Cauchy horizons in Gowdy space times

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January 28, 2004

Abstract

We analyse exhaustively the structure of *non-degenerate* Cauchy horizons in Gowdy space-times, and we establish existence of a large class of non-polarized Gowdy space-times with such horizons.

Added in proof: Our results here, together with deep new results of H. Ringström (talk at the Miami Waves conference, January 2004), establish strong cosmic censorship in (toroidal) Gowdy space-times.

1 Introduction

In 1981 Vince Moncrief pointed out the interest of studying the Gowdy metrics as a toy model in mathematical general relativity, and proved the fundamental global existence result for those metrics [15]. He developed approximate methods to study their dynamics, and discovered the leading

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order behavior of a large class of solutions of the associated equations [11]. Together with Beverly Berger he initiated the numerical investigation of those models [2]. In spite of considerable effort by many researchers, the global properties of those models are far from being understood. It is a pleasure to dedicate to him this contribution to the topic.

In Gowdy space times the essential part of the Einstein equations reduces to a nonlinear wave-map-type system of equations [10] for a map x from $(M, g_{\alpha\beta})$ to the hyperbolic plane $(\mathscr{H}, h_{ab})$, where $M = [T, 0) \times \mathrm{S}^1$ with the flat metric $g = -dt^2 + d\theta^2$. The solutions are critical points of the Lagrangean

$$\mathscr{L}[x] = \frac{1}{2}\int_M t g^{\alpha\beta} h_{ab} \partial_\alpha x^a \partial_\beta x^b \, d\theta dt \,. \tag{1.1}$$

It is sometimes convenient to use coordinates $P, Q \in \mathbb{R}$ on the hyperbolic plane, in which the hyperbolic metric h_{ab} takes the form

$$h = dP^2 + e^{2P} dQ^2. \tag{1.2}$$

Let $X_t = \frac{\partial x}{\partial t}, X_\theta = \frac{\partial x}{\partial \theta}$, D denote the Levi-Civita connection of h_{ab}, and $D_\theta \equiv \frac{D}{D\theta} := D_{X_\theta}$, $D_t \equiv \frac{D}{Dt} := D_{X_t}$. The Euler-Lagrange equations for (1.1) take the form

$$\frac{DX_t}{Dt} - \frac{DX_\theta}{D\theta} = -\frac{X_t}{t} \tag{1.3}$$

or, in coordinates,

$$\Box x^a + \Gamma^a_{bc} \circ x \partial_\mu x^b \partial^\mu x^c = -\frac{\partial_t x^a}{t},$$

where the Γ's are the Christoffel symbols of h_{ab}, and $\Box = \partial_t^2 - \partial_\theta^2$.

As already mentioned, V. Moncrief established global existence of smooth solutions on $(-\infty, 0)$ of the Cauchy problem for (1.3) [15]. This implies [6, 15] that a maximal globally hyperbolic Gowdy space-time $(\mathscr{M}, g)$ with toroidal Cauchy surfaces can be covered by a global coordinate system $(t, \theta, x^a) \in (-\infty, 0) \times S^1 \times S^1 \times S^1$, in which the metric takes the following form[1]

$$\begin{aligned} g \;&=\; e^{-\gamma/2}|t|^{-1/2}(-dt^2 + d\theta^2) + |t|e^P (dx^1)^2 + 2|t|e^P Q \, dx^1 dx^2 \\ &\quad +|t| \left(e^P Q^2 + e^{-P}\right)(dx^2)^2 \,, \end{aligned} \tag{1.4}$$

[1] The form (1.4) has been claimed by Gowdy [10] under the hypothesis of two commuting Killing vectors, with $\mathbb{T}^3$ spatial topology. This is not quite correct as there are a few further global constants involved [6] even in the current case of vanishing twist. However those constants can be eliminated after passing from the torus $\mathbb{T}^3$ to its universal cover $\mathbb{R}^3$, or when working in local coordinates, and this suffices for the local considerations of the proofs given here.

where the function γ solves the equations

$$\partial_t \gamma = -t \left(|X_t|^2 + |X_\theta|^2\right) \ , \quad \partial_\theta \gamma = -2h(tX_t, X_\theta) \ . \tag{1.5}$$

The main question of interest is the curvature blow-up – or lack thereof – at the boundary $t = 0$ of the associated space-time. An exhaustive analysis of this has been carried out in [9] for the so-called *polarised case*, where the image of the map x is contained in a geodesic in the hyperbolic space. There exist only two results in the literature which prove curvature blow up without the polarisation condition in a Cauchy problem context: The first is due to one of the current authors (PTC), who proves uniform curvature blow-up [7] under the condition that the solution at $t = t_0$ satisfies[2]

$$\sup_\theta t_0^2(|X_t|^2 + |X_\theta|^2)(t_0, \theta) < \frac{1}{6^{3/2}} \ . \tag{1.6}$$

The second one[3] is due to Ringström [22], who assumes, at $t = t_0$, smallness of the derivatives of Q together with a bound

$$0 < \eta \le |t_0| P_t(t_0, \theta) \le 1 - \eta \ , \tag{1.7}$$

for some strictly positive constant η. He further requires $|t_0|$ to be sufficiently small.

The purpose of this paper is to study Cauchy horizons in Gowdy space-times. Recall that existence of Cauchy horizons is precisely what one wants to avoid in order to maintain predictability of the Einstein equations. We wish therefore to describe properties of those Gowdy space-times which possess Cauchy horizons, as a step towards proving that generic initial data for those space-times will *not* lead to formation of Cauchy horizons. Our results here, together with those in [4, 14, 20–22, 24], give strong indications that this will be the case, though no definitive statement is available so far.

To put our results in proper context, we start by recalling some results from [7] concerning the geometric properties of the associated maximal globally hyperbolic solution $(\mathscr{M}, {}^4g)$ of the Einstein equations. The following gives a meaning to the statement that the set $\{t = 0\}$ can be thought as a (perhaps singular) boundary of the space-time:

[2]The threshold $6^{-3/2}$ in (1.6) has been recently raised to $1/2$ in [4, 24].

[3]There is no explicit statement about curvature blow-up in [22]; however, this follows immediately from the results in [22] together with the calculations in the proof of Theorem 3.5.1 of [7]. The results in [22] have been strengthened and generalised in [24].

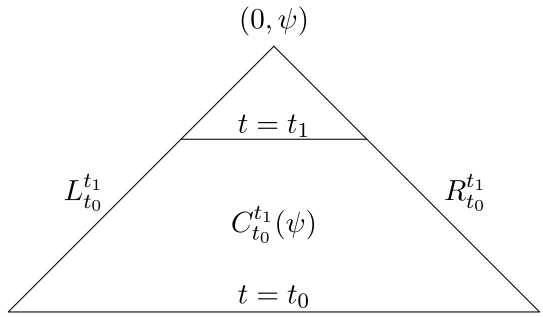

Figure 1: The truncated domains of dependence $C_{t_0}^{t}(\psi)$.

PROPOSITION 1.1 *Every future directed inextendible causal curve* $(a,b) \ni s \to \Gamma(s)$ *in* $(\mathscr{M}, {}^4g)$ *reaches the boundary* $t = 0$ *in finite proper time. Further the limit*

$$\lim_{s\to b} \theta(\Gamma(s)) \ ,$$

where θ *is the coordinate appearing in* (1.4)*, exists.*

In order to continue, we need some terminology. For $t_0 < 0$ let the set $C_{t_0}^0(\psi)$ be defined as (compare Figure 1)

$$C_{t_0}^0(\psi) = \{t_0 \le t < 0\ ,\ -|t| \le \theta - \psi \le |t|\}\ . \tag{1.8}$$

We shall say that $\lim_{C_{t_0}^0(\psi)} f = \alpha$ if

$$\lim_{t\to 0} \sup_{-|t|\le\theta-\psi\le|t|} |f(t,\theta) - \alpha| = 0\ . \tag{1.9}$$

Such limits look a little awkward at first sight; however, they arise naturally when considering the behavior of the geometry along causal curves with endpoints on the boundary $t = 0$. In any case, the existence of such limits can often be established in the problem at hand [4, 7].

