On "hyperboloidal" Cauchy data for vacuum Einstein equations and obstructions to smoothness of Scri

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September 6, 1999

Abstract

The relationship between the geometric properties of "hyperboloidal" Cauchy data for vacuum Einstein equations at the conformal boundary of the initial data surface and between the space–time geometry is analyzed in detail. We prove that a necessary condition for existence of a smooth or a polyhomogeneous Scri (*i.e.*, a Scri around which the metric is expandanble in terms of $r^{-j}\log^i r$ rather than in terms of r^{-j}) is the vanishing of the shear of the conformal boundary of the initial data surface. We derive the "boundary constraints" which have to be satisfied by an initial data set for compatibility with Friedrich's conformal framework. We show that a sufficient condition for existence of a smooth Scri (not necessarily complete) is the vanishing of the shear of the conformal boundary of the initial data surface and smoothness up to boundary of the conformally rescaled initial data. We also show that the occurrence of some log terms in an asymptotic expansion at the conformal boundary of solutions of the constraint equations is related to the non–vanishing of the Weyl tensor at the conformal boundary.

1 Introduction

In the papers of Bondi *et al.* [3], Sachs [18] and Penrose [17] which laid the foundations of our present understanding of the gravitational radiation, some rather strong hypotheses about the asymptotic behaviour of the gravitational field in lightlike directions were postulated. In a

*On leave of absence from the Institute of Mathematics, Polish Academy of Sciences, Warsaw. Supported in part by KBN grant # 2 1047 91 01.

recent study by ourselves [1] (*cf.* also [2]) of the asymptotic properties of solutions of constraint equations on spacelike hypersurfaces intersecting "Scri" transversally it has, however, been observed that generic Cauchy data constructed in such a setting by the "conformal method" failed to be smoothly extendable, after appropriate rescalings, to the conformal boundary[1] due to the occurrence of some logarithmic terms in an asymptotic expansion of the fields near the conformal boundary. This behaviour was observed even though the "seed fields" used in the conformal method were smoothly extendable to the conformal boundary. The aim of this paper is to analyze in detail the obstructions to smoothness of the solutions of the constraint equation constructed as in [1]. In order to present our results it is useful to define two tensor fields $\sigma^\pm$ defined on the connected component under consideration ∂M_i of the conformal boundary ∂M of the initial data surface:

$$\sigma^\pm_{AB} \equiv \left(\lambda_{AB} - \frac{h^{CD}\lambda_{CD}}{2}h_{AB}\right) \pm \left(K_{AB} - \frac{h^{CD}K_{CD}}{2}h_{AB}\right)\Big|_{\partial M_i}, \tag{1.1}$$

which we shall call the *shear tensors of* ∂M_i, where h_{AB} is the induced metric on ∂M_i, K_{AB} is (the restriction to ∂M of) the extrinsic curvature (K_{ij}) of M in the unphysical space–time (V, γ), and λ_{AB} is the extrinsic curvature of ∂M_i in (M, g). In the case of Cauchy data constructed as described in Section 2, our results linking the geometry of the boundary of the initial data surface with the geometry of the resulting space–time can be summarized as follows:

1. Suppose that neither σ^+ nor σ^- vanishes. Then there exists no development of the initial data with a smooth or polyhomogeneous Scri.
2. Suppose that $\sigma^+ \equiv 0$ or $\sigma^- \equiv 0$; changing the time orientation if necessary we may without loss of generality assume that $\sigma^+ \equiv 0$:
   (a) Suppose moreover that $K^{\log}_{1A}|_{\partial M_i} \not\equiv 0$, where $K^{\log}_{1A}$ denotes "the logarithmic part" of K_{1A}, *cf.* Section 3 for details. If there exists a development with a polyhomogeneous Scri, then it is essentially polyhomogeneous, *i.e.*, no development with a smooth Scri exists.
   (b) Suppose instead that $K^{\log}_{1A}|_{\partial M_i} \equiv 0$.
      i. Suppose moreover that $K^{\log}_{11}|_{\partial M_i} \not\equiv 0$. If there exists a polyhomogeneous Scri, we cannot exclude the possibility that it is inessentially so, *i.e.*, that it can be made smooth by an appropriate change of gauges.
      ii. Suppose finally that we moreover have $K^{\log}_{11}|_{\partial M_i} \equiv 0$. Then there exists a development which admits a smooth conformal boudary.

It should be stressed that the results linking the log terms with the non–vanishing of the Weyl tensor proved here show that the occurrence of shear and of at least some of the log terms in asymptotic expansions of physical fields at Scri is not an artefact of a bad gauge

[1] It has similarly been observed by Christodoulou and Klainerman [6] that generic space–times constructed by evolving "small" Cauchy data asymptotically flat in spacelike directions seem to display asymptotic behaviour incompatible with the Bondi–Sachs–Penrose (BSP) conditions (and this result has apparently been recently rigorously established: D. Christodoulou, private communication). Since the Christodoulou–Klainerman theorem is proved under quite different conditions its conclusions are, however, somewhat different from the ones obtained here.

choice, including the choice of a conformal factor, or of a pathological choice of the initial data hypersurface (within the class of uniformly bounded from above and uniformly bounded away from zero, locally C^2, conformal factors and C^1 deformations of the initial data hypersurface which fix ∂M): if Scri is *not* shear–free (by which we mean that eq. (3.11) does not hold), then no conformal transformation will make it shear free. Similarly if the Weyl tensor does not vanish at ∂M, then no gauge change in the above sense will make it vanish (*cf.* Section 3 for a more detailed discussion).

Let us mention that results somewhat similar to those obtained here have been recently derived in a Bondi–Sachs type setting in [7].

The overall picture that emerges from the results of [6, 1, 7] and from the results here is that the usual hypotheses of smoothness of Scri are overly restrictive. These results seem to indicate that the correct setup for an analysis of the gravitational radiation is that of *polyhomogeneous* rather than smooth functions on the conformally completed manifold, *i.e.*, functions that have asymptotic expansions in terms of powers of x and $\log x$ rather than of x only. It should, however, be stressed that even though the fact that the physical fields $(\tilde{g}, \tilde{K})$ satisfy the constraint equations guarantees the existence of some vacuum development $(V, \tilde{\gamma})$, it is by no means obvious that in the case, say, $\sigma^+ \equiv 0$, the existence of some kind of compactification of $(M, \tilde{g}, \tilde{K})$ implies the existence of some useful conformal completion of $(V, \tilde{\gamma})$. The question of existence of conformal completions of $(V, \tilde{\gamma})$ with three dimensional boundaries is the most important unsolved problem of the present theory.

This paper is organized as follows: In Section 2 we review the results of [1] about existence and structure of solutions of constraint equations satisfying "rough" hyperboloidal conditions[2]. In Section 3 we study the relationship between the Weyl tensor at the conformal boundary and some of the log terms in the Cauchy data, *cf.* Theorem 3.1. We should stress that in that Section we do *not* assume that the data have been constructed by the methods of [1], rather we study a quite general class of initial data satisfying boundary conditions, which are inspired by those that would be satisfied by data constructed in [1]. Nevertheless the class of data considered is significantly larger than that described in Section 2, and we have actually tried to tailor the boundary conditions to be the weakest possible for our analysis to go through without some significant supplementary difficulties. The argument in Section 3 is essentially a careful reexamination of the argument presented by Geroch in [12] assuming a smooth Scri. In that Section we also derive some results which are necessary for further applications, *cf.* Theorem 3.2. It seemed of interest to rederive the results of Section 3 in a more direct manner, using directly the constraint equations: this is done in Section 4. In that Section we show that the rather indirect four–dimensional methods of Section 3 can be replaced by an analysis which is conceptually more straightforward but much more tedious from a calculational point of view, using 3+1 and 2+1+1 decompositions of the fields. In Section 5 we adress the question of existence of a time evolution of the initial data which possesses a polyhomogeneous or smooth Scri. Theorem 5.1 is our main existence result, while Proposition 5.1 is our main non–existence result. Various conventions and definitions are given in Appendices A and E. In Appendix B we show that initial data with a "shear–free" conformal boundary can be embedded in a space–time which is asymptotically Minkowskian in light–like directions. In Appendix C the

[2] We use the qualification "rough" for the boundary conditions of [1] to differentiate them from those that are needed in Friedrich's evolution theorems. It is precisely the object of this paper to demonstrate with what conditions should the "rough" boundary conditions of [1] be supplemented to obtain data suitable for Friedrich's theorems.

notion of the extrinsic curvature of a null hypersurface is shortly discussed. Some calculational arguments needed in section 3 can be found in Appendix D.

Acknowledgements: P.T.C. wishes to thank David Singleton and Robert Bartnik for several useful discussions and comments. He also wishes to thank the Department of Mathematics of Université de Tours for friendly hospitality during the final stage of work on this paper.

2 The conformal method

In this section we shall outline the Lichnerowicz—Choquet–Bruhat—York method for constructing solutions of the constraint equations (2.1)–(2.2) of general relativity, with asymptotic conditions appropriate to the "hyperboloidal" initial value problem [8, 9], *cf.* also [2, 1]. Let $\bar{M}$ be a smooth, connected, Hausdorff, compact three dimensional manifold with boundary, set $M = \mathrm{int}\bar{M}$, $\partial\bar{M} = \partial M \equiv \bar{M}\setminus M$. Given "seed fields" (g_{ij}, A^{ij}), where g_{ij} is any smooth Riemannian metric on M extending smoothly to ∂M, and A^{ij} is any symmetric, traceless tensor field on M extending smoothly to ∂M, a solution $(\tilde{g}, \tilde{K})$ of the general relativistic constraint equations

$$R(\tilde{g}) = \tilde{g}_{ij}\tilde{g}_{kl}\tilde{K}^{ik}\tilde{K}^{jl} - (\tilde{g}_{ij}\tilde{K}^{ij})^2\,, \tag{2.1}$$

$$\tilde{D}_i(\tilde{K}^{ij} - \tilde{g}_{kl}\tilde{K}^{kl}\tilde{g}^{ij}) = 0\,, \tag{2.2}$$

where $\tilde{D}_i$ is the Levi–Civita connection of $\tilde{g}_{ij}$, and $R(f)$ is the Ricci scalar of a metric f, satisfying the condition

$$\tilde{K} \equiv \tilde{g}_{ij}\tilde{K}^{ij} = 3\epsilon, \qquad \epsilon = \pm 1\,, \tag{2.3}$$

can be constructed by the following procedure: Let x be any defining function for ∂M, *i.e.* a function satisfying $x \in C^\infty(\bar{M})$, $x \geq 0$, $x(p) = 0 \Longleftrightarrow p \in \partial M$, with dx non–vanishing at ∂M. Let $A^{ij} \in C^\infty(\bar{M})$ be symmetric traceless and let X be any solution of the equation

$$D_i\left[x^{-3}\left(D^iX^j + D^jX^i - \frac{2}{3}D_kX^kg^{ij}\right)\right] = -D_i\left(x^{-2}A^{ij}\right)\,, \tag{2.4}$$

define

$$L^{ij} \equiv \frac{\omega^2}{x^3}\left(D^iX^j + D^jX^i - \frac{2}{3}D_kX^kg^{ij}\right) + \frac{\omega^2}{x^2}A^{ij}\,, \tag{2.5}$$

where ω is a solution of the equation

$$\omega\Delta_g\omega - \frac{3}{2}|D\omega|^2_g + \frac{1}{4}\omega^2(R(g) - |L|^2_g) + \frac{3}{2} = 0\,, \tag{2.6}$$

satisfying $\omega \geq 0$, ωx^{-1} — uniformly bounded on M from above and uniformly bounded away from zero, where $|L|^2_g = g_{ij}g_{kl}L^{ik}L^{jl}$. Setting

$$\begin{aligned}\tilde{g}_{ij} &= \omega^{-2}g_{ij}\,,\\ \tilde{K}^{ij} &= \omega^3L^{ij} + \epsilon\,\tilde{g}^{ij}\,,\end{aligned}$$

one obtains a solution of (2.2)–(2.1) satisfying (2.3). Moreover when X is chosen appropriately (*cf.* point 1 below) the fields $(\tilde{g}_{ij}, \tilde{K}^{ij})$ satisfy (in a rough way[3]) the asymptotic conditions

[3] For a precise description of various boundary conditions imposed, *cf.* [1]. In the smooth seed fields context discussed here, the boundary condition on X is that $x^{-2}X$ be bounded; this ensures existence and uniqueness; *cf.* also footnote 2.

appropriate to the "asymptotically hyperboloidal" setting (*cf.* [9, 1]). In [1] the following has been shown:

1. For any (g, A) as above one can find a unique solution $X \in x^2 C^\infty(\bar{M}) + x^4 \log x\, C^\infty(\bar{M})$ of (2.4) such that

$$\left(\frac{x}{\omega}\right)^2 L^{ij} = U^{ij} + x^2 \log x\, U^{ij}_{\log}, \qquad U^{ij}, U^{ij}_{\log} \in C^\infty(\bar{M}) .$$

   Given any g there exists an open dense set (in the $C^\infty(\bar{M})$ topology) of A's for which $U^{ij}_{\log}|_{\partial M} \not\equiv 0$ (however, there exists an infinite dimensional closed subspace of A's for which $U^{ij}_{\log}|_{\partial M} \equiv 0$). If $U^{ij}_{\log}|_{\partial M} \equiv 0$, then $U^{ij}_{\log} \equiv 0$ and thus $x^2\omega^{-2}L^{ij} \in C^\infty(\bar{M})$. Let us note for further reference that in an orthonormal frame e_i, as described in Appendix A, if we write, in a neighbourhood of ∂M,

$$L^{ij} = L^{ij}_0(v) + x L^{ij}_1(v) + \cdots ,$$

   where $\{v\}$ are coordinates on ∂M, then we have $L^{1i}_0 \equiv 0$, while both $L^{AB}_0(v) - \frac{1}{2}L^{CD}_0 h_{CD} h^{AB}$ and $L^{AB}_1(v) - \frac{1}{2}L^{CD}_1 h_{CD} h^{AB}$ are freely specifiable tensor fields on ∂M.

2. For any (g, A) as above one can find a solution $\omega \in \mathcal{A}_{phg}$, where $\mathcal{A}_{phg}$ denotes the space of polyhomogeneous functions on $\bar{M}$ (*cf.* Appendix E), of equation (2.6). More precisely, there exists a sequence $\{N_j\}_{j=0}^\infty$ with $N_0 = N_1 = N_2 = N_3 = 0$, $N_4 = 1$ and functions $\omega_{i,j} \in C^\infty(\bar{M})$ such that

$$\omega \sim \sum_{i\geq 0} \sum_{j=0}^{N_i} \omega_{i,j} x^i \log^j x \tag{2.7}$$

   (*cf.* Appendix E for the meaning of "$\sim$"). For an open dense set of (g, A)'s we have $\omega_{4,1}|_{\partial M} \not\equiv 0$. If $\omega_{4,1}|_{\partial M} \equiv 0$, then $\omega \in C^\infty(\bar{M})$.

When $L^{ij} \equiv 0$, the obstruction to smoothness of ω has been analyzed in detail in [2]. In that reference it has been shown in particular that $\omega_{4,1}|_{\partial M}$ vanishes if the Weyl tensor of the unphysical space–time metric is bounded near ∂M (*cf.* [2] for details). It is the object of this paper to extend that analysis to the case $L^{ij} \not\equiv 0$.

3 Weyl tensor vs log terms

In this Section we shall show that under some conditions the vanishing of the Weyl tensor at ∂M is equivalent to the vanishing of some of the log terms in the solutions of the constraint equations. The methods of this Section do not require any understanding of the origin of the log terms, which will be discussed in more detail in the next Section. It should be stressed that we do not assume the condition $\mathrm{tr}\tilde{K} = \mathrm{const}$ (which has been made in [1], as described in Section 2). To obtain sharp results we will consider functions which satisfy some detailed differentiability requirements near the conformal boundary, *cf.* Definition 3.1 below. The reader who is not interested in sharp results might assume that all the functions appearing in the formulas below are polyhomogeneous, thus all the coefficients of expansions in terms of powers of x and $\log x$ are smooth functions of the boundary variables.

