FLUID MECHANICS

Second Edition

by

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Translated from the Russian by

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CONTENTS

PREFACE TO THE SECOND ENGLISH EDITION

The content and treatment in this edition remain in accordance with what was said in the preface to the first edition (see below). My chief care in revising and augmenting has been to comply with this principle.

Despite the lapse of thirty years, the previous edition has, with very slight exceptions, not gone out of date. Its material has been only fairly slightly supplemented and modified. About ten new sections have been added.

In recent decades, fluid mechanics has undergone extremely rapid development, and there has accordingly been a great increase in the literature of the subject. The development has been mainly in applications, however, and in an increasing complexity of the problems accessible to theoretical calculation (with or without computers). These include, in particular, various problems of instability and its development, including non-linear regimes. All such topics are beyond the scope of our book; in particular, stability problems are discussed, as previously, mainly in terms of results.

There is also no treatment of non-linear waves in dispersive media, which is by now a significant branch of mathematical physics. The purely hydrodynamic subject of this theory consists in waves with large amplitude on the surface of a liquid. Its principal physical applications are in plasma physics, non-linear optics, various problems of electrodynamics, and so on, and in that respect they belong in other volumes of the *Course*.

There have been important changes in our understanding of the mechanism whereby turbulence occurs. Although a consistent theory of turbulence is still a thing of the future, there is reason to suppose that the right path has finally been found. The basic ideas now available and the results obtained are discussed in three sections (§§30–32) written jointly with M. I. Rabinovich, to whom I am deeply grateful for this valuable assistance. A new area in continuum mechanics over the last few decades is that of liquid crystals. This combines features of the mechanics of liquid and elastic media. Its principles are discussed in the new edition of *Theory of Elasticity*.

This book has a special place among those I had occasion to write jointly with L. D. Landau. He gave it a part of his soul. That branch of theoretical physics, new to him at the time, caught his fancy, and in a very typical way he set about thinking through it *ab initio* and deriving its basic results. This led to a number of original papers which appeared in various journals, but several of his conclusions or ideas were not published elsewhere than in the book, and in some instances even his priority was not established till later. In the new edition, I have added an appropriate reference to his authorship in all such cases that are known to me.

In the revision of this book, as in other volumes of the *Course*, I have had the help and advice of many friends and colleagues. I should like to mention in particular numerous discussions with G. I. Barenblatt, L. P. Pitaevskiĭ, Ya. G. Sinaĭ, and Ya. B. Zel'dovich. Several useful comments came from A. A. Andronov, S. I. Anisimov, V. A. Belokon', A. L. Fabrikant, V. P. Kraĭnov, A. G. Kulikovskiĭ, M. A. Liberman, R. V. Polovin, and A. V. Timofeev. To all of them I express my sincere gratitude.

Institute of Physical Problems E. M. LIFSHITZ
August 1984

PREFACE TO THE FIRST ENGLISH EDITION

The present book deals with fluid mechanics, i.e. the theory of the motion of liquids and gases.

The nature of the book is largely determined by the fact that it describes fluid mechanics as a branch of theoretical physics, and it is therefore markedly different from other textbooks on the same subject. We have tried to develop as fully as possible all matters of physical interest, and to do so in such a way as to give the clearest possible picture of the phenomena and their interrelation. Accordingly, we discuss neither approximate methods of calculation in fluid mechanics, nor empirical theories devoid of physical significance. On the other hand, accounts are given of some topics not usually found in textbooks on the subject: the theory of heat transfer and diffusion in fluids; acoustics; the theory of combustion; the dynamics of superfluids; and relativistic fluid dynamics.

In a field which has been so extensively studied as fluid mechanics it was inevitable that important new results should have appeared during the several years since the last Russian edition was published. Unfortunately, our preoccupation with other matters has prevented us from including these results in the English edition. We have merely added one further chapter, on the general theory of fluctuations in fluid dynamics.

We should like to express our sincere thanks to Dr Sykes and Dr Reid for their excellent translation of the book, and to Pergamon Press for their ready agreement to our wishes in various matters relating to its publication.

Moscow 1958

L. D. Landau
E. M. Lifshitz

EVGENIĬ MIKHAĬLOVICH LIFSHITZ (1915–1985)†

Soviet physics suffered a heavy loss on 29 October 1985 with the death of the outstanding theoretical physicist Academician Evgeniĭ Mikhaĭlovich Lifshitz.

Lifshitz was born on 21 February 1915 in Khar'kov. In 1933 he graduated from the Khar'kov Polytechnic Institute. He worked at the Khar'kov Physicotechnical Institute from 1933 to 1938 and at the Institute of Physical Problems of the USSR Academy of Sciences in Moscow from 1939 until his death. He was elected an associate member of the USSR Academy of Sciences in 1966 and a full member in 1979.

Lifshitz's scientific activity began very early. He was among L. D. Landau's first students and at 19 he co-authored with him a paper on the theory of pair production in collisions. This paper, which has not lost its significance to this day, outlined many methodological features of modern relativistically invariant techniques of quantum field theory. It includes, in particular, a consistent allowance for retardation.

Modern ferromagnetism theory is based on the "Landau-Lifshitz" equation, which describes the dynamics of the magnetic moment in a ferromagnet. A 1935 article on this subject is one of the best known papers on the physics of magnetic phenomena. The derivation of the equation is accompanied by development of a theory of ferromagnetic resonance and of the domain structure of ferromagnets.

In a 1937 paper on the Boltzmann kinetic equation for electrons in a magnetic field, E. M. Lifshitz developed a drift approximation extensively used much later, in the 50s, in plasma theory.

A paper published in 1939 on deuteron dissociation in collisions remains a brilliant example of the use of quasi-classical methods in quantum mechanics.

A most important step towards the development of a theory of second-order phase transitions, following the work by L. D. Landau, was a paper by Lifshitz dealing with the change of the symmetry of a crystal, of its space group, in transitions of this type (1941). Many years later the results of this paper came into extensive use, and the terms "Lifshitz criterion" and "Lifshitz point," coined on its basis have become indispensable components of modern statistical physics.

A decisive role in the detection of an important physical phenomenon, second sound in superfluid helium, was played by a 1944 paper by E. M. Lifshitz. It is shown in it that second sound is effectively excited by a heater having an alternating temperature. This was precisely the method used to observe second sound in experiment two years later.

A new approach to the theory of molecular-interaction forces between condensed bodies was developed by Lifshitz in 1954–1959. It is based on the profound physical idea that these forces are manifestations of stresses due to quantum and thermal fluctuations of an electromagnetic field in a medium. This idea was pursued to develop a very elegant and general theory in which the interaction forces are expressed in terms of electrodynamic material properties such as the complex dielectric permittivity. This theory of E. M.

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† By A. F. Andreev, A. S. Borovik-Romanov, V. L. Ginzburg, L. P. Gor'kov, I. E. Dzyaloshinskiĭ, Ya. B. Zel'dovich, M. I. Kaganov, L. P. Pitaevskiĭ, E. L. Feĭnberg, and I. M. Khalatnikov; published in Russian in *Uspekhi fizicheskikh nauk* **148**, 549–550, 1986. This translation is by J. G. Adashko (first published in *Soviet Physics Uspekhi* **29**, 294–295, 1986), and is reprinted by kind permission of the American Institute of Physics.

Lifshitz stimulated many studies and was confirmed by experiment. It gained him the M. V. Lomonosov Prize in 1958.

E. M. Lifshitz made a fundamental contribution in one of the most important branches of modern physics, the theory of gravitation. His research into this field started with a classical 1946 paper on the stability of cosmological solutions of Einstein's theory of gravitation. The perturbations were divided into distinctive classes—scalar, with variation of density, vector, describing vortical motion, and finally tensor, describing gravitational waves. This classification is still of decisive significance in the analysis of the origin of the universe. From there, E. M. Lifshitz tackled the exceedingly difficult question of the general character of the singularities of this theory. Many years of labor led in 1972 to a complete solution of this problem in papers written jointly with V. A. Belinskiĭ and I. M. Khalatnikov, which earned their authors the 1974 L. D. Landau Prize. The singularity was found to have a complicated oscillatory character and could be illustratively represented as contraction of space in two directions with simultaneous expansion in the third. The contraction and expansion alternate in time according to a definite law. These results elicited a tremendous response from specialists, altered radically our ideas concerning relativistic collapse, and raised a host of physical and mathematical problems that still await solution.

His life-long occupation was the famous Landau and Lifshitz *Course of Theoretical Physics*, to which he devoted about 50 years. (The first edition of *Statistical Physics* was written in 1937. A new edition of *Theory of Elasticity* went to press shortly before his last illness.) The greater part of the *Course* was written by Lifshitz together with his teacher and friend L. D. Landau. After the automobile accident that made Landau unable to work, Lifshitz completed the edition jointly with Landau's students. He later continued to revise the previously written volumes in the light of the latest advances in science. Even in the hospital, he discussed with visiting friends the topics that should be subsequently included in the *Course.*

The *Course of Theoretical Physics* became world famous. It was translated in its entirety into six languages. Individual volumes were published in 10 more languages. In 1972 L. D. Landau and E. M. Lifshitz were awarded the Lenin Prize for the volumes published by then.

The *Course of Theoretical Physics* remains a monument to E. M. Lifshitz as a scientist and a pedagogue. It has educated many generations of physicists, is being studied, and will continue to teach students in future generations.

A versatile physicist, E. M. Lifshitz dealt also with applications. He was awarded the USSR State Prize in 1954.

A tremendous amount of E.M. Lifshitz's labor and energy was devoted to Soviet scientific periodicals. From 1946 to 1949 and from 1955 to his death he was deputy editor-in-chief of the *Journal of Experimental and Theoretical Physics.* His extreme devotion to science, adherence to principles, and meticulousness greatly helped to make this journal one of the best scientific periodicals in the world.

E. M. Lifshitz accomplished much in his life. He will remain in our memory as a remarkable physicist and human being. His name will live forever in the history of Soviet physics.

NOTATION

ρ density
p pressure
T temperature
s entropy per unit mass
ε internal energy per unit mass
$w = \varepsilon + p/\rho$ heat function (enthalpy)
$\gamma = c_p/c_v$ ratio of specific heats at constant pressure and constant volume
η dynamic viscosity
$\nu = \eta/\rho$ kinematic viscosity
κ thermal conductivity
$\chi = \kappa/\rho c_p$ thermometric conductivity
R Reynolds number
c velocity of sound
M ratio of fluid velocity to velocity of sound (Mach number)

Vector and tensor (three-dimensional) suffixes are denoted by Latin letters $i, k, l, \ldots$. Summation over repeated ("dummy") suffixes is everywhere implied. The unit tensor is δ_{ik}:

References to other volumes in the *Course of Theoretical Physics:*

Fields = Vol. 2 (*The Classical Theory of Fields*, fourth English edition, 1975).
QM = Vol. 3 (*Quantum Mechanics*, third English edition, 1977).
SP 1 = Vol. 5 (*Statistical Physics*, Part 1, third English edition, 1980).
ECM = Vol. 8 (*Electrodynamics of Continuous Media*, second English edition, 1984).
SP 2 = Vol. 9 (*Statistical Physics*, Part 2, English edition, 1980).
PK = Vol. 10 (*Physical Kinetics*, English edition, 1981).

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CHAPTER I

IDEAL FLUIDS

§1. The equation of continuity

Fluid dynamics concerns itself with the study of the motion of fluids (liquids and gases). Since the phenomena considered in fluid dynamics are macroscopic, a fluid is regarded as a continuous medium. This means that any small volume element in the fluid is always supposed so large that it still contains a very great number of molecules. Accordingly, when we speak of infinitely small elements of volume, we shall always mean those which are "physically" infinitely small, i.e. very small compared with the volume of the body under consideration, but large compared with the distances between the molecules. The expressions *fluid particle* and *point in a fluid* are to be understood in a similar sense. If, for example, we speak of the displacement of some fluid particle, we mean not the displacement of an individual molecule, but that of a volume element containing many molecules, though still regarded as a point.

The mathematical description of the state of a moving fluid is effected by means of functions which give the distribution of the fluid velocity $\mathbf{v} = \mathbf{v}(x, y, z, t)$ and of any two thermodynamic quantities pertaining to the fluid, for instance the pressure $p(x, y, z, t)$ and the density $\rho(x, y, z, t)$. All the thermodynamic quantities are determined by the values of any two of them, together with the equation of state; hence, if we are given five quantities, namely the three components of the velocity $\mathbf{v}$, the pressure p and the density ρ, the state of the moving fluid is completely determined.

All these quantities are, in general, functions of the coordinates x, y, z and of the time t. We emphasize that $\mathbf{v}(x, y, z, t)$ is the velocity of the fluid at a given point (x, y, z) in space and at a given time t, i.e. it refers to fixed points in space and not to specific particles of the fluid; in the course of time, the latter move about in space. The same remarks apply to ρ and p.

We shall now derive the fundamental equations of fluid dynamics. Let us begin with the equation which expresses the conservation of matter. We consider some volume V_0 of space. The mass of fluid in this volume is $\int \rho \, \mathrm{d}V$, where ρ is the fluid density, and the integration is taken over the volume V_0. The mass of fluid flowing in unit time through an element $\mathrm{d}\mathbf{f}$ of the surface bounding this volume is $\rho \mathbf{v} \cdot \mathrm{d}\mathbf{f}$; the magnitude of the vector $\mathrm{d}\mathbf{f}$ is equal to the area of the surface element, and its direction is along the normal. By convention, we take $\mathrm{d}\mathbf{f}$ along the outward normal. Then $\rho \mathbf{v} \cdot \mathrm{d}\mathbf{f}$ is positive if the fluid is flowing out of the volume, and negative if the flow is into the volume. The total mass of fluid flowing out of the volume V_0 in unit time is therefore

$$\oint \rho \mathbf{v} \cdot \mathrm{d}\mathbf{f},$$

where the integration is taken over the whole of the closed surface surrounding the volume in question.

Next, the decrease per unit time in the mass of fluid in the volume V_0 can be written

$$-\frac{\partial}{\partial t}\int \rho \, \mathrm{d}V.$$

Equating the two expressions, we have

$$\frac{\partial}{\partial t}\int \rho \, \mathrm{d}V = -\oint \rho \, \mathbf{v}\cdot \mathrm{d}\mathbf{f}. \qquad (1.1)$$

The surface integral can be transformed by Green's formula to a volume integral:

$$\oint \rho \, \mathbf{v}\cdot \mathrm{d}\mathbf{f} = \int \mathrm{div}\,(\rho\mathbf{v})\,\mathrm{d}V.$$

Thus

$$\int\left[\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho\mathbf{v})\right]\mathrm{d}V = 0.$$

Since this equation must hold for any volume, the integrand must vanish, i.e.

$$\partial\rho/\partial t + \mathrm{div}\,(\rho\mathbf{v}) = 0. \qquad (1.2)$$

This is the *equation of continuity*. Expanding the expression div $(\rho\mathbf{v})$, we can also write (1.2) as

$$\partial\rho/\partial t + \rho\,\mathrm{div}\,\mathbf{v} + \mathbf{v}\cdot\mathbf{grad}\,\rho = 0. \qquad (1.3)$$

The vector

$$\mathbf{j} = \rho\mathbf{v} \qquad (1.4)$$

is called the *mass flux density*. Its direction is that of the motion of the fluid, while its magnitude equals the mass of fluid flowing in unit time through unit area perpendicular to the velocity.

§2. Euler's equation

Let us consider some volume in the fluid. The total force acting on this volume is equal to the integral

$$-\oint p \, \mathrm{d}\mathbf{f}$$

of the pressure, taken over the surface bounding the volume. Transforming it to a volume integral, we have

$$-\oint p \, \mathrm{d}\mathbf{f} = -\int \mathbf{grad}\, p \, \mathrm{d}V.$$

Hence we see that the fluid surrounding any volume element $\mathrm{d}V$ exerts on that element a force $-\mathrm{d}V\,\mathbf{grad}\,p$. In other words, we can say that a force $-\mathbf{grad}\,p$ acts on unit volume of the fluid.

We can now write down the equation of motion of a volume element in the fluid by equating the force $-\mathbf{grad}\,p$ to the product of the mass per unit volume (ρ) and the acceleration $\mathrm{d}\mathbf{v}/\mathrm{d}t$:

$$\rho\,\mathrm{d}\mathbf{v}/\mathrm{d}t = -\mathbf{grad}\,p. \qquad (2.1)$$

The derivative $\mathrm{d}\mathbf{v}/\mathrm{d}t$ which appears here denotes not the rate of change of the fluid velocity at a fixed point in space, but the rate of change of the velocity of a given fluid particle as it moves about in space. This derivative has to be expressed in terms of quantities referring to points fixed in space. To do so, we notice that the change $\mathrm{d}\mathbf{v}$ in the velocity of the given fluid particle during the time $\mathrm{d}t$ is composed of two parts, namely the change during $\mathrm{d}t$ in the velocity at a point fixed in space, and the difference between the velocities (at the same instant) at two points $\mathrm{d}\mathbf{r}$ apart, where $\mathrm{d}\mathbf{r}$ is the distance moved by the given fluid particle during the time $\mathrm{d}t$. The first part is $(\partial\mathbf{v}/\partial t)\mathrm{d}t$, where the derivative $\partial\mathbf{v}/\partial t$ is taken for constant x, y, z, i.e. at the given point in space. The second part is

$$\mathrm{d}x\frac{\partial\mathbf{v}}{\partial x}+\mathrm{d}y\frac{\partial\mathbf{v}}{\partial y}+\mathrm{d}z\frac{\partial\mathbf{v}}{\partial z} = (\mathrm{d}\mathbf{r}\cdot\mathbf{grad})\mathbf{v}.$$

Thus

$$\mathrm{d}\mathbf{v} = (\partial\mathbf{v}/\partial t)\mathrm{d}t + (\mathrm{d}\mathbf{r}\cdot\mathbf{grad})\mathbf{v},$$

or, dividing both sides by $\mathrm{d}t$,†

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\mathbf{grad})\mathbf{v}. \tag{2.2}$$

Substituting this in (2.1), we find

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\mathbf{grad})\mathbf{v} = -\frac{1}{\rho}\mathbf{grad}\, p. \tag{2.3}$$

This is the required equation of motion of the fluid; it was first obtained by L. Euler in 1755. It is called *Euler's equation* and is one of the fundamental equations of fluid dynamics.

If the fluid is in a gravitational field, an additional force $\rho\mathbf{g}$, where $\mathbf{g}$ is the acceleration due to gravity, acts on any unit volume. This force must be added to the right-hand side of equation (2.1), so that equation (2.3) takes the form

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\mathbf{grad})\mathbf{v} = -\frac{\mathbf{grad}\, p}{\rho} + \mathbf{g}. \tag{2.4}$$

In deriving the equations of motion we have taken no account of processes of energy dissipation, which may occur in a moving fluid in consequence of internal friction (viscosity) in the fluid and heat exchange between different parts of it. The whole of the discussion in this and subsequent sections of this chapter therefore holds good only for motions of fluids in which thermal conductivity and viscosity are unimportant; such fluids are said to be *ideal*.

The absence of heat exchange between different parts of the fluid (and also, of course, between the fluid and bodies adjoining it) means that the motion is adiabatic throughout the fluid. Thus the motion of an ideal fluid must necessarily be supposed adiabatic.

In adiabatic motion the entropy of any particle of fluid remains constant as that particle moves about in space. Denoting by s the entropy per unit mass, we can express the condition for adiabatic motion as

$$\mathrm{d}s/\mathrm{d}t = 0, \tag{2.5}$$

† The derivative $\mathrm{d}/\mathrm{d}t$ thus defined is called the *substantial* time derivative, to emphasize its connection with the moving substance.

where the total derivative with respect to time denotes, as in (2.1), the rate of change of entropy for a given fluid particle as it moves about. This condition can also be written

$$\partial s/\partial t + \mathbf{v}\cdot\mathbf{grad}\, s = 0. \tag{2.6}$$

This is the general equation describing adiabatic motion of an ideal fluid. Using (1.2), we can write it as an "equation of continuity" for entropy:

$$\partial(\rho s)/\partial t + \operatorname{div}(\rho s\mathbf{v}) = 0. \tag{2.7}$$

The product $\rho s\mathbf{v}$ is the *entropy flux density.*

The adiabatic equation usually takes a much simpler form. If, as usually happens, the entropy is constant throughout the volume of the fluid at some initial instant, it retains everywhere the same constant value at all times and for any subsequent motion of the fluid. In this case we can write the adiabatic equation simply as

$$s = \text{constant}, \tag{2.8}$$

and we shall usually do so in what follows. Such a motion is said to be *isentropic.*

We may use the fact that the motion is isentropic to put the equation of motion (2.3) in a somewhat different form. To do so, we employ the familiar thermodynamic relation

$$\mathrm{d}w = T\mathrm{d}s + V\mathrm{d}p,$$

where w is the heat function per unit mass of fluid (enthalpy), $V = 1/\rho$ is the specific volume, and T is the temperature. Since s = constant, we have simply

$$\mathrm{d}w = V\mathrm{d}p = \mathrm{d}p/\rho,$$

and so $(\mathbf{grad}\, p)/\rho = \mathbf{grad}\, w$. Equation (2.3) can therefore be written in the form

$$\partial\mathbf{v}/\partial t + (\mathbf{v}\cdot\mathbf{grad})\mathbf{v} = -\mathbf{grad}\, w. \tag{2.9}$$

It is useful to notice one further form of Euler's equation, in which it involves only the velocity. Using a formula well known in vector analysis,

$$\tfrac{1}{2}\mathbf{grad}\, v^2 = \mathbf{v}\times\mathbf{curl}\,\mathbf{v} + (\mathbf{v}\cdot\mathbf{grad})\mathbf{v},$$

we can write (2.9) in the form

$$\partial\mathbf{v}/\partial t - \mathbf{v}\times\mathbf{curl}\,\mathbf{v} = -\mathbf{grad}\,(w + \tfrac{1}{2}v^2). \tag{2.10}$$

If we take the curl of both sides of this equation, we obtain

$$\frac{\partial}{\partial t}(\mathbf{curl}\,\mathbf{v}) = \mathbf{curl}\,(\mathbf{v}\times\mathbf{curl}\,\mathbf{v}), \tag{2.11}$$

which involves only the velocity.

The equations of motion have to be supplemented by the boundary conditions that must be satisfied at the surfaces bounding the fluid. For an ideal fluid, the boundary condition is simply that the fluid cannot penetrate a solid surface. This means that the component of the fluid velocity normal to the bounding surface must vanish if that surface is at rest:

$$v_n = 0. \tag{2.12}$$

In the general case of a moving surface, v_n must be equal to the corresponding component of the velocity of the surface.

At a boundary between two immiscible fluids, the condition is that the pressure and the velocity component normal to the surface of separation must be the same for the two fluids, and each of these velocity components must be equal to the corresponding component of the velocity of the surface.

As has been said at the beginning of §1, the state of a moving fluid is determined by five quantities: the three components of the velocity $\mathbf{v}$ and, for example, the pressure p and the density ρ. Accordingly, a complete system of equations of fluid dynamics should be five in number. For an ideal fluid these are Euler's equations, the equation of continuity, and the adiabatic equation.

PROBLEM

Write down the equations for one-dimensional motion of an ideal fluid in terms of the variables a, t, where a (called a *Lagrangian variable*†) is the x coordinate of a fluid particle at some instant $t = t_0$.

SOLUTION. In these variables the coordinate x of any fluid particle at any instant is regarded as a function of t and its coordinate a at the initial instant: $x = x(a, t)$. The condition of conservation of mass during the motion of a fluid element (the equation of continuity) is accordingly written $\rho\,dx = \rho_0\,da$, or

$$\rho\left(\frac{\partial x}{\partial a}\right)_t = \rho_0,$$

where $\rho_0(a)$ is a given initial density distribution. The velocity of a fluid particle is, by definition, $v = (\partial x/\partial t)_a$, and the derivative $(\partial v/\partial t)_a$ gives the rate of change of the velocity of the particle during its motion. Euler's equation becomes

$$\left(\frac{\partial v}{\partial t}\right)_a = -\frac{1}{\rho_0}\left(\frac{\partial p}{\partial a}\right)_t,$$

and the adiabatic equation is

$$(\partial s/\partial t)_a = 0.$$

§3. Hydrostatics

For a fluid at rest in a uniform gravitational field, Euler's equation (2.4) takes the form

$$\mathbf{grad}\,p = \rho\mathbf{g}. \tag{3.1}$$

This equation describes the mechanical equilibrium of the fluid. (If there is no external force, the equation of equilibrium is simply $\mathbf{grad}\,p = 0$, i.e. $p =$ constant; the pressure is the same at every point in the fluid.)

Equation (3.1) can be integrated immediately if the density of the fluid may be supposed constant throughout its volume, i.e. if there is no significant compression of the fluid under the action of the external force. Taking the z-axis vertically upward, we have

$$\partial p/\partial x = \partial p/\partial y = 0, \qquad \partial p/\partial z = -\rho g.$$

Hence

$$p = -\rho g z + \text{constant}.$$

If the fluid at rest has a free surface at height h, to which an external pressure p_0, the same at every point, is applied, this surface must be the horizontal plane $z = h$. From the condition $p = p_0$ for $z = h$, we find that the constant is $p_0 + \rho g h$, so that

$$p = p_0 + \rho g(h - z). \tag{3.2}$$

† Although such variables are usually called Lagrangian, the equations of motion in these coordinates were first obtained by Euler, at the same time as equations (2.3).

For large masses of liquid, and for a gas, the density ρ cannot in general be supposed constant; this applies especially to gases (for example, the atmosphere). Let us suppose that the fluid is not only in mechanical equilibrium but also in thermal equilibrium. Then the temperature is the same at every point, and equation (3.1) may be integrated as follows. We use the familiar thermodynamic relation

$$\mathrm{d}\Phi = -s\mathrm{d}T + V\mathrm{d}p,$$

where Φ is the thermodynamic potential (Gibbs free energy) per unit mass. For constant temperature

$$\mathrm{d}\Phi = V\mathrm{d}p = \mathrm{d}p/\rho.$$

Hence we see that the expression $(\mathbf{grad}\, p)/\rho$ can be written in this case as $\mathbf{grad}\, \Phi$, so that the equation of equilibrium (3.1) takes the form

$$\mathbf{grad}\, \Phi = \mathbf{g}.$$

For a constant vector $\mathbf{g}$ directed along the negative z-axis we have

$$\mathbf{g} \equiv -\mathbf{grad}\,(gz).$$

Thus

$$\mathbf{grad}\,(\Phi + gz) = 0,$$

whence we find that throughout the fluid

$$\Phi + gz = \text{constant}; \tag{3.3}$$

gz is the potential energy of unit mass of fluid in the gravitational field. The condition (3.3) is known from statistical physics to be the condition for thermodynamic equilibrium of a system in an external field.

We may mention here another simple consequence of equation (3.1). If a fluid (such as the atmosphere) is in mechanical equilibrium in a gravitational field, the pressure in it can be a function only of the altitude z (since, if the pressure were different at different points with the same altitude, motion would result). It then follows from (3.1) that the density

$$\rho = -\frac{1}{g}\frac{\mathrm{d}p}{\mathrm{d}z} \tag{3.4}$$

is also a function of z only. The pressure and density together determine the temperature, which is therefore again a function of z only. Thus, in mechanical equilibrium in a gravitational field, the pressure, density and temperature distributions depend only on the altitude. If, for example, the temperature is different at different points with the same altitude, then mechanical equilibrium is impossible.

Finally, let us derive the equation of equilibrium for a very large mass of fluid, whose separate parts are held together by gravitational attraction—a star. Let ϕ be the Newtonian gravitational potential of the field due to the fluid. It satisfies the differential equation

$$\triangle\phi = 4\pi G\rho, \tag{3.5}$$

where G is the Newtonian constant of gravitation. The gravitational acceleration is $-\mathbf{grad}\,\phi$, and the force on a mass ρ is $-\rho\,\mathbf{grad}\,\phi$. The condition of equilibrium is therefore

$$\mathbf{grad}\, p = -\rho\, \mathbf{grad}\, \phi.$$

Dividing both sides by ρ, taking the divergence of both sides, and using equation (3.5), we obtain

$$\operatorname{div}\left(\frac{1}{\rho}\mathbf{grad}\,p\right) = -4\pi G\rho. \tag{3.6}$$

It must be emphasized that the present discussion concerns only mechanical equilibrium; equation (3.6) does not presuppose the existence of complete thermal equilibrium.