The following relates properties of the Gowdy map x to curvature blow-up in space-time (the result follows immediately from the arguments in the proofs of Theorem 3.5.1 and Proposition 3.5.2 in [7]):

PROPOSITION 1.2 *Let* $\psi \in S^1$ *be such that*

$$\lim_{C_{t_0}^0(\psi)} |t^2 D_\theta X_\theta| = \lim_{C_{t_0}^0(\psi)} |t^2 D_\theta X_t| = \lim_{C_{t_0}^0(\psi)} |t X_\theta| = 0\ . \tag{1.10}$$

If there exists $1 \neq v(\psi) \in \mathbb{R}$ *such that*

$$\lim_{C^0_{t_0}(\psi)} |tX_t| = v(\psi) \ , \tag{1.11}$$

then the curvature scalar $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ *blows up on every inextendible causal curve in* $(\mathscr{M}, {}^4g)$ *with end point in* $\{0\} \times \{\psi\} \times S^1 \times S^1$*.*

Remark 1.3 Condition (1.11) can be weakened to

$$\textit{either} \quad \limsup |tX_t| < 1 \quad \textit{or} \quad \liminf |tX_t| > 1 \ ,$$

where the limits sup and inf are understood in a way analogous to (1.9).

Remark 1.4 Suppose that there exists a sequence of points $(t_i, \theta_i) \in C^0_{t_0}(\psi)$ with $t_i \to 0$ and $\Big||tX_t|(t_i, \theta_i) - 1\Big| > \epsilon$ for some ϵ. If (1.10) holds along the same sequence, one then obtains curvature blow up along this sequence, which suffices to obtain inextendibility of the space-time at $(0, \psi)$. From this point of view the essential conditions are thus (1.10), while the existence of the velocity function $v(\psi)$ defined by (1.11) is actually irrelevant in the following sense: v matters only at points at which it exists *and* equals one.

Recall, now, that we are interested in conditions which would guarantee that Cauchy horizons *do not* occur. Consider, thus, a Gowdy space-time with a Cauchy horizon $\mathcal{H}$. It is then easily seen (see Proposition 3.1 below) that there exists a point ψ such that all curvature invariants remain bounded along the timelike curves $t \to (t, \psi, x^1, x^2)$. Proposition 1.2 suggests then that the asymptotic velocity v should equal one at ψ. This is, however, not entirely clear, because of the additional hypotheses (1.10) made. Now, one would like to have a sharp Cauchy horizon criterion without any unjustified restrictions. Here we prove the following related statement:

Proposition 1.5 *Consider a maximal globally hyperbolic Gowdy space-time which is smoothly extendible across a Cauchy horizon* $\mathcal{H}$*. Then there exists a point* $\psi \in S^1$*, coordinates* (P, Q) *on the hyperbolic space, and a number* $P_\infty(\psi)$ *such that*

$$\lim_{C^0_{t_0}(\psi)} \left(P(t, \psi) + \ln|t|\right) = P_\infty(\psi) \ , \quad \lim_{C^0_{t_0}(\psi)} Q = 0 \ . \tag{1.12}$$

Further there exists a sequence $t_i \to 0$ *such that*

$$\left(|t|\partial_t P\right)(t_i, \psi) \to 1 \ . \tag{1.13}$$

To continue, one would like to understand precisely the geometry of the set across which extensions are possible. Polarised solutions are known which have $v = 1$ on an interval $[\theta_l, \theta_r]$, and which are extendible across $\{0\} \times (\theta_l, \theta_r) \times S^1 \times S^1$ (cf., e.g., [8]). This begs the question of existence of non-polarised solutions with $v = 1$ on an interval $[\theta_l, \theta_r]$. In Section 2 we provide a simple construction of non-polarised Gowdy metrics, extendible across such a smooth Cauchy horizon; this is rather similar to a construction of Moncrief [18] (compare [3]).

Based on those examples, one would naively expect that extendibility always requires the existence of such an interval $[\theta_l, \theta_r]$. Further, in those examples, one has much better control of the geometry than (1.10); one has $\mathrm{AVTD}_\infty^{(P,Q)}$ behavior near $\{0\} \times [\theta_l, \theta_r]$, as defined at the beginning of Section 3. Again one could hope that $\mathrm{AVTD}_\infty^{(P,Q)}$ behavior always holds across Cauchy horizons, but this has not been proved so far. The main result of our paper is the following theorem, which partially settles the issues raised so far:

THEOREM 1.6 *Consider a smooth Gowdy space-time that is extendible across a Cauchy horizon $\mathcal{H}$. Then:*

(i) *$\mathcal{H}$ is a Killing horizon;* i.e.*, there exists a Killing vector field which is tangent to the null geodesics threading $\mathcal{H}$.*

(ii) *$\mathcal{H}$ is non-degenerate if and only if there exists an interval $[\theta_l, \theta_r] \subset [a, b]$ such that for $\theta \in [\theta_l, \theta_r]$ the velocity function $v(\theta)$ exists and is equal to 1 there, with*

$$|tP_t - 1| + |tP_\theta| + |te^P Q_t| + |te^P Q_\theta| \leq C t^{\alpha_p} , \qquad (1.14)$$

for some $C, \alpha_p > 0$, in a certain coordinate system (P, Q) on hyperbolic space.

(iii) *In fact, when $\mathcal{H}$ is non-degenerate there exists a (P, Q) coordinate system on the hyperbolic space $\mathscr{H}_2$ in which the solution is $AVTD_\infty^{(P,Q)}$ near $\{0\} \times [\theta_l, \theta_r]$, with $Q_\infty = 0$ over $[\theta_l, \theta_r]$, and with the conclusions of Theorem 3.2 below holding on $[\theta_l, \theta_r]$.*

REMARK 1.7 Assuming smooth Cauchy data, we note that there exists $k_0 \in \mathbb{N}$ such that if the metric is C^{k_0} extendible across a non-degenerate $\mathcal{H}$, then it is smoothly extendible there. An explicit estimate for k_0 can be found by chasing differentiability in the constructions in the proof below to ensure that (3.5) holds. Smoothness of the extension follows then from Theorem 3.2.

Similar results can be established for Gowdy metrics with (sufficiently high) finite differentiability, whether in a classical or in a Sobolev sense.

REMARK 1.8 Extendibility above is meant in the class of Lorentzian manifolds; no field equations are assumed to be satisfied by the extension. We note that a necessary condition, in the non-degenerate case, for extendibility in the class of vacuum metrics is analyticity of the metric.

Theorem 1.6 shows that degenerate horizons $\mathcal{H}$, if any, would correspond to a single point in the (t, θ) coordinates. For analytic metrics non-existence of degenerate horizons $\mathcal{H}$ in the Gowdy class of metrics follows from the arguments of the proof of Theorem 2 of [19], but the information provided by those considerations does not seem to suffice to exclude the possibility that $\kappa = 0$ for smooth but not necessarily analytic metrics. While we find it unlikely that degenerate horizons could exist here, we can exclude them only under some supplementary hypotheses on the behavior of the Gowdy map x:

PROPOSITION 1.9 *Let ψ be such that $v(\psi) = 1$ and suppose that either*

a) there exists $\alpha_p > 0$ such that we have

$$|t\partial_\theta P| + |te^P \partial_t Q| \leq C|t|^{\alpha_p} \tag{1.15}$$

on $C^0_{t_0}(\psi)$, or

b) Equation (1.10) holds, or

c) the solution is $AVTD_2^{(P,Q)}$ on $C^0_{t_0}(\psi)$.

Then:

(i) *If there exists a sequence $\psi_i \to \psi$ such that $v(\psi_i) \neq 1$, then there exists no extension of $\mathscr{M}$ across $\{0\} \times \psi \times S^1 \times S^1$.*

(ii) *Moreover, if*

$$-\left(\frac{d}{d\theta} v\left(\psi\right)\right)^2 + e^{2\, P_\infty(\psi)} \left(\frac{d^2}{d\theta^2} Q_\infty\left(\psi\right)\right)^2 \neq 0\,,$$

then the curvature scalar $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ blows up on every inextendible causal curve in $(\mathscr{M}, {}^4g)$ with accumulation point in $\{0\} \times \{\psi\} \times S^1 \times S^1$.

We note that the power-law condition (1.15) is justified for initial data near those corresponding to a flat Kasner solution $(P_0, Q_0) \equiv (-\ln|t|, 0)$ by the results in [4].

This paper is organised as follows: In Section 2 we present our construction of solutions with Cauchy horizons. In Section 3 we define $\mathrm{AVTD}_k^{(P,Q)}$ Gowdy maps, and we give proofs of the results described above. In Section 4 we investigate further the blow-up of the Kretschmann scalar $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$. In Appendix A the Riemann tensor of the Gowdy metrics is given.

