The Hamiltonian mass and asymptotically anti-de Sitter space-times

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Abstract: We give a Hamiltonian definition of mass for asymptotically hyperboloidal Riemannian manifolds, or for spacelike hypersurfaces in space-times with metrics which are asymptotic to the anti-de Sitter one.

1 Introduction

In classical mechanics the energy is most conveniently defined, up to a constant, via Hamilton's equations of motion,

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}\ , \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}\ , \tag{1}$$

or, equivalently,

$$-dH = \frac{dp_i}{dt}dq^i - \frac{dq^i}{dt}dp_i\ . \tag{2}$$

The textbook generalization of (2) to the theory of a set of fields φ^A on Minkowski space-time is,

$$-\delta H = \int_{\{x^0=\mathrm{const}\}} \frac{\partial \pi_A}{\partial x^0}\delta\varphi^A - \frac{\partial \varphi^A}{\partial x^0}\delta\pi_A\ , \tag{3}$$

the symbol δ denoting a variation of fields. If the field equations arise from a Lagrange function $\mathcal{L}(\varphi^A, {\varphi^A}_{,\mu})$, then a Hamiltonian is given by the formula

$$H = \int_{\{x^0=\mathrm{const}\}} \pi_A \frac{\partial \varphi^A}{\partial x^0} - \mathcal{L}\ , \tag{4}$$

with π_A related to the field φ^A through the equation

$$\pi_A = \frac{\partial \mathcal{L}}{\partial {\varphi^A}_{,0}}\ . \tag{5}$$

On every path connected component of the phase space any other Hamiltonian differs from H given by (4) by a constant.

The generalization of those standard facts to geometric field theories, due to Kijowski and Tulczyjew [14], is perhaps somewhat less familiar: here the hypersurface $\{x^0 = \text{const.}\}$ is replaced by an arbitrary hypersurface $\mathscr{S}$ in the space-time manifold $\mathscr{M}$, the field momentum π_A is replaced by a collection of momenta ${\pi_A}^\mu$ which, again for a Lagrangian theory of first order, are related to the field as

$$ {\pi_A}^\mu = \frac{\partial \mathcal{L}}{\partial {\varphi^A}_{,\mu}} \ , \tag{6} $$

while the partial time derivative $\partial/\partial x^0$ is, typically,[1] replaced in (3)-(4) by the Lie derivative $\mathscr{L}_X$ along the flow of some chosen vector field X. Equations (4)-(5) become [14]

$$ -\delta H = \int_{\mathscr{S}} \mathscr{L}_X \pi_A \delta\varphi^A - \mathscr{L}_X \varphi^A \delta\pi_A + \int_{\partial\mathscr{S}} X^{[\mu}\pi_A^{\nu]}\delta\varphi^A \, dS_{\mu\nu} \ , \tag{7} $$

$$ H = \int_{\mathscr{S}} \left({\pi_A}^\mu \mathscr{L}_X \varphi^A - \mathcal{L} X^\mu \right) dS_\mu \ , \tag{8} $$

and to obtain a Hamiltonian dynamical system one needs to handle the boundary terms appearing in (7), *e.g.* by imposing boundary conditions on the fields. Specializing to vacuum general relativity, and using as a field variable the metric density

$$ \mathfrak{g}^{\mu\nu} := \frac{1}{16\pi}\sqrt{-\det g}\; g^{\mu\nu} \ , \tag{9} $$

one is then led to the following equations [12, 13] (*cf.* also [4, 6])

$$ -\delta H \ = \ \int_{\mathscr{S}} \left(\mathscr{L}_X {p^\lambda}_{\mu\nu} \delta\mathfrak{g}^{\mu\nu} - \mathscr{L}_X \mathfrak{g}^{\mu\nu} \delta {p^\lambda}_{\mu\nu} \right) dS_\lambda + \int_{\partial\mathscr{S}} X^{[\mu} {p^{\nu]}}_{\alpha\beta} \delta\mathfrak{g}^{\alpha\beta} \, dS_{\mu\nu} \ , \tag{10} $$

$$ H(X, \mathscr{S}) \ = \ \int_{\mathscr{S}} (p^\mu_{\alpha\beta} \mathcal{L}_X \mathfrak{g}^{\alpha\beta} - X^\mu \mathcal{L}) \, dS_\mu \, . \tag{11} $$