If the body is not rotating, it will be spherical when in equilibrium, and the density and pressure distributions will be spherically symmetrical. Equation (3.6) in spherical polar coordinates then takes the form

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r}\right) = -4\pi G\rho. \tag{3.7}$$

§4. The condition that convection be absent

A fluid can be in mechanical equilibrium (i.e. exhibit no macroscopic motion) without being in thermal equilibrium. Equation (3.1), the condition for mechanical equilibrium, can be satisfied even if the temperature is not constant throughout the fluid. However, the question then arises of the stability of such an equilibrium. It is found that the equilibrium is stable only when a certain condition is fulfilled. Otherwise, the equilibrium is unstable, and this leads to the appearance in the fluid of currents which tend to mix the fluid in such a way as to equalize the temperature. This motion is called *convection*. Thus the condition for a mechanical equilibrium to be stable is the condition that convection be absent. It can be derived as follows.

Let us consider a fluid element at height z, having a specific volume $V(p, s)$, where p and s are the equilibrium pressure and entropy at height z. Suppose that this fluid element undergoes an adiabatic upward displacement through a small interval ξ; its specific volume then becomes $V(p', s)$, where p' is the pressure at height $z+\xi$. For the equilibrium to be stable, it is necessary (though not in general sufficient) that the resulting force on the element should tend to return it to its original position. This means that the element must be heavier than the fluid which it "displaces" in its new position. The specific volume of the latter is $V(p', s')$, where s' is the equilibrium entropy at height $z+\xi$. Thus we have the stability condition

$$V(p', s') - V(p', s) > 0.$$

Expanding this difference in powers of $s'-s = \xi\,\mathrm{d}s/\mathrm{d}z$, we obtain

$$\left(\frac{\partial V}{\partial s}\right)_p \frac{\mathrm{d}s}{\mathrm{d}z} > 0. \tag{4.1}$$

The formulae of thermodynamics give

$$\left(\frac{\partial V}{\partial s}\right)_p = \frac{T}{c_p}\left(\frac{\partial V}{\partial T}\right)_p,$$

where c_p is the specific heat at constant pressure. Both c_p and T are positive, so that we can write (4.1) as

$$\left(\frac{\partial V}{\partial T}\right)_p \frac{\mathrm{d}s}{\mathrm{d}z} > 0. \tag{4.2}$$

The majority of substances expand on heating, i.e. $(\partial V/\partial T)_p > 0$. The condition that convection be absent then becomes

$$\mathrm{d}s/\mathrm{d}z > 0, \tag{4.3}$$

i.e. the entropy must increase with height.

From this we easily find the condition that must be satisfied by the temperature gradient $\mathrm{d}T/\mathrm{d}z$. Expanding the derivative $\mathrm{d}s/\mathrm{d}z$, we have

$$\frac{\mathrm{d}s}{\mathrm{d}z} = \left(\frac{\partial s}{\partial T}\right)_p \frac{\mathrm{d}T}{\mathrm{d}z} + \left(\frac{\partial s}{\partial p}\right)_T \frac{\mathrm{d}p}{\mathrm{d}z} = \frac{c_p}{T}\frac{\mathrm{d}T}{\mathrm{d}z} - \left(\frac{\partial V}{\partial T}\right)_p \frac{\mathrm{d}p}{\mathrm{d}z} > 0.$$

Finally, substituting from (3.4) $\mathrm{d}p/\mathrm{d}z = -g/V$, we obtain

$$-\mathrm{d}T/\mathrm{d}z < g\beta T/c_p, \tag{4.4}$$

where $\beta = (1/V)(\partial V/\partial T)_p$ is the thermal expansion coefficient. For a column of gas in equilibrium which can be taken as a thermodynamically perfect gas, $\beta T = 1$ and (4.4) becomes

$$-\mathrm{d}T/\mathrm{d}z < g/c_p. \tag{4.5}$$

Convection occurs if these conditions are not satisfied, i.e. if the temperature decreases upwards with a gradient whose magnitude exceeds the value given by (4.4) and (4.5).†

§5. Bernoulli's equation

The equations of fluid dynamics are much simplified in the case of steady flow. By *steady flow* we mean one in which the velocity is constant in time at any point occupied by fluid. In other words, **v** is a function of the coordinates only, so that $\partial \mathbf{v}/\partial t = 0$. Equation (2.10) then reduces to

$$\tfrac{1}{2}\,\mathbf{grad}\, v^2 - \mathbf{v}\times\mathbf{curl}\,\mathbf{v} = -\mathbf{grad}\, w. \tag{5.1}$$

We now introduce the concept of *streamlines*. These are lines such that the tangent to a streamline at any point gives the direction of the velocity at that point; they are determined by the following system of differential equations:

$$\frac{\mathrm{d}x}{v_x} = \frac{\mathrm{d}y}{v_y} = \frac{\mathrm{d}z}{v_z}. \tag{5.2}$$

In steady flow the streamlines do not vary with time, and coincide with the paths of the fluid particles. In non-steady flow this coincidence no longer occurs: the tangents to the streamlines give the directions of the velocities of fluid particles at various points in space at a given instant, whereas the tangents to the paths give the directions of the velocities of given fluid particles at various times.

We form the scalar product of equation (5.1) with the unit vector tangent to the streamline at each point; this unit vector is denoted by **l**. The projection of the gradient on any direction is, as we know, the derivative in that direction. Hence the projection of **grad** w is $\partial w/\partial l$. The vector $\mathbf{v}\times\mathbf{curl}\,\mathbf{v}$ is perpendicular to **v**, and its projection on the direction of **l** is therefore zero.

---

† For water at 20°C, the right-hand side of (4.4) is about one degree per 6.7 km; for air, the right-hand side of (4.5) is about one degree per 100 m.

Thus we obtain from equation (5.1)

$$\frac{\partial}{\partial l}(\tfrac{1}{2}v^2 + w) = 0.$$

It follows from this that $\frac{1}{2}v^2 + w$ is constant along a streamline:

$$\tfrac{1}{2}v^2 + w = \text{constant}. \tag{5.3}$$

In general the constant takes different values for different streamlines. Equation (5.3) is called *Bernoulli's equation.*†

If the flow takes place in a gravitational field, the acceleration **g** due to gravity must be added to the right-hand side of equation (5.1). Let us take the direction of gravity as the z-axis, with z increasing upwards. Then the cosine of the angle between the directions of **g** and **l** is equal to the derivative $-\mathrm{d}z/\mathrm{d}l$, so that the projection of **g** on **l** is

$$-g\,\mathrm{d}z/\mathrm{d}l.$$

Accordingly, we now have

$$\frac{\partial}{\partial l}(\tfrac{1}{2}v^2 + w + gz) = 0.$$

Thus Bernoulli's equation states that along a streamline

$$\tfrac{1}{2}v^2 + w + gz = \text{constant}. \tag{5.4}$$

§6. The energy flux

Let us choose some volume element fixed in space, and find how the energy of the fluid contained in this volume element varies with time. The energy of unit volume of fluid is

$$\tfrac{1}{2}\rho v^2 + \rho\varepsilon,$$

where the first term is the kinetic energy and the second the internal energy, ε being the internal energy per unit mass. The change in this energy is given by the partial derivative

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2 + \rho\varepsilon).$$

To calculate this quantity, we write

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2) = \tfrac{1}{2}v^2\frac{\partial\rho}{\partial t} + \rho\mathbf{v}\cdot\frac{\partial\mathbf{v}}{\partial t},$$

or, using the equation of continuity (1.2) and the equation of motion (2.3),

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2) = -\tfrac{1}{2}v^2\,\mathrm{div}(\rho\mathbf{v}) - \mathbf{v}\cdot\mathbf{grad}\,p - \rho\mathbf{v}\cdot(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}.$$

In the last term we replace $\mathbf{v}\cdot(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ by $\frac{1}{2}\mathbf{v}\cdot\mathbf{grad}\,v^2$, and $\mathbf{grad}\,p$ by $\rho\,\mathbf{grad}\,w - \rho T\,\mathbf{grad}\,s$ (using the thermodynamic relation $\mathrm{d}w = T\mathrm{d}s + (1/\rho)\mathrm{d}p$), obtaining

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2) = -\tfrac{1}{2}v^2\,\mathrm{div}\,(\rho\mathbf{v}) - \rho\mathbf{v}\cdot\mathbf{grad}\,(\tfrac{1}{2}v^2 + w) + \rho T\mathbf{v}\cdot\mathbf{grad}\,s.$$

† It was derived for an incompressible fluid (§10) by D. Bernoulli in 1738.

In order to transform the derivative $\partial(\rho\varepsilon)/\partial t$, we use the thermodynamic relation

$$\mathrm{d}\varepsilon = T\mathrm{d}s - p\mathrm{d}V = T\mathrm{d}s + (p/\rho^2)\mathrm{d}\rho.$$

Since $\varepsilon + p/\rho = \varepsilon + pV$ is simply the heat function w per unit mass, we find

$$\mathrm{d}(\rho\varepsilon) = \varepsilon\mathrm{d}\rho + \rho\mathrm{d}\varepsilon = w\mathrm{d}\rho + \rho T\mathrm{d}s,$$

and so

$$\frac{\partial(\rho\varepsilon)}{\partial t} = w\frac{\partial\rho}{\partial t} + \rho T\frac{\partial s}{\partial t} = -w\,\mathrm{div}\,(\rho\mathbf{v}) - \rho T\mathbf{v}\cdot\mathbf{grad}\,s.$$

Here we have also used the general adiabatic equation (2.6).

Combining the above results, we find the change in the energy to be

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2 + \rho\varepsilon) = -(\tfrac{1}{2}v^2 + w)\,\mathrm{div}\,(\rho\mathbf{v}) - \rho\mathbf{v}\cdot\mathbf{grad}(\tfrac{1}{2}v^2 + w),$$

or, finally,

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2 + \rho\varepsilon) = -\mathrm{div}\,[\rho\mathbf{v}(\tfrac{1}{2}v^2 + w)]. \tag{6.1}$$

In order to see the meaning of this equation, let us integrate it over some volume:

$$\frac{\partial}{\partial t}\int(\tfrac{1}{2}\rho v^2 + \rho\varepsilon)\mathrm{d}V = -\int\mathrm{div}\,[\rho\mathbf{v}(\tfrac{1}{2}v^2 + w)]\,\mathrm{d}V,$$

or, converting the volume integral on the right into a surface integral,

$$\frac{\partial}{\partial t}\int(\tfrac{1}{2}\rho v^2 + \rho\varepsilon)\mathrm{d}V = -\oint\rho\mathbf{v}(\tfrac{1}{2}v^2 + w)\cdot\mathrm{d}\mathbf{f}. \tag{6.2}$$

The left-hand side is the rate of change of the energy of the fluid in some given volume. The right-hand side is therefore the amount of energy flowing out of this volume in unit time. Hence we see that the expression

$$\rho\mathbf{v}(\tfrac{1}{2}v^2 + w) \tag{6.3}$$

may be called the *energy flux density* vector. Its magnitude is the amount of energy passing in unit time through unit area perpendicular to the direction of the velocity.

The expression (6.3) shows that any unit mass of fluid carries with it during its motion an amount of energy $w + \frac{1}{2}v^2$. The fact that the heat function w appears here, and not the internal energy ε, has a simple physical significance. Putting $w = \varepsilon + p/\rho$, we can write the flux of energy through a closed surface in the form

$$-\oint\rho\mathbf{v}(\tfrac{1}{2}v^2 + \varepsilon)\cdot\mathrm{d}\mathbf{f} - \oint p\mathbf{v}\cdot\mathrm{d}\mathbf{f}.$$

The first term is the energy (kinetic and internal) transported through the surface in unit time by the mass of fluid. The second term is the work done by pressure forces on the fluid within the surface.

§7. The momentum flux

We shall now give a similar series of arguments for the momentum of the fluid. The momentum of unit volume is $\rho\mathbf{v}$. Let us determine its rate of change, $\partial(\rho\mathbf{v})/\partial t$. We shall use tensor notation. We have

$$\frac{\partial}{\partial t}(\rho v_i) = \rho\frac{\partial v_i}{\partial t} + \frac{\partial \rho}{\partial t}v_i.$$

Using the equation of continuity (1.2) in the form

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho v_k)}{\partial x_k},$$

and Euler's equation (2.3) in the form

$$\frac{\partial v_i}{\partial t} = -v_k\frac{\partial v_i}{\partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_i},$$

we obtain

$$\begin{aligned}\frac{\partial}{\partial t}(\rho v_i) &= -\rho v_k\frac{\partial v_i}{\partial x_k} - \frac{\partial p}{\partial x_i} - v_i\frac{\partial(\rho v_k)}{\partial x_k}\\ &= -\frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_k}(\rho v_i v_k).\end{aligned}$$

We write the first term on the right in the form

$$\frac{\partial p}{\partial x_i} = \delta_{ik}\frac{\partial p}{\partial x_k},$$

and finally obtain

$$\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial \Pi_{ik}}{\partial x_k}, \tag{7.1}$$

where the tensor Π_{ik} is defined as

$$\Pi_{ik} = p\delta_{ik} + \rho v_i v_k. \tag{7.2}$$

This tensor is clearly symmetrical.

To see the meaning of the tensor Π_{ik}, we integrate equation (7.1) over some volume:

$$\frac{\partial}{\partial t}\int \rho v_i \mathrm{d}V = -\int \frac{\partial \Pi_{ik}}{\partial x_k}\mathrm{d}V.$$

The integral on the right is transformed into a surface integral by Green's formula:†

$$\frac{\partial}{\partial t}\int \rho v_i \mathrm{d}V = -\oint \Pi_{ik}\mathrm{d}f_k. \tag{7.3}$$

The left-hand side is the rate of change of the ith component of the momentum contained in the volume considered. The surface integral on the right is therefore the

† The rule for transforming an integral over a closed surface into one over the volume bounded by that surface can be formulated as follows: the surface element $\mathrm{d}f_i$ must be replaced by the operator $\mathrm{d}V\cdot\partial/\partial x_i$, which is to be applied to the whole of the integrand.

amount of momentum flowing out through the bounding surface in unit time. Consequently, $\Pi_{ik}\mathrm{d}f_k$ is the ith component of the momentum flowing through the surface element $\mathrm{d}\mathbf{f}$. If we write $\mathrm{d}f_k$ in the form $n_k\,\mathrm{d}f$, where $\mathrm{d}f$ is the area of the surface element, and $\mathbf{n}$ is a unit vector along the outward normal, we find that $\Pi_{ik}n_k$ is the flux of the ith component of momentum through unit surface area. We may notice that, according to (7.2), $\Pi_{ik}n_k = pn_i + \rho v_i v_k n_k$. This expression can be written in vector form

$$p\mathbf{n} + \rho\mathbf{v}(\mathbf{v}\cdot\mathbf{n}). \tag{7.4}$$

Thus Π_{ik} is the ith component of the amount of momentum flowing in unit time through unit area perpendicular to the x_k-axis. The tensor Π_{ik} is called the *momentum flux density tensor*. The energy flux is determined by a vector, energy being a scalar; the momentum flux, however, is determined by a tensor of rank two, the momentum itself being a vector.

The vector (7.4) gives the momentum flux in the direction of $\mathbf{n}$, i.e. through a surface perpendicular to $\mathbf{n}$. In particular, taking the unit vector $\mathbf{n}$ to be directed parallel to the fluid velocity, we find that only the longitudinal component of momentum is transported in this direction, and its flux density is $p + \rho v^2$. In a direction perpendicular to the velocity, only the transverse component (relative to $\mathbf{v}$) of momentum is transported, its flux density being just p.

§8. The conservation of circulation

The integral

$$\Gamma = \oint \mathbf{v}\cdot\mathrm{d}\mathbf{l},$$

taken along some closed contour, is called the *velocity circulation* round that contour.

Let us consider a closed contour drawn in the fluid at some instant. We suppose it to be a "fluid contour", i.e. composed of the fluid particles that lie on it. In the course of time these particles move about, and the contour moves with them. Let us investigate what happens to the velocity circulation. In other words, let us calculate the time derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint \mathbf{v}\cdot\mathrm{d}\mathbf{l}.$$

We have written here the total derivative with respect to time, since we are seeking the change in the circulation round a "fluid contour" as it moves about, and not round a contour fixed in space.

To avoid confusion, we shall temporarily denote differentiation with respect to the coordinates by the symbol δ, retaining the symbol d for differentiation with respect to time. Next, we notice that an element $\mathrm{d}\mathbf{l}$ of the length of the contour can be written as the difference $\delta\mathbf{r}$ between the position vectors $\mathbf{r}$ of the points at the ends of the element. Thus we write the velocity circulation as $\oint \mathbf{v}\cdot\delta\mathbf{r}$. In differentiating this integral with respect to time, it must be borne in mind that not only the velocity but also the contour itself (i.e. its shape) changes. Hence, on taking the time differentiation under the integral sign, we must differentiate not only $\mathbf{v}$ but also $\delta\mathbf{r}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint \mathbf{v}\cdot\delta\mathbf{r} = \oint \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\cdot\delta\mathbf{r} + \oint \mathbf{v}\cdot\frac{\mathrm{d}\delta\mathbf{r}}{\mathrm{d}t}.$$

Since the velocity **v** is just the time derivative of the position vector **r**, we have

$$\mathbf{v}\cdot\frac{\mathrm{d}\delta\mathbf{r}}{\mathrm{d}t} = \mathbf{v}\cdot\delta\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{v}\cdot\delta\mathbf{v} = \delta(\tfrac{1}{2}v^2).$$

The integral of a total differential along a closed contour, however, is zero. The second integral therefore vanishes, leaving

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint\mathbf{v}\cdot\delta\mathbf{r} = \oint\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\cdot\delta\mathbf{r}.$$

It now remains to substitute for the acceleration $\mathrm{d}\mathbf{v}/\mathrm{d}t$ its expression from (2.9):

$$\mathrm{d}\mathbf{v}/\mathrm{d}t = -\,\mathbf{grad}\,w.$$

Using Stokes' formula, we then have

$$\oint\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\cdot\delta\mathbf{r} = \oint\mathbf{curl}\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right)\cdot\delta\mathbf{f} = 0,$$

since $\mathbf{curl}\,\mathbf{grad}\,w \equiv 0$. Thus, going back to our previous notation, we find†

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint\mathbf{v}\cdot\mathrm{d}\mathbf{l} = 0,$$

or

$$\oint\mathbf{v}\cdot\mathrm{d}\mathbf{l} = \text{constant}. \tag{8.1}$$

We have therefore reached the conclusion that, in an ideal fluid, the velocity circulation round a closed "fluid" contour is constant in time (*Kelvin's theorem* (1869) or the *law of conservation of circulation*).

It should be emphasized that this result has been obtained by using Euler's equation in the form (2.9), and therefore involves the assumption that the flow is isentropic. The theorem does not hold for flows which are not isentropic.‡

By applying Kelvin's theorem to an infinitesimal closed contour δC and transforming the integral according to Stokes' theorem, we get

$$\oint\mathbf{v}\cdot\mathrm{d}\mathbf{l} = \int\mathbf{curl}\,\mathbf{v}\cdot\mathrm{d}\mathbf{f} \cong \delta\mathbf{f}\cdot\mathbf{curl}\,\mathbf{v} = \text{constant}, \tag{8.2}$$

where $\mathrm{d}\mathbf{f}$ is a fluid surface element spanning the contour δC. The vector **curl v** is often called the *vorticity* of the fluid flow at a given point. The constancy of the product (8.2) can be intuitively interpreted as meaning that the vorticity moves with the fluid.

PROBLEM

Show that, in flow which is not isentropic, any moving particle carries with it a constant value of the product $(1/\rho)\,\mathbf{grad}\,s\cdot\mathbf{curl}\,\mathbf{v}$ (H. Ertel 1942).

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† This result remains valid in a uniform gravitational field, since in that case $\mathbf{curl}\,\mathbf{g} \equiv 0$.

‡ Mathematically, it is necessary that there should be a one-to-one relation between p and ρ (which for isentropic flow is $s(p,\rho) = $ constant); then $-(1/\rho)\,\mathbf{grad}\,p$ can be written as the gradient of some function, a result which is needed in deriving Kelvin's theorem.

SOLUTION. When the flow is not isentropic, the right-hand side of Euler's equation (2.3) cannot be replaced by $-\mathbf{grad}\, w$, and (2.11) becomes

$$\partial\boldsymbol{\omega}/\partial t = \mathbf{curl}\,(\mathbf{v}\times\boldsymbol{\omega}) + (1/\rho^2)\,\mathbf{grad}\,\rho\times\mathbf{grad}\,p,$$

where for brevity $\boldsymbol{\omega} = \mathbf{curl}\,\mathbf{v}$. We multiply scalarly by $\mathbf{grad}\, s$; since $s = s(p, \rho)$, $\mathbf{grad}\, s$ is a linear function of $\mathbf{grad}\, p$ and $\mathbf{grad}\,\rho$, and $\mathbf{grad}\, s\cdot(\mathbf{grad}\,\rho\times\mathbf{grad}\, p) = 0$. The expression on the right-hand side can then be transformed as follows:

$$\begin{aligned}\mathbf{grad}\, s\cdot\partial\boldsymbol{\omega}/\partial t &= \mathbf{grad}\, s\cdot\mathbf{curl}\,(\mathbf{v}\times\boldsymbol{\omega})\\ &= -\mathrm{div}\,[\mathbf{grad}\, s\times(\mathbf{v}\times\boldsymbol{\omega})]\\ &= -\mathrm{div}\,[\mathbf{v}(\boldsymbol{\omega}\cdot\mathbf{grad}\, s)] + \mathrm{div}\,[\boldsymbol{\omega}(\mathbf{v}\cdot\mathbf{grad}\, s)]\\ &= -(\boldsymbol{\omega}\cdot\mathbf{grad}\, s)\,\mathrm{div}\,\mathbf{v} - \mathbf{v}\cdot\mathbf{grad}\,(\boldsymbol{\omega}\cdot\mathbf{grad}\, s) + \boldsymbol{\omega}\cdot\mathbf{grad}\,(\mathbf{v}\cdot\mathbf{grad}\, s).\end{aligned}$$

From (2.6), $\mathbf{v}\cdot\mathbf{grad}\, s = -\partial s/\partial t$, and therefore

$$\frac{\partial}{\partial t}(\boldsymbol{\omega}\cdot\mathbf{grad}\, s) + \mathbf{v}\cdot\mathbf{grad}\,(\boldsymbol{\omega}\cdot\mathbf{grad}\, s) + (\boldsymbol{\omega}\cdot\mathbf{grad}\, s)\,\mathrm{div}\,\mathbf{v} = 0.$$

The first two terms can be combined as $\mathrm{d}(\boldsymbol{\omega}\cdot\mathbf{grad}\, s)/\mathrm{d}t$, where $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \mathbf{v}\cdot\mathbf{grad}$; in the last term, we put from (1.3) $\rho\,\mathrm{div}\,\mathbf{v} = -\mathrm{d}\rho/\mathrm{d}t$. The result is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{\omega}\cdot\mathbf{grad}\, s}{\rho}\right) = 0,$$

which gives the required conservation law.

§9. Potential flow

From the law of conservation of circulation we can derive an important result. Let us at first suppose that the flow is steady, and consider a streamline of which we know that **curl v** is zero at some point. We draw an arbitrary infinitely small closed contour to encircle the streamline at that point. In the course of time, this contour moves with the fluid, but always encircles the same streamline. Since the product (8.2) must remain constant, it follows that **curl v** must be zero at every point on the streamline.

Thus we reach the conclusion that, if at any point on a streamline the vorticity is zero, the same is true at all other points on that streamline. If the flow is not steady, the same result holds, except that instead of a streamline we must consider the path described in the course of time by some particular fluid particle;† we recall that in non-steady flow these paths do not in general coincide with the streamlines.

At first sight it might seem possible to base on this result the following argument. Let us consider steady flow past some body. Let the incident flow be uniform at infinity; its velocity **v** is a constant, so that $\mathbf{curl\, v} \equiv 0$ on all streamlines. Hence we conclude that **curl v** is zero along the whole of every streamline, i.e. in all space.

A flow for which $\mathbf{curl\, v} = 0$ in all space is called a *potential flow* or *irrotational flow*, as opposed to *rotational flow*, in which the curl of the velocity is not everywhere zero. Thus we should conclude that steady flow past any body, with a uniform incident flow at infinity, must be potential flow.

Similarly, from the law of conservation of circulation, we might argue as follows. Let us suppose that at some instant we have potential flow throughout the volume of the fluid. Then the velocity circulation round any closed contour in the fluid is zero.‡ By Kelvin's

† To avoid misunderstanding, we may mention here that this result has no meaning in turbulent flow. We may also remark that a non-zero vorticity may occur on a streamline after the passage of a shock wave. We shall see that this is because the flow is no longer isentropic (§114).

‡ Here we suppose for simplicity that the fluid occupies a simply-connected region of space. The same final result would be obtained for a multiply-connected region, but restrictions on the choice of contours would have to be made in the derivation.

theorem, we could then conclude that this will hold at any future instant, i.e. we should find that, if there is potential flow at some instant, then there is potential flow at all subsequent instants (in particular, any flow for which the fluid is initially at rest must be a potential flow). This is in accordance with the fact that, if **curl v** = 0, equation (2.11) is satisfied identically.

In fact, however, all these conclusions are of only very limited validity. The reason is that the proof given above that **curl v** = 0 all along a streamline is, strictly speaking, invalid for a line which lies in the surface of a solid body past which the flow takes place, since the presence of this surface makes it impossible to draw a closed contour in the fluid encircling such a streamline. The equations of motion of an ideal fluid therefore admit solutions for which *separation* occurs at the surface of the body: the streamlines, having followed the surface for some distance, become separated from it at some point and continue into the fluid. The resulting flow pattern is characterized by the presence of a "surface of tangential discontinuity" proceeding from the body; on this surface the fluid velocity, which is everywhere tangential to the surface, has a discontinuity. In other words, at this surface one layer of fluid "slides" on another. Figure 1 shows a surface of discontinuity which separates moving fluid from a region of stationary fluid behind the body. From a mathematical point of view, the discontinuity in the tangential velocity component corresponds to a surface on which the curl of the velocity is non-zero.

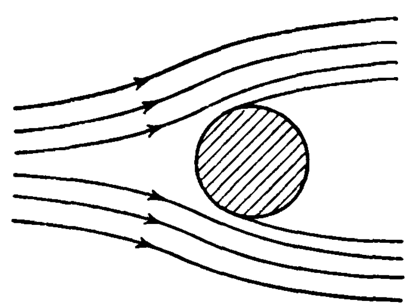

FIG. 1

When such discontinuous flows are included, the solution of the equations of motion for an ideal fluid is not unique: besides continuous flow, they admit also an infinite number of solutions possessing surfaces of tangential discontinuity starting from any prescribed line on the surface of the body past which the flow takes place. It should be emphasized, however, that none of these discontinuous solutions is physically significant, since tangential discontinuities are absolutely unstable, and therefore the flow would in fact become turbulent (see Chapter III).

The actual physical problem of flow past a given body has, of course, a unique solution. The reason is that ideal fluids do not really exist; any actual fluid has a certain viscosity, however small. This viscosity may have practically no effect on the motion of most of the fluid, but, no matter how small it is, it will be important in a thin layer of fluid adjoining the body. The properties of the flow in this *boundary layer* decide the choice of one out of the infinity of solutions of the equations of motion for an ideal fluid. It is found that, in the general case of flow past bodies of arbitrary form, solutions with separation must be taken, which in turn will result in turbulence.

In spite of what we have said above, the study of the solutions of the equations of motion for continuous steady potential flow past bodies is in some cases meaningful. Although, in the general case of flow past bodies of arbitrary form, the actual flow pattern bears almost no relation to the pattern of potential flow, for bodies of certain special ("streamlined"—§46) shapes the flow may differ very little from potential flow; more precisely, it will be potential flow except in a thin layer of fluid at the surface of the body and in a relatively narrow "wake" behind the body.