Let $(M, \tilde g, \tilde K)$ be an initial data set, thus M is an (open) manifold, $\tilde g$ is a Riemannian metric on M and $\tilde K$ is a symmetric tensor field on M, with $(\tilde g, \tilde K)$ satisfying the constraint equations (2.1)–(2.2). We shall also assume that $M = \mathrm{int}\bar M$, where $\bar M$ is a compact manifold with boundary, and that there exists a function $\omega \in C^0(\bar M) \cap C^3_{\mathrm{loc}}(M)$ such that

$$\tilde g = \omega^{-2} g \quad \text{on} \quad M\,,$$

$$\omega|_{\partial M} = 0\,,$$

and g is a Riemannian metric on $\bar M$ such that $g \in C^0(\bar M) \cap C^3_{\mathrm{loc}}(M)$. If we assume moreover that $\tilde g_{ij} \in H^{k+1}_{\mathrm{loc}}(M)$, $\tilde K^{ij} \in H^k_{\mathrm{loc}}(M)$, with some $k \geq 2$, then it follows from well known theorems [14, 13, 4, 5] that there exists a vacuum space–time $(V, \tilde\gamma)$, with $\tilde\gamma \in C^{k-1}_{\mathrm{loc}}(V)$, together with an embedding $i : M \longrightarrow V$ such that $i^*\tilde\gamma = \tilde g$, and $\tilde K$ represents the extrinsic curvature tensor of $i(M)$ in $(V, \tilde\gamma)$. Given an orthonormal frame $\tilde e_i$, $i = 1, 2, 3$, for $\tilde g$ the four vectors $(\tilde e_a)$, $a = 0, \ldots, 3$, where $\tilde e_0 = \tilde\eta$, the unit normal to $i(M)$ in $(V, \tilde\gamma)$, form an orthonormal tetrad for $\tilde\gamma$. Let Ω be *any* function on V such that

$$\Omega \circ i = \omega\,, \tag{3.1}$$

define

$$\gamma = \Omega^2 \tilde\gamma\,.$$

In what follows we shall identify M with $i(M)$ without explicitly mentioning this fact, and we shall always assume that (3.1) holds.

Following Geroch [12] we define

$$S_{ab} = R_{ab}(\gamma) - \frac{1}{6} R(\gamma)\gamma_{ab} \tag{3.2}$$

$$\mathcal{G}^a{}_b = R^a{}_b(\tilde\gamma) - \frac{1}{6} R(\tilde\gamma)\delta^a{}_b \tag{3.3}$$

where $R_{ab}(f)$, $R(f)$ denote the Ricci tensor and the Ricci scalar of the metric f; we have [12]

$$R_{abcd} = C_{abcd} + \gamma_{a[c}S_{d]b} - \gamma_{b[c}S_{d]a}\,, \tag{3.4}$$

where R_{abcd} and C_{abcd} are the Riemann tensor and the Weyl tensor of the metric γ_{ab}, and

$$\begin{aligned} S_{ab} &= -\frac{2}{\Omega}\nabla_a\nabla_b\Omega + \frac{\nabla^c\Omega\nabla_c\Omega}{\Omega^2}\gamma_{ab} + \frac{1}{\Omega^2}\mathcal{G}_{ab}\,, && (3.5)\\ \Big(\mathcal{G}_{ab} &\equiv \gamma_{ac}\mathcal{G}^c{}_b\Big)\,, \\ C_{abcd}\nabla^d\Omega + \Omega\nabla_{[a}S_{b]c} &= \nabla_{[a}\Big(\Omega^{-1}\mathcal{G}_{b]c}\Big) - \Omega^{-2}\gamma_{c[a}\mathcal{G}_{b]d}\nabla^d\Omega\,, && (3.6)\\ \nabla^m\Big(\Omega^{-1}C_{abcm}\Big) &= -\Omega^{-2}\nabla_{[a}\Big(\Omega^{-1}\mathcal{G}_{b]c}\Big) + \Omega^{-4}\gamma_{c[a}\mathcal{G}_{b]d}\nabla^d\Omega && (3.7)\end{aligned}$$

(in this section[4] and in Appendices A and D all indices are tetrad indices, and are raised and lowered with $\gamma_{ab} = \mathrm{diag}(-1, 1, 1, 1)$). If $\tilde\gamma$ is vacuum, then clearly $\mathcal{G}^c{}_b = 0$.

We wish to stress that above and throughout the remainder of this Section we have *not* assumed that any kind of well behaved conformal completion of (V, γ) exists, and that equations (3.5)–(3.7) hold whatever the extension Ω of ω to V.

[4] On the other hand it should be noticed that in Sections 4, 5 and in Appendix B *coordinate* indices are used.

The trace of equation (3.5) gives

$$S \equiv \gamma^{ab} S_{ab} = \frac{4|\nabla\Omega|^2_\gamma}{\Omega^2} - \frac{2}{\Omega}\Box\Omega + \frac{1}{\Omega^2}\gamma^{ab}\mathcal{G}_{ab} \,. \tag{3.8}$$

Let us assume that near a connected component ∂M_i of ∂M, we have

$$\omega \in C^1(\bar{M}), \qquad \gamma^{ab}\mathcal{G}_{ab} = o(1)$$

and that we have chosen Ω so that along M we have

$$\Box\Omega = o(\omega^{-1}) \ .$$

The requirement that $S = o(\Omega^{-2})$ gives

$$\lim_{M \ni p \to \partial M_i} |\nabla\Omega|^2_\gamma = 0 \quad \Longrightarrow \quad \omega_0|_{\partial M_i} = \pm |D\omega|_g\Big|_{\partial M_i} \,, \tag{3.9}$$

where

$$\omega_0 \equiv \nabla_0\Omega|_M \,.$$

It follows that if we wish S_{ab} to be $o(\Omega^{-2})$ we need to choose the boundary value of $\nabla_0\Omega|_{\partial M_i}$ so that (3.9) holds. The trace free part of (3.5) reads

$$\begin{aligned} S_{ab} - \frac{1}{4} S\gamma_{ab} \quad &= \quad -\frac{2}{\Omega}\left[\nabla_a\nabla_b\Omega - \frac{1}{4}\Box\Omega\,\gamma_{ab}\right] \\ &\quad + \frac{1}{\Omega^2}\left(\mathcal{G}_{ab} - \frac{1}{4}\gamma^{cd}\mathcal{G}_{cd}\,\gamma_{ab}\right) . \end{aligned} \tag{3.10}$$

Equation (3.10) shows that if $\mathcal{G}_{ab} = o(\Omega)$, then a necessary condition for $(V, \tilde{\gamma})$ to admit a Scri with a $C^2(\bar{V})$ metric is, that there exists on M functions ω_0, ω_{00} such that if we set

$$\nabla_0\Omega|_M = \omega_0 \,, \qquad \nabla_0\nabla_0\Omega|_M = \omega_{00} \,,$$

then we have

$$\lim_{M \ni p \to \partial M_i} \left(\nabla_a\nabla_b\Omega - \frac{1}{4}\Box\Omega\,\gamma_{ab}\right) = 0 \,. \tag{3.11}$$

By an abuse of terminology we shall refer to (3.11) as the statement, that a smooth Scri is shear free in the "gauge" $\Box\Omega = 0$.

Let us recall, that the couples (γ, Ω) and $(\phi^2\gamma, \phi\Omega)$ determine the same physical metric $\tilde{\gamma} = \Omega^{-2}\gamma$. If ϕ is a twice differentiable function uniformly bounded from above on M and uniformly bounded away from zero on M, then a straightforward calculation shows that the condition (3.11) is invariant under such rescalings. As will be seen in Proposition 3.3, if the space–time under consideration admits a Scri which is a null hypersurface, then the geometric meaning of (3.11) is the vanishing of the conformal extrinsic curvature of Scri, as defined in appendix C.

We wish to show that from the Cauchy problem point of view the requirement that equation (3.11) holds will lead to a *constraint* on the Cauchy data: a simple $3+1$ decomposition of (3.5) gives

$$\begin{aligned} S_{ij} - \frac{1}{3} g^{kl} S_{kl}\, g_{ij} \quad &= \quad -\frac{2}{\Omega}\left[\left(K_{ij} - \frac{1}{3} K g_{ij}\right)\nabla_0\Omega + \left(D_i D_j\Omega - \frac{1}{3}\Delta_g\Omega\, g_{ij}\right)\right] + \\ &\quad + \frac{1}{\Omega^2}\left(\mathcal{G}_{ij} - \frac{1}{3} g^{kl}\mathcal{G}_{kl}\, g_{ij}\right) , \end{aligned} \tag{3.12}$$

where K_{ij} is the extrinsic curvature of M in the (unphysical) space–time (V, γ), $K = g^{kl} K_{kl}$ and where Δ_g is the Laplacian of the metric g_{ij}. Equation (3.12) implies:

Proposition 3.1 *Suppose that $g \in C^1(\bar M)$, $\omega \in C^2(\bar M)$, $K \in C^0(\bar M)$, $\mathcal{G}_{ij} = o(\Omega)$. A necessary condition for the existence of a conformal completion $\bar V$ of (V,γ) such that $\gamma \in C^2(\bar V)$ is that*

$$\Big(K_{ij} - \frac{1}{3}K g_{ij}\Big)\Big|_{\partial M} = -\epsilon\Big[\frac{1}{|D\omega|_g}\Big(D_iD_j\omega - \frac{1}{3}\Delta_g\omega\, g_{ij}\Big)\Big]\Big|_{\partial M}, \tag{3.13}$$

$$\epsilon \equiv \Big(\frac{\omega_0}{|D\omega|_g}\Big)\Big|_{\partial M} = \pm 1,$$

where D_i is the Levi–Civita connection of g_{ij}.

It is worthwile to analyze the covariance properties of eq. (3.13): recall that the quadruplets $(\omega, g_{ij}, K^{ij}, \nabla_0\Omega\big|_M)$ and $(\bar\omega, \bar g_{ij}, \bar K^{ij}, \bar\nabla_0\bar\Omega\big|_M)$, where

$$\bar\omega = \phi\,\omega\ , \tag{3.14}$$

$$\bar g_{ij} = \phi^2 g_{ij}\ , \tag{3.15}$$

$$\bar K^{ij} = \phi^{-3}(K^{ij} - \frac{K}{3}g^{ij}) + (\phi^{-1}\frac{K}{3} - \phi^{-2}\phi_0)g^{ij}\ , \tag{3.16}$$

$$\bar\nabla_0\bar\Omega\Big|_M = \nabla_0\Omega\Big|_M + \phi^{-1}\phi_0\omega\ , \tag{3.17}$$

with $\phi \in C^2_{\mathrm{loc}}(M)$, $\phi_0 \in C^1_{\mathrm{loc}}(M)$, determine the same physical Cauchy data set $(\tilde g, \tilde K)$. If ϕ is assumed to be uniformly bounded from above on M and uniformly bounded away from zero on M, then a straightforward calculation shows that the condition (3.13) is invariant under (3.14)–(3.17). Thus, (3.13) is indeed a condition on the physical Cauchy data.

Equation (3.13) is a non–trivial condition, because it is *not* satisfied by generic initial data as constructed in [1]:

Proposition 3.2 *For generic $(g, A) \in C^\infty(\bar M) \oplus C^\infty(\bar M)$, the solution of the constraint equations constructed by the conformal method will* not *satisfy (3.13).*

Remarks:

1. Equation (3.12) shows that S_{ij} will blow up at $\mathcal{I} = \bar V \setminus V$ as Ω^{-1} when (3.13) does not hold. This implies immediately that initial data for which (3.13) does not hold will lead to a space–time $(V, \tilde\gamma)$ which does *not* admit compactifications (V, γ) with a metric $\gamma \in C^2(\bar V)$. It will be seen in Proposition 5.1 below that a stronger result holds: whenever (3.13) fails no conformal completion of the space–time with a polyhomogeneous Scri can exist, *cf.* Section 5 for details.

2. If V admits a boundary such that $\mathcal{I} = \bar V \setminus V$ is a null hypersurface with $\partial M_i \subset \mathcal{I}$, then the equation (3.18) below is equivalent to the vanishing of the conformal extrinsic curvature of $\mathcal{I}$ on ∂M_i (*cf.* Appendix C). This together with Proposition 3.3 shows that in vacuum the equation (3.11) is equivalent to the vanishing of the conformal extrinsic curvature of $\mathcal{I}$ on ∂M_i.

Proof: $2+1$ decomposing (3.13) one finds

$$\left(K_{AB}-\frac{1}{2}h^{CD}K_{CD}h_{AB}\right)\Big|_{\partial M_i}=\epsilon\left(\lambda_{AB}-\frac{1}{2}\lambda\, h_{AB}\right), \tag{3.18}$$

where

$$\lambda \equiv h^{AB}\lambda_{AB} ,$$

the indices A, B, *etc.* run from 2 to 3, h_{AB} is the metric induced on ∂M_i by g, and λ_{AB} is the extrinsic curvature of ∂M_i in (M,g). As can be seen from what is said in Section 2 (and is analyzed in some more detail in [1]), both sides of (3.18) assume (independently) arbitrary values as (g,A) run over $C^\infty(\bar M)\oplus C^\infty(\bar M)$, and the result follows. □

In the Appendix B we show that Cauchy data for which (3.18) holds are asymptotically flat in a quite satisfactory sense. The question whether or not and in which sense a space–time in which (3.13) fails to hold deserves the name of asymptotically flat and/or asymptotically Minkowskian remains to be investigated (*cf.* [7] for a discussion of this question in a Bondi–Sachs framework).

In the remainder of this paper we shall be concerned with the case in which (3.13) holds. In order to do this we need to impose on g and K differentiability conditions consistent with the occurrence of log terms in the solutions of the constraint equations. In order to cover situations in which finite differentiability of space–time fields is appropriate we introduce the following:

Definition 3.1 *For $0<k\in I\!N$ the function $f\in C^k_{\rm loc}(M)\cap C^0(\bar M)$ is said to satisfy the k–condition at ∂M_i, where ∂M_i is a connected component of the boundary ∂M of M, if in local coordinates $y=(x,v^A)$ as described in Appendix E we have*

$$0\le|\alpha|\le k-1 \qquad \lim_{x\to 0}\partial^\alpha_y(x^k\log x\, f)=0\,,$$

$$|\alpha|=k \qquad \lim_{x\to 0}\partial^\alpha_v(x^k\log x\, f)=0\,,$$

$$\lim_{x\to 0}\frac{\partial^k_x(x^k\log x\, f)}{\log x}=k!f\Big|_{\partial M_i}.$$

Remarks:

1. A function f satisfying

$$\forall \quad 0\le|\alpha|\le k \qquad |\partial^\alpha_y f|=o(x^{-|\alpha|}\log^{-1}x)\ ,$$

   will also satisfy the k–condition. In particular if $f\in C^k(\bar M)$, then f satisfies the k–condition.

2. Functions displaying the above behaviour at ∂M_i appear naturally when studying the regularity of solutions of constraint equations near ∂M_i, under finite differentiability hypotheses on the "seed fields" used in the conformal method (*cf.* [1]).

3. Note that if f satisfies the k–condition, then $x^k \log x\, f \in C^{k-1}(\bar M)$, and $x^k f \in C^k(\bar M)$; moreover the function $x^k \log x f$ behaves under differentiation and passing to the limits at ∂M_i in a way somewhat similar as it would if $f \in C^k(\bar M)$. An example of a function which is not in $C^1(\bar M)$ but which satisfies the k–condition for all $k \geq 1$ is given *e.g.* by the functions $\log(x)^{-\alpha}$, with any $\alpha > 0$. Functions of the form $\log(x)^{-\alpha}\phi(v/x)$, $\alpha > 1$, where ϕ is any smooth function compactly supported in a coordinate patch on ∂M (with v – coordinates on a coordinate patch of ∂M extended to a neighbourhood of ∂M in any smooth way) also satisfy the k–condition for any k.

