CHAPTER 10

PARTICLE IN A GRAVITATIONAL FIELD

§ 81. Gravitational fields in nonrelativistic mechanics

Gravitational fields, or fields of gravity, have the basic property that all bodies move in them in the same manner, independently of mass, provided the initial conditions are the same.

For example, the laws of free fall in the gravity field of the earth are the same for all bodies; whatever their mass, all acquire one and the same acceleration.

This property of gravitational fields provides the possibility of establishing an analogy between the motion of a body in a gravitational field and the motion of a body not located in any external field, but which is considered from the point of view of a noninertial system of reference. Namely, in an inertial reference system, the free motion of all bodies is uniform and rectilinear, and if, say, at the initial time their velocities are the same, they will be the same for all times. Clearly, therefore, if we consider this motion in a given noninertial system, then relative to this system all the bodies will move in the same way.

Thus the properties of the motion in a noninertial system are the same as those in an inertial system in the presence of a gravitational field. In other words, a noninertial reference system is equivalent to a certain gravitational field. This is called the *principle of equivalence*.

Let us consider, for example, motion in a uniformly accelerated reference system. A body of arbitrary mass, freely moving in such a system of reference, clearly has relative to this system a constant acceleration, equal and opposite to the acceleration of the system itself. The same applies to motion in a uniform constant gravitational field, e.g. the field of gravity of the earth (over small regions, where the field can be considered uniform). Thus a uniformly accelerated system of reference is equivalent to a constant, uniform external field. In the same way, nonuniformity accelerated linear motion of the reference system is clearly equivalent to a uniform but gravitational field.

However, the fields to which noninertial reference systems are equivalent are not completely identical with "actual" gravitational fields which occur also in inertial frames. For there is a very essential difference with respect to their behavior at infinity. At infinite distances from the bodies producing the field, "actual" gravitational fields always go to zero. Contrary to this, the fields to which noninertial frames are equivalent increase without limit at infinity, or, in any event, remain finite in value. Thus, for example, the centrifugal force which appears in a rotating reference system increases without limit as we move away from the axis of rotation; the field to which a reference system in accelerated linear motion is equivalent is the same over all space and also at infinity.

The fields to which noninertial systems are equivalent vanish as soon as we transform to an inertial system. In contrast to this, "actual" gravitational fields (existing also in an inertial reference frame) cannot be eliminated by any choice of reference system. This is already clear from what has been said above concerning the difference in conditions at infinity between "actual" gravitational fields and fields to which noninertial systems are equivalent; since the latter do not approach zero at infinity, it is clear that it is impossible, by any choice of reference frame, to eliminate an "actual" field, since it vanishes at infinity.

All that can be done by a suitable choice of reference system is to eliminate the gravitational field in a given region of space, sufficiently small so that the field can be considered uniform over it. This can be done by choosing a system in accelerated motion, the acceleration of which is equal to that which would be acquired by a particle placed in the region of the field which we are considering.

The motion of a particle in a gravitational field is determined, in nonrelativistic mechanics, by a Lagrangian having (in an inertial reference frame) the form

$$L = \frac{mv^2}{2} - m\phi, \tag{81.1}$$

where ϕ is a certain function of the coordinates and time which characterizes the field and is called the *gravitational potential*.† Correspondingly, the equation of motion of the particle is

$$\dot{\mathbf{v}} = -\text{grad}\, \phi. \tag{81.2}$$

It does not contain the mass or any other constant characterizing the properties of the particle; this is the mathematical expression of the basic property of gravitational fields.

§ 82. The gravitational field in relativistic mechanics

The fundamental property of gravitational fields that all bodies move in them in the same way, remains valid also in relativistic mechanics. Consequently there remains also the analogy between gravitational fields and noninertial reference systems. Therefore in studying the properties of gravitational fields in relativistic mechanics, we naturally also start from this analogy.

In an inertial reference system, in cartesian coordinates, the interval ds is given by the relation:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2.$$

Upon transforming to any other inertial reference system (i.e. under Lorentz transformation), the interval, as we know, retains the same form. However, if we transform to a noninertial system of reference, ds^2 will no longer be a sum of squares of the four coordinate differentials.

So, for example, when we transform to a uniformly rotating system of coordinates,

$$x = x' \cos \Omega t - y' \sin \Omega t, \qquad y = x' \sin \Omega t + y' \cos \Omega t, \qquad z = z'$$

(Ω is the angular velocity of the rotation, directed along the Z axis), the interval takes on the

† In what follows we shall seldom have to use the electromagnetic potential ϕ, so that the designation of the gravitational potential by the same symbol cannot lead to misunderstanding.

form

$$ds^2 = [c^2 - \Omega^2(x'^2 + y'^2)]dt^2 - dx'^2 - dy'^2 - dz'^2 + 2\Omega y'\, dx'\, dt - 2\Omega x'\, dy'\, dt.$$

No matter what the law of transformation of the time coordinate, this expression cannot be represented as a sum of squares of the coordinate differentials.

Thus in a noninertial system of reference the square of an interval appears as a quadratic form of general type in the coordinate differentials, that is, it has the form

$$ds^2 = g_{ik}\, dx^i dx^k, \tag{82.1}$$

where the g_{ik} are certain functions of the space coordinates x^1, x^2, x^3 and the time coordinate x^0. Thus, when we use a noninertial system, the four-dimensional coordinate system x^0, x^1, x^2, x^3 is curvilinear. The quantities g_{ik}, determining all the geometric properties in each curvilinear system of coordinates, represent, we say, the *space-time metric*.

The quantities g_{ik} can clearly always be considered symmetric in the indices i and k ($g_{ki} = g_{ik}$), since they are determined from the symmetric form (82.1), where g_{ik} and g_{ki} enter as factors of one and the same product $dx^i dx^k$. In the general case, there are ten different quantities g_{ik}—four with equal, and $4 \cdot 3/2 = 6$ with different indices. In an inertial reference system, when we use cartesian space coordinates $x^{1,2,3} = x, y, z$, and the time, $x^0 = ct$, the quantities g_{ik} are

$$g_{00} = 1, \qquad g_{11} = g_{22} = g_{33} = -1, \qquad g_{ik} = 0 \quad \text{for} \quad i \neq k. \tag{82.2}$$

We call a four-dimensional system of coordinates with these values of g_{ik} *galilean*.

In the previous section it was shown that a noninertial system of reference is equivalent to a certain field of force. We now see that in relativistic mechanics, these fields are determined by the quantities g_{ik}.

The same applies also to "actual" gravitational fields. Any gravitational field is just a change in the metric of space-time, as determined by the quantities g_{ik}. This important fact means that the geometrical properties of space-time (its metric) are determined by physical phenomena, and are not fixed properties of space and time.

The theory of gravitational fields, constructed on the basis of the theory of relativity, is called the *general theory of relativity*. It was established by Einstein (and finally formulated by him in 1915), and represents probably the most beautiful of all existing physical theories. It is remarkable that it was developed by Einstein in a purely deductive manner and only later was substantiated by astronomical observations.

As in nonrelativistic mechanics, there is a fundamental difference between "actual" gravitational fields and fields to which noninertial reference systems are equivalent. Upon transforming to a noninertial reference system, the quadratic form (82.1), i.e. the quantities g_{ik}, are obtained from their galilean values (82.2) by a simple transformation of coordinates, and can be reduced over all space to their galilean values by the inverse coordinate transformation. That such forms for g_{ik} are very special is clear from the fact that it is impossible by a mere transformation of the four coordinates to bring the ten quantities g_{ik} to a preassigned form.

An "actual" gravitational field cannot be eliminated by any transformation of coordinates. In other words, in the presence of a gravitational field space-time is such that the quantities g_{ik} determining its metric cannot, by any coordinate transformation, be brought to their galilean values over all space. Such a space-time is said to be *curved*, in contrast to *flat* space-time, where such a reduction is possible.

By an appropriate choice of coordinates, we can, however, bring the quantities g_{ik} to galilean form at any individual point of the non-galilean space-time: this amounts to the reduction to diagonal form of a quadratic form with constant coefficients (the values of g_{ik} at the given point). Such a coordinate system is said to be *galilean for the given point*.†

We note that, after reduction to diagonal form at a given point, the matrix of the quantities g_{ik} has one positive and three negative principal values.‡ From this it follows, in particular, that the determinant g, formed from the quantities g_{ik}, is always negative for a real space-time:

$$g < 0. \tag{82.3}$$

A change in the metric of space-time also means a change in the purely spatial metric. To a galilean g_{ik} in flat space-time, there corresponds a euclidean geometry of space. In a gravitational field, the geometry of space becomes non-euclidean. This applies both to "true" gravitational fields, in which space-time is "curved", as well as to fields resulting from the fact that the reference system is non-inertial, which leave the space-time flat.

The problem of spatial geometry in a gravitational field will be considered in more detail in § 84. It is useful to give here a simple argument which shows pictorially that space will become non-euclidean when we change to a non-inertial system of reference. Let us consider two reference frames, of which one (K) is inertial, while the other (K') rotates uniformly with respect to K around their common z axis. A circle in the x, y plane of the K system (with its center at the origin) can also be regarded as a circle in the x', y' plane of the K' system. Measuring the length of the circle and its diameter with a yardstick in the K system, we obtain values whose ratio is equal to π, in accordance with the euclidean character of the geometry in the inertial reference system. Now let the measurement be carried out with a yardstick at rest relative to K'. Observing this process from the K system, we find that the yardstick laid along the circumference suffers a Lorentz contraction, whereas the yardstick placed radially is not changed. It is therefore clear that the ratio of the circumference to the diameter, obtained from such a measurement, will be greater than π.

In the general case of an arbitrary, varying gravitational field, the metric of space is not only non-euclidean, but also varies with the time. This means that the relations between different geometrical distances change with time. As a result, the relative position of "test bodies" introduced into the field cannot remain unchanged in any coordinate system.§ Thus if the particles are placed around the circumference of a circle and along a diameter, since the ratio of the circumference to the diameter is not equal to π and changes with time, it is clear that if the separations of the particles along the diameter remain unchanged the separations around the circumference must change, and conversely. Thus in the general theory of relativity it is impossible in general to have a system of bodies which are fixed relative to one another.

This result essentially changes the very concept of a system of reference in the general theory of relativity, as compared to its meaning in the special theory. In the latter we meant

† To avoid misunderstanding, we state immediately that the choice of such a coordinate system does not mean that the gravitational field has been eliminated over the corresponding infinitesimal volume of four-space. Such an elimination is also always possible, by virtue of the principle of equivalence, and has a greater significance (see § 87).

‡ This set of signs is called the *signature* of the matrix.

§ Strictly speaking, the number of particles should be greater than four. Since we can construct a tetrahedron from any six line segments, we can always, by a suitable definition of the reference system, make a system of four particles form an invariant tetrahedron. *A fortiori*, we can fix the particles relative to one another in systems of three or two particles.

by a reference system a set of bodies at rest relative to one another in unchanging relative positions. Such systems of bodies do not exist in the presence of a variable gravitational field, and for the exact determination of the position of a particle in space we must, strictly speaking, have an infinite number of bodies which fill all the space like some sort of "medium". Such a system of bodies with arbitrarily running clocks fixed on them constitutes a reference system in the general theory of relativity.

In connection with the arbitrariness of the choice of a reference system, the laws of nature must be written in the general theory of relativity in a form which is appropriate to any four-dimensional system of coordinates (or, as one says, in "*covariant*" form). This, of course, does not imply the physical equivalence of all these reference systems (like the physical equivalence of all inertial reference systems in the special theory). On the contrary, the specific appearances of physical phenomena, including the properties of the motion of bodies, become different in all systems of reference.

§ 83. Curvilinear coordinates

Since, in studying gravitational fields we are confronted with the necessity of considering phenomena in an arbitrary reference frame, it is necessary to develop four-dimensional geometry in arbitrary curvilinear coordinates. Sections 83, 85 and 86 are devoted to this.