Another important case of potential flow occurs for small oscillations of a body immersed in fluid. It is easy to show that, if the amplitude a of the oscillations is small compared with the linear dimension l of the body ($a \ll l$), the flow past the body will be potential flow. To show this, we estimate the order of magnitude of the various terms in Euler's equation

$$\partial \mathbf{v}/\partial t + (\mathbf{v}\cdot\mathbf{grad})\mathbf{v} = -\mathbf{grad}\, w.$$

The velocity $\mathbf{v}$ changes markedly (by an amount of the same order as the velocity $\mathbf{u}$ of the oscillating body) over a distance of the order of the dimension l of the body. Hence the derivatives of $\mathbf{v}$ with respect to the coordinates are of the order of u/l. The order of magnitude of $\mathbf{v}$ itself (at fairly small distances from the body) is determined by the magnitude of $\mathbf{u}$. Thus we have $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v} \sim u^2/l$. The derivative $\partial \mathbf{v}/\partial t$ is of the order of ωu, where ω is the frequency of the oscillations. Since $\omega \sim u/a$, we have $\partial \mathbf{v}/\partial t = u^2/a$. It now follows from the inequality $a \ll l$ that the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ is small compared with $\partial \mathbf{v}/\partial t$ and can be neglected, so that the equation of motion of the fluid becomes $\partial \mathbf{v}/\partial t = -\mathbf{grad}\, w$. Taking the curl of both sides, we obtain $\partial(\mathbf{curl}\,\mathbf{v})/\partial t = 0$, whence $\mathbf{curl}\,\mathbf{v} = \text{constant}$. In oscillatory motion, however, the time average of the velocity is zero, and therefore $\mathbf{curl}\,\mathbf{v} = \text{constant}$ implies that $\mathbf{curl}\,\mathbf{v} = 0$. Thus the motion of a fluid executing small oscillations is potential flow to a first approximation.

We shall now obtain some general properties of potential flow. We first recall that the derivation of the law of conservation of circulation, and therefore all its consequences, were based on the assumption that the flow is isentropic. If the flow is not isentropic, the law does not hold, and therefore, even if we have potential flow at some instant, the vorticity will in general be non-zero at subsequent instants. Thus only isentropic flow can in fact be potential flow.

In potential flow, the velocity circulation along any closed contour is zero:

$$\oint \mathbf{v}\cdot d\mathbf{l} = \int \mathbf{curl}\,\mathbf{v}\cdot d\mathbf{f} = 0. \tag{9.1}$$

It follows from this that, in particular, closed streamlines cannot exist in potential flow.† For, since the direction of a streamline is at every point the direction of the velocity, the circulation along such a line can never be zero.

In rotational flow the velocity circulation is not in general zero. In this case there may be closed streamlines, but it must be emphasized that the presence of closed streamlines is not a necessary property of rotational flow.

Like any vector field having zero curl, the velocity in potential flow can be expressed as the gradient of some scalar. This scalar is called the *velocity potential*; we shall denote it by ϕ:

$$\mathbf{v} = \mathbf{grad}\,\phi. \tag{9.2}$$

† This result, like (9.1), may not be valid for motion in a multiply-connected region of space. In potential flow in such a region, the velocity circulation may be non-zero if the closed contour round which it is taken cannot be contracted to a point without crossing the boundaries of the region.

Writing Euler's equation in the form (2.10)

$$\partial \mathbf{v}/\partial t + \tfrac{1}{2}\,\mathbf{grad}\, v^2 - \mathbf{v}\times\mathbf{curl}\,\mathbf{v} = -\mathbf{grad}\, w$$

and substituting $\mathbf{v} = \mathbf{grad}\,\phi$, we have

$$\mathbf{grad}\left(\frac{\partial\phi}{\partial t} + \tfrac{1}{2}v^2 + w\right) = 0,$$

whence

$$\partial\phi/\partial t + \tfrac{1}{2}v^2 + w = f(t), \tag{9.3}$$

where $f(t)$ is an arbitrary function of time. This equation is a first integral of the equations of potential flow. The function $f(t)$ in equation (9.3) can be put equal to zero without loss of generality, because the potential is not uniquely defined: since the velocity is the space derivative of ϕ, we can add to ϕ any function of the time.

For steady flow we have (taking the potential ϕ to be independent of time) $\partial\phi/\partial t = 0$, $f(t) =$ constant, and (9.3) becomes Bernoulli's equation:

$$\tfrac{1}{2}v^2 + w = \text{constant}. \tag{9.4}$$

It must be emphasized here that there is an important difference between the Bernoulli's equation for potential flow and that for other flows. In the general case, the "constant" on the right-hand side is a constant along any given streamline, but is different for different streamlines. In potential flow, however, it is constant throughout the fluid. This enhances the importance of Bernoulli's equation in the study of potential flow.

§10. Incompressible fluids

In a great many cases of the flow of liquids (and also of gases), their density may be supposed invariable, i.e. constant throughout the volume of the fluid and throughout its motion. In other words, there is no noticeable compression or expansion of the fluid in such cases. We then speak of *incompressible flow.*

The general equations of fluid dynamics are much simplified for an incompressible fluid. Euler's equation, it is true, is unchanged if we put $\rho =$ constant, except that ρ can be taken under the gradient operator in equation (2.4):

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\mathbf{grad})\mathbf{v} = -\mathbf{grad}\left(\frac{p}{\rho}\right) + \mathbf{g}. \tag{10.1}$$

The equation of continuity, on the other hand, takes for constant ρ the simple form

$$\operatorname{div}\mathbf{v} = 0. \tag{10.2}$$

Since the density is no longer an unknown function as it was in the general case, the fundamental system of equations in fluid dynamics for an incompressible fluid can be taken to be equations involving the velocity only. These may be the equation of continuity (10.2) and equation (2.11):

$$\frac{\partial}{\partial t}(\mathbf{curl}\,\mathbf{v}) = \mathbf{curl}(\mathbf{v}\times\mathbf{curl}\,\mathbf{v}). \tag{10.3}$$

Bernoulli's equation too can be written in a simpler form for an incompressible fluid. Equation (10.1) differs from the general Euler's equation (2.9) in that it has $\mathbf{grad}\,(p/\rho)$ in place of **grad** w. Hence we can write down Bernoulli's equation immediately by simply replacing the heat function in (5.4) by p/ρ:

$$\tfrac{1}{2}v^2 + p/\rho + gz = \text{constant}. \tag{10.4}$$

For an incompressible fluid, we can also write p/ρ in place of w in the expression (6.3) for the energy flux, which then becomes

$$\rho \mathbf{v}\left(\tfrac{1}{2}v^2 + \frac{p}{\rho}\right). \tag{10.5}$$

For we have, from a well-known thermodynamic relation, the expression $d\varepsilon = T ds - p dV$ for the change in internal energy; for $s = \text{constant}$ and $V = 1/\rho = \text{constant}$, $d\varepsilon = 0$, i.e. $\varepsilon = \text{constant}$. Since constant terms in the energy do not matter, we can omit ε in $w = \varepsilon + p/\rho$.

The equations are particularly simple for potential flow of an incompressible fluid. Equation (10.3) is satisfied identically if **curl v** $= 0$. Equation (10.2), with the substitution $\mathbf{v} = \mathbf{grad}\,\phi$, becomes

$$\triangle \phi = 0, \tag{10.6}$$

i.e. Laplace's equation† for the potential ϕ. This equation must be supplemented by boundary conditions at the surfaces where the fluid meets solid bodies. At fixed solid surfaces, the fluid velocity component v_n normal to the surface must be zero, whilst for moving surfaces it must be equal to the normal component of the velocity of the surface (a given function of time). The velocity v_n, however, is equal to the normal derivative of the potential ϕ: $v_n = \partial\phi/\partial n$. Thus the general boundary conditions are that $\partial\phi/\partial n$ is a given function of coordinates and time at the boundaries.

For potential flow, the velocity is related to the pressure by equation (9.3). In an incompressible fluid, we can replace w in this equation by p/ρ:

$$\partial\phi/\partial t + \tfrac{1}{2}v^2 + p/\rho = f(t). \tag{10.7}$$

We may notice here the following important property of potential flow of an incompressible fluid. Suppose that some solid body is moving through the fluid. If the result is potential flow, it depends at any instant only on the velocity of the moving body at that instant, and not, for example, on its acceleration. For equation (10.6) does not explicitly contain the time, which enters the solution only through the boundary conditions, and these contain only the velocity of the moving body.

From Bernoulli's equation, $\tfrac{1}{2}v^2 + p/\rho = \text{constant}$, we see that, in steady flow of an incompressible fluid (not in a gravitational field), the greatest pressure occurs at points where the velocity is zero. Such a point usually occurs on the surface of a body past which the fluid is moving (at the point O in Fig. 2), and is called a *stagnation point*. If $\mathbf{u}$ is the velocity of the incident current (i.e. the fluid velocity at infinity), and p_0 the pressure at infinity, the pressure at the stagnation point is

$$p_{\max} = p_0 + \tfrac{1}{2}\rho u^2. \tag{10.8}$$

If the velocity distribution in a moving fluid depends on only two coordinates (x and y, say), and the velocity is everywhere parallel to the xy-plane, the flow is said to be *two-*

† The velocity potential was first introduced by Euler, who obtained an equation of the form (10.6) for it; this form later became known as Laplace's equation.

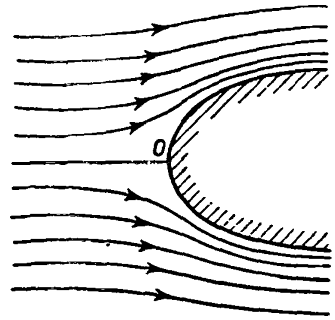

FIG. 2

dimensional or *plane flow*. To solve problems of two-dimensional flow of an incompressible fluid, it is sometimes convenient to express the velocity in terms of what is called the *stream function.* From the equation of continuity $\operatorname{div} \mathbf{v} \equiv \partial v_x/\partial x + \partial v_y/\partial y = 0$ we see that the velocity components can be written as the derivatives

$$v_x = \partial\psi/\partial y, \qquad v_y = -\partial\psi/\partial x \tag{10.9}$$

of some function $\psi(x, y)$, called the stream function. The equation of continuity is then satisfied automatically. The equation that must be satisfied by the stream function is obtained by substituting (10.9) in equation (10.3). We then obtain

$$\frac{\partial}{\partial t}\triangle\psi - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\triangle\psi + \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\triangle\psi = 0. \tag{10.10}$$

If we know the stream function we can immediately determine the form of the streamlines for steady flow. For the differential equation of the streamlines (in two-dimensional flow) is $\mathrm{d}x/v_x = \mathrm{d}y/v_y$ or $v_y\,\mathrm{d}x - v_x\,\mathrm{d}y = 0$; it expresses the fact that the direction of the tangent to a streamline is the direction of the velocity. Substituting (10.9), we have

$$\frac{\partial\psi}{\partial x}\mathrm{d}x + \frac{\partial\psi}{\partial y}\mathrm{d}y = \mathrm{d}\psi = 0,$$

whence ψ = constant. Thus the streamlines are the family of curves obtained by putting the stream function $\psi(x, y)$ equal to an arbitrary constant.

If we draw a curve between two points A and B in the xy-plane, the mass flux Q across this curve is given by the difference in the values of the stream function at these two points, regardless of the shape of the curve. For, if v_n is the component of the velocity normal to the curve at any point, we have

$$Q = \rho\oint_A^B v_n\,\mathrm{d}l = \rho\oint_A^B (-v_y\,\mathrm{d}x + v_x\,\mathrm{d}y) = \rho\int_A^B \mathrm{d}\psi,$$

or

$$Q = \rho(\psi_B - \psi_A). \tag{10.11}$$

There are powerful methods of solving problems of two-dimensional potential flow of an incompressible fluid past bodies of various profiles, involving the application of the theory of functions of a complex variable.† The basis of these methods is as follows. The potential and the stream function are related to the velocity components by‡

$$v_x = \partial\phi/\partial x = \partial\psi/\partial y, \qquad v_y = \partial\phi/\partial y = -\partial\psi/\partial x.$$

These relations between the derivatives of ϕ and ψ, however, are the same, mathematically, as the well-known Cauchy–Riemann conditions for a complex expression

$$w = \phi + i\psi \tag{10.12}$$

to be an analytic function of the complex argument $z = x + iy$. This means that the function $w(z)$ has at every point a well-defined derivative

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = v_x - iv_y. \tag{10.13}$$

The function w is called the *complex potential*, and $\mathrm{d}w/\mathrm{d}z$ the *complex velocity*. The modulus and argument of the latter give the magnitude v of the velocity and the angle θ between the direction of the velocity and that of the x-axis:

$$\mathrm{d}w/\mathrm{d}z = ve^{-i\theta}. \tag{10.14}$$

At a solid surface past which the flow takes place, the velocity must be along the tangent. That is, the profile contour of the surface must be a streamline, i.e. ψ = constant along it; the constant may be taken as zero, and then the problem of flow past a given contour reduces to the determination of an analytic function $w(z)$ which takes real values on the contour. The statement of the problem is more involved when the fluid has a free surface; an example is found in Problem 9.

The integral of an analytic function round any closed contour C is well known to be equal to $2\pi i$ times the sum of the residues of the function at its simple poles inside C; hence

$$\oint w'\,\mathrm{d}z = 2\pi i \sum_k A_k,$$

where A_k are the residues of the complex velocity. We also have

$$\oint w'\,\mathrm{d}z = \oint (v_x - iv_y)(\mathrm{d}x + i\mathrm{d}y)$$

$$= \oint (v_x\mathrm{d}x + v_y\mathrm{d}y) + i\oint (v_x\mathrm{d}y - v_y\mathrm{d}x).$$

The real part of this expression is just the velocity circulation Γ round the contour C. The imaginary part, multiplied by ρ, is the mass flux across C; if there are no sources of fluid within the contour, this flux is zero and we then have simply

$$\Gamma = 2\pi i \sum_k A_k; \tag{10.15}$$

all the residues A_k are in this case purely imaginary.

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† A more detailed account of these methods and their numerous applications may be found in many books which treat fluid dynamics from a more mathematical standpoint. Here, we shall describe only the basic idea.

‡ The existence of the stream function depends, however, only on the flow's being two-dimensional, not necessarily a potential flow.

Finally, let us consider the conditions under which the fluid may be regarded as incompressible. When the pressure changes adiabatically by Δp, the density changes by $\Delta\rho = (\partial\rho/\partial p)_s\Delta p$. According to Bernoulli's equation, however, Δp is of the order of ρv^2 in steady flow. We shall show in §64 that the derivative $(\partial p/\partial \rho)_s$ is the square of the velocity c of sound in the fluid, so that $\Delta\rho \sim \rho v^2/c^2$. The fluid may be regarded as incompressible if $\Delta\rho/\rho \ll 1$. We see that a necessary condition for this is that the fluid velocity be small compared with that of sound:

$$v \ll c. \tag{10.16}$$

However, this condition is sufficient only in steady flow. In non-steady flow, a further condition must be fulfilled. Let τ and l be a time and a length of the order of the times and distances over which the fluid velocity undergoes significant changes. If the terms $\partial\mathbf{v}/\partial t$ and $(1/\rho)\,\mathbf{grad}\,p$ in Euler's equation are comparable, we find, in order of magnitude, $v/\tau \sim \Delta p/l\rho$ or $\Delta p \sim l\rho v/\tau$, and the corresponding change in ρ is $\Delta\rho \sim l\rho v/\tau c^2$. Now comparing the terms $\partial\rho/\partial t$ and $\rho\,\mathrm{div}\,\mathbf{v}$ in the equation of continuity, we find that the derivative $\partial\rho/\partial t$ may be neglected (i.e. we may suppose ρ constant) if $\Delta\rho/\tau \ll \rho v/l$, or

$$\tau \gg l/c. \tag{10.17}$$

If the conditions (10.16) and (10.17) are both fulfilled, the fluid may be regarded as incompressible. The condition (10.17) has an obvious meaning: the time l/c taken by a sound signal to traverse the distance l must be small compared with the time τ during which the flow changes appreciably, so that the propagation of interactions in the fluid may be regarded as instantaneous.

PROBLEMS

Problem 1. Determine the shape of the surface of an incompressible fluid subject to a gravitational field, contained in a cylindrical vessel which rotates about its (vertical) axis with a constant angular velocity Ω.

Solution. Let us take the axis of the cylinder as the z-axis. Then $v_x = -y\Omega$, $v_y = x\Omega$, $v_z = 0$. The equation of continuity is satisfied identically, and Euler's equation (10.1) gives

$$x\Omega^2 = \frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad y\Omega^2 = \frac{1}{\rho}\frac{\partial p}{\partial y}, \qquad \frac{1}{\rho}\frac{\partial p}{\partial z} + g = 0.$$

The general integral of these equations is

$$p/\rho = \tfrac{1}{2}\Omega^2(x^2+y^2) - gz + \text{constant}.$$

At the free surface $p = $ constant, so that the surface is a paraboloid:

$$z = \tfrac{1}{2}\Omega^2(x^2+y^2)/g,$$

the origin being taken at the lowest point of the surface.

Problem 2. A sphere, with radius R, moves with velocity $\mathbf{u}$ in an incompressible ideal fluid. Determine the potential flow of the fluid past the sphere.

Solution. The fluid velocity must vanish at infinity. The solutions of Laplace's equation $\triangle\phi = 0$ which vanish at infinity are well known to be $1/r$ and the derivatives, of various orders, of $1/r$ with respect to the coordinates (the origin is taken at the centre of the sphere). On account of the complete symmetry of the sphere, only one constant vector, the velocity $\mathbf{u}$, can appear in the solution, and, on account of the linearity of both Laplace's equation and the boundary condition, ϕ must involve $\mathbf{u}$ linearly. The only scalar which can be formed from $\mathbf{u}$ and the derivatives of $1/r$ is the scalar product $\mathbf{u}\cdot\mathbf{grad}(1/r)$. We therefore seek ϕ in the form

$$\phi = \mathbf{A}\cdot\mathbf{grad}(1/r) = -(\mathbf{A}\cdot\mathbf{n})/r^2,$$

where $\mathbf{n}$ is a unit vector in the direction of $\mathbf{r}$. The constant $\mathbf{A}$ is determined from the condition that the normal components of the velocities $\mathbf{v}$ and $\mathbf{u}$ must be equal at the surface at the sphere, i.e. $\mathbf{v}\cdot\mathbf{n} = \mathbf{u}\cdot\mathbf{n}$ for $r = R$. This condition gives $\mathbf{A} = \frac{1}{2}\mathbf{u}R^3$, so that

$$\phi = -\frac{R^3}{2r^2}\mathbf{u}\cdot\mathbf{n}, \qquad \mathbf{v} = \frac{R^3}{2r^3}[3\mathbf{n}(\mathbf{u}\cdot\mathbf{n}) - \mathbf{u}].$$

The pressure distribution is given by equation (10.7):

$$p = p_0 - \tfrac{1}{2}\rho v^2 - \rho\partial\phi/\partial t,$$

where p_0 is the pressure at infinity. To calculate the derivative $\partial\phi/\partial t$, we must bear in mind that the origin (which we have taken at the centre of the sphere) moves with velocity $\mathbf{u}$. Hence

$$\partial\phi/\partial t = (\partial\phi/\partial\mathbf{u})\cdot\dot{\mathbf{u}} - \mathbf{u}\cdot\mathbf{grad}\,\phi.$$

The pressure distribution over the surface of the sphere is given by the formula

$$p = p_0 + \tfrac{1}{8}\rho u^2(9\cos^2\theta - 5) + \tfrac{1}{2}\rho R\mathbf{n}\cdot \mathrm{d}\mathbf{u}/\mathrm{d}t,$$

where θ is the angle between $\mathbf{n}$ and $\mathbf{u}$.

PROBLEM 3. The same as Problem 2, but for an infinite cylinder moving perpendicular to its axis.†

SOLUTION. The flow is independent of the axial coordinate, so that we have to solve Laplace's equation in two dimensions. The solutions which vanish at infinity are the first and higher derivatives of $\log r$ with respect to the coordinates, where $\mathbf{r}$ is the radius vector perpendicular to the axis of the cylinder. We seek a solution in the form

$$\phi = \mathbf{A}\cdot\mathbf{grad}\log r = \mathbf{A}\cdot\mathbf{n}/r,$$

and from the boundary conditions we obtain $\mathbf{A} = -R^2\mathbf{u}$, so that

$$\phi = -\frac{R^2}{r}\mathbf{u}\cdot\mathbf{n}, \qquad \mathbf{v} = \frac{R^2}{r^2}[2\mathbf{n}(\mathbf{u}\cdot\mathbf{n}) - \mathbf{u}].$$

The pressure at the surface of the cylinder is given by

$$p = p_0 + \tfrac{1}{2}\rho u^2(4\cos^2\theta - 3) + \rho R\mathbf{n}\cdot\mathrm{d}\mathbf{u}/\mathrm{d}t.$$

PROBLEM 4. Determine the potential flow of an incompressible ideal fluid in an ellipsoidal vessel rotating about a principal axis with angular velocity Ω, and determine the total angular momentum of the fluid.

SOLUTION. We take Cartesian coordinates x, y, z along the axes of the ellipsoid at a given instant, the z-axis being the axis of rotation. The velocity of points in the vessel wall is

$$\mathbf{u} = \mathbf{\Omega}\times\mathbf{r},$$

so that the boundary condition $v_n = \partial\phi/\partial n = u_n$ is

$$\partial\phi/\partial n = \Omega(xn_y - yn_x),$$

or, using the equation of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$,

$$\frac{x}{a^2}\frac{\partial\phi}{\partial x} + \frac{y}{b^2}\frac{\partial\phi}{\partial y} + \frac{z}{c^2}\frac{\partial\phi}{\partial z} = xy\Omega\left(\frac{1}{b^2} - \frac{1}{a^2}\right).$$

The solution of Laplace's equation which satisfies this boundary condition is

$$\phi = \Omega\frac{a^2 - b^2}{a^2 + b^2}xy. \tag{1}$$

The angular momentum of the fluid in the vessel is

$$M = \rho\int(xv_y - yv_x)\,\mathrm{d}V.$$

† The solution of the more general problems of potential flow past an ellipsoid and an elliptical cylinder may be found in: N. E. Kochin, I. A. Kibel' and N. V. Roze, *Theoretical Hydromechanics* (*Teoreticheskaya gidromekhanika*), Part 1, chapter VII, Moscow 1963; H. Lamb, *Hydrodynamics*, 6th ed., §§103–116, Cambridge 1932.

Integrating over the volume V of the ellipsoid, we have

$$M = \frac{\Omega\rho V}{5}\frac{(a^2-b^2)^2}{a^2+b^2}.$$

Formula (1) gives the absolute motion of the fluid relative to the instantaneous position of the axes x, y, z which are fixed to the rotating vessel. The motion relative to the vessel (i.e. relative to a rotating system of coordinates x, y, z) is found by subtracting the velocity $\mathbf{\Omega}\times\mathbf{r}$ from the absolute velocity; denoting the relative velocity of the fluid by $\mathbf{v}'$, we have

$$v'_x = \frac{\partial\phi}{\partial x} + y\Omega = \frac{2\Omega a^2}{a^2+b^2}y, \qquad v'_y = -\frac{2\Omega b^2}{a^2+b^2}x, \qquad v'_z = 0.$$

The paths of the relative motion are found by integrating the equations $\dot{x} = v'_x$, $\dot{y} = v'_y$, and are the ellipses $x^2/a^2 + y^2/b^2 =$ constant, which are similar to the boundary ellipse.

PROBLEM 5. Determine the flow near a stagnation point (Fig. 2).

SOLUTION. A small part of the surface of the body near the stagnation point may be regarded as plane. Let us take it as the xy-plane. Expanding ϕ for x, y, z small, we have as far as the second-order terms

$$\phi = ax + by + cz + Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx;$$

a constant term in ϕ is immaterial. The constant coefficients are determined so that ϕ satisfies the equation $\triangle\phi = 0$ and the boundary conditions $v_z = \partial\phi/\partial z = 0$ for $z = 0$ and all x, y, $\partial\phi/\partial x = \partial\phi/\partial y = 0$ for $x = y = z = 0$ (the stagnation point). This gives $a = b = c = 0$; $C = -A - B$, $E = F = 0$. The term Dxy can always be removed by an appropriate rotation of the x and y axes. We then have

$$\phi = Ax^2 + By^2 - (A+B)z^2. \tag{1}$$

If the flow is axially symmetrical about the z-axis (symmetrical flow past a solid of revolution), we must have $A = B$, so that

$$\phi = A(x^2 + y^2 - 2z^2).$$

The velocity components are $v_x = 2Ax$, $v_y = 2Ay$, $v_z = -4Az$. The streamlines are given by equations (5.2), from which we find $x^2z = c_1$, $y^2z = c_2$, i.e. the streamlines are cubical hyperbolae.

If the flow is uniform in the y-direction (e.g. flow in the z-direction past a cylinder with its axis in the y-direction), we must have $B = 0$ in (1), so that

$$\phi = A(x^2 - z^2).$$

The streamlines are the hyperbolae $xz =$ constant.

PROBLEM 6. Determine the potential flow near an angle formed by two intersecting planes.

SOLUTION. Let us take polar coordinates r, θ in the cross-sectional plane (perpendicular to the line of intersection), with the origin at the vertex of the angle; θ is measured from one of the arms of the angle. Let the angle be α radians; for $\alpha < \pi$ the flow takes place within the angle, for $\alpha > \pi$ outside it. The boundary condition that the normal velocity component vanish means that $\partial\phi/\partial\theta = 0$ for $\theta = 0$ and $\theta = \alpha$. The solution of Laplace's equation satisfying these conditions can be written†

$$\phi = Ar^n \cos n\theta, \qquad n = \pi/\alpha,$$

so that

$$v_r = nAr^{n-1}\cos n\theta, \qquad v_\theta = -nAr^{n-1}\sin n\theta.$$

For $n < 1$ (flow outside an angle; Fig. 3), v_r becomes infinite as $1/r^{1-n}$ at the origin. For $n > 1$ (flow inside an angle; Fig. 4), v becomes zero for $r = 0$.

The stream function, which gives the form of the streamlines, is $\psi = Ar^n \sin n\theta$. The expressions obtained for ϕ and ψ are the real and imaginary parts of the complex potential $w = Az^n$.‡

PROBLEM 7. A spherical hole with radius a is suddenly formed in an incompressible fluid filling all space. Determine the time taken for the hole to be filled with fluid (Besant 1859; Rayleigh 1917).

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† We take the solution which involves the lowest positive power of r, since r is small.

‡ If the boundary planes are supposed infinite, Problems 5 and 6 involve degeneracy, in that the values of the constants A and B in the solutions are indeterminate. In actual cases of flow past finite bodies, they are determined by the general conditions of the problem.

SOLUTION. The flow after the formation of the hole will be spherically symmetrical, the velocity at every point being directed to the centre of the hole. For the radial velocity $v_r \equiv v < 0$ we have Euler's equation in spherical polar coordinates:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}. \tag{1}$$

The equation of continuity gives

$$r^2 v = F(t), \tag{2}$$

where $F(t)$ is an arbitrary function of time; this equation expresses the fact that, since the fluid is incompressible, the volume flowing through any spherical surface is independent of the radius of that surface.

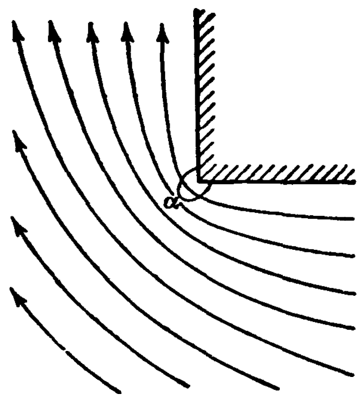

FIG. 3

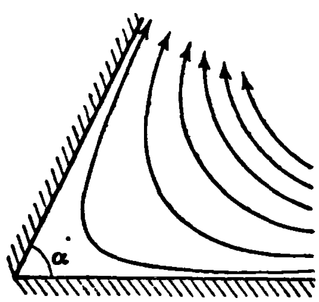

FIG. 4

Substituting v from (2) in (1), we have

$$\frac{F'(t)}{r^2} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}.$$

Integrating this equation over r from the instantaneous radius $R = R(t) \leqslant a$ of the hole to infinity, we obtain

$$-\frac{F'(t)}{R} + \tfrac{1}{2}V^2 = \frac{p_0}{\rho} \tag{3}$$

where $V = \mathrm{d}R(t)/\mathrm{d}t$ is the rate of change of the radius of the hole, and p_0 is the pressure at infinity; the fluid velocity at infinity is zero, and so is the pressure at the surface of the hole. From equation (2) for points on the surface of the hole we find

$$F(t) = R^2(t)\, V(t),$$

and, substituting this expression for $F(t)$ in (3), we obtain the equation

$$-\frac{3V^2}{2} - \tfrac{1}{2}R\frac{\mathrm{d}V^2}{\mathrm{d}R} = \frac{p_0}{\rho}. \tag{4}$$

The variables are separable; integrating with the boundary condition $V = 0$ for $R = a$ (the fluid being initially at rest), we have

$$V \equiv \frac{\mathrm{d}R}{\mathrm{d}t} = -\sqrt{\left[\frac{2p_0}{3\rho}\left(\frac{a^3}{R^3}-1\right)\right]}.$$

Hence we have for the required total time for the hole to be filled

$$\tau = \sqrt{\frac{3\rho}{2p_0}}\int_0^a \frac{\mathrm{d}R}{\sqrt{[(a/R)^3-1]}}.$$

This integral reduces to a beta function, and we have finally

$$\tau = \sqrt{\frac{3a^2\rho\pi}{2p_0}}\frac{\Gamma(5/6)}{\Gamma(1/3)} = 0{\cdot}915a\sqrt{\frac{\rho}{p_0}}.$$

Problem 8. A sphere immersed in an incompressible fluid expands according to a given law $R = R(t)$. Determine the fluid pressure at the surface of the sphere.