Added in proof: It has been shown recently by H. Ringström [24] that (1.10) holds at all points $\theta \in S^1$ for all solutions of Gowdy equations. The arguments of H. Ringström [24] also show that the collection of Gowdy initial data for which the set $\{\theta : v(\theta) = \pm 1\}$ has no interior is of second category. Consider any solution in that last class, and suppose it contains a Cauchy horizon. Proposition 1.9 shows that the horizon is non-degenerate, which is impossible by Theorem 1.6. It follows that generic (in the second category sense) Gowdy metrics on T^3 have no Cauchy horizons, which establishes strong cosmic censorship within this class of metrics.

2 Two families of solutions with $v \equiv 0$ and with $v \equiv 1$

There are by now at least two systematic ways [20, 22] of constructing reasonably general families of solutions with controlled asymptotic behavior satisfying (compare (1.7))

$$0 < \eta \le |t|P_t(t,\theta) \le 1 - \eta \qquad \forall\, \theta \in S^1\ ,\ t_0 \le t < 0\ . \tag{2.1}$$

The proofs given there only cover situations in which[4] $|t|P_t$ is bounded away from zero. It turns out that there is a trivial way of constructing families of solutions for which tX_t approaches zero asymptotically, as follows: Let

$$y\left(x^0, r = \sqrt{(x^1)^2 + (x^2)^2}\right)$$

be any rotationally-symmetric solution of the standard wave-map equation from the three-dimensional Minkowski space-time $(\mathbb{R}^{1,2}, \eta)$ into the two-dimensional hyperbolic space $(\mathscr{H}_2, h)$. Set

$$x(t,\theta) = y(x^0 = \theta, r = -t)\ . \tag{2.2}$$

[4]However, it seems that certain families of solutions with vanishing v can also be constructed using Fuchsian techniques (A. Rendall private communication).

Note that the replacement $|t| \to r$, $\theta \to x^0$ brings the action (1.1) to that for a rotationally invariant wave-map as above, so it should be clear that x satisfies the Gowdy evolution equation (1.3). Now, the solution x will not be periodic in θ in general (and it seems that there are no non-trivial smooth wave maps y at all which are periodic in x^0). However, it is straightforward to construct periodic solutions for which (2.2) will hold on an open neighborhood of some interval $\{0\} \times [b, 2\pi - b] \subset \{0\} \times [0, 2\pi] \approx \{0\} \times S^1$, with $0 < b < \pi$, as follows: let x be as in (2.2), and for $\theta \in [b/2, 2\pi - b/2]$ let

$$f(\theta) = x(-b/2, \theta) \ , \quad g(\theta) = \partial_t x(-b/2, \theta) \ .$$

Extend f and g to 2π–periodic functions $(\hat{f}, \hat{g})$ in any way. Let $x_{\hat{f},\hat{g}}$ be the solution of the Gowdy evolution equation (1.3) with initial data $(\hat{f}, \hat{g})$ at $t = -b/2$. Uniqueness in domains of dependence of solutions of (1.3) shows that $x_{\hat{f},\hat{g}}$ will coincide with x in the domain of dependence of the set $\{t = -b/2\, ,\ \theta \in [b/2, 2\pi - b/2]\}$:

$$x_{\hat{f},\hat{g}} = x \ \text{ on } \ \{-b/2 \le t \le 0\, ,\ b + t \le \theta \le 2\pi - b - t\} \ .$$

Clearly one can also construct solutions which will satisfy (2.2) on the union of any finite number of disjoint domains of dependence as above.

The wave maps y can be obtained by prescribing arbitrary rotationally invariant, say smooth, Cauchy data $y(0, r)$ and $\partial_t y(0, r)$, and the solutions are always global [5]. In this way one obtains a family of solutions x of the Gowdy equations parameterised, locally, by four free smooth functions (this, by the way, shows that the "function counting method" might be rather misleading, as this family of solutions does certainly *not* form an open set in the set of all solutions). The y's are globally smooth both in the t and θ variables, which implies that the x's display the usual "asymptotically velocity dominated" behavior with

$$\lim_{t\to 0} tX_t = \lim_{t\to 0} X_t = 0 \ , \tag{2.3}$$

the convergence being uniform in θ for $\theta \in [a, b]$. In particular one has curvature blow-up for causal curves ending on $\{0\} \times [a, b] \times S^1 \times S^1$ by Proposition 1.2.

Equation (2.3) shows that x has zero velocity in the relevant range of θ's. It follows from the results proved in [4] that all solutions which have $v \equiv 0$ on an interval I are obtained from the procedure above on an neighborhood of $\{0\} \times I$.

The solutions just described can be used to construct solutions with $v = 1$ on open intervals. (Here and elsewhere v is always the quantity defined as in (1.11), whenever it exists, with X_t being associated with the (P, Q) representation of the solution as in (1.4).) Indeed, let $x = (P, Q)$ be one of the zero-velocity solutions defined by (2.2), and define a new solution $\hat{x}$ by performing the "Gowdy-to-Ernst" transformation [21]:

$$\hat{P} := -P - \ln|t| \;, \quad e^{\hat{P}} \partial_t \hat{Q} := -e^P \partial_\theta Q \;, \quad e^{\hat{P}} \partial_\theta \hat{Q} := -e^P \partial_t Q \;. \tag{2.4}$$

The new map satisfies again the Gowdy equation (1.3). It immediately follows that we have

$$\Big| |tX_t| - 1 \Big| + |tX_\theta| \leq C|t| \;,$$

so that we obtain power-law blow-up, together with the new asymptotic velocity equal to one on any interval on which the old velocity $v = 0$. All the resulting space-times have a smooth Cauchy horizon across any interval on which $v = \pm 1$, which can be checked by standard calculations using (3.6) (compare [8, 16–18]).

As already mentioned in the introduction, a similar technique, mapping singular solutions to ones with Cauchy horizons and vice-versa, has been used by V. Moncrief in [18], compare [3]. While V. Moncrief used this method to produce singular solutions out of ones with Cauchy horizons, our approach is exactly the reverse one.

3 Proofs

We start with some terminology. We will be mainly interested in solutions of (1.3) with the following behavior (for justification, see [4, 9, 14, 20] and (4.9)-(4.8) below):

$$P(t,\theta) = -v_1(\theta) \ln|t| + P_\infty(\theta) + o(1) \;, \tag{3.1}$$

$$Q(t,\theta) = Q_\infty(\theta) + \begin{cases} |t|^{2v_1(\theta)} \Big(\psi_Q(\theta) + o(1)\Big) \;, & 0 < v_1(\theta) \notin \mathbb{N} \;; \\ |t|^{2v_1(\theta)} \Big(Q_{\ln}(\theta) \ln|t| + \psi_Q(\theta) + o(1)\Big) \;, & 0 < v_1(\theta) \in \mathbb{N} \;; \\ Q_{\ln}(\theta) \ln|t| + o(1) \;, & v_1(\theta) \in -\mathbb{N}^* \;; \\ o(1) \;, & -\mathbb{N}^* \not\ni v_1(\theta) \leq 0 \;. \end{cases} \tag{3.2}$$

The function

$$v := |v_1| \tag{3.3}$$

will be called the *velocity function*, while Q_∞ will be called the *position function*. We shall say that a solution is in the $\mathrm{AVTD}_k^{(P,Q)}$ class if (3.1)-(3.2) hold with functions v_1, P_∞, Q_∞ and ψ_Q which are of C_k differentiability class (on closed intervals the derivatives are understood as one-sided ones at the end points). For $k > 0$ we will assume that the behavior (3.1)-(3.2) is preserved under differentiation in the following way:

$$\forall\ 0 \le i + j \le k \qquad \partial_\theta^j (t\partial_t)^i \Big(P(t,\theta) + v_1(\theta) \ln |t| - P_\infty(\theta) \Big) = o(1) \ , \quad (3.4)$$

similarly for Q.

We continue with the

Proof of Proposition 1.1: The fact that inextendible causal curves meet the set $t = 0$ in finite time is proved in Proposition 3.5.1 in [7]. Next, since t is a time function on $\mathscr{M}$, we can parameterize Γ by t:

$$\Gamma(t) = (t, \theta(t), x^a(t)) \ .$$

Timelikeness of Γ together with the form (1.4) of the metric gives

$$\left| \frac{d\theta}{dt} \right| \le 1 \ ,$$

which implies that θ is uniformly Lipschitz along Γ, and the existence of the limit $\psi := \lim_{t \to 0} \theta(t)$ immediately follows.

