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A Hamiltonian framework for field theories in the radiating regime

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Preface

There is no preface

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1. Introduction

In any physical theory one of the fundamental notions is that of energy of the objects at hand: in mechanics one considers the energy of, say, moving masses; in field theories one is interested in the energy of field configurations. A unified treatment of this question, which applies both to mechanics and to field theory, proceeds through a Hamiltonian formalism. We will shortly review below how such a procedure is carried out in the theory of scalar fields on Minkowski space-time; let us, at this stage, mention that an important issue, often ignored in the textbooks, is that of the boundary conditions satisfied by the set of fields under consideration. While this issue can be safely ignored — for many purposes — when considering the usual field theories, such as scalar fields or electromagnetism, on the $\{t = \text{const}\}$ hypersurfaces, where t is a Minkowski-time, it sometimes plays a critical role when other classes of hypersurfaces are considered. In the case of gravity the situation is worse: even for $\{t = \text{const}\}$ asymptotically Minkowskian slices the boundary terms are crucial. (This is one of the main differences between the Arnowitt-Deser-Misner (ADM) mass for gravity (*cf.* Section 5.4 below), which is given by a boundary integral, and the usual energy expression for field theories in Minkowski space-time, where the Hamiltonian is usually a volume integral.) Now, in field theory the energy plays its most important role in the radiation regime, where it can be radiated away by the field. This leads one to the need of considering hypersurfaces which extend to the radiation zone; this requirement is made precise by considering hypersurface which asymptote to null hypersurfaces in an appropriate way. (Technically, this will correspond to hypersurfaces with specific boundary behaviour in a conformal compactification of Minkowski space-time. In such compactifications the radiation zone becomes a neighbourhood of a conformal boundary $\mathscr{I}$.) The aim of this work is to analyse the issues which arise when attempting to obtain a Hamiltonian description of radiating fields, with emphasis on the geometric character of the objects involved. More precisely, we develop a geometric Hamiltonian formalism adequate for a canonical description of field theories in the radiation regime, extending previous work of Kijowski and Tulczyjew [85]. While our main objective here is the radiation zone, we note that several aspects of our construction are new even in more standard contexts. The formalism is first applied in detail to the toy model of a massless scalar field in Minkowski space-time at null infinity. This has some interest of its own; more importantly, it allows us also to adress the difficulties which arise in a simpler setting. Our real interest is the gravitational field, and we apply, next, our formalism to general relativity at null infinity. In particular we derive Hamiltonian formulae for energy, momentum, angular-momentum, as well as for the Hamiltonians for boosts and "supertranslations". Now, the — generally accepted — notion of energy in general relativity appropriate in the radiation regime is the one which has been introduced by Trautman [108], and further studied by Bondi [24]; we will refer to this mass as the Trautman–Bondi mass; the original motivation for this work was to show how this quantity arises in a Hamiltonian framework. One of the main results of the work here is a natural Hamiltonian definition of global Lorentz charges — that is, angular momentum and boost integrals — for cuts of $\mathscr{I}$, which is free from the "supertranslation ambiguities".

We shall now expand the quick overview, just given, of our work here. Let us start with a brief review of the Hamiltonian description of the dynamics of the massless scalar wave equation on Minkowski space-time,

$$\Box\phi = 0 \,. \tag{1.1}$$

Consider the collection of solutions of (3.1) with initial data which are, say, smooth and compactly[1] supported on the hypersurface $\mathscr{S}_0 = \{x^0 = 0\} \subset \mathbb{R}^{1,3}$, where $\mathbb{R}^{1,3}$ stands for the four dimensional Minkowski space-time. As is well known (and discussed in more detail below), this theory can be described as a dynamical system by considering the restrictions (ϕ_t, π_t) of $(\phi, \partial\phi/\partial t)$ to the hypersurfaces $\mathscr{S}_t = \{x^0 = t\}$. In this approach the family (ϕ_t, π_t), $t \in \mathbb{R}$, can be thought of as a smooth curve in the set $C_0^\infty(\mathbb{R}^3) \oplus C_0^\infty(\mathbb{R}^3)$ of smooth compactly supported functions on $\mathbb{R}^3$. The associated dynamical system is Hamiltonian, and in the standard formulation all the Hamiltonians generating the equations of motion are of the form

$$\mathscr{H} = \frac{1}{2}\int_{\mathbb{R}^3} (\pi^2 + |\nabla\phi|^2) d^3x + C \,. \tag{1.2}$$

This follows from the facts that: 1) $\mathscr{H}$ given by (3.2) is differentiable and satisfies the appropriate generating equations (*cf.* Chapter 4.1 below); 2) the difference of any two Hamiltonians has vanishing differential, and therefore must be a constant because the space of field configurations is path connected.

The constant in (3.2) can be gotten rid of by requiring that the energy vanishes for the trivial configuration $\phi = \pi = 0$. This requirement leads then to a uniquely defined quantity, usually identified with the total energy contained in a field configuration.

[1] The condition of smooth compactly supported initial data is made only for simplicity, and can be considerably relaxed. In particular, later on in this work, we shall consider field configurations which do not satisfy this hypothesis.

When one attempts to replace the hypersurfaces $\mathscr{S}_t$ above with hypersurfaces which extend to null infinity, various difficulties arise (they will be presented and adressed in the work below). In this context some Hamiltonian formulations of the dynamics have been presented [8, 11, 12, 68, 75], and in all those formulations the Hamiltonian turns out to be the energy calculated "at spatial infinity". More precisely, in the above mentioned descriptions of a scalar field in Minkowski space-time the Hamiltonian is, essentially[2], the energy calculated on hypersurfaces $\{x^0 = \text{const}\}$ as in Equation (3.2), where x^0 refers to the Minkowskian time coordinate. (Similarly, in the analysis of the gravitational field in [8, 11, 12, 68, 75] the Hamiltonian is, essentially, the ADM mass.) Now we are interested in a definition of the energy in the radiating regime, where one expects that the correct energy should *not* be given by the integrals of the energy-momentum tensor on the level sets of the Minkowskian time, but on hypersurfaces extending into the radiation zone; this would be the scalar field equivalent of the Trautman–Bondi mass in general relativity [24, 108], and we will use those names when refering to that mass. It has been argued [113] that no such formulation is possible, because in a Hamiltonian system the energy is conserved, while the Trautman–Bondi mass is not. We shall show that the argument of [113] does not apply when things are suitably formulated, and that there exists a Hamiltonian description of the dynamics of the scalar field in the radiating regime in which the Hamiltonian is the Trautman–Bondi mass; see Sections 4.4 and 4.5 for a simple exposition of these ideas.