There is actually a problem here, related to the fact that there is no invariant Lagrangian depending upon the metric and its first derivatives only. This can be taken care of by introducing a background metric b, and removing from the usual Hilbert Lagrangian R a complete divergence:

$$ \mathfrak{g}^{\mu\nu} R_{\mu\nu} - 2\Lambda \frac{\sqrt{-\det g_{\mu\nu}}}{16\pi} = -\partial_\alpha \left(\mathfrak{g}^{\mu\nu} p^\alpha_{\mu\nu} \right) + \mathcal{L} \ , $$

where $\mathcal{L}$ depends now upon the physical metric, its first derivatives, as well as upon the background metric and its derivatives up to order two. Here

$$ p^\alpha_{\mu\nu} := \left(B^\alpha_{\mu\nu} - \delta^\alpha_{(\mu} B^\kappa_{\nu)\kappa} \right) - \left(\Gamma^\alpha_{\mu\nu} - \delta^\alpha_{(\mu} \Gamma^\kappa_{\nu)\kappa} \right) \ , \tag{12} $$

with $B^\alpha_{\mu\nu}$ – the Levi-Civita connection of the metric b. Assuming X to be a Killing vector field of the background b, somewhat lengthy calculations [4, 7] lead from (11) to[2]

$$ H(X, \mathscr{S}, b) \ = \ \frac{1}{2} \int_{\partial\mathscr{S}} \mathbb{U}^{\alpha\beta} dS_{\alpha\beta} \ , \tag{13} $$

[1] More general situations can also be considered, as described in [6].

[2] The integral over $\partial\mathscr{S}$ should be understood by a limiting process, as the limit as R tends to infinity of integrals over the sets $t = 0$, $r = R$. $dS_{\alpha\beta}$ is defined as $\frac{\partial}{\partial x^\alpha} \lrcorner \frac{\partial}{\partial x^\beta} \lrcorner dx^0 \wedge \cdots \wedge dx^n$, with $\lrcorner$ denoting contraction; g stands for the space-time metric unless explicitly indicated otherwise. Square brackets denote antisymmetrization with an appropriate numerical factor (1/2 for two indices), and $\mathring{\nabla}$ denotes covariant differentiation *with respect to the background metric b*. The summation convention is used throughout. We use Greek indices for coordinate components and lower-case Latin indices for the tetrad ones; upper-case Latin indices run from 2 to n and are associated to frames on ${}^{n-1}M$.

$$\mathbb{U}^{\nu\lambda} = \mathbb{U}^{\nu\lambda}{}_{\beta}X^{\beta} + \frac{1}{8\pi}\left(\sqrt{|\det g_{\rho\sigma}|}\; g^{\alpha[\nu} - \sqrt{|\det b_{\rho\sigma}|}\; b^{\alpha[\nu}\right)\delta^{\lambda]}_{\beta}\mathring{\nabla}_{\alpha}X^{\beta} , \tag{14}$$

$$\mathbb{U}^{\nu\lambda}{}_{\beta} = \frac{2|\det b_{\mu\nu}|}{16\pi\sqrt{|\det g_{\rho\sigma}|}} g_{\beta\gamma}\mathring{\nabla}_{\kappa}\left(e^2 g^{\gamma[\nu}g^{\lambda]\kappa}\right) , \tag{15}$$

$$e = \sqrt{|\det g_{\rho\sigma}|}/\sqrt{|\det b_{\mu\nu}|} . \tag{16}$$

In (13) we have added b to the list of arguments of H to emphasize its potential dependence upon the background b.

2 Spacelike hypersurfaces in asymptotically anti-de Sitter space-times

We consider from now on a strictly negative cosmological constant; see [9] for an alternative Hamiltonian treatment of that case. Let $\mathscr{S}$ be an n-dimensional spacelike hypersurface in a $n+1$-dimensional Lorentzian space-time $(\mathscr{M}, g)$. Suppose that $\mathscr{M}$ contains an open set $\mathscr{U}$ which is covered by a finite number of coordinate charts (t, r, v^A), with $r \in [R, \infty)$, and with (v^A) — local coordinates on some compact $n-1$ dimensional manifold ${}^{n-1}M$, such that $\mathscr{S} \cap \mathscr{U} = \{t = 0\}$. Assume that the metric g approaches a background metric b of the form

$$b = -a^{-2}(r)dt^2 + a^2(r)dr^2 + r^2 h , \qquad h = h_{AB}(v^C)dv^A dv^B , \tag{17}$$

with $a(r) = 1/\sqrt{r^2/\ell^2 + k}$, where $k = 0, \pm 1$, h is a Riemannian Einstein metric on ${}^{n-1}M$ with Ricci scalar $n(n-1)k$, and ℓ is a strictly positive constant related to the cosmological constant Λ by the formula $2\Lambda = -n(n-1)/\ell^2$. For example, if h is the standard round metric on S^2 and $k = 1$, then b is the anti-de Sitter metric. It seems that the most convenient way to make the approach rates precise is to introduce an orthonormal frame for b,

$$e_0 = a(r)\partial_t , \qquad e_1 = \frac{1}{a(r)}\partial_r , \quad e_A = \frac{1}{r}\mathring{e}_A , \tag{18}$$