□

We are ready now to pass to the proof of the following result, mentioned in the Introduction:

Proposition 3.1 *Consider a Gowdy space-time with a Cauchy horizon $\mathcal{H}$. Then there exists $\psi \in S^1$ such that all curvature invariants remain bounded along the timelike curves $t \to (t, \psi, x^1, x^2)$, $-1 \le t < 0$.*

Proof: Let $p \in \mathcal{H}$ and let $\hat{\Gamma}$ be any timelike curve starting from p and entering the globally hyperbolic region $\mathscr{M}$. We set $\Gamma := \hat{\Gamma} \cap \mathscr{M}$; by Proposition 1.1 there exists $\psi \in S^1$ such that θ approaches ψ along Γ. By hypothesis the extended metric is smooth around p, hence there exists a neighborhood $\mathscr{O}$ of p on which all curvature invariants are bounded. In particular all curvature invariants are bounded on $\mathscr{O} \cap \mathscr{M}$. Since the metric is $U(1) \times U(1)$ invariant on $\mathscr{M}$, all the curvature invariants will also be bounded on the set obtained by moving $\mathscr{O} \cap \mathscr{M}$ by isometries. This last set contains all the timelike curves as in the statement of the Proposition, with $-\epsilon < t < 0$, and the result easily follows. □

One of the ingredients of the proof of Theorem 1.6 is the following result, proved in [4]:

THEOREM 3.2 *Suppose that there exist constants* $C, \alpha_p > 0$ *such that*

$$|t\partial_\theta P| + |te^P \partial_t Q| \le C|t|^{\alpha_p} \tag{3.5}$$

and consider any point $\psi \in [a, b]$ *such that* $v_1(\psi) = 1$. *Then the functions* $(\bar{P}, \bar{Q}) := (P + \ln |t|, Q)|_{C^0_{t_0}(\psi)}$ *can be extended to an* $AVTD^{(P,Q)}_\infty$ *map from* $\mathbb{R}^2$ *to* $\mathscr{H}_2$. *If* $\lim_{C^0_{t_0}(\psi)} t\partial^j_\theta P_t = 0$ *for all* $j \in \mathbb{N}$, *then for all* $i, k \in \mathbb{N}$ *we have*

$$\lim_{C^0_{t_0}(\psi)} \partial_t^{2i+1}\partial_\theta^k (P + \ln |t|) = \lim_{C^0_{t_0}(\psi)} \partial_t^{2i+1}\partial_\theta^k Q = \lim_{C^0_{t_0}(\psi)} \partial_\theta^{k+1} Q = 0\ . \tag{3.6}$$

Further, if $v_1 = 1$ *on an interval* $[\theta_l, \theta_r]$, *then the restriction* $(\tilde{P}, \tilde{Q}) := (P + \ln |t|, Q)|_{\Omega(\theta_l,\theta_r,t_0)}$ *can be extended to a smooth map from* $\mathbb{R}^2$ *to* $\mathscr{H}_2$, *with* (3.6) *holding for all* $\psi \in [\theta_l, \theta_r]$.

REMARK 3.3 The vanishing of the last term in (3.6) for all $k \ge 0$ is somewhat surprising. As already pointed out in the introduction, the power-law condition (3.5) is justified for initial data near those corresponding to a flat Kasner solution $(P_0, Q_0) \equiv (-\ln |t|, 0)$ by the results in [4].

We are ready now to pass to the

PROOF OF THEOREM 1.6: Recall that $\mathcal{H}$ is a Killing horizon if there exists a Killing vector field which is tangent to the generators of $\mathcal{H}$. The fact that $\mathcal{H}$ is a Killing horizon follows from the proof of Proposition 1 of [1] (see in particular Lemma 1.1 and Lemma 1.2 there); further, any other Killing vector is spacelike on $\mathcal{H}$. Thus, if a Gowdy space-time is extendible across a connected Cauchy horizon, then one of the Killing vectors, say X, is tangent to the generators of the event horizon. Since our claims are local, without loss of generality we may assume that $\mathcal{H}$ is connected. Locally near a point in $\mathcal{H}$ we can construct a null Gauss coordinate system $(T, W, \hat{x}^A)$, $A = 1, 2$, as in [19], leading to the following local form of the metric

$$g = 2dT\, dW + T\phi\, dW^2 + 2T\beta_A d\hat{x}^A dW + \mu_{AB} d\hat{x}^A d\hat{x}^B\ , \tag{3.7}$$

with some smooth functions ϕ, β_A, and μ_{AB} which have obvious tensorial properties with respect to $\hat{x}^A$–coordinate-transformations. More precisely, let $(W, \hat{x}^A)$ be local coordinates on $\mathcal{H}$ such that $\partial_W = X$, and such that ∂_2 is another linearly independent Killing vector. Let k be the null vector field defined along $\mathcal{H}$, pointing towards the globally hyperbolic region, such that

$$g(\partial_W, k) = 1 \ , \quad g(\partial_A, k) = 0$$

on $\mathcal{H}$. The local coordinates $(W, \hat{x}^A)$ are extended from $\mathcal{H}$ to a neighborhood thereof by requiring $(W, \hat{x}^A)$ to be constant along the geodesics issued from $\mathcal{H}$ with initial velocity k. Letting T be the affine parameter along those geodesics, with $T = 0$ on $\mathcal{H}$, one obtains (3.7). Now, isometries map geodesics to geodesics and preserve affine parameterisations, which easily implies

$$\partial_W g_{\mu\nu} = 0 = \partial_2 g_{\mu\nu} \ ,$$

throughout the domain of definition of the coordinates. Equation (2.9) of [19] for R_{3b} shows that on any connected component of $\mathcal{H}$ there exists a constant $\kappa \geq 0$ such that

$$\phi|_{\mathcal{H}} = \kappa \ .$$

Whatever the range of the W and $\hat{x}^2$ coordinate in the original extension, we can without loss of generality assume that the functions above are defined for $W, \hat{x}^2$ in $\mathbb{R}$ – or in S^1 – since those functions are independent of W and $\hat{x}^2$ anyway. There might be difficulties if we are trying to build a manifold by patching together the resulting local coordinates, but we do not need to patch things back together, so this is irrelevant for the local calculations that follow.

For $T > 0$ we replace the coordinates T and W by new coordinates $(\hat{t}, \hat{x}^3)$, $\hat{t} > 0$, defined as

$$W = \hat{x}^3 + \alpha \ln \hat{t} \ , \quad T = \beta \hat{t}^2 \ , \tag{3.8}$$

with constants $\alpha \in \mathbb{R}$, $\beta > 0$, leading to

$$\begin{aligned} g &= \alpha\beta\left(4 + \kappa\alpha + O(\hat{t}^2)\right) d\hat{t}^2 + 2\beta\left(2 + \alpha\kappa + O(\hat{t}^2)\right) \hat{t}\, d\hat{t} d\hat{x}^3 \\ &\quad + 2\hat{t}\beta_A d\hat{x}^A(\hat{t} d\hat{x}^3 + \alpha d\hat{t}) + \beta\hat{t}^2(\kappa + O(\hat{t}^2))(d\hat{x}^3)^2 \\ &\quad + \mu_{AB} d\hat{x}^A d\hat{x}^B \ . \end{aligned} \tag{3.9}$$

We can choose α, β so that

$$\alpha\beta(4 + \kappa\alpha) < 0 \ ,$$

and we assume that some such choice has been made.