Recall, next, that in a manner rather analogous to the scalar field on Minkowski space-time, the Einstein equations induce a dynamical system on the phase space of those gravitational initial data which are asymptotically flat in spacelike directions; this is discussed in more detail in Section 5.4 below. In this case all the Hamiltonians corresponding to space-time motions which reduce to unit time translations to the future in the asymptotically flat regions, are of the form

$$\mathscr{H} = M_{ADM} + C \ ,$$
$$M_{ADM} = \frac{1}{16\pi}\int_{S_\infty}(g_{ij,j} - g_{jj,i})dS_i \tag{1.3}$$

(the integral over the union "S_∞" of all the "spheres at infinity" is understood as a limit as R tends to infinity of integrals over the union of spheres of radius R in all the asymptotically flat regions), with the constant C usually set to

[2] The qualification "essentially" here is due to the fact that the equality of the Hamiltonian with the energy of Minkowskian-time slices $\{x^0 = \text{const}\}$ is correct for smooth compactly supported initial data, and is expected to be true for the more general data considered in those works. However, no such rigourous results are known even in the case of the scalar field on Minkowski space-time. A similar comment applies to the gravitational field.

zero[3]. Here g_{ij} denotes the metric on the "$t = 0$ hypersurface", with indices i, j in (3.3) running from 1 to 3, and being summed over.

In complete analogy to the scalar field case the situation in the radiation regime is much less satisfactory. Here one expects the Trautman–Bondi energy [24, 98, 107, 108] to be the physically relevant measure of the total energy contained in a hypersurface $\mathscr{S}$ that intersects $\mathscr{I}^+$ in an appropriate way. To our knowledge no satisfactory theoretical justification of such a statement has been given so far, though some partial results can be found in [23, 41, 113]. The strongest hint in this direction seems to be given by the uniqueness theorem of [41], which asserts that the Trautman–Bondi energy is the unique functional, up to a multiplicative constant, in an appropriate class of functionals, which is monotonic in time under time translations of $\mathscr{S}$. While monotonicity is certainly a reasonable condition, it is not clear to us that the requirement of monotonicity is a sufficient criterion for excluding all other possibilities. We emphasize that this problem has nothing to do with the gravitational field, as it occurs already for a massless scalar field in Minkowski space-time.

The purpose of this monograph is to show that there exists a Hamiltonian description of dynamics of a massless scalar field, as well as of the dynamics of the gravitational field, in the radiation regime. We construct such a framework, and exhibit two different ways in which the Trautman–Bondi energy arises. The first such occurrence is by taking an appropriate limit of the Hamiltonians on the phase space $\widehat{\mathscr{P}}_{[-1,0]}$ of Sections 6.5 or 5.9 below. This gives a unique result, up to one normalization constant, on each connected component of the phase space. Next, we show that the Trautman–Bondi energy is one of the Hamiltonians on yet another phase space (the phase space $\mathscr{P}_{[-1,0]}$ of Sections 6.4 or 5.8). For reasons which we discuss in detail below the freedom of adding a constant to any Hamiltonian leads to essential ambiguities, related to the nature of $\mathscr{P}_{[-1,0]}$, which we describe in an exhaustive way. While those ambiguities are somewhat reminiscent of the ones that arise in the "Noether charge" approach (*cf., e.g.,* [28, 41]), the arbitrariness left turns out to be considerably smaller. We are unaware of any natural prescription which would remove that arbitrariness. We give arguments, parallel to those in [41], which indicate that the requirement of monotonicity with respect to time translations to the future singles out a unique Hamiltonian — the Trautman–Bondi energy. The analysis here is similar to, but not identical with the one in [41], because we work in the class of smoothly conformally compactifiable space-times which have complete spacelike hypersurfaces. We note that the class of functionals, which are Hamiltonians, is considerably

[3] While the choice $C = 0$ is rather reasonable, it is not clear whether this is the best one in all situations: the topology of the initial surface can be varied at will, so that the space of initial data is certainly not connected in any reasonable topology, and one is free to choose different values of C on different connected components of the phase space. This freedom could have some physical significance, *e.g.* when a path integral is performed.

smaller than the class of the functionals considered in [41], but the problem here is more constrained in view of the global conditions imposed.

It should be pointed out that some of our constructions are somewhat related to those of [8]. Those authors consider fields which, when extended to $\mathscr{I}^+$, are defined on the semi-global sets $\mathscr{I}^+_{(-\infty,\tau)}$, *cf.* Equation (6.20) below. As already pointed out, in the approach of [8] the dynamics is (up to severe mathematical difficulties) Hamiltonian, with the Hamiltonian equal to (again ignoring some mathematical problems) the ADM mass, and *not* the Trautman–Bondi mass. Moreover, our approach allows us to avoid altogether those difficulties [8], which are related to global existence questions for the general relativistic Cauchy problem, as well as to convergence of various integrals on $\mathscr{I}^+_{(-\infty,\tau)}$.

This work is organized as follows: In Chapter 4 we start by recalling some elementary facts concerning Hamiltonian dynamical systems, and we give some toy examples illustrating some of the ideas developed in the remainder of this work. In Chapter 5 we describe our geometric Hamiltonian framework, adequate both to the usual asymptotically-flat-at-spatial-infinity regime and to the radiation regime, which generalizes the framework of [85]. We note that our framework clarifies some questions which arise already in standard contexts, in particular the question of interpretation of general relativistic initial data sets with vanishing lapse function. As far as the radiation zone is concerned, it turns out that the case of the massless scalar field on Minkowski space-time already exhibits several essential features of the problems that arise there, while avoiding various technicalities which occur when one wishes to describe the Einstein gravity. Therefore we continue, in Chapter 6, with a detailed description of the application of our formalism to the case of the massless scalar field. The formalism of Chapter 5 is applied to the case of Einstein gravity in Chapters 5 and 6. The inspection of the table of contents should give the reader a faithful impression of the contents of the various sections.

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2. Preliminaries

2.1 Hamiltonian dynamics

There exist different approaches to the definition of a Hamiltonian system (*cf., e.g.,* [2, 30, 85, 88]). An exhaustive treatment in the infinite dimensional case would involve delicate considerations concerning the manifold structure of the spaces at hand; in particular, one would have to introduce the notion of tangent vectors, differential forms, as well as an appropriate notion of non-degeneracy and closedness of the symplectic form. We do not wish to enter into such questions, and the purpose of this chapter is to present a simple minded approach which avoids all those issues. We shall show in the following chapters that the dynamics of the massless scalar field in the radiation regime, as well as that of the gravitational field in the radiation regime, satisfies the requirements of the definitions given in this chapter.

We begin with the definition of an *autonomous* Hamiltonian system, where both the dynamics and the associated Hamiltonian function H are time-independent. Consider a vector space $\mathscr{P}$ in which the notion of a differentiable curve can be defined. (In this work, unless indicated otherwise, when $\mathscr{P}$ is a space of differentiable functions on a set $\mathscr{O}$, then we will say that a curve $f_\lambda \in \mathscr{P}$, $\lambda \in \mathbb{R}$, is differentiable if the family of functions f_λ is jointly differentiable in $\lambda \in \mathbb{R}$ and $x \in \mathscr{O}$.) Recall that a family of maps $T_\lambda : \mathscr{P} \to \mathscr{P}$, $\lambda \in \mathbb{R}$, is called a differentiable dynamical system on $\mathscr{P}$ if

1. $T_0 = \mathrm{id}$, the identity map on $\mathscr{P}$, if
2. $T_a \circ T_b = T_{a+b}$, and if
3. for every $p \in \mathscr{P}$ the orbit of T_λ defined as the map $\lambda \to T_\lambda(p) \in \mathscr{P}$ is a differentiable curve on $\mathscr{P}$.