Solution. Let the required pressure be $P(t)$. Calculations exactly similar to those of Problem 7, except that the pressure at $r = R$ is $P(t)$ and not zero, give instead of (3) the equation

$$-\frac{F'(t)}{R(t)} + \tfrac{1}{2}V^2 = \frac{p_0}{\rho} - \frac{P(t)}{\rho}$$

and accordingly instead of (4) the equation

$$\frac{p_0 - P(t)}{\rho} = -\frac{3V^2}{2} - RV\frac{\mathrm{d}V}{\mathrm{d}R}.$$

Bearing in mind the fact that $V = \mathrm{d}R/\mathrm{d}t$, we can write the expression for $P(t)$ in the form

$$P(t) = p_0 + \tfrac{1}{2}\rho\left[\frac{\mathrm{d}^2(R^2)}{\mathrm{d}t^2} + \left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2\right].$$

Problem 9. Determine the form of a jet emerging from an infinitely long slit in a plane wall.

Solution. Let the wall be along the x-axis in the xy-plane, and the aperture be the segment $-\frac{1}{2}a \leqslant x \leqslant \frac{1}{2}a$ of that axis, the fluid occupying the half-plane $y > 0$. Far from the wall ($y \to \infty$) the fluid velocity is zero, and the pressure is p_0, say.

At the free surface of the jet (BC and $B'C'$ in Fig. 5a) the pressure $p = 0$, while the velocity takes the constant value $v_1 = \sqrt{(2p_0/\rho)}$, by Bernoulli's equation. The wall lines are streamlines, and continue into the free boundary of the jet. Let ψ be zero on the line ABC; then, on the line $A'B'C'$, $\psi = -Q/\rho$, where $Q = \rho a_1 v_1$ is the rate at which the fluid emerges in the jet (a_1, v_1 being the jet width and velocity at infinity). The potential ϕ varies from $-\infty$ to $+\infty$ both along ABC and along $A'B'C'$; let ϕ be zero at B and B'. Then, in the plane of the complex variable w, the region of flow is an infinite strip of width Q/ρ (Fig. 5b). (The points in Fig. 5b, c, d are named to correspond with those in Fig. 5a.)

We introduce a new complex variable, the logarithm of the complex velocity:

$$\zeta = -\log\left[\frac{1}{v_1 e^{\frac{1}{2}i\pi}}\frac{\mathrm{d}w}{\mathrm{d}z}\right] = \log\frac{v_1}{v} + i(\tfrac{1}{2}\pi + \theta); \tag{1}$$

here $v_1 e^{\frac{1}{2}i\pi}$ is the complex velocity of the jet at infinity. On $A'B'$ we have $\theta = 0$; on AB, $\theta = -\pi$; on BC and $B'C'$, $v = v_1$, while at infinity in the jet $\theta = -\frac{1}{2}\pi$. In the plane of the complex variable ζ, therefore, the region of flow is a semi-infinite strip of width π in the right half-plane (Fig. 5c). If we can now find a conformal transformation which carries the strip in the w-plane into the half-strip in the ζ-plane (with the points corresponding as in Fig. 5), we shall have determined w as a function of $\mathrm{d}w/\mathrm{d}z$, and w can then be found by a simple quadrature.

In order to find the desired transformation, we introduce one further auxiliary complex variable, u, such that the region of flow in the u-plane is the upper half-plane, the points B and B' corresponding to $u = \pm 1$, the points C and C' to $u = 0$, and the infinitely distant points A and A' to $u = \pm\infty$ (Fig. 5d). The dependence of w on this auxiliary variable is given by the conformal transformation which carries the upper half of the u-plane into the strip in the w-plane. With the above correspondence of points, this transformation is

$$w = -\frac{Q}{\rho\pi}\log u. \tag{2}$$

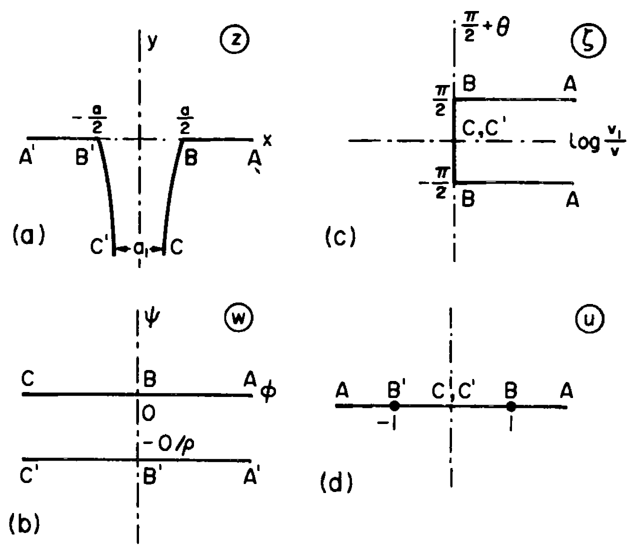

FIG. 5

In order to find the dependence of ζ on u, we have to find a conformal transformation of the half-strip in the ζ-plane into the upper half of the u-plane. Regarding this half-strip as a triangle with one vertex at infinity, we can find the desired transformation by means of the well-known Schwarz–Christoffel formula; it is

$$\zeta = -i \sin^{-1} u. \tag{3}$$

Formulae (2) and (3) give the solution of the problem, since they furnish the dependence of dw/dz on w in parametric form.

Let us now determine the form of the jet. On BC we have $w = \phi$, $\zeta = i(\frac{1}{2}\pi + \theta)$, while u varies from 1 to 0. From (2) and (3) we obtain

$$\phi = -\frac{Q}{\rho\pi} \log(-\cos\theta), \tag{4}$$

and from (1) we have

$$d\phi/dz = v_1 e^{-i\theta},$$

or

$$dz \equiv dx + i\,dy = \frac{1}{v_1} e^{i\theta} d\phi = \frac{a_1}{\pi} e^{i\theta} \tan\theta \, d\theta,$$

whence we find, by integration with the conditions $y = 0$, $x = \frac{1}{2}a$ for $\theta = -\pi$, the form of the jet, expressed parametrically. In particular, the compression of the jet is $a_1/a = \pi/(2+\pi) = 0{\cdot}61$.

§11. The drag force in potential flow past a body

Let us consider the problem of potential flow of an incompressible ideal fluid past some solid body. This problem is, of course, completely equivalent to that of the motion of a fluid when the same body moves through it. To obtain the latter case from the former, we need only change to a system of coordinates in which the fluid is at rest at infinity. We shall, in fact, say in what follows that the body is moving through the fluid.

Let us determine the nature of the fluid velocity distribution at great distances from the moving body. The potential flow of an incompressible fluid satisfies Laplace's equation, $\triangle\phi = 0$. We have to consider solutions of this equation which vanish at infinity, since the fluid is at rest there. We take the origin somewhere inside the moving body; the coordinate system moves with the body, but we shall consider the fluid velocity distribution at a particular instant. As we know, Laplace's equation has a solution $1/r$, where r is the distance from the origin. The gradient and higher space derivatives of $1/r$ are also solutions. All these solutions, and any linear combination of them, vanish at infinity. Hence the general form of the required solution of Laplace's equation at great distances from the body is

$$\phi = -\frac{a}{r} + \mathbf{A}\cdot\mathbf{grad}\frac{1}{r} + \ldots,$$

where a and $\mathbf{A}$ are independent of the coordinates; the omitted terms contain higher-order derivatives of $1/r$. It is easy to see that the constant a must be zero. For the potential $\phi = -a/r$ gives a velocity

$$\mathbf{v} = -\mathbf{grad}(a/r) = a\mathbf{r}/r^3.$$

Let us calculate the corresponding mass flux through some closed surface, say a sphere with radius R. On this surface the velocity is constant and equal to a/R^2; the total flux through it is therefore $\rho(a/R^2)4\pi R^2 = 4\pi\rho a$. But the flux of an incompressible fluid through any closed surface must, of course, be zero. Hence we conclude that $a = 0$.

Thus ϕ contains terms of order $1/r^2$ and higher. Since we are seeking the velocity at large distances, the terms of higher order may be neglected, and we have

$$\phi = \mathbf{A}\cdot\mathbf{grad}(1/r) = -\mathbf{A}\cdot\mathbf{n}/r^2, \tag{11.1}$$

and the velocity $\mathbf{v} = \mathbf{grad}\,\phi$ is

$$\mathbf{v} = (\mathbf{A}\cdot\mathbf{grad})\,\mathbf{grad}\frac{1}{r} = \frac{3(\mathbf{A}\cdot\mathbf{n})\mathbf{n} - \mathbf{A}}{r^3}, \tag{11.2}$$

where $\mathbf{n}$ is a unit vector in the direction of $\mathbf{r}$. We see that at large distances the velocity diminishes as $1/r^3$. The vector $\mathbf{A}$ depends on the actual shape and velocity of the body, and can be determined only by solving completely the equation $\triangle\phi = 0$ at all distances, taking into account the appropriate boundary conditions at the surface of the moving body.

The vector $\mathbf{A}$ which appears in (11.2) is related in a definite manner to the total momentum and energy of the fluid in its motion past the body. The total kinetic energy of the fluid (the internal energy of an incompressible fluid is constant) is $E = \frac{1}{2}\int \rho v^2\,\mathrm{d}V$, where the integration is taken over all space outside the body. We take a region of space V bounded by a sphere with large radius R, whose centre is at the origin, and first integrate only over V, later letting R tend to infinity. We have identically

$$\int v^2\,\mathrm{d}V = \int u^2\,\mathrm{d}V + \int(\mathbf{v}+\mathbf{u})\cdot(\mathbf{v}-\mathbf{u})\,\mathrm{d}V,$$

where $\mathbf{u}$ is the velocity of the body. Since $\mathbf{u}$ is independent of the coordinates, the first integral on the right is simply $u^2(V - V_0)$, where V_0 is the volume of the body. In the second integral, we write the sum $\mathbf{v}+\mathbf{u}$ as $\mathbf{grad}\,(\phi + \mathbf{u}\cdot\mathbf{r})$; using the facts that $\operatorname{div}\mathbf{v} = 0$ (equation of continuity) and $\operatorname{div}\mathbf{u} \equiv 0$, we have

$$\int v^2\,\mathrm{d}V = u^2(V - V_0) + \int \operatorname{div}\left[(\phi + \mathbf{u}\cdot\mathbf{r})(\mathbf{v}-\mathbf{u})\right]\mathrm{d}V.$$

The second integral is now transformed into an integral over the surface S of the sphere and the surface S_0 of the body:

$$\int v^2 \, dV = u^2(V - V_0) + \oint\limits_{S+S_0} (\phi + \mathbf{u}\cdot\mathbf{r})(\mathbf{v}-\mathbf{u})\cdot d\mathbf{f}.$$

On the surface of the body, the normal components of $\mathbf{v}$ and $\mathbf{u}$ are equal by virtue of the boundary conditions; since the vector $d\mathbf{f}$ is along the normal to the surface, it is clear that the integral over S_0 vanishes identically. On the remote surface S we substitute the expressions (11.1), (11.2) for ϕ and $\mathbf{v}$, and neglect terms which vanish as $R \to \infty$. Writing the surface element on the sphere S in the form $d\mathbf{f} = \mathbf{n}R^2 do$, where do is an element of solid angle, we obtain

$$\int v^2 \, dV = u^2(\tfrac{4}{3}\pi R^3 - V_0) + \int [3(\mathbf{A}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{n}) - (\mathbf{u}\cdot\mathbf{n})^2 R^3] \, do.$$

Finally, effecting the integration† and multiplying by $\frac{1}{2}\rho$, we obtain the following expression for the total energy of the fluid:

$$E = \tfrac{1}{2}\rho(4\pi \mathbf{A}\cdot\mathbf{u} - V_0 u^2). \qquad (11.3)$$

As has been mentioned already, the exact calculation of the vector $\mathbf{A}$ requires a complete solution of the equation $\triangle\phi = 0$, taking into account the particular boundary conditions at the surface of the body. However, the general nature of the dependence of $\mathbf{A}$ on the velocity $\mathbf{u}$ of the body can be found directly from the facts that the equation is linear in ϕ, and the boundary conditions are linear in both ϕ and $\mathbf{u}$. It follows from this that $\mathbf{A}$ must be a linear function of the components of $\mathbf{u}$. The energy E given by formula (11.3) is therefore a quadratic function of the components of $\mathbf{u}$, and can be written in the form

$$E = \tfrac{1}{2} m_{ik} u_i u_k, \qquad (11.4)$$

where m_{ik} is some constant symmetrical tensor, whose components can be calculated from those of $\mathbf{A}$; it is called the *induced-mass tensor.*

Knowing the energy E, we can obtain an expression for the total momentum $\mathbf{P}$ of the fluid. To do so, we notice that infinitesimal changes in E and $\mathbf{P}$ are related by‡ $dE = \mathbf{u}\cdot d\mathbf{P}$;

† The integration over o is equivalent to averaging the integrand over all directions of the vector $\mathbf{n}$ and multiplying by 4π. To average expressions of the type $(\mathbf{A}\cdot\mathbf{n})(\mathbf{B}\cdot\mathbf{n}) \equiv A_i n_i B_k n_k$, where $\mathbf{A}$, $\mathbf{B}$ are constant vectors, we notice that

$$\overline{(\mathbf{A}\cdot\mathbf{n})(\mathbf{B}\cdot\mathbf{n})} = A_i B_k \overline{n_i n_k} = \tfrac{1}{3}\delta_{ik} A_i B_k = \tfrac{1}{3}\mathbf{A}\cdot\mathbf{B}.$$

‡ For, let the body be accelerated by some external force $\mathbf{F}$. The momentum of the fluid will thereby be increased; let it increase by $d\mathbf{P}$ during a time dt. This increase is related to the force by $d\mathbf{P} = \mathbf{F}\,dt$, and on scalar multiplication by the velocity $\mathbf{u}$ we have $\mathbf{u}\cdot d\mathbf{P} = \mathbf{F}\cdot\mathbf{u}\,dt$, i.e. the work done by the force $\mathbf{F}$ acting through the distance $\mathbf{u}\,dt$, which in turn must be equal to the increase dE in the energy of the fluid.

It should be noticed that it would not be possible to calculate the momentum directly as the integral $\int \rho\mathbf{v}\,dV$ over the whole volume of the fluid. The reason is that this integral, with the velocity $\mathbf{v}$ distributed in accordance with (11.2), diverges, in the sense that the result of the integration, though finite, depends on how the integral is taken: on effecting the integration over a large region, whose dimensions subsequently tend to infinity, we obtain a value depending on the shape of the region (sphere, cylinder, etc.). The method of calculating the momentum which we use here, starting from the relation $\mathbf{u}\cdot d\mathbf{P} = dE$, leads to a completely definite final result, given by formula (11.6), which certainly satisfies the physical relation between the rate of change of the momentum and the forces acting on the body.

it follows from this that, if E is expressed in the form (11.4), the components of $\mathbf{P}$ must be

$$P_i = m_{ik} u_k. \tag{11.5}$$

Finally, a comparison of formulae (11.3), (11.4) and (11.5) shows that $\mathbf{P}$ is given in terms of $\mathbf{A}$ by

$$\mathbf{P} = 4\pi\rho\mathbf{A} - \rho V_0 \mathbf{u}. \tag{11.6}$$

It must be noticed that the total momentum of the fluid is a perfectly definite finite quantity.

The momentum transmitted to the fluid by the body in unit time is $\mathrm{d}\mathbf{P}/\mathrm{d}t$. With the opposite sign it evidently gives the reaction $\mathbf{F}$ of the fluid, i.e. the force acting on the body:

$$\mathbf{F} = -\mathrm{d}\mathbf{P}/\mathrm{d}t. \tag{11.7}$$

The component of $\mathbf{F}$ parallel to the velocity of the body is called the *drag force*, and the perpendicular component is called the *lift force*.

If it were possible to have potential flow past a body moving uniformly in an ideal fluid, we should have $\mathbf{P}$ = constant, since $\mathbf{u}$ = constant, and so $\mathbf{F} = 0$. That is, there would be no drag and no lift; the pressure forces exerted on the body by the fluid would balance out (a result known as *d'Alembert's paradox*). The origin of this paradox is most clearly seen by considering the drag. The presence of a drag force in uniform motion of a body would mean that, to maintain the motion, work must be continually done by some external force, this work being either dissipated in the fluid or converted into kinetic energy of the fluid, and the result being a continual flow of energy to infinity in the fluid. There is, however, by definition no dissipation of energy in an ideal fluid, and the velocity of the fluid set in motion by the body diminishes so rapidly with increasing distance from the body that there can be no flow of energy to infinity.

However, it must be emphasized that all these arguments relate only to the motion of a body in an infinite volume of fluid. If, for example, the fluid has a free surface, a body moving uniformly parallel to this surface will experience a drag. The appearance of this force (called *wave drag*) is due to the occurrence of a system of waves propagated on the free surface, which continually remove energy to infinity.

Suppose that a body is executing an oscillatory motion under the action of an external force $\mathbf{f}$. When the conditions discussed in §10 are fulfilled, the fluid surrounding the body moves in a potential flow, and we can use the relations previously obtained to derive the equations of motion of the body. The force $\mathbf{f}$ must be equal to the time derivative of the total momentum of the system, and the total momentum is the sum of the momentum $M\mathbf{u}$ of the body (M being the mass of the body) and the momentum $\mathbf{P}$ of the fluid:

$$M\,\mathrm{d}\mathbf{u}/\mathrm{d}t + \mathrm{d}\mathbf{P}/\mathrm{d}t = \mathbf{f}.$$

Using (11.5), we then obtain

$$M\,\mathrm{d}u_i/\mathrm{d}t + m_{ik}\,\mathrm{d}u_k/\mathrm{d}t = f_i,$$

which can also be written

$$\frac{\mathrm{d}u_k}{\mathrm{d}t}(M\delta_{ik} + m_{ik}) = f_i. \tag{11.8}$$

This is the equation of motion of a body immersed in an ideal fluid.

Let us now consider what is in some ways the converse problem. Suppose that the fluid executes some oscillatory motion on account of some cause external to the body. This motion will set the body in motion also.† We shall derive the equation of motion of the body.

We assume that the velocity of the fluid varies only slightly over distances of the order of the dimension of the body. Let **v** be what the fluid velocity at the position of the body would be if the body were absent; that is, **v** is the velocity of the unperturbed flow. According to the above assumption, **v** may be supposed constant throughout the volume occupied by the body. We denote the velocity of the body by **u** as before.

The force which acts on the body and sets it in motion can be determined as follows. If the body were wholly carried along with the fluid (i.e. if $\mathbf{v} = \mathbf{u}$), the force acting on it would be the same as the force which would act on the liquid in the same volume if the body were absent. The momentum of this volume of fluid is $\rho V_0\mathbf{v}$, and therefore the force on it is $\rho V_0\, \mathrm{d}\mathbf{v}/\mathrm{d}t$. In reality, however, the body is not wholly carried along with the fluid; there is a motion of the body relative to the fluid, in consequence of which the fluid itself acquires some additional motion. The resulting additional momentum of the fluid is $m_{ik}(u_k - v_k)$, since in (11.5) we must now replace **u** by the velocity $\mathbf{u} - \mathbf{v}$ of the body relative to the fluid. The change in this momentum with time results in the appearance of an additional reaction force on the body of $-m_{ik}\, \mathrm{d}(u_k - v_k)/\mathrm{d}t$. Thus the total force on the body is

$$\rho V_0 \frac{\mathrm{d}v_i}{\mathrm{d}t} - m_{ik}\frac{\mathrm{d}}{\mathrm{d}t}(u_k - v_k).$$

This force is to be equated to the time derivative of the body momentum. Thus we obtain the following equation of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}(Mu_i) = \rho V_0 \frac{\mathrm{d}v_i}{\mathrm{d}t} - m_{ik}\frac{\mathrm{d}}{\mathrm{d}t}(u_k - v_k).$$

Integrating both sides with respect to time, we have

$$(M\delta_{ik} + m_{ik})u_k = (m_{ik} + \rho V_0\delta_{ik})v_k. \qquad (11.9)$$

We put the constant of integration equal to zero, since the velocity **u** of the body in its motion caused by the fluid must vanish when **v** vanishes. The relation obtained determines the velocity of the body from that of the fluid. If the density of the body is equal to that of the fluid ($M = \rho V_0$), we have $\mathbf{u} = \mathbf{v}$, as we should expect.

PROBLEMS

PROBLEM 1. Obtain the equation of motion for a sphere executing an oscillatory motion in an ideal fluid, and for a sphere set in motion by an oscillating fluid.

SOLUTION. Comparing (11.1) with the expression for ϕ for flow past a sphere obtained in §10, Problem 2, we see that

$$\mathbf{A} = \tfrac{1}{2}R^3\mathbf{u},$$

where R is the radius of the sphere. The total momentum transmitted to the fluid by the sphere is, according to (11.6), $\mathbf{P} = \frac{2}{3}\pi\rho R^3\mathbf{u}$, so that the tensor m_{ik} is

$$m_{ik} = \tfrac{2}{3}\pi\rho R^3\delta_{ik}.$$

† For example, we may be considering the motion of a body in a fluid through which a sound wave is propagated, the wavelength being large compared with the dimension of the body.

The drag on the moving sphere is

$$\mathbf{F} = -\tfrac{2}{3}\pi\rho R^3 d\mathbf{u}/dt,$$

and the equation of motion of the sphere oscillating in the fluid is

$$\tfrac{4}{3}\pi R^3(\rho_0 + \tfrac{1}{2}\rho)\frac{d\mathbf{u}}{dt} = \mathbf{f},$$

where ρ_0 is the density of the sphere. The coefficient of $d\mathbf{u}/dt$ is the *virtual mass* of the sphere; it consists of the actual mass of the sphere and the induced mass, which in this case is half the mass of the fluid displaced by the sphere.

If the sphere is set in motion by the fluid, we have for its velocity, from (11.9),

$$\mathbf{u} = \frac{3\rho}{\rho + 2\rho_0}\mathbf{v}.$$

If the density of the sphere exceeds that of the fluid ($\rho_0 > \rho$), $u < v$, i.e. the sphere "lags behind" the fluid; if $\rho_0 < \rho$, on the other hand, the sphere "goes ahead".

PROBLEM 2. Express the moment of the forces acting on a body moving in a fluid in terms of the vector **A**.

SOLUTION. As we know from mechanics, the moment **M** of the forces acting on a body is determined from its Lagrangian function (in this case, the energy E) by the relation $\delta E = \mathbf{M}\cdot\delta\boldsymbol{\theta}$, where $\delta\boldsymbol{\theta}$ is the vector of an infinitesimal rotation of the body, and δE is the resulting change in E. Instead of rotating the body through an angle $\delta\boldsymbol{\theta}$ (and correspondingly changing the components m_{ik}), we may rotate the fluid through an angle $-\delta\boldsymbol{\theta}$ relative to the body (and correspondingly change the velocity **u**). We have $\delta\mathbf{u} = -\delta\boldsymbol{\theta}\times\mathbf{u}$, so that

$$\delta E = \mathbf{P}\cdot\delta\mathbf{u} = -\delta\boldsymbol{\theta}\cdot\mathbf{u}\times\mathbf{P}.$$

Using the expression (11.6) for **P**, we then obtain the required formula:

$$\mathbf{M} = -\mathbf{u}\times\mathbf{P} = 4\pi\rho\mathbf{A}\times\mathbf{u}.$$

§12. Gravity waves

The free surface of a liquid in equilibrium in a gravitational field is a plane. If, under the action of some external perturbation, the surface is moved from its equilibrium position at some point, motion will occur in the liquid. This motion will be propagated over the whole surface in the form of waves, which are called *gravity waves*, since they are due to the action of the gravitational field. Gravity waves appear mainly on the surface of the liquid; they affect the interior also, but less and less at greater and greater depths.

We shall here consider gravity waves in which the velocity of the moving fluid particles is so small that we may neglect the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ in comparison with $\partial\mathbf{v}/\partial t$ in Euler's equation. The physical significance of this is easily seen. During a time interval of the order of the period τ of the oscillations of the fluid particles in the wave, these particles travel a distance of the order of the amplitude a of the wave. Their velocity v is therefore of the order of a/τ. It varies noticeably over time intervals of the order of τ and distances of the order of λ in the direction of propagation (where λ is the wavelength). Hence the time derivative of the velocity is of the order of v/τ, and the space derivatives are of the order of v/λ. Thus the condition $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v} \ll \partial\mathbf{v}/\partial t$ is equivalent to

$$\frac{1}{\lambda}\left(\frac{a}{\tau}\right)^2 \ll \frac{a}{\tau}\cdot\frac{1}{\tau},$$

or

$$a \ll \lambda, \tag{12.1}$$

i.e. the amplitude of the oscillations in the wave must be small compared with the wavelength. We have seen in §9 that, if the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ in the equation of motion may be neglected, we have potential flow. Assuming the fluid incompressible, we can therefore use equations (10.6) and (10.7). The term $\frac{1}{2}v^2$ in the latter equation may be neglected, since it contains the square of the velocity; putting $f(t) = 0$ and including a term $\rho g z$ on account of the gravitational field, we obtain

$$p = -\rho g z - \rho \partial\phi/\partial t. \tag{12.2}$$

We take the z-axis vertically upwards, as usual, and the xy-plane in the equilibrium surface of the liquid.

Let us denote by ζ the z coordinate of a point on the surface; ζ is a function of x, y and t. In equilibrium $\zeta = 0$, so that ζ gives the vertical displacement of the surface in its oscillations. Let a constant pressure p_0 act on the surface. Then we have at the surface, by (12.2),

$$p_0 = -\rho g \zeta - \rho \partial\phi/\partial t.$$

The constant p_0 can be eliminated by redefining the potential ϕ, adding to it a quantity $p_0 t/\rho$ independent of the coordinates. We then obtain the condition at the surface as

$$g\zeta + (\partial\phi/\partial t)_{z=\zeta} = 0. \tag{12.3}$$

Since the amplitude of the wave oscillations is small, the displacement ζ is small. Hence we can suppose, to the same degree of approximation, that the vertical component of the velocity of points on the surface is simply the time derivative of ζ:

$$v_z = \partial\zeta/\partial t.$$

But $v_z = \partial\phi/\partial z$, so that

$$(\partial\phi/\partial z)_{z=\zeta} = \partial\zeta/\partial t = -\left(\frac{1}{g}\frac{\partial^2\phi}{\partial t^2}\right)_{z=\zeta}.$$

Since the oscillations are small, we can take the value of the derivatives at $z = 0$ instead of $z = \zeta$. Thus we have finally the following system of equations to determine the motion in a gravitational field:

$$\triangle\phi = 0, \tag{12.4}$$

$$\left(\frac{\partial\phi}{\partial z} + \frac{1}{g}\frac{\partial^2\phi}{\partial t^2}\right)_{z=0} = 0. \tag{12.5}$$

We shall here consider waves on the surface of a liquid whose area is unlimited, and we shall also suppose that the wavelength is small in comparison with the depth of the liquid; we can then regard the liquid as infinitely deep. We shall therefore omit the boundary conditions at the sides and bottom.

Let us consider a gravity wave propagated along the x-axis and uniform in the y-direction; in such a wave, all quantities are independent of y. We shall seek a solution which is a simple periodic function of time and of the coordinate x, i.e. we put

$$\phi = f(z)\cos(kx - \omega t).$$

Here ω is what is called the *circular frequency* (we shall say simply the *frequency*) of the wave; k is called the *wave number*; $\lambda = 2\pi/k$ is the *wavelength*.