Let $X_1 = \partial_W = \partial_{\hat{x}^3}$, $X_2 = \partial_{\hat{x}^2}$, and define

$$\lambda_{ab} = g(X_a, X_b) \ , \quad a,b = 1,2 \ .$$

From (3.9) we have

$$\lambda_{11} = \beta\kappa\hat{t}^2 + O(\hat{t}^4) \ , \quad \lambda_{12} = \beta_1\hat{t}^2 + O(\hat{t}^4) \ , \quad \lambda_{22} = \mu_{22} \ , \tag{3.10}$$

Note that X_2 is spacelike on $\mathcal{H}$. It follows that

$$\partial_\mu\left(\sqrt{\det\lambda}\right) = \begin{cases} \sqrt{\beta\kappa\mu_{22}} + O(\hat{t}^2), & \partial_\mu = \partial_{\hat{t}},\ \kappa \neq 0, \\ O(\hat{t}), & \partial_\mu = \partial_{\hat{t}},\ \kappa = 0, \\ \frac{1}{2}\sqrt{\frac{\beta\kappa}{\mu_{22}}}\partial_{\hat{x}^1}\left(\mu_{22}\right)\hat{t} + O(\hat{t}^3), & \partial_\mu = \partial_{\hat{x}^1},\ \kappa \neq 0, \\ O(\hat{t}^2), & \partial_\mu = \partial_{\hat{x}^1},\ \kappa = 0, \\ 0, & \partial_\mu = \partial_{\hat{x}^i},\ i = 2,3, \end{cases} \tag{3.11}$$

and

$$t := -\sqrt{\det\lambda} = \begin{cases} -\sqrt{\beta\kappa\mu_{22}}\hat{t} + O(\hat{t}^3), & \kappa \neq 0, \\ O(\hat{t}^2), & \kappa = 0. \end{cases} \tag{3.12}$$

Now, away from the set $t = 0$ the space-time metric g can be written[1] in the form (1.4). The functions t and θ are defined uniquely up to a single multiplicative constant, *cf., e.g.,* [6]; the normalisation (3.12) gets rid of that freedom. By a rotation of the Killing vectors ∂_{x^a} we can always achieve

$$\partial_{x^1} = X_1 = \partial_{\hat{x}^3} \ , \quad \partial_{x^2} = X_2 = \partial_{\hat{x}^2} \ .$$

Recall that a Killing horizon is said to be degenerate if $\partial_\mu\left(g(X,X)\right)|_{\mathcal{H}} = 0$. Since μ_{22} does not vanish at $\hat{t} = 0$, (3.12) shows that $\mathcal{H}$ is degenerate if and only if κ vanishes. From now on we assume that $\kappa \neq 0$. Inspection of (3.9) shows that a convenient choice of α and β is

$$2 + \alpha\kappa = 0 \ , \quad \alpha\beta(4 + \kappa\alpha) = -1 \ ,$$

so that $\beta = \kappa/4$. Ordering the entries as $(\hat{t}, \hat{x}^3, \hat{x}^1, \hat{x}^2)$, (3.9) takes the following matrix form

$$g = \begin{pmatrix} -1 + O(\hat{t}^2) & O(\hat{t}^3) & O(\hat{t}) & O(\hat{t}) \\ O(\hat{t}^3) & \beta\kappa\hat{t}^2 + O(\hat{t}^4) & O(\hat{t}^2) & O(\hat{t}^2) \\ O(\hat{t}) & O(\hat{t}^2) & \mu_{11} & \mu_{12} \\ O(\hat{t}) & O(\hat{t}^2) & \mu_{12} & \mu_{22} \end{pmatrix} .$$

This gives $\det g = -\beta\kappa\hat{t}^2 \det\mu + O(\hat{t}^4)$ and

$$g^{-1} = \begin{pmatrix} -1+O(\hat{t}^2) & O(\hat{t}) & O(\hat{t}) & O(\hat{t}) \\ O(\hat{t}) & (\beta\kappa\hat{t}^2)^{-1}+O(1) & O(1) & O(1) \\ O(\hat{t}) & O(1) & \mu^{11}+O(\hat{t}^2) & \mu^{12}+O(\hat{t}^2) \\ O(\hat{t}) & O(1) & \mu^{12}+O(\hat{t}^2) & \mu^{22}+O(\hat{t}^2) \end{pmatrix} , \tag{3.13}$$

where μ^{AB} is the matrix inverse to μ_{AB}. From (3.13) and (3.11) we find for $t<0$

$$\begin{aligned} g(\nabla t,\nabla t) &= g^{\hat{t}\hat{t}}\left(\frac{\partial t}{\partial\hat{t}}\right)^2 + 2g^{\hat{t}\hat{x}^1}\frac{\partial t}{\partial\hat{t}}\frac{\partial t}{\partial\hat{x}^1} + g^{\hat{x}^1\hat{x}^1}\left(\frac{\partial t}{\partial\hat{x}^1}\right)^2 \\ &= \left(-1+O(\hat{t}^2)\right)\left(\frac{\partial t}{\partial\hat{t}}\right)^2 + O(\hat{t})\frac{\partial t}{\partial\hat{t}}\frac{\partial t}{\partial\hat{x}^1} + \left(\mu^{11}+O(\hat{t}^2)\right)\left(\frac{\partial t}{\partial\hat{x}^1}\right)^2 \\ &= -\beta\kappa\mu_{22} + O(\hat{t}^2) \ . \end{aligned} \tag{3.14}$$

Comparing (1.4) and (3.13) we also obtain

$$0 = g(\nabla t,\nabla\theta) = -\left(\sqrt{\beta\kappa\mu_{22}} + O(\hat{t}^2)\right)\frac{\partial\theta}{\partial\hat{t}} + O(\hat{t})\frac{\partial\theta}{\partial\hat{x}^1} ,$$

and

$$\begin{aligned} \beta\kappa\mu_{22} + O(\hat{t}^2) &= -g(\nabla t,\nabla t) = g(\nabla\theta,\nabla\theta) \\ &= g^{\hat{t}\hat{t}}\left(\frac{\partial\theta}{\partial\hat{t}}\right)^2 + 2g^{\hat{t}\hat{x}^1}\frac{\partial\theta}{\partial\hat{t}}\frac{\partial\theta}{\partial\hat{x}^1} + g^{\hat{x}^1\hat{x}^1}\left(\frac{\partial\theta}{\partial\hat{x}^1}\right)^2 \\ &= -\left(1+O(\hat{t}^2)\right)\left(\frac{\partial\theta}{\partial\hat{t}}\right)^2 + O(\hat{t})\frac{\partial\theta}{\partial\hat{t}}\frac{\partial\theta}{\partial\hat{x}^1} \\ &\quad + \left(\mu^{11}+O(\hat{t})\right)\left(\frac{\partial\theta}{\partial\hat{x}^1}\right)^2 \ . \end{aligned}$$

It follows that $\partial\theta/\partial\hat{x}^1$ is uniformly bounded from above and away from zero for $t>0$, with $\partial_{\hat{t}}\theta = O(\hat{t})$. This implies that θ extends by continuity to a Lipschitz function on $\mathcal{H}$, and also implies the existence of the claimed interval of θ's.

Comparing (3.10) with (1.4) we find

$$|t|e^P = \beta\kappa\hat{t}^2 + O(\hat{t}^4) \ , \quad |t|e^P Q = \beta_1\hat{t}^2 + O(\hat{t}^4) \ .$$

For $\kappa \neq 0$ this gives

$$e^P = \sqrt{\frac{\beta\kappa}{\mu_{22}}}\hat{t} + O(\hat{t}^3) = \frac{|t|}{\mu_{22}} + O(|t|^3) \ , \quad Q = \frac{\beta_1}{\beta\kappa} + O(\hat{t}^2) \ ,$$

and what has been shown so far about t and θ further gives

$$tP_t = 1 + O(t^2)\ ,\ tP_\theta = O(|t|)\ ,\ te^P Q_t = O(|t|^3)\ ,\ te^P Q_\theta = O(|t|^2)\ .$$

We have thus obtained a representation of the solution for which

$$v_1 = -1\ , \quad v = 1\ , \quad |tX_\theta| = O(|t|^3)\ , \tag{3.15}$$

and the sufficiency part of point (ii) is established. At this stage one can directly derive a full asymptotic expansion of the solution using the Gowdy equations (1.3); this will lead to $\mathrm{AVTD}_\infty^{(P,Q)}$ behavior with a perhaps non-constant function Q_∞. An alternative way consists in swapping the order of the Killing vectors; (3.15) will then still hold except for a change of sign of v_1, so that the hypotheses of Theorem 3.2 are satisfied, and point (iii) follows.

