\partial_{k_1}\ldots\partial_{k_\ell} f| \leq C$ for all $0 \leq \ell \leq k$.

The electric and magnetic fields E and B are the orthogonal projections to TM of their space-time analogues

$$E^\mu = F^\mu{}_\nu n^\nu \ , \quad B^\mu = *F^\mu{}_\nu n^\nu \ , \tag{4.1.2}$$

where F is the Maxwell two-form, and where n is a unit normal to M, when embedded in an electro-vacuum space-time.[2] We assume that in the manifestly asymptotically flat coordinates we have

$$E^i = O_{k-1}(r^{-\gamma-1}) \ , \quad B^i = O_{k-1}(r^{-\gamma-1}) \ \ , \ \gamma > 3/4 \ . \tag{4.1.3}$$

The Einstein-Maxwell scalar constraint equation reads, for maximal initial data,

$$^{(3)}R = 16\pi\mu + |K|^2_g + 2\Big(|E|^2_g + |B|^2_g\Big) \ , \tag{4.1.4}$$

where the function $\mu \geq 0$ represents the non-electromagnetic energy density and $|\cdot|_g$ denotes the norm of a vector with respect to the metric g.

To obtain our inequality we start by bounding (4.1.4) from below, as follows. By [31] there exists a coordinate system, with controlled asymptotic behaviour, in which the metric takes the form [3]

$$g = e^{-2U+2\alpha}\left(d\rho^2 + dz^2\right) + \rho^2 e^{-2U}\left(d\varphi + \rho\, W_\rho d\rho + W_z dz\right)^2 \ ; \tag{4.1.5}$$

such coordinates are global in M_1, with M_2 being represented by the "puncture" $\{\rho = z = 0\}$.

Consider an orthonormal frame e_i such that e_3 is proportional to the rotational Killing vector field

$$\eta := \partial_\varphi \ .$$

Let θ^i denote the dual co-frame; for definiteness we take

$$\theta^1 = e^{-U+\alpha} d\rho \ , \quad \theta^2 = e^{-U+\alpha} dz \ , \quad \theta^3 = \rho e^{-U}\left(d\varphi + \rho W_\rho d\rho + W_z dz\right) \ .$$

We assume that the initial data are invariant under the flow of η; this implies the space-time equations $\mathscr{L}_\eta F = 0$ and $\mathscr{L}_\eta * F = 0$, where a star stands for the Hodge dual, and $\mathscr{L}$ denotes a Lie-derivative. Then, Maxwell's equations

$$dF = d * F = 0 \ , \tag{4.1.6}$$

[2] The existence of an electro-vacuum and axisymmetric evolution of the data follows from its smoothness by [22] and [25]. This will, in particular, allow us to use the (space-time) computations in [135].

[3] It should be noted that this ρ-coordinate does not, in general, agree with the Weyl ρ-coordinate of the previous chapters.

together with the hypothesis of simple-connectedness of $M/\mathrm{U}(1)$, imply the existence of functions χ and ψ such that

$$\partial_\alpha \chi = F_{\mu\alpha}\eta^\mu \ , \qquad \partial_\alpha \psi = *F_{\mu\alpha}\eta^\mu \ . \tag{4.1.7}$$

In the orthonormal basis $\{n, e_i\}$ we have

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} ,$$

therefore

$$(\partial_\alpha \chi) = \sqrt{g_{\varphi\varphi}}\,(F_{3\alpha}) = \rho e^{-U}(E_3, B_2, -B_1, 0) \ ,$$

which together with the analogous expression for $(\partial_\alpha \psi)$ yields

$$\begin{aligned} |E|_g^2 + |B|_g^2 &= \frac{e^{2U}}{\rho^2}\Big((\partial_n \chi)^2 + |D\chi|_g^2 + (\partial_n \psi)^2 + |D\psi|_g^2\Big) \\ &\geq \frac{e^{2U}}{\rho^2}\Big(|D\chi|_g^2 + |D\psi|_g^2\Big) \\ &= \frac{e^{4U-2\alpha}}{\rho^2}\Big((\partial_\rho \chi)^2 + (\partial_z \chi)^2 + (\partial_\rho \psi)^2 + (\partial_z \psi)^2\Big) \ , \end{aligned} \tag{4.1.8}$$

where ∂_n denotes the derivative in the direction normal to the initial data hypersurface.