Substituting in the equation $\triangle\phi = 0$, we have

$$\mathrm{d}^2 f/\mathrm{d}z^2 - k^2 f = 0.$$

The solution which decreases as we go into the interior of the liquid (i.e. as $z \to -\infty$) is

$$\phi = Ae^{kz}\cos(kx - \omega t). \tag{12.6}$$

We have also to satisfy the boundary condition (12.5). Substituting (12.6), we obtain

$$\omega^2 = kg \tag{12.7}$$

as the relation between the wave number and the frequency of a gravity wave (the *dispersion relation*).

The velocity distribution in the moving liquid is found by simply taking the space derivatives of ϕ:

$$v_x = -Ake^{kz}\sin(kx - \omega t), \qquad v_z = Ake^{kz}\cos(kx - \omega t). \tag{12.8}$$

We see that the velocity diminishes exponentially as we go into the liquid. At any given point in space (i.e. for given x, z) the velocity vector rotates uniformly in the xz-plane, its magnitude remaining constant.

Let us also determine the paths of fluid particles in the wave. We temporarily denote by x, z the coordinates of a moving fluid particle (and not of a point fixed in space), and by x_0, z_0 the values of x and z at the equilibrium position of the particle. Then $v_x = \mathrm{d}x/\mathrm{d}t$, $v_z = \mathrm{d}z/\mathrm{d}t$, and on the right-hand side of (12.8) we may approximate by writing x_0, z_0 in place of x, z, since the oscillations are small. An integration with respect to time then gives

$$\left.\begin{aligned} x - x_0 &= -A\frac{k}{\omega}e^{kz_0}\cos(kx_0 - \omega t), \\ z - z_0 &= -A\frac{k}{\omega}e^{kz_0}\sin(kx_0 - \omega t). \end{aligned}\right\} \tag{12.9}$$

Thus the fluid particles describe circles about the points (x_0, z_0) with a radius which diminishes exponentially with increasing depth.

The velocity of propagation U of the wave is, as we shall show in §67, $U = \partial\omega/\partial k$. Substituting here $\omega = \sqrt{(kg)}$, we find that the velocity of propagation of gravity waves on an unbounded surface of infinitely deep liquid is

$$U = \tfrac{1}{2}\sqrt{(g/k)} = \tfrac{1}{2}\sqrt{(g\lambda/2\pi)}. \tag{12.10}$$

It increases with the wavelength.

Long gravity waves

Having considered gravity waves whose length is small compared with the depth of the liquid, let us now discuss the opposite limiting case of waves whose length is large compared with the depth. These are called *long* waves.

Let us examine first the propagation of long waves in a channel. The channel is supposed to be along the x-axis, and of infinite length. The cross-section of the channel may have any shape, and may vary along its length. We denote the cross-sectional area of the liquid in the channel by $S = S(x, t)$. The depth and width of the channel are supposed small in comparison with the wavelength.

We shall here consider longitudinal waves, in which the liquid moves along the channel. In such waves the velocity component v_x along the channel is large compared with the components v_y, v_z.

We denote v_x by v simply, and omit small terms. The x-component of Euler's equation can then be written in the form

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

and the z-component in the form

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g;$$

we omit terms quadratic in the velocity, since the amplitude of the wave is again supposed small. From the second equation we have, since the pressure at the free surface ($z = \zeta$) must be p_0,

$$p = p_0 + g\rho(\zeta - z).$$

Substituting this expression in the first equation, we obtain

$$\partial v/\partial t = -g\partial\zeta/\partial x. \tag{12.11}$$

The second equation needed to determine the two unknowns v and ζ can be derived similarly to the equation of continuity; it is essentially the equation of continuity for the case in question. Let us consider a volume of liquid bounded by two plane cross-sections of the channel at a distance $\mathrm{d}x$ apart. In unit time a volume $(Sv)_x$ of liquid flows through one plane, and a volume $(Sv)_{x+\mathrm{d}x}$ through the other. Hence the volume of liquid between the two planes changes by

$$(Sv)_{x+\mathrm{d}x} - (Sv)_x = \frac{\partial(Sv)}{\partial x}\mathrm{d}x.$$

Since the liquid is incompressible, however, this change must be due simply to the change in the level of the liquid. The change per unit time in the volume of liquid between the two planes considered is $(\partial S/\partial t)\mathrm{d}x$. We can therefore write

$$\frac{\partial S}{\partial t}\mathrm{d}x = -\frac{\partial(Sv)}{\partial x}\mathrm{d}x,$$

or

$$\frac{\partial S}{\partial t} + \frac{\partial(Sv)}{\partial x} = 0. \tag{12.12}$$

This is the required equation of continuity.

Let S_0 be the equilibrium cross-sectional area of the liquid in the channel. Then $S = S_0 + S'$, where S' is the change in the cross-sectional area caused by the wave. Since the change in the liquid level is small, we can write S' in the form $b\zeta$, where b is the width of the channel at the surface of the liquid. Equation (12.12) then becomes

$$b\frac{\partial\zeta}{\partial t} + \frac{\partial(S_0 v)}{\partial x} = 0. \tag{12.13}$$

Differentiating (12.13) with respect to t and substituting $\partial v/\partial t$ from (12.11), we obtain

$$\frac{\partial^2\zeta}{\partial t^2} - \frac{g}{b}\frac{\partial}{\partial x}\left(S_0\frac{\partial\zeta}{\partial x}\right) = 0. \tag{12.14}$$

If the channel cross-section is the same at all points, then S_0 = constant and

$$\frac{\partial^2\zeta}{\partial t^2} - \frac{gS_0}{b}\frac{\partial^2\zeta}{\partial x^2} = 0. \tag{12.15}$$

This is called a *wave equation*: as we shall show in §64, it corresponds to the propagation of waves with a velocity U which is independent of frequency and is the square root of the coefficient of $\partial^2\zeta/\partial x^2$. Thus the velocity of propagation of long gravity waves in channels is

$$U = \sqrt{(gS_0/b)}. \tag{12.16}$$

In an entirely similar manner, we can consider long waves in a large tank, which we suppose infinite in two directions (those of x and y). The depth of liquid in the tank is denoted by h. The component v_z of the velocity is now small. Euler's equations take a form similar to (12.11):

$$\frac{\partial v_x}{\partial t} + g\frac{\partial\zeta}{\partial x} = 0, \qquad \frac{\partial v_y}{\partial t} + g\frac{\partial\zeta}{\partial y} = 0. \tag{12.17}$$

The equation of continuity is derived in the same way as (12.12) and is

$$\frac{\partial h}{\partial t} + \frac{\partial(hv_x)}{\partial x} + \frac{\partial(hv_y)}{\partial y} = 0.$$

We write the depth h as $h_0 + \zeta$, where h_0 is the equilibrium depth. Then

$$\frac{\partial\zeta}{\partial t} + \frac{\partial(h_0 v_x)}{\partial x} + \frac{\partial(h_0 v_y)}{\partial y} = 0. \tag{12.18}$$

Let us assume that the tank has a horizontal bottom (h_0 = constant). Differentiating (12.18) with respect to t and substituting (12.17), we obtain

$$\frac{\partial^2\zeta}{\partial t^2} - gh_0\left(\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2}\right) = 0. \tag{12.19}$$

This is again a (two-dimensional) wave equation; it corresponds to waves propagated with a velocity

$$U = \sqrt{(gh_0)}. \tag{12.20}$$

PROBLEMS

PROBLEM 1. Determine the velocity of propagation of gravity waves on an unbounded surface of liquid with depth h.

SOLUTION. At the bottom of the liquid, the normal velocity component must be zero, i.e. $v_z = \partial\phi/\partial z = 0$ for $z = -h$. From this condition we find the ratio of the constants A and B in the general solution

$$\phi = [Ae^{kz} + Be^{-kz}]\cos(kx - \omega t).$$

The result is

$$\phi = A\cos(kx - \omega t)\cosh k(z+h).$$

From the boundary condition (12.5) we find the relation between k and ω to be

$$\omega^2 = gk\tanh kh.$$

The velocity of propagation of the wave is

$$U = \frac{1}{2}\sqrt{\frac{g}{k\tanh kh}}\left[\tanh kh + \frac{kh}{\cosh^2 kh}\right].$$

For $kh \gg 1$ we have the result (12.10), and for $kh \ll 1$ the result (12.20).

PROBLEM 2. Determine the relation between frequency and wavelength for gravity waves on the surface separating two liquids, the upper liquid being bounded above by a fixed horizontal plane, and the lower liquid being similarly bounded below. The density and depth of the lower liquid are ρ and h, those of the upper liquid are ρ' and h', and $\rho > \rho'$.

SOLUTION. We take the xy-plane as the equilibrium plane of separation of the two liquids. Let us seek a solution having in the two liquids the forms

$$\left.\begin{aligned} \phi &= A\cosh k(z+h)\cos(kx-\omega t), \\ \phi' &= B\cosh k(z-h')\cos(kx-\omega t), \end{aligned}\right\} \tag{1}$$

so that the conditions at the upper and lower boundaries are satisfied; see the solution to Problem 1. At the surface of separation, the pressure must be continuous; by (12.2), this gives the condition

$$\rho g\zeta + \rho\frac{\partial\phi}{\partial t} = \rho' g\zeta + \rho'\frac{\partial\phi'}{\partial t} \quad \text{for} \quad z = \zeta,$$

or

$$\zeta = \frac{1}{g(\rho-\rho')}\left(\rho'\frac{\partial\phi'}{\partial t} - \rho\frac{\partial\phi}{\partial t}\right). \tag{2}$$

Moreover, the velocity component v_z must be the same for each liquid at the surface of separation. This gives the condition

$$\partial\phi/\partial z = \partial\phi'/\partial z \quad \text{for} \quad z = 0. \tag{3}$$

Now $v_z = \partial\phi/\partial z = \partial\zeta/\partial t$ and, substituting (2), we have

$$g(\rho-\rho')\frac{\partial\phi}{\partial z} = \rho'\frac{\partial^2\phi'}{\partial t^2} - \rho\frac{\partial^2\phi}{\partial t^2}. \tag{4}$$

Substituting (1) in (3) and (4) gives two homogeneous linear equations for A and B, and the condition of compatibility gives

$$\omega^2 = \frac{kg(\rho-\rho')}{\rho\coth kh + \rho'\coth kh'}.$$

For $kh \gg 1$, $kh' \gg 1$ (both liquids very deep),

$$\omega^2 = kg\frac{\rho-\rho'}{\rho+\rho'},$$

while for $kh \ll 1$, $kh' \ll 1$ (long waves),

$$\omega = k\sqrt{\frac{g(\rho-\rho')hh'}{\rho h' + \rho' h}}.$$

Lastly, if $kh \gtrsim 1$ and $kh' \ll 1$,

$$\omega^2 = k^2 g h'(\rho-\rho')/\rho.$$

PROBLEM 3. Determine the relation between frequency and wavelength for gravity waves propagated simultaneously on the surface of separation and on the upper surface of two liquid layers, the lower (density ρ) being infinitely deep, and the upper (density ρ') having depth h' and a free upper surface.

SOLUTION. We take the xy-plane as the equilibrium plane of separation of the two liquids. Let us seek a solution having in the two liquids the forms

$$\left.\begin{aligned} \phi &= Ae^{kz}\cos(kx-\omega t), \\ \phi' &= [Be^{-kz} + Ce^{kz}]\cos(kx-\omega t). \end{aligned}\right\} \tag{1}$$

At the surface of separation, i.e. for $z = 0$, we have the conditions (see Problem 2)

$$\frac{\partial\phi}{\partial z} = \frac{\partial\phi'}{\partial z}, \qquad g(\rho-\rho')\frac{\partial\phi}{\partial z} = \rho'\frac{\partial^2\phi'}{\partial t^2} - \rho\frac{\partial^2\phi}{\partial t^2}, \tag{2}$$

and at the upper surface, i.e. for $z = h'$, the condition

$$\frac{\partial\phi'}{\partial z} + \frac{1}{g}\frac{\partial^2\phi'}{\partial t^2} = 0. \tag{3}$$

The first equation (2), on substitution of (1), gives $A = C - B$, and the remaining two conditions then give two equations for B and C; from the condition of compatibility we obtain a quadratic equation for ω^2, whose roots are

$$\omega^2 = kg\frac{(\rho-\rho')(1-e^{-2kh'})}{\rho+\rho'+(\rho-\rho')e^{-2kh'}}, \qquad \omega^2 = kg.$$

For $h' \to \infty$ these roots correspond to waves propagated independently on the surface of separation and on the upper surface.

PROBLEM 4. Determine the characteristic frequencies of oscillation (see §69) of a liquid with depth h in a rectangular tank with width a and length b.

SOLUTION. We take the x and y axes along two sides of the tank. Let us seek a solution in the form of a stationary wave:

$$\phi = f(x,y)\cosh k(z+h)\cos\omega t.$$

We obtain for f the equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + k^2 f = 0,$$

and the condition at the free surface gives, as in Problem 1, the relation

$$\omega^2 = gk\tanh kh.$$

We take the solution of the equation for f in the form

$$f = \cos px \cos qy, \qquad p^2 + q^2 = k^2.$$

At the sides of the tank we must have the conditions

$$v_x = \partial\phi/\partial x = 0 \quad \text{for} \quad x = 0, a;$$

$$v_y = \partial\phi/\partial y = 0 \quad \text{for} \quad y = 0, b.$$

Hence we find $p = m\pi/a$, $q = n\pi/b$, where m, n are integers. The possible values of k^2 are therefore

$$k^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right).$$

§13. Internal waves in an incompressible fluid

There is a kind of gravity wave which can be propagated inside an incompressible fluid. Such waves are due to an inhomogeneity of the fluid caused by the gravitational field. The pressure (and therefore the entropy s) necessarily varies with height; hence any displacement of a fluid particle in height destroys the mechanical equilibrium, and consequently causes an oscillatory motion. For, since the motion is adiabatic, the particle carries with it to its new position its old entropy s, which is not the same as the equilibrium value at the new position.

We shall suppose below that the wavelength is small in comparison with distances over which the gravitational field causes a marked change in density†; and we shall regard the fluid itself as incompressible. This means that we can neglect the change in its density caused by the pressure change in the wave. The change in density caused by thermal expansion cannot be neglected, since it is this that causes the phenomenon in question.

Let us write down a system of hydrodynamic equations for this motion. We shall use a suffix 0 to distinguish the values of quantities in mechanical equilibrium, and a prime to mark small deviations from those values. Then the equation of conservation of the entropy $s = s_0 + s'$ can be written, to the first order of smallness,

$$\partial s'/\partial t + \mathbf{v}\cdot\mathbf{grad}\, s_0 = 0, \tag{13.1}$$

where s_0, like the equilibrium values of other quantities, is a given function of the vertical coordinate z.

Next, in Euler's equation we again neglect the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ (since the oscillations are small); taking into account also the fact that the equilibrium pressure distribution is given by $\mathbf{grad}\, p_0 = \rho_0\mathbf{g}$, we have to the same accuracy

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{\mathbf{grad}\, p}{\rho} + \mathbf{g} = -\frac{\mathbf{grad}\, p'}{\rho_0} + \frac{\mathbf{grad}\, p_0}{\rho_0{}^2}\rho'.$$

Since, from what has been said above, the change in density is due only to the change in entropy, and not to the change in pressure, we can put

$$\rho' = \left(\frac{\partial \rho_0}{\partial s_0}\right)_p s',$$

and we then obtain Euler's equation in the form

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\mathbf{g}}{\rho_0}\left(\frac{\partial \rho_0}{\partial s_0}\right)_p s' - \mathbf{grad}\frac{p'}{\rho_0}. \tag{13.2}$$

We can take ρ_0 under the gradient operator, since, as stated above, we always neglect the change in the equilibrium density over distances of the order of a wavelength. The density may likewise be supposed constant in the equation of continuity, which then becomes

$$\operatorname{div}\mathbf{v} = 0. \tag{13.3}$$

We shall seek a solution of equations (13.1)–(13.3) in the form of a plane wave:

$$\mathbf{v} = \text{constant} \times e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)},$$

and similarly for s' and p'. Substitution in the equation of continuity (13.3) gives

$$\mathbf{v}\cdot\mathbf{k} = 0, \tag{13.4}$$

† The density and pressure gradients are related by

$$\mathbf{grad}\, p = (\partial p/\partial \rho)_s\, \mathbf{grad}\, \rho = c^2\, \mathbf{grad}\, \rho,$$

where c is the speed of sound in the fluid. The hydrostatic equation $\mathbf{grad}\, p = \rho g$ thus gives $\mathbf{grad}\, \rho = (\rho/c^2)\mathbf{g}$. The density in the gravitational field therefore varies considerably over distances $l \cong c^2/g$. For air and water, $l \cong 10$ km and 200 km respectively.

i.e. the fluid velocity is everywhere perpendicular to the *wave vector* **k** (a transverse wave). Equations (13.1) and (13.2) give

$$i\omega s' = \mathbf{v}\cdot\mathbf{grad}\, s_0, \qquad -i\omega\mathbf{v} = \frac{1}{\rho_0}\left(\frac{\partial\rho_0}{\partial s_0}\right)_p s'\mathbf{g} - \frac{i\mathbf{k}}{\rho_0}p'.$$

The condition $\mathbf{v}\cdot\mathbf{k} = 0$ gives with the second of these equations

$$ik^2 p' = \left(\frac{\partial\rho_0}{\partial s_0}\right)_p s'\mathbf{g}\cdot\mathbf{k},$$

and, eliminating **v** and s' from the two equations, we obtain the desired dispersion relation,

$$\omega^2 = \omega_0{}^2 \sin^2\theta, \tag{13.5}$$

where

$$\omega_0{}^2 = -\frac{g}{\rho}\left(\frac{\partial\rho}{\partial s}\right)_p \frac{\mathrm{d}s}{\mathrm{d}z}. \tag{13.6}$$

Here and henceforward we omit the suffix zero to the equilibrium values of thermodynamic quantities; the z-axis is vertically upwards, and θ is the angle between this axis and the direction of **k**. If the expression on the right of (13.6) is positive, the condition for the stability of the equilibrium distribution $s(z)$ (the condition that convection be absent—see §4) is fulfilled.

We see that the frequency depends only on the direction of the wave vector, and not on its magnitude. For $\theta = 0$ we have $\omega = 0$; this means that waves of the type considered, with the wave vector vertical, cannot exist.

If the fluid is in both mechanical equilibrium and complete thermodynamic equilibrium, its temperature is constant and we can write

$$\frac{\mathrm{d}s}{\mathrm{d}z} = \left(\frac{\partial s}{\partial p}\right)_T \frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g\left(\frac{\partial s}{\partial p}\right)_T.$$

Finally, using the well-known thermodynamic relations

$$\left(\frac{\partial s}{\partial p}\right)_T = \frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial T}\right)_p, \qquad \left(\frac{\partial\rho}{\partial s}\right)_p = \frac{T}{c_p}\left(\frac{\partial\rho}{\partial T}\right)_p,$$

where c_p is the specific heat per unit mass, we find

$$\omega_0 = \sqrt{\frac{Tg}{c_p\rho}}\left|\left(\frac{\partial\rho}{\partial T}\right)_p\right|. \tag{13.7}$$

In particular, for a perfect gas,

$$\omega_0 = \frac{g}{\sqrt{(c_p T)}}. \tag{13.8}$$

The dependence of the frequency on the direction of the wave vector has the result that the wave propagation velocity $\mathbf{U} = \partial\omega/\partial\mathbf{k}$ is not parallel to **k**. Representing $\omega(\mathbf{k})$ in the form

$$\omega = \omega_0\sqrt{[1 - (\mathbf{k}\cdot\boldsymbol{\nu}/k)^2]},$$

where $\mathbf{v}$ is a unit vector in the vertically upward direction, and differentiating, we find

$$\mathbf{U} = -(\omega_0^2/\omega k)\,(\mathbf{n}\cdot\mathbf{v})\,[\mathbf{v}-(\mathbf{n}\cdot\mathbf{v})\mathbf{n}] \tag{13.9}$$

(where $\mathbf{n} = \mathbf{k}/k$). This is perpendicular to $\mathbf{k}$, and its magnitude is

$$U = (\omega_0/k)\cos\theta.$$

Its vertical component is

$$\mathbf{U}\cdot\mathbf{v} = -(\omega_0/k)\cos\theta\sin\theta.$$

§14. Waves in a rotating fluid

Another kind of internal wave can be propagated in an incompressible fluid uniformly rotating as a whole. These waves are due to the Coriolis forces which occur in rotation.

We shall consider the fluid in coordinates rotating with it. With this treatment, the mechanical equations of motion must include additional (centrifugal and Coriolis) terms. Correspondingly, forces (per unit mass of fluid) must be added on the right of Euler's equation. The centrifugal force can be written as $\mathbf{grad}\,\frac{1}{2}(\mathbf{\Omega}\times\mathbf{r})^2$, where $\mathbf{\Omega}$ is the angular velocity vector of the fluid rotation. This term can be combined with the force $-(1/\rho)$ $\mathbf{grad}\ p$ by using an effective pressure

$$P = p - \tfrac{1}{2}\rho(\mathbf{\Omega}\times\mathbf{r})^2. \tag{14.1}$$

The Coriolis force is $2\mathbf{v}\times\mathbf{\Omega}$, and occurs only when the fluid has a motion relative to the rotating coordinates, $\mathbf{v}$ being the velocity in those coordinates. We can transfer this term to the left-hand side of Euler's equation, writing the equation as

$$\partial\mathbf{v}/\partial t + (\mathbf{v}\cdot\mathbf{grad})\mathbf{v} + 2\mathbf{\Omega}\times\mathbf{v} = -(1/\rho)\mathbf{grad}\,P. \tag{14.2}$$

The equation of continuity is unchanged; for an incompressible fluid, it is simply $\operatorname{div}\mathbf{v} = 0$.

We shall again assume the wave amplitude to be small, and neglect the term quadratic in the velocity in (14.2), which becomes

$$\partial\mathbf{v}/\partial t + 2\mathbf{\Omega}\times\mathbf{v} = -(1/\rho)\,\mathbf{grad}\,p', \tag{14.3}$$

where p' is the variable part of the pressure in the wave, and ρ is a constant. The pressure can be eliminated by taking the curl of both sides. The right-hand side gives zero, and on the left-hand side, since the fluid is incompressible,

$$\begin{aligned}\mathbf{curl}\,(\mathbf{\Omega}\times\mathbf{v}) &= \mathbf{\Omega}\operatorname{div}\mathbf{v} - (\mathbf{\Omega}\cdot\mathbf{grad})\mathbf{v}\\ &= -(\mathbf{\Omega}\cdot\mathbf{grad})\mathbf{v}.\end{aligned}$$

Taking the direction of $\mathbf{\Omega}$ as the z-axis, we write the resulting equation as

$$\frac{\partial}{\partial t}\mathbf{curl}\,\mathbf{v} = 2\Omega\frac{\partial\mathbf{v}}{\partial z}. \tag{14.4}$$

We seek the solution as a plane wave

$$\mathbf{v} = \mathbf{A}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{14.5}$$

which, since $\operatorname{div}\mathbf{v} = 0$, satisfies the transversality condition

$$\mathbf{k}\cdot\mathbf{A} = 0. \tag{14.6}$$

Substitution of (14.5) in (14.4) gives

$$\omega \mathbf{k} \times \mathbf{v} = 2i\Omega k_z \mathbf{v}. \tag{14.7}$$

The dispersion relation for these waves is found by eliminating $\mathbf{v}$ from this vector equation. Vector multiplication on both sides by $\mathbf{k}$ gives

$$-\omega k^2 \mathbf{v} = 2i\Omega k_z \mathbf{k} \times \mathbf{v}$$

and a comparison of the two equations yields the dependence of ω on $\mathbf{k}$:

$$\omega = 2\Omega k_z / k = 2\Omega \cos\theta, \tag{14.8}$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{\Omega}$.

With (14.4), (14.7) takes the form

$$\mathbf{n} \times \mathbf{v} = i\mathbf{v},$$

where $n = \mathbf{k}/k$. If we use the complex wave amplitude in the form $\mathbf{A} = \mathbf{a} + i\mathbf{b}$ with real vectors $\mathbf{a}$ and $\mathbf{b}$, it follows that $\mathbf{n} \times \mathbf{b} = \mathbf{a}$: the vectors $\mathbf{a}$ and $\mathbf{b}$ (both lying in the plane perpendicular to $\mathbf{k}$) are at right angles and equal in magnitude. By taking their directions as the x and y axes, and separating real and imaginary parts in (14.5), we find

$$v_x = a\cos(\omega t - \mathbf{k}\cdot\mathbf{r}), \qquad v_y = -a\sin(\omega t - \mathbf{k}\cdot\mathbf{r}).$$

The wave is thus circularly polarized: at each point in space, the vector $\mathbf{v}$ rotates in the course of time, remaining constant in magnitude.†

The wave propagation velocity is

$$\mathbf{U} = \partial\omega/\partial\mathbf{k} = (2\Omega/k)[\boldsymbol{\nu} - \mathbf{n}(\mathbf{n}\cdot\boldsymbol{\nu})], \tag{14.9}$$

where $\boldsymbol{\nu}$ is a unit vector along $\mathbf{\Omega}$; as with internal gravity waves, it is perpendicular to the wave vector. Its magnitude and its component along $\mathbf{\Omega}$ are

$$U = (2\Omega/k)\sin\theta, \qquad \mathbf{U}\cdot\boldsymbol{\nu} = (2\Omega/k)\sin^2\theta = U\sin\theta.$$

These are called *inertial waves*. Since the Coriolis forces do no work on the moving fluid, the energy in the waves is entirely kinetic energy.

One particular form of axially symmetrical (not plane) inertial waves can be propagated along the axis of rotation of the fluid; see Problem 1.

There is one more comment to be made, regarding steady motions in a rotating fluid rather than wave propagation in it.

Let l be a characteristic length for such motion, and u a characteristic velocity. In order of magnitude, the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ in (14.2) is u^2/l, and $2\mathbf{\Omega}\times\mathbf{v}$ is Ωu. The former can be neglected in comparison with the latter if $u/l\Omega \ll 1$, and the equation of steady motion then reduces to

$$2\mathbf{\Omega} \times \mathbf{v} = -(1/\rho)\,\mathbf{grad}\,P \tag{14.10}$$

or

$$2\Omega v_y = (1/\rho)\partial P/\partial x, \qquad 2\Omega v_x = -(1/\rho)\partial P/\partial y, \qquad \partial P/\partial z = 0,$$

† This motion is relative to rotating coordinates. For fixed coordinates, it is combined with the rotation of the whole fluid.

where x and y are Cartesian coordinates in the plane perpendicular to the axis of rotation. Hence we see that P, and therefore v_x and v_y, are independent of the longitudinal coordinate z. Next, eliminating P from the first two equations, we get

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0,$$

and the equation div $\mathbf{v} = 0$ then shows that $\partial v_z/\partial z = 0$. Thus steady motion (in rotating coordinates) in a rapidly rotating fluid is a superposition of two independent motions: two-dimensional flow in the transverse plane and axial flow independent of z (J. Proudman 1916).

PROBLEMS

Problem 1. Determine the motion in an axially symmetrical wave propagated along the axis of an incompressible fluid rotating as a whole (W. Thomson 1880).

Solution. We take cylindrical polar coordinates r, ϕ, z, with the z-axis parallel to $\mathbf{\Omega}$. In an axially symmetrical wave, all quantities are independent of the angle variable ϕ. The dependence on time and on the coordinate z is given by a factor $\exp[i(kz-\omega t)]$. Taking components in (14.3), we get

$$-i\omega v_r - 2\Omega v_\phi = -(1/\rho)\partial p'/\partial r, \tag{1}$$

$$-i\omega v_\phi + 2\Omega v_r = 0, \qquad -i\omega v_z = -(ik/\rho)p'. \tag{2}$$

These are to be combined with the equation of continuity

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + ikv_z = 0. \tag{3}$$

Expressing v_ϕ and p' in terms of v_r by means of (2) and (3) and substituting in (1), we find the equation

$$\frac{\mathrm{d}^2F}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}F}{\mathrm{d}r} + \left[\frac{4\Omega^2k^2}{\omega^2} - k^2 - \frac{1}{r^2}\right]F = 0 \tag{4}$$

for the function $F(r)$ which determines the radial dependence of v_r:

$$v_r = F(r)e^{i(\omega t - kz)}.$$

The solution that vanishes for $r = 0$ is

$$F = \text{constant} \times J_1[kr\sqrt{\{(4\Omega^2/\omega^2)-1\}}], \tag{5}$$

where J_1 is a Bessel function of order 1.