with $\mathring{e}_A$ — an h-orthonormal frame on $({}^{n-1}M, h)$, so that $b_{ab} = b(e_a, e_b) = \eta_{ab}$ — the usual Minkowski matrix $\mathrm{diag}(-1, +1, \cdots, +1)$. We then require that the frame components g_{ab} of g with respect to the frame (18) satisfy

$$e^{ab} = O(r^{-\beta}) , \quad e_a(e^{bc}) = O(r^{-\beta}) , \quad b_{ab}e^{ab} = O(r^{-\gamma}) , \tag{19}$$

where $e^{ab} = g^{ab} - b^{ab}$, with

$$\beta > n/2 , \qquad \gamma > n . \tag{20}$$

(The $n+1$ dimensional generalizations of the Kottler metrics (sometimes referred to as "Schwarzschild-anti de Sitter" metrics) are of the form (17) with

$$a(r) = 1/\sqrt{r^2/\ell^2 + k - 2\eta/r}$$

for a constant η, and thus satisfy (19) with $\beta = n$, and with $\gamma = 2n$.) One can check (*cf.* [7]) that we have the following asymptotic behaviour of the frame components of the b-Killing vector fields,

$$X^a = O(r) , \quad \mathring{\nabla}_a X^b = O(r) .$$

Assuming that $\mathscr{L}_X p^\lambda{}_{\mu\nu}$ and $\mathscr{L}_X \mathfrak{g}^{\mu\nu}$ have the same asymptotic behaviour as $\delta p^\lambda{}_{\mu\nu}$ and $\delta\mathfrak{g}^{\mu\nu}$ (which is equivalent to requiring that the dynamics preserves the phase space), it is then easily seen that under the asymptotic conditions (19)-(20) the volume integrals appearing in (10)-(11) are convergent, the undesirable boundary integral in the variational formula (10) vanishes, so that the integrals (13) do indeed provide Hamiltonians on the space of fields satisfying (19)-(20). (Assuming (19)-(20) and $X = \partial_t$, the numerical value of the integral (13) coincides with that of an expression proposed by Abbott and Deser [1]). This singles out the charges (13) amongst various alternative expressions because Hamiltonians are uniquely defined, up to the addition of a constant, on each path connected component of the phase space. The key advantage of the Hamiltonian approach is precisely this uniqueness property, which does not seem to have a counterpart in the Noether charge analysis [15] (*cf.*, however [11, 16]), or in Hamilton-Jacobi type arguments [3].

To define the integrals (13) we have fixed a model background metric b, as well as an orthonormal frame as in (18); this last equation required the corresponding coordinate system (t, r, v^A) as in (17). Hence, the background structure necessary for our analysis consists of a *background metric* and a *background coordinate system*. This leads to a *potential coordinate dependence* of the integrals (13): let g be any metric such that its frame components g^{ab} tend to η^{ab} as r tends to infinity, in such a way that the integrals $H(\mathscr{S}, X, b)$ given by (13) converge. Consider another coordinate system $(\hat{t}, \hat{r}, \hat{v}^A)$ with the associated background metric $\hat{b}$:

$$\hat{b} = -a^{-2}(\hat{r})d\hat{t}^2 + a^2(\hat{r})d\hat{r}^2 + \hat{r}^2\hat{h} \ , \qquad \hat{h} = h_{AB}(\hat{v}^C)d\hat{v}^A d\hat{v}^B \ ,$$

together with an associated frame $\hat{e}^a$,

$$\hat{e}_0 = a(\hat{r})\partial_{\hat{t}} \ , \qquad \hat{e}_1 = \frac{1}{a(\hat{r})}\partial_{\hat{r}} \ , \quad \hat{e}_A = \frac{1}{\hat{r}}\hat{\mathring{e}}_A \ , \tag{21}$$

and suppose that in the new hatted coordinates the integrals defining the Hamiltonians $H(\widehat{\mathscr{S}}, \hat{X}, \hat{b})$ converge again. An obvious way of obtaining such coordinate systems is to make a coordinate transformation

$$t \to \hat{t} = t + \delta t \ , \ r \to \hat{r} = r + \delta r \ , \ v^A \to \hat{v}^A = v^A + \delta v^A \ , \tag{22}$$

with $(\delta t, \delta r, \delta v^A)$ decaying sufficiently fast:

$$\begin{aligned} \hat{t} = t + O(r^{-1-\beta}) \ , &\quad e_a(\hat{t}) = \ell\, \delta^0_a + O(r^{-1-\beta}) \ , \\ \hat{r} = r + O(r^{1-\beta}) \ , &\quad e_a(\hat{r}) = \tfrac{\delta^1_a}{\ell} + O(r^{1-\beta}) \ , \\ \hat{v}^A = v^A + O(r^{-1-\beta}) \ , &\quad e_a(\hat{v}^A) = \delta^A_a + O(r^{-1-\beta}) \ , \end{aligned} \tag{23}$$

and with analogous conditions on second derivatives; this guarantees that the hatted analogue of Equations (19) and (20) will also hold. In [7] the following is proved:

- All backgrounds satisfying the requirements above and preserving $\mathscr{S}$ (so that $\hat{t} = t$) differ from each other by a coordinate transformation of the form (23). Equivalently, coordinate transformations compatible with our fall-off conditions are compositions of (23) with an isometry of the background. (This is the most difficult part of the work in [7].)