Remark 4.1.1 *For a stationary development of the data, if we let τ denote the stationary vector, by Proposition 3.2.1 the field equations imply the integrability conditions*

$$d\tau^\flat \wedge \tau^\flat \wedge \eta^\flat = d\eta^\flat \wedge \tau^\flat \wedge \eta^\flat = 0 \ . \tag{4.1.9}$$

It then follows that n is spanned by the Killing vectors τ and η, and (4.1.8) *becomes an equality.*

Writing

$$4\pi T_{\mu\nu}\eta^\mu = F_{\mu\alpha}F_\nu{}^\alpha \eta^\mu - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\eta_\nu = \partial_\alpha \psi F_\nu{}^\alpha - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\eta_\nu \ ,$$

we are now able to justify Remark 4.0.3. First, the vanishing of the electric and magnetic charges is an immediate consequence of their definition

$$Q_E = -\frac{1}{4\pi}\int_{S_\infty} *F \ , \quad Q_B = \frac{1}{4\pi}\int_{S_\infty} F \ , \tag{4.1.10}$$

and Maxwell's equations (4.1.6). Next, the vanishing of angular momentum will be established under the supplementary condition that

$$\psi F_{\mu\nu} = o(r^{-2}) \; . \tag{4.1.11}$$

The symmetry of the problem implies that $\vec{J}$ is aligned along the axis of rotation. Letting J_z denote the component of angular-momentum along the rotation axis, by the Komar identity we obtain, recall that here we are assuming that $\partial M = \emptyset$,

$$\begin{aligned}
16\pi J_z &= \int_{S_\infty} \nabla^\mu \eta^\nu dS_{\mu\nu} = \frac{1}{2}\int_M \nabla_\mu \nabla^\mu \eta^\nu dS_\nu \\
&= \frac{1}{2}\int_M R_\mu{}^\nu \eta^\mu dS_\nu = 4\pi \int_M T_\mu{}^\nu \eta^\mu dS_\nu \\
&= -\int_M \psi \underbrace{\nabla_\alpha F^{\nu\alpha}}_{0} dS_\nu + \int_M \nabla_\alpha(\psi F^{\nu\alpha}) dS_\nu - \frac{1}{4}\int_M F_{\alpha\beta}F^{\alpha\beta}\eta^\nu dS_\nu \\
&= 2\int_{S_\infty} \psi F^{\nu\alpha} dS_{\nu\alpha} - \frac{1}{4}\int_M F_{\alpha\beta}F^{\alpha\beta} \underbrace{\eta^\mu n_\mu}_{0} d^3\mu_g = 0 \; ,
\end{aligned} \tag{4.1.12}$$

where we have used the fact that η is tangent to M, and where the first integral in the last line vanishes by (4.1.11).

As discussed in [50], in vacuum the one-form[4]

$$\begin{aligned}
\lambda &:= 2\epsilon_{ijk} K^j{}_\ell \eta^k \eta^\ell dx^i \\
&= 2\epsilon(\partial_A, \partial_B, \partial_\varphi) K(dx^B, \partial_\varphi) dx^A \\
&= 2g(\eta,\eta)\epsilon(e_a, e_b, e_3) K(\theta^b, e_3)\theta^a
\end{aligned} \tag{4.1.13}$$

is closed. Here, as before, the upper case indices $A, B = 1, 2$ correspond to the coordinates (ρ, z), while the lower case indices $a, b = 1, 2$ are frame indices. In electro-vacuum we have instead (see, e.g., [135])

$$d\Big(\lambda - 2(\chi d\psi - \psi d\chi)\Big) = 0 \; , \tag{4.1.14}$$

and, since we have assumed that $M/\mathrm{U}(1)$ is simply connected, there exists a function v such that

$$\lambda = 2(dv + \chi d\psi - \psi d\chi) \; . \tag{4.1.15}$$

[4]We take this opportunity to point out a factor of 2 missing in the left-hand-side of Eq. (2.6) in [38], which affects numerical factors in some subsequent equations, but has no other consequences.

Then, writing K_{b3} for $K(e_b, e_3)$, and using $\epsilon_{ab} := \epsilon(e_a, e_b, e_3) \in \{0, \pm 1\}$, we have

$$2\rho^2 e^{-2U}(K_{23}\theta^1 - K_{13}\theta^2) = \lambda_\rho d\rho + \lambda_z dz \;;$$

equivalently

$$K_{13} = -\frac{e^{3U-\alpha}}{2\rho^2}\lambda_z \;, \quad K_{23} = \frac{e^{3U-\alpha}}{2\rho^2}\lambda_\rho \;, \tag{4.1.16}$$

so that

$$e^{2(\alpha-U)}\,|K|_g^2 \geq 2e^{2(\alpha-U)}(K_{13}^2 + K_{23}^2) = \frac{e^{4U}}{2\rho^4}|\lambda|_\delta^2 \;. \tag{4.1.17}$$

In [31] (compare [15, 50, 65]) it has been shown that

$$\begin{aligned} m &= \frac{1}{16\pi}\int \Big[{}^{(3)}R + \frac{1}{2}\rho^2 e^{-4\alpha+2U}\left(\rho W_{\rho,z} - W_{z,\rho}\right)^2\Big] e^{2(\alpha-U)} d^3x \\ &\quad + \frac{1}{8\pi}\int (DU)^2\, d^3x \qquad (4.1.18) \\ &\geq \frac{1}{16\pi}\int \Big[{}^{(3)}R e^{2(\alpha-U)} + 2\,(DU)^2\Big] d^3x \,. \qquad (4.1.19) \end{aligned}$$