The motion comprises regions between coaxial cylinders with radius r_n such that

$$kr_n\sqrt{\{(4\Omega^2/\omega^2)-1\}} = x_n,$$

where $x_1, x_2, \ldots$ are the successive zeros of $J_1(x)$. On these cylindrical surfaces $v_r = 0$, and the fluid therefore does not cross them.

For these waves in an infinite fluid, ω is independent of k. The possible values of the frequency are, however, restricted by the condition $\omega < 2\Omega$; if this is not satisfied, (4) has no solution satisfying the necessary conditions of finiteness.

If the rotating fluid is bounded by a cylindrical wall with radius R, we have to use the condition $v_r = 0$ at the wall. This gives the relation

$$ka\sqrt{\{(4\Omega^2/\omega^2)-1\}} = x_n$$

between ω and k for a wave with a given n (the number of coaxial regions in it).

Problem 2. Derive an equation describing an arbitrary small perturbation of the pressure in a rotating fluid.

Solution. Equation (14.3) in components is

$$\frac{\partial v_x}{\partial t} - 2\Omega v_y = -\frac{1}{\rho}\frac{\partial p'}{\partial x}, \qquad \frac{\partial v_y}{\partial t} + 2\Omega v_x = -\frac{1}{\rho}\frac{\partial p'}{\partial y}, \qquad \frac{\partial v_z}{\partial t} = -\frac{1}{\rho}\frac{\partial p'}{\partial z}. \tag{1}$$

Differentiating these with respect to x, y, and z, adding, and using div $\mathbf{v} = 0$, we find

$$\frac{1}{\rho}\triangle p' = 2\Omega\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right).$$

Differentiation with respect to t, again using equations (1), gives

$$\frac{1}{\rho}\frac{\partial}{\partial t}\triangle p' = 4\Omega^2\frac{\partial v_z}{\partial z},$$

and by a further differentiation with respect to t we arrive at the final equation

$$\frac{\partial^2}{\partial t^2}\triangle p' + 4\Omega^2\frac{\partial^2 p'}{\partial z^2} = 0. \tag{2}$$

For periodic perturbations with frequency ω, this becomes

$$\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} + \left(1 - \frac{4\Omega^2}{\omega^2}\right)\frac{\partial^2 p'}{\partial z^2} = 0. \tag{3}$$

For waves having the form (14.5), this of course gives the known dispersion relation (14.8), with $\omega < 2\Omega$ and a negative coefficient of $\partial^2 p'/\partial z^2$ in (3). Perturbations from a point source are propagated along generators of a cone whose axis is along $\mathbf{\Omega}$ and whose vertical angle is 2θ, where $\sin\theta = \omega/2\Omega$.

When $\omega > 2\Omega$, the coefficient of $\partial^2 p'/\partial z^2$ in (3) is positive, and this equation becomes Laplace's equation by an obvious change in the z scale. In this case, a point source of perturbation affects the whole volume of the fluid, to an extent that decreases away from the source according to a power law.

CHAPTER II

VISCOUS FLUIDS

§15. The equations of motion of a viscous fluid

Let us now study the effect of energy dissipation, occurring during the motion of a fluid, on that motion itself. This process is the result of the thermodynamic irreversibility of the motion. This irreversibility always occurs to some extent, and is due to internal friction (viscosity) and thermal conduction.

In order to obtain the equations describing the motion of a viscous fluid, we have to include some additional terms in the equation of motion of an ideal fluid. The equation of continuity, as we see from its derivation, is equally valid for any fluid, whether viscous or not. Euler's equation, on the other hand, requires modification.

We have seen in §7 that Euler's equation can be written in the form

$$\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial \Pi_{ik}}{\partial x_k},$$

where Π_{ik} is the momentum flux density tensor. The momentum flux given by formula (7.2) represents a completely reversible transfer of momentum, due simply to the mechanical transport of the different particles of fluid from place to place and to the pressure forces acting in the fluid. The viscosity (internal friction) causes another, irreversible, transfer of momentum from points where the velocity is large to those where it is small.

The equation of motion of a viscous fluid may therefore be obtained by adding to the "ideal" momentum flux (7.2) a term $-\sigma'_{ik}$ which gives the irreversible "viscous" transfer of momentum in the fluid. Thus we write the momentum flux density tensor in a viscous fluid in the form

$$\Pi_{ik} = p\delta_{ik} + \rho v_i v_k - \sigma'_{ik} = -\sigma_{ik} + \rho v_i v_k. \tag{15.1}$$

The tensor

$$\sigma_{ik} = -p\delta_{ik} + \sigma'_{ik} \tag{15.2}$$

is called the *stress tensor*, and σ'_{ik} the *viscous stress tensor*. σ_{ik} gives the part of the momentum flux that is not due to the direct transfer of momentum with the mass of moving fluid.†

The general form of the tensor σ'_{ik} can be established as follows. Processes of internal friction occur in a fluid only when different fluid particles move with different velocities, so that there is a relative motion between various parts of the fluid. Hence σ'_{ik} must depend on the space derivatives of the velocity. If the velocity gradients are small, we may suppose

† We shall see below that σ'_{ik} contains a term proportional to δ_{ik}, i.e. of the same form as the term $p\delta_{ik}$. When the momentum flux tensor is put in such a form, therefore, we should specify what is meant by the pressure p; see the end of §49.

that the momentum transfer due to viscosity depends only on the first derivatives of the velocity. To the same approximation, σ'_{ik} may be supposed a linear function of the derivatives $\partial v_i/\partial x_k$. There can be no terms in σ'_{ik} independent of $\partial v_i/\partial x_k$, since σ'_{ik} must vanish for $\mathbf{v}$ = constant. Next, we notice that σ'_{ik} must also vanish when the whole fluid is in uniform rotation, since it is clear that in such a motion no internal friction occurs in the fluid. In uniform rotation with angular velocity $\mathbf{\Omega}$, the velocity $\mathbf{v}$ is equal to the vector product $\mathbf{\Omega}\times\mathbf{r}$. The sums

$$\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}$$

are linear combinations of the derivatives $\partial v_i/\partial x_k$, and vanish when $\mathbf{v}=\mathbf{\Omega}\times\mathbf{r}$. Hence σ'_{ik} must contain just these symmetrical combinations of the derivatives $\partial v_i/\partial x_k$.

The most general tensor of rank two satisfying the above conditions is

$$\sigma'_{ik}=\eta\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}-\tfrac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right)+\zeta\delta_{ik}\frac{\partial v_l}{\partial x_l}, \tag{15.3}$$

with coefficients η and ζ independent of the velocity. In making this statement we use the fact that the fluid is isotropic, as a result of which its properties must be described by scalar quantities only (in this case, η and ζ). The terms in (15.3) are arranged so that the expression in parentheses has the property of vanishing on contraction with respect to i and k.† The constants η and ζ are called *coefficients of viscosity*, and ζ often the *second viscosity*. As we shall show in §§16 and 49, they are both positive:

$$\eta>0, \qquad \zeta>0. \tag{15.4}$$

The equations of motion of a viscous fluid can now be obtained by simply adding the expressions $\partial\sigma'_{ik}/\partial x_k$ to the right-hand side of Euler's equation

$$\rho\left(\frac{\partial v_i}{\partial t}+v_k\frac{\partial v_i}{\partial x_k}\right)=-\frac{\partial p}{\partial x_i}.$$

Thus we have

$$\rho\left(\frac{\partial v_i}{\partial t}+v_k\frac{\partial v_i}{\partial x_k}\right)=-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_k}\left\{\eta\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}-\tfrac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right)\right\}+\frac{\partial}{\partial x_i}\left(\zeta\frac{\partial v_l}{\partial x_l}\right). \tag{15.5}$$

This is the most general form of the equations of motion of a viscous fluid. The quantities η and ζ are functions of pressure and temperature. In general, p and T, and therefore η and ζ, are not constant throughout the fluid, so that η and ζ cannot be taken outside the gradient operator.

In most cases, however, the viscosity coefficients do not change noticeably in the fluid, and they may be regarded as constant. We then have equations (15.5), in vector form, as

$$\rho\left[\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}\right]=-\mathbf{grad}\,p+\eta\triangle\mathbf{v}+(\zeta+\tfrac{1}{3}\eta)\mathbf{grad}\,\mathrm{div}\,\mathbf{v}. \tag{15.6}$$

This is called the *Navier–Stokes equation*. It becomes considerably simpler if the fluid may be regarded as incompressible, so that div $\mathbf{v}=0$, and the last term on the right of (15.6)

† That is, on taking the sum of the components with $i=k$.

is zero. In discussing viscous fluids, we shall almost always regard them as incompressible, and accordingly use the equation of motion in the form†

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \mathbf{grad})\mathbf{v} = -\frac{1}{\rho}\,\mathbf{grad}\,p + \frac{\eta}{\rho}\triangle\,\mathbf{v}. \tag{15.7}$$

The stress tensor in an incompressible fluid takes the simple form

$$\sigma_{ik} = -p\delta_{ik} + \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right). \tag{15.8}$$

We see that the viscosity of an incompressible fluid is determined by only one coefficient. Since most fluids may be regarded as practically incompressible, it is this viscosity coefficient η which is generally of importance. The ratio

$$\nu = \eta/\rho \tag{15.9}$$

is called the *kinematic viscosity* (while η itself is called the *dynamic viscosity*). We give below the values of η and ν for various fluids, at a temperature of 20° C:

	η (g/cm sec)	ν (cm^2/sec)
Water	0·010	0·010
Air	0·00018	0·150
Alcohol	0·018	0·022
Glycerine	8·5	6·8
Mercury	0·0156	0·0012

It may be mentioned that the dynamic viscosity of a gas at a given temperature is independent of the pressure. The kinematic viscosity, however, is inversely proportional to the pressure.

The pressure can be eliminated from equation (15.7) in the same way as from Euler's equation. Taking the curl of both sides, we obtain, instead of equation (2.11) as for an ideal fluid,

$$\frac{\partial}{\partial t}(\mathbf{curl}\,\mathbf{v}) = \mathbf{curl}\,(\mathbf{v} \times \mathbf{curl}\,\mathbf{v}) + \nu\triangle\,(\mathbf{curl}\,\mathbf{v})$$

Since the fluid is incompressible, the equation can be transformed by expanding the product in the first term on the right and using the equation div $\mathbf{v} = 0$:

$$\frac{\partial}{\partial t}(\mathbf{curl} \cdot \mathbf{v}) + (\mathbf{v} \cdot \mathbf{grad})\,\mathbf{curl}\,\mathbf{v} - (\mathbf{curl}\,\mathbf{v} \cdot \mathbf{grad})\,\mathbf{v}$$
$$= \nu\triangle\,\mathbf{curl}\,\mathbf{v}. \tag{15.10}$$

† Equation (15.7) was first stated as a result of studies on models by C. L. Navier (1827). A derivation, similar to the modern one, for equations (15.6) (without the ζ term) and (15.7) was given by G. G. Stokes (1845).

When the velocity distribution is known, the pressure distribution in the fluid can be found by solving the Poisson-type equation

$$\triangle p = -\rho \frac{\partial v_i}{\partial x_k}\frac{\partial v_k}{\partial x_i} = -\rho \frac{\partial^2 v_i v_k}{\partial x_k \partial x_i}, \tag{15.11}$$

which is obtained by taking the divergence of (15.7).

We may also give the equation satisfied by the stream function $\psi(x,y)$ in two-dimensional flow of an incompressible viscous fluid. It is derived by substituting (10.9) in (15.10):

$$\frac{\partial}{\partial t}\triangle\psi - \frac{\partial \psi}{\partial x}\frac{\partial \triangle \psi}{\partial y} + \frac{\partial \psi}{\partial y}\frac{\partial \triangle \psi}{\partial x} - \nu \triangle\triangle\psi = 0. \tag{15.12}$$

We must also write down the boundary conditions on the equations of motion of a viscous fluid. There are always forces of molecular attraction between a viscous fluid and the surface of a solid body, and these forces have the result that the layer of fluid immediately adjacent to the surface is brought completely to rest, and "adheres" to the surface. Accordingly, the boundary conditions on the equations of motion of a viscous fluid require that the fluid velocity should vanish at fixed solid surfaces:

$$\mathbf{v} = 0. \tag{15.13}$$

It should be emphasized that both the normal and the tangential velocity component must vanish, whereas for an ideal fluid the boundary conditions require only the vanishing of v_n.†

In the general case of a moving surface, the velocity **v** must be equal to the velocity of the surface.

It is easy to write down an expression for the force acting on a solid surface bounding the fluid. The force acting on an element of the surface is just the momentum flux through this element. The momentum flux through the surface element **df** is

$$\Pi_{ik}\mathrm{d}f_k = (\rho v_i v_k - \sigma_{ik})\mathrm{d}f_k.$$

Writing $\mathrm{d}f_k$ in the form $\mathrm{d}f_k = n_k \mathrm{d}f$, where **n** is a unit vector along the normal, and recalling that $\mathbf{v} = 0$ at a solid surface,‡ we find that the force **P** acting on unit surface area is

$$P_i = -\sigma_{ik} n_k = p n_i - \sigma'_{ik} n_k. \tag{15.14}$$

The first term is the ordinary pressure of the fluid, while the second is the force of friction, due to the viscosity, acting on the surface. We must emphasize that **n** in (15.14) is a unit vector along the outward normal to the fluid, i.e. along the inward normal to the solid surface.

† We may note that, in general, Euler's equations cannot be satisfied with the extra boundary condition (in comparison with the case of an ideal fluid) that the tangential velocity be zero. Mathematically, this occurs becuase the equation is first-order in the derivatives with respect to the coordinates, whereas the Navier–Stokes equation is second-order.

‡ In determining the force acting on the surface, each surface element must be considered in a frame of reference in which it is at rest. The force is equal to the momentum flux only when the surface is fixed.

If we have a surface of separation between two immiscible fluids, the conditions at the surface are that the velocities of the fluids must be equal and the forces which they exert on each other must be equal and opposite. The latter condition is written

$$n_{1,k}\sigma_{1,ik} + n_{2,k}\sigma_{2,ik} = 0,$$

where the suffixes 1 and 2 refer to the two fluids. The normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are in opposite directions, i.e. $\mathbf{n}_1 = -\mathbf{n}_2 \equiv \mathbf{n}$, so that we can write

$$n_i\sigma_{1,ik} = n_i\sigma_{2,ik}. \tag{15.15}$$

At a free surface of the fluid the condition

$$\sigma_{ik}n_k \equiv \sigma'_{ik}n_k - pn_i = 0 \tag{15.16}$$

must hold.

Equations of motion in curvilinear coordinates

We give below, for reference, the equations of motion for a viscous incompressible fluid in frequently used curvilinear coordinates. In cylindrical polar coordinates r, ϕ, z the components of the stress tensor are

$$\begin{aligned}
\sigma_{rr} &= -p + 2\eta\frac{\partial v_r}{\partial r}, & \sigma_{r\phi} &= \eta\left(\frac{1}{r}\frac{\partial v_r}{\partial \phi} + \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r}\right),\\
\sigma_{\phi\phi} &= -p + 2\eta\left(\frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r}\right), & \sigma_{\phi z} &= \eta\left(\frac{\partial v_\phi}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \phi}\right),\\
\sigma_{zz} &= -p + 2\eta\frac{\partial v_z}{\partial z}, & \sigma_{zr} &= \eta\left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right).
\end{aligned} \tag{15.17}$$

The three components of the Navier–Stokes equation are

$$\begin{aligned}
\frac{\partial v_r}{\partial t} + (\mathbf{v}\cdot\mathbf{grad})v_r - \frac{v_\phi^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\triangle v_r - \frac{2}{r^2}\frac{\partial v_\phi}{\partial \phi} - \frac{v_r}{r^2}\right),\\
\frac{\partial v_\phi}{\partial t} + (\mathbf{v}\cdot\mathbf{grad})v_\phi + \frac{v_r v_\phi}{r} &= -\frac{1}{\rho r}\frac{\partial p}{\partial \phi} + \nu\left(\triangle v_\phi + \frac{2}{r^2}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r^2}\right),\\
\frac{\partial v_z}{\partial t} + (\mathbf{v}\cdot\mathbf{grad})v_z &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\triangle v_z,
\end{aligned} \tag{15.18}$$

where

$$(\mathbf{v}\cdot\mathbf{grad})f = v_r\frac{\partial f}{\partial r} + \frac{v_\phi}{r}\frac{\partial f}{\partial \phi} + v_z\frac{\partial f}{\partial z},$$

$$\triangle f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}.$$

The equation of continuity is

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} = 0. \tag{15.19}$$

In spherical polar coordinates r, ϕ, θ we have for the stress tensor

$$\begin{aligned}
\sigma_{rr} &= -p+2\eta\frac{\partial v_r}{\partial r},\\
\sigma_{\phi\phi} &= -p+2\eta\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}+\frac{v_r}{r}+\frac{v_\theta\cot\theta}{r}\right),\\
\sigma_{\theta\theta} &= -p+2\eta\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}\right),\\
\sigma_{r\theta} &= \eta\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta}+\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right),\\
\sigma_{\theta\phi} &= \eta\left(\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}+\frac{1}{r}\frac{\partial v_\phi}{\partial\theta}-\frac{v_\phi\cot\theta}{r}\right),\\
\sigma_{\phi r} &= \eta\left(\frac{\partial v_\phi}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi}-\frac{v_\phi}{r}\right),
\end{aligned} \tag{15.20}$$

while the Navier–Stokes equations are

$$\begin{aligned}
&\frac{\partial v_r}{\partial t}+(\mathbf{v}\cdot\mathbf{grad})v_r-\frac{v_\theta^2+v_\phi^2}{r}\\
&\qquad= -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left[\triangle v_r-\frac{2}{r^2\sin^2\theta}\frac{\partial(v_\theta\sin\theta)}{\partial\theta}-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}-\frac{2v_r}{r^2}\right],\\
&\frac{\partial v_\theta}{\partial t}+(\mathbf{v}\cdot\mathbf{grad})v_\theta+\frac{v_r v_\theta}{r}-\frac{v_\phi^2\cot\theta}{r}\\
&\qquad= -\frac{1}{\rho r}\frac{\partial p}{\partial\theta}+\nu\left[\triangle v_\theta-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial\phi}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r^2\sin^2\theta}\right],\\
&\frac{\partial v_\phi}{\partial t}+(\mathbf{v}\cdot\mathbf{grad})v_\phi+\frac{v_r v_\phi}{r}+\frac{v_\theta v_\phi\cot\theta}{r}\\
&\qquad= -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\phi}+\nu\left[\triangle v_\phi+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial\phi}-\frac{v_\phi}{r^2\sin^2\theta}\right],
\end{aligned} \tag{15.21}$$

where

$$(\mathbf{v}\cdot\mathbf{grad})f = v_r\frac{\partial f}{\partial r}+\frac{v_\theta}{r}\frac{\partial f}{\partial\theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial f}{\partial\phi},$$

$$\triangle f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} = 0.$$

The equation of continuity is

$$\frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(v_\theta\sin\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} = 0. \tag{15.22}$$

§16. Energy dissipation in an incompressible fluid

The presence of viscosity results in the dissipation of energy, which is finally transformed into heat. The calculation of the energy dissipation is especially simple for an incompressible fluid.

The total kinetic energy of an incompressible fluid is

$$E_{\text{kin}} = \tfrac{1}{2}\rho \int v^2 \, \mathrm{d}V.$$

We take the time derivative of this energy, writing $\partial(\frac{1}{2}\rho v^2)/\partial t = \rho v_i \partial v_i/\partial t$ and substituting for $\partial v_i/\partial t$ the expression for it given by the Navier–Stokes equation:

$$\frac{\partial v_i}{\partial t} = -v_k \frac{\partial v_i}{\partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial \sigma'_{ik}}{\partial x_k}.$$

The result is

$$\begin{aligned}\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2) &= -\rho \mathbf{v}\cdot(\mathbf{v}\cdot\mathbf{grad})\mathbf{v} - \mathbf{v}\cdot\mathbf{grad}\, p + v_i \frac{\partial \sigma'_{ik}}{\partial x_k} \\ &= -\rho(\mathbf{v}\cdot\mathbf{grad})\left(\tfrac{1}{2}v^2 + \frac{p}{\rho}\right) + \operatorname{div}(\mathbf{v}\cdot\boldsymbol{\sigma}') - \sigma'_{ik}\frac{\partial v_i}{\partial x_k}.\end{aligned}$$

Here $\mathbf{v}\cdot\boldsymbol{\sigma}'$ denotes the vector whose components are $v_i\sigma'_{ik}$. Since $\operatorname{div}\mathbf{v} = 0$ for an incompressible fluid, we can write the first term on the right as a divergence:

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2) = -\operatorname{div}\left[\rho\mathbf{v}\left(\tfrac{1}{2}v^2 + \frac{p}{\rho}\right) - \mathbf{v}\cdot\boldsymbol{\sigma}'\right] - \sigma'_{ik}\frac{\partial v_i}{\partial x_k}. \tag{16.1}$$

The expression in brackets is just the energy flux density in the fluid: the term $\rho\mathbf{v}(\frac{1}{2}v^2 + p/\rho)$ is the energy flux due to the actual transfer of fluid mass, and is the same as the energy flux in an ideal fluid (see (10.5)). The second term, $\mathbf{v}\cdot\boldsymbol{\sigma}'$, is the energy flux due to processes of internal friction. For the presence of viscosity results in a momentum flux σ'_{ik}; a transfer of momentum, however, always involves a transfer of energy, and the energy flux is clearly equal to the scalar product of the momentum flux and the velocity.

If we integrate (16.1) over some volume V, we obtain

$$\frac{\partial}{\partial t}\int \tfrac{1}{2}\rho v^2 \,\mathrm{d}V = -\oint\left[\rho\mathbf{v}\left(\tfrac{1}{2}v^2 + \frac{p}{\rho}\right) - \mathbf{v}\cdot\boldsymbol{\sigma}'\right]\cdot \mathrm{d}\mathbf{f} - \int \sigma'_{ik}\frac{\partial v_i}{\partial x_k}\mathrm{d}V. \tag{16.2}$$

The first term on the right gives the rate of change of the kinetic energy of the fluid in V owing to the energy flux through the surface bounding V. The integral in the second term is consequently the decrease per unit time in the kinetic energy owing to dissipation.

If the integration is extended to the whole volume of the fluid, the surface integral vanishes (since the velocity vanishes at infinity†), and we find the energy dissipated per unit time in the whole fluid to be

$$\dot{E}_{\text{kin}} = -\int \sigma'_{ik}\frac{\partial v_i}{\partial x_k}\mathrm{d}V = -\tfrac{1}{2}\int \sigma'_{ik}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)\mathrm{d}V,$$

† We are considering the motion of the fluid in a system of coordinates such that the fluid is at rest at infinity. Here, and in similar cases, we speak, for the sake of definiteness, of an infinite volume of fluid, but this implies no loss of generality. For a fluid enclosed in a finite volume, the surface integral again vanishes, because the velocity at the surface vanishes.

since the tensor σ'_{ik} is symmetrical. In incompressible fluids, the tensor σ'_{ik} is given by (15.8), so that we have finally for the energy dissipation in an incompressible fluid

$$\dot{E}_{\text{kin}} = -\tfrac{1}{2}\eta \int \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)^2 \mathrm{d}V. \tag{16.3}$$

The dissipation leads to a decrease in the mechanical energy, i.e. we must have $\dot{E}_{\text{kin}} < 0$. The integral in (16.3), however, is always positive. We therefore conclude that the viscosity coefficient η is always positive.

PROBLEM

Transform the integral (16.3) for potential flow into an integral over the surface bounding the region of flow.

SOLUTION. Putting $\partial v_i/\partial x_k = \partial v_k/\partial x_i$ and integrating once by parts, we find

$$\dot{E}_{\text{kin}} = -2\eta \int \left(\frac{\partial v_i}{\partial x_k}\right)^2 \mathrm{d}V = -2\eta \int v_i \frac{\partial v_i}{\partial x_k} \mathrm{d}f_k,$$

or

$$\dot{E}_{\text{kin}} = -\eta \int \mathbf{grad}\, v^2 \cdot \mathrm{d}\mathbf{f}.$$

§17. Flow in a pipe

We shall now consider some simple problems of motion of an incompressible viscous fluid.

Let the fluid be enclosed between two parallel planes moving with a constant relative velocity **u**. We take one of these planes as the xz-plane, with the x-axis in the direction of **u**. It is clear that all quantities depend only on y, and that the fluid velocity is everywhere in the x-direction. We have from (15.7) for steady flow

$$\mathrm{d}p/\mathrm{d}y = 0, \quad \mathrm{d}^2 v/\mathrm{d}y^2 = 0.$$

(The equation of continuity is satisfied identically.) Hence p = constant, $v = ay + b$. For $y = 0$ and $y = h$ (h being the distance between the planes) we must have respectively $v = 0$ and $v = u$. Thus

$$v = yu/h. \tag{17.1}$$

The fluid velocity distribution is therefore linear. The mean fluid velocity is

$$\bar{v} = \frac{1}{h}\int_0^h v\,\mathrm{d}y = \tfrac{1}{2}u. \tag{17.2}$$

From (15.14) we find that the normal component of the force on either plane is just p, as it should be, while the tangential friction force on the plane $y = 0$ is

$$\sigma_{xy} = \eta\,\mathrm{d}v/\mathrm{d}y = \eta u/h; \tag{17.3}$$

the force on the plane $y = h$ is $-\eta u/h$.

Next, let us consider steady flow between two fixed parallel planes in the presence of a pressure gradient. We choose the coordinates as before; the x-axis is in the direction of motion of the fluid. The Navier–Stokes equations give, since the velocity clearly depends only on y,

$$\frac{\partial^2 v}{\partial y^2} = \frac{1}{\eta}\frac{\partial p}{\partial x}, \qquad \frac{\partial p}{\partial y} = 0.$$

The second equation shows that the pressure is independent of y, i.e. it is constant across the depth of the fluid between the planes. The right-hand side of the first equation is therefore a function of x only, while the left-hand side is a function of y only; this can be true only if both sides are constant. Thus $\mathrm{d}p/\mathrm{d}x$ = constant, i.e. the pressure is a linear function of the coordinate x along the direction of flow. For the velocity we now obtain

$$v = \frac{1}{2\eta}\frac{\mathrm{d}p}{\mathrm{d}x}y^2 + ay + b.$$

The constants a and b are determined from the boundary conditions, $v = 0$ for $y = 0$ and $y = h$. The result is

$$v = -\frac{1}{2\eta}\frac{\mathrm{d}p}{\mathrm{d}x}y(y-h). \tag{17.4}$$

Thus the velocity varies parabolically across the fluid, reaching its maximum value in the middle. The mean fluid velocity (averaged over the depth of the fluid) is

$$\bar{v} = -\frac{h^2}{12\eta}\frac{\mathrm{d}p}{\mathrm{d}x}. \tag{17.5}$$

The frictional force acting on one of the fixed planes is

$$\sigma_{xy} = \eta(\partial v/\partial y)_{y=0} = -\tfrac{1}{2}h\,\mathrm{d}p/\mathrm{d}x. \tag{17.6}$$

Finally, let us consider steady flow in a pipe with arbitrary cross-section (the same along the whole length of the pipe, however). We take the axis of the pipe as the x-axis. The fluid velocity is evidently along the x-axis at all points, and is a function of y and z only. The equation of continuity is satisfied identically, while the y and z components of the Navier–Stokes equation again give $\partial p/\partial y = \partial p/dz = 0$, i.e. the pressure is constant over the cross-section of the pipe. The x-component of equation (15.7) gives

$$\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = \frac{1}{\eta}\frac{\mathrm{d}p}{\mathrm{d}x}. \tag{17.7}$$

Hence we again conclude that $\mathrm{d}p/\mathrm{d}x$ = constant; the pressure gradient may therefore be written $-\Delta p/l$, where Δp is the pressure difference between the ends of the pipe and l is its length.