We set

$$\mathcal{X}(p) = \frac{dT_\lambda(p)}{d\lambda}\Big|_{\lambda=0} . \tag{2.1}$$

We will call $\mathcal{X}$ *a vector field generating the dynamics*; by this we *only* mean that Equation (4.1) holds. In particular, no hypotheses are made about the possibility of recovering T_λ from $\mathcal{X}$ — here the fundamental object is T_λ.

Let Ω be a bilinear antisymmetric map on $\mathscr{P}$ with values in $\mathbb{R}$; the Ω's we will consider will satisfy some non-degeneracy conditions but we do not need to make those precise here. We shall say that the dynamical system is

Hamiltonian if there exists a function H on $\mathscr{P}$ such that for all differentiable curves p_σ on $\mathscr{P}$ we have

$$\frac{dH(p_\sigma)}{d\sigma}\Big|_{\sigma=0} = -\Omega\left(\mathfrak{X}, \frac{dp_\sigma}{d\sigma}\Big|_{\sigma=0}\right) . \tag{2.2}$$

(We note that this definition implicitly requires $H(p_\sigma)$ to be differentiable at $\sigma = 0$ whenever p_σ is.) The function H will be called *a Hamiltonian* for the dynamical system $(\mathscr{P}, T_\lambda)$.

As an illustration, let $\mathscr{P} = C_0^\infty(\mathbb{R}^3) \oplus C_0^\infty(\mathbb{R}^3)$ be the space of pairs of smooth compactly supported functions on $\mathbb{R}^3$. Let f be a solution of the massless scalar field equation on Minkowski space-time,

$$\Box f = 0 ,$$

satisfying

$$f(t=0) = \varphi , \qquad \frac{\partial f}{\partial t}(t=0) = \pi , \qquad (\varphi, \pi) \in \mathscr{P} ,$$

and set

$$\mathscr{P} \ni (\varphi, \pi) \to T_\lambda(\varphi, \pi) = \Big(f(t=\lambda), \frac{\partial f}{\partial t}(t=\lambda)\Big) \in \mathscr{P} .$$

Here t is a Minkowskian time coordinate in Minkowski space-time. If we equip $\mathscr{P}$ with the standard ("symplectic") form,

$$\Omega((\varphi_1, \pi_1), (\varphi_2, \pi_2)) = \int_{\mathbb{R}^3} (\varphi_1\pi_2 - \varphi_2\pi_1) d^3x ,$$

then the dynamical system (P, T_λ) is Hamiltonian in the above sense, with

$$H(\varphi, \pi) = \frac{1}{2}\int_{\mathbb{R}^3} (|D\varphi|^2 + \pi^2)\, d^3x . \tag{2.3}$$

Here $|D\varphi|$ denotes the length of the space-gradient of φ.

It turns out that even for the massless scalar field we need to generalize the set-up above, allowing *non-autonomous* (time-dependent) Hamiltonian systems. More precisely,

1. it will be necessary to consider a dynamical system generated (in the sense of Equation (4.5) below) by a time-dependent "vector field" $\mathfrak{X}_t$,
2. with the corresponding time-dependent flow $T_{t,s}$ only locally defined.

More precisely, we will consider a family $T_{t,s}$ of maps

$$\mathbb{R} \times \mathbb{R} \times \mathscr{P} \supset \mathscr{U} \ni (t, s, p) \to T_{t,s}(p) \in \mathscr{P} , \tag{2.4}$$

defined on an open connected subset $\mathscr{U}$ of $\mathbb{R} \times \mathbb{R} \times \mathscr{P}$. Those maps describe where a trajectory of the dynamical systems passing through a point p at time s will arrive at time t. We shall further require:

1. For all $p \in \mathscr{P}$ the maps $T_{t,s}$ are defined on an open set of t's and s's, containing zero: $\mathscr{U} \supset \{0\} \times \{0\} \times \mathscr{P}$.
2. The composition formula

$$T_{t_3,t_2} \circ T_{t_2,t_1} = T_{t_3,t_1}$$

   holds whenever all the objects in the above equation are simultaneously defined.
3. For all $(t,p) \in \mathbb{R} \times \mathscr{P}$ the curves $s \to T_{t+s,t}(p)$ are differentiable at $s = 0$.

We set

$$\mathfrak{X}_t(p) = \frac{dT_{t+s,t}}{ds}\Big|_{s=0} . \tag{2.5}$$

We mention that in one of the cases considered below for the massless scalar field (the phase space $\widehat{\mathscr{P}}_{[-1,0]}$ of Section 6.5) the set $\mathscr{U}$ will be of the form

$$\mathscr{U} = \{t \in (-1,\infty), s \in (-1,\infty), p \in \mathscr{P}\} . \tag{2.6}$$

In the scalar field case the restrictions on t and s that follow from (4.6) arise because we will mainly be interested in those solutions of the massless scalar field equation which are defined to the future of a given hyperboloid in Minkowski space-time. In the gravitational field case there is a further fundamental reason for allowing a $\mathscr{U}$ not necessarily equal to $\mathbb{R} \times \mathbb{R} \times \mathscr{P}$, related to the blow up in finite time of solutions of the Einstein equations.

The equivalent of (4.2) reads

$$\frac{dH(t,p_\sigma)}{d\sigma}\Big|_{\sigma=0} = -\Omega\Big(\mathfrak{X}_t, \frac{dp_\sigma}{d\sigma}\Big|_{\sigma=0}\Big) , \tag{2.7}$$

for all curves p_σ differentiable at $\sigma = 0$, and for those t for which $(t, 0, p = p_0) \in \mathscr{U}$ (so that $\mathfrak{X}_t$ is defined). Equation (4.7) is the desired generalization of the notion of a Hamiltonian dynamical system to the time-dependent case, with the Hamiltonian H being defined on an appropriate subset of $\mathbb{R} \times \mathscr{P}$.

Recall that there is another standard way of dealing with time-dependent Hamiltonians [88, Chapter V, p. 328], which consists in enlarging the phase space by adding to it t and its conjugate variable p^0. While this can be done in our case, we have found the approach above to be simpler.

The global time-independent formulation is a special case of the local time-dependent one if one sets $T_{t,s} \equiv T_{t-s}$, $\mathscr{U} \equiv \mathbb{R} \times \mathbb{R} \times \mathscr{P}$.

Let us close this section with the simple observation, that functionals satisfying Equation (4.7) are unique up to a constant *when the phase space $\mathscr{P}$ is connected, and locally path connected* via differentiable paths. Indeed, if H_1 and H_2 satisfy Equation (4.7) then $H_1 - H_2$ has vanishing derivative along any one-differentiable one-parameter family of fields, so that $H_1 - H_2$ is locally constant by local path connectedness, hence constant by connectedness of $\mathscr{P}$.