Inserting (4.1.8) and (4.1.17) into (4.1.19) we obtain

$$m \geq \frac{1}{8\pi}\int \Big[(DU)^2 + \frac{e^{4U}}{\rho^4}\,(Dv + \chi D\psi - \psi D\chi)^2 + \frac{e^{2U}}{\rho^2}\left((D\chi)^2 + (D\psi)^2\right)\Big] d^3x \,, \tag{4.1.20}$$

where, from now on, we use the symbol Df to denote the gradient of a function f with respect to the flat metric $\delta = d\rho^2 + dz^2 + \rho^2 d\varphi^2$, and we will use both $(v)^2$ and $|v|^2$ in alternative to $|v|_\delta^2$, the square of the norm of a vector $v = v^A\partial_A$ with respect to δ.

Remark 4.1.2 *As a consequence of [49] and Remark 4.1.1 we see that for* stationary *data with vanishing non-electromagnetic energy density ($\mu = 0$)*

$$m = \frac{1}{8\pi}\int \Big[(DU)^2 + \frac{e^{4U}}{\rho^4}\,(Dv + \chi D\psi - \psi D\chi)^2 + \frac{e^{2U}}{\rho^2}\left((D\chi)^2 + (D\psi)^2\right)\Big] d^3x \;.$$

It follows from (4.1.7) that ψ and χ are constant on each connected component $\mathscr{A}_j$ of the "axis"

$$\mathscr{A} := \{\rho = 0\} \setminus \{z = 0\} \;;$$

(4.1.15)-(4.1.16) then show that so is v. We set

$$v_j := v|_{\mathscr{A}_j} \;, \quad \psi_j := \psi|_{\mathscr{A}_j} \;, \quad \chi_j := \chi|_{\mathscr{A}_j} \;, \quad j = 1, 2 \;, \tag{4.1.21}$$

where $\mathscr{A}_1 = \{\rho = 0\} \cap \{z < 0\}$ and $\mathscr{A}_2 = \{\rho = 0\} \cap \{z > 0\}$.

We have the following, from which Theorem 4.0.1 immediately follows:

THEOREM 4.1.3 *Let (M, g, K, v, χ, ψ) be a three dimensional smooth data set invariant under an action of* U(1)*, where M is the union of a compact set and of two asymptotically flat regions M_1 and M_2, in the sense of* (4.1.1)*, and where v, ψ and χ are global potentials as in* (4.1.7),(4.1.15) *and* (4.1.16)*, satisfying* (4.1.3)*. Let m and $\vec{J}$ denote the ADM mass and angular momentum of M_1, and let Q_E and Q_B be the global electric and magnetic charges of M_1. If* (4.1.20) *holds and $M/\mathrm{U}(1)$ is simply connected, then*

$$m \geq \sqrt{\frac{|\vec{J}|^2}{m^2} + Q_E^2 + Q_B^2} \ .$$

REMARK 4.1.4 *We stress the fact that, in the previous result, no constraints are assumed. Consequently a clear abuse is created in adopting the electromagnetic terminology. Nonetheless the parallelism is obvious, as this result is a "natural" technical generalization of Theorem 4.0.1.*

For future reference we take the chance to provide formulae for the "Maxwell" 2-form and the global "charges" in terms of the axial *potentials: in an orthonormal basis $\{n, e_i\}$ as before, this time referring to an embedding of M in a space-time not necessarily satisfying Einstein equations, the Maxwell 2-form is given by*

$$F_{\mu\nu} = \frac{e^U}{\rho} \begin{pmatrix} 0 & \partial_2\psi & -\partial_1\psi & -\partial_1\chi \\ -\partial_2\psi & 0 & 0 & -\partial_2\chi \\ \partial_1\psi & 0 & 0 & 0 \\ \partial_1\chi & \partial_2\chi & 0 & 0 \end{pmatrix} \ ;$$

also via equations (4.1.7) *we get (compare [135])*

$$\begin{aligned} Q_E &= -\frac{1}{4\pi}\int_{S_\infty} *F = -\frac{-2\pi}{4\pi}\int_{S_\infty/U(1)} i_\eta * F \\ &= \frac{1}{2}\int_{S_\infty/U(1)} d\psi = \frac{\psi_2 - \psi_1}{2} \ , \end{aligned} \tag{4.1.22}$$

with a similar computation yielding

$$Q_B = \frac{\chi_1 - \chi_2}{2} \ . \tag{4.1.23}$$

PROOF: If the mass is infinite there is nothing to prove, otherwise by (4.1.19) we need to find a lower bound on

$$I := \int_{\mathbb{R}^3} \Big[(DU)^2 + \frac{e^{4U}}{\rho^4}\left(Dv + \chi D\psi - \psi D\chi\right)^2 + \frac{e^{2U}}{\rho^2}\left((D\chi)^2 + (D\psi)^2\right) \Big] d^3x \ . \tag{4.1.24}$$