Let us consider the transformation from one coordinate system, x^0, x^1, x^2, x^3, to another x'^0, x'^1, x'^2, x'^3:

$$x^i = f^i(x'^0, x'^1, x'^2, x'^3),$$

where the f^i are certain functions. When we transform the coordinates, their differentials transform according to the relation

$$dx^i = \frac{\partial x^i}{\partial x'^k} dx'^k. \tag{83.1}$$

Every aggregate of four quantities A^i ($i = 0, 1, 2, 3$), which under a transformation of coordinates transform like the coordinate differentials, is called a *contravariant* four-vector:

$$A^i = \frac{\partial x^i}{\partial x'^k} A'^k. \tag{83.2}$$

Let ϕ be some scalar. Under a coordinate transformation, the four quantities $\partial\phi/\partial x^i$ transform according to the formula

$$\frac{\partial \phi}{\partial x^i} = \frac{\partial \phi}{\partial x'^k}\frac{\partial x'^k}{\partial x^i}, \tag{83.3}$$

which is different from formula (83.2). Every aggregate of four quantities A_i which, under a coordinate transformation, transform like the derivatives of a scalar, is called a *covariant* four-vector:

$$A_i = \frac{\partial x'^k}{\partial x^i} A'_k. \tag{83.4}$$

Because two types of vectors appear in curvilinear coordinates, there are three types of tensors of the second rank. We call a *contravariant tensor* of the second rank, A^{ik}, an aggregate of sixteen quantities which transform like the products of the components of two contravariant vectors, i.e. according to the law

$$A^{ik} = \frac{\partial x^i}{\partial x'^l}\frac{\partial x^k}{\partial x'^m} A'^{lm}. \tag{83.5}$$

A *covariant tensor* of rank two, transforms according to the formula

$$A_{ik} = \frac{\partial x'^l}{\partial x^i}\frac{\partial x'^m}{\partial x^k} A'_{lm}, \tag{83.6}$$

and a *mixed tensor* transforms as follows:

$$A^i_k = \frac{\partial x^i}{\partial x'^l}\frac{\partial x'^m}{\partial x^k} A'^l_m. \tag{83.7}$$

The definitions given here are the natural generalization of the definitions of four-vectors and four-tensors in galilean coordinates (§ 6), according to which the differentials dx^i constitute a contravariant four-vector and the derivatives $\partial\phi/\partial x^i$ form a covariant four-vector.†

The rules for forming four-tensors by multiplication or contraction of products of other four-tensors remain the same in curvilinear coordinates as they were in galilean coordinates. For example, it is easy to see that, by virtue of the transformation laws (83.2) and (83.4), the scalar product of two four-vectors A^iB_i is invariant:

$$A^iB_i = \frac{\partial x^i}{\partial x'^l}\frac{\partial x'^m}{\partial x^i} A'^lB'_m = \frac{\partial x'^m}{\partial x'^l}A'^lB'_m = A'^lB'_l.$$

The unit four-tensor δ^i_k is defined the same as before in curvilinear coordinates: its components are again $\delta^i_k = 0$ for $i \neq k$, and are equal to 1 for $i = k$. If A^k is a four-vector, then multiplying by δ^i_k we get:

$$A^k\delta^i_k = A^i,$$

i.e. another four-vector; this proves that δ^i_k is a tensor.

The square of the line element in curvilinear coordinates is a quadratic form in the differentials dx^i:

$$ds^2 = g_{ik}dx^idx^k, \tag{83.8}$$

where the g_{ik} are functions of the coordinates; g_{ik} is symmetric in the indices i and k:

$$g_{ik} = g_{ki}. \tag{83.9}$$

Since the (contracted) product of g_{ik} and the contravariant tensor $dx^i\,dx^k$ is a scalar, the g_{ik} form a covariant tensor; it is called the *metric tensor*.

Two tensors A_{ik} and B^{ik} are said to be *reciprocal* to each other if

$$A_{ik}B^{kl} = \delta^l_i.$$

In particular the contravariant metric tensor is the tensor g^{ik} reciprocal to the tensor g_{ik}, that is,

$$g_{ik}g^{kl} = \delta^l_i. \tag{83.10}$$

The same physical quantity can be represented in contra- or co-variant components. It is obvious that the only quantities that can determine the connection between the different

† Nevertheless, while in a galilean system the coordinates x^i themselves (and not just their differentials) also form a four-vector, this is, of course, not the case in curvilinear coordinates.

forms are the components of the metric tensor. This connection is given by the formulas:

$$A^i = g^{ik}A_k, \qquad A_i = g_{ik}A^k. \tag{83.11}$$

In a galilean coordinate system the metric tensor has components:

$$g_{ik}^{(0)} = g^{ik(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{83.12}$$

Then formula (83.11) gives the familiar relation $A^0 = A_0$, $A^{1,2,3} = -A_{1,2,3}$, etc.†

These remarks also apply to tensors. The transition between the different forms of a given physical tensor is accomplished by using the metric tensor according to the formulas:

$$A^i{}_k = g^{il}A_{lk}, \qquad A^{ik} = g^{il}g^{km}A_{lm},$$

etc.

In § 6 we defined (in galilean coordinates) the completely antisymmetric unit pseudotensor e^{iklm}. Let us transform it to an arbitrary system of coordinates, and now denote it by E^{iklm}. We keep the notation e^{iklm} for the quantities defined as before by $e^{0123} = 1$ (or $e_{0123} = -1$).

Let the x'^i be galilean, and the x^i be arbitrary curvilinear coordinates. According to the general rules for transformation of tensors, we have:

$$E^{iklm} = \frac{\partial x^i}{\partial x'^p}\frac{\partial x^k}{\partial x'^r}\frac{\partial x^l}{\partial x'^s}\frac{\partial x^m}{\partial x'^t}e^{prst},$$

or

$$E^{iklm} = J\,e^{iklm},$$

where J is the determinant formed from the derivatives $\partial x^i/\partial x'^p$, i.e. it is just the Jacobian of the transformation from the galilean to the curvilinear coordinates:

$$J = \frac{\partial(x^0, x^1, x^2, x^3)}{\partial(x'^0, x'^1, x'^2, x'^3)}.$$

This Jacobian can be expressed in terms of the determinant of the metric tensor g_{ik} (in the system x^i). To do this we write the formula for the transformation of the metric tensor:

$$g^{ik} = \frac{\partial x^i}{\partial x'^l}\frac{\partial x^k}{\partial x'^m}g^{lm(0)},$$

and equate the determinants of the two sides of this equation. The determinant of the reciprocal tensor $|g^{ik}| = 1/g$. The determinant $|g^{lm(0)}| = -1$. Thus we have $1/g = -J^2$, and so $J = 1/\sqrt{-g}$.

Thus, in curvilinear coordinates the antisymmetric unit tensor of rank four must be defined as

$$E^{iklm} = \frac{1}{\sqrt{-g}}\,e^{iklm}. \tag{83.13}$$

† Whenever, in giving analogies, we use galilean coordinate systems, one should realize that such a system can be selected only in a flat space. In the case of a curved four-space, one should speak of a coordinate system that is galilean over a given infinitesimal element of four-volume, which can always be found. None of the derivations are affected by this change.

The indices of this tensor are lowered by using the formula

$$e^{prst} g_{ip} g_{kr} g_{ls} g_{mt} = -g\, e_{iklm},$$

so that its covariant components are

$$E_{iklm} = \sqrt{-g}\, e_{iklm}. \tag{83.14}$$

In a galilean coordinate system x'^i the integral of a scalar with respect to $d\Omega' = dx'^0\, dx'^1\, dx'^2\, dx'^3$ is also a scalar, i.e. the element $d\Omega'$ behaves like a scalar in the integration (§ 6). On transforming to curvilinear coordinates x^i, the element of integration $d\Omega'$ goes over into

$$d\Omega' \to \frac{1}{J}\, d\Omega = \sqrt{-g}\, d\Omega.$$

Thus, in curvilinear coordinates, when integrating over a four-volume the quantity $\sqrt{-g}\,d\Omega$ behaves like an invariant.†

All the remarks at the end of § 6 concerning elements of integration over hypersurfaces, surfaces and lines remain valid for curvilinear coordinates, with the one difference that the definition of dual tensors changes. The element of "area" of the hypersurface spanned by three infinitesimal displacements is the contravariant antisymmetric tensor dS^{ikl}; the vector dual to it is gotten by multiplying by the tensor $\sqrt{-g}\, e_{iklm}$, so it is equal to

$$\sqrt{-g}\, dS_i = -\tfrac{1}{6} e_{iklm}\, dS^{klm} \sqrt{-g}. \tag{83.15}$$

Similarly, if df^{ik} is the element of (two-dimensional) surface spanned by two infinitesimal displacements, the dual tensor is defined as‡

$$\sqrt{-g}\, df^*_{ik} = \tfrac{1}{2}\sqrt{-g}\, e_{iklm}\, df^{lm}. \tag{83.16}$$

We keep the designations dS_i and df^*_{ki} as before for $\frac{1}{6} e_{iklm}\, dS^{klm}$ and $\frac{1}{2} e_{iklm}\, df^{lm}$ (and not their products by $\sqrt{-g}$); the rules (6.14–19) for transforming the various integrals into one another remain the same, since their derivation was formal in character and not related to the tensor properties of the different quantities. Of particular importance is the rule for transforming the integral over a hypersurface into an integral over a four-volume (Gauss' theorem), which is accomplished by the substitution:

$$dS_i \to d\Omega\, \frac{\partial}{\partial x^i}. \tag{83.17}$$

† If ϕ is a scalar, the quantity $\sqrt{-g}\,\phi$, which gives an invariant when integrated over $d\Omega$, is called a *scalar density*. Similarly, we speak of *vector* and *tensor densities* $\sqrt{-g}\,A^i$, $\sqrt{-g}\,A^{ik}$, etc. These quantities give a vector or tensor on multiplication by the four-volume element $d\Omega$ (the integral $\int A^i \sqrt{-g}\, d\Omega$ over a finite region cannot, generally speaking, be a vector, since the laws of transformation of the vector A^i are different at different points).

‡ It is understood that the elements dS^{klm} and df^{ik} are constructed on the infinitesimal displacements dx^i, dx'^i, dx''^i in the same way as in § 6, no matter what the geometrical significance of the coordinates x^i. Then the formal significance of the elements dS_i and df^*_{ik} is the same as before. In particular, as before $dS_0 = dx_1\, dx_2\, dx_3 \equiv dV$. We keep the earlier definition of dV for the product of differentials of the three space coordinates; we must, however, remember that the element of geometrical spatial volume is given in curvilinear coordinates not by dV, but by $\sqrt{\gamma}\, dV$, where γ is the determinant of the spatial metric tensor (which will be defined in the next section).

§ 84. Distances and time intervals

We have already said that in the general theory of relativity the choice of a coordinate system is not limited in any way; the triplet of space coordinates x^1, x^2, x^3, can be any quantities defining the position of bodies in space, and the time coordinate x^0 can be defined by an arbitrarily running clock. The question arises of how, in terms of the values of the quantities x^1, x^2, x^3, x^0, we can determine actual distances and time intervals.

First we find the relation of the proper time, which from now on we shall denote by τ, to the coordinate x^0. To do this we consider two infinitesimally separated events, occurring at one and the same point in space. Then the interval ds between the two events is, as we know, just $c\,d\tau$, where $d\tau$ is the (proper) time interval between the two events. Setting $dx^1 = dx^2 = dx^3 = 0$ in the general expression $ds^2 = g_{ik}\,dx^i\,dx^k$, we consequently find

$$ds^2 = c^2\,d\tau^2 = g_{00}(dx^0)^2,$$

from which

$$d\tau = \frac{1}{c}\sqrt{g_{00}}\,dx^0, \tag{84.1}$$

or else, for the time between any two events occurring at the same point in space,

$$\tau = \frac{1}{c}\int \sqrt{g_{00}}\,dx^0. \tag{84.2}$$

This relation determines the actual time interval (or as it is also called, the *proper time* for the given point in space) for a change of the coordinate x^0. We note in passing that the quantity g_{00}, as we see from these formulas, is positive:

$$g_{00} > 0. \tag{84.3}$$

It is necessary to emphasize the difference between the meaning of (84.3) and the meaning of the signature [the signs of three principal values of the tensor g_{ik} (§ 82)]. A tensor g_{ik} which does not satisfy the second of these conditions cannot correspond to any real gravitational field, i.e. cannot be the metric of a real space-time. Nonfulfilment of the condition (84.3) would mean only that the corresponding system of reference cannot be realized with real bodies; if the condition on the principal values is fulfilled, then a suitable transformation of the coordinates can make g_{00} positive (an example of such a system is given by the rotating system of coordinates, see § 89).

We now determine the element dl of *spatial distance*. In the special theory of relativity we can define dl as the interval between two infinitesimally separated events occurring at one and the same time. In the general theory of relativity, it is usually impossible to do this, i.e. it is impossible to determine dl by simply setting $dx^0 = 0$ in ds. This is related to the fact that in a gravitational field the proper time at different points in space has a different dependence on the coordinate x^0.

To find dl, we now proceed as follows.

Suppose a light signal is directed from some point B in space (with coordinates $x^\alpha + dx^\alpha$) to a point A infinitely near to it (and having coordinates x^α) and then back over the same path. Obviously, the time (as observed from the one point B) required for this, when multiplied by c, is twice the distance between the two points.