2.2 The role of boundary conditions in Hamiltonian field theory

In this section we want to give a short and informal overview of the ideas, which we later use to describe radiation phenomena in the Hamiltonian field theory. Let us analyse more in detail the definition of the Hamiltonian flow, given by formula (4.7), in the case of the scalar wave equation. We introduce the following notation: whenever we have a differentiable family of functions $f(x;\sigma)$, where the variables x describe the position of a point in the physical space, or in space-time, and σ is an abstract parameter, then the derivative of this family with respect to the parameter, calculated at $\sigma = 0$ will be denoted by δ:

$$\delta f(x) := \frac{\partial f(x;\sigma)}{\partial \sigma}\Big|_{\sigma=0} . \tag{2.8}$$

This notation is standard in the calculus of variation. In functional spaces considered in this monograph (as, *e.g.*, in the space $\mathscr{P} = \{(\varphi, \pi)\}$ of Cauchy data for the wave equation, considered in the previous section), *differentiable curves* are simply differentiable one-parameter families of functions. Whenever we meet a "variation of a function", we understand that a one-parameter family $f(x;\sigma)$ of functions over physical space (or space-time) has been chosen and the derivative (4.8) has been calculated. In this notation, the left hand sides of (4.2) or (4.7) become simply δH.

Another derivative, which we shall often use in our work, is the Lie derivative $\mathscr{L}_X$ or (in the case of field theories which are more general than the scalar field theory) the covariant derivative $\mathscr{D}_X$, along a space-time vector field X defining the evolution which we want to describe. It is sometimes convenient to use coordinates adapted to this vector field so that we have $X = \partial_0$, and the corresponding Lie derivative reduces to the time derivative. Whenever it does not lead to any misunderstanding, we shall denote it by a "dot". As will be seen in the next Chapter, the use of adapted coordinates may be avoided and the entire Hamiltonian field theory formulated in geometric terms, both for flows of vector fields and for motions of hypersurfaces.

In adapted coordinates used in the previous section, the components of the vector (4.5), defined on the space of Cauchy data $\mathscr{P} = \{(\varphi, \pi)\}$ for the scalar field theory are simply denoted by $(\mathscr{L}_X\varphi, \mathscr{L}_X\pi) = (\dot{\varphi}, \dot{\pi})$, where a dot denotes now the usual derivative with respect to the Minkowskian time coordinate, in a Minkowskian coordinate system. The field equation

$$\Box f = \Delta f - \ddot{f} = 0 ,$$

expressed in terms of Cauchy data (φ, π), gives the following system of equations:

$$\dot{\varphi} = \pi , \tag{2.9}$$

$$\dot{\pi} = \Delta\varphi . \tag{2.10}$$

Using the definition of the form Ω in space $\mathscr{P}$, given in the previous section, we may rewrite the Equation (4.7) for the scalar field in the following way:

$$\begin{aligned} -\delta H &= \int_{\mathbb{R}^3} (\dot{\pi}\delta\varphi - \dot{\varphi}\delta\pi) d^3x \\ &= \int_{\mathbb{R}^3} ((\mathscr{L}_X \pi)\delta\varphi - (\mathscr{L}_X \varphi)\delta\pi) d^3x \ . \end{aligned} \tag{2.11}$$

Let us analyse in more detail the mechanism which leads to this equation. For this purpose we calculate explicitly the variation of the Hamiltonian (4.3). Because derivatives with respect to the parameter σ commute with derivatives with respect to space variables, we have:

$$\begin{aligned} -\delta H(\varphi, \pi) &= -\delta \left\{ \frac{1}{2} \int_{\mathbb{R}^3} (|D\varphi|^2 + \pi^2)\, d^3x \right\} \\ &= - \int_{\mathbb{R}^3} (D\varphi \cdot D\delta\varphi + \pi\delta\pi)\, d^3x \\ &= \int_{\mathbb{R}^3} (\Delta\varphi\delta\varphi - \pi\delta\pi)\; d^3x - \int_{\mathbb{R}^3} D(D\varphi\delta\varphi)\, d^3x \ . \end{aligned} \tag{2.12}$$

Hence, equation (4.12) is equivalent to (4.11) if and only if the last integral vanishes. Due to the Stokes theorem, it may be converted into a "surface integral at infinity" of the vector field $-(D\varphi)\delta\varphi$. It vanishes if sufficiently fast fall-off conditions are imposed on the field f; for definiteness, as in the previous section we assume that f is compactly supported on each hypersurface of constant Minkowskian time, but much weaker asymptotic conditions are of course sufficient.

Now, consider the evolution of the same scalar field f but in a finite volume $V \subset \mathbb{R}^3$, with non-empty boundary. Let H_V be the total amount of the usual field energy contained in V:

$$H_V(\varphi, \pi) = \frac{1}{2} \int_V (|D\varphi|^2 + \pi^2)\, d^3x \ . \tag{2.13}$$

Similar calculations as above lead to the following result:

$$-\delta H_V(\varphi, \pi) = \int_V (\Delta\varphi\delta\varphi - \pi\delta\pi)\; d^3x + \int_{\partial V} (\pi^a \delta\varphi)\, d\sigma_a \ , \tag{2.14}$$

where we have introduced the notation

$$\pi^a := -D^a \varphi \ , \tag{2.15}$$

and used the Stokes theorem to convert the last integral into a surface integral. To recover the definition of a Hamiltonian (4.7) — or, equivalently, Equation (4.11) — we must annihilate the surface integral by imposing some boundary conditions on the elements of our phase space $\mathscr{P} = \{(\varphi, \pi)\}$. It should be stressed that the need for imposing boundary conditions does not arise only because we wish to have a formula such as (4.7); without boundary conditions the time evolution does not define a dynamical system, since then the initial value problem is not well posed: the scalar field f in the future may be changed due to incoming or outgoing radiation, even if the Cauchy data at $t = 0$ remain the same. Boundary conditions are *necessary* to obtain a deterministic system. Physically, they can be thought of as our control of the radiation passing through the boundary ∂V of the region V, in which we perform experiments.

In the example just given the boundary of the space-time region $\mathbb{R} \times V$ is a timelike hypersurface. In such situations a simple way to annihilate the surface integral in (4.14) is to choose some function $\psi : \mathbb{R} \times \partial V \to \mathbb{R}$ and to impose on the scalar field f the Dirichlet boundary condition

$$f\big|_{\mathbb{R}\times\partial V} = \psi \ , \tag{2.16}$$

for all functions used in the sequel[1]. This implies

$$\varphi\big|_{\partial V} \equiv f\big|_{\{0\}\times\partial V} = \psi\big|_{\{0\}\times\partial V} \implies \delta\varphi\big|_{\partial V} = 0 \ , \tag{2.17}$$

within the class of functions satisfying condition (4.16). If ψ is time-independent, we obtain in this way a well defined, autonomous Hamiltonian system. For a time-dependent boundary condition one could think, at a first glance, that the very notion of a phase-space $\mathscr{P}_t = \{(\varphi, \pi) : \varphi\big|_{\partial V} = \psi(t, \cdot)\}$ does depend upon time and, therefore, no Hamiltonian description is possible. The remedy to this difficulty is, however, straightforward: choose any time-dependent function $\phi = \phi(t, x)$, which satisfies the boundary condition (4.16), and parameterize the data (φ, π) by the following functions:

$$\widetilde{\varphi} := \varphi - \phi \ , \qquad \widetilde{\pi} := \pi - \dot{\phi} \ . \tag{2.18}$$