Thus the velocity distribution for flow in a pipe is determined by a two-dimensional equation of the form $\triangle v$ = constant. This equation has to be solved with the boundary condition $v = 0$ at the circumference of the cross-section of the pipe. We shall solve the equation for a pipe with circular cross-section. Taking the origin at the centre of the circle and using polar coordinates, we have by symmetry $v = v(r)$. Using the expression for the Laplacian in polar coordinates, we have

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}v}{\mathrm{d}r}\right) = -\frac{\Delta p}{\eta l}.$$

Integration gives

$$v = -\frac{\Delta p}{4\eta l}r^2 + a\log r + b. \tag{17.8}$$

The constant a must be put equal to zero, since the velocity must remain finite at the centre of the pipe. The constant b is determined from the requirement that $v = 0$ for $r = R$, where R is the radius of the pipe. We then find

$$v = \frac{\Delta p}{4\eta l}(R^2 - r^2). \tag{17.9}$$

Thus the velocity distribution across the pipe is parabolic.

It is easy to determine the mass Q of fluid passing per unit time through any cross-section of the pipe (called the *discharge*). A mass $\rho \cdot 2\pi r v\, dr$ passes per unit time through an annular element $2\pi r\, dr$ of the cross-sectional area. Hence

$$Q = 2\pi\rho \int_0^R rv\, dr.$$

Using (17.9), we obtain

$$Q = \frac{\pi \Delta p}{8\nu l} R^4. \tag{17.10}$$

The mass of fluid is thus proportional to the fourth power of the radius of the pipe.†

PROBLEMS

PROBLEM 1. Determine the flow in a pipe of annular cross-section, the internal and external radii being R_1, R_2.

SOLUTION. Determining the constants a and b in the general solution (17.8) from the conditions that $v = 0$ for $r = R_1$ and $r = R_2$, we find

$$v = \frac{\Delta p}{4\eta l}\left[R_2{}^2 - r^2 + \frac{R_2{}^2 - R_1{}^2}{\log(R_2/R_1)}\log\frac{r}{R_2}\right].$$

The discharge is

$$Q = \frac{\pi\Delta p}{8\nu l}\left[R_2{}^4 - R_1{}^4 - \frac{(R_2{}^2 - R_1{}^2)^2}{\log(R_2/R_1)}\right].$$

PROBLEM 2. The same as Problem 1, but for a pipe of elliptical cross-section.

SOLUTION. We seek a solution of equation (17.7) in the form $v = Ay^2 + Bz^2 + C$. The constants A, B, C are determined from the requirement that this expression must satisfy the boundary condition $v = 0$ on the circumference of the ellipse (i.e. $Ay^2 + Bz^2 + C = 0$ must be the same as the equation $y^2/a^2 + z^2/b^2 = 1$, where a and b are the semi-axes of the ellipse). The result is

$$v = \frac{\Delta p}{2\eta l}\frac{a^2b^2}{a^2 + b^2}\left(1 - \frac{y^2}{a^2} - \frac{z^2}{b^2}\right).$$

† The dependence of Q on Δp and R given by this formula was established empirically by G. Hagen (1839) and J. L. M. Poiseuille (1840) and theoretically justified by G. G. Stokes (1845).

Parallel viscous flow between fixed walls is often called *Poiseuille flow* in the literature; equation (17.4) relates to two-dimensional Poiseuille flow.

The discharge is

$$Q = \frac{\pi \Delta p}{4\nu l} \frac{a^3 b^3}{a^2 + b^2}.$$

Problem 3. The same as Problem 1, but for a pipe whose cross-section is an equilateral triangle with side a.

Solution. The solution of equation (17.7) which vanishes on the bounding triangle is

$$v = \frac{\Delta p}{l} \frac{2}{\sqrt{3} a \eta} h_1 h_2 h_3,$$

where h_1, h_2, h_3 are the lengths of the perpendiculars from a given point in the triangle to its three sides. For each of the expressions $\triangle h_1, \triangle h_2, \triangle h_3$ (where $\triangle = \partial^2/\partial z^2 + \partial^2/\partial y^2$) is zero; this is seen at once from the fact that each of the perpendiculars h_1, h_2, h_3 may be taken as the axis of y or z, and the result of applying the Laplacian to a coordinate is zero. We therefore have

$$\triangle (h_1 h_2 h_3) = 2(h_1 \, \mathbf{grad}\, h_2 \cdot \mathbf{grad}\, h_3 + h_2 \, \mathbf{grad}\, h_3 \cdot \mathbf{grad}\, h_1 + h_3 \, \mathbf{grad}\, h_1 \cdot \mathbf{grad}\, h_2)$$

But $\mathbf{grad}\, h_1 = \mathbf{n}_1$, $\mathbf{grad}\, h_2 = \mathbf{n}_2$, $\mathbf{grad}\, h_3 = \mathbf{n}_3$, where $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are unit vectors along the perpendiculars h_1, h_2, h_3. Any two of $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are at an angle $2\pi/3$, so that $\mathbf{grad}\, h_1 \cdot \mathbf{grad}\, h_2 = \mathbf{n}_1 \cdot \mathbf{n}_2 = \cos(2\pi/3) = -\frac{1}{2}$, and so on. We thus obtain the relation

$$\triangle (h_1 h_2 h_3) = -(h_1 + h_2 + h_3) = -\tfrac{1}{2}\sqrt{3} a,$$

and we see that equation (17.7) is satisfied. The discharge is

$$Q = \frac{\sqrt{3} a^4 \Delta p}{320 \nu l}.$$

Problem 4. A cylinder with radius R_1 moves parallel to its axis with velocity u inside a coaxial cylinder with radius R_2. Determine the motion of a fluid occupying the space between the cylinders.

Solution. We take cylindrical polar coordinates, with the z-axis along the axis of the cylinders. The velocity is everywhere along the z-axis and depends only on r (as does the pressure): $v_z = v(r)$. We obtain for v the equation

$$\triangle v = \frac{1}{r} \frac{d}{dr}\left(r \frac{dv}{dr} \right) = 0;$$

the term $(\mathbf{v} \cdot \mathbf{grad})\mathbf{v} = v\, \partial v/\partial z$ vanishes identically. Using the boundary conditions $v = u$ for $r = R_1$ and $v = 0$ for $r = R_2$, we find

$$v = u \frac{\log(r/R_2)}{\log(R_1/R_2)}.$$

The frictional force per unit length of either cylinder is $2\pi \eta u / \log(R_2/R_1)$.

Problem 5. A layer of fluid with thickness h is bounded above by a free surface and below by a fixed plane inclined at an angle α to the horizontal. Determine the flow due to gravity.

Solution. We take the fixed plane as the xy-plane, with the x-axis in the direction of flow (Fig. 6). We seek a solution depending only on z. The Navier–Stokes equations with $v_x = v(z)$ in a gravitational field are

$$\eta \frac{d^2 v}{dz^2} + \rho g \sin \alpha = 0, \qquad \frac{dp}{dz} + \rho g \cos \alpha = 0.$$

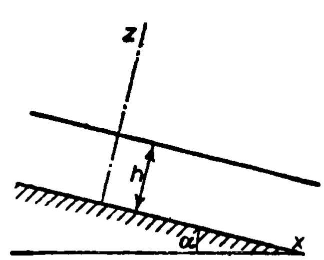

Fig. 6

At the free surface ($z = h$) we must have $\sigma_{xz} = \eta\, dv/dz = 0$, $\sigma_{zz} = -p = -p_0$ (p_0 being the atmospheric pressure). For $z = 0$ we must have $v = 0$. The solution satisfying these conditions is

$$p = p_0 + \rho g(h-z)\cos\alpha, \qquad v = \frac{\rho g \sin\alpha}{2\eta} z(2h-z).$$

The discharge, per unit length in the y-direction, is

$$Q = \rho \int_0^h v\,dz = \frac{\rho g h^3 \sin\alpha}{3\nu}.$$

PROBLEM 6. Determine the way in which the pressure falls along a tube of circular cross-section in which a viscous perfect gas is flowing isothermally (bearing in mind that the dynamic viscosity η of a perfect gas is independent of the pressure).

SOLUTION. Over any short section of the pipe the gas may be supposed incompressible, provided that the pressure gradient is not too great, and we can therefore use formula (17.10), according to which

$$-\frac{dp}{dx} = \frac{8\eta Q}{\pi\rho R^4}.$$

Over greater distances, however, ρ varies, and the pressure is not a linear function of x. According to the equation of state, the gas density $\rho = mp/T$, where m is the mass of a molecule, so that

$$-\frac{dp}{dx} = \frac{8\eta QT}{\pi m R^4}\cdot\frac{1}{p}.$$

(The discharge Q of the gas through the tube is obviously the same, whether or not the gas is incompressible.) From this we find

$$p_2^{\,2} - p_1^{\,2} = \frac{16\eta QT}{\pi m R^4} l,$$

where p_2, p_1 are the pressures at the ends of a section of the tube with length l.

§18. Flow between rotating cylinders

Let us now consider the motion of a fluid between two infinite coaxial cylinders with radii R_1, R_2 ($R_2 > R_1$), rotating about their axis with angular velocities Ω_1, Ω_2.† We take cylindrical polar coordinates r, ϕ, z, with the z-axis along the axis of the cylinders. It is evident from symmetry that

$$v_z = v_r = 0, \qquad v_\phi = v(r), \qquad p = p(r).$$

The Navier–Stokes equation in cylindrical polar coordinates gives in this case two equations:

$$dp/dr = \rho v^2/r, \tag{18.1}$$

$$\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2} = 0. \tag{18.2}$$

The latter equation has solutions of the form r^n; substitution gives $n = \pm 1$, so that

$$v = ar + \frac{b}{r}.$$

The constants a and b are found from the boundary conditions, according to which the fluid velocity at the inner and outer cylindrical surfaces must be equal to that of the

† Flow between rotating cylinders is often called *Couette flow* in the literature (M. Couette 1890). In the limit $R_1 \to R_2$, it becomes the flow (17.1) between moving parallel planes, referred to as two-dimensional Couette flow.

corresponding cylinder: $v = R_1\Omega_1$ for $r = R_1$, $v = R_2\Omega_2$ for $r = R_2$. As a result we find the velocity distribution to be

$$v = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} r + \frac{(\Omega_1 - \Omega_2) R_1^2 R_2^2}{R_2^2 - R_1^2} \frac{1}{r}. \tag{18.3}$$

The pressure distribution is then found from (18.1) by straightforward integration.

For $\Omega_1 = \Omega_2 = \Omega$ we have simply $v = \Omega r$, i.e. the fluid rotates rigidly with the cylinders. When the outer cylinder is absent ($\Omega_2 = 0$, $R_2 = \infty$) we have $v = \Omega_1 R_1^2/r$.

Let us also determine the moment of the frictional forces acting on the cylinders. The frictional force acting on unit area of the inner cylinder is along the tangent to the surface and, from (15.14), is equal to the component $\sigma'_{r\phi}$ of the stress tensor. Using formulae (15.17), we find

$$\begin{aligned}[\sigma'_{r\phi}]_{r=R_1} &= \eta\left[\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)\right]_{r=R_1} \\ &= -2\eta\frac{(\Omega_1 - \Omega_2)R_2^2}{R_2^2 - R_1^2}.\end{aligned}$$

The moment of this force is found by multiplying by R_1, and the total moment M_1 acting on unit length of the cylinder by multiplying the result by $2\pi R_1$. We thus have

$$M_1 = -\frac{4\pi\eta(\Omega_1 - \Omega_2)R_1^2 R_2^2}{R_2^2 - R_1^2}. \tag{18.4}$$

The moment of the forces acting on the outer cylinder is $M_2 = -M_1$. When $\Omega_2 = 0$ and the gap between the cylinders is small ($\delta \equiv R_2 - R_1 \ll R_2$), (18.4) becomes

$$M_2 = \eta RSu/\delta, \tag{18.5}$$

where $S \cong 2\pi R$ is the surface area of the cylinder per unit length, and $u = \Omega_1 R$ is its peripheral velocity.†

The following general remark may be made concerning the solutions of the equations of motion of a viscous fluid which we have obtained in §§17 and 18. In all these cases the non-linear term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ in the equations which determine the velocity distribution is identically zero, so that we are actually solving linear equations, a fact which very much simplifies the problem. For this reason all the solutions also satisfy the equations of motion for an incompressible ideal fluid, say in the form (10.2) and (10.3). This is why formulae (17.1) and (18.3) do not contain the viscosity coefficient at all. This coefficient appears only in formulae, such as (17.9), which relate the velocity to the pressure gradient in the fluid, since the presence of a pressure gradient is due to the viscosity; an ideal fluid could flow in a pipe even if there were no pressure gradient.

§19. **The law of similarity**

In studying the motion of viscous fluids we can obtain a number of important results from simple arguments concerning the dimensions of various physical quantities. Let us

† The solution of the more complex problem of the motion of a viscous fluid in a narrow space between cylinders whose axes are parallel but not coincident may be found in: N. E. Kochin, I. A. Kibel' and N. V. Roze. *Theoretical Hydromechanics* (*Teoreticheskaya gidromekhanika*), Part 2, p. 534, Moscow 1963; A. Sommerfeld, *Mechanics of Deformable Bodies*, §36, New York 1950.

consider any particular type of motion, for instance the motion of a body of some definite shape through a fluid. If the body is not a sphere, its direction of motion must also be specified: e.g. the motion of an ellipsoid in the direction of its greatest or least axis. Alternatively, we may be considering flow in a region with boundaries having a definite form (a pipe with given cross-section, etc.).

In such a case we say that bodies of the same shape are *geometrically similar*; they can be obtained from one another by changing all linear dimensions in the same ratio. Hence, if the shape of the body is given, it suffices to specify any one of its linear dimensions (the radius of a sphere or of a cylindrical pipe, one semi-axis of a spheroid with given eccentricity, and so on) in order to determine its dimensions completely.

We shall at present consider steady flow. If, for example, we are discussing flow past a solid body (which case we shall take below, for definiteness), the velocity of the main stream must therefore be constant. We shall suppose the fluid incompressible.

Of the parameters which characterize the fluid itself, only the kinematic viscosity $\nu = \eta/\rho$ appears in the equations of hydrodynamics (the Navier–Stokes equations); the unknown functions which have to be determined by solving the equations are the velocity $\mathbf{v}$ and the ratio p/ρ of the pressure p to the constant density ρ. Moreover, the flow depends, through the boundary conditions, on the shape and dimensions of the body moving through the fluid and on its velocity. Since the shape of the body is supposed given, its geometrical properties are determined by one linear dimension, which we denote by l. Let the velocity of the main stream be u. Then any flow is specified by three parameters, ν, u and l. These quantities have the following dimensions:

$$\nu = \text{cm}^2/\text{sec}, \qquad l = \text{cm}, \qquad u = \text{cm}/\text{sec}.$$

It is easy to verify that only one dimensionless quantity can be formed from the above three, namely ul/ν. This combination is called the *Reynolds number* and is denoted by R:

$$\mathrm{R} = \rho ul/\eta = ul/\nu. \tag{19.1}$$

Any other dimensionless parameter can be written as a function of R.

We shall now measure lengths in terms of l, and velocities in terms of u, i.e. we introduce the dimensionless quantities $(\mathbf{r}/l$, $\mathbf{v}/u$. Since the only dimensionless parameter is the Reynolds number, it is evident that the velocity distribution obtained by solving the equations of incompressible flow is given by a function having the form

$$\mathbf{v} = u\mathbf{f}(\mathbf{r}/l, \mathrm{R}). \tag{19.2}$$

It is seen from this expression that, in two different flows of the same type (for example, flow past spheres with different radii by fluids with different viscosities), the velocities $\mathbf{v}/u$ are the same functions of the ratio $\mathbf{r}/l$ if the Reynolds number is the same for each flow. Flows which can be obtained from one another by simply changing the unit of measurement of coordinates and velocities are said to be *similar*. Thus flows of the same type with the same Reynolds number are similar. This is called the *law of similarity* (O. Reynolds 1883).

A formula similar to (19.2) can be written for the pressure distribution in the fluid. To do so, we must construct from the parameters ν, l, u some quantity with the dimensions of pressure divided by density; this quantity can be u^2, for example. Then we can say that $p/\rho u^2$ is a function of the dimensionless variable $\mathbf{r}/l$ and the dimensionless parameter R. Thus

$$p = \rho u^2 f(\mathbf{r}/l, \mathrm{R}). \tag{19.3}$$

Finally, similar considerations can also be applied to quantities which characterize the flow but are not functions of the coordinates. Such a quantity is, for instance, the drag force F acting on the body. We can say that the dimensionless ratio of F to some quantity formed from ν, u, l, ρ and having the dimensions of force must be a function of the Reynolds number alone. Such a combination of ν, u, l, ρ can be $\rho u^2 l^2$, for example. Then

$$F = \rho u^2 l^2 f(\mathrm{R}). \tag{19.4}$$

If the force of gravity has an important effect on the flow, then the latter is determined not by three but by four parameters, l, u, ν and the acceleration g due to gravity. From these parameters we can construct not one but two independent dimensionless quantities. These can be, for instance, the Reynolds number and the *Froude number*, which is

$$\mathrm{F} = u^2/lg. \tag{19.5}$$

In formulae (19.2)–(19.4) the function f will now depend on not one but two parameters (R and F), and two flows will be similar only if both these numbers have the same values.

Finally, we may say a little regarding non-steady flows. A non-steady flow of a given type is characterized not only by the quantities ν, u, l but also by some time interval τ characteristic of the flow, which determines the rate of change of the flow. For instance, in oscillations, according to a given law, of a solid body, of a given shape, immersed in a fluid, τ may be the period of oscillation. From the four quantities ν, u, l, τ we can again construct two independent dimensionless quantities, which may be the Reynolds number and the number

$$\mathrm{S} = u\tau/l, \tag{19.6}$$

sometimes called the *Strouhal number*. Similar motion takes place in these cases only if both these numbers have the same values.

If the oscillations of the fluid occur spontaneously (and not under the action of a given external exciting force), then for motion of a given type S will be a definite function of R:

$$\mathrm{S} = f(\mathrm{R}).$$

§20. Flow with small Reynolds numbers

The Navier–Stokes equation is considerably simplified in the case of flow with small Reynolds numbers. For steady flow of an incompressible fluid, this equation is

$$(\mathbf{v}\cdot\mathbf{grad})\mathbf{v} = -(1/\rho)\mathbf{grad}\,p + (\eta/\rho)\triangle\mathbf{v}.$$

The term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ is of the order of magnitude of u^2/l, u and l having the same meaning as in §19. The quantity $(\eta/\rho)\triangle\mathbf{v}$ is of the order of magnitude of $\eta u/\rho l^2$. The ratio of the two is just the Reynolds number. Hence the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ may be neglected if the Reynolds number is small, and the equation of motion reduces to a linear equation

$$\eta\triangle\mathbf{v} - \mathbf{grad}\,p = 0. \tag{20.1}$$

Together with the equation of continuity

$$\mathrm{div}\,\mathbf{v} = 0 \tag{20.2}$$

it completely determines the motion. It is useful to note also the equation

$$\triangle\,\mathbf{curl}\,\mathbf{v} = 0, \tag{20.3}$$

which is obtained by taking the curl of equation (20.1).

As an example, let us consider rectilinear and uniform motion of a sphere in a viscous fluid (G. G. Stokes 1851). The problem of the motion of a sphere, it is clear, is exactly equivalent to that of flow past a fixed sphere, the fluid having a given velocity $\mathbf{u}$ at infinity. The velocity distribution in the first problem is obtained from that in the second problem by simply subtracting the velocity $\mathbf{u}$; the fluid is then at rest at infinity, while the sphere moves with velocity $-\mathbf{u}$. If we regard the flow as steady, we must, of course, speak of the flow past a fixed sphere, since, when the sphere moves, the velocity of the fluid at any point in space varies with time.

Since $\operatorname{div}(\mathbf{v}-\mathbf{u}) = \operatorname{div}\mathbf{v} = 0$, $\mathbf{v}-\mathbf{u}$ can be expressed as the curl of some vector $\mathbf{A}$:

$$\mathbf{v}-\mathbf{u} = \mathbf{curl}\ \mathbf{A},$$

with $\mathbf{curl}\ \mathbf{A}$ equal to zero at infinity. The vector $\mathbf{A}$ must be axial, in order for its curl to be polar, like the velocity. In flow past a sphere, a completely symmetrical body, there is no preferred direction other than that of $\mathbf{u}$. This parameter $\mathbf{u}$ must appear linearly in $\mathbf{A}$, because the equation of motion and its boundary conditions are linear. The general form of a vector function $\mathbf{A}(\mathbf{r})$ satisfying all these requirements is $\mathbf{A} = f'(r)\mathbf{n}\times\mathbf{u}$, where $\mathbf{n}$ is a unit vector parallel to the position vector $\mathbf{r}$ (the origin being taken at the centre of the sphere), and $f'(r)$ is a scalar function of r. The product $f'(r)\mathbf{n}$ can be represented as the gradient of another function $f(r)$. We shall thus look for the velocity in the form

$$\mathbf{v} = \mathbf{u}+\mathbf{curl}\,(\mathbf{grad}\,f\times\mathbf{u}) = \mathbf{u}+\mathbf{curl}\,\mathbf{curl}\,(f\mathbf{u}); \tag{20.4}$$

the last expression is obtained by noting that $\mathbf{u}$ is constant.

To determine the function f, we use equation (20.3). Since

$$\begin{aligned}\mathbf{curl}\,\mathbf{v} = \mathbf{curl}\,\mathbf{curl}\,\mathbf{curl}(f\mathbf{u}) &= (\mathbf{grad}\,\operatorname{div} - \triangle)\,\mathbf{curl}(f\mathbf{u})\\ &= -\triangle\,\mathbf{curl}(f\mathbf{u}),\end{aligned}$$

(20.3) takes the form $\triangle^2\,\mathbf{curl}(f\mathbf{u}) = \triangle^2(\mathbf{grad}\,f\times\mathbf{u}) = (\triangle^2\mathbf{grad}\,f)\times\mathbf{u} = 0$. It follows from this that

$$\triangle^2\,\mathbf{grad}\,f = 0. \tag{20.5}$$

A first integration gives

$$\triangle^2 f = \text{constant}.$$

It is easy to see that the constant must be zero, since the velocity difference $\mathbf{v}-\mathbf{u}$ must vanish at infinity, and so must its derivatives. The expression $\triangle^2 f$ contains fourth derivatives of f, whilst the velocity is given in terms of the second derivatives of f. Thus we have

$$\triangle^2 f \equiv \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\right)\triangle f = 0.$$

Hence

$$\triangle f = 2a/r + c.$$

The constant c must be zero if the velocity $v-u$ is to vanish at infinity. From $\triangle f = 2a/r$ we obtain

$$f = ar + b/r. \tag{20.6}$$

The additive constant is omitted, since it is immaterial (the velocity being given by derivatives of f).

Substituting in (20.4), we have after a simple calculation

$$\mathbf{v} = \mathbf{u} - a\frac{\mathbf{u}+\mathbf{n}(\mathbf{u}\cdot\mathbf{n})}{r} + b\frac{3\mathbf{n}(\mathbf{u}\cdot\mathbf{n})-\mathbf{u}}{r^3}. \tag{20.7}$$

The constants a and b have to be determined from the boundary conditions: at the surface of the sphere ($r = R$), $\mathbf{v} = 0$, i.e.

$$-\mathbf{u}\left(\frac{a}{R}+\frac{b}{R^3}-1\right)+\mathbf{n}(\mathbf{u}\cdot\mathbf{n})\left(-\frac{a}{R}+\frac{3b}{R^3}\right) = 0.$$

Since this equation must hold for all $\mathbf{n}$, the coefficients of $\mathbf{u}$ and $\mathbf{n}(\mathbf{u}\cdot\mathbf{n})$ must each vanish. Hence $a = \frac{3}{4}R$, $b = \frac{1}{4}R^3$. Thus we have finally

$$f = \tfrac{3}{4}Rr + \tfrac{1}{4}R^3/r, \tag{20.8}$$

$$\mathbf{v} = -\tfrac{3}{4}R\frac{\mathbf{u}+\mathbf{n}(\mathbf{u}\cdot\mathbf{n})}{r} - \tfrac{1}{4}R^3\frac{\mathbf{u}-3\mathbf{n}(\mathbf{u}\cdot\mathbf{n})}{r^3} + \mathbf{u}, \tag{20.9}$$

or, in spherical polar components with the axis parallel to $\mathbf{u}$,

$$\left.\begin{aligned} v_r &= u\cos\theta\left[1-\frac{3R}{2r}+\frac{R^3}{2r^3}\right], \\ v_\theta &= -u\sin\theta\left[1-\frac{3R}{4r}-\frac{R^3}{4r^3}\right]. \end{aligned}\right\} \tag{20.10}$$

This gives the velocity distribution about the moving sphere. To determine the pressure, we substitute (20.4) in (20.1):

$$\begin{aligned} \mathbf{grad}\,p = \eta\triangle\mathbf{v} &= \eta\triangle\,\mathbf{curl}\,\mathbf{curl}\,(f\mathbf{u}) \\ &= \eta\triangle\,(\mathbf{grad}\,\mathrm{div}\,(f\mathbf{u}) - \mathbf{u}\triangle f). \end{aligned}$$

But $\triangle^2 f = 0$, and so

$$\mathbf{grad}\,p = \mathbf{grad}[\eta\triangle\,\mathrm{div}(f\mathbf{u})] = \mathbf{grad}(\eta\mathbf{u}\cdot\mathbf{grad}\,\triangle f).$$

Hence

$$p = \eta\mathbf{u}\cdot\mathbf{grad}\,\triangle f + p_0, \tag{20.11}$$

where p_0 is the fluid pressure at infinity. Substitution for f leads to the final expression

$$p = p_0 - \tfrac{3}{2}\eta\frac{\mathbf{u}\cdot\mathbf{n}}{r^2}R. \tag{20.12}$$

Using the above formulae, we can calculate the force $\mathbf{F}$ exerted on the sphere by the moving fluid (or, what is the same thing, the drag on the sphere as it moves through the fluid). To do so, we take spherical polar coordinates with the axis parallel to $\mathbf{u}$; by symmetry, all quantities are functions only of r and of the polar angle θ. The force $\mathbf{F}$ is evidently parallel to the velocity $\mathbf{u}$. The magnitude of this force can be determined from (15.14). Taking from this formula the components, normal and tangential to the surface, of the force on an element of the surface of the sphere, and projecting these components on the direction of $\mathbf{u}$, we find

$$F = \oint(-p\cos\theta + \sigma'_{rr}\cos\theta - \sigma'_{r\theta}\sin\theta)\mathrm{d}f, \tag{20.13}$$

where the integration is taken over the whole surface of the sphere.

Substituting the expressions (20.10) in the formulae

$$\sigma'_{rr} = 2\eta\frac{\partial v_r}{\partial r}, \qquad \sigma'_{r\theta} = \eta\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)$$

(see (15.20)), we find that at the surface of the sphere

$$\sigma'_{rr} = 0, \qquad \sigma'_{r\theta} = -(3\eta/2R)u\sin\theta,$$

while the pressure (20.12) is $p = p_0 - (3\eta/2R)u\cos\theta$. Hence the integral (20.13) reduces to $F = (3\eta u/2R)\oint \mathrm{d}f$. In this way we finally arrive at *Stokes' formula* for the drag on a sphere moving slowly in a fluid:†

$$F = 6\pi\eta R u. \tag{20.14}$$

The drag is proportional to the velocity and linear size of the body. This could have been foreseen from dimensional arguments: the fluid density ρ does not appear in the approximate equations (20.1), (20.2), and so the force F which they give must be expressed only in terms of η, u and R; from these, only one combination with the dimensions of force can be formed, namely the product $\eta R u$.

A similar dependence occurs for slowly moving bodies with other shapes. The direction of the drag on a body of arbitrary shape is not the same as that of the velocity; the general form of the dependence of $\mathbf{F}$ on $\mathbf{u}$ can be written

$$F_i = \eta a_{ik} u_k, \tag{20.15}$$

where a_{ik} is a tensor of rank two, independent of the velocity. It is important to note that this tensor is symmetrical, a result which holds in the linear approximation with respect to the velocity, and is a particular case of a general law valid for slow motion accompanied by dissipative processes (see *SP*1, §121).

Refinement of Stokes' formula

The above solution of the problem of flow past a sphere is not valid at large distances, even if the Reynolds number is small. To see this, let us estimate the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ neglected in (20.1). At large distances, $\mathbf{v} \cong \mathbf{u}$; the velocity derivatives there are of the order of uR/r^2, as is seen from (20.9). Hence $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v} \sim u^2R/r^2$. The terms retained in (20.1) are of the order of $\eta Ru/\rho r^3$, as can be seen from the same expression (20.9) for the velocity or (20.12) for the pressure. The condition $\eta Ru/\rho r^3 \gg u^2R/r^2$ is satisfied only for distances such that

$$r \ll \nu/u. \tag{20.16}$$

At greater distances, the terms neglected are not negligible, and the velocity distribution so found is incorrect.