Let $(\tilde{U}, \tilde{v}, \tilde{\chi}, \tilde{\psi})$ be the harmonic map associated with the extreme Kerr-Newman with angular momentum along the z–axis equal to $(v_2 - v_1)/8$ and electric charge $(\psi_2 - \psi_1)/2$. We wish to show that the action $I := I(U, v, \chi, \psi)$ is larger than or equal to that of $(\tilde{U}, \tilde{v}, \tilde{\chi}, \tilde{\psi})$, which shall be denoted by $\tilde{I}$. As we shall see I differs form an harmonic map action H (4.1.42) by a boundary term; the idea is then to use a result of [77], that the action H is minimized by the solution of the Dirichlet problem which is expected to be $(\tilde{U}, \tilde{v}, \tilde{\chi}, \tilde{\psi})$; however, that result does not apply directly because of the singularity of the equations at the axis $\rho = 0$; moreover, we are working in an unbounded domain. We will overcome such problems by constructing a controlled sequence of integrals over compact domains which avoid the singular set and saturate $\mathbb{R}^3$. Such strategy was developed in [38]; here we generalize it to the electro-vacuum setting with considerable simplifications of the intermediary steps.

So, let $\sigma > 0$, $r = \sqrt{\rho^2 + z^2}$ and let $f_\sigma \in C^\infty(\mathbb{R}^3)$ be any family of functions satisfying

- $\partial_\varphi f_\sigma \equiv 0$;
- $0 \leq f_\sigma \leq 1$;
- $f_\sigma = 0$ on the set $\{r \leq \sigma/2\} \cup \{r \geq 2/\sigma\}$;
- $f_\sigma = 1$ on the set $\{r \geq \sigma\} \cap \{r \leq 1/\sigma\}$;
- $|Df_\sigma| \leq C/\sigma$ for $\sigma/2 \leq r \leq \sigma$; and
- $|Df_\sigma| \leq C\sigma$ for $1/\sigma \leq r \leq 2/\sigma$.

Let $\theta = U, v, \chi, \psi$ and write

$$\theta_\sigma := f_\sigma \theta + (1 - f_\sigma)\,\tilde{\theta} = f_\sigma(\theta - \tilde{\theta}) + \tilde{\theta}\ .$$

We claim that $I^\sigma := I(U_\sigma, v_\sigma, \chi_\sigma, \psi_\sigma)$ satisfies

Lemma 4.1.5 $\lim_{\sigma\to 0} I^\sigma = I$.

Proof: Indeed, for

$$\lambda_\sigma := Dv_\sigma + \chi_\sigma D\psi_\sigma - \psi_\sigma D\chi_\sigma\ , \tag{4.1.25}$$

we have

$$
\begin{aligned}
\int_{\mathbb{R}^3} \frac{e^{4U_\sigma}}{\rho^4}|\lambda_\sigma|^2 \;=\; & \underbrace{\int_{\{0\le\sigma/2\}} \frac{e^{4\tilde U}}{\rho^4}|\tilde\lambda|^2}_{I} + \underbrace{\int_{\{\sigma/2\le r\le\sigma\}} \frac{e^{4U_\sigma}}{\rho^4}|\lambda_\sigma|^2}_{II} + \\
& + \underbrace{\int_{\{\sigma\le r\le 1/\sigma\}} \frac{e^{4U}}{\rho^4}|\lambda|^2}_{III} + \underbrace{\int_{\{1/\sigma\le r\le 2/\sigma\}} \frac{e^{4U_\sigma}}{\rho^4}|\lambda_\sigma|^2}_{IV} \\
& + \underbrace{\int_{\{2/\sigma\le r\}} \frac{e^{4\tilde U}}{\rho^4}|\tilde\lambda|^2}_{V} \,.
\end{aligned}
$$

Since the maps under consideration have finite energy, the integrals I and V converge to zero, by the dominated convergence theorem. III converges to the integral over $\mathbb{R}^3$ of $\frac{e^{4U}}{\rho^4}|\lambda|^2$ by, e.g., the monotone convergence theorem.

We will now show that

$$
II = \int_{\{\sigma/2\le r\le\sigma\}} \frac{e^{4U_\sigma}}{\rho^4}\left(Dv_\sigma + \chi_\sigma D\psi_\sigma - \psi_\sigma D\chi_\sigma\right)^2 \to_{\sigma\to 0} 0 \,. \tag{4.1.26}
$$

The key identity is

$$
\begin{aligned}
\lambda_\sigma \;=\; & f_\sigma\lambda + (1-f_\sigma)\tilde\lambda + Df_\sigma(v-\tilde v) + Df_\sigma(\tilde\chi\psi - \tilde\psi\chi) \\
& + f_\sigma(1-f_\sigma)\left\{(\psi-\tilde\psi)D(\chi-\tilde\chi) - (\chi-\tilde\chi)D(\psi-\tilde\psi)\right\} \,,
\end{aligned} \tag{4.1.27}
$$

which will allow to establish (4.1.26) by a step-by-step estimation of the integrals obtained by replacing λ_σ by each of its five summands. We will exemplify this by dealing with the most delicate case.