Consider, finally, a solution with $v = 1$ at a point ψ, or on an interval $[\theta_l, \theta_r]$, with (1.14) holding there. By Theorem 3.2 the solution is $\mathrm{AVTD}_\infty^{(P,Q)}$ on $C^0_{t_0}(\psi)$, or on $[t_0, 0) \times [\theta_l, \theta_r]$, and by integration of (1.5) one easily finds

$$\gamma = -\ln|t| + O(1)\ . \tag{3.16}$$

As already pointed out, such solutions are smoothly extendible by the calculations in [8]; alternatively, one can run backwards the calculations done above. In any case (3.14) holds. From (1.4) we have

$$g(\nabla t, \nabla t) = -e^{\gamma/2}|t|^{1/2}\ ,$$

and comparing (3.16) with (3.14) the non-degeneracy of the horizon follows.
□

The reader will have noticed that the last part of the proof of Theorem 1.6 shows the following:

PROPOSITION 3.4 *$AVTD_\infty^{(P,Q)}$ space-times do not contain degenerate horizons.*

We continue with the

PROOF OF PROPOSITION 1.5: The result follows immediately from the calculations in the proof of Theorem 1.6, with the following modifications: we choose X_2 to be the Killing vector which is tangent to the generators of $\mathcal{H}$, and we let X_1 be any other Killing vector. If ψ is a point as in

Proposition 3.1, then $g(X_1, X_1)$ tends to a strictly positive limit along the curve Γ defined in the proof of Proposition 3.1. Further this limit is the same for any causal curve which accumulates at the point p of the proof of Proposition 3.1. It follows that along any of the curves $t \to (t, \psi, x^a)$ the limit $\lim_{t\to 0} g(X_1, X_1)$ exists, and in fact one has that the limit

$$\lim_{C^0_{t_0}(\psi)} g(X_1, X_1)$$

exists. Using (1.4), this can be translated into the statement that the function $te^{P(t,\psi)}$ has a finite limit as t goes to zero, which justifies the first equation in (1.12). Since X_2 is normal to $\mathcal{H}$ the function $g(X_1, X_2)$ vanishes on $\mathcal{H}$, which implies the second equation in (1.12). Finally if $|t|\partial_t P$ avoids the value one, then either $|t|\partial_t P < 1 - \epsilon$ or $|t|\partial_t P > 1 + \epsilon$ for some $\epsilon > 0$ for t large enough, which is incompatible with the first equation in (1.12). □

We close this section with the

Proof of Proposition 1.9: Without loss of generality we can choose the coordinates (P, Q) on hyperbolic space so that $v_1(\psi) = 1$. It follows from the results in [4] that under any of the conditions of Proposition 1.9 the conclusions of Theorem 3.2 hold; thus x is $\mathrm{AVTD}^{(P,Q)}_{\infty}$ in $C^0_{t_0}(\psi)$. Under the hypotheses of point (i), suppose that the solution is extendible. Since the velocity function is not equal to one in an open interval containing ψ, Theorem 1.6 shows that the horizon must be degenerate. This contradicts Proposition 3.4, and establishes point (i).

Point (ii) can then be established by inspection of the curvature tensor, which we give for completeness in Appendix A. The relevant calculations have been done using Grtensor [13]. The interested reader will find MAPLE worksheets and input files on URL `http://grtensor.phy.queensu.ca/gowdy`. It follows from the the $\mathrm{AVTD}^{(P,Q)}_{\infty}$ character of the solution on $C^0_{t_0}(\psi)$ that there exist bounded functions γ_∞ and Z such that

$$\gamma = -\ln|t| + \gamma_\infty(\theta) + o(1) ,$$

$$P(t, \theta) = -v(\theta)\ln|t| + P_\infty(\theta) + tZ(t, \theta) .$$

The field equations further show that $\lim_{t\to 0} t\partial_t Z(t, \psi) = 0$ (in fact $Z = O(|t| \ln|t|)$; see the next section for more detailed expansions), and one finds

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} \sim 4\,\frac{\left(-\left(\frac{d}{d\theta}v\left(\psi\right)\right)^2 + e^{2\,P_\infty(\psi)}\left(\frac{d^2}{d\theta^2}Q_\infty\left(\psi\right)\right)^2\right)e^{\gamma(t,\psi)}}{t} , \quad (3.17)$$

as $t \to 0$, provided that the coefficient in the biggest parenthesis in the numerator does not vanish. As $e^{\gamma(t,\theta)} \sim 1/|t|$ the curvature scalar $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ diverges then like $1/t^2$. (As already pointed out in the introduction, in the next section we will study in detail the remaining possible behaviors of $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma}$.) □

4 The blow-up structure of the Kretschmann scalar

Throughout this section we assume that ψ is such that $v_1(\psi) = 1$. To analyse the behavior of $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ in the case

$$-\left(\frac{d}{d\theta}v\left(\psi\right)\right)^2 + e^{2\,P_\infty(\psi)}\left(\frac{d^2}{d\theta^2}Q_\infty\left(\psi\right)\right)^2 = 0\ , \tag{4.1}$$

the first question to answer is how far to push an expansion of P and Q to isolate the potentially unbounded terms in the curvature. A tedious but straightforward inspection of the curvature tensor, as given in the appendix, shows that terms of the form $f(\theta)t^3 \ln^j|t|$ in Q, with $j \neq 0$, might lead to a logarithmic blow up of the tetrad components $R_{\ (1)\ \ (3)\ \ (1)\ \ (4)}$ and $R_{\ (2)\ \ (3)\ \ (2)\ \ (4)}$ (here the tetrad given at the beginning of Appendix A is used), and that any higher power of t will lead to a vanishing contribution to the tetrad components of the Riemann tensor. This means that one needs to have the exact form of all the coefficients in an asymptotic expansion of Q up to order $O(t^3)$. Similarly one finds that one needs to have the exact form of all the coefficients in an asymptotic expansion of P up to order $O(t^2)$. Those expansion coefficients can be found by collecting all terms with the same powers of t and of $\ln|t|$, and setting the result to zero, in the Gowdy equations,

$$\begin{aligned}\partial_t^2 P - \partial_\theta^2 P &= -\frac{\partial_t P}{t} + e^{2P}\left((\partial_t Q)^2 - (\partial_\theta Q)^2\right)\ , \\ \partial_t^2 Q - \partial_\theta^2 Q &= -\frac{\partial_t Q}{t} - 2\left(\partial_t P \partial_t Q - \partial_\theta P \partial_\theta Q\right)\ .\end{aligned} \tag{4.2}$$

Recall that we are dealing with $\mathrm{AVTD}_\infty^{(P,Q)}$ solutions. It is easily seen from (3.1)-(3.2) with $v_1(\psi) = 1$ and from (4.2) that (compare (3.6))

$$(\partial_\theta Q_\infty)(\psi) = 0\ . \tag{4.3}$$

Assuming this together with (4.1), we note the following simple observations concerning the behavior of the solution at $\theta = \psi$:

(i) The term Q_∞ does not give any undifferentiated contribution to P.

(ii) A term $f(\theta)|t|^{\alpha(\theta)}\ln^j|t|$ in Q with α larger than or equal to, say $7/4$, generates in P, for $v(\theta)$ close to 1, terms of the form

$$|t|^{2v(\theta)+\alpha(\theta)-2}\Big(\hat{f}(\theta)\ln^{j+1}|t| + \text{ lower powers of } \ln|t|\Big)$$
$$+ \text{ higher powers of } |t| \text{ (multiplied perhaps by higher powers of } \ln|t|) \; . \tag{4.4}$$

(iii) A term $f(\theta)|t|^{\beta(\theta)}\ln^j|t|$ in P with $\beta \geq 0$ generates in Q, for $v(\theta)$ close to 1, terms of the form $\tilde{f}(\theta)|t|^{2v(\theta)+\beta(\theta)}\ln^{j+1}|t|$, as well as terms with the same power of $|t|$ but lower powers of $\ln|t|$, or terms with higher powers of $|t|$.