- Under the coordinate transformations (23) the integrals (13) remain unchanged:

$$H(\mathscr{S}, X, b) = H(\widehat{\mathscr{S}}, \hat{X}, \hat{b}) \ .$$

Here, if $X = X^\mu(t,r,v^A)\partial_\mu$, then the vector field $\hat X$ is defined using the *same* functions X^μ of the *hatted* variables.

- The conditions (20) are optimal[3], in the sense that allowing $\beta = n/2$ leads to a background-dependent numerical value of the Hamiltonian.
- For some topologies of ${}^{n-1}M$, isometries of b lead to interesting, non-trivial transformation properties of the mass integrals $H(\mathscr{S},X,b)$, which have to be accounted for when defining a single number called mass. More precisely, if ${}^{n-1}M$ is negatively curved, a geometric invariant is obtained by setting
$$m = H(\mathscr{S},\partial_t,b)\ . \tag{24}$$
If ${}^{n-1}M$ is a flat torus, then any choice of normalization of the volume of ${}^{n-1}M$ leads again to an invariant via (24). If ${}^{n-1}M = S^{n-1}$, then the group G of isometries of b preserving $\{t=0\}$ is the Lorentz group $O(n,1)$, which acts on the space $\mathscr{K}^\perp$ of b-Killing vectors normal to $\{t=0\}$ through its usual defining representation, in particular $\mathscr{K}^\perp$ is equipped in a natural way with a G-invariant Lorentzian scalar product $\eta^{(\mu)(\nu)}$. Choosing a basis $X_{(\mu)}$ of $\mathscr{K}^\perp$ and setting
$$m_{(\mu)} = H(\mathscr{S},X_{(\mu)},b)\ , \tag{25}$$
the invariant mass is obtained by calculating the Lorentzian norm of $m_{(\mu)}$:
$$m^2 := |\eta^{(\mu)(\nu)} m_{(\mu)} m_{(\nu)}|\ . \tag{26}$$

3 The mass of asymptotically hyperboloidal Riemannian manifolds

In the asymptotically flat case the mass is an object that can be defined purely in Riemannian terms [2], *i.e.*, without making any reference to a space-time, and this remains true in the asymptotically hyperboloidal case. The situation is somewhat more delicate here, because the transcription of the notion of a *space-time background Killing vector field* to a purely Riemannian setting requires more care. The Riemannian information carried by space-time Killing vector fields of the form $X = Ve_0$, where e_0 is a unit normal to the hypersurface $\mathscr{S}$, is encoded in the function V, which for vacuum backgrounds satisfies the set of equations
$$\Delta_b V + \lambda V = 0\ , \tag{27}$$
$$\mathring{D}_i \mathring{D}_j V = V(\mathrm{Ric}(b)_{ij} - \lambda b_{ij})\ , \tag{28}$$
where $\mathring{D}$ is the Levi-Civita covariant derivative of b and λ is a constant. We can forget now that $\mathscr{S}$ is a hypersurface in some space-time, and consider an n-dimensional Riemannian manifold $(\mathscr{S},g)$ together with the set, denoted by $\mathcal{N}_b$, of solutions of (27)-(28); we shall assume that $\mathcal{N}_b \neq \emptyset$. If one imposes boundary conditions in the spirit of (18)-(20) on the Riemannian metric g, *except that the condition there on the space-time trace $b_{ab}g^{ab}$ is not needed any more*, then well defined global geometric invariants can be extracted — in a way similar to that discussed at the end of the previous section — from the integrals
$$H(V,b) := \lim_{R\to\infty} \int_{r=R} \mathbb{U}^i(V)\,dS_i \tag{29}$$

[3] Strictly speaking, it is the Riemannian counterpart of (20) that is optimal, see [5].

where $V \in \mathcal{N}_b$ and [5]

$$\mathbb{U}^i(V) := 2\sqrt{\det g}\ \left(V g^{i[k} g^{j]l} \mathring{D}_j g_{kl} + D^{[i} V g^{j]k}(g_{jk} - b_{jk})\right) \ . \tag{30}$$

If ${}^{n-1}M$ is an $(n-1)$-dimensional sphere, and if the manifold $\mathscr{S}$ admits a spin structure, then a positive energy theorem holds [5, 8, 17, 18]; this isn't true anymore for general ${}^{n-1}M$'s, *cf., e.g.*, [10].

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