The existence of multiple ends manifests itself in the asymptotic behavior

$$
U = 2\log r + O(1) \,,\; r\to 0 \,, \tag{4.1.28}
$$

established in [31, Theorem 2.6]. Then, using the decay rates of the extreme Kerr-Newman map, compiled in Table A.1 of the appendix, we get [5]

$$
\begin{aligned}
e^{4U_\sigma} &= e^{4f_\sigma U} e^{4(1-f_\sigma)\tilde U} \\
&\lesssim r^{8f_\sigma} r^{4(1-f_\sigma)} \\
&= r^{4(f_\sigma+1)} \le r^4 \,,\; r\to 0.
\end{aligned} \tag{4.1.29}
$$

[5] We will write $f \lesssim g$ if and only if $f = O(g)$.

Recall that near $r = 0$ the coordinates (ρ, z) can be obtained from the usual cylindrical coordinates in the other asymptotically flat region, which we denote by $(\hat{\rho}, \hat{z})$, by an inversion $(\hat{\rho}, \hat{z}) = (\frac{\rho}{r^2}, \frac{z}{r^2})$, compare [31, Theorem 2.9, p. 2580]. This leads to estimates for small r, equivalently for large $\hat{r}$, such as

$$|\lambda|_\delta = \frac{1}{r^2}|\lambda|_{\hat{\delta}} \lesssim \frac{1}{r^2}\hat{\rho}^2\hat{r}^{-\beta} \lesssim \frac{1}{r^2}\frac{\rho^2}{r^4}r^\beta = \rho^2 r^{\beta-6}\ ,\ r \to 0\ , \tag{4.1.30}$$

where $\hat{\delta} = d\hat{\rho}^2 + d\hat{z}^2 + \hat{\rho}^2 d\varphi^2$.

The same procedure yields

$$|D\chi|_\delta, |D\psi|_\delta = \rho O(r^{\gamma-3})\ ,\ r \to 0\ , \tag{4.1.31}$$

and we see that, for small r,

$$|Dv|_\delta \leq |\lambda|_\delta + |\chi D\psi - \psi D\chi|_\delta \lesssim \rho^2 r^{\beta-6} + \rho\, r^{2\gamma-4}\ . \tag{4.1.32}$$

From this and the known asymptotic behaviour of extreme Kerr-Newman one obtains, when $\beta \geq 2\gamma + 1$,[6]

$$v - \tilde{v} = O(r^{2\gamma-2})\ , \tag{4.1.33}$$

and we are now able to see that the contribution of the term $Df_\mu(v - \tilde{v})$ in the region $\rho \geq z$, where r is comparable to ρ, is estimated by

$$\int_{\sigma/2}^{\sigma} \frac{r^4}{\rho^4 r^2}\left(r^{2\gamma-2}\right)^2 r^2 dr \lesssim \sigma^{4\gamma-3} \to_{\sigma\to 0} 0 \text{ provided that } \gamma > 3/4\ .$$

This explains our ranges of β and γ in (4.1.1) and (4.1.3).

Since v and $\tilde{v}$ have the same axis data, Taylor expanding on ρ along the axis yields

$$(v - \tilde{v})(\rho, z) = \underbrace{(v - \tilde{v})(0, z)}_{=0} + \partial_\rho(v - \tilde{v})(c(\rho), z)\rho\ ,\ |c(\rho)| \leq |\rho|\ . \tag{4.1.34}$$

Also, again for $\beta \geq 2\gamma + 1$,

$$\partial_\rho v = \rho O(r^{2\gamma-4})\ ,$$

[6] For $\beta < 2\gamma + 1$ the dominating behaviour in (4.1.32) is governed by λ, which leads to $v - \tilde{v} = O(r^{\beta-3})$ and the necessity to impose $\beta > 5/2$, as in vacuum [38, p. 2602].

with the same estimate holding for the ρ-derivative of the difference. Then, in $\{\rho \le z\}$,

$$v - \tilde{v} = \rho^2 O(r^{2\gamma-4}) \ , \ r \to 0 \ . \tag{4.1.35}$$

We see that, in this region, the integral under consideration is estimated by

$$\int_{\{\theta:\rho<z\}} \int_{\sigma/2}^{\sigma} \frac{r^4}{\rho^4 r^2} \left(\rho^2 r^{2\gamma-4}\right)^2 r^2 \sin\theta \, dr d\theta \lesssim \sigma^{4\gamma-3} \to_{\sigma\to 0} 0 \ ,$$

and (4.1.26) follows.