Let us write the interval, separating the space and time coordinates:

$$ds^2 = g_{\alpha\beta}\,dx^\alpha\,dx^\beta + 2g_{0\alpha}\,dx^0\,dx^\alpha + g_{00}\,(dx^0)^2 \tag{84.4}$$

where it is understood that we sum over repeated Greek indices from 1 to 3. The interval between the events corresponding to the departure and arrival of the signal from one point to the other is equal to zero. Solving the equation $ds^2 = 0$ with respect to dx^0, we find two roots:

$$dx^{0(1)} = \frac{1}{g_{00}}\{-g_{0\alpha}dx^\alpha - \sqrt{(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00})\,dx^\alpha dx^\beta}\},$$
$$dx^{0(2)} = \frac{1}{g_{00}}\{-g_{0\alpha}dx^\alpha + \sqrt{(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00})\,dx^\alpha dx^\beta}\}, \tag{84.5}$$

corresponding to the propagation of the signal in the two directions between A and B. If x^0 is the moment of arrival of the signal at A, the times when it left B and when it will return to B are, respectively, $x^0 + dx_0^{(1)}$ and $x^0 + dx_0^{(2)}$. In the schematic diagram of Fig. 18 the solid lines are the world lines corresponding to the given coordinates x^α and $x^\alpha + dx^\alpha$, while the dashed lines are the world lines of the signals.† It is clear that the total interval of "time" between the departure of the signal and its return to the original point is equal to

$$dx^{0(2)} - dx^{0(1)} = \frac{2}{g_{00}}\sqrt{(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00})\,dx^\alpha dx^\beta}.$$

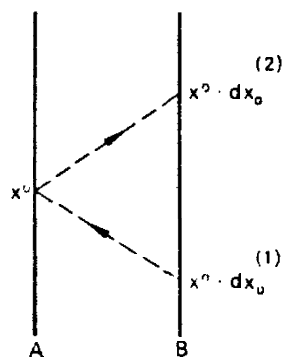

FIG. 18

The corresponding interval of proper time is obtained, according to (84.1), by multiplying by $\sqrt{g_{00}}/c$, and the distance dl between the two points by multiplying once more by $c/2$. As a result, we obtain

$$dl^2 = \left(-g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}\right) dx^\alpha dx^\beta.$$

This is the required expression, defining the distance in terms of the space coordinate elements. We rewrite it in the form

$$dl^2 = \gamma_{\alpha\beta}\, dx^\alpha dx^\beta, \tag{84.6}$$

where

$$\gamma_{\alpha\beta} = \left(-g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}\right). \tag{84.7}$$

† In Fig. 18, it is assumed that $dx_0^{(2)} > 0$, $dx_0^{(1)} < 0$, but this is not necessary: $dx^{0(1)}$ and $dx^{0(2)}$ may have the same sign. The fact that in this case the value $x^0(A)$ at the moment of arrival of the signal at A might be less than the value $x^0(B)$ at the moment of its departure from B contains no contradiction, since the rates of clocks at different points in space are not assumed to be synchronized in any way.

is the three-dimensional metric tensor, determining the metric, i.e., the geometric properties of the space. The relations (84.7) give the connection between the metric of real space and the metric of the four-dimensional space-time.†

However, we must remember that the g_{ik} generally depend on x^0, so that the space metric (84.6) also changes with time. For this reason, it is meaningless to integrate dl; such an integral would depend on the world line chosen between the two given space points. Thus, generally speaking, in the general theory of relativity the concept of a definite distance between bodies loses its meaning, remaining valid only for infinitesimal distances. The only case where the distance can be defined also over a finite domain is that in which the g_{ik} do not depend on the time, so that the integral $\int dl$ along a space curve has a definite meaning.

It is worth noting that the tensor $-\gamma_{\alpha\beta}$ is the reciprocal of the contravariant three-dimensional tensor $g^{\alpha\beta}$. In fact, from $g^{ik}g_{kl} = \delta^i_l$, we have, in particular,

$$g^{\alpha\beta}g_{\beta\gamma}+g^{\alpha 0}g_{0\gamma} = \delta^\alpha_\gamma, \qquad g^{\alpha\beta}g_{\beta 0}+g^{\alpha 0}g_{00} = 0, \qquad g^{0\beta}g_{\beta 0}+g^{00}g_{00} = 1. \tag{84.8}$$

Determining $g^{\alpha 0}$ from the second equation and substituting in the first, we obtain:

$$-g^{\alpha\beta}\gamma_{\beta\gamma} = \delta^\alpha_\gamma.$$

This result can be formulated differently, by the statement that the quantities $-g^{\alpha\beta}$ form the contravariant three-dimensional metric tensor corresponding to the metric (84.6):

$$\gamma^{\alpha\beta} = -g^{\alpha\beta}. \tag{84.9}$$

We also state that the determinants g and γ, formed respectively from the quantities g_{ik} and $\gamma_{\alpha\beta}$, are related to one another by

$$-g = g_{00}\gamma. \tag{84.10}$$

In some of the later applications it will be convenient to introduce the three-dimensional vector **g**, whose covariant components are defined as

$$g_\alpha = -\frac{g_{0\alpha}}{g_{00}}. \tag{84.11}$$

Considering **g** as a vector in the space with metric (84.6), we must define its contravariant components as $g^\alpha = \gamma^{\alpha\beta}g_\beta$. Using (84.9) and the second of equations (84.8), it is easy to see that

$$g^\alpha = \gamma^{\alpha\beta}g_\beta = -g^{0\alpha}. \tag{84.12}$$

We also note the formula

$$g^{00} = \frac{1}{g_{00}} - g_\alpha g^\alpha, \tag{84.13}$$

which follows from the third of equations (84.8).

† The quadratic form (84.6) must clearly be positive definite. For this, its coefficients must, as we know from the theory of forms, satisfy the conditions

$$\gamma_{11} > 0, \qquad \begin{vmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{vmatrix} > 0, \qquad \begin{vmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{vmatrix} > 0.$$

Expressing γ_{ik} in terms of g_{ik}, it is easy to show that these conditions take the form

$$\begin{vmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{vmatrix} < 0, \qquad \begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} > 0, \qquad g < 0.$$

These conditions, together with the condition (84.3), must be satisfied by the components of the metric tensor in every system of reference which can be realized with the aid of real bodies.

We now turn to the definition of the concept of simultaneity in the general theory of relativity. In other words, we discuss the question of the possibility of synchronizing clocks located at different points in space, i.e. the setting up of a correspondence between the readings of these clocks.

Such a synchronization must obviously be achieved by means of an exchange of light signals between the two points. We again consider the process of propagation of signals between two infinitely near points A and B, as shown in Fig. 18. We should regard as simultaneous with the moment x^0 at the point A that reading of the clock at point B which is halfway between the moments of departure and return of the signal to that point, i.e. the moment

$$x^0+\Delta x^0 = x^0+\tfrac{1}{2}(dx^{0(2)}+dx^{0(1)}).$$

Substituting (84.5), we thus find that the difference in the values of the "time" x^0 for two simultaneous events occurring at infinitely near points is given by

$$\Delta x^0 = -\frac{g_{0\alpha}dx^\alpha}{g_{00}} \equiv g_\alpha dx^\alpha. \tag{84.14}$$

This relation enables us to synchronize clocks in any infinitesimal region of space. Carrying out a similar synchronization from the point A, we can synchronize clocks, i.e. we can define simultaneity of events, along any open curve.†

However, synchronization of clocks along a closed contour turns out to be impossible in general. In fact, starting out along the contour and returning to the initial point, we would obtain for Δx^0 a value different from zero. Thus it is, *a fortiori*, impossible to synchronize clocks over all space. The exceptional cases are those reference systems in which all the components $g_{0\alpha}$ are equal to zero.‡

It should be emphasized that the impossibility of synchronization of all clocks is a property of the arbitrary reference system, and not of the space-time itself. In any gravitational field, it is always possible (in infinitely many ways) to choose the reference system so that the three quantities $g_{0\alpha}$ become identically equal to zero, and thus make possible a complete synchronization of clocks (see § 97).

Even in the special theory of relativity, proper time elapses differently for clocks moving relative to one another. In the general theory of relativity, proper time elapses differently even at different points of space in the same reference system. This means that the interval of proper time between two events occurring at some point in space, and the interval of time between two events simultaneous with these at another point in space, are in general different from one another.

§ 85. Covariant differentiation

In galilean coordinates§ the differentials dA_i of a vector A_i form a vector, and the derivatives $\partial A_i/\partial x^k$ of the components of a vector with respect to the coordinates form a tensor. In curvilinear coordinates this is not so; dA_i is not a vector, and $\partial A_i/\partial x^k$ is not a

† Multiplying (84.14) by g_{00} and bringing both terms to one side, we can state the condition for synchronization in the form $dx_0 = g_{0i}dx^i = 0$: the "covariant differential" dx_0 between two infinitely near simultaneous events must be equal to zero.

‡ We should also assign to this class those cases where the $g_{0\alpha}$ can be made equal to zero by a simple transformation of the time coordinate, which does not involve any choice of the system of objects serving for the definition of the space coordinates.

§ In general, whenever the quantities g_{ik} are constant.

tensor. This is due to the fact that dA_i is the difference of vectors located at different (infinitesimally separated) points of space; at different points in space vectors transform differently, since the coefficients in the transformation formulas (83.2), (83.4) are functions of the coordinates.

It is also easy to verify these statements directly. To do this we determine the transformation formulas for the differentials dA_i in curvilinear coordinates. A covariant vector is transformed according to the formula

$$A_i = \frac{\partial x'^k}{\partial x^i} A'_k;$$

therefore

$$dA_i = \frac{\partial x'^k}{\partial x^i} dA'_k + A'_k d\frac{\partial x'^k}{\partial x^i} = \frac{\partial x'^k}{\partial x^i} dA'_k + A'_k \frac{\partial^2 x'^k}{\partial x^i \partial x^l} dx^l.$$

Thus dA_i does not transform at all like a vector (the same also applies, of course, to the differential of a contravariant vector). Only if the second derivatives $\partial^2 x'^k/\partial x^i \partial x^l = 0$, i.e. if the x'^k are linear functions of the x^k, do the transformation formulas have the form

$$dA_i = \frac{\partial x'^k}{\partial x^i} dA'_k,$$

that is, dA_i transforms like a vector.

We now undertake the definition of a tensor which in curvilinear coordinates plays the same role as $\partial A_i/\partial x^k$ in galilean coordinates. In other words, we must transform $\partial A_i/\partial x^k$ from galilean to curvilinear coordinates.

In curvilinear coordinates, in order to obtain a differential of a vector which behaves like a vector, it is necessary that the two vectors to be subtracted from each other be located at the same point in space. In other words, we must somehow "translate" one of the vectors (which are separated infinitesimally from each other) to the point where the second is located, after which we determine the difference of two vectors which now refer to one and the same point in space. The operation of translation itself must be defined so that in galilean coordinates the difference shall coincide with the ordinary differential dA_i. Since dA_i is just the difference of the components of two infinitesimally separated vectors, this means that when we use galilean coordinates the components of the vector should not change as a result of the translation operation. But such a translation is precisely the translation of a vector parallel to itself. Under a *parallel translation* of a vector, its components in galilean coordinates do not change. If, on the other hand, we use curvilinear coordinates, then in general the components of the vector will change under such a translation. Therefore in curvilinear coordinates, the difference in the components of the two vectors after translating one of them to the point where the other is located will not coincide with their difference before the translation (i.e. with the differential dA_i).

Thus to compare two infinitesimally separated vectors we must subject one of them to a parallel translation to the point where the second is located. Let us consider an arbitrary contravariant vector; if its value at the point x^i is A^i, then at the neighboring point x^i+dx^i it is equal to A^i+dA^i. We subject the vector A^i to an infinitesimal parallel displacement to the point x^i+dx^i; the change in the vector which results from this we denote by δA^i. Then the difference DA^i between the two vectors which are now located at the same point is

$$DA^i = dA^i - \delta A^i. \tag{85.1}$$

The change δA^i in the components of a vector under an infinitesimal parallel displacement depends on the values of the components themselves, where the dependence must clearly be linear. This follows directly from the fact that the sum of two vectors must transform according to the same law as each of the constituents. Thus δA^i has the form

$$\delta A^i = -\Gamma^i_{kl} A^k dx^l, \tag{85.2}$$

where the Γ^i_{kl} are certain functions of the coordinates. Their form depends, of course, on the coordinate system; for a galilean coordinate system $\Gamma^i_{kl} = 0$.

From this it is already clear that the quantities Γ^i_{kl} do not form a tensor, since a tensor which is equal to zero in one coordinate system is equal to zero in every other one. In a curvilinear space it is, of course, impossible to make all the Γ^i_{kl} vanish over all of space. The principle of equivalence implies, however, that by a suitable choice of the coordinate system the gravitational field can be eliminated; that is, the quantities Γ^i_{kl}, which, as we shall see later, act as the gravitational field strength, can become zero over a given infinitesimal region (see the end of this section†). The quantities Γ^i_{kl} are called *Christoffel symbols*. In addition to the quantities Γ^i_{kl} we shall later also use quantities $\Gamma_{i,kl}$‡ defined as follows:

$$\Gamma_{i,kl} = g_{im}\Gamma^m_{kl}. \tag{85.3}$$

Conversely,

$$\Gamma^i_{kl} = g^{im}\Gamma_{m,kl}. \tag{85.4}$$

It is also easy to relate the change in the components of a covariant vector under a parallel displacement to the Christoffel symbols. To do this we note that under a parallel displacement, a scalar is unchanged. In particular, the scalar product of two vectors does not change under a parallel displacement.