4. This is a condition concerning space–derivatives of f only, no time derivatives are involved.

Before stating the main result of this Section, Theorem 3.1, we need to describe a property of the initial data sets considered:

Proposition 3.3 *Consider a Cauchy data set $(M, \tilde g, \tilde K)$ for a vacuum space–time $(V, \tilde\gamma)$ such that*

$$\tilde g_{ij} = \omega^{-2} g_{ij}\,, \qquad g_{ij} = \hat g_{ij} + x^3 \log x\, \hat g^{\log}_{ij}\,,$$
$$\tilde L^{ij} = \omega^3 L^{ij}\,, \qquad L^{ij} = \hat L^{ij} + x^2 \log x\, \hat L^{ij}_{\log}\,,$$
$$\tilde K = \hat K + x^2 \log x\, \hat K_{\log}\,,$$
$$\omega = \hat\omega + x^4 \log x\, \hat\omega^{\log}\,,$$

where

$$\tilde K \equiv \tilde g_{ij}\tilde K^{ij}\,, \quad \tilde L^{ij} \equiv \tilde K^{ij} - \frac{\tilde K}{3}\tilde g^{ij}\,,$$

with $\hat g_{ij} \in C^3(\bar M)$, $\hat\omega \in C^4(\bar M)$, $\hat L^{ij}, \hat K \in C^2(\bar M)$, and $\hat\omega^{\log}$ satisfies the 4–condition, $\hat g^{\log}_{ij}$ satisfies the 3–condition, and $\hat L^{ij}_{\log}$, $\hat K_{\log}$ satisfy the 2–condition at ∂M_i. Suppose moreover that there exists $\epsilon = \pm 1$ such that

$$\left(\lambda_{AB} - \frac{\lambda}{2} h_{AB}\right)\Big|_{\partial M_i} = \epsilon\left(K_{AB} - \frac{\kappa}{2} h_{AB}\right)\Big|_{\partial M_i}\,, \tag{3.19}$$

$$\lambda \equiv h^{AB}\lambda_{AB}\,, \qquad \kappa \equiv h^{CD} K_{CD}\,. \tag{3.20}$$

Then:

1. *there exists a function $\Omega \in C^4_{\rm loc}(V)$ satisfying $\Omega\Big|_M = \omega$ such that*

$$\lim_{M \ni p \to \partial M_i} |\nabla \Omega|^2_{\gamma} = 0\ ,$$

$$\lim_{M \ni p \to \partial M_i} \left(\nabla_a \nabla_b \Omega - \frac{1}{4}\Box\Omega\, \gamma_{ab}\right) = 0\,;$$

2. *on M there exist functions $\hat{R}_{abcd} \in C^1(\bar{M})$ and $R^{\log}_{abcd}$ satisfying the 1–condition such that*

$$R_{abcd}\Big|_M = \hat{R}_{abcd} + \Omega \log \Omega \, R^{\log}_{abcd} \tag{3.21}$$

(in particular

$$C_{abcd}\Big|_M = \hat{C}_{abcd} + \Omega \log \Omega \, C^{\log}_{abcd} \,, \tag{3.22}$$

$$S_{ab}\Big|_M = \hat{S}_{ab} + \Omega \log \Omega \, S^{\log}_{ab} \,, \tag{3.23}$$

with $\hat{C}_{abcd}, \hat{S}_{ab} \in C^1(\bar{M})$, and $C^{\log}_{abcd}$, $S^{\log}_{ab}$ satisfy the 1–condition). $\hat{R}_{abcd}$ and $R^{\log}_{abcd}$, respectively $\hat{C}_{abcd}$ and $C^{\log}_{abcd}$, can be chosen to have all the symmetries of a Riemann tensor, respectively of a Weyl tensor.

Remarks:

1. It should be stressed that we do not claim that the decompositions (3.21)–(3.23) are preserved under evolution (although we find it likely that this will be the case, at least under the conditions of Section 2). Thus, in all further calculations we assume that (3.21)–(3.23) hold on M only.
2. Since we are working in a frame formalism in this section, when $\hat{g}^{\log}_{ij} = 0$ the positioning of indices on *e.g.* $\hat{L}^{ij}_{\log}$ upstairs or downstairs does not affect the decomposition of L^{ij} into the log part and the regular part. If $\hat{g}^{\log}_{ij} \neq 0$ (or if one uses a coordinate formalism as in some other sections) one should use *e.g.* an orthonormal frame for $\hat{g}$ in all the calculations (or use $\hat{g}$ to raise and lower indices). This will not affect the equality of leading order part of the covariant and contravariant log part of a tensor, but will only lead to differences between some lower order terms in the covariant as compared to contravariant log part. Throughout this paper we shall freely move the index "log" in the subscript or superscript position, writing *e.g.* $\hat{L}^{\log}_{ij}$ for $\hat{g}_{ik}\hat{g}_{j\ell}\hat{L}^{k\ell}_{\log}$, etc.

In the remainder of this paper we shall assume that the choice of the extension Ω of ω from M to V has been made in a way consistent with Proposition 3.3.

The proof of Proposition 3.3 is a straightforward $2+2$ decomposition, and can be found in the Appendix D. Our main result is the following:

Theorem 3.1 *Suppose that the hypotheses of Proposition 3.3 hold and let $(V, \tilde{\gamma})$ be a Cauchy development of $(M, \tilde{g}, \tilde{K})$ such that $\tilde{\gamma} \in C^3_{\rm loc}(V)$. Let ∂M_i be a connected component of ∂M which is diffeomorphic to S^2. Then in any orthonormal frame such that $e_0 \perp M$ we have*

$$\lim_{M \ni p \to \partial M_i} C_{abcd}(p) = 0 \,. \tag{3.24}$$

if and only if

$$\lim_{M \ni p \to \partial M_i} C^{010A}_{\log}(p) = -\frac{1}{2} \lim_{M \ni p \to \partial M_i} S^{1A}_{\log}(p) = 0 \,. \tag{3.25}$$

Remarks:

1. We do not assume $\tilde{K} = \text{const}$, in particular there might be log terms in an asymptotic expansion of $\tilde{K}$ at ∂M_i.

2. Cauchy data $(\tilde{g}, \tilde{K})$ such that (3.18) holds, constructed by the conformal method from generic smooth seed fields, as described in Section 2, will behave near ∂M in a way consistent with the above assumptions. In such a case one moreover has

$$\hat{L}^{AB}_{\log} = -\frac{\hat{L}^{11}_{\log}}{2} h^{AB} \,, \tag{3.26}$$

with $\hat{g}^{\log}_{ij} = 0$, $\hat{g}_{ij} \in C^\infty(\bar{M})$, $\hat{L}^{ij} \in C^\infty(\bar{M})$, while $\hat{\omega}^{\log}$ and $\hat{L}^{ij}_{\log}$ are "polyhomogeneous" near ∂M_i.

3. For $(\tilde{g}, \tilde{K}) \in H^{\mathrm{loc}}_5(M) \oplus H^{\mathrm{loc}}_4(M)$ there exist vacuum developments satisfying $\gamma \in C^3_{\mathrm{loc}}(V)$.

4. The proof below is a careful reexamination of Geroch's version of the Weyl tensor vanishing theorem of Penrose [12].

5. The condition that $(V, \tilde{\gamma})$ is vacuum can be replaced by the conditions

$$\mathcal{G}_{ab}\Big|_M = o(x^2) \,, \tag{3.27}$$

$$\Big[\nabla_{[a}(\Omega^{-1}\mathcal{G}_{b]c})\Big]\Big|_M = o(x^2) \,, \tag{3.28}$$

$$\Big[\gamma_{c[a}\, \mathcal{G}_{b]d}\, \nabla^d \Omega\Big]\Big|_M = o(x^4) \,. \tag{3.29}$$

6. It should be stressed that in Theorem 3.1 we do not assume any uniform bounds on derivatives normal to M of R_{abcd}, $\hat{R}_{abcd}$ and $R^{\log}_{abcd}$.

7. Our arguments below also show that for other topologies (3.25) and (3.24) are not equivalent: from what is said here and from the results of [1] it follows that there exist solutions of constraint equations with $\partial M_i \neq S^2$ which satisfy the hypotheses of Proposition 3.3 (in fact (g, K, ω, ω_0) can even be chosen to be smooth up to ∂M_i) and for which the space–time Weyl tensor does not vanish on ∂M_i.

Theorem 3.1 follows immediately from Proposition 3.3 and from the following:

Lemma 3.1 *Under the hypotheses of Proposition 3.3, the only non–vanishing components of $C_{abcd}|_{\partial M_i}$ are (up to permutations) $C_{0A0B}|_{\partial M_i}$, $C_{0A1B}|_{\partial M_i}$ and $C_{1A1B}|_{\partial M_i}$; with*

$$\lim_{M \ni p \to \partial M_i} C_{0A0B} = \epsilon \lim_{M \ni p \to \partial M_i} C_{0A1B} = \lim_{M \ni p \to \partial M_i} C_{1A1B} \,,$$

$$\epsilon \equiv \lim_{M \ni p \to \partial M_i} \Big[|D\omega|_g^{-1} \nabla_0 \Omega\Big] = \pm 1 \,.$$

$C_{0A0B}|_{\partial M_i}$ is a symmetric traceless tensor field on ∂M_i and satisfies the equation

$$\mathcal{D}_A \left(C^{0A0B}|_{\partial M_i}\right) = -\lim_{M \ni p \to \partial M_i} \Big[|D\omega|_g C^{10B0}_{\log}\Big] = \frac{1}{2} \lim_{M \ni p \to \partial M_i} \Big[|D\omega|_g S^{1B}_{\log}\Big] \,, \tag{3.30}$$

where $\mathcal{D}_A$ is the Levi-Civita connexion of the metric h induced by g on ∂M_i.

Proof: Let Ω, $\hat{R}_{abcd}$, etc., be as given by Proposition 3.3. From equation (3.6) with $a, b = i, j$ and from (3.22)–(3.23), (3.28)–(3.29) one obtains

$$\lim_{M\ni p\to\partial M_i} C_{ijcd}\nabla^d\Omega = 0\,. \tag{3.31}$$

Consider the Bianchi identity

$$-\nabla_0 R_{a0} + \nabla_i R_{ai} = \frac{1}{2}\nabla_a R \quad\Longleftrightarrow\quad -\nabla_0 S_{a0} + \nabla_i S_{ai} = \nabla_a S\,. \tag{3.32}$$

(3.32) with $a = j$ and (3.23) immediately give

$$\nabla_0 S_{0j}\Big|_M = O(\log x)\,.$$

(3.32) inserted in (3.6) with $a = c = 0$ yields

$$\begin{aligned}[C_{0i0j}\nabla^j\Omega]\Big|_M &= -\left[\frac{\Omega}{2}\left(\nabla_0 S_{i0} - \nabla_i S_{0i}\right)\right]\Big|_M + o(1)\\ &= o(1)\,.\end{aligned}$$

From this we can conclude that

$$\lim_{M\ni p\to\partial M_i} C_{0i0j}\nabla^j\Omega = 0\,, \tag{3.33}$$

which implies

$$\lim_{M\ni p\to\partial M_i} C_{0i01} = 0\,. \tag{3.34}$$

Define $\epsilon = \pm 1$ by $(\nabla^a\Omega e_a)\Big|_{\partial M_i} = [|D\omega|_g(\epsilon e^0 + e^1)]\Big|_{\partial M_i}$ ($\Rightarrow \lim_{M\ni p\to\partial M_i}\nabla_0\Omega = \epsilon|D\omega|_g\Big|_{\partial M_i}$). From (3.31) we obtain

$$\lim_{M\ni p\to\partial M_i} C_{ij01} = 0\,, \tag{3.35}$$

$$\lim_{M\ni p\to\partial M_i} C_{ij0A} = \epsilon\lim_{M\ni p\to\partial M_i} C_{ij1A}\,. \tag{3.36}$$

(3.34)–(3.35) imply

$$\lim_{M\ni p\to\partial M_i} C_{ab01} = 0\,. \tag{3.37}$$

Consider *e.g.*

$$\begin{aligned}\lim_{M\ni p\to\partial M_i} C_{0323} &= \lim_{M\ni p\to\partial M_i}[-C_{0020} + C_{0121} + C_{0222} + C_{0323}]\\ &= \lim_{M\ni p\to\partial M_i} C_0{}^a{}_{2a} = 0\,,\end{aligned}$$

which shows that

$$\lim_{M\ni p\to\partial M_i} C_{0ABC} = 0\,. \tag{3.38}$$

(3.36) also gives now

$$\lim_{M\ni p\to\partial M_i} C_{1ABC} = 0\,. \tag{3.39}$$

The identity (A.13) of Appendix A implies

$$\lim_{M\ni p\to\partial M_i} C_{1A1B} = \lim_{M\ni p\to\partial M_i} C_{0A0B}\,, \tag{3.40}$$

so that one also obtains

$$\begin{aligned}
\lim_{M\ni p\to\partial M_i} C_{2323} &= \lim_{M\ni p\to\partial M_i}[-C_{0303}+C_{1313}+C_{2323}+C_{3333}]\\
&= \lim_{M\ni p\to\partial M_i}[C^a{}_{3a3}] = 0 \qquad \Longrightarrow
\end{aligned}$$

$$\lim_{M\ni p\to\partial M_i} C_{ABCD} = 0\,. \tag{3.41}$$

Define

$$\hat C_{AB} \;\equiv \lim_{M\ni p\to\partial M_i} C_{0A0B} = \lim_{M\ni p\to\partial M_i} \hat C_{0A0B}\,.$$

From (3.34)–(3.41) it follows that all nontrivial information about $\lim_{M\ni p\to\partial M_i} C_{abcd}$ is contained in $\hat C_{AB}$, since the only non–vanishing components of $\lim_{M\ni p\to\partial M_i} C_{abcd}$ are C_{0A0B}, C_{1A0B}, C_{1A1B} (and appropriate permutations of the indices thereof), with

$$\lim_{M\ni p\to\partial M_i} C_{0A0B} = \epsilon \lim_{M\ni p\to\partial M_i} C_{0A1B} = \lim_{M\ni p\to\partial M_i} C_{1A1B} = \hat C_{AB}\,.$$

Let us also note that we have

$$\begin{aligned}
h^{AB}\hat C_{AB} &= \lim_{M\ni p\to\partial M_i}[-C_{0000}+C_{1010}+C_{2020}+C_{3030}]\\
&= \lim_{M\ni p\to\partial M_i} C^a{}_{0a0} = 0\,.
\end{aligned}$$

so that $\hat C_{AB}$ is a symmetric traceless tensor on ∂M_i. From (3.7) and (3.22) it follows

$$\lim_{M\ni p\to\partial M_i}[\Omega\nabla_a(\Omega^{-1}\hat C^{a0B0})] = -\lim_{M\ni p\to\partial M_i}\left[C^{a0B0}_{\log}\nabla_a\Omega\right], \tag{3.42}$$

since this equation does not involve derivatives of C^{abcd} in directions normal to M. For $p\in\partial M_i$ let $\gamma_p=\gamma_p(s)$ be an integral curve of e^1 with $\gamma_p(0)=p$, thus for any function f we have $\frac{d}{ds}f(\gamma(s)) = e_1(f)(\gamma(s))$. From l'Hospital rule we have

$$\begin{aligned}
\lim_{s\to 0}\left(\frac{\nabla_a\Omega\,\hat C^{a0B0}}{\Omega}\right)(\gamma(s)) &= \lim_{s\to 0}\left[\frac{e_1(e_a(\Omega))\,\hat C^{a0B0} + \nabla_a\Omega\, e_1(C^{a0B0})}{e_1(\Omega)}\right](\gamma(s)) = \\
&= \lim_{s\to 0}\left[e_1(\hat C^{10B0}) + \frac{e_1(e_a(\Omega))\,\hat C^{a0B0}}{e_1(\Omega)}\right](\gamma(s))\,.
\end{aligned} \tag{3.43}$$

(3.43) shows that (3.42) does not involve derivatives of $\hat C^{a0B0}$ in directions transverse to ∂M_i. A somewhat lengthy but otherwise straightforward $2+2$ decomposition of (3.43) gives

$$\begin{aligned}
\lim_{s\to 0}\left[\Omega\nabla_a(\Omega^{-1}\hat C^{a0B0})\right] &= \mathcal{D}_A\hat C^{AB}\\
&\quad -\lim_{s\to 0}\left\{\frac{1}{e_1(\Omega)}\left[e_1(e_A(\Omega)) + \nabla_0\Omega\,\omega_A{}^0{}_1 + \nabla_1\Omega\,\omega_A{}^1{}_1\right]\right\}\hat C^{AB}\,,
\end{aligned}$$

where $\mathcal{D}_A$ is the Levi–Civita connexion of the metric h induced by g on ∂M_i. Since $\lim_{M\ni p\to\partial M_i} \nabla_1\nabla_A\Omega = 0$, (3.42) gives

$$\mathcal{D}_A\hat{C}^{AB} = -\lim_{M\ni p\to\partial M_i}\left[|D\omega|_g C^{10B0}_{\log}\right]. \tag{3.44}$$