The remaining terms in I^σ can be controlled in a similar, although considerably more direct and simpler, fashion. For instance, when controlling the $|DU_\sigma|^2$ term one of the steps requires to estimate the integral

$$\begin{aligned} \int_{\{\sigma/2\le r\le\sigma\}} |DU_\sigma|^2 &= \int_{\{\sigma/2\le r\le\sigma\}} |(U-\tilde{U})Df_\sigma + f_\sigma DU + (1-f_\sigma)D\tilde{U}|^2 \\ &\lesssim \int_{\{\sigma/2\le r\le\sigma\}} \left((U-\tilde{U})^2 r^{-2} + |DU|^2 + |D\tilde{U}|^2\right) \ , \end{aligned}$$

where in fact the second and third term go to zero by the Lebesgue dominated convergence theorem while the vanishing of the first follows by direct estimation using (4.1.28) and the decay rates presented in Table A.1. □

We now show that:

LEMMA 4.1.6 *$I^\sigma \ge \tilde{I}$ for all σ small enough.*

PROOF: This time consider, for $0 < \epsilon < 1$,

$$\hat{f}_\epsilon = \begin{cases} 0, & \rho \le \epsilon \, ; \\ \frac{\log \frac{\rho}{\epsilon}}{\log \frac{\sqrt{\epsilon}}{\epsilon}}, & \epsilon \le \rho \le \sqrt{\epsilon} \, ; \\ 1, & \rho \ge \sqrt{\epsilon} \, . \end{cases}$$

Set, for $\theta = U, v, \chi, \psi$,

$$\theta_{\sigma,\epsilon} = \hat{f}_\epsilon \theta_\sigma + (1-\hat{f}_\epsilon)\tilde{\theta} \ ,$$

and let $I^{\sigma,\epsilon}$ denote the action of $(U_{\sigma,\epsilon}, v_{\sigma,\epsilon}, \chi_{\sigma,\epsilon}, \psi_{\sigma,\epsilon})$ and

$$\lambda_{\sigma,\epsilon} = Dv_{\sigma,\epsilon} + \chi_{\sigma,\epsilon} D\psi_{\sigma,\epsilon} - \psi_{\sigma,\epsilon} D\chi_{\sigma,\epsilon} \ .$$

We claim that

$$\int_{\{\rho\le\sqrt{\epsilon}\}}\Big[\,(DU_{\sigma,\epsilon})^2+\frac{e^{4U_{\sigma,\epsilon}}}{\rho^4}\,(\lambda_{\sigma,\epsilon})^2+\frac{e^{2U_{\sigma,\epsilon}}}{\rho^2}\,\big((D\chi_{\sigma,\epsilon})^2+(D\psi_{\sigma,\epsilon})^2\big)\,\Big]d^3x\to_{\epsilon\to 0}0\;. \tag{4.1.36}$$

Equivalently,

$$I^{\sigma,\epsilon}\to_{\epsilon\to 0}I^{\sigma}\;. \tag{4.1.37}$$

In order to see this, note that the integral over the set $\{0\le\rho\le\epsilon\}$, where $\theta_{\sigma,\epsilon}=\tilde{\theta}$, approaches zero as ϵ tends to zero by the Lebesgue dominated convergence theorem; the same happens away from the set $\{\sigma/2<r<2/\sigma\}$. So it remains to consider the integral over

$$\mathscr{W}_{\sigma,\epsilon}:=\{\epsilon\le\rho\le\sqrt{\epsilon}\}\cap\{\sigma/2<r<2/\sigma\}\;.$$

The computations leading to (4.1.27) now give

$$\begin{aligned}\lambda_{\sigma,\epsilon}\;&=\;\hat{f}_\epsilon\lambda_\sigma+(1-\hat{f}_\epsilon)\tilde{\lambda}+D\hat{f}_\epsilon(v_\sigma-\tilde{v})+D\hat{f}_\epsilon(\tilde{\chi}\psi_\sigma-\tilde{\psi}\chi_\sigma)\\&\quad+\hat{f}_\epsilon(1-\hat{f}_\epsilon)\Big\{(\psi_\sigma-\tilde{\psi})D(\chi_\sigma-\tilde{\chi})-(\chi_\sigma-\tilde{\chi})D(\psi_\sigma-\tilde{\psi})\Big\}\;.\end{aligned} \tag{4.1.38}$$

Since $I^\sigma\to I$, we see that I^σ must be finite, at least for all small enough σ. Fix such a $\sigma>0$. As before the first two terms in the right-hand side of (4.1.38) constitute no problem. To control the others note that, for all ϵ such that $\sqrt{\epsilon}<\sigma/2$, we have, in the (ρ,z,φ) coordinates,

$$\mathscr{W}_{\sigma,\epsilon}\subseteq[\epsilon,\sqrt{\epsilon}\,]\times([z_0(\sigma),z_1(\sigma)]\cup[z_2(\sigma),z_3(\sigma)])\times[0,2\pi]\;, \tag{4.1.39}$$

for a good choice of z_i's satisfying $z_i(\sigma)\neq 0$; e.g., the z–coordinate value, in increasing order, of the points in the intersection of $\rho=\epsilon$ with both $r=\sigma/2$ and $r=2/\sigma$. We then see that