Let A_i and B^i be any covariant and contravariant vectors. Then from $\delta(A_i B^i) = 0$, we have

$$B^i\,\delta A_i = -A_i\,\delta B^i = \Gamma^i_{kl} B^k A_i\,dx^l$$

or, changing the indices,

$$B^i\,\delta A_i = \Gamma^k_{il} A_k B^i\,dx^l.$$

From this, in view of the arbitrariness of the B^i,

$$\delta A_i = \Gamma^k_{il} A_k\,dx^l, \tag{85.5}$$

which determines the change in a covariant vector under a parallel displacement.

Substituting (85.2) and $dA^i = (\partial A^i/\partial x^l)\,dx^l$ in (85.1), we have

$$DA^i = \left(\frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl} A^k\right) dx^l. \tag{85.6}$$

Similarly, we find for a covariant vector,

$$DA_i = \left(\frac{\partial A_i}{\partial x^l} - \Gamma^k_{il} A_k\right) dx^l. \tag{85.7}$$

The expressions in parentheses in (85.6) and (85.7) are tensors, since when multiplied by the vector dx^l they give a vector. Clearly, these are the tensors which give the desired generalization of the concept of a derivative to curvilinear coordinates. These tensors are called the *covariant derivatives* of the vectors A^i and A_i respectively. We shall denote them by $A^i_{;k}$ and $A_{i;k}$. Thus,

$$DA^i = A^i_{;l}\,dx^l; \qquad DA_i = A_{i;l}\,dx^l, \tag{85.8}$$

† This is precisely the coordinate system which we have in mind in arguments where we, for brevity's sake, speak of a "galilean" system; still all the proofs remain applicable not only to flat, but also to curved space.

‡ In place of Γ^i_{kl} and $\Gamma_{i,kl}$, the symbols $\left\{ {kl \atop i} \right\}$ and $\left[{kl \atop i} \right]$ are sometimes used.

while the covariant derivatives themselves are:

$$A^i_{;l} = \frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl} A^k, \tag{85.9}$$

$$A_{i;l} = \frac{\partial A_i}{\partial x^l} - \Gamma^k_{il} A_k. \tag{85.10}$$

In galilean coordinates, $\Gamma^i_{kl} = 0$, and covariant differentiation reduces to ordinary differentiation.

It is also easy to calculate the covariant derivative of a tensor. To do this we must determine the change in the tensor under an infinitesimal parallel displacement. For example, let us consider any contravariant tensor, expressible as a product of two contravariant vectors $A^i B^k$. Under parallel displacement,

$$\delta(A^i B^k) = A^i \delta B^k + B^k \delta A^i = -A^i \Gamma^k_{lm} B^l dx^m - B^k \Gamma^i_{lm} A^l dx^m.$$

By virtue of the linearity of this transformation we must also have, for an arbitrary tensor A^{ik},

$$\delta A^{ik} = -(A^{im}\Gamma^k_{ml} + A^{mk}\Gamma^i_{ml})\, dx^l. \tag{85.11}$$

Substituting this in

$$DA^{ik} = dA^{ik} - \delta A^{ik} \equiv A^{ik}_{;l}\, dx^l,$$

we get the covariant derivative of the tensor A^{ik} in the form

$$A^{ik}_{;l} = \frac{\partial A^{ik}}{\partial x^l} + \Gamma^i_{ml} A^{mk} + \Gamma^k_{ml} A^{im}. \tag{85.12}$$

In completely similar fashion we obtain the covariant derivative of the mixed tensor A^i_k and the covariant tensor A_{ik} in the form

$$A^i_{k;l} = \frac{\partial A^i_k}{\partial x^l} - \Gamma^m_{kl} A^i_m + \Gamma^i_{ml} A^m_k, \tag{85.13}$$

$$A_{ik;l} = \frac{\partial A_{ik}}{\partial x^l} - \Gamma^m_{il} A_{mk} - \Gamma^m_{kl} A_{im}. \tag{85.14}$$

One can similarly determine the covariant derivative of a tensor of arbitrary rank. In doing this one finds the following rule of covariant differentiation. To obtain the covariant derivative of the tensor $A^{\cdot\cdot}_{\cdot\cdot}$ with respect to x^l, we add to the ordinary derivative $\partial A^{\cdot\cdot}_{\cdot\cdot}/\partial x^l$ for each covariant index $i(A^{\cdot\cdot}_{\cdot i})$ a term $-\Gamma^k_{il} A^{\cdot\cdot}_{\cdot k}$, and for each contravariant index $i(A^{\cdot i}_{\cdot\cdot})$ a term $+\Gamma^i_{kl} A^{\cdot k}_{\cdot\cdot}$.

One can easily verify that the covariant derivative of a product is found by the same rule as for ordinary differentiation of products. In doing this we must consider the covariant derivative of a scalar ϕ as an ordinary derivative, that is, as the covariant vector $\phi_k = \partial\phi/\partial x^k$, in accordance with the fact that for a scalar $\delta\phi = 0$, and therefore $D\phi = d\phi$. For example, the covariant derivative of the product $A_i B_k$ is

$$(A_i B_k)_{;l} = A_{i;l} B_k + A_i B_{k;l}.$$

If in a covariant derivative we raise the index signifying the differentiation, we obtain the so-called *contravariant derivative*. Thus,

$$A_i^{;k} = g^{kl} A_{i;l}, \qquad A^{i;k} = g^{kl} A^i_{;l}.$$

We prove that the Christoffel symbols Γ^i_{kl} must be symmetric in the subscripts. Since the covariant derivative of a vector $A_{i;k}$ is a tensor, the difference $A_{i;k} - A_{k;i}$ is also a tensor. Let the vector A_i be the gradient of a scalar, that is, $A_i = \partial\phi/\partial x^i$. Since $\partial A_i/\partial x^k = \partial^2\phi/\partial x^k \partial x^i = \partial A_k/\partial x^i$, with the help of (85.10) we have

$$A_{k;i} - A_{i;k} = (\Gamma^l_{ik} - \Gamma^l_{ki}) \frac{\partial\phi}{\partial x^l}.$$

But, as already mentioned, the principle of equivalence implies that there exists a galilean coordinate system in which the Γ^i_{kl} are zero at a given point, and therefore the left side of our equation is also zero. But since $A_{k;i} - A_{i;k}$ is a tensor, then being zero in one system it must also be zero in any coordinate system. Thus we find that

$$\Gamma^i_{kl} = \Gamma^i_{lk}. \tag{85.15}$$

Clearly, also,

$$\Gamma_{i,kl} = \Gamma_{i,lk}. \tag{85.16}$$

In general, there are altogether forty different quantities Γ^i_{kl}; for each of the four values of the index i there are ten different pairs of values of the indices k and l (counting pairs obtained by interchanging k and l as the same).

In concluding this section we present the formulas for transforming the Christoffel symbols from one coordinate system to another. These formulas can be obtained by comparing the laws of transformation of the two sides of the equations defining the covariant derivatives, and requiring that these laws be the same for both sides. A simple calculation gives

$$\Gamma^i_{kl} = \Gamma'^m_{np} \frac{\partial x^i}{\partial x'^m} \frac{\partial x'^n}{\partial x^k} \frac{\partial x'^p}{\partial x^l} + \frac{\partial^2 x'^m}{\partial x^k \partial x^l} \frac{\partial x^i}{\partial x'^m}. \tag{85.17}$$

From this formula it is clear that the quantity Γ^i_{kl} behaves like a tensor only under linear transformations [for which the second term in (85.17) drops out].

Formula (85.17) enables us to prove easily the assertion made above that (subject to the condition (85.15)) it is always possible to choose a coordinate system in which all the Γ^i_{kl} become zero at a previously assigned point (such a system is said to be *locally-inertial* or *locally-geodesic* (see § 87).†

In fact, let the given point be chosen as the origin of coordinates, and let the values of the Γ^i_{kl} at it be initially (in the coordinates x^i) equal to $(\Gamma^i_{kl})_0$. In the neighborhood of this point, we now make the transformation

$$x'^i = x^i + \tfrac{1}{2}(\Gamma^i_{kl})_0 x^k x^l. \tag{85.18}$$

Then

$$\left(\frac{\partial^2 x'^m}{\partial x^k \partial x^l} \frac{\partial x^i}{\partial x'^m}\right)_0 = (\Gamma^i_{kl})_0$$

and according to (85.17), all the Γ'^m_{np} become equal to zero.

We note that for the transformation (85.18).

$$\left(\frac{\partial x'^i}{\partial x^k}\right)_0 = \delta^i_k,$$

so that it does not change the value of any tensor (including the tensor g_{ik}) at the given point, so that we can make the Christoffel symbols vanish at the same time as we bring the g_{ik} to galilean form.

† It can also be shown that, by a suitable choice of the coordinate system, one can make all the Γ^i_{kl} go to zero not just at a point but all along a given world line. (The proof of this statement can be found in the book by P. K. Rashevskii, *Riemannian Geometry and Tensor Analysis*,, Nauka, 1964, § 91.)

§ 86. The relation of the Christoffel symbols to the metric tensor

Let us show that the covariant derivative of the metric tensor g_{ik} is zero. To do this we note that the relation

$$DA_i = g_{ik}DA^k$$

is valid for the vector DA_i, as for any vector. On the other hand, $A_i = g_{ik}A^k$, so that

$$DA_i = D(g_{ik}A^k) = g_{ik}DA^k + A^k Dg_{ik}.$$

Comparing with $DA_i = g_{ik}DA^k$, and remembering that the vector A^k is arbitrary,

$$Dg_{ik} = 0.$$

Therefore the covariant derivative

$$g_{ik;l} = 0. \tag{86.1}$$

Thus g_{ik} may be considered as a constant during covariant differentiation.

The equation $g_{ik;l} = 0$ can be used to express the Christoffel symbols Γ^i_{kl} in terms of the metric tensor g_{ik}. To do this we write in accordance with the general definition (85.14):

$$g_{ik;l} = \frac{\partial g_{ik}}{\partial x^l} - g_{mk}\Gamma^m_{il} - g_{im}\Gamma^m_{kl} = \frac{\partial g_{ik}}{\partial x^l} - \Gamma_{k,il} - \Gamma_{i,kl} = 0.$$

Thus the derivatives of g_{ik} are expressed in terms of the Christoffel symbols.† We write the values of the derivatives of g_{ik}, permuting the indices i, k, l:

$$\frac{\partial g_{ik}}{\partial x^l} = \Gamma_{k,il} + \Gamma_{i,kl},$$

$$\frac{\partial g_{li}}{\partial x^k} = \Gamma_{i,kl} + \Gamma_{l,ik},$$

$$-\frac{\partial g_{kl}}{\partial x^i} = -\Gamma_{l,ki} - \Gamma_{k,li}.$$

Taking half the sum of these equations, we find (remembering that $\Gamma_{i,kl} = \Gamma_{i,lk}$)

$$\Gamma_{i,kl} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^l} + \frac{\partial g_{il}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^i}\right). \tag{86.2}$$

From this we have for the symbols $\Gamma^i_{kl} = g^{im}\Gamma_{m,kl}$,

$$\Gamma^i_{kl} = \frac{1}{2} g^{im}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right). \tag{86.3}$$

These formulas give the required expressions for the Christoffel symbols in terms of the metric tensor.

We now derive an expression for the contracted Christoffel symbol Γ^i_{ki} which will be important later on. To do this we calculate the differential dg of the determinant g made up from the components of the tensor g_{ik}; dg can be obtained by taking the differential of each component of the tensor g_{ik} and multiplying it by its coefficient in the determinant, i.e. by the corresponding minor. On the other hand, the components of the tensor g^{ik} reciprocal to g_{ik} are equal to the minors of the determinant of the g_{ik}, divided by the determinant.

† Choosing a locally-geodesic system of coordinates therefore means that at the given point all the first derivatives of the components of the metric tensor vanish.

Therefore the minors of the determinant g are equal to gg^{ik}. Thus,

$$dg = gg^{ik}dg_{ik} = -gg_{ik}dg^{ik} \tag{86.4}$$

(since $g_{ik}g^{ik} = \delta_i^i = 4$, $g^{ik}\,dg_{ik} = -g_{ik}\,dg^{ik}$).