To finish the proof, let us note that (3.22) inserted in (3.6) gives

$$\left(C^{\log}_{0A0d}\nabla^d\Omega\right)\Big|_{\partial M_i} = -\frac{1}{2}\lim_{M\ni p\to\partial M_i}\left(\frac{1}{\log\Omega}\nabla_0 S_{A0}\right). \tag{3.45}$$

From the Bianchi identity (3.32) we have

$$\nabla_0 S_{A0} = \nabla_i S_{Ai} - \nabla_A S\,,$$

so that from (3.23) one obtains

$$\begin{aligned}\lim_{M\ni p\to\partial M_i}\left(\frac{1}{\log\Omega}\nabla_0 S_{A0}\right) &= \lim_{M\ni p\to\partial M_i}\left(\frac{1}{\log\Omega}\nabla_1 S_{A1}\right)\\ &= \left(|D\omega|_g S^{\log}_{1A}\right)\Big|_{\partial M_i}.\end{aligned} \tag{3.46}$$

Equations (3.45) and (3.46) give

$$C^{\log}_{0A01}\Big|_{\partial M_i} = -\frac{1}{2}S^{\log}_{1A}\Big|_{\partial M_i}.$$

which achieves the proof. □

Proof of Theorem 3.1: Clearly, eq. (3.44) shows that $\hat{C}^{AB}$ can vanish only if $C^{10B0}_{\log}$ does. If $\partial M_i \approx S^2$, the converse is also true because in that case the operator $L^{AB}\to\mathcal{D}_A L^{AB}$ which maps symmetric trace free tensors to vector fields is elliptic with trivial kernel. □

Theorem 3.1 holds under more general conditions than those discussed in Section 2, since it allows for non-constant $\tilde{K}$, and for $x^3\log x$ terms in g_{ij}; moreover the structure of the log terms in K_{ij} is much more general than that which is obtained in the setting of Section 2. When $g_{ij}\in C^3(\bar{M})$, the $C^{01A0}_{\log}|_{\partial M_i}$ terms responsible for the non–vanishing of the Weyl tensor at ∂M_i can be related to the log terms in K_{ij} as follows:

Corollary 3.1 *Under the conditions of Theorem 3.1, suppose moreover that $\hat{g}^{\log}_{ij}\equiv 0$ (thus $g_{ij}=\hat{g}_{ij}\in C^3(\bar{M})$), and*

$$\omega_0 = \hat{\omega}_0 + \frac{1}{3!}\omega_0\,\omega^3\log\omega\,\omega^{\log}_0\,, \tag{3.47}$$

$\hat{\omega}_0\in C^3(\bar{M})$, $\omega^{\log}_0$ satisfying the 3-condition. If we write

$$\omega = \hat{\omega} + \frac{1}{4!}\omega^4\log\omega\,\omega^{\log} \tag{3.48}$$

$$K_{ij} = \hat{K}_{ij} + \frac{1}{2}\omega_0\omega^2\log\omega K^{\log}_{ij}\,, \tag{3.49}$$

($\hat{\omega}\in C^4(\bar{M})$, $\hat{K}_{ij}\in C^1(\bar{M})$, $\omega_{\log}$ satisfying the 4-condition, $K^{\log}_{ij}$ — satisfies the 1-condition), then

$$C^{010A}_{\log}\Big|_{\partial M_i} = \frac{1}{2}\left(|D\omega|_g\omega_0 K^{1A}_{\log}\right)\Big|_{\partial M_i}; \tag{3.50}$$

thus (3.24) holds iff $K^{1A}_{\log}\Big|_{\partial M_i} = 0$.

Proof: Inserting the equation (3.49) into (3.5) one finds

$$S^{\log}_{0A}\Big|_{\partial M_i} = -\left[|D\omega|_g\omega_0 L^{\log}_{1A}\right]\Big|_{\partial M_i} . \tag{3.51}$$

□

Theorem 3.1 and Corollary 3.1 show how the $K^{\log}_{1A}$ terms affect the space–time geometry at ∂M. In the remainder of this Section we shall consider the role of the other log terms. Let us start with an identity between the log terms in ω and K_{ij} which holds under the conditions (3.27)–(3.29).

Lemma 3.2 *Under the hypotheses of Corollary 3.1 we have the identity*

$$\omega^{\log}\Big|_{\partial M_i} = \frac{1}{2}\left(K^{\log} - 3K^{\log}_{11} - \frac{4}{3}\tilde{K}^{\log}\right)\Big|_{\partial M_i} , \tag{3.52}$$

with $\omega^{\log}$ as in (3.48), $K^{\log}_{11}$ as in (3.49), and

$$\tilde{K}^{\log} = 3\,(K^{\log} + \omega_0^{\log}) . \tag{3.53}$$

Remark: Equation (3.53) is consistent with (D.11) which gives

$$\tilde{K} = \hat{K} + \frac{1}{3!}\omega_0\,\omega^3\log\omega\tilde{K}^{\log} , \qquad \hat{K} \equiv 3\hat{\omega}_0 + \omega g^{ij}\hat{K}_{ij} .$$

Thus $\tilde{K}^{\log}$ consists of the log terms in the trace of the extrinsic curvature on M of the physical space–time metric. This implies in particular that $\tilde{K}^{\log}$ is invariant under the transformations (3.14)–(3.17) (this can also be checked by a direct calculation under appropriate hypotheses on ϕ and ϕ_0).

Proof: Equation (3.7) gives

$$\lim_{M\ni p\to\partial M_i}\left(\Omega\nabla_a(\Omega^{-1}\hat{C}^{a0cd})\right) = -\lim_{M\ni p\to\partial M_i}\left(\nabla_i\omega\, C^{i0cd}_{\log}\right) , \tag{3.54}$$

By Lemma 3.1 and a calculation similar to that of the proof of Lemma 3.1 one finds that the right hand side of (3.54) with $c = 0$, $d = 1$ vanishes identically (this holds irrespectively of whether or not $C^{0A0B}_{\log}|_{\partial M_i} = 0$). From (3.54) one therefore obtains

$$C^{0101}_{\log}\Big|_{\partial M_i} = 0 .$$

(3.4) gives

$$C_{0101} = R_{0101} + \frac{1}{2}(S_{11} - S_{00}) .$$

On the other hand we have

$$\begin{aligned} R_{0101} &= -S_{11} - \frac{1}{6}R + R_{i1i1} \\ &\Longrightarrow \\ C_{0101} &= -\frac{1}{2}\left(S_{00} + S_{11} + \frac{1}{3}R\right) + R_{i1i1} . \end{aligned}$$

From (A.12) one obtains

$$C^{\log}_{0101}\Big|_{\partial M_i} = -\Big(\frac{1}{2}h^{AB}S^{\log}_{AB} + S^{\log}_{11}\Big)\Big|_{\partial M_i},$$

and (3.65)–(3.66) below give

$$0 = C^{\log}_{0101}\Big|_{\partial M_i} = \left[\frac{|D\omega|^2_g}{2}\Big(K^{\log}_{11} + K^{\log} + \frac{4}{3}\,\omega^{\log}_0 + \frac{2}{3}\,\omega^{\log}\Big)\right]\Big|_{\partial M_i}. \tag{3.55}$$

(3.52) follows from (3.53) and (3.55). □

Recall that the choice of $\omega_0^{\log}$ is arbitrary, so that for fixed $\tilde{K}^{\log}$ the function $K^{\log}$ is arbitrary. It follows that (3.52) imposes a relation between the functions $\omega^{\log}$ and $K^{\log}_{11}$ (determined by the seed fields if the conformal construction has been used) modulo choices of gauges. If, *e.g.*, we have $\tilde{K}^{\log} = 0$, we can choose $\omega_0^{\log} = 0$ so that $K^{\log} = 0$ and then eq. (3.55) reads

$$\omega^{\log}\Big|_{\partial M_i} = -\frac{3}{2}K^{\log}_{11}\Big|_{\partial M_i}. \tag{3.56}$$

This last equation shows in particular, that under the conditions of Section 2 and with this choice of gauges the vanishing of all log terms in K_{ij} implies no log terms in ω as well.

Theorem 3.2 *Under the hypotheses of Proposition 3.3 let $(V,\tilde{\gamma})$ be a Cauchy development of $(M,\tilde{g},\tilde{K})$ such that $\tilde{\gamma}\in C^3_{\rm loc}(V)$; suppose moreover that $g_{ij}\in C^3(\bar{M})$ and*

$$\lim_{M\ni p\to\partial M_i} C_{abcd} = 0\ , \tag{3.57}$$

in an orthonormal frame such that $e_0 \perp M$. Then the function ω can be extended from M to a function Ω on V (locally of the same differentiability class as ω) such that

$$C^{\log}_{abcd}\Big|_{\partial M_i} = 0 \tag{3.58}$$

if and only if

$$\tilde{K}^{\log}\Big|_{\partial M_i} = 0\,, \tag{3.59}$$

$$K^{\log}_{1A}\Big|_{\partial M_i} = 0\,, \tag{3.60}$$

$$\Big(K^{\log}_{AB} - \frac{1}{2}h^{CD}K^{\log}_{CD}\,h_{AB}\Big)\Big|_{\partial M_i} = 0\,. \tag{3.61}$$

Moreover if (3.58) holds then we also have

$$R^{\log}_{abcd}\Big|_{\partial M_i} = 0\,. \tag{3.62}$$

Remark: When the topology of the boundary is spherical and $(\tilde{g},\tilde{K})$ are constructed by the conformal method starting from smooth "seed fields", as described in Section 2, then the proof below shows that $\Omega^{-1}C^{\log}_{abcd}\in C_0(\bar{M})$ and is polyhomogeneous if and only if (3.60)–(3.61) hold.

Proof: Regardless of the topology of ∂M it follows from equation (3.30) that $C^{\log}_{01A0}\Big|_{\partial M_i} = S^{\log}_{1A}\Big|_{\partial M_i} = 0$ whenever (3.57) holds, so that if we choose ω to be of the form (3.48) and ω_0 to be of the form (3.47) then (3.60) will hold by (3.51). If we then choose ω_{00} so that

$$\omega_{00} = \hat{\omega}_{00} + \frac{1}{2!}\omega_0^2\,\omega^2\log\omega\,\omega_{00}^{\log}\,, \tag{3.63}$$

with $\hat{\omega}_{00} \in C^2(\bar{M})$, $\omega_{00}^{\log}$ satisfying the 2–condition, then the decomposition (3.49) inserted in (3.5) implies

$$\begin{aligned} S^{\log}_{0i}\Big|_{\partial M_i} &= -\Big[|D\omega|_g\omega_0\Big(K^{\log}_{i1}+\omega_0^{\log}n_i\Big)\Big]\Big|_{\partial M_i}\,, &(3.64)\\ S^{\log}_{00}\Big|_{\partial M_i} &= -\Big[|D\omega|_g^2\Big\{\omega_{00}^{\log}+\frac{1}{3}\Big(\omega^{\log}-\omega_0^{\log}\Big)\Big\}\Big]\Big|_{\partial M_i}\,, &(3.65)\\ S^{\log}_{ij}\Big|_{\partial M_i} &= \Big[|D\omega|_g^2\Big\{-K^{\log}_{ij}-\omega^{\log}n_in_j+\frac{1}{3}\Big(\omega^{\log}-\omega_0^{\log}\Big)g_{ij}\Big\}\Big]\Big|_{\partial M_i}\,. &(3.66)\end{aligned}$$

Let us consider the $2+2$ decomposed form of (3.58). From (3.4) and (A.12) one obtains

$$R_{1A1B} = C_{1A1B} + S_{AB} + S_{11}h_{AB} \implies C^{\log}_{1A1B}\Big|_{\partial M_i} = -\Big(S^{\log}_{AB}+S^{\log}_{11}h_{AB}\Big)\Big|_{\partial M_i}\,.$$

Equation (3.66) yields

$$C^{\log}_{1A1B}\Big|_{\partial M_i} = \Big[|D\omega|_g^2\Big(K^{\log}_{AB}+(K^{\log}_{11}+\frac{1}{3}\omega^{\log}+\frac{2}{3}\omega_0^{\log})\,h_{AB}\Big)\Big]\Big|_{\partial M_i}\,, \tag{3.67}$$

and (3.55) shows that (3.58) with $abcd = 1A1B$ is equivalent to (3.61). From (3.4), (A.11) and (3.66) with a little work one finds

$$C^{\log}_{1A0B}\Big|_{\partial M_i} = \Big[|D\omega|_g\omega_0\Big(K^{\log}_{AB}+\frac{1}{2}(K^{\log}_{11}+\omega_0^{\log})\,h_{AB}\Big)\Big]\Big|_{\partial M_i}\,,$$

so that (3.61) and (3.53) imply that $C^{\log}_{1A0B}|_{\partial M_i}$ will vanish if and only if (3.59) holds. Manipulations with the Bianchi identities and eq. (3.6) similar to those outlined at the end of the proof of Lemma 3.1 can be used to derive the following

$$\Big[|D\omega|_g\Big(S^{\log}_{Ab}+2C^{\log}_{1Ab1}\Big)\Big]\Big|_{\partial M_i} = 2\Big[C^{\log}_{1Ab0}\omega_0\Big]\Big|_{\partial M_i} \qquad\implies\qquad C^{\log}_{1A01}\Big|_{\partial M_i} = -\frac{1}{2}\,S^{\log}_{A0}\Big|_{\partial M_i}\,;$$

$$\Big[C^{\log}_{ABbd}\nabla^d\Omega\Big]\Big|_{\partial M_i} = 0 \qquad\implies\qquad C^{\log}_{ABC0}\Big|_{\partial M_i} = \epsilon\,C^{\log}_{ABC1}\Big|_{\partial M_i},\; C^{\log}_{AB01}\Big|_{\partial M_i} = 0\,.$$

Calculations similar to the ones at the beginning of this proof show that all the remaining components of $C^{\log}_{abcd}|_{\partial M_i}$ vanish when (3.59)–(3.61) hold. Then eq. (3.64) shows that $S^{\log}_{i0}|_{\partial M_i} = 0$ provided we choose

$$\omega_0^{\log}\Big|_{\partial M_i} = -K^{\log}_{11}\Big|_{\partial M_i}\,. \tag{3.68}$$

Similarly it follows from (3.64) that $S^{\log}_{00}|_{\partial M_i} = 0$ will hold if we set

$$\omega_{00}^{\log}\Big|_{\partial M_i} = \frac{1}{3}\Big(\omega_0^{\log}-\omega^{\log}\Big)\Big|_{\partial M_i}\,. \tag{3.69}$$

Equations (3.61) and (3.55) show now that the right hand side of (3.66) vanishes identically, and the proof is completed. $\square$

4 A direct approach

Throughout this section, in contrast to section 3, all indices are *coordinate indices*, and not tetrad indices. In this section we shall derive, by a direct analysis of the constraint equations, the obstructions to smoothness of the solutions of the constraint equations. The relationship between the "log terms" and the boundary values of the space–time Weyl tensor derived in the previous Section will also be rederived here by a straightforward $2+2$ decomposition calculation. We will assume that the fields g_{ij}, L_{ij}, ω have the regularity described in Proposition 3.3 and that

$$\tilde{K} \equiv 3\epsilon, \qquad \epsilon = \pm \ . \tag{4.1}$$

(If $\tilde{K} \equiv$ constant $\neq \pm 3$ the results described below go through with trivial modifications.) As in [2] (*cf.* Lemma 2.1 of [2]) it is possible to prove that the following gauge conditions can be imposed:

Proposition 4.1 *Assume that $g \in C^\infty(\bar{M})$, that (4.1) holds, and that the hypotheses of Proposition 3.3 are satisfied. We may without loss of generality choose the coordinate system and the conformal factor so that*

1. *the coordinates (x, y^2, y^3) form a Gaussian coordinate system near ∂M_i, and*
2. *$tr\lambda = 0$ on ∂M_i.*

The above and the scalar constraint equation imply then

$$\omega = x + O(x^3) \quad near \quad \partial M_i \ .$$

Remark: It is worthwile pointing out that with the above conditions we have $|Dx|_g = 1$. Moreover we have the following useful identity on ∂M_i:

$$\lambda_{AB}\lambda^B{}_C = \frac{1}{2}\lambda : \lambda h_{AC} \ , \tag{4.2}$$

where we have used the notation

$$\nu : \mu = \nu_{AB}\mu^{AB} \ ,$$

where ν, μ are covariant 2-tensors on ∂M_i. The identity (4.2) is easily proved using the fact that λ_{AB} is symmetric and therefore may be diagonalized. Equation (4.2) implies in particular

$$\mu^{AB}\mu_{BC}\nu^C{}_A = 0 \ . \tag{4.3}$$

for any symmetric trace free 2-tensors μ, ν on ∂M_i. Similarly we will use the notation $\lambda : \mu : \nu = \lambda_A{}^B\mu_B{}^C\nu_C{}^A$, etc.