The new variables fulfill a homogeneous, time independent, boundary condition

$$\widetilde{\varphi}\big|_{\partial V} = 0 \ , \tag{2.19}$$

and Equations (4.18) provide an identification of all the phase spaces $\mathscr{P}_t$ with the time-independent phase space $\widetilde{\mathscr{P}} = \{(\widetilde{\varphi}, \widetilde{\pi}) : \widetilde{\varphi}\big|_{\partial V} = 0\}$. The Hamiltonian description of the field theory is, therefore, applicable (*i.e.*, formula (4.11) remains valid) also in case of time-dependent boundary data. The only price we pay for this is an explicit time-dependence of the Hamiltonian, arising from the (given *a priori*) "reference function" ϕ and its derivatives, when

[1] In the case of null boundaries, or for the description of radiation, this method does not apply; a simple example illustrating this will be given in Section 4.4 below. The point of the examples in this section is not to show in a simple case how we handle the radiation problem, but to give an indication of the kind of problems that arise when domains with boundary are considered.

we substitute $\varphi := \widetilde{\varphi} + \phi$ and $\pi := \widetilde{\pi} + \dot{\phi}$ in formula (4.13). Physically, the non-autonomous properties of the field dynamics within V is due to time-dependent external forces, applied on the boundary ∂V, in order to control the boundary data of the field in a prescribed way.

Imposing Dirichlet boundary conditions is *by no means* a unique way to annihilate the surface integral in (4.14), obtaining thus a Hamiltonian dynamical system. A frequently used alternative consists in imposing Neumann conditions, *i.e.*, prescribing the value of $\pi^a \, d\sigma_a$ on the boundary. For this purpose we write

$$\pi^a \delta\varphi = \delta(\pi^a \delta\varphi) - \varphi\delta\pi^a \ ,$$

and transfer the first term to the left-hand side of (4.14). (The manipulations involved are somewhat reminiscent of those which arise when Legendre transformations are carried on.) This leads us to the formula

$$-\delta\widetilde{H}_V(\varphi,\pi) = \int_V (\Delta\varphi\delta\varphi - \pi\delta\pi) \, d^3x - \int_{\partial V} (\varphi\delta\pi^a) \, d\sigma_a \ , \tag{2.20}$$

where the new Hamiltonian, describing the mixed Cauchy–Neumann evolution equals:

$$\widetilde{H}_V(\varphi,\pi) := H_V(\varphi,\pi) + \int_{\partial V} (\pi^a\varphi) \, d\sigma_a = H_V(\varphi,\pi) + \int_V D_a(\pi^a\varphi) \, d^3x \ . \tag{2.21}$$

Equation (4.20) leads to the formula (4.11) for the Hamiltonian evolution of the Cauchy data, because the Neumann boundary condition on $\pi^a d\sigma_a$ implies:

$$\delta(\pi^a \, d\sigma_a)\big|_{\partial V} \equiv 0 \ ,$$

within the class of functions allowed by the condition and, therefore, the surface integral vanishes.

These issues will be discussed in following chapters under more general circumstances, and the mathematical structures associated with the above formulae will be described. One of the points of the examples given was to stress that in some situations there might be many ways to translate the field evolution into the language of Hamiltonian dynamics. Different physical situations lead to different boundary conditions. For the gravitational field, governed by Einstein equations, possible choices of boundary conditions on bounded domains with boundary have been discussed in [83], [82] and [84] from a symplectic point of view. It is expected that those considerations might shed some light on the associated analytic problem; see [60] for some rigorous analytic results that do not involve symplectic considerations.

2.3 Tangential translations as a Hamiltonian system

In the usual treatment of relativistic field theories on Minkowski space-time the field energy provides the Hamiltonian for the time evolution; here the dynamics is associated with the space-time vector field $X = \partial_t$, which is transversal to the Cauchy surfaces. Under some circumstances one might be interested in evolution of the fields under motions associated with vector fields tangent to the Cauchy surface; we will encounter such situations throughout this monograph. Again in textbook treatments, the "generator of tangential space translations" is the momentum of the field configuration. Let us analyse more carefully such a "dynamics" for motions of the massless scalar field under the group T_λ of space translations, generated by the field $X = \partial_a$:

$$\mathscr{P} \ni (\varphi, \pi) \to T_\lambda(\varphi, \pi) := (T_\lambda(\varphi), T_\lambda(\pi)) \in \mathscr{P} \;, \tag{2.22}$$

where

$$T_\lambda(\varphi)(x) := \varphi(x + \lambda e_a) \;, \tag{2.23}$$

$$T_\lambda(\pi)(x) := \pi(x + \lambda e_a) \;, \tag{2.24}$$

and e_a denotes the versor of a-th axis in $\mathbb{R}^3$. The vector field (4.5), corresponding to this evolution, equals:

$$(\mathscr{L}_X \varphi, \mathscr{L}_X \pi) = (\partial_a \varphi, \partial_a \pi) \;.$$

Let us check that the a-th component of the field momentum,

$$P_a(\varphi, \pi) = \int_{\mathbb{R}^3} (\pi \partial_a \varphi) \, d^3x \;, \tag{2.25}$$

provides indeed a Hamiltonian for the dynamics, by calculating its variation:

$$-\delta P_a(\varphi, \pi) = -\int_{\mathbb{R}^3} (\pi \partial_a \delta\varphi + (\partial_a \varphi)\delta\pi) \, d^3x \tag{2.26}$$

$$= \int_{\mathbb{R}^3} ((\partial_a \pi)\delta\varphi - (\partial_a \varphi)\delta\pi) \, d^3x - \int_{\mathbb{R}^3} \partial_a(\pi\delta\varphi) \, d^3x \tag{2.27}$$

$$= \int_{\mathbb{R}^3} ((\mathscr{L}_X \pi)\delta\varphi - (\mathscr{L}_X \varphi)\delta\pi) d^3x \;. \tag{2.28}$$

The last equation is satisfied, because the integral of a total divergence vanishes when sufficiently fast asymptotic fall-of conditions are imposed on φ; recall that we are assuming that φ is compactly supported. In the case of a bounded domain V, the resulting boundary integral

$$I_{\partial V} = \int_{\partial V} (\pi \delta\varphi) \, d\sigma_a \;,$$

provides in general an obstruction to the Hamiltonian character of the associated dynamics.

Replacing ∂_a by an an arbitrary complete vector field $X = X^a \partial_a$, tangent to the Cauchy space $\mathbb{R}^3$, and the group of rigid space translations by the one-parameter group of diffeomorphisms $\mathscr{G}^X$ generated by X, we may generalize the above example. A "time evolution" associated with $\mathscr{G}^X$ can be defined by Lie transporting both φ and π along the flow of the field X; we stress that the "time" involved has nothing to do with the physical time, and is simply a parameter along the integral curve of X. With this definition the vector tangent to the evolution curve is given by the Lie derivative of both objects with respect to X:

$$(\mathscr{L}_X\varphi, \mathscr{L}_X\pi) = (X^a\partial_a\varphi, \partial_a(X^a\pi)) \ . \tag{2.29}$$

(The last expression for the Lie derivative of π is due to the fact that the momentum is not a scalar function but a scalar density. This is discussed in more detail in the next chapter.)