Consider the linear counterpart of (4.2),

$$\partial_t^2 f - \partial_\theta^2 f = -\frac{\partial_t f}{t} \; . \tag{4.5}$$

As shown in [12] (compare [9, Equation (4a)]), for every smooth solution of (4.5) on $(-\infty,0)\times S^1$ there exist functions $f_{\ln}(t^2,\theta)$, $\mathring{f}(t^2,\theta)$, which are smooth up to boundary on the set $(t^2,\theta)\in[0,\infty)\times S^1$, such that

$$f(t,\theta) = f_{\ln}(t^2,\theta)\ln|t| + \mathring{f}(t^2,\theta) \; .$$

Further for any $f_{\ln}(0,\theta)$ and $\mathring{f}(0,\theta)$ there exists a solution as above, and we have the asymptotic expansion

$$\begin{aligned} f &= f_{\ln}(0,\theta)\ln|t| + \mathring{f}(0,\theta) \\ &\quad + \frac{\partial_\theta^2 f_{\ln}(0,\theta)}{4} t^2\ln|t| + \frac{\partial_\theta^2 \mathring{f}(0,\theta) - \partial_\theta^2 f_{\ln}(0,\theta)}{4} t^2 + O(t^4\ln|t|) \; . \end{aligned}$$

Under the current hypotheses the linearisation of the P equation differs from (4.5) by terms decaying sufficiently fast so that the leading order behavior of P is correctly reflected by the above. That is not the case anymore for Q, because of the $1/t$ behavior of the $\partial_t P$ term, so that the leading terms in the Q equation linearised with respect to Q are

$$\partial_t^2 \tilde{f} - \partial_\theta^2 \tilde{f} = +\frac{\partial_t \tilde{f}}{t} \; . \tag{4.6}$$

The associated indicial exponents are zero and two, from which it is not too difficult to prove the following behavior of solutions of (4.6)

$$\tilde{f}(t,\theta) = \tilde{f}_0(t^2,\theta) + \tilde{f}_{\ln}(t^2,\theta)t^2\ln|t| \; ,$$

with freely prescribable functions $\tilde{f}_0(0,\theta)$ and $\partial^2_\theta \tilde{f}_0(0,\theta)$, together with an associated asymptotic expansion

$$\tilde{f} \;=\; \tilde{f}_0(0,\theta) + \frac{\partial^2_\theta \tilde{f}_0(0,\theta)}{2} t^2 \ln|t| + \frac{\partial^2_\theta \tilde{f}_0(0,\theta)}{2}(0,\theta)t^2 + O(t^4 \ln|t|) \; .$$

Since the full solution (P,Q) is already known to belong to the AVTD$^{(P,Q)}_\infty$ class, using what has been said one can proceed as follows: one starts with the leading order behavior of P,

$$P = -v_1(\theta)\ln t + P_\infty(\theta) + O_{\ln}(|t|^2) \; , \tag{4.7}$$

where $f = O_{\ln}(|t|^s)$ denotes a function which satisfies

$$|\partial^i_t \partial^j_\theta f| \le C_{i,j}|t|^{s-i}|\ln(|t|)|^{N_{i,j}} \; ,$$

for some constants $C_{i,j}, N_{i,j}$. Inserting (4.7) into the equation for Q in (4.2) one finds that at $\theta = \psi$ we have the expansions (recall (4.1), (4.3))

$$Q(t,\theta) \;=\; Q_\infty(\theta) + \frac{t^2}{2}\partial^2_\theta Q_\infty(\theta)\ln|t| + \psi_Q(\theta)t^2 + t^4 W(t,\theta) \; . \tag{4.8}$$

Inserting (4.8) in the first equation in (4.2) one then obtains

$$\begin{aligned} P(t,\theta) \;=\; & -v(\theta)\ln|t| + P_\infty(\theta) - \frac{t^2}{4}\partial^2_\theta v(\theta)\ln|t| \\ & + \frac{e^{2P_\infty(\theta)}\Big(\partial^2_\theta Q_\infty(\theta)\Big)^2}{4}\Big(\ln^2|t| - 2\ln|t| + \frac{3}{2}\Big)t^2 \\ & + e^{2P_\infty(\theta)}\Big(4\psi_Q(\theta) + \partial^2_\theta Q_\infty(\theta)\Big)\frac{\partial^2_\theta Q_\infty(\theta)}{4}\Big(\ln|t| - 1\Big)t^2 \\ & + \left\{\frac{\partial^2_\theta P_\infty(\theta) + \partial^2_\theta v(\theta)}{4} + e^{2P_\infty(\theta)}\Big(\psi_Q(\theta) + \frac{\partial^2_\theta Q_\infty(\theta)}{4}\Big)^2\right\} t^2 \\ & + t^4 Z(t,\theta) \; . \end{aligned} \tag{4.9}$$

All the functions appearing above which depend only upon θ are smooth, with $f = Z$ or W satisfying estimates of the form

$$(t\partial_t)^i(\partial_\theta)^j f = O(|\ln|t||^{j+N})$$

for some N. One can now insert those expansions in the Riemann tensor and obtain its behavior for $|t|$ small. In the case $\frac{dv(\psi)}{d\theta} \neq 0$ but $(\frac{d}{d\theta}v(\psi))^2 = e^{2\,P_\infty(\psi)}(\frac{d^2}{d\theta^2}Q_\infty(\psi))^2$, a GRTENSOR calculation gives, again at $\theta = \psi$,

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} \sim 3\,t\,(\ln(|t|))^4\, e^{\gamma(t,\theta)}\left(\frac{d^2}{d\theta^2}Q_\infty(\theta)\right)^4 \left(e^{P_\infty(\theta)}\right)^4 \tag{4.10}$$

so that with $e^{\gamma(t,\theta)} \sim 1/|t|$ then $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ now diverges like $(\ln(|t|))^4$.

Next, if $v(\psi) = 1$ and $\frac{dv(\psi)}{d\theta} = \frac{d^2}{d\theta^2}Q_\infty(\psi) = 0$ we obtain

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} \sim |t|(\ln(|t|))^2 e^{\gamma(t,\psi)} R(\psi) \qquad (4.11)$$

as $t \to 0$, provided that $R(\psi)$ does not vanish. Here

$$R(\theta) = -3\left(e^{P_\infty(\theta)}\frac{d^3}{d\theta^3}Q_\infty(\theta) - \frac{d^2}{d\theta^2}v(\theta)\right)\left(e^{P_\infty(\theta)}\frac{d^3}{d\theta^3}Q_\infty(\theta) + \frac{d^2}{d\theta^2}v(\theta)\right). \qquad (4.12)$$

Since $e^{\gamma(t,\theta)} \sim 1/|t|$, we find that $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ diverges like $(\ln(|t|))^2$.

One can likewise check what happens if $R(\psi) = 0$:

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} \sim -3\,(\frac{d^2}{d\theta^2}v(\psi))(2\,(\frac{d}{d\theta}P_\infty(\psi))^2 + 2\,\frac{d^2}{d\theta^2}P_\infty(\psi) + 4\,e^{P_\infty(\psi)}\frac{d}{d\theta}\psi_Q(\psi)$$
$$+8\,e^{P_\infty(\psi)}\psi_Q(\theta)\frac{d}{d\theta}P_\infty(\psi) + \frac{d^2}{d\theta^2}v(\psi))|t|\ln(|t|)e^{\gamma(t,\psi)}\,,$$

and one obtains $\ln|t|$ behavior of the Kretschmann scalar unless the coefficient above vanishes; in that last case the Kretschmann scalar is bounded.

A Appendix. The curvature tensor of Gowdy metrics

We use the following tetrad

$$e1^a = \left[e^{(1/4\,\gamma(t,\,\theta))}\,t^{(1/4)},\ 0,\ 0,\ 0\right]$$
$$e2^a = \left[0,\ e^{(1/4\,\gamma(t,\,\theta))}\,t^{(1/4)},\ 0,\ 0\right]$$
$$e3^a = \left[0,\ 0,\ -\mathrm{Q}(t,\,\theta)\sqrt{\frac{e^{\mathrm{P}(t,\,\theta)}}{t}},\ \sqrt{\frac{e^{\mathrm{P}(t,\,\theta)}}{t}}\right]$$
$$e4^a = \left[0,\ 0,\ \frac{1}{\sqrt{t\,e^{\mathrm{P}(t,\,\theta)}}},\ 0\right]$$