We shall start our analysis with the vector constraint equation.

4.1 The vector constraint equation

Consider the momentum constraint equation for the physical data,

$$\tilde{D}^i(\tilde{K}_{ij} - \tilde{K}\tilde{g}_{ij}) = 0.$$

Setting

$$\tilde{K}_{ij} = \omega^{-1}L_{ij} + \frac{1}{3}\tilde{K}\tilde{g}_{ij} \tag{4.4}$$

with $\tilde{K}$ $\equiv$constant, the momentum constraint equation takes the form

$$\omega D^i L_{ij} - 2\omega^{,i}L_{ij} = 0 \ . \tag{4.5}$$

A 2+1 decomposition calculation yields

$$\begin{aligned}(x\partial_x - 2 - x\lambda_F{}^F)L_{11} + x\mathcal{D}^A L_{1A} + x\lambda^{AB}L_{AB} &= o(x^2) &(4.6)\\ (x\partial_x - 2 - x\lambda_F{}^F)L_{1B} + x\mathcal{D}^A L_{AB} &= o(x^2) &(4.7)\end{aligned}$$

We have:

Proposition 4.2 *Let the hypotheses and the gauge conditions of Proposition 4.1 hold. According to Proposition 3.3 we can write*

$$L_{ij} = \hat{L}_{ij} + x^2 \log x \hat{L}^{\log}_{ij} \ , \tag{4.8}$$

with $\hat{L}_{ij} \in C^2(\bar{M})$ and $\hat{L}^{\log}_{ij}$ satisfying the 2–condition at ∂M_i. Suppose moreover that

$$\hat{L}^{\log}_{AB}\Big|_{\partial M_i} = -\frac{1}{2}\hat{L}^{\log}_{11}h_{AB} \ . \tag{4.9}$$

Then

$$\begin{aligned} L_{1j}\Big|_{\partial M_i} &= 0 \ , &(4.10)\\ \partial_x L_{11}\Big|_{\partial M_i} &= \lambda^{AC}L_{AC} \ , &(4.11)\\ \partial_x L_{1B}\Big|_{\partial M_i} &= \mathcal{D}^C L_{CB} \ , &(4.12)\end{aligned}$$

and the terms $\hat{L}^{\log}_{1j}$ in the decomposition of L_{ij} take the following values on ∂M_i:

$$\begin{aligned} \hat{L}^{\log}_{11}\Big|_{\partial M_i} &= -(\mathcal{D}^A\mathcal{D}^B L_{AB} + \partial_x\lambda_{AB}L^{AB} + \lambda^{AB}\partial_x L_{AB})\\ &= -(\mathcal{D}^A\mathcal{D}^B L_{AB} + R_{AB}L^{AB} + \lambda^{AB}\partial_x L_{AB}) \ , &(4.13)\\ \hat{L}^{\log}_{1A}\Big|_{\partial M_i} &= -\partial_x(\mathcal{D}^B L_{AB})\\ &= -\Big[\mathcal{D}^B(\partial_x L_{AB}) + 2\lambda^{BC}\mathcal{D}_B L_{AC} + 2L_{AB}\mathcal{D}^C\lambda_{CB}\\ &\qquad + L^{BC}\mathcal{D}_A\lambda_{BC}\Big] \ , &(4.14)\end{aligned}$$

where R_{AB} denotes the pull-back of the Ricci tensor of g_{ij} to ∂M_i.

Remarks:

1. The tensor $\hat{L}^{\log}_{ij}\Big|_{\partial M_i}$ defined above in eq. (4.8) differs from the trace–free part of $K^{\log}_{ij}\Big|_{\partial M_i}$ defined in eq. (3.49) by a normalization factor of $\epsilon/2$.
2. The hypothesis (4.9) holds for solutions of the constraint equations constructed by the conformal method starting from seed fields which are smooth up to the boundary.

Proof: The result is obtained by inserting the decomposition (4.8) in (4.6)–(4.7) and comparing terms with appropriate powers of x. To get (4.13) one uses the identity

$$R_{AB} = \partial_x \lambda_{AB} + r_{AB} + 2\lambda_{AC}\lambda^C{}_B - \lambda_{AB}\lambda^C{}_C. \tag{4.15}$$

□

Note that $\hat{L}^{\log}_{AB}$ is expressed in terms of $\hat{L}^{\log}_{11}$. This means that the vanishing of $\hat{L}^{\log}_{ij}$ is equivalent to the vanishing of the three functions $\hat{L}^{\log}_{11}, \hat{L}^{\log}_{1A}$ on ∂M_i.

In the remainder of this subsection we shall consider those solutions of the constraint equations which have been constructed by the conformal method starting from seed fields $g, A, x \in C^\infty(\bar{M})$, as described in Section 2. For simplicity we shall only consider those A_{ij}'s for which we have

$$A_{ij}D^i x\Big|_{\partial M_i} = 0. \tag{4.16}$$

((4.10) shows that any tensor L_{ij} satisfying (4.5) necessarily satisfies (4.16). This observation and a simple argument using uniqueness of solutions of (2.4) shows that the space of transverse traceless tensors constructed as described in Section 2 is exhausted by those tensors which are obtained from tensors satisfying (4.16).) Under (4.16) it follows by comparing terms with appropriate powers of x in (2.4) that the desired solution X of (2.4) satisfies

$$X = O(x^3)\ . \tag{4.17}$$

Equations (2.5) and (4.17) show that we have

$$(L_{AB} - \frac{1}{2}h^{CD}L_{CD}h_{AB})\Big|_{\partial M_i} = (A_{AB} - \frac{1}{2}h^{CD}A_{CD}h_{AB})\Big|_{\partial M_i}\ , \tag{4.18}$$

$$(\partial_x L_{AB} - \frac{1}{2}h^{CD}\partial_x L_{CD}h_{AB})\Big|_{\partial M_i} = (\partial_x A_{AB} - \frac{1}{2}h^{CD}\partial_x A_{CD}h_{AB})\Big|_{\partial M_i}\ . \tag{4.19}$$

It is natural to ask the question, for which seed fields will one obtain solutions of the constraint equations such that $L_{ij} \in C^\infty(\bar{M})$ – as discussed in Section 2, this will occur if $\hat{L}^{\log}_{ij}|_{\partial M_i} = 0$. The appropriate conditions on A_{ij} can be derived by inserting (4.18)–(4.19) into (4.13)–(4.14), they do not, however, seem to be very enlightening in general. Some simplifications occur if in addition to (4.16) we assume that "Scri is shear free", in the sense that

$$L_{AB}\Big|_{\partial M_i} = \epsilon\lambda_{AB}\ , \tag{4.20}$$

with $\epsilon = \pm 1$. Under these conditions the equation $\hat{L}^{\log}_{11}|_{\partial M_i} = 0$ can be rewritten as

$$\hat{L}^{\log}_{11}\Big|_{\partial M_i} = -\left[\lambda^{AB}\partial_x A_{AB} + \epsilon(\mathcal{D}^A\mathcal{D}^B\lambda_{AB} + R_{AB}\lambda^{AB})\right]\Big|_{\partial M_i} = 0\ . \tag{4.21}$$

With some work from (4.14) one finds that

$$\hat{L}^{\log}_{1A}\Big|_{\partial M_i} = -\mathcal{D}^B p_{AB} \ , \tag{4.22}$$

where

$$p_{AB} \equiv \partial_x A_{AB} + \frac{3}{2}\epsilon\lambda_{CD}\lambda^{CD} h_{AB} \ . \tag{4.23}$$

From $L^i{}_i = 0$ and from (4.20) it is simple to show that p_{AB} is trace–free, so that p_{AB} is a symmetric traceless tensor on the two dimensional manifold ∂M_i. If ∂M_i is a sphere, $\hat{L}^{\log}_{1A}|_{\partial M_i} = 0$ implies $p_{AB} = 0$; for other topologies this will, however, not be true in general.

4.2 The scalar constraint equation

Under the conditions of Proposition 4.1 we can write

$$\omega = x + x^3\omega_3 + x^4\log(x)\omega_{4,1} + l.o. \ , \tag{4.24}$$

where $l.o.$ denotes lower order terms. Inserting the expansion (4.24) in equation (2.6) one obtains

$$\begin{aligned}
\omega_3\Big|_{\partial M_i} &= \frac{1}{12}[-4\partial_x(\lambda_F{}^F) - L_{AB}L^{AB} + R(g)] && (4.25)\\
\omega_{4,1}\Big|_{\partial M_i} &= \frac{1}{16}[2\partial_x^2(\lambda_F{}^F) + 2L^{AB}\partial_x L_{AB} + 4L^C_A L_C{}^B\lambda^A{}_B - \partial_x R] \\
&= \frac{1}{16}[2\partial_x^2(\lambda_F{}^F) + 2L^{AB}\partial_x L_{AB} - \partial_x R] \ . && (4.26)
\end{aligned}$$

where we have used (4.3) and the fact that $L_{1j}|_{\partial M_i} = 0$ which implies that $h^{AB}L_{AB}|_{\partial M_i} = 0$. A 2+1 decomposition of the second Bianchi identity $D_iR = 2D^jR_{ij}$ using $\lambda_F{}^F = 0$ on ∂M_i gives the relation

$$\partial_x R\Big|_{\partial M_i} = 2(\partial_x R_{11} - \mathcal{D}^A\mathcal{D}^B\lambda_{AB} + \lambda^{AB}R_{AB}) \ . \tag{4.27}$$

Note that since $r_{AB} = \frac{1}{2}rh_{AB}$, $\lambda_F{}^F|_{\partial M_i} = 0$ and $\lambda_A{}^B\lambda_B{}^C\lambda_C{}^A|_{\partial M_i} = 0$, one finds from the identity (4.15) that $\lambda^{AB}R_{AB}|_{\partial M_i} = \lambda : \partial_x\lambda$. Further, using $R_{11} = \partial_x(\lambda_F{}^F) - \lambda : \lambda$ we find

$$\partial_x R_{11}\Big|_{\partial M_i} = \partial_x^2(\lambda_F{}^F) - 2\lambda^{AB}\partial_x\lambda_{AB} \ ,$$

where we have used (4.3). Thus we get

$$\omega_{4,1} = \frac{1}{8}[R_{AB}\lambda^{AB} + \mathcal{D}^A\mathcal{D}^B\lambda_{AB} + \partial_x L : L]\Big|_{\partial M_i} \ , \tag{4.28}$$

where in the last step we have used the fact that $g^{ij}L_{ij} \equiv 0$ and $L_{1j}|_{\partial M_i} = 0$, which implies $h^{AB}L_{AB}|_{\partial M_i} = 0$ and therefore $L_{AC}L^C{}_B|_{\partial M_i} = \frac{1}{2}L_{CD}L^{CD}h_{AB}$. In particular, if "Scri is shear free" in the sense that (4.20) holds, then

$$\omega_{4,1} = -\frac{\epsilon}{8}L^{\log}_{11}\Big|_{\partial M_i} \ . \tag{4.29}$$

This gives an alternative proof of Lemma 3.2, namely that under the present conditions the vanishing of $L^{\log}_{11}|_{\partial M_i}$ implies that the boundary value of the log coefficient in ω vanishes as well.

4.3 The Weyl tensor

Throughout this section we shall assume that the hypotheses and the gauge conditions of Proposition 4.1 are satisfied. By a transformation of the form (3.16) with an appropriate ϕ_0 we can also achieve

$$K = 0 \ , \tag{4.30}$$

and we shall also assume in this section that (4.30) holds. Due to $Ein(\tilde\gamma) = 0$, $C_{i0k}{}^0$ is the negative of the Ricci trace of the spatial part of $Riem(\tilde\gamma)$:

$$\begin{aligned} -C_{i0k}{}^0 = -\tilde C_{i0k}{}^0 &= -\tilde R_{i0k}{}^0 \\ &= -\tilde\gamma^\alpha_\beta \tilde R_{i\alpha k}{}^\beta + \tilde g^j_l \tilde R_{ijk}{}^l \\ &= \tilde g^j_l \tilde R_{ijk}{}^l \ , \end{aligned}$$

where $C_{abc}{}^d$, respectively $\tilde C_{abc}{}^d$, denotes the Weyl tensor of the metric γ, respectively $\tilde\gamma$. Using the conformal transformation rules and the Gauss equation (A.12) we get the formula

$$\begin{aligned} C_{ij} \equiv C_{0i0j} &= R_{ij}(g) - \frac{1}{3}R(g)g_{ij} + \frac{1}{\omega}(D_iD_j\omega - \frac{1}{3}D_kD^k\omega g_{ij} + \epsilon L_{ij}) \\ &\quad -L_i{}^kL_{kj} + \frac{1}{3}L^{kl}L_{kl}g_{ij} \ . \end{aligned} \tag{4.31}$$

By l'Hospital's rule we have

$$\left[\frac{1}{\omega}\left(D_AD_B\omega + \epsilon L_{AB}\right)\right]\Big|_{\partial M_i} = -\partial_x\lambda_{AB} + \epsilon\partial_x L_{AB}$$

so that, using $\lambda_A{}^A|_{\partial M_i} = 0$, $L_A{}^A|_{\partial M_i} = 0$ and the fact that the square of a symmetric traceless two dimensional tensor is proportional to the metric one is led to

$$C_{AB}\Big|_{\partial M_i} = R_{AB} - \partial_x\lambda_{AB} + \epsilon\partial_x L_{AB} + \kappa h_{AB} \ , \tag{4.32}$$

for some function κ. Now we have $C_0{}^i{}_{0i} = C_0{}^\alpha{}_{0\alpha} = 0$, and since $C_{0101}|_{\partial M_i} \equiv 0$ by Section 3 it follows[5] that κ can be determined by the requirement that the right hand side of (4.32) is trace–free. The embedding equation (4.15) allows us to rewrite eq. (4.32) as

$$C_{AB}\Big|_{\partial M_i} = \epsilon\left(\partial_x L_{AB} - \frac{1}{2}h^{CD}\partial_x L_{CD}h_{AB}\right) \ . \tag{4.33}$$

Thus, the value of C_{AB} on ∂M_i is just the trace free part of $\partial_x L_{AB}$. Equations (4.33) and (4.22)–(4.23) give an alternative proof of the assertion, that under the shear–free condition (4.20) the boundary values of the log terms in L_{1A} are determined by $C_{abcd}|_{\partial M_i}$.

Let us finally mention that with some work it follows from (4.31) that $\hat C^{\log}_{ij}$ takes the form

$$\begin{aligned} \hat C^{\log}_{11}\Big|_{\partial M_i} &= 8\omega_{4,1} + \epsilon\hat L^{\log}_{11} \ , \\ \hat C^{\log}_{1A}\Big|_{\partial M_i} &= \epsilon\hat L^{\log}_{1A} \ , \\ \hat C^{\log}_{AB}\Big|_{\partial M_i} &= 4\omega_{4,1}h_{AB} - \epsilon\hat L^{\log}_{AB} \end{aligned}$$

[5] This identity can of course be derived directly by a somewhat lengthy $2+2$ decomposition calculation.

(*cf.* also (3.50)). We see from (4.29) and from Proposition 4.2 that under the hypotheses of this section the fields $\hat{C}^{\rm log}_{11}$ and $\hat{C}^{\rm log}_{AB}$ vanish identically on ∂M_i (*cf.* also the proof of Lemma 3.2).