Let P_V be given by the formula

$$P_V^X(\varphi,\pi) := \int_V (\pi X^a\partial_a\varphi)\, d^3x \ ; \tag{2.30}$$

calculating its variation we obtain:

$$-\delta P_V^X(\varphi,\pi) = \int_V ((\mathscr{L}_X\pi)\delta\varphi - (\mathscr{L}_X\varphi)\delta\pi)d^3x + \int_{\partial V} (\pi^a\delta\varphi)\, d\sigma_a \ , \tag{2.31}$$

where we have set

$$\pi^a := -X^a\pi \ . \tag{2.32}$$

The dynamics associated with the dragging of the scalar field along a vector field X is, again, Hamiltonian if the surface integral vanishes. This occurs without the need of imposing any boundary conditions on the initial data when X is tangent to the boundary ∂V: in such a case the surface integral vanishes identically and P_V^X becomes a Hamiltonian. This holds *e.g.*, for the one-parameter group of rotations, whenever they leave V invariant. In that case, P_V^X is usually identified with the total amount of angular momentum carried by the field within V.

2.4 The Hamiltonian description of a mixed Cauchy – characteristic initial value problem

The main purpose of this monograph is to give a description of radiation phenomena in terms of Hamiltonian dynamics. Those phenomena are best captured by adding to space-time a conformal boundary, called *Scri*, and denoted by the symbol $\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^-$; outgoing radiation can then be studied in a neighbourhood of $\mathscr{I}^+$, while ingoing radiation is related to the behaviour of the fields in a neighbourhood of $\mathscr{I}^-$. $\mathscr{I}^+$ is a null-like, three-dimensional manifold, with structure similar to that of a light cone. To illustrate methods which will be used to describe field dynamics on $\mathscr{I}^+$, let us consider a toy example, in which the "*asymptotic* light cone $\mathscr{I}^+$" is replaced by a standard, finite light cone. Now, a Hamiltonian description of the field dynamics within the future-oriented light cone

$$\mathscr{C}^+ = \{(t, x) : \|x\| < t\}$$

must take into account the incoming radiation, which enters $\mathscr{C}^+$ through its boundary. In this monograph we have concentrated on a description of the outgoing radiation. Hence, we use the past oriented cone $\mathscr{C}^- = \{(t, x) : \|x\| < -t\}$, to make our toy model better adapted to this purpose. Both cases are, however, symmetric: replacing t with $-t$ one obtains a toy model for outgoing radiation from the incoming radiation one, and vice versa.

To eliminate technicalities and make the model as simple as possible, let us restrict ourselves to the two-dimensional Minkowski space. This means that space is one dimensional: $x \in \mathbb{R}^1$. We consider again a massless scalar field, solving the wave equation. We want to describe its Cauchy data on the surfaces $\{t = \text{const}\}$ in the interior of the cone $\mathscr{C}^-$. To be able to identify these surfaces for different times, let us introduce new coordinates $(\xi^\mu) = (\tau, \xi)$ (where $\mu = 0, 1$), related to the Minkowskian coordinates $(x^\mu) = (t, x)$ in the following way:

$$t = -e^{-\tau} , \tag{2.33}$$

$$x = \xi e^{-\tau} , \tag{2.34}$$

where $\tau \in \mathbb{R}^1$ and $|\xi| \leq 1$. Within this range, the new coordinates parameterize the entire cone $\mathscr{C}^-$. To derive the Hamiltonian description of the wave equation in these coordinates, we use the textbook procedure, based on the standard, relativistic-invariant Lagrangian

$$\mathbf{L} = L \, d^2 x , \tag{2.35}$$

where

$$L = -\frac{1}{2} g^{\mu\nu} (\partial_\mu f)(\partial_\nu f) = \frac{1}{2} \left\{ (\partial_t f)^2 - (\partial_x f)^2 \right\} . \tag{2.36}$$

We rewrite this Lagrangian in the coordinates (τ, ξ), using the following formulae which may be easily derived from (4.33) and (4.34):

$$\begin{aligned} \partial_t &= e^\tau \left(\partial_\tau + \xi \partial_\xi \right) , \\ \partial_x &= e^\tau \partial_\xi . \end{aligned}$$

Moreover, we have

$$d^2 x = dt dx = e^{-2\tau} d\tau d\xi = e^{-2\tau} d^2 \xi .$$

Expressing the Lagrangian (4.35) in terms of new coordinates we thus obtain:

$$\mathbf{L} = \mathscr{L} \, d^2 \xi , \tag{2.37}$$

where

$$\mathscr{L} = \frac{1}{2}\left\{(\partial_\tau f + \xi\partial_\xi f)^2 - (\partial_\xi f)^2\right\} . \tag{2.38}$$

The standard procedure, valid for an arbitrary Lagrangian density $\mathscr{L} = \mathscr{L}(f, f_\mu, \xi^\mu)$, where $f_\mu := \partial_\mu f$, proceeds as follows: We introduce generalized momenta:

$$\pi^\mu := \frac{\partial \mathscr{L}}{\partial f_\mu} , \tag{2.39}$$

and calculate the variation of the Lagrangian:

$$\delta\mathscr{L} = \frac{\partial \mathscr{L}}{\partial f}\delta f + \pi^\mu \delta f_\mu = \partial_\mu\left(\pi^\mu \delta f\right) + \left(\frac{\partial \mathscr{L}}{\partial f} - \partial_\mu \pi^\mu\right)\delta f . \tag{2.40}$$

The field equation $\Box f = 0$ is equivalent to the vanishing of the Euler–Lagrange term in (4.40):

$$\frac{\partial \mathscr{L}}{\partial f} - \partial_\mu \pi^\mu = 0 , \tag{2.41}$$

and, therefore, is equivalent to the following equation which must be fulfilled by the variation of $\mathscr{L}$:

$$\delta\mathscr{L} = \partial_\mu\left(\pi^\mu \delta f\right) = (\pi\delta\varphi)^{\cdot} + \partial_\xi\left(\pi^1\delta\varphi\right) , \tag{2.42}$$

where we have denoted by φ the restriction of the field f to the Cauchy surface $\Sigma = \{\tau = \text{const.}\}$ and by a dot — the derivative with respect to the new time variable τ. Moreover, we have introduced the momentum $\pi := \pi^0$, which provides the remaining piece of Cauchy data on the surface[2]. Integrating the field equation (4.42) over a volume V in the Cauchy surface $\Sigma = \{\tau = \text{const.}\}$, we obtain the following identity, valid for fields satisfying the wave equation:

$$\begin{aligned}\delta\int_V \mathscr{L}\, d\xi &= \int_V (\pi\delta\varphi)^{\cdot}\ d\xi + \int_{\partial V}\left(\pi^1\delta\varphi\right)\ d\sigma_1\\ &= \int_V \left(\dot\pi\delta\varphi - \dot\varphi\delta\pi + \delta(\pi\dot\varphi)\right)\ d\xi + \left[\pi^1\delta\varphi\right]_{\partial V} ,\end{aligned} \tag{2.43}$$

where the integral over the 0-dimensional boundary ∂V is equal to the difference of values of the integrand between the two ends of ∂V. This identity is equivalent to the following formula:

$$\begin{aligned}-\delta H_V(\varphi,\pi) &= \int_V \left(\dot\pi\delta\varphi - \dot\varphi\delta\pi\right)\ d\xi + \left[\pi^1\delta\varphi\right]_{\partial V}\\ &= \int_V ((\mathscr{L}_X\pi)\delta\varphi - (\mathscr{L}_X\varphi)\delta\pi) + \left[\pi^1\delta\varphi\right]_{\partial V} ,\end{aligned} \tag{2.44}$$