Writing a partial derivative as a subscript, a GRTensor calculation with Maple gives the following components of the Riemann tensor in this frame:

$$R_{\ (1)\ (2)\ (1)\ (2)} = \frac{1}{4}e^{(1/2\,\gamma)}(-(e^P)^2\,t^2\,{Q_t}^2 - 2\,t^3\,(e^P)^2\,{Q_t}^2\,P_t - 2\,t^3\,(e^P)^2\,Q_t\,Q_{t,\,t} - t^2\,{P_t}^2$$
$$-\,2\,t^3\,P_t\,P_{t,\,t} - t^2\,(e^P)^2\,{Q_\theta}^2 - 2\,t^3\,(e^P)^2\,{Q_\theta}^2\,P_t - {P_\theta}^2\,t^2 - 1 + 2\,t^3\,P_{\theta,\,\theta}\,P_t$$
$$+\,4\,t^3\,(e^P)^2\,Q_t\,Q_\theta\,P_\theta + 2\,t^3\,(e^P)^2\,Q_t\,Q_{\theta,\,\theta})\Big/t^{(3/2)}$$

$$R_{(1)\ (2)\ (3)\ (4)} = -\frac{1}{2}\, e^P \sqrt{t}\, e^{(1/2\,\gamma)}\, (-P_t\, Q_\theta + Q_t\, P_\theta)$$

$$R_{(1)\ (3)\ (1)\ (3)} = -\frac{1}{8} e^{(1/2\,\gamma)} (-5\, t\, P_t - 4\, t^2\, P_{t,\,t} + t^3\, (e^P)^2\, {Q_t}^2\, P_t + t^3\, {P_t}^3 + t^3\, (e^P)^2\, {Q_\theta}^2\, P_t$$
$$+ 3\, t^3\, P_t\, {P_\theta}^2 + 5\, (e^P)^2\, t^2\, {Q_t}^2 + t^2\, {P_t}^2 - t^2\, (e^P)^2\, {Q_\theta}^2 - {P_\theta}^2\, t^2 - 1 + 2\, t^3\, (e^P)^2\, Q_t\, Q_\theta\, P_\theta)$$
$$\Big/ t^{(3/2)}$$

$$R_{(1)\ (3)\ (1)\ (4)} = \frac{1}{8} e^P\, e^{(1/2\,\gamma)} (-8\, t\, P_t\, Q_t + {Q_t}^3\, t^2\, (e^P)^2 + Q_t\, t^2\, {P_t}^2 + 3\, Q_t\, t^2\, (e^P)^2\, {Q_\theta}^2$$
$$+ Q_t\, t^2\, {P_\theta}^2 - 5\, Q_t - 4\, t\, Q_{t,\,t} + 2\, t^2\, Q_\theta\, P_\theta\, P_t) \Big/ \sqrt{t}$$

$$R_{(1)\ (3)\ (2)\ (3)} = \frac{1}{8} e^{(1/2\,\gamma)} (4\, t\, P_{t,\,\theta} - 3\, t^2\, {P_t}^2\, P_\theta - 2\, t^2\, P_t\, (e^P)^2\, Q_t\, Q_\theta - 4\, t\, (e^P)^2\, Q_t\, Q_\theta$$
$$- P_\theta\, t^2\, (e^P)^2\, {Q_t}^2 - P_\theta\, t^2\, (e^P)^2\, {Q_\theta}^2 - {P_\theta}^3\, t^2 + 3\, P_\theta) \Big/ \sqrt{t}$$

$$R_{(1)\ (3)\ (2)\ (4)} = -\frac{1}{8} e^P\, e^{(1/2\,\gamma)} (6\, t\, Q_t\, P_\theta - 2\, Q_t\, t^2\, P_\theta\, P_t - 3\, {Q_t}^2\, t^2\, (e^P)^2\, Q_\theta + 4\, t\, Q_{t,\,\theta}$$
$$- t^2\, Q_\theta\, {P_t}^2 - t^2\, {Q_\theta}^3\, (e^P)^2 - t^2\, Q_\theta\, {P_\theta}^2 + 3\, Q_\theta + 2\, Q_\theta\, t\, P_t) \Big/ \sqrt{t}$$

$$R_{(1)\ (4)\ (1)\ (4)} = \frac{1}{8} e^{(1/2\,\gamma)} (1 - 5\, t\, P_t - 4\, t^2\, P_{t,\,t} + 3\, (e^P)^2\, t^2\, {Q_t}^2 - t^2\, {P_t}^2 + t^2\, (e^P)^2\, {Q_\theta}^2 + {P_\theta}^2\, t^2$$
$$+ t^3\, (e^P)^2\, {Q_t}^2\, P_t + t^3\, {P_t}^3 + t^3\, (e^P)^2\, {Q_\theta}^2\, P_t + 3\, t^3\, P_t\, {P_\theta}^2 + 2\, t^3\, (e^P)^2\, Q_t\, Q_\theta\, P_\theta) \Big/$$
$$t^{(3/2)}$$

$$R_{(1)\ (4)\ (2)\ (3)} = -\frac{1}{8} e^P\, e^{(1/2\,\gamma)} (2\, t\, Q_t\, P_\theta - 2\, Q_t\, t^2\, P_\theta\, P_t - 3\, {Q_t}^2\, t^2\, (e^P)^2\, Q_\theta + 4\, t\, Q_{t,\,\theta}$$
$$- t^2\, Q_\theta\, {P_t}^2 - t^2\, {Q_\theta}^3\, (e^P)^2 - t^2\, Q_\theta\, {P_\theta}^2 + 3\, Q_\theta + 6\, Q_\theta\, t\, P_t) \Big/ \sqrt{t}$$

$$R_{(1)\ (4)\ (2)\ (4)} = -\frac{1}{8} e^{(1/2\,\gamma)} (4\, t\, P_{t,\,\theta} - 3\, t^2\, {P_t}^2\, P_\theta - 2\, t^2\, P_t\, (e^P)^2\, Q_t\, Q_\theta - 4\, t\, (e^P)^2\, Q_t\, Q_\theta$$
$$- P_\theta\, t^2\, (e^P)^2\, {Q_t}^2 - P_\theta\, t^2\, (e^P)^2\, {Q_\theta}^2 - {P_\theta}^3\, t^2 + 3\, P_\theta) \Big/ \sqrt{t}$$

$$R_{(2)\ (3)\ (2)\ (3)} = -\frac{1}{8} e^{(1/2\,\gamma)} (t^3\, (e^P)^2\, {Q_t}^2\, P_t + t^3\, {P_t}^3 + t^3\, (e^P)^2\, {Q_\theta}^2\, P_t + 3\, t^3\, P_t\, {P_\theta}^2$$
$$- (e^P)^2\, t^2\, {Q_t}^2 - t^2\, {P_t}^2 + 5\, t^2\, (e^P)^2\, {Q_\theta}^2 + {P_\theta}^2\, t^2 - t\, P_t + 1 - 4\, t^2\, P_{\theta,\,\theta}$$
$$+ 2\, t^3\, (e^P)^2\, Q_t\, Q_\theta\, P_\theta) \Big/ t^{(3/2)}$$

$$R_{(2)\ (3)\ (2)\ (4)} = \frac{1}{8} e^P\, e^{(1/2\,\gamma)} ({Q_t}^3\, t^2\, (e^P)^2 + Q_t\, t^2\, {P_t}^2 + 3\, Q_t\, t^2\, (e^P)^2\, {Q_\theta}^2 + Q_t\, t^2\, {P_\theta}^2 - Q_t$$
$$- 8\, t\, P_\theta\, Q_\theta + 2\, t^2\, Q_\theta\, P_\theta\, P_t - 4\, t\, Q_{\theta,\,\theta}) \Big/ \sqrt{t}$$

$$R_{\,(2)\;(4)\;(2)\;(4)} = \frac{1}{8} e^{(1/2\,\gamma)}((e^P)^2\,t^2\,{Q_t}^2 + t^2\,{P_t}^2 + 3\,t^2\,(e^P)^2\,{Q_\theta}^2 - {P_\theta}^2\,t^2 + t^3\,(e^P)^2\,{Q_t}^2\,P_t$$
$$+ t^3\,{P_t}^3 + t^3\,(e^P)^2\,{Q_\theta}^2\,P_t + 3\,t^3\,P_t\,{P_\theta}^2 - 1 - t\,P_t - 4\,t^2\,P_{\theta,\theta} + 2\,t^3\,(e^P)^2\,Q_t\,Q_\theta\,P_\theta)$$
$$\Big/ t^{(3/2)}$$

$$R_{\,(3)\;(4)\;(3)\;(4)} = -\frac{1}{4}\,\frac{e^{(1/2\,\gamma)}\,(t^2\,{P_t}^2 - 1 + (e^P)^2\,t^2\,{Q_t}^2 - {P_\theta}^2\,t^2 - t^2\,(e^P)^2\,{Q_\theta}^2)}{t^{(3/2)}}$$

ACKNOWLEDGEMENTS: PTC wishes to thank A. Gerber and J.R. Licois for help with computer algebra calculations. He acknowledges the hospitality of the Erwin Schrödinger Institute, Vienna, during the final stage of work on this paper. Portions of this work were made possible by use of GRTENSOR [25].

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[25] This is a package which runs within Maple. It is entirely distinct from packages distributed with Maple and must be obtained independently. The GRTensor software and documentation is distributed freely on the World-Wide-Web from the address *http://grtensor.org*