From (86.3), we have

$$\Gamma^i_{ki} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mk}}{\partial x^i} + \frac{\partial g_{mi}}{\partial x^k} - \frac{\partial g_{ki}}{\partial x^m}\right).$$

Changing the positions of the indices m and i in the third and first terms in parentheses, we see that these two terms cancel each other, so that

$$\Gamma^i_{ki} = \frac{1}{2}g^{im}\frac{\partial g_{im}}{\partial x^k},$$

or, according to (86.4),

$$\Gamma^i_{ki} = \frac{1}{2g}\frac{\partial g}{\partial x^k} = \frac{\partial \ln\sqrt{-g}}{\partial x^k}. \tag{86.5}$$

It is useful to note also the expression for the quantity $g^{kl}\Gamma^i_{kl}$; we have

$$g^{kl}\Gamma^i_{kl} = \frac{1}{2}g^{kl}g^{im}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{lm}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right) = g^{kl}g^{lm}\left(\frac{\partial g_{mk}}{\partial x^l} - \frac{1}{2}\frac{\partial g_{kl}}{\partial x^m}\right).$$

With the help of (86.4) this can be transformed to

$$g^{kl}\Gamma^i_{kl} = -\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\,g^{ik})}{\partial x^k}. \tag{86.6}$$

For various calculations it is important to remember that the derivatives of the contravariant tensor g^{ik} are related to the derivatives of g_{ik} by the relations

$$g_{il}\frac{\partial g^{ik}}{\partial x^m} = -g^{ik}\frac{\partial g_{il}}{\partial x^m} \tag{86.7}$$

(which are obtained by differentiating the equality $g_{il}g^{ik} = \delta^k_l$). Finally we point out that the derivatives of g^{ik} can also be expressed in terms of the quantities Γ^i_{kl}. Namely, from the identity $g^{ik}{}_{;l} = 0$ it follows directly that

$$\frac{\partial g^{ik}}{\partial x^l} = -\Gamma^i_{ml}g^{mk} - \Gamma^k_{ml}g^{im}. \tag{86.8}$$

With the aid of the formulas which we have obtained we can put the expression for $A^i{}_{;i}$, the generalized divergence of a vector in curvilinear coordinates, in convenient form. Using (86.5), we have

$$A^i{}_{;i} = \frac{\partial A^i}{\partial x^i} + \Gamma^i_{li}A^l = \frac{\partial A^i}{\partial x^i} + A^l\frac{\partial \ln\sqrt{-g}}{\partial x^l}$$

or, finally,

$$A^i{}_{;i} = \frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\,A^i)}{\partial x^i}. \tag{86.9}$$

We can derive an analogous expression for the divergence of an antisymmetric tensor A^{ik}. From (85.12), we have

$$A^{ik}{}_{;k} = \frac{\partial A^{ik}}{\partial x^k} + \Gamma^i_{mk}A^{mk} + \Gamma^k_{mk}A^{im}.$$

But, since $A^{mk} = -A^{km}$,

$$\Gamma^i_{mk}A^{mk} = -\Gamma^i_{km}A^{km} = 0.$$

Substituting the expression (86.5) for Γ^k_{mk}, we obtain

$$A^{ik}{}_{;k} = \frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\,A^{ik})}{\partial x^k}. \tag{86.10}$$

Now suppose A_{ik} is a symmetric tensor; we calculate the expression $A^k_{i;k}$ for its mixed components. We have

$$A^k_{i;k} = \frac{\partial A^k_i}{\partial x^k} + \Gamma^k_{lk}A^l_i - \Gamma^l_{ik}A_l{}^k = \frac{1}{\sqrt{-g}}\frac{\partial(A^k_i\sqrt{-g})}{\partial x^k} - \Gamma^l_{ki}A^k_l.$$

The last term here is equal to

$$-\frac{1}{2}\left(\frac{\partial g_{il}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^l}\right)A^{kl}.$$

Because of the symmetry of the tensor A^{kl}, two of the terms in parentheses cancel each other, leaving

$$A^k_{i;k} = \frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\,A^k_i)}{\partial x^k} - \frac{1}{2}\frac{\partial g_{kl}}{\partial x^i}A^{kl}. \tag{86.11}$$

In cartesian coordinates, $\partial A_i/\partial x^k - \partial A_k/\partial x^i$ is an antisymmetric tensor. In curvilinear coordinates this tensor is $A_{i;k} - A_{k;i}$. However, with the help of the expression for $A_{i;k}$ and since $\Gamma^i_{kl} = \Gamma^i_{lk}$, we have

$$A_{i;k} - A_{k;i} = \frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i}. \tag{86.12}$$

Finally, we transform to curvilinear coordinates the sum $\partial^2\phi/\partial x_i\partial x^i$ of the second derivatives of a scalar ϕ. It is clear that in curvilinear coordinates this sum goes over into $\phi_{;i}^{;i}$. But $\phi_{;i} = \partial\phi/\partial x^i$, since covariant differentiation of a scalar reduces to ordinary differentiation. Raising the index i, we have

$$\phi^{;i} = g^{ik}\frac{\partial\phi}{\partial x^k},$$

and using formula (86.9), we find

$$\phi_{;i}^{;i} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(\sqrt{-g}\,g^{ik}\frac{\partial\phi}{\partial x^k}\right). \tag{86.13}$$

It is important to note that Gauss' theorem (83.17) for the transformation of the integral of a vector over a hypersurface into an integral over a four-volume can, in view of (86.9), be written as

$$\oint A^i\sqrt{-g}\,dS_i = \int A^i{}_{;i}\sqrt{-g}\,d\Omega. \tag{86.14}$$

§ 87. Motion of a particle in a gravitational field

The motion of a free material particle is determined in the special theory of relativity from the principle of least action,

$$\delta S = -mc\,\delta\int ds = 0, \tag{87.1}$$

according to which the particle moves so that its world line is an extremal between a given pair of world points, in our case a straight line (in ordinary three-dimensional space this corresponds to uniform rectilinear motion).

The motion of a particle in a gravitational field is determined by the principle of least action in this same form (87.1), since the gravitational field is nothing but a change in the metric of space-time, manifesting itself only in a change in the expression for ds in terms of the dx^i. Thus, in a gravitational field the particle moves so that its world point moves along an extremal or, as it is called, a *geodesic line* in the four-space x^0, x^1, x^2, x^3; however, since in the presence of the gravitational field space-time is not galilean, this line is not a "straight line", and the real spatial motion of the particle is neither uniform nor rectilinear.

Instead of starting once again directly from the principle of least action (see the problem at the end of this section), it is simpler to obtain the equations of motion of a particle in a gravitational field by an appropriate generalization of the differential equations for the free motion of a particle in the special theory of relativity, i.e. in a galilean four-dimensional coordinate system. These equations are $du^i/ds = 0$ or $du^i = 0$, where $u^i = dx^i/ds$ is the four-velocity. Clearly, in curvilinear coordinates this equation is generalized to the equation

$$Du^i = 0. \tag{87.2}$$

From the expression (85.6) for the covariant differential of a vector, we have

$$du^i + \Gamma^i_{kl} u^k \, dx^l = 0.$$

Dividing this equation by ds, we have

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{kl} \frac{dx^k}{ds}\frac{dx^l}{ds} = 0. \tag{87.3}$$

This is the required equation of motion. We see that the motion of a particle in a gravitational field is determined by the quantities Γ^i_{kl}. The derivative d^2x^i/ds^2 is the four-acceleration of the particle. Therefore we may call the quantity $-m\Gamma^i_{kl}u^k u^l$ the "four-force", acting on the particle in the gravitational field. Here, the tensor g_{ik} plays the role of the "potential" of the gravitational field—its derivatives determine the field "intensity" Γ^i_{kl}.†

In § 85 it was shown that by a suitable choice of the coordinate system one can always make all the Γ^i_{kl} zero at an arbitrary point of space-time. We now see that the choice of such a locally-inertial system of reference means the elimination of the gravitational field in the given infinitesimal element of space-time, and the possibility of making such a choice is an expression of the principle of equivalence in the relativistic theory of gravitation.‡

As before, we define the four-momentum of a particle in a gravitational field as

$$p^i = mcu^i. \tag{87.4}$$

Its square is

$$p_i p^i = m^2c^2. \tag{87.5}$$

† We also give the form of the equations of motion expressed in terms of covariant components of the four-acceleration. From the condition $Du_i = 0$, we find

$$\frac{du_i}{ds} - \Gamma_{k,il} u^k u^l = 0.$$

Substituting for $\Gamma_{k,il}$ from (86.2), two of the terms cancel and we are left with

$$\frac{du_i}{ds} - \frac{1}{2}\frac{\partial g_{kl}}{\partial x^i} u^k u^l = 0.$$

‡ In the footnote on p. 241 we also noted the possibility of choosing a reference system which is "inertial along a given world line." In particular, if this line is the time axis (along which x^1, x^2, x^3 = const), then the gravitational field will be eliminated for all times in the given spatial element.

Substituting $-\partial S/\partial x^i$ for p_i, we find the Hamilton–Jacobi equation for a particle in a gravitational field:

$$g^{ik}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k}-m^2c^2=0. \tag{87.6}$$

The equation of a geodesic in the form (87.3) is not applicable to the propagation of a light signal, since along the world line of the propagation of a light ray the interval ds, as we know, is zero, so that all the terms in equation (87.3) become infinite. To get the equations of motion in the form needed for this case, we use the fact that the direction of propagation of a light ray in geometrical optics is determined by the wave vector tangent to the ray. We can therefore write the four-dimensional wave vector in the form $k^i = dx^i/d\lambda$, where λ is some parameter varying along the ray. In the special theory of relativity, in the propagation of light in vacuum the wave vector does not vary along the path, that is, $dk^i = 0$ (see § 53). In a gravitational field this equation clearly goes over into $Dk^i = 0$ or

$$\frac{dk^i}{d\lambda}+\Gamma^i_{kl}k^k k^l=0 \tag{87.7}$$

(these equations also determine the parameter λ).†

The absolute square of the wave four-vector (see § 48) is zero, that is,

$$k_i k^i = 0. \tag{87.8}$$

Substituting $\partial\psi/\partial x^i$ in place of k_i (ψ is the eikonal), we find the eikonal equation in a gravitational field

$$g^{ik}\frac{\partial\psi}{\partial x^i}\frac{\partial\psi}{\partial x^k}=0. \tag{87.9}$$

In the limiting case of small velocities, the relativistic equations of motion of a particle in a gravitational field must go over into the corresponding non-relativistic equations. In this we must keep in mind that the assumption of small velocity implies the requirement that the gravitational field itself be weak; if this were not so a particle located in it would acquire a high velocity.

Let us examine how, in this limiting case, the metric tensor g_{ik} determining the field is related to the nonrelativistic potential ϕ of the gravitational field.

In nonrelativistic mechanics the motion of a particle in a gravitational field is determined by the Lagrangian (81.1). We now write it in the form

$$L=-mc^2+\frac{mv^2}{2}-m\phi, \tag{87.10}$$

adding the constant $-mc^2$.‡ This must be done so that the nonrelativistic Lagrangian in the absence of the field, $L = -mc^2+mv^2/2$, shall be the same exactly as that to which the corresponding relativistic function $L = -mc^2\sqrt{1-v^2/c^2}$ reduces in the limit as $v/c \to 0$.

Consequently, the nonrelativistic action function S for a particle in a gravitational field has the form

$$S=\int L\,dt=-mc\int\left(c-\frac{v^2}{2c}+\frac{\phi}{c}\right)dt.$$

† Geodesics, along which $ds \equiv 0$, are said to be *null* or *isotropic*.

‡ The potential ϕ is, of course, defined only to within an arbitrary additive constant. We assume throughout that one makes the natural choice of this constant so that the potential vanishes far from the bodies producing the field.

Comparing this with the expression $S = -mc\int ds$, we see that in the limiting case under consideration

$$ds = \left(c - \frac{v^2}{2c} + \frac{\phi}{c}\right) dt.$$

Squaring and dropping terms which vanish for $c \to \infty$, we find

$$ds^2 = (c^2 + 2\phi)\, dt^2 - d\mathbf{r}^2. \tag{87.11}$$

where we have used the fact that $\mathbf{v}\, dt = d\mathbf{r}$.

Thus in the limiting case the component g_{00} of the metric tensor is

$$g_{00} = 1 + \frac{2\phi}{c^2}. \tag{87.12}$$

As for the other components, from (87.11) it would follow that $g_{\alpha\beta} = \delta_{\alpha\beta}$, $g_{0\alpha} = 0$. Actually, however, the corrections to them are, generally speaking, of the same order of magnitude as the corrections to g_{00} (for more detail, see § 106). The impossibility of determining these corrections by the method given above is related to the fact that the corrections to the $g_{\alpha\beta}$, though of the same order of magnitude as the correction to g_{00}, would give rise to terms in the Lagrangian of a higher order of smallness (because in the expression for ds^2 the components $g_{\alpha\beta}$ are not multiplied by c^2, while this is the case for g_{00}).

PROBLEM

Derive the equation of motion (87.3) from the principle of least action (87.1).