$$\begin{aligned}\int_{\mathscr{W}_{\sigma,\epsilon}}\frac{e^{4U_{\sigma,\epsilon}}}{\rho^4}\Big(D\hat{f}_\epsilon(v_\sigma-\tilde{v})\Big)^2d^3x\;&\le\;2\pi\sum_{i=0,1}\int_{z_{2i}(\sigma)}^{z_{2i+1}(\sigma)}\int_\epsilon^{\sqrt{\epsilon}}\frac{e^{4U_{\sigma,\epsilon}}}{\rho^4}\Big(D\hat{f}_\epsilon\Big)^2(v_\sigma-\tilde{v})^2\,\rho\,d\rho\,dz\\&\le\;2\pi\sum_{i=0,1}\int_{z_{2i}(\sigma)}^{z_{2i+1}(\sigma)}\int_\epsilon^{\sqrt{\epsilon}}\frac{C(\sigma)}{\rho^3}\frac{1}{\rho^2(\log\epsilon)^2}(v_\sigma-\tilde{v})^2\,d\rho\,dz\end{aligned}$$

Since $z_i(\sigma)\neq 0$ we see that v_σ and $\tilde{v}$ are smooth on the set $\{\rho\le\sqrt{\epsilon}\;,\;z\in\cup_i[z_{2i},z_{2i+1}]\}$. Then, Taylor expanding on ρ along the axis, while noting that v_σ

and $\tilde{v}$ have the same axis data and that f_σ is, by construction, axisymmetric, yields (compare (4.1.34))

$$v_\sigma - \tilde{v} = O(\rho^2) \; , \; \rho \to 0 \; , \; \text{in } \{\rho \le \sqrt{\epsilon} \; , \; z \in \cup_i [z_{2i}, z_{2i+1}]\} \; , \tag{4.1.40}$$

hence

$$\begin{aligned} 2\pi \sum_{i=0,1} \int_{z_{2i}(\sigma)}^{z_{2i+1}(\sigma)} \int_{\epsilon}^{\sqrt{\epsilon}} \frac{C(\sigma)}{\rho^3} \frac{1}{\rho^2 (\log \epsilon)^2} (v - \tilde{v})^2 \, d\rho \, dz \; &\lesssim \; \frac{C(\sigma)}{(\log \epsilon)^2} \int_{\epsilon}^{\sqrt{\epsilon}} \frac{1}{\rho^5} \rho^4 \, d\rho \\ &\lesssim \; \frac{C(\sigma)}{(\log \epsilon)^2} \log \epsilon \to_{\epsilon \to 0} 0 \; . \end{aligned}$$

The remaining terms are controlled in an analogous way, with the $(DU_{\sigma,\epsilon})^2$ term behaving exactly as in vacuum [38]. This ends the proof of (4.1.36).

Using the rescaling $U = u + \ln \rho$ we have

$$I_\Omega(U, v, \chi, \psi) = H_\Omega(u, v, \chi, \psi) + B_\Omega(U) \; , \tag{4.1.41}$$

where

$$H_\Omega = \int_\Omega \Big[(Du)^2 + e^{4u} \left(Dv + \chi D\psi - \psi D\chi\right)^2 + e^{2u} \left((D\chi)^2 + (D\psi)^2\right) \Big] d^3x \; , \tag{4.1.42}$$

is the energy, over $\Omega \subset\subset \mathbb{R}^3 \setminus \mathscr{A}$, of the harmonic map

$$\Phi = (u, v, \chi, \psi) : \mathbb{R}^3 \setminus \mathscr{A} \longrightarrow \mathbb{H}^2_{\mathbb{C}} \; , \tag{4.1.43}$$

which differs from I by the boundary term

$$B_\Omega(U) = \int_{\partial\Omega} \frac{\partial \ln \rho}{\partial N} (2U - \ln \rho) dS \; , \tag{4.1.44}$$

where N is the outward pointing unit normal to $\partial\Omega$. Consequently for both I and H the associated variational equations are the harmonic map equations, with target space the two-dimensional complex hyperbolic space. Hence the target manifold satisfies the convexity conditions of [77] (see Remark (i), p. 5 there). For compact Ω away from the axis we can thus conclude from [77] that action minimisers of H_Ω with Dirichlet boundary conditions exist, are smooth, and satisfy the variational equations. It is also well known (see [38] and references therein) that solutions of the Dirichlet boundary value problem are unique when the target manifold has negative sectional curvature, which is the case here. All this implies that $(\tilde{U}, \tilde{v}, \tilde{\chi}, \tilde{\psi})$, with its own boundary data, minimizes the action integral H , and consequently of I, over the sets

$$\mathscr{C}_{\sigma,\epsilon} := \{\rho \geq \epsilon\} \cap \{\sigma/2 \leq r \leq 2/\sigma\} \ .$$