5 The evolution problem

The aim of this Section is to discuss the question of existence of developments $(V, \tilde{\gamma})$ of the "hyperboloidal" Cauchy data $(\tilde{g}, \tilde{K})$ which admit some kind of conformal completion $(\bar{V}, \gamma)$. Let us start with smooth up to boundary initial data:

Theorem 5.1 *Let $\partial M_i \approx S^2$, the two dimensional sphere, and let $(\tilde{g}, \tilde{K})$ be a solution of the vacuum constraint equations for which there exist smooth functions ω and ω_0, with $\omega/x \in C^\infty(\bar{M})$, such that the fields*

$$g_{ij} = \omega^2 \tilde{g}_{ij} \ , \tag{5.1}$$

$$K^{ij} = \omega^{-3}(\tilde{K}^{ij} - \frac{1}{3}\tilde{g}_{kl}\tilde{K}^{kl}\tilde{g}^{ij}) + \frac{\tilde{g}_{kl}\tilde{K}^{kl} - 3\,\omega_0}{3\,\omega} g^{ij} \ , \tag{5.2}$$

are in $C^\infty(\bar{M})$. Suppose moreover that there exists $\epsilon = \pm 1$ such that

$$\Big(\lambda_{AB} - \frac{h^{CD}\lambda_{CD}}{2} h_{AB}\Big)\Big|_{\partial M_i} = \epsilon\Big(K_{AB} - \frac{h^{CD}K_{CD}}{2} h_{AB}\Big)\Big|_{\partial M_i} . \tag{5.3}$$

Then

1. *there exists a Lorentzian manifold with boundary $(\bar{V}, \gamma)$ and a function $\Omega \in C^\infty(\bar{V})$ (defining function for ∂V) such that $(\mathrm{int}\,\bar{V}, \Omega^{-2}\gamma)$ is a development of $(M, \tilde{g}, \tilde{K})$, with ∂M_i being a smooth submanifold of $\bar{V} \setminus V$.*

2. *If moreover $M = B(1)$ (the unit ball in $\mathbb{R}^3$), then there exists $\epsilon_0 > 0$ such that if*

$$||\omega - \breve{\omega}||_{C^8(\bar{M})} + ||g_{ij} - \breve{g}_{ij}||_{C^7(\bar{M})} + ||\omega_0||_{C^7(\bar{M})} + ||K^{ij}||_{C^6(\bar{M})} < \epsilon_0 \ , \tag{5.4}$$

where $\breve{g}_{ij}$ is the standard flat metric on $B(1)$ and $\breve{\omega} = (1 - r^2)/2$, then $(\mathrm{int}\,\bar{V}, \Omega^{-2}\gamma)$ is timelike and null future geodesically complete.

Remarks:

1. The norm in (5.4) can be considerably weakened at the price, however, of a rather more complicated statement of the corresponding result, *cf.* [9] for details.

2. Part 1 above still holds when $\partial M_i \not\approx S^2$ provided one adds the assumption that $C_{0A0B}|_{\partial M_i} = 0$.

Proof: The calculations of Section 3 show that $\nabla_0\Omega|_M \in C^\infty(\bar{M})$ and $\nabla_0\nabla_0\Omega|_M \in C^\infty(\bar{M})$ can be chosen so that $S_{ab}|_M \in C^\infty(\bar{M})$, $\Omega^{-1}C^a{}_{bcd}|_M \in C^\infty(\bar{M})$. Point 1 follows then from Theorem 6.5 of [8], point 2 follows from Theorem 3.5 of [9] and from the results of [11]. □

The condition (5.3) can be rewritten in terms of the seed fields (g, A), so that from (4.21)–(4.23), (3.56) and Theorem 5.1 one obtains:

Corollary 5.1 *Let $\partial M \approx S^2$, the two dimensional sphere, and let $(\tilde g, \tilde K)$ be constructed as described in Section 2 starting with smooth seed fields (g, A). Suppose moreover that there exists $\epsilon = \pm 1$ such that*

$$\left(\lambda_{AB} - \frac{\lambda}{2} h_{AB}\right)\Big|_{\partial M} = \epsilon \left[A_{AB} - \frac{1}{2} h^{CD} A_{CD} h_{AB}\right]\Big|_{\partial M} . \tag{5.5}$$

Without loss of generality we may choose the "gauge"

$$\lambda\Big|_{\partial M_i} = 0$$

with x, v^A — Gauss coordinates near the boundary. Suppose that moreover in this gauge we have

$$A_{1j}\Big|_{\partial M} = 0 \ ,$$

$$\left[\mathcal{D}^A \mathcal{D}^B \lambda_{AB} + R_{AB} \lambda^{AB}\right]\Big|_{\partial M} = 0 \ ,$$

$$\left(\partial_x A_{AB} - \frac{1}{2} h^{CD} \partial_x A_{CD} h_{AB}\right)\Big|_{\partial M} = 0 \ .$$

Then the fields ω, and K constructed by the conformal method are in $C^\infty(\bar M)$. Further, $\omega^{-1} C^a{}_{bcd}$ is in $C^\infty(\bar M)$ and the conclusions of Theorem 5.1 hold.

Remark: It is clear from the methods of [1] that under the hypotheses of Corollary 5.1 the condition (5.4) can be replaced by the condition that (g_{ij}, A^{ij}) are sufficiently close to $(\breve g_{ij}, 0)$ in some possibly higher norm.

We shall now consider the case of initial data sets for which the log terms occur. Note first, that if K^{ij} has log terms in its expansion at ∂M, then even if we start with $g_{ij} \in C^\infty(\bar M)$ then by evolution g_{ij} will pick up log terms at the next moment of time. It follows that the appropriate setup to discuss the evolution problem is that of polyhomogeneous space–time metrics γ. Before proceeding any further, let us introduce some terminology. Let $\bar{\mathcal M}$ be an n–dimensional connected, paracompact, Hausdorff manifold with boundary, thus $\mathcal M \equiv \mathrm{int}\bar{\mathcal M}$ and $\partial\mathcal M \equiv \partial\bar{\mathcal M} \equiv \bar{\mathcal M} \setminus \mathcal M$ are smooth manifolds. A semi–Riemannian metric f on $\bar{\mathcal M}$ will be called *polyhomogeneous* if $f \in C^\infty_{\mathrm{loc}}(\mathcal M) \cap C^0(\bar{\mathcal M})$ and if in local coordinates $y = (x, v)$ near $\partial\mathcal M$, where x is a defining function for $\partial\mathcal M$ and v are coordinates on $\partial\mathcal M$, the functions $f_{\mu\nu} \equiv f(\partial/\partial y^\mu, \partial/\partial y^\nu)$ are polyhomogeneous. Given a polyhomogeneous metric f on $\mathcal M$ it is natural to enlarge the atlas on $\mathcal M$ to include all those local coordinate transformations which preserve the polyhomogeneous character of the metric near $\partial\mathcal M$.

We shall say that the initial data $(M, \tilde g, \tilde K)$ are polyhomogeneous if M is as described above and if there exists polyhomogeneous functions ω and ω_0, with $\omega/x \in C^0(\bar M)$, such that the fields (g, K) given by (5.1)–(5.1) are polyhomogeneous.

We shall say that a development $(V, \tilde\gamma)$ of polyhomogeneous initial data $(M, \tilde g, \tilde K)$ admits a polyhomogeneous Scri if

1. there exists a manifold with boundary $\bar V$ such that $V = \mathrm{int}\bar V$, and such that ∂M is a smooth submanifold of ∂V;

2. there exist a polyhomogeneous function $\Omega \in C^1(\bar V)$ satisfying $\Omega|_{\partial M} = 0$, $\nabla\Omega(p) \neq 0$ for $p \in \partial V$, such that the metric $\Omega^2\tilde\gamma$ is polyhomogeneous;

3. there exists a time function t defined in a neighbourhood of $\bar M$ in $\bar V$ such that $t|_M = 0$, compatible with the polyhomogeneous structure, with the associated lapse function being uniformly bounded away from zero.

It should be pointed out, that all the results described in Section 2 go through under polyhomogeneity conditions on the seed fields (g, A). More precisely, if g is any polyhomogeneous metric on $\bar M$ and A^{ij} is any uniformy bounded polyhomogeneous symmetric tensor field on $\bar M$, then there exist solutions (X, ω) of (2.4) and (2.6) such that L^{ij} given by (2.5) is polyhomogeneous and uniformly bounded on $\bar M$, and ω/x is polyhomogeneous, uniformly bounded, and uniformly bounded away from zero on $\bar M$.

Given a polyhomogeneous initial data set it is natural to ask the question whether or not there exists an obstruction to the existence of a development of the initial data with a polyhomogeneous Scri. We have the following:

Proposition 5.1 *Let $(M, \tilde g, \tilde K)$ be a polyhomogeneous initial data set with $g_{ij} \in C^1(\bar M)$ and $K^{ij} \in C^0(\bar M)$. If both for $\epsilon = 1$ and for $\epsilon = -1$ we have*

$$\Big(\lambda_{AB} - \frac{\lambda}{2}h_{AB}\Big)\Big|_{\partial M_i} \neq \epsilon\Big(K_{AB} - \frac{\kappa}{2}h_{AB}\Big)\Big|_{\partial M_i}, \tag{5.6}$$

then there exists no development $(V, \tilde\gamma)$ of $(M, \tilde g, \tilde K)$ which admits a polyhomogeneous Scri.

Proof: Suppose that a polyhomogeneous Scri exists; therefore one can find a time function t defined in a neighbourhood of $\bar M$ in $\bar V$ such that $t|_M = 0$, and the associated lapse function α and the shift vector β are, in local coordinates, polyhomogeneous, with α uniformly bounded away from zero in a neighbourhood of $\bar M$. We have the evolution equations (we use the notation of [19], in particular indices are *coordinate* indices)

$$\frac{\partial g_{ij}}{\partial t} - \mathcal{L}_\beta g_{ij} = -2\alpha K_{ij}\ , \tag{5.7}$$

$$\begin{aligned}\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_\beta K_{ij} = {}& -D_iD_j\alpha + 2\frac{\alpha}{\Omega}\left(D_iD_j\Omega\nabla_0\Omega K_{ij}\right)\\ &+\alpha\left[-\frac{3|\nabla\Omega|^2_\gamma - \Omega\Box\,\Omega}{\Omega^2}g_{ij} + {}^3R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j\right]\ ,\end{aligned} \tag{5.8}$$

where α is the lapse function, β is the shift vector, $\mathcal{L}_\beta$ denotes a Lie derivative and ${}^3R_{ij}$ is the Ricci tensor of g_{ij}. Note that by hypothesis all the functions α, β, g_{ij}, Ω and K^{ij} are polyhomogeneous and uniformly bounded. If (3.19) does not hold, then the terms $D_iD_j\Omega$ in the trace free part of eq. (5.8) with $ij = AB$ will give a contribution which blows up as Ω^{-1} near ∂M, while all the remaining terms in the trace free part of the $ij = AB$ equation are at most blowing up as some power of $\log x$, which gives a contradiction. $\Box$

Proposition 5.1 suggests very strongly, that there exists *no* polyhomogeneous Scri's which are *not* shear free, even without the extra regularity conditions $g \in C^1(\bar M)$, $K^{ij} \in C^0(\bar M)$

which we have imposed above. Such a result has been observed in [7] in a Bondi–Sachs type setting. Recall that the condition (5.3) has been derived in Section 3 precisely under the condition $g \in C^1(\bar{M})$, $K^{ij} \in C^0(\bar{M})$. It is rather clear that an analysis of (5.8) under general polyhomogeneity conditions on (g, K) will lead to an analogous "no–go" theorem, with possibly a somewhat modified version of condition (5.6).

Proposition 5.1 shows in particular that generic initial data constructed as in Section 2 with smooth seed fields (g, A) will *not* admit a development which admits a polyhomogeneous Scri (this will hold even when ω and L are smooth up to boundary as long as (5.6) holds). The above "no–go" result does of course not exclude the possibility of existence of some non–polyhomogeneous conformal completions of $(V, \tilde{\gamma})$.

It may be of some interest to stress that if V is a spacetime which is the maximal devlopment of initial data for which (5.3) holds, then in the class of polyhomogeneous conformal completions one obtains a conformal completion $\bar{V}$ for which $\bar{V} \setminus V$ is a two dimensional smooth manifold, namely ∂M. The physical significance of such space–times and completions remains to be investigated.

Consider now a polyhomogeneous initial data set for which (5.5) holds — if this condition were not preserved by evolution via Einstein equations, it would follow from Proposition 5.1 that the resulting space–time could not admit a polyhomogeneous Scri. It is therefore natural to ask the question, whether (5.5) is preserved by evolution. We shall show that this is the case for initial data satisfying $g_{ij} \in \mathcal{A}^{phg} \cap C^1(\bar{M})$, $K^{ij} \in \mathcal{A}^{phg} \cap C^0(\bar{M})$, and satisfying moreover the conditions

$$\tilde{K} = 3\epsilon, \qquad K = 0, \qquad \lambda^A{}_A\Big|_{\partial M_i} = 0.$$

It can be shown [7] that for any polyhomogeneous metric Gauss coordinates can be constructed, the coefficients of the transformed metric being again polyhomogeneous in the Gauss coordinate system. In what follows we shall use this coordinate system on M in a neighbourhood of ∂M_i. Let n^i be the three dimensional normal to the sets $\{\Omega = c_1\} \cap \{t = c_2\}$, for some constants c_1, c_2, thus $n^i = D^i\Omega/|D\Omega|$ and the extrinsic curvature of those sets is given by $\lambda_{AB} = -\mathcal{L}_n g_{AB}$, where $\mathcal{L}$ is a Lie derivative. From the evolution equations (5.7)–(5.8) with a little work one finds, assuming (5.5),

$$\frac{\partial}{\partial t}(\lambda_{AB} - \epsilon K_{AB})\Big|_{\partial M_i, t=0} = \epsilon \mathcal{L}_{\alpha n + \epsilon \beta}(\lambda_{AB} - \epsilon K_{AB}) + \epsilon \mathcal{D}_A \mathcal{D}_B \alpha - \frac{1}{2}\mathcal{L}_{[\frac{\partial}{\partial t} - \beta, n]} g_{AB} + \kappa g_{AB} \ , \quad (5.9)$$

with some function κ. (Fortunately, as explained below, we do not need to use (5.7)–(5.8) to evaluate κ, which would require quite a messy calculation.) From $K_{ij}n^j|_{\partial M_i} = 0$ it follows that

$$\left[\frac{\partial}{\partial t}, n\right]\Big|_{\partial M_i, t=0} = \frac{\partial n}{\partial t}\Big|_{\partial M_i, t=0} = 0 \ .$$

It should be noted that in (5.9) the lapse function α and the shift vector β are arbitrary. We can therefore require that they are polyhomogeneous, and that

$$\alpha\Big|_{\partial M_i, t=0} = 1 \ , \qquad (5.10)$$

$$n(\alpha)\Big|_{\partial M_i, t=0} = 0 \ , \qquad (5.11)$$

$$(\beta + \epsilon n)\Big|_{\partial M_i, t=0} = 0 \ . \tag{5.12}$$

This leads to

$$\frac{\partial}{\partial t}(\lambda_{AB} - \epsilon K_{AB})\Big|_{\partial M_i, t=0} = \kappa g_{AB} \ . \tag{5.13}$$

(It might be of some interest to point out that with the choice (5.12) we have

$$\frac{\partial}{\partial t}\Big|_{\partial M_i, t=0} = (\alpha\eta + \beta)\Big|_{\partial M_i, t=0} = -\left(\frac{\epsilon\alpha}{|D\omega|_g}\nabla\Omega\right)\Big|_{\partial M_i, t=0} \ ,$$

so that at $t = 0$ the vector field $\frac{\partial}{\partial t}|_{\partial M_i, t=0}$ is proportional to $\nabla\Omega$ on ∂M_i, as should be if we want the integral curves of $\frac{\partial}{\partial t}$ to follow the level set $\Omega = 0$.) Consider now the tensor field

$$\sigma_{AB} \equiv \lambda_{AB} - \frac{1}{2}h^{CD}\lambda_{CD}h_{AB} - \epsilon\left(K_{AB} - \frac{1}{2}h^{CD}K_{CD}h_{AB}\right) \ .$$

By definition we have $h^{AB}\sigma_{AB} \equiv 0$, so that

$$h^{AB}\frac{\partial\sigma_{AB}}{\partial t} = -\frac{\partial h^{AB}}{\partial t}\sigma_{AB} \ .$$

Since $\sigma_{AB}|_{\partial M_i, t=0} \equiv 0$ it follows that $h^{AB}\frac{\partial\sigma_{AB}}{\partial t}|_{\partial M_i, t=0} = 0$, which together with the definition of σ_{AB} and (5.13) yields

$$\kappa = -\frac{1}{2}\frac{\partial(h^{CD}\lambda_{CD} - \epsilon h^{CD}K_{CD})}{\partial t}\Big|_{\partial M_i, t=0} \ . \tag{5.14}$$

Under the conditions (5.10)–(5.12) one finally finds from (5.13)–(5.14) that

$$\frac{\partial\sigma_{AB}}{\partial t}\Big|_{\partial M_i, t=0} = 0 \ . \tag{5.15}$$

(5.15) shows that the requirement of preservation of the vanishing of the shear of the conformal boundary does not lead to any supplementary conditions on the initial data.