Solution: We have:

$$\delta ds^2 = 2ds\,\delta ds = \delta(g_{ik}\, dx^i\, dx^k) = dx^i\, dx^k \frac{\partial g_{ik}}{\partial x^l}\delta x^l + 2g_{ik}\, dx^i\, d\delta x^k.$$

Therefore

$$\delta S = -mc \int \left\{\frac{1}{2}\frac{dx^i}{ds}\frac{dx^k}{ds}\frac{\partial g_{ik}}{\partial x^l}\delta x^l + g_{ik}\frac{dx^i}{ds}\frac{d\delta x^k}{ds}\right\} ds$$

$$= -mc \int \left\{\frac{1}{2}\frac{dx^i}{ds}\frac{dx^k}{ds}\frac{\partial g_{ik}}{\partial x^l}\delta x^l - \frac{d}{ds}\left(g_{ik}\frac{dx^i}{ds}\right)\delta x^k\right\} ds$$

(in integrating by parts, we use the fact that $\delta x^k = 0$ at the limits). In the second term in the integral, we replace the index k by the index l. We then find, by equating to zero the coefficient of the arbitrary variation δx^l:

$$\frac{1}{2} u^i u^k \frac{\partial g_{ik}}{\partial x^l} - \frac{d}{ds}(g_{il} u^i) = \frac{1}{2} u^i u^k \frac{\partial g_{ik}}{\partial x^l} - g_{il}\frac{du^i}{ds} - u^i u^k \frac{\partial g_{il}}{\partial x^k} = 0.$$

Noting that the third term can be written as

$$-\frac{1}{2} u^i u^k \left(\frac{\partial g_{il}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^i}\right),$$

and introducing the Christoffel symbols $\Gamma_{l,ik}$ in accordance with (86.2), we have:

$$g_{il}\frac{du^i}{ds} + \Gamma_{l,ik} u^i u^k = 0.$$

Equation (87.3) is obtained from this by raising the index l.

§ 88. The constant gravitational field

A gravitational field is said to be *constant* if one can choose a system of reference in which all the components of the metric tensor are independent of the time coordinate x^0; the latter is then called the *world time*.

The choice of a world time is not completely unique. Thus, if we add to x^0 an arbitrary function of the space coordinates, the g_{ik} will still not contain x^0; this transformation corresponds to the arbitrariness in the choice of the time origin at each point in space.† In addition, of course, the world time can be multiplied by an arbitrary constant, i.e. the units for measuring it are arbitrary.

Strictly speaking, only the field produced by a single body can be constant. In a system of several bodies, their mutual gravitational attraction will give rise to motion, as a result of which the field produced by them cannot be constant.

If the body producing the field is fixed (in the reference system in which the g_{ik} do not depend on x^0), then both directions of time are equivalent. For a suitable choice of the time origin at all the points in space, the interval ds should in this case not be changed when we change the sign of x^0, and therefore all the components $g_{0\alpha}$ of the metric tensor must be identically equal to zero. Such constant gravitational fields are said to be *static*.

However, for the field produced by a body to be constant, it is not necessary for the body to be at rest. Thus the field of an axially symmetric body rotating uniformly about its axis will also be constant. However in this case the two time directions are no longer equivalent by any means—if the sign of the time is changed, the sign of the angular velocity is changed. Therefore in such constant gravitational fields (we shall call them *stationary* fields) the components $g_{0\alpha}$ of the metric tensor are in general different from zero.

The meaning of the world time in a constant gravitational field is that an interval of world time between events at a certain point in space coincides with the interval of world time between any other two events at any other point in space, if these events are respectively simultaneous (in the sense explained in § 84) with the first pair of events. But to the same interval of world time x^0 there correspond, at different points of space, different intervals of proper time τ.

The relation between world time and proper time, formula (84.1), can now be written in the form

$$\tau = \frac{1}{c}\sqrt{g_{00}}\,x^0, \tag{88.1}$$

applicable to any finite time interval.

If the gravitational field is weak, then we may use the approximate expression (87.12), and (88.1) gives

$$\tau = \frac{x^0}{c}\left(1+\frac{\phi}{c^2}\right). \tag{88.2}$$

† It is easy to see that under such a transformation the spatial metric, as expected, does not change. In fact, under the substitution

$$x^0 \to x^0 + f(x^1, x^2, x^3)$$

with an arbitrary function $f(x^1, x^2, x^3)$, the components g_{ik} change to

$$g_{\alpha\beta} \to g_{\alpha\beta} + g_{00} f_{,\alpha} f_{,\beta} + g_{0\alpha} f_{,\beta} + g_{0\beta} f_{,\alpha},$$

$$g_{0\alpha} \to g_{0\alpha} + g_{00} f_{,\alpha}, \qquad g_{00} \to g_{00},$$

where $f_{,\alpha} \equiv \partial f/\partial x^\alpha$. This obviously does not change the tensor (84.7).

Thus proper time elapses the more slowly the smaller the gravitational potential at a given point in space, i.e., the larger its absolute value (later, in § 96, it will be shown that the potential ϕ is negative). If one of two identical clocks is placed in a gravitational field for some time, the clock which has been in the field will thereafter appear to be slow.

As was already indicated above, in a static gravitational field the components $g_{0\alpha}$ of the metric tensor are zero. According to the results of § 84, this means that in such a field synchronization of clocks is possible over all space. We note also that the element of spatial distance in a static field is simply:

$$dl^2 = -g_{\alpha\beta}\, dx^\alpha\, dx^\beta. \tag{88.3}$$

In a stationary field the $g_{0\alpha}$ are different from zero and the synchronization of clocks over all space is impossible. Since the g_{ik} do not depend on x^0, formula (84.14) for the difference between the values of world time for two simultaneous events occurring at different points in space can be written in the form

$$\Delta x^0 = -\int \frac{g_{0\alpha} dx^\alpha}{g_{00}} \tag{88.4}$$

for any two points on the line along which the synchronization of clocks is carried out. In the synchronization of clocks along a closed contour, the difference in the value of the world time which would be recorded upon returning to the starting point is equal to the integral

$$\Delta x^0 = -\oint \frac{g_{0\alpha} dx^\alpha}{g_{00}} \tag{88.5}$$

taken along the closed contour.†

Let us consider the propagation of a light ray in a constant gravitational field. We have seen in § 53 that the frequency of the light is the time derivative of the eikonal ψ (with opposite sign). The frequency expressed in terms of the world time x^0/c is therefore $\omega_0 = -c(\partial\psi/\partial x^0)$. Since the eikonal equation (87.9) in a constant field does not contain x^0 explicitly, the frequency ω_0 remains constant during the propagation of the light ray. The frequency measured in terms of the proper time is $\omega = -(\partial\psi/\partial\tau)$; this frequency is different at different points of space.

From the relation

$$\frac{\partial\psi}{\partial\tau} = \frac{\partial\psi}{\partial x^0}\frac{\partial x^0}{\partial\tau} = \frac{\partial\psi}{\partial x^0}\frac{c}{\sqrt{g_{00}}},$$

we have

$$\omega = \frac{\omega_0}{\sqrt{g_{00}}}. \tag{88.6}$$

In a weak gravitational field we obtain from this, approximately,

$$\omega = \omega_0\left(1 - \frac{\phi}{c^2}\right). \tag{88.7}$$

We see that the light frequency increases with increasing absolute value of the potential of the gravitational field, i.e. as we approach the bodies producing the field; conversely, as the light recedes from these bodies the frequency decreases. If a ray of light, emitted at a point

† The integral (88.5) is identically zero if the sum $g_{0\alpha} dx^\alpha/g_{00}$ is an exact differential of some function of the space coordinates. However, such a case would simply mean that we are actually dealing with a static field, and that all the $g_{0\alpha}$ could be made equal to zero by a transformation of the form $x^0 \to x^0 + f(x^\alpha)$.

where the gravitational potential is ϕ_1, has (at that point) the frequency ω, then upon arriving at a point where the potential is ϕ_2, it will have a frequency (measured in units of the proper time at that point) equal to

$$\frac{\omega}{1-\frac{\phi_1}{c^2}}\left(1-\frac{\phi_2}{c^2}\right) = \omega\left(1+\frac{\phi_1-\phi_2}{c^2}\right).$$

A line spectrum emitted by some atoms located, for example, on the sun, looks the same there as the spectrum emitted by the same atoms located on the earth would appear on it. If, however, we observe on the earth the spectrum emitted by the atoms located on the sun, then, as follows from what has been said above, its lines appear to be shifted with respect to the lines of the same spectrum emitted on the earth. Namely, each line with frequency ω will be shifted through the interval $\Delta\omega$ given by the formula

$$\Delta\omega = \frac{\phi_1-\phi_2}{c^2}\,\omega, \tag{88.8}$$

where ϕ_1 and ϕ_2 are the potentials of the gravitational field at the points of emission and observation of the spectrum respectively. If we observe on the earth a spectrum emitted on the sun or the stars, then $|\phi_1| > |\phi_2|$, and from (88.8) it follows that $\Delta\omega < 0$, i.e. the shift occurs in the direction of lower frequency. The phenomenon we have described is called the "*red shift*".

The occurrence of this phenomenon can be explained directly on the basis of what has been said above about world time. Because the field is constant, the interval of world time during which a certain vibration in the light wave propagates from one given point of space to another is independent of x^0. Therefore it is clear that the number of vibrations occurring in a unit interval of world time will be the same at all points along the ray. But to one and the same interval of world time there corresponds a larger and larger interval of proper time, the further away we are from the bodies producing the field. Consequently, the frequency, i.e. the number of vibrations per unit proper time, will decrease as the light recedes from these masses.

During the motion of a particle in a constant field, its energy, defined as

$$-c\,\frac{\partial S}{\partial x^0},$$

the derivative of the action with respect to the world time, is conserved; this follows, for example, from the fact that x^0 does not appear explicitly in the Hamilton–Jacobi equation. The energy defined in this way is the time component of the covariant four-vector of momentum $p_k = mcu_k = mcg_{kl}u^l$. In a static field, $ds^2 = g_{00}(dx^0)^2 - dl^2$, and we have for the energy, which we here denote by $\mathscr{E}_0$,

$$\mathscr{E}_0 = mc^2 g_{00}\frac{dx^0}{ds} = mc^2 g_{00}\frac{dx^0}{\sqrt{g_{00}(dx^0)^2 - dl^2}}.$$

We introduce the velocity

$$v = \frac{dl}{d\tau} = \frac{c\,dl}{\sqrt{g_{00}}\,dx^0}$$

of the particle, measured in terms of the proper time, that is, by an observer located at the given point. Then we obtain for the energy

$$\mathscr{E}_0 = \frac{mc^2\sqrt{g_{00}}}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{88.9}$$

This is the quantity which is conserved during the motion of the particle.

It is easy to show that the expression (88.9) remains valid also for a stationary field. if only the velocity v is measured in terms of the proper time, as determined by clocks synchronized along the trajectory of the particle. If the particle departs from point A at the moment of world time x^0 and arrives at the infinitesimally distant point B at the moment x^0+dx^0, then to determine the velocity we must now take, not the time interval $(x^0+dx^0)-x^0 = dx^0$, but rather the difference between x^0+dx^0 and the moment $x^0-(g_{0\alpha}/g_{00})dx^\alpha$ which is simultaneous at the point B with the moment x^0 at the point A:

$$(x^0+dx^0)-\left(x^0-\frac{g_{0\alpha}}{g_{00}}dx^\alpha\right) = dx^0+\frac{g_{0\alpha}}{g_{00}}dx^\alpha.$$

Multiplying by $\sqrt{g_{00}}/c$, we obtain the corresponding interval of proper time, so that the velocity is

$$v^\alpha = \frac{c\,dx^\alpha}{\sqrt{h}(dx^0-g_\alpha dx^\alpha)}, \tag{88.10}$$

where we have introduced the notation

$$h = g_{00}, \qquad g_\alpha = -\frac{g_{0\alpha}}{g_{00}} \tag{88.11}$$

for the three-dimensional vector g (which was already mentioned in § 84) and for the three-dimensional scalar g_{00}. The covariant components of the velocity **v** form a three-dimensional vector in the space with metric $\gamma_{\alpha\beta}$, and correspondingly the square of this vector is to be taken as†

$$v_\alpha = \gamma_{\alpha\beta}v^\beta, \qquad v^2 = v_\alpha v^\alpha. \tag{88.12}$$

We note that with such a definition. the interval ds is expressed in terms of the velocity in the usual fashion:

$$\begin{aligned} ds^2 &= g_{00}(dx^0)^2+2g_{0\alpha}dx^0\,dx^\alpha+g_{\alpha\beta}dx^\alpha\,dx^\beta \\ &= h(dx^0-g_\alpha\,dx^\alpha)^2-dl^2 \\ &= h(dx^0-g_\alpha\,dx^\alpha)^2\left(1-\frac{v^2}{c^2}\right). \end{aligned} \tag{88.13}$$

The components of the four-velocity

$$u^i = \frac{dx^i}{ds}$$

† In our further work we shall repeatedly introduce, in addition to four-vectors and four-tensors, three-dimensional vectors and tensors defined in the space with metric $\gamma_{\alpha\beta}$; in particular the vectors **g** and **v**, which we have already used, are of this type. Just as in four dimensions the tensor operations (in particular, raising and lowering of indices) are done using the metric tensor g_{ik}; so, in three dimensions these are done using the tensor $\gamma_{\alpha\beta}$. To avoid misunderstandings that may arise, we shall denote three-dimensional quantities by symbols other than those used for four-dimensional quantities.

are

$$u^\alpha = \frac{v^\alpha}{c\sqrt{1-\frac{v^2}{c^2}}}, \qquad u^0 = \frac{1}{\sqrt{h}\sqrt{1-\frac{v^2}{c^2}}} + \frac{g_\alpha v^\alpha}{c\sqrt{1-\frac{v^2}{c^2}}}. \tag{88.14}$$

The energy is

$$\mathscr{E}_0 = mc^2 g_{0i}u^i = mc^2 h(u^0 - g_\alpha u^\alpha),$$

and after substituting (88.14), takes the form (88.9).