In particular, since the maps $(\theta_{\sigma,\epsilon})$ and $(\tilde{\theta})$ coincide on $\partial\mathscr{C}_{\sigma,\epsilon}$ we conclude that

$$I_{\mathscr{C}_{\sigma,\epsilon}}(U_{\sigma,\epsilon}, v_{\eta,\epsilon}, \chi_{\eta,\epsilon}, \psi_{\eta,\epsilon}) > I_{\mathscr{C}_{\sigma,\epsilon}}(\tilde{U}, \tilde{v}, \tilde{\chi}, \tilde{\psi}) \ .$$

In fact the maps under consideration coincide on the closure of the complement of $\mathscr{C}_{\sigma,\epsilon}$ and therefore

$$I^{\sigma,\epsilon} > \tilde{I} \ . \tag{4.1.45}$$

Recalling (4.1.37) we obtain

$$I^{\sigma} = \lim_{\epsilon \to 0} I^{\sigma,\epsilon} \geq \lim_{\epsilon \to 0} \tilde{I} = \tilde{I} \ . \tag{4.1.46}$$

□

Returning to the proof of Theorem 4.0.1, Lemmata 4.1.5 and 4.1.6 yield

$$I = \lim_{\sigma \to 0} I^{\sigma} \geq \tilde{I} \ ,$$

and the result is a consequence of Remark 4.1.2 followed by a duality rotation.

□

4.2 Concluding remarks

The study of Dain inequalities is still in an early stage with important questions still needing to be settled even for pure vacuum; also, some impressive generalization can be easily formulated and justified by the heuristic argument presented in the introduction. We finish this chapter by addressing some of this issues:

1. **Extreme Kerr-Newman as a minimum of the mass functional.** Our class of data does not include extreme Kerr-Newman and consequently eliminates a priori the possibility of establishing it as the unique minimum for the mass functional. This difficulty is not present in Dain's original work where another class of data is considered; however, this is done at the

cost of a considerably longer list of (stronger) technical assumptions, some of which are derivable properties of asymptotical flatness and the existence of multiple ends as was first observed and established in [31]. To obtain the desired result within the spirit of the program initiated by Chruściel one could start by generalizing the results in [31] for data allowing for both asymptotically flat and asymptotically cylindrical ends, and then try to adapt the arguments presented here.

From what as been said, we expected inequality (4.0.2) to be strict within the class of data considered in this work.

2. **Multiple asymptotically flat ends.** Even for vacuum the question of multiple ends requires further work. Although a Dain inequality was already established in [38] it depends on a function of the angular momenta for which an explicit expression remains unknown for all $N > 2$, where N is the number of asymptotically flat ends. This is clearly an unsatisfactory situation since the Penrose-like argument, presented in the introduction, provides evidence that the unknown function should simply be the square root of the total angular momentum. In fact, for the two body problem, $N = 3$, such expectation as been recently supported by numerical evidences [52].

   One also expects the ideas in [38] to generalize to electro-vacuum by using the methods developed here, but in this case it seems hard to speculate what the exact expression for the lower bound function, this time of both angular momenta and Maxwell charges, should be. This problem is related to the fact that the Majumdar-Papapetrou metrics provide the existence of regular and extreme multiple black hole solutions; analogous difficulties have been found for the Penrose inequality [137].

3. **Asymptotically electro-vacuum data.** The heuristic argument leading to the Dain inequality presented here works for other data, involving far more general matter models: axisymmetric asymptotically electro-vacuum initial data whose domain of outer communications becomes electro-vacuum asymptotically with time. Establishing a Dain inequality in such generality would be quite impressive but, at this moment, such a goal seems unreachable.

Appendix A

Decay rates for extreme Kerr-Newman

$r \to 0$	$r \to +\infty$
$\tilde{U} = \log(r) + O(1)$	$\tilde{U} = -\frac{m}{r} + O(r^{-2})$
$\tilde{\chi} = \rho^2 O(r^{-2}) = O(1)$	$\tilde{\chi} = \rho^2 O(r^{-3}) = O(r^{-1})$
$\tilde{\psi} = O(1)$	$\tilde{\psi} = \rho O(r^{-2}) = O(r^{-1})$
$\partial_\rho \tilde{\chi} = \rho O(r^{-2})$	$\partial_\rho \tilde{\chi} = \rho O(r^{-3})$
$\partial_\rho \tilde{\psi} = \rho O(r^{-2})$	$\partial_\rho \tilde{\psi} = \rho O(r^{-2})$
$\lvert D\tilde{\chi}\rvert_\delta = \rho O(r^{-2})$	$\lvert D\tilde{\chi}\rvert_\delta = \rho O(r^{-3})$
$\lvert D\tilde{\psi}\rvert_\delta = O(r^{-1})$	$\lvert D\tilde{\psi}\rvert_\delta = O(r^{-1})$
$\tilde{v} = O(1)$	$\tilde{v} = O(1)$
$\partial_\rho \tilde{v} = \rho O(r^{-2})$	$\partial_\rho \tilde{v} = \rho O(r^{-2})$

Table A.1: Decay rates for Extreme Kerr-Newman

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