A Appendix

For $p \in M$ let $e_a = e_a{}^\mu\partial_\mu$, $a = 1, \ldots, 4$, be an orthonormal tetrad, with

$$e_0{}^\mu = \eta^\mu \ , \qquad e_1{}^\mu = n^\mu \ ,$$

where $\eta^\mu\partial_\mu$ is a unit timelike vector field orthogonal to M in (V, γ), and $n = n^\mu\partial_\mu$ is any unit vector field tangent to M such that for $p \in \partial M_i$, n is the inward pointing vector orthogonal to ∂M_i (such tetrads can easily be constructed by *e.g.* a Gramm–Schmidt procedure, with the differentiability class of $e_i{}^\mu\partial_\mu = e_i{}^k\partial_k$, $i, k = 1, 2, 3$, being that of $g_{ij} = \gamma_{ij}$). By definition we have

$$\gamma_{ab} = \gamma_{\mu\nu}e_a{}^\mu e_b{}^\nu = \mathrm{diag}(-1, 1, 1, 1) \ ,$$

$\gamma^{ab} = \gamma_{ab}$; set $e^a = \gamma^{ab} e_b$. Define

$$\omega^a{}_{bc} = \langle \nabla_{e_c} e_b, e^a \rangle_\gamma \Longleftrightarrow \nabla_{e_a} e_b = \omega^c{}_{ba} e_c \,, \tag{A.1}$$

where $\langle \ , \ \rangle_f$ is the scalar product in the metric f, and ∇ is the Levi–Civita derivative operator of γ. It follows

$$\begin{aligned} \nabla_a (X^b e_b) &\equiv \nabla_{e_a} (X^b e_b) = \Big(e_a(X^b) + \omega^b{}_{ca} X^c \Big) e_b \\ &=: (\nabla_a X^b) e_b \,. \end{aligned} \tag{A.2}$$

Let g be the metric induced from (V, γ) on M, let h be the metric induced from (M, g) on ∂M_i, we shall denote by D_i, respectively by $\mathcal{D}_A$, the Levi–Civita derivative operator of g, respectively of h. As is well known (*cf. e.g.* [15]), the tetrad formalism is most convenient in $3+1$ and $2+1$ decomposition calculations because we have

$$\begin{aligned} D_i (X^j e_j) &\equiv D_{e_i} (X^j e_j) \\ &= \Big[e_i (X^j) + \omega^j{}_{ki} X^k \Big] e_j \\ &=: (D_i X^j) e_j \,, \end{aligned} \tag{A.3}$$

$$\begin{aligned} \mathcal{D}_A (X^B e_B) &\equiv \mathcal{D}_{e_A} (X^B e_B) \\ &= \Big[e_A (X^B) + \omega^B{}_{CA} X^C \Big] e_B \\ &=: (\mathcal{D}_A X^B) e_b \,, \end{aligned} \tag{A.4}$$

where the $\omega^{\cdot}_{\cdot\cdot}$ coefficients in (A.3)–(A.4) are the same as those appearing in (A.2). From (A.1) we have

$$K_{ij} = \omega_{0ij} = \omega_{0ji} = \langle \nabla_{e_i} e_j, \eta \rangle_\gamma \,, \tag{A.5}$$

where K_{ij} is the extrinsic curvature tensor of M in (V, γ) (the sign conventions here agree with those of [40] when η is chosen to be future pointing), and

$$\begin{aligned} \lambda_{AB} &= \omega_{1AB} = \omega_{1BA} = \langle \nabla_{e_A} e_B, n \rangle_\gamma \\ &= \langle D_{e_A} e_B, n \rangle_g \,, \end{aligned} \tag{A.6}$$

where λ_{AB} is the extrinsic curvature of ∂M_i in (M, g) (note the unconventional sign in (A.6) due to the fact that n is the *inward* pointing normal to ∂M_i. This choice of n is convenient for our purposes because if x is a defining function for ∂M, then $n = Dx / |Dx|_g$.) The following equations are often used:

$$\nabla_i X_j = D_i X_j + X_0 K_{ij} \,, \tag{A.7}$$

$$D_A X_B = \mathcal{D}_A X_B - X_1 \lambda_{AB} \,, \tag{A.8}$$

$$\nabla_i X_0 = D_i X_0 + K_i{}^j X_j \,, \tag{A.9}$$

$$\Box\, \Omega = -\Omega_{00} + \Delta_g \Omega + K \Omega_0 \,, \tag{A.10}$$

$$\Omega_{00} \equiv \nabla_0 \nabla_0 \Omega \,, \quad \Omega_0 \equiv e_0(\Omega) \,, \quad K \equiv K_i{}^i \,,$$

$$\Box \equiv \nabla^a \nabla_a \,, \quad \Delta_g \equiv D^i D_i \,.$$

We have the following embedding equations (*cf. e.g.* [19])

$$R_{ijk0}(\gamma) = D_j K_{ik} - D_i K_{jk} \,, \tag{A.11}$$

$$R_{ijkl}(\gamma) = R_{ijkl}(g) + K_{ik} K_{jl} - K_{il} K_{jk} \,, \tag{A.12}$$

$R_{abcd}(f)$ being the Riemann tensor of a metric f. We shall need the following well known result:

Proposition A.1 *Let γ be a conformally vacuum metric on a four dimensional manifold. Then the Weyl tensor C_{abcd} of γ satisfies*

$$C_{ijkl} = 2(\gamma_{i[k}C_{l]j} + \gamma_{j[l}C_{k]i}) , \qquad \text{(A.13)}$$

where

$$C_{ij} \equiv C_{0i0j} .$$

Proof: Since (A.13) is conformally invariant without loss of generality we may assume that γ is vacuum. A $3+1$ decomposition calculation gives (*cf. e.g.* [19])

$$C_{ij} = R_{ij}(g) + \frac{K}{3}L_{ij} - L_{ik}L^k{}_j + \frac{2}{9}K^2 g_{ij} ,$$

$$L_{ij} \equiv K_{ij} - \frac{K}{3}g_{ij}$$

($g_{ij} = \gamma_{ij}$), while from (A.12) one obtains

$$R_{ijkl}(g) = 2g_{i[k}C_{l]j} + 2g_{j[l}C_{k]i} + A_{ijkl} ,$$

$$A_{ijkl} = 2g_{i[k}L_{l]e}L^e{}_j + 2g_{j[l}L_{k]e}L^e{}_i + 2L_{i[k}L_{l]j} - |L|^2_g g_{i[k}g_{l]j} .$$

Tracelessness and symmetry of L_{ij} together with the identity

$$\delta^i_{[k}\delta^j_l L_e{}^e L_{f]}{}^f = 0 \qquad \text{(A.14)}$$

imply $A_{ijkl} \equiv 0$, and the result follows. □

B Asymptotically Minkowskian coordinates

The object of this section is to show that polyhomogeneous initial data sets for which (3.19) holds are asymptotically flat, in the sense that the Cauchy data set can be realised by an embedding of the initial data surface in an asymptotically Minkowskian (not necessarily vacuum) space–time. The notion of asymptotic flatness used here is rather rough, and it is clear from the formulae below that the metrics considered posses finer properties than (B.3), we shall however not attempt to formalize that.

Throughout this Section, for simplicity of calculations, we shall assume that changing the time orientation if necessary we have

$$\tilde{K} = -3, \qquad K = 0 . \qquad \text{(B.1)}$$

(There is little doubt that the results below still hold without these conditions.)

We shall say that $(V, \tilde{\gamma})$ is *asymptotically Minkowskian in lightlike directions* if there exist constants R, C_1, C_2 and a coordinate system on V covering the set

$$\mathcal{O} \equiv \mathcal{O}_{R,C_1,C_2} \equiv \{(t,x) : r \geq R,\ C_1 < t - r < C_2\} , \qquad \text{(B.2)}$$

and constants C and N such that on $\mathcal{O}$ we have

$$|\tilde{\gamma}_{\mu\nu} - \eta_{\mu\nu}| + |\partial_\sigma \tilde{\gamma}_{\mu\nu}| \le C r^{-1} \log^N r \ . \tag{B.3}$$

We have the following:

Proposition B.1 *Let $(M, \tilde{g}, \tilde{K})$ be a polyhomogeneous data set (as defined in Section 5) with $g \in C^1(\bar{M})$ and $K \in C^0(\bar{M})$. Suppose moreover that we have $\partial M_i \approx S^2$, and that (B.1) holds, and that (changing time orientation if necessary) we have*

$$\left(\lambda_{AB} - \frac{h^{CD}\lambda_{CD}}{2} h_{AB} + K_{AB} - \frac{h^{CD}K_{CD}}{2} h_{AB}\right)\Big|_{\partial M_i} = 0 \, . \tag{B.4}$$

Then there exists a polyhomogeneous space–time $(V, \tilde{\gamma})$ which is asymptotically Minkowskian in lightlike directions and an embedding $i : M \to V$ such that $i^*\tilde{\gamma} = \tilde{g}$, and $\tilde{K}$ is the extrinsic curvature tensor of $i(M)$ in $(V, \tilde{\gamma})$.

Remarks:

1. We do not claim that $(V, \tilde{\gamma})$ can be chosen vacuum, though we believe that this is indeed the case.
2. The method of proof of Proposition B.1 works without the conditions $g \in C^1(\bar{M})$, $K \in C^0(\bar{M})$ as long as g and K are polyhomogeneous provided that the coefficients of the $\log^i x$ terms in K_{AB} are related to some of the coefficients of $x \log^i x$ in g_{ij}.

Proof: Let, as usual, $h_{AB} = g_{AB}|_{\partial M_i}$. As is well known, there exists a conformal diffeomorphism $\Phi : S^2 \to S^2$ such that $\Phi^* h_{AB} = \breve{\phi}^2 \breve{h}_{AB}$, where $\breve{h}_{AB}$ is the standard round metric on S^2,

$$\breve{h}_{AB} = d\theta^2 + \sin^2\theta \, d\phi^2 \ .$$

Let us still denote by g_{ij} the metric obtained by pull back with the map $(x, v) \to (x, \Phi(v))$, we thus have $g_{AB}|_{\partial M_i} = \breve{\phi}^2 \breve{h}_{AB}$. Let ϕ denote any function on $\bar{M}$ such that $\phi|_{\partial M_i} = \breve{\phi}$, with $\partial\phi/\partial x|_{\partial M_i}$ so chosen that the extrinsic curvature of ∂M_i in the metric $\phi^{-2} g$ is pure trace. Using once again the symbol g_{ij} for the metric $\phi^{-2} g_{ij}$ we have thus obtained $g_{AB}|_{\partial M_i} = \breve{h}_{AB}$. As has been shown in [7], for any polyhomogeneous metric Gauss coordinates near the boundary can be introduced, so that we have $g^{xA} = g(dx, dx^A) \equiv 0$, $g^{xx} = g(dx, dx) \equiv 1$ for $0 \le x \le x_0$, for some x_0. We thus have

$$g_{ij} dx^i dx^j = dx^2 + g_{AB} dx^A dx^B \ ,$$

with

$$g_{AB}|_{\partial M_i} = \breve{h}_{AB}, \qquad \lambda\Big|_{\partial M_i} = -\frac{1}{2} \breve{h}^{AB} \frac{\partial g_{AB}}{\partial x}\Big|_{\partial M_i} = 0 \, .$$

It now follows from (2.6) that

$$f \equiv \frac{\omega}{x} = 1 + O(x^2) \ .$$

Introducing $r = 1/x$ we have

$$\tilde{g}_{ij} dx^i dx^j = \omega^{-2} g_{ij} dx^i dx^j = f^{-2} r^{-2} dr^2 + f^{-2} r^2 g_{AB} dx^A dx^B \ . \tag{B.5}$$

Define
$$\tau \equiv t - \sqrt{1+r^2}\ ,$$
and set
$$\tilde{\gamma}_{tt} \equiv -1, \qquad \tilde{\gamma}_{ti} \equiv 0, \qquad \tilde{\gamma}_{ij}(\tau = 0) = \chi_{ij}\ , \tag{B.6}$$
with
$$\chi_{rr} = \frac{r^2}{1+r^2} + \tilde{g}_{rr} = \frac{r^2}{1+r^2} + \frac{1}{f^2 r^2}\ , \tag{B.7}$$
$$\chi_{rA} = 0\ , \tag{B.8}$$
$$\chi_{AB} = \tilde{g}_{AB} = \frac{r^2}{f^2} g_{AB}\ . \tag{B.9}$$

It follows that the embedding $(r, x^A) \xrightarrow{i} (t = \sqrt{1+r^2}, r, x^A)$ has the desired property that $i^*\tilde{\gamma} = \tilde{g}$, for any $\tilde{\gamma}$ which on $i(M)$ takes the values given by (B.6)–(B.9). To analyze the extrinsic curvature of the slice $\tau = 0$ it is useful to pass to the coordinates (τ, r, x^A) so that the metric
$$d\tilde{s}^2 = \tilde{\gamma}_{\mu\nu} dx^\mu dx^\nu = -dt^2 + \tilde{\gamma}_{ij} dx^i dx^j \tag{B.10}$$
takes the form
$$d\tilde{s}^2 = -d\tau^2 - 2\frac{r}{\sqrt{1+r^2}} d\tau dr + \tilde{g}_{ij} dx^i dx^j\ . \tag{B.11}$$
The extrinsic curvature tensor of $i(M)$ will take the desired values if we set
$$\frac{\partial \tilde{\gamma}_{ij}}{\partial \tau}(\tau = 0) = \psi_{ij}\ ,$$
with
$$\begin{aligned}\psi_{ij} &= -2\sqrt{\chi_{rr}} f r \tilde{K}_{ij} + \beta^k \frac{\partial \tilde{g}_{ij}}{\partial x^k} + \frac{\partial \beta^k}{\partial x^i} \tilde{g}_{kj} + \frac{\partial \beta^k}{\partial x^j} \tilde{g}_{ki} \\ &= 2\sqrt{\chi_{rr}} f r \left[\tilde{g}_{ij} - \frac{r}{f}(K_{ij} - \frac{1}{3} g^{kl} K_{kl} g_{ij}) \right] - \frac{f^2 r^3}{\sqrt{1+r^2}} \frac{\partial \tilde{g}_{ij}}{\partial r} \\ &\qquad + \frac{\partial \beta^k}{\partial x^i} \tilde{g}_{kj} + \frac{\partial \beta^k}{\partial x^j} \tilde{g}_{ki}\ ,\end{aligned} \tag{B.12}$$
$$\beta^A = 0\ , \tag{B.13}$$
$$\beta^r = -\frac{f^2 r^3}{\sqrt{1+r^2}} = -r^2 + O(1)\ , \tag{B.14}$$
where $\tilde{g}^{ij}$ is the matrix inverse to $\tilde{g}_{ij}$. One easily finds
$$\psi_{rr} = 2\sqrt{\chi_{rr}} f r \tilde{g}_{rr} + 2\beta^r \frac{\partial \tilde{g}_{rr}}{\partial r} + \frac{\partial \beta^r}{\partial r} \tilde{g}_{rr} + O(1/r^2) = O(1/r^2)\ , \tag{B.15}$$
$$\psi_{rA} = O(1/r)\ , \tag{B.16}$$
$$\psi_{AB} = -2r^2 (K_{AB} - \frac{1}{2} \frac{\partial h_{AB}}{\partial x}) + O(r \log^N r) = O(r \log^N r) \tag{B.17}$$

for some constant N (note that $\psi_{AB} = O(r \log^N r)$ wouldn't hold if (B.4) weren't satisfied). Let finally $\phi \in C^\infty(I\!R)$ be any function satisfying supp$\phi \subset (-\epsilon, \epsilon)$, $0 \le \phi \le 1$, $\phi|_{[-\epsilon/2,\epsilon/2]} = 1$, set $\tilde\phi(t,r) = \phi(t - \sqrt{1+r^2})$. Define

$$\tilde\gamma_{\mu\nu} dx^\mu dx^\nu = -dt^2 + (1-\tilde\phi)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] + \tilde\phi\left[\chi_{ij} + (t - \sqrt{1+r^2})\psi_{ij}\right] dx^i dx^j \ .$$