In the limiting case of a weak gravitational field and low velocities, by substituting $g_{00} = 1+(2\phi/c^2)$ in (88.9), we get approximately:

$$\mathscr{E}_0 = mc^2 + \frac{mv^2}{2} + m\phi, \tag{88.15}$$

where $m\phi$ is the potential energy of the particle in the gravitational field, which is in agreement with the Lagrangian (87.10).

PROBLEMS

1. Determine the force acting on a particle in a constant gravitational field.

Solution: For the components of Γ_{kl}^i which we need, we find the following expressions:

$$\begin{aligned}
\Gamma_{00}^\alpha &= \frac{1}{2}h^{;\alpha},\\
\Gamma_{0\beta}^\alpha &= \frac{h}{2}(g^\alpha_{;\beta} - g_\beta^{;\alpha}) - \frac{1}{2}g_\beta h^{;\alpha},\\
\Gamma_{\beta\gamma}^\alpha &= \lambda_{\beta\gamma}^\alpha + \frac{h}{2}[g_\beta(g_\gamma^{;\alpha} - g^\alpha_{;\gamma}) + g_\gamma(g_\beta^{;\alpha} - g^\alpha_{;\beta})] + \frac{1}{2}g_\beta g_\gamma h^{;\alpha}.
\end{aligned} \tag{1}$$

In these expressions all the tensor operations (covariant differentiation, raising and lowering of indices) are carried out in the three-dimensional space with metric $\gamma_{\alpha\beta}$, on the three-dimensional vector g^α and the three-dimensional scalar h (88.11); $\lambda^\alpha_{\beta\gamma}$ is the three-dimensional Christoffel symbol, constructed from the components of the tensor $\gamma_{\alpha\beta}$ in just the same way as Γ^i_{kl} is constructed from the components of g_{ik}; in the computations we use (84.9–12).

Substituting (1) in the equation of motion

$$\frac{du^\alpha}{ds} = -\Gamma^\alpha_{00}(u^0)^2 - 2\Gamma^\alpha_{0\beta}u^0u^\beta - \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma$$

and using the expression (88.14) for the components of the four-velocity, we find after some simple transformations:

$$\frac{d}{ds}\frac{v^\alpha}{c\sqrt{1-\frac{v^2}{c^2}}} = -\frac{h^{;\alpha}}{2h\left(1-\frac{v^2}{c^2}\right)} - \frac{\sqrt{h}(g^\alpha_{;\beta} - g_\beta^{;\alpha})v^\beta}{c\left(1-\frac{v^2}{c^2}\right)} - \frac{\lambda^\alpha_{\beta\gamma}v^\beta v^\gamma}{c^2\left(1-\frac{v^2}{c^2}\right)}. \tag{2}$$

The force $\mathbf{f}$ acting on the particle is the derivative of its momentum $\mathbf{p}$ with respect to the (synchronized) proper time, as defined by the three-dimensional covariant differential:

$$f^\alpha = c\sqrt{1-\frac{v^2}{c^2}}\frac{Dp^\alpha}{ds} = c\sqrt{1-\frac{v^2}{c^2}}\frac{d}{ds}\frac{mv^\alpha}{\sqrt{1-\frac{v^2}{c^2}}} + \lambda^\alpha_{\beta\gamma}\frac{mv^\beta v^\gamma}{\sqrt{1-\frac{v^2}{c^2}}}.$$

From (2) we therefore have (for convenience we lower the index α):

$$f_\alpha = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}\left\{-\frac{\partial}{\partial x^\alpha}\ln\sqrt{h}+\sqrt{h}\left(\frac{\partial g_\beta}{\partial x^\alpha}-\frac{\partial g_\alpha}{\partial x^\beta}\right)\frac{v^\beta}{c}\right\},$$

or, in the usual three-dimensional notation,†

$$\mathbf{f} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}\left\{-\nabla\ln\sqrt{h}+\sqrt{h}\,\frac{\mathbf{v}}{c}\times(\operatorname{curl}\mathbf{g})\right\}. \tag{3}$$

We note that if the body is at rest, then the force acting on it [the first term in (3)] has a potential. For low velocities of motion the second term in (3) has the form $mc\sqrt{h}\,\mathbf{v}\times(\operatorname{curl}\mathbf{g})$ analogous to the Coriolis force which would appear (in the absence of the field) in a coordinate system rotating with angular velocity

$$\Omega = \frac{c}{2}\sqrt{h}\operatorname{curl}\mathbf{g}.$$

2. Derive Fermat's principle for the propagation of a ray in a constant gravitational field.

Solution: Fermat's principle (§ 53) states:

$$\delta\int k_\alpha\,dx^\alpha = 0,$$

where the integral is taken along the ray, and the integral must be expressed in terms of the frequency ω_0 (which is constant along the ray) and the coordinate differentials. Noting that $k_0 = -\partial\psi/\partial x^0 = (\omega_0/c)$, we write:

$$\frac{\omega_0}{c} = k_0 = g_{0i}k^i = g_{00}k^0+g_{0\alpha}k^\alpha = h(k^0-g_\alpha k^\alpha).$$

Substituting this in the relation $k_ik^i = g_{ik}k^ik^k = 0$, written in the form

$$h(k^0-g_\alpha k^\alpha)^2-\gamma_{\alpha\beta}k^\alpha k^\beta = 0,$$

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† In three-dimensional curvilinear coordinates, the unit antisymmetric tensor is defined as

$$\eta_{\alpha\beta\gamma} = \sqrt{\gamma}\,e_{\alpha\beta\gamma},\qquad \eta^{\alpha\beta\gamma} = \frac{1}{\sqrt{\gamma}}e^{\alpha\beta\gamma},$$

where $e_{123} = e^{123} = 1$, and the sign changes under transposition of indices [compare (83.13–14)]. Accordingly the vector $\mathbf{c} = \mathbf{a}\times\mathbf{b}$, defined as the vector dual to the antisymmetric tensor $c_{\beta\gamma} = a_\beta b_\gamma - a_\gamma b_\beta$, has components:

$$c_\alpha = \tfrac{1}{2}\sqrt{\gamma}\,e_{\alpha\beta\gamma}c^{\beta\gamma} = \sqrt{\gamma}\,e_{\alpha\beta\gamma}a^\beta b^\gamma,\qquad c^\alpha = \frac{1}{2\sqrt{\gamma}}e^{\alpha\beta\gamma}c_{\beta\gamma} = \frac{1}{\sqrt{\gamma}}e^{\alpha\beta\gamma}a_\beta b_\gamma.$$

Conversely,

$$c_{\alpha\beta} = \sqrt{\gamma}\,e_{\alpha\beta\gamma}c^\gamma,\qquad c^{\alpha\beta} = \frac{1}{\sqrt{\gamma}}e^{\alpha\beta\gamma}c_\gamma.$$

In particular, curl **a** should be understood in this same sense as the vector dual to the tensor $a_{\beta;\alpha}-a_{\alpha;\beta} = (\partial a_\beta/\partial x^\alpha)-(\partial a_\alpha/\partial x^\beta)$, so that its contravariant components are

$$(\operatorname{curl}\mathbf{a})^\alpha = \frac{1}{2\sqrt{\gamma}}e^{\alpha\beta\gamma}\left(\frac{\partial a_\gamma}{\partial x^\beta}-\frac{\partial a_\beta}{\partial x^\gamma}\right).$$

In this same connection we repeat that for the three-dimensional divergence of a vector [see (86.9)]:

$$\operatorname{div}\mathbf{a} = \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^\alpha}(\sqrt{\gamma}\,a^\alpha).$$

To avoid misunderstandings when comparing with formulas frequently used for the three-dimensional vector operations in orthogonal curvilinear coordinates (see, for example, *Electrodynamics of Continuous Media*, appendix), we point out that in these formulas the components of the vectors are understood to be the quantities $\sqrt{g_{11}}\,A^1(=\sqrt{A_1A^1})$, $\sqrt{g_{22}}\,A^2$, $\sqrt{g_{33}}\,A^3$.

we obtain:

$$\frac{1}{h}\left(\frac{\omega_0}{c}\right)^2 - \gamma_{\alpha\beta} k^\alpha k^\beta = 0.$$

Noting that the vector k^α must have the direction of the vector dx^α, we then find:

$$k^\alpha = \frac{\omega_0}{c\sqrt{h}} \frac{dx^\alpha}{dl},$$

where dl (84.6) is the element of spatial distance along the ray. In order to obtain the expression for k_α, we write

$$k^\alpha = g^{\alpha i} k_i = g^{\alpha 0} k_0 + g^{\alpha\beta} k_\beta = -g^\alpha \frac{\omega_0}{c} - \gamma^{\alpha\beta} k_\beta,$$

so that

$$k_\alpha = -\gamma_{\alpha\beta}\left(k^\beta + \frac{\omega_0}{c} g^\beta\right) = -\frac{\omega_0}{c}\left(\frac{\gamma_{\alpha\beta}}{\sqrt{h}} \frac{dx^\beta}{dl} + g_\alpha\right).$$

Finally, multiplying by dx^α, we obtain Fermat's principle in the form (dropping the constant factor ω_0/c):

$$\delta \int \left(\frac{dl}{\sqrt{h}} + g_\alpha dx^\alpha\right) = 0.$$

In a static field, we have simply:

$$\delta \int \frac{dl}{\sqrt{h}} = 0.$$

We call attention to the fact that in a gravitational field the ray does not propagate along the shortest line in space, since the latter would be defined by the equation $\delta \int dl = 0$.

§ 89. Rotation

As a special case of a stationary gravitational field, let us consider a uniformly rotating reference system. To calculate the interval ds we carry out the transformation from a system at rest (inertial system) to the uniformly rotating one. In the coordinates r', ϕ', z', t of the system at rest (we use cylindrical coordinates r', ϕ', z'), the interval has the form

$$ds^2 = c^2\, dt^2 - dr'^2 - r'^2\, d\phi'^2 - dz'^2. \tag{89.1}$$

Let the cylindrical coordinates in the rotating system be r, ϕ, z. If the axis of rotation coincides with the axes Z and Z', then we have $r' = r$, $z' = z$, $\phi' = \phi + \Omega t$, where Ω is the angular velocity of rotation. Substituting in (89.1), we find the required expression for ds^2 in the rotating system of reference:

$$ds^2 = (c^2 - \Omega^2 r^2)\, dt^2 - 2\Omega r^2\, d\phi\, dt - dz^2 - r^2\, d\phi^2 - dr^2. \tag{89.2}$$

It is necessary to note that the rotating system of reference can be used only out to distances equal to c/Ω. In fact, from (89.2) we see that for $r > c/\Omega$, g_{00} becomes negative, which is not admissible. The inapplicability of the rotating reference system at large distances is related to the fact that there the velocity would become greater than the velocity of light, and therefore such a system cannot be made up from real bodies.

As in every stationary field, clocks on the rotating body cannot be uniquely synchronized at all points. Proceeding with the synchronization along any closed curve, we find, upon returning to the starting point, a time differing from the initial value by an amount [see

(88.5)]

$$\Delta t = -\frac{1}{c}\oint \frac{g_{0\alpha}}{g_{00}}\,dx^\alpha = \frac{1}{c^2}\oint \frac{\Omega r^2 d\phi}{1-\frac{\Omega^2 r^2}{c^2}}$$

or, assuming that $\Omega r/c \ll 1$ (i.e. that the velocity of the rotation is small compared with the velocity of light),

$$\Delta t = \frac{\Omega}{c^2}\int r^2 d\phi = \pm\frac{2\Omega}{c^2}S, \tag{89.3}$$

where S is the projected area of the contour on a plane perpendicular to the axis of rotation (the sign + or − holding according as we traverse the contour in, or opposite to, the direction of rotation).