[2] Actually, $\pi d\xi$ is a pull-back of a differential odd-form $\pi^\mu\partial_\mu\rfloor d\xi^0 \wedge d\xi^1$ to the surface $\tau = $ const. This proves that it does not depend upon the choice of the coordinate x^0, but only upon the choice of the Cauchy surface Σ. The structure of the canonical field momentum π^μ is discussed in the next chapter.

where

$$H_V(\varphi, \pi) := \int_V (\pi\dot{\varphi} - \mathscr{L}) \ . \tag{2.45}$$

The procedure used applies to an arbitrary Lagrangian. Its geometric context, together with the structure of the momentum π^μ will be discussed thoroughly in the next chapter. Here, we perform calculations for the specific Lagrangian (4.37) and obtain:

$$\pi \ = \pi^0 = \frac{\partial \mathscr{L}}{\partial f_0} = \partial_\tau f + \xi \frac{\partial f}{\partial \xi} = \dot{\varphi} + \xi \partial_\xi \varphi \ , \tag{2.46}$$

$$\pi^1 = \frac{\partial \mathscr{L}}{\partial f_1} = \xi \left(\partial_\tau f + \xi \frac{\partial f}{\partial \xi} \right) - \frac{\partial f}{\partial \xi} = \xi \dot{\varphi} - (1 - \xi^2) \partial_\xi \varphi \ . \tag{2.47}$$

These equations imply, according to (4.41), the following field dynamics:

$$\dot{\varphi} = \pi - \xi \partial_\xi \varphi \ , \tag{2.48}$$

$$\dot{\pi} = \partial_\xi \left(\xi \dot{\varphi} - (1 - \xi^2) \partial_\xi \varphi \right) = \partial_\xi (\xi \pi) - \partial_\xi^2 \varphi \ . \tag{2.49}$$

Using (4.48), the Hamiltonian (4.45) may by written explicitely in terms of Cauchy data:

$$\begin{aligned} H_V(\varphi, \pi) &:= \int_V \{\pi(\pi - \xi \partial_\xi \varphi) - \mathscr{L}\} \, d\xi \\ &= \frac{1}{2} \int_V \left\{ \pi^2 - 2\pi \xi \partial_\xi \varphi + (\partial_\xi \varphi)^2 \right\} d\xi \\ &= \frac{1}{2} \int_V \left\{ (\pi - \xi \partial_\xi \varphi)^2 + (1 - \xi^2)(\partial_\xi \varphi)^2 \right\} d\xi \ . \end{aligned} \tag{2.50}$$

The main lesson stemming from the above formulae is the following: Consider, first, a domain V that lies strictly inside the interior of the cone, *i.e.*, $V = [a, b]$, with $-1 < a, b < 1$. Then $\mathbb{R} \times \partial V$ is timelike and the situation is similar to the one described in Section 4.2: boundary conditions on ∂V have to be added and the mixed Cauchy- (on V) and boundary- (on ∂V) problem is well posed. Restricting the class of admissible functions to those fulfilling Dirichlet condition (4.16), we obtain (4.17) and, therefore, the boundary term in (4.44) vanishes. This implies the Hamiltonian form of the field evolution within the cone. A treatment similar to the one used in Section 4.2 is also applicable in the space of functions fulfilling Neumann conditions on ∂V.

The situation changes drastically if we pass to sections of the light cone, setting $V = [-1, 1]$. Having chosen Cauchy data (φ, π) on V at a given instant of time, say τ_0, we still have the freedom to chose boundary data on ∂V, *i.e.*, the values $\varphi(\tau, -1)$ and $\varphi(\tau, 1)$, but *only for* $\tau \leq \tau_0$. Indeed, with this whole set of Cauchy and boundary data, the scalar field f is uniquely determined within the entire light cone $\mathscr{C}^-$. Consequently, the values $\varphi(\tau, -1)$ and $\varphi(\tau, 1)$ for $\tau > \tau_0$ are uniquely determined by the Cauchy data and *cannot be chosen* *freely*. Even if we take trivial boundary data in the past, the boundary term in (4.43) might cease to vanish at an instant of time $\tau_0 + \epsilon$, and things can be arranged so that this happens arbitrarily close to τ_0. At a first glance there is no way to obtain a Hamiltonian evolution of Cauchy data in this case.

The remedy for these difficulties consists in treating the data on the boundary of the light cone not as *boundary* data, but as a further piece of Cauchy data. For this purpose, we extend the parameterization Equations (4.33)-(4.34) to $|\xi| > 1$ setting:

$$t = -x := -e^{-\tau+\xi-1} \qquad \text{for } \xi > 1 \;, \tag{2.51}$$

$$t = x \quad := -e^{-(\tau+\xi-1)} \qquad \text{for } \xi < -1 \;, \tag{2.52}$$

and we consider the data (φ, π) on the entire surface $\Sigma = \{\xi \in \mathbb{R}^1\}$. Within the interior of the light cone, *i.e.*, for $|\xi| < 1$, the dynamics is governed by the Hamiltonian (4.50):

$$H_{[-1,1]}(\varphi, \pi) = \frac{1}{2} \int_{-1}^{1} \left\{ (\pi - \xi \partial_\xi \varphi)^2 + (1 - \xi^2)(\partial_\xi \varphi)^2 \right\} d\xi \;, \tag{2.53}$$

which, as we have already seen, satisfies the correct Hamiltonian equation for the wave equation, modulo the boundary term in (4.44) which will be taken care of by the considerations that follow. Outside of the interval $\xi \in [-1, 1]$, the dynamics reduces to translations tangent to the hypersurface on which the data are given, as discussed in the previous section. The only difference here is that the relevant part of the phase spaces "lives" on a null rather than a spacelike hypersurface, which plays no role in the considerations of Section 4.3. More precisely, equation (4.51) implies that we have $X = \partial_\tau = \partial_\xi$ for $\xi < -1$, whereas (4.52) implies: $X = \partial_\tau = -\partial_\xi$ for $\xi > 1$. Consequently, we have:

$$\mathscr{L}_X \varphi = -\partial_\xi \varphi \;, \qquad \mathscr{L}_X \pi = -\partial_\xi \pi \qquad \text{for } \xi > 1 \;, \tag{2.54}$$

and

$$\mathscr{L}_X \varphi = \partial_\xi \varphi \;, \qquad \mathscr{L}_X \pi = \partial_\xi \pi \;, \qquad \text{for } \xi < -1 \;, \tag{2.55}$$

which are special cases of the formula (4.29). According to the standard procedure, described in the next chapter (see also footnote 2, page 39), the momentum π on Σ is taken as the pull-back to the Cauchy surface, of the differential (odd) form $\pi^\mu \partial_\mu \rfloor d\xi^0 \wedge d\xi^1$, with π^μ is given by formula (4.39). According to (4.30), the contribution to the Hamiltonian of the field contained in the region $[1, \infty)$ equals

$$H_{[1,\infty)}(\varphi, \pi) = \int_{1}^{\infty} (-\pi \partial_\xi \varphi) \, d^3x \;, \tag{2.56}$$

whereas the corresponding contribution from the region $[1, \infty)$ is equal to

$$H_{(-\infty,-1]}(\varphi, \pi) = \int_{-\infty}^{-1} (+\pi \partial_\xi \varphi) \, d^3x \;. \tag{2.57}$$