It is easily seen that ϵ can be chosen small enough so that the metric $\tilde\gamma$ satisfies (B.3) in the quasi–Minkowskian coordinates $(t, x = r\sin\theta\sin\phi, y = r\sin\theta\cos\phi, z = r\cos\theta)$ on the set $\{r \ge 1/x_0, t \in (-\infty,\infty)\}$. □

C The extrinsic curvature of a null hypersurface

Let (V,γ) be a Lorentzian manifold, let $\mathcal{N}$ be a null differentiable hypersurface in V, *i.e.*, $\mathcal{N}$ is a differentiable submanifold of codimension 1 such that for all $p \in \mathcal{N}$ there exists a neighbourhood $\mathcal{O}_p$ of p and a vector field k defined on $\mathcal{O}_p$ such that $\forall q \in \mathcal{O}_p$ we have $0 \ne k_q \in T_q\mathcal{N}$ and $\gamma(k,k) = 0$. It follows that k_q is defined up to the multiplication by a non-zero number. For $X, Y \in T_p\mathcal{N}$ define

$$N(X,Y) = \gamma(X, \nabla_Y k) \ . \tag{C.1}$$

It is easily seen that N is a symmetric tensor field on $\mathcal{N}$ satisfying

$$\forall X \in T_p\mathcal{N} \qquad N(k,X) = 0 \ . \tag{C.2}$$

In the space of 2–covariant symmetric tensors let us define the equivalence relation

$$N_1 \sim_1 N_2 \qquad \text{iff} \qquad \forall\ X,Y \in T_p\mathcal{N} \quad N_1(X,Y)_p = \alpha_p N_1(X,Y)_p\,, \quad \alpha_p \in I\!R\ . \tag{C.3}$$

It follows that the equivalence class $[N]_1$, N given by (C.1), is an object well defined by the geometry of V and by $\mathcal{N}$. $[N]_1$ will be called the *extrinsic curvature tensor* of $\mathcal{N}$.

Consider next a Lorentzian metric γ_1 on V such that $\gamma_1 = \phi^2\gamma$, where ϕ is a differentiable function on V; set

$$X, Y \in T_p\mathcal{N} \qquad N_1(X,Y) = \gamma_1(X, \nabla^1_Y k)\ , \tag{C.4}$$

where ∇^1 is the Levi–Civita covariant derivative of the metric γ_1. One finds

$$N_1(X,Y) = \phi^2 N(X,Y) + \beta\gamma(X,Y)\,, \tag{C.5}$$

$$\beta = -\phi\, k^\mu \partial_\mu \phi\,.$$

In the space of 2–covariant symmetric tensors let us define a second equivalence relation

$$N_1 \sim_2 N_2 \qquad \text{iff} \qquad \forall\ X,Y \in T_p\mathcal{N} \quad N_2(X,Y)_p = \alpha_p N_1(X,Y)_p + \beta_p\gamma(X,Y)\,, \quad \alpha_p, \beta_p \in I\!R\ . \tag{C.6}$$

From what has been said it follows that the equivalence class $[N]_2$, N given by (C.1), is is an object well defined by the conformal geometry of V and by $\mathcal{N}$. $[N]_2$ will be called the *conformal extrinsic curvature tensor* of $\mathcal{N}$.

D Proof of Proposition 3.3

Consider the equation

$$S_{ab} = \frac{|D\Omega|_g^2 - (\nabla_0\Omega)^2}{\Omega^2}\,\gamma_{ab} - \frac{2\nabla_a\nabla_b\Omega}{\Omega}\;; \tag{D.1}$$

the equations

$$\lim_{M\ni p\to\partial M_i} \Big[(\nabla_0\Omega)^2 - |D\Omega|_g^2\Big] \;=\; 0\,, \tag{D.2}$$

$$\lim_{M\ni p\to\partial M_i} \left[\frac{(\nabla_0\Omega)^2 - |D\Omega|_g^2}{\Omega}\,\gamma_{ab} - 2\,\nabla_a\nabla_b\Omega\right] \;=\; 0\,, \tag{D.3}$$

are clearly necessary conditions for

$$S_{ab}\Big|_M = \hat S_{ab} + \Omega\log\Omega\, S^{\log}_{ab}\,, \tag{D.4}$$

with $\hat S_{ab}\in C^1(\bar M)$, and $S^{\log}_{ab}$ satisfying the 1–condition (we will show that they are also sufficient). (D.2) is easily satisfied by setting

$$\nabla_0\Omega\Big|_M = \omega_0\,,\qquad \omega_0\Big|_{\partial M_i} = \epsilon|D\omega|_g\Big|_{\partial M_i}\,, \tag{D.5}$$

$$\epsilon = \pm 1\,,$$

ω_0 otherwise arbitrary, with an ϵ which shall be determined later. A $3+1$ decomposition of the trace–free part of (D.3),

$$\lim_{M\ni p\to\partial M_i}\left(\nabla_a\nabla_b\Omega - \frac14\Box\Omega\,\gamma_{ab}\right) = 0\,,$$

gives the equations

$$\lim_{M\ni p\to\partial M_i} \nabla_0\nabla_0\Omega \;=\; -\frac13 \lim_{M\ni p\to\partial M_i}(\Delta_g\omega + K\omega_0)\,, \tag{D.6}$$

$$\lim_{M\ni p\to\partial M_i} D_i\omega_0 \;=\; -\lim_{M\ni p\to\partial M_i}(K_i{}^jD_j\omega)\,, \tag{D.7}$$

$$\lim_{M\ni p\to\partial M_i}\left(D_iD_j\omega - \frac13\Delta_g\omega\, g_{ij}\right) \;=\; -\lim_{M\ni p\to\partial M_i}\left[\omega_0\left(K_{ij} - \frac13 K g_{ij}\right)\right]. \tag{D.8}$$

(D.6) will be satisfied if we choose

$$\nabla_0\nabla_0\Omega\Big|_M = \omega_{00}\,,\qquad \omega_{00}\Big|_{\partial M_i} = -\frac13(\Delta_g\omega + K\epsilon|D\omega|_g)\Big|_{\partial M_i}\,, \tag{D.9}$$

ω_{00} otherwise arbitrary. To analyze (D.7), recall that we have

$$\tilde K^{ij} = \omega^3K^{ij} + \omega_0\tilde g^{ij}\,, \tag{D.10}$$

in particular

$$\tilde K = \omega K + 3\omega_0 \quad\Rightarrow\quad \tilde K\Big|_{\partial M_i} = 3\omega_0\Big|_{\partial M_i}\,. \tag{D.11}$$

(D.10) gives

$$\tilde{D}_i(\tilde{K}^{ij} - \tilde{K}\tilde{g}^{ij}) = 0 \quad \Longleftrightarrow \quad \omega D_i(K^{ij} - Kg^{ij}) = 2\Big(K^{ij}D_i\omega + D^j\omega_0\Big)\,, \tag{D.12}$$

which shows that (D.7) is automatically satisfied. A (2+1) decomposition of (D.8) with $i = A$, $j = 1$ gives

$$\mathcal{D}_A D_1\omega\Big|_{\partial M_i} = -(\omega_0 K_{1A})\Big|_{\partial M_i}\,. \tag{D.13}$$

where $\mathcal{D}_A$ is the Levi–Civita connection of the metric h_{AB} induced on ∂M_i by g. Equation (D.13) is satisfied beacuse of (D.7) and (D.11). To show that (D.8) with $i = 1$, $j = 1$ holds, recall that we have the scalar constraint equation,

$$\omega\Delta_g\omega - \frac{3}{2}|D\omega|_g^2 + \frac{1}{4}\omega^2\Big(R(g) - |L|_g^2\Big) + \frac{1}{6}\tilde{K}^2 = 0\,, \tag{D.14}$$

from which it follows that

$$|D\omega|_g\Big|_{\partial M_i} = \frac{1}{3}|\tilde{K}|\Big|_{\partial M_i} \tag{D.15}$$

(which is consistent with (D.11)). Applying e^1 to (D.14) one obtains

$$\Big(2D_1D_1\omega + \lambda D_1\omega - \frac{1}{3}\tilde{K}D_1\tilde{K}\Big)\Big|_{\partial M_i} = 0\,, \tag{D.16}$$

where

$$\lambda \equiv h^{AB}\lambda_{AB}\,,$$

with λ_{AB} – the extrinsic curvature of ∂M_i in (M, g). With a little work from (D.16) one finds that

$$D_1D_1\omega\Big|_{\partial M_i} = -\frac{1}{2}\Big[\lambda D_1\omega + \Big(3K_{11} - K\Big)\omega_0\Big]\Big|_{\partial M_i}\,, \tag{D.17}$$

which is precisely (D.8) with $i = 1$, $j = 1$. It follows that the only non–trivial condition imposed by (D.8) is the equation

$$\Big[\omega_0\Big(K_{AB} - \frac{\kappa}{2}h_{AB}\Big)\Big]\Big|_{\partial M_i} = \Big[|D\omega|_g\Big(\lambda_{AB} - \frac{\lambda}{2}h_{AB}\Big)\Big]\Big|_{\partial M_i}\,, \tag{D.18}$$

$$\kappa \equiv h^{AB}K_{AB}\,,$$

which will be satisfied if we choose ϵ according to the sign occuring in (3.20). Now the trace of (D.3) gives

$$\lim_{M\ni p\to\partial M_i}\Big(\Box\,\Omega - 2\,\frac{|D\omega|_g^2 - (\omega_0)^2}{\omega}\Big) = 0\,,$$

which with a little effort can be seen to hold if (D.9) holds.

We have thus shown that (D.8) will hold whenever (D.18) holds provided we choose ω_0 and ω_{00} to satisfy (D.5) and (D.9). It is then elementary to show that (D.4) will be satisfied provided ω_0 and ω_{00} are moreover required to satisfy

$$\omega_0 = \hat{\omega}_0 + \frac{1}{3!}x^3\log x\,\hat{\omega}_0^{\log}\,,$$

$$\omega_{00} = \hat\omega_{00} + \frac{1}{2!}x^2 \log x\, \hat\omega_{00}^{\log} ,$$

with $\hat\omega_0 \in C^3(\bar M)$, $\hat\omega_{00} \in C^2(\bar M)$, $\hat\omega_0^{\log}$ satisfying the 3–condition, $\hat\omega_{00}^{\log}$ satisfying the 2–condition. [This is easily satisfied as we have a complete freedom of choice of the extensions of ω off M. We can *e.g.* choose ω_0 to be any function in $C^\infty_{\rm loc}(M)\cap C^3(\bar M)$ satisfying (D.5), and choose ω_{00} to be any function in $C^\infty_{\rm loc}(M)\cap C^2(\bar M)$ satisfying (D.9) (with such a choice we will have $\hat\omega_0^{\log}\equiv 0$, $\hat\omega_{00}^{\log}\equiv 0$; some other choices might be more convenient for some other purposes).] Now the embedding equations (A.11) and (A.12) together with the conditions on g and K show that

$$R_{ijk0}\Big|_M = \hat R_{ijk0} + x\log x\, \hat R^{\log}_{ijk0} , \qquad (D.19)$$

$$R_{ijkl}\Big|_M = \hat R_{ijkl} + x\log x\, \hat R^{\log}_{ijkl} , \qquad (D.20)$$

with $\hat R_{ijk0}$, $\hat R^{\log}_{ijk0}$, $\hat R_{ijkl}$, $\hat R^{\log}_{ijkl}$ satisfying the appropriate conditions. Since

$$R_{i0j0} = -R_{ij} + g^{kl}R_{ikjl} ,$$

equation (3.21) follows from (3.23) (which has already been established) and from (D.19)–(D.20). $\Box$

E Conventions, function spaces

The summation convention is used throughout this paper. Conventions on signature, covariant derivatives and extrinsic curvatures can be found in Appendix A.

Let $\bar M$ be a compact, connected Hausdorff manifold with boundary. Thus $M\equiv {\rm int}\bar M$ is a smooth, paracompact, Hausdorff manifold of dimension 3, and $\partial M\equiv\partial\bar M$ is a smooth manifold which is a (necessarily finite) sum of its connected components ∂M_i. Throughout this paper x will denote a defining function for ∂M, *i.e.* a function satisfying $x|_{\partial M}=0$, $x\ge 0$, $|dx|\Big|_{\partial M}>0$, and the implication $x(p)=0\Rightarrow p\in\partial M$ holds.

We can always choose a finite number of coordinate charts $\phi_j:\mathcal O_j\to I\!R^{n,+}\equiv\{y\in I\!R^n : y^1\ge 0\}$, $j=1,\dots,J$, covering a neighbourhood of ∂M such that $y^1=x$. When referring to local coordinates we shall implicitly assume that $y^1=x$, and we shall use the letter v to denote the coordinates y^2, y^3;

$$v^A = y^A , \quad A = 2,3 .$$

Thus

$$y = (x,v) .$$

The standard Schwartz multi-index notation is used throughout, thus if $\alpha=(\alpha_1,\dots\alpha_n)$, then

$$\partial^\alpha = \partial^\alpha_y = \partial^{\alpha_1}_{y^1}\cdots\partial^{\alpha_n}_{y^n} = \partial^{\alpha_1}_x\partial^{\alpha_2}_{y^2}\cdots\partial^{\alpha_n}_{y^n} = \partial^{\alpha_1}_x\partial^\beta_v ,$$

where $\beta=(\alpha_2,\dots,\alpha_n)$.

For $k\in I\!N_0\equiv I\!N\cup\{0\}$ the spaces $C^k_{\rm loc}(M)$ are the spaces of functions k–times differentiable on M. The spaces $C^k(\bar M)$ are the Banach spaces of functions differentiable k-times on M such that f and its derivatives can be extended to *continuous* functions on $\bar{M}$, and equipped with the supremum norm.

Let f_i be a sequence of functions, $f_i \in C^\infty_{\rm loc}(M)$, such that for every $N \in \mathbb{N}$ and for all $|\alpha| \le N$ we have

$$|\partial^\alpha_y f_i| \le C_{i,N} x^{s_{i,N}} ,$$

for some sequence $s_{i,N} \to_{i\to\infty} \infty$ and some constants $C_{i,N}$. We shall write

$$f \sim \sum_{i=0}^{\infty} f_i$$

if for every $n, M \in \mathbb{N}$ there exists $N \in \mathbb{N}$ and a constant $C(n,M)$ such that for all $|\alpha| \le M$

$$\left|\partial^\alpha_y \left(f - \sum_{i=0}^{N} f_i\right)\right| \le C(n,M)\, x^n .$$

f will be said to be polyhomogeneous, $f \in \mathcal{A}_{phg}$, if $f \in C^\infty_{\rm loc}(M)$ and if there exists a sequence $\{N_j\}_{j=0}^\infty$, $N_j \in \mathbb{N}_0$, and a sequence of functions $f_{jk} \in C^\infty(\bar{M})$ such that

$$f \sim \sum_{j=0}^{\infty} \sum_{k=0}^{N_j} f_{jk}\, x^j \,\ln^k x . \tag{E.1}$$

Let F be a function space over M. A tensor field $X = (X^A{}_\beta)$, where A, β are some multi-indices, $|A| = r$, $|\beta| = s$, will be said to belong to F if in local coordinates as described at the beginning of this section the components $X^A{}_\beta$ of X are in F. We shall write that $X \in x^\alpha F$ when $x^{-\alpha} X \in F$, etc. We shall write that $X \in F_1 + F_2$, where F_a, $a = 1,2$ are two function spaces, if there exist $X_a \in F_a$ such that $X = X_1 + X_2$.

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