Let us assume that a ray of light propagates along a certain closed contour. Let us calculate to terms of order v/c the time t that elapses between the starting out of the light ray and its return to the initial point. The velocity of light, by definition, is always equal to c, if the times are synchronized along the given closed curve and if at each point we use the proper time. Since the difference between proper and world time is of order v^2/c^2, then in calculating the required time interval t to terms of order v/c this difference can be neglected. Therefore we have

$$t = \frac{L}{c} \pm \frac{2\Omega}{c^2}S,$$

where L is the length of the contour. Corresponding to this, the velocity of light, measured as the ratio L/t, appears equal to

$$c \pm 2\Omega\frac{S}{L}. \tag{89.4}$$

This formula, like the first approximation for the Doppler effect, can also be easily derived in a purely classical manner.

PROBLEM

Calculate the element of spatial distance in a rotating coordinate system.

Solution: With the help of (84.6) and (84.7), we find

$$dl^2 = dr^2 + dz^2 + \frac{r^2 d\phi^2}{1-\Omega^2\frac{r^2}{c^2}},$$

which determines the spatial geometry in the rotating reference system. We note that the ratio of the circumference of a circle in the plane z = constant (with center on the axis of rotation) to its radius r is

$$2\pi/\sqrt{1-\Omega^2 r^2/c^2},$$

i.e. larger than 2π.

§ 90. The equations of electrodynamics in the presence of a gravitational field

The electromagnetic field equations of the special theory of relativity can be easily generalized so that they are applicable in an arbitrary four-dimensional curvilinear system of coordinates, i.e., in the presence of a gravitational field.

The electromagnetic field tensor in the special theory of relativity is defined as $F_{ik} = (\partial A_k/\partial x^i) - (\partial A_i/\partial x^k)$. Clearly it must now be defined correspondingly as $F_{ik} = A_{k;i} - A_{i;k}$. But because of (86.12),

$$F_{ik} = A_{k;i} - A_{i;k} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}, \tag{90.1}$$

and therefore the relation of F_{ik} to the potential A_i does not change. Consequently the first pair of Maxwell equations (26.5) also does not change its form†

$$\frac{\partial F_{ik}}{\partial x^l} + \frac{\partial F_{li}}{\partial x^k} + \frac{\partial F_{kl}}{\partial x^i} = 0. \tag{90.2}$$

In order to write the second pair of Maxwell equations, we must first determine the current four-vector in curvilinear coordinates. We do this in a fashion completely analogous to that which we followed in § 28. The spatial volume element, constructed on the space coordinate elements dx^1, dx^2, and dx^3, is $\sqrt{\gamma}\, dV$, where γ is the determinant of the spatial metric tensor (84.7) and $dV = dx^1\, dx^2\, dx^3$ (see the footnote on p. 232). We introduce the charge density ϱ according to the definition $de = \varrho\sqrt{\gamma}\, dV$, where de is the charge located within the volume element $\sqrt{\gamma}\, dV$. Multiplying this equation on both sides by dx^i, we have:

$$de\, dx^i = \varrho\, dx^i \sqrt{\gamma}\, dx^1\, dx^2\, dx^3 = \frac{\varrho}{\sqrt{g_{00}}} \sqrt{-g}\, d\Omega\, \frac{dx^i}{dx^0}$$

[where we have used the formula $-g = \gamma\, g_{00}$ (84.10)]. The product $\sqrt{-g}\, d\Omega$ is the invariant element of four-volume, so that the current four-vector is defined by the expression

$$j^i = \frac{\varrho c}{\sqrt{g_{00}}} \frac{dx^i}{dx^0} \tag{90.3}$$

(the quantities dx^i/dx^0 are the rates of change of the coordinates with the "time" x^0, and *do not* constitute a four-vector). The component j^0 of the current four-vector, multiplied by $\sqrt{g_{00}}/c$, is the spatial density of charge.

For point charges the density ϱ is expressed as a sum of δ-functions, as in formula (28.1). We must, however, correct the definition of these functions for the case of curvilinear coordinates. By $\delta(\mathbf{r})$ we shall again mean the product $\delta(x^1)\,\delta(x^2)\,\delta(x^3)$, regardless of the geometrical meaning of the coordinates x^1, x^2, x^3; then the integral over dV (and not over $\sqrt{\gamma}\, dV$) is unity: $\int \delta(\mathbf{r})\, dV = 1$. With this same definition of the δ-functions, the charge density is

$$\varrho = \sum_a \frac{e_a}{\sqrt{\gamma}} \delta(\mathbf{r} - \mathbf{r}_a),$$

and the current four-vector is

$$j^i = \sum_a \frac{e_a c}{\sqrt{-g}} \delta(\mathbf{r} - \mathbf{r}_a) \frac{dx^i}{dx^0}. \tag{90.4}$$

Conservation of charge is expressed by the equation of continuity, which differs from

† It is easily seen that the equation can also be written in the form

$$F_{ik;l} + F_{li;k} + F_{kl;i} = 0,$$

from which its covariance is obvious.

(29.4) only in replacement of the ordinary derivatives by covariant derivatives:

$$j^i_{;i} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} (\sqrt{-g}\, j^i) = 0 \tag{90.5}$$

[using formula (86.9)].

The second pair of Maxwell equations (30.2) is generalized similarly; replacing the ordinary derivatives by covariant derivatives, we find:

$$F^{ik}_{;k} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^k} (\sqrt{-g}\, F^{ik}) = -\frac{4\pi}{c} j^i \tag{90.6}$$

[using formula (86.10)].

Finally the equations of motion of a charged particle in gravitational and electromagnetic fields is obtained by replacing the four-acceleration du^i/ds in (23.4) by Du^i/ds:

$$mc \frac{Du^i}{ds} = mc \left(\frac{du^i}{ds} + \Gamma^i_{kl} u^k u^l \right) = \frac{e}{c} F^{ik} u_k. \tag{90.7}$$

PROBLEM

Write the Maxwell equations in a given gravitational field in three-dimensional form (in the three-dimensional space with metric $\gamma_{\alpha\beta}$), introducing the three-vectors **E**, **D** and the antisymmetric three-tensors $B_{\alpha\beta}$ and $H_{\alpha\beta}$ according to the definitions:

$$E_\alpha = F_{0\alpha}, \qquad B_{\alpha\beta} = F_{\alpha\beta},$$
$$D^\alpha = -\sqrt{g_{00}}\, F^{0\alpha}, \qquad H^{\alpha\beta} = \sqrt{g_{00}}\, F^{\alpha\beta}. \tag{1}$$

Solution: The quantities introduced above are not independent. Writing out the equations

$$F_{0\alpha} = g_{0l} g_{\alpha m} F^{lm}, \qquad F^{\alpha\beta} = g^{\alpha l} g^{\beta m} F_{lm},$$

and introducing the three-dimensional metric tensor $\gamma_{\alpha\beta} = -g_{\alpha\beta} + h g_\alpha g_\beta$ [with **g** and h from (88.11)], and using formulas (84.9) and 84.12), we get:

$$D_\alpha = \frac{E_\alpha}{\sqrt{h}} + g^\beta H_{\alpha\beta}, \qquad B^{\alpha\beta} = \frac{H^{\alpha\beta}}{\sqrt{h}} + g^\beta E^\alpha - g^\alpha E^\beta. \tag{2}$$

We introduce the vectors **B** and **H**, dual to the tensors $B_{\alpha\beta}$ and $H_{\alpha\beta}$, in accordance with the definition:

$$B^\alpha = -\frac{1}{2\sqrt{\gamma}} e^{\alpha\beta\gamma} B_{\beta\gamma}, \qquad H_\alpha = -\frac{1}{2} \sqrt{\gamma}\, e_{\alpha\beta\gamma} H^{\beta\gamma} \tag{3}$$

(see the footnote on p. 252; the minus sign is introduced so that in galilean coordinates the vectors **H** and **B** coincide with the ordinary magnetic field intensity). Then (2) can be written in the forms

$$\mathbf{D} = \frac{\mathbf{E}}{\sqrt{h}} + \mathbf{H} \times \mathbf{g}, \qquad \mathbf{B} = \frac{\mathbf{H}}{\sqrt{h}} + \mathbf{g} \times \mathbf{E}. \tag{4}$$

Introducing definition (1) in (90.2), we get the equations:

$$\frac{\partial B_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial B_{\gamma\alpha}}{\partial x^\beta} + \frac{\partial B_{\beta\gamma}}{\partial x^\alpha} = 0,$$
$$\frac{\partial B_{\alpha\beta}}{\partial x^0} + \frac{\partial E_\alpha}{\partial x^\beta} - \frac{\partial E_\beta}{\partial x^\alpha} = 0,$$

or, changing to the dual quantities (3):

$$\operatorname{div} \mathbf{B} = 0, \qquad \operatorname{curl} \mathbf{E} = -\frac{1}{c\sqrt{\gamma}} \frac{\partial}{\partial t} (\sqrt{\gamma}\, \mathbf{B}) \tag{5}$$

ΩR
$+ 2\Omega \frac{\eta R^2}{2\eta R} = c + \Omega R$

($x^0 = ct$; the definitions of the operations div and curl are given in the footnote on p. 252). Similarly we find from (90.6) the equations

$$\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^\alpha}(\sqrt{\gamma}D^\alpha) = 4\pi\varrho,$$

$$\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^\beta}(\sqrt{\gamma}\,H^{\alpha\beta}) + \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^0}(\sqrt{\gamma}\,D^\alpha) = -4\pi\varrho\frac{dx^\alpha}{dx^0},$$

or, in three-dimensional notation:

$$\operatorname{div}\mathbf{D} = 4\pi\varrho, \qquad \operatorname{curl}\mathbf{H} = \frac{1}{c\sqrt{\gamma}}\frac{\partial}{\partial t}(\sqrt{\gamma}\,\mathbf{D}) + \frac{4\pi}{c}\mathbf{s}, \tag{6}$$

where s is the vector with components $s^\alpha = \varrho\, dx^\alpha/dt$.

We also write the continuity equation (90.5) in three-dimensional form:

$$\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial t}(\sqrt{\gamma}\,\varrho) + \operatorname{div}\mathbf{s} = 0. \tag{7}$$

The reader should note the analogy (purely formal, of course) of equations (5) and (6) to the Maxwell equations for the electromagnetic field in material media. In particular, in a static gravitational field the quantity $\sqrt{\gamma}$ drops out of the terms containing time derivatives, and relation (4) reduces to $\mathbf{D} = \mathbf{E}/\sqrt{h}$, $\mathbf{B} = \mathbf{H}/\sqrt{h}$. We may say that with respect to its effect on the electromagnetic field a static gravitational field plays the role of a medium with electric and magnetic permeabilities $\varepsilon = \mu = 1/\sqrt{h}$.

CHAPTER 11

THE GRAVITATIONAL FIELD EQUATIONS

§ 91. The curvature tensor

Let us go back once more to the concept of parallel displacement of a vector. As we said in § 85, in the general case of a curved four-space, the infinitesimal parallel displacement of a vector is defined as a displacement in which the components of the vector are not changed in a system of coordinates which is galilean in the given infinitesimal volume element.

If $x^i = x^i(s)$ is the parametric equation of a certain curve (s is the arc length measured from some point), then the vector $u^i = dx^i/ds$ is a unit vector tangent to the curve. If the curve we are considering is a geodesic, then along it $Du^i = 0$. This means that if the vector u^i is subjected to a parallel displacement from a point x^i on a geodesic curve to the point $x^i + dx^i$ on the same curve, then it coincides with the vector $u^i + du^i$ tangent to the curve at the point $x^i + dx^i$. Thus when the tangent to a geodesic moves along the curve, it is displaced parallel to itself.

On the other hand, during the parallel displacement of two vectors, the "angle" between them clearly remains unchanged. Therefore we may say that during the parallel displacement of any vector along a geodesic curve, the angle between the vector and the tangent to the geodesic remains unchanged. In other words, during the parallel displacement of a vector, its component along the geodesic must be the same at all points of the path.

Now the very important result appears that in a curved space the parallel displacement of a vector from one given point to another gives different results if the displacement is carried out over different paths. In particular, it follows from this that if we displace a vector parallel to itself along some closed contour, then upon returning to the starting point, it will not coincide with its original value.

In order to make this clear, let us consider a curved two-dimensional space, i.e., any curved surface. Figure 19 shows a portion of such a surface, bounded by three geodesic curves. Let us subject the vector 1 to a parallel displacement along the contour made up of

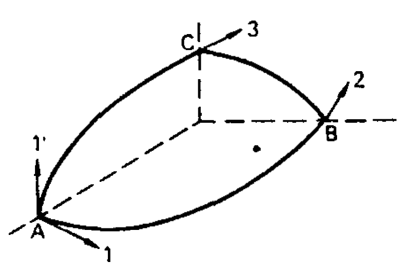

FIG. 19.