Let us prove that the functional H, equal to the sum of these contributions,

$$H := H_{(-\infty,-1]} + H_{[-1,1]} + H_{[1,\infty)} \ , \tag{2.58}$$

satisfies the equation (4.2) defining a Hamiltonian for the joint dynamical system, given by (4.55) for $\xi < -1$, by (4.54) for $\xi > 1$ and by Equations (4.48)-(4.49) for $-1 < \xi < 1$. Indeed, a variation of H gives us the sum of two formulae of the type (4.31), for $\xi < -1$ and $\xi > 1$ respectively, together with formula (4.44) for $-1 < \xi < 1$. This means that we have:

$$\begin{aligned} -\delta H(\varphi, \pi) = & \int_{\Sigma} \left(\mathscr{L}_X \pi \delta\varphi - \mathscr{L}_X \varphi \delta\pi \right) \, d\xi \\ & + \left[\pi^1 \delta\varphi \right]_{-\infty}^{-1} + \left[\pi^1 \delta\varphi \right]_{-1}^{1} + \left[\pi^1 \delta\varphi \right]_{1}^{\infty} \ , \end{aligned} \tag{2.59}$$

with appropriate values for $(\mathscr{L}_X\varphi, \mathscr{L}_X\pi)$ in the respective regions of Σ. But the intermediate boundary terms at $\xi = -1$ and $\xi = 1$ cancel because of the continuity of ξ and π^1. Assuming sufficiently strong fall-of conditions for the Cauchy data (*e.g.*, assuming that they are compactly supported on Σ) we also obtain a cancelation of the boundary terms at both infinities. What remains is the desired Hamiltonian formula for the total dynamics on Σ:

$$-\delta H(\varphi, \pi) = \int_{\Sigma} \left(\mathscr{L}_X \pi \delta\varphi - \mathscr{L}_X \varphi \delta\pi \right) \, d\xi \ . \tag{2.60}$$

The continuity of π^1, which if fundamental for the cancelation of the intermediate boundary terms in (4.59), may be roughly explained as follows: π^1 is a component of the vector density π^μ, corresponding to the family of hypersurfaces $\xi^1 = \xi = \text{const}$. This vector density is defined, and continuous, on the entire space-time (which in our case is the light cone $\mathscr{C}^-$). At boundary points $\xi = -1$ and $\xi = 1$ there is no jump in the field of tangents to the surfaces $\xi = \text{const.}$ and, therefore, π^1 is continuous as well. We note that there are various delicate issues concerning the *exterior* orientation of the hypersurfaces involved, which are discussed in the next chapter and in Appendix A.

2.5 The Trautman–Bondi energy for the scalar field

Formula (4.60) enables us to describe the dynamics of a massless scalar field within the light cone in terms of a Hamiltonian dynamical system in an abstract space $\mathbb{R}^2$, parameterized by the "generalized time parameter" τ and the "generalized space parameter" ξ. Cauchy data are given on the Cauchy surfaces $\Sigma = \{\xi^0 = \tau = \text{const.}\}$. The phase space $\mathscr{P} = \{(\varphi, \pi)\}$ is defined as the collection of compactly supported fields on Σ. The function φ is supposed to be continuous and piecewise smooth. On the other hand, the momentum

$\pi = \pi^0$ might fail to be continuous at boundary points $\xi = -1$ and $\xi = 1$, because the Cauchy surfaces $\Sigma = \{\xi^0 = \tau = \text{const}\}$ "change direction" in a non-continuous way there. Consequently, we assume that π is piecewise continuous in the three regions of Σ separately. Moreover, $\pi^0 \equiv \pi$ has to fulfill the following constraint:

$$\pi = -\dot{\varphi} = \partial_\xi \varphi \quad \text{for } \xi < -1 \; , \tag{2.61}$$

$$\pi = \dot{\varphi} \quad = -\partial_\xi \varphi \quad \text{for } \xi > 1 \; . \tag{2.62}$$

This can be seen from Equation (4.47) and from the fact that π^1 coincides with $\pi = \pi^0$ for $|\xi| > 1$; this last property holds because the hypersurfaces $\{\xi^1 = \text{const}\}$ coincide with the hypersurfaces $\{\xi^0 = \text{const}\}$ there.

The above constraints imply the following formulae for the Hamiltonian on the light cone:

$$H_{(-\infty,-1]} = \int_{-\infty}^{-1} (\partial_\xi \varphi)^2 \, d^3x \; , \tag{2.63}$$

and

$$H_{[1,\infty)} = \int_{1}^{\infty} (\partial_\xi \varphi)^2 \, d^3x \; . \tag{2.64}$$

(The reader is referred to Equations (6.38)-(6.40) for an explicit calculation of the associated variational formulae in a similar context.)

The evolution with respect to the field $X = \partial_\tau$ is determined by the Hamiltonian formula (4.60), where the Hamiltonian is defined by (4.58). The Hamiltonian system obtained this way is autonomous, because the Hamiltonian does not depend explicitly on time. Hence, it is conserved during the evolution. But formulae (4.63) and (4.64) prove that $H_{(-\infty,-1]}$ and $H_{[1,\infty)}$ are monotonically increasing functions of time. Indeed, their values are equal to the integral of a non-negative function $(\partial_\xi \varphi)^2$ over a portion of the boundary $\partial\mathscr{C}^-$ of the cone which grows when time increases. This implies that $H_{[-1,1]}$ must be a monotonically decreasing function of time. We see that, due to radiation, the energy is being transferred from the "Cauchy zone": $[-1, 1]$, to the "radiation zone": $[-\infty, -1] \cup [1, \infty]$.

The real radiation problem is obtained when the boundary $\partial\mathscr{C}^-$ is moved to infinity. By analogy with similar constructions done in general relativity, in such a case the amount of energy $H_{[-1,1]}$ contained in the Cauchy zone will be called the Trautman-Bondi energy of the scalar field. As we shall see in the sequel, the properties of $H_{[-1,1]}$ are analogous to the decreasing properties of the Trautman-Bondi mass in general relativity, irrespective of the limiting transition mentioned.

It should be pointed out that in the simple model of this section we have considered field configurations defined globally on $\mathscr{C}^-$. For various reasons, discussed below, it is useful to consider situations in which this is not the case. This introduces some supplementary complications, which are taken care of in the remainder of this monograph.

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