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† With a view to later applications, it may be mentioned that calculations with (20.7) and the constants a and b undetermined give

$$F = 8\pi\eta a u. \tag{20.14a}$$

The drag can also be calculated for a slowly moving ellipsoid with any shape. The relevant formulae are given by H. Lamb, *Hydrodynamics*, 6th ed., §339, Cambridge 1932. Here we shall give the limiting expressions for a plane circular disk with radius R moving perpendicular to its plane:

$$F = 16\eta R u,$$

and for a similar disk moving in its plane:

$$F = 32\eta R u/3.$$

To find the velocity distribution at large distances from the body, we have to include the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ omitted from (20.1). Since at these distances $\mathbf{v}$ is almost the same as $\mathbf{u}$, we can approximately replace $\mathbf{v}\cdot\mathbf{grad}$ by $\mathbf{u}\cdot\mathbf{grad}$. We then find for the velocity at large distances the linear equation

$$(\mathbf{u}\cdot\mathbf{grad})\mathbf{v} = -(1/\rho)\,\mathbf{grad}\,p + \nu\Delta\mathbf{v} \tag{20.17}$$

(C. W. Oseen 1910). We shall not pause to give here the procedure for solving this equation for flow past a sphere,† but merely mention that the velocity distribution thus obtained can be used to derive a more accurate formula for the drag on the sphere, which includes the next term in the expansion of the drag in powers of the Reynolds number $\mathrm{R} = uR/\nu$:

$$F = 6\pi\eta uR(1 + 3uR/8\nu). \tag{20.18}$$

In solving the problem of flow past an infinite cylinder at right angles to its axis, Oseen's equation has to be used from the start; the equation (20.1) has in this case no solution satisfying the boundary conditions at the surface of the cylinder and also at infinity. The drag per unit length of the cylinder is found to be

$$F = \frac{4n\eta u}{\frac{1}{2} - C - \log(uR/4\nu)} = \frac{4\pi\eta u}{\log(3{\cdot}70\nu/uR)}, \tag{20.19}$$

where $C = 0{\cdot}577\ldots$ is Euler's constant (H. Lamb 1911).‡

Another comment should be made regarding the problem of flow past a sphere. The replacement of $\mathbf{v}$ by $\mathbf{u}$ in the non-linear term in (20.17) is valid at large distances from the sphere, $r \gg R$. It is therefore natural that Oseen's equation, while correctly refining the picture of flow at large distances, does not do the same at short distances. This is evident from the fact that the solution of (20.17) which satisfies the necessary conditions at infinity does not satisfy the exact condition that the velocity be zero on the surface of the sphere, which is met only by the zero-order term in the expansion of the velocity in powers of the Reynolds number and not even by the first-order term.

It might therefore seem at first sight that the solution of Oseen's equation cannot be used for a valid calculation of the correction term in the drag. This is not so, however, for the following reason. The contribution to $\mathbf{F}$ from the motion of the fluid at short distances (for which $u \ll \nu/r$) has to be expandable in powers of $\mathbf{u}$. The first non-zero correction term in the vector $\mathbf{F}$ arising from this contribution therefore has to be proportional to $\mathbf{u}u^2$, and gives a second-order correction relative to the Reynolds number; it thus does not affect the first-order correction in (20.18).

Further corrrections to Stokes' formula and a valid refinement of the flow pattern at short distances can not be obtained by a direct solution of (20.17). Although these refinements themselves are not very important, there is considerable methodological interest in deriving and analysing a consistent perturbation theory for solving problems of viscous flow at small Reynolds numbers (S. Kaplun and P. A. Lagerstrom 1957; I.

† It is given by N. E. Kochin, I. A. Kibel' and N. V. Roze, *Theoretical Hydromechanics* (*Teoreticheskaya gidromekhanika*), Part 2, chapter II, §§25–26, Moscow 1963; H. Lamb, *Hydrodynamics*, 6th ed., §§342–3, Cambridge 1932.

‡ The impossibility of calculating the drag in the cylinder problem by means of (20.1) is evident from dimensional arguments. As already mentioned, the result would have to be expressed in terms of η, u and R, but in this case we are concerned with the force per unit length of the cylinder, and the only quantity having the right dimensions would be ηu, which is independent of the size of the body and therefore does not vanish as $R \to 0$; this is physically absurd.

Proudman and J. R. A. Pearson 1957). We shall describe the existing situation and give all expressions needed to illustrate it, without going through the calculations in detail.†

To show explicitly the small parameter R, the Reynolds number, we use the dimensionless velocity and position vector $\mathbf{v}' = \mathbf{v}/u$, $\mathbf{r}' = \mathbf{r}/R$, and in the rest of this section denote them by **v** and **r** without the primes. The exact solution of the equation of motion (which we take in the form (15.10) with the pressure eliminated) is then

$$\mathrm{R}\ \mathbf{curl}\ (\mathbf{v}\times\mathbf{curl}\ \mathbf{v}) + \triangle\ \mathbf{curl}\ \mathbf{v} = 0. \tag{20.20}$$

We distinguish two regions of space around the sphere: the near region with $r \ll 1/\mathrm{R}$, and the far region with $r \gg 1$. These together cover all space, overlapping in the intermediate range

$$1/\mathrm{R} \gg r \gg 1. \tag{20.21}$$

In a consistent perturbation theory, the initial approximation in the near region is the Stokes approximation, i.e. the solution of the equation $\triangle\ \mathbf{curl}\ \mathbf{v} = 0$ obtained from (20.20) by neglecting the term which contains the factor R. This solution is given by formulae (20.10); in dimensionless variables, it is

$$v_r^{(1)} = \cos\theta\left(1 - \frac{3}{2r} + \frac{1}{2r^3}\right), \qquad v_\theta^{(1)} = -\sin\theta\left(1 - \frac{3}{4r} - \frac{1}{4r^3}\right),$$
$$r \ll 1/\mathrm{R}, \tag{20.22}$$

the superscript (1) denoting the first approximation.

The first approximation in the far region is simply the constant $\mathbf{v}^{(1)} = \boldsymbol{\nu}$ corresponding to the unperturbed uniform incoming flow ($\boldsymbol{\nu}$ being a unit vector in the direction of the flow). Substitution of $\mathbf{v} = \boldsymbol{\nu} + \mathbf{v}^{(2)}$ in (20.20) gives for $\mathbf{v}^{(2)}$ Oseen's equation

$$\mathrm{R}\ \mathbf{curl}\ (\boldsymbol{\nu}\times\mathbf{curl}\ \mathbf{v}^{(2)}) + \triangle\ \mathbf{curl}\ \mathbf{v}^{(2)} = 0. \tag{20.23}$$

The solution must satisfy the condition that the velocity $\mathbf{v}^{(2)}$ be zero at infinity and the condition for joining to the solution (20.22) in the intermediate range. The latter excludes, in particular, solutions that increase too rapidly with decreasing r.‡ The appropriate solution is

$$v_r^{(1)} + v_r^{(2)} = \cos\theta + \frac{3}{2r^2\mathrm{R}}\{1 - [1 + \tfrac{1}{2}r\mathrm{R}(1+\cos\theta)]e^{-\frac{1}{2}r\mathrm{R}(1-\cos\theta)}\},$$

$$v_\theta^{(1)} + v_\theta^{(2)} = -\sin\theta + \frac{3}{4r}\sin\theta\, e^{-\frac{1}{2}r\mathrm{R}(1-\cos\theta)},$$
$$r \gg 1. \tag{20.24}$$

† These may be found in M. Van Dyke, *Perturbation Methods in Fluid Mechanics*, New York 1964. The calculations there are given not in terms of the velocity $\mathbf{v}(\mathbf{r})$ but in the more compact, less visualizable, terminology of the stream function. For axially symmetrical flow, including flow past a sphere, the stream function $\psi(r,\theta)$ in spherical polar coordinates is defined by

$$v_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta},$$
$$v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r},\ v_\phi = 0.$$

These satisfy identically the continuity equation (15.22).

‡ To determine the numerical coefficients in the solution, we have also to take account of the condition that the total amount of fluid passing through any closed surface around the sphere must be zero.

Note that the variable for the far region is really the product $\rho = r\mathrm{R}$, not the radial coordinate r itself. When this variable is used, R disappears from (20.20), in accordance with the fact that when $r \gtrsim 1/\mathrm{R}$ the viscous and inertia terms in the equation become comparable in order of magnitude. The number R occurs in the solution only through the boundary condition for joining to that in the near region. The expansion of $\mathbf{v}(\mathbf{r})$ in the far region is therefore an expansion in powers of R for given values of $\rho = r\mathrm{R}$, since the second terms in (20.24), when expressed in terms of ρ, contain R as a factor.

To test the correctness of joining for the solutions (20.22) and (20.24), we observe that in the intermediate range (20.21) $r\mathrm{R} \ll 1$ and the expressions (20.24) can be expanded in powers of this variable. As far as the first two terms (apart from the uniform flow), we have

$$\left.\begin{aligned} v_r &= \cos\theta\left(1-\frac{3}{2r}\right)+\frac{3\mathrm{R}}{16}(1-\cos\theta)(1+3\cos\theta), \\ v_\theta &= -\sin\theta\left(1-\frac{3}{4r}\right)-\frac{3\mathrm{R}}{8}\sin\theta\,(1-\cos\theta). \end{aligned}\right\} \tag{20.25}$$

In the same range, on the other hand, $r \gg 1$ and therefore we can omit the terms in $1/r^3$ in (20.22); the remaining terms are the same as the first terms in (20.25), and the second terms there will be made use of later.

On going to the next approximation in the near region, we write $\mathbf{v} = \mathbf{v}^{(1)} + \mathbf{v}^{(2)}$ and obtain from (20.20) an equation for the correction in the second approximation:

$$\triangle\,\mathbf{curl}\,\mathbf{v}^{(2)} = -\mathrm{R}\,\mathbf{curl}\,(\mathbf{v}^{(1)}\times\mathbf{curl}\,\mathbf{v}^{(1)}). \tag{20.26}$$

The solution of this equation must satisfy the condition of vanishing on the surface of the sphere and that of joining to the solution in the far region; the latter means that the leading terms in the function $\mathbf{v}^{(2)}(\mathbf{r})$ when $r \gg 1$ must agree with the second terms in (20.25). The appropriate solution is

$$\begin{aligned} v_r^{(2)} &= \frac{3\mathrm{R}}{8}v_r^{(1)}+\frac{3\mathrm{R}}{32}\left(1-\frac{1}{r}\right)^2\left(2+\frac{1}{r}+\frac{1}{r^2}\right)(1-3\cos^2\theta), \\ v_\theta^{(2)} &= \frac{3\mathrm{R}}{8}v_\theta^{(1)}+\frac{3\mathrm{R}}{32}\left(1-\frac{1}{r}\right)\left(4+\frac{1}{r}+\frac{1}{r^2}+\frac{2}{r^3}\right)\sin\theta\cos\theta, \\ & \qquad r \ll 1/\mathrm{R}. \end{aligned} \tag{20.27}$$

In the intermediate region, only the terms without a factor $1/r$ remain in these expressions, and they do in fact agree with the second terms in (20.25).

From the velocity distribution (20.27), we can calculate the correction to Stokes' formula for the drag. The second terms in (20.27), because of their angular dependence, do not contribute to the drag; the first terms give the correction $3\mathrm{R}/8$ shown in (20.18). According to the above discussion, the exact velocity distribution near the sphere leads in this approximation to the same result for the drag as the solution of Oseen's equation.

The next approximation can be obtained by continuing the procedure described. It involves logarithmic terms in the velocity distribution; in the expression (20.18) for the drag, the brackets are replaced by

$$1+\frac{3}{8}\mathrm{R}-\frac{9}{40}\mathrm{R}^2\log(1/\mathrm{R}),$$

the logarithm being assumed large.†

† See I. Proudman and J. R. A. Pearson, *Journal of Fluid Mechanics* 2, 237, 1957.

PROBLEMS

PROBLEM 1. Determine the motion of a fluid occupying the space between two concentric spheres with radii R_1, R_2 ($R_2 > R_1$), rotating uniformly about different diameters with angular velocities Ω_1, Ω_2; the Reynolds numbers $\Omega_1 R_1^2/\nu$, $\Omega_2 R_2^2/\nu$ are small compared with unity.

SOLUTION. On account of the linearity of the equations, the motion between two rotating spheres may be regarded as a superposition of the two motions obtained when one sphere is at rest and the other rotates. We first put $\Omega_2 = 0$, i.e. only the inner sphere is rotating. It is reasonable to suppose that the fluid velocity at every point is along the tangent to a circle in a plane perpendicular to the axis of rotation with its centre on the axis. On account of the axial symmetry, the pressure gradient in this direction is zero. Hence the equation of motion (20.1) becomes $\triangle \mathbf{v} = 0$. The angular velocity vector $\mathbf{\Omega}_1$ is an axial vector. Arguments similar to those given previously show that the velocity can be written as

$$\mathbf{v} = \mathbf{curl}[f(r)\mathbf{\Omega}_1] = \mathbf{grad} f \times \mathbf{\Omega}_1.$$

The equation of motion then gives $\mathbf{grad} \triangle f \times \mathbf{\Omega}_1 = 0$. Since the vector $\mathbf{grad} \triangle f$ is parallel to the position vector, and the vector product $\mathbf{r} \times \mathbf{\Omega}_1$ cannot be zero for given $\mathbf{\Omega}_1$ and arbitrary $\mathbf{r}$, we must have $\mathbf{grad} \triangle f = 0$, so that

$$\triangle f = \text{constant}.$$

Integrating, we find

$$f = ar^2 + \frac{b}{r}, \qquad \mathbf{v} = \left(\frac{b}{r^3} - 2a\right)\mathbf{\Omega}_1 \times \mathbf{r}.$$

The constants a and b are found from the conditions that $\mathbf{v} = 0$ for $r = R_2$ and $\mathbf{v} = \mathbf{u}$ for $r = R_1$, where $\mathbf{u} = \mathbf{\Omega}_1 \times \mathbf{r}$ is the velocity of points on the rotating sphere. The result is

$$\mathbf{v} = \frac{R_1^3 R_2^3}{R_2^3 - R_1^3}\left(\frac{1}{r^3} - \frac{1}{R_2^3}\right)\mathbf{\Omega}_1 \times \mathbf{r}.$$

The fluid pressure is constant ($p = p_0$). Similarly, we have for the case where the outer sphere rotates and the inner one is at rest ($\Omega_1 = 0$)

$$\mathbf{v} = \frac{R_1^3 R_2^3}{R_2^3 - R_1^3}\left(\frac{1}{R_1^3} - \frac{1}{r^3}\right)\mathbf{\Omega}_2 \times \mathbf{r}.$$

In the general case where both spheres rotate, we have

$$\mathbf{v} = \frac{R_1^3 R_2^3}{R_2^3 - R_1^3}\left\{\left(\frac{1}{r^3} - \frac{1}{R_2^3}\right)\mathbf{\Omega}_1 \times \mathbf{r} + \left(\frac{1}{R_1^3} - \frac{1}{r^3}\right)\mathbf{\Omega}_2 \times \mathbf{r}\right\}.$$

If the outer sphere is absent ($R_2 = \infty, \Omega_2 = 0$), i.e. we have simply a sphere with radius R rotating in an infinite fluid, then

$$\mathbf{v} = (R^3/r^3)\mathbf{\Omega} \times \mathbf{r}.$$

Let us calculate the moment of the frictional forces acting on the sphere in this case. If we take spherical polar coordinates with the polar axis parallel to $\mathbf{\Omega}$, we have $v_r = v_\theta = 0$, $v_\phi = v = (R^2\Omega/r^2)\sin\theta$. The frictional force on unit area of the sphere is

$$\sigma'_{r\phi} = \eta\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)_{r=R} = -3\eta\Omega\sin\theta.$$

The total moment on the sphere is

$$M = \int_0^\pi \sigma'_{r\phi} R \sin\theta . 2\pi R^2 \sin\theta \, d\theta,$$

whence we find

$$M = -8\pi\eta R^3 \Omega.$$

If the inner sphere is absent, $\mathbf{v} = \mathbf{\Omega}_2 \times \mathbf{r}$, i.e. the fluid simply rotates rigidly with the sphere surrounding it.

PROBLEM 2. Determine the velocity of a spherical drop of fluid (with viscosity η') moving under gravity in a fluid with viscosity η (W. Rybczyński 1911).

SOLUTION. We use a system of coordinates in which the drop is at rest. For the fluid outside the drop we again seek a solution of equation (20.5) in the form (20.6), so that the velocity has the form (20.7). For the fluid inside the drop, we have to find a solution which does not have a singularity at $r = 0$ (and the second derivatives of f, which determine the velocity, must also remain finite). This solution is

$$f = \tfrac{1}{4}Ar^2 + \tfrac{1}{8}Br^4,$$

and the corresponding velocity is

$$\mathbf{v} = -A\mathbf{u} + Br^2[\mathbf{n}(\mathbf{u}\cdot\mathbf{n}) - 2\mathbf{u}].$$

At the surface of the sphere† the following conditions must be satisfied. The normal velocity components outside ($\mathbf{v}_e$) and inside ($\mathbf{v}_i$) the drop must be zero:

$$v_{i,r} = v_{e,r} = 0.$$

The tangential velocity component must be continuous:

$$v_{i,\theta} = v_{e,\theta},$$

as must be the component $\sigma_{r\theta}$ of the stress tensor:

$$\sigma_{i,r\theta} = \sigma_{e,r\theta}.$$

The condition that the stress tensor components σ_{rr} be equal need not be written down; it would determine the required velocity u, which is more simply found in the manner shown below. From the above four conditions we obtain four equations for the constants a, b, A, B, whose solutions are

$$a = R\frac{2\eta + 3\eta'}{4(\eta + \eta')}, \qquad b = R^3\frac{\eta'}{4(\eta + \eta')}, \qquad A = -BR^2 = \frac{\eta}{2(\eta + \eta')}.$$

By (20.14a), we have for the drag

$$F = 2\pi u\eta R(2\eta + 3\eta')/(\eta + \eta').$$

As $\eta' \to \infty$ (corresponding to a solid sphere) this formula becomes Stokes' formula. In the limit $\eta' \to 0$ (corresponding to a gas bubble) we have $F = 4\pi u\eta R$, i.e the drag is two-thirds of that on a solid sphere.

Equating F to the force of gravity on the drop, $\frac{4}{3}\pi R^3(\rho - \rho')g$, we find

$$u = \frac{2R^2 g(\rho - \rho')(\eta + \eta')}{3\eta(2\eta + 3\eta')}.$$

PROBLEM 3. Two parallel plane circular disks (with radius R) lie one above the other a small distance apart; the space between them is filled with fluid. The disks approach at a constant velocity u, displacing the fluid. Determine the resistance to their motion (O. Reynolds).

SOLUTION. We take cylindrical polar coordinates, with the origin at the centre of the lower disk, which we suppose fixed. The flow is axially symmetric and, since the fluid layer is thin, predominantly radial: $v_z \ll v_r$, and also $\partial v_r/\partial r \ll \partial v_r/\partial z$. Hence the equations of motion become

$$\eta\frac{\partial^2 v_r}{\partial z^2} = \frac{\partial p}{\partial r}, \qquad \frac{\partial p}{\partial z} = 0, \tag{1}$$

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0, \tag{2}$$

with the boundary conditions

$$\text{at } z = 0: \quad v_r = v_z = 0;$$

$$\text{at } z = h: \quad v_r = 0, \quad v_z = -u;$$

$$\text{at } r = R: \quad p = p_0,$$

† We may neglect the change of shape of the drop in its motion, since this change is of a higher order of smallness. However, it must be borne in mind that, in order that the moving drop should in fact be spherical, the forces due to surface tension at its boundary must exceed the forces due to pressure differences, which tend to make the drop non-spherical. This means that we must have $\eta u/R \ll \alpha/R$, where α is the surface-tension coefficient, or, substituting $u \sim R^2 g\rho/\eta$,

$$R \ll \sqrt{(\alpha/\rho g)}.$$

where h is the distance between the disks, and p_0 the external pressure. From equations (1) we find

$$v_r = \frac{1}{2\eta}\frac{\mathrm{d}p}{\mathrm{d}r}z(z-h).$$

Integrating equation (2) with respect to z, we obtain

$$u = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\int_0^h rv_r\,\mathrm{d}z = -\frac{h^3}{12\eta r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}p}{\mathrm{d}r}\right),$$

whence

$$p = p_0 + \frac{3\eta u}{h^3}(R^2 - r^2).$$

The total resistance to the moving disk is

$$F = 3\pi\eta u R^4/2h^3.$$

§21. The laminar wake

In steady flow of a viscous fluid past a solid body, the flow at great distances behind the body has certain characteristics which can be investigated independently of the particular shape of the body.

Let us denote by $\mathbf{U}$ the constant velocity of the incident current; we take the direction of $\mathbf{U}$ as the x-axis, with the origin somewhere inside the body. The actual fluid velocity at any point may be written $\mathbf{U}+\mathbf{v}$; $\mathbf{v}$ vanishes at infinity.

It is found that, at great distances behind the body, the velocity $\mathbf{v}$ is noticeably different from zero only in a relatively narrow region near the x-axis. This region, called the *laminar wake*,† is reached by fluid particles which move along streamlines passing fairly close to the body. Hence the flow in the wake is essentially rotational. The reason is that rotational flow of a viscous fluid past a solid body is due to the surface of the body.‡ This is easily seen if we recall that, in the pattern of potential flow for an ideal fluid, only the normal velocity component is zero on the surface of the body, not the tangential component $\mathbf{v}_t$. The boundary condition of adhesion for a real fluid makes $\mathbf{v}_t$ also zero, however. If the pattern of potential flow were maintained, this would cause a non-zero discontinuity of $\mathbf{v}_t$, i.e. the occurrence of a surface vorticity. The viscosity smooths out the discontinuity, and the rotational state penetrates into the fluid, from which it passes by convection into the wake region.

On the other hand, the viscosity has almost no effect at any point on streamlines that do not pass near the body, and the vorticity, which is zero in the incident current, remains practically zero on these streamlines, as it would in an ideal fluid. Thus the flow at great distances from the body may be regarded as potential flow everywhere except in the wake.

We shall now derive formulae relating the properties of the flow in the wake to the forces acting on the body. The total momentum transported by the fluid through any closed surface surrounding the body is equal to the integral of the momentum flux density tensor over that surface, $\oint \Pi_{ik}\mathrm{d}f_k$. The components of the tensor Π_{ik} are

$$\Pi_{ik} = p\delta_{ik} + \rho(U_i + v_i)(U_k + v_k).$$

† In contradistinction to the turbulent wake; see §37.

‡ The fact that the relation **curl** $\mathbf{v} = 0$ does not remain valid along a streamline which passes over a solid surface has already been noted (§9).

We write the pressure in the form $p = p_0 + p'$, where p_0 is the pressure at infinity. The integration of the constant term $p_0\delta_{ik} + \rho U_i U_k$ gives zero, since the vector integral $\oint \mathbf{df}$ over a closed surface is zero. The integral $\oint \rho v_k \mathrm{d}f_k$ also vanishes: since the total mass of fluid in the volume considered is constant, the total mass flux through the surface surrounding the volume must be zero. Finally, the velocity $\mathbf{v}$ far from the body is small compared with $\mathbf{U}$. Hence, if the surface in question is sufficiently far from the body, we can neglect the term $\rho v_i v_k$ in Π_{ik} as compared with $\rho U_k v_i$. Thus the total momentum flux is

$$\oint (p'\delta_{ik} + \rho U_k v_i)\mathrm{d}f_k.$$

Let us now take the fluid volume concerned to be the volume between two infinite planes x = constant, one of them far in front of the body and the other far behind it. The integral over the infinitely distant "lateral" surface vanishes (since $p' = \mathbf{v} = 0$ at infinity), and it is therefore sufficient to integrate only over the two planes. The momentum flux thus obtained is evidently the difference between the total momentum flux entering through the forward plane and that leaving through the backward plane. This difference, however, is just the quantity of momentum transmitted to the body by the fluid per unit time, i.e. the force $\mathbf{F}$ exerted on the body.

Thus the components of the force $\mathbf{F}$ are

$$F_x = \left(\iint\limits_{x=x_2} - \iint\limits_{x=x_1}\right)(p' + \rho U v_x)\mathrm{d}y\,\mathrm{d}z,$$

$$F_y = \left(\iint\limits_{x=x_2} - \iint\limits_{x=x_1}\right)\rho U v_y \mathrm{d}y\,\mathrm{d}z,$$

$$F_z = \left(\iint\limits_{x=x_2} - \iint\limits_{x=x_1}\right)\rho U v_z \mathrm{d}y\,\mathrm{d}z,$$

where the integration is taken over the infinite planes $x = x_1$ (far behind the body) and $x = x_2$ (far in front of it). Let us first consider the expression for F_x.

Outside the wake we have potential flow, and therefore Bernoulli's equation

$$p + \tfrac{1}{2}\rho(\mathbf{U}+\mathbf{v})^2 = \text{constant} \equiv p_0 + \tfrac{1}{2}\rho U^2$$

holds, or, neglecting the term $\frac{1}{2}\rho v^2$ in comparison with $\rho\mathbf{U}\cdot\mathbf{v}$,

$$p' = -\rho U v_x.$$

We see that in this approximation the integrand in F_x vanishes everywhere outside the wake. In other words, the integral over the plane $x = x_2$ (which lies in front of the body and does not intersect the wake) is zero, and the integral over the plane $x = x_1$ need be taken only over the area covered by the cross-section of the wake. Inside the wake, however, the pressure change p' is of the order of ρv^2, i.e. small compared with $\rho U v_x$. Thus we reach the result that the drag on the body is

$$F_x = -\rho U \iint v_x \,\mathrm{d}y\,\mathrm{d}z, \tag{21.1}$$

where the integration is taken over the cross-sectional area of the wake far behind the body. The velocity v_x in the wake is, of course, negative: the fluid moves more slowly than it would if the body were absent. Attention is called to the fact that the integral in (21.1) gives the amount by which the discharge through the wake falls short of its value in the absence of the body.

Let us now consider the force (whose components are F_y, F_z) which tends to move the body transversely. This force is called the *lift*. Outside the wake, where we have potential flow, we can write $v_y = \partial\phi/\partial y$, $v_z = \partial\phi/\partial z$; the integral over the plane $x = x_2$, which does not meet the wake, is zero:

$$\iint v_y \,\mathrm{d}y\,\mathrm{d}z = \iint \frac{\partial\phi}{\partial y}\mathrm{d}y\,\mathrm{d}z = 0, \qquad \iint \frac{\partial\phi}{\partial z}\mathrm{d}y\,\mathrm{d}z = 0,$$

since $\phi = 0$ at infinity. We therefore find for the lift

$$F_y = -\rho U \iint v_y \,\mathrm{d}y\,\mathrm{d}z, \qquad F_z = -\rho U \iint v_z \,\mathrm{d}y\,\mathrm{d}z. \tag{21.2}$$

The integration in these formulae is again taken only over the cross-sectional area of the wake. If the body has an axis of symmetry (not necessarily complete axial symmetry), and the flow is parallel to this axis, then the flow past the body has an axis of symmetry also. In this case the lift is, of course, zero.

Let us return to the flow in the wake. An estimate of the magnitudes of various terms in the Navier–Stokes equation shows that the term $\nu \triangle \mathbf{v}$ can in general be neglected at distances r from the body such that $rU/\nu \gg 1$ (cf. the derivation of the opposite condition (20.16)); these are the distances at which the flow outside the wake may be regarded as potential flow. It is not possible to neglect that term inside the wake even at these distances, however, since the transverse derivatives $\partial^2\mathbf{v}/\partial y^2$, $\partial^2\mathbf{v}/\partial z^2$ are large compared with $\partial^2\mathbf{v}/\partial x^2$.

Let Y be of the order of magnitude of the width of the wake, i.e. the distances from the x-axis at which the velocity $\mathbf{v}$ falls off markedly. The order of magnitude of the terms in the Navier–Stokes equation is then

$$(\mathbf{v}\cdot\mathbf{grad})\mathbf{v} \sim U\,\partial v/\partial x \sim Uv/x, \quad \nu\triangle\mathbf{v} \sim \nu\,\partial^2 v/\partial y^2 \sim \nu v/Y^2.$$

If these two magnitudes are comparable, we find

$$Y = \sqrt{(\nu x/U)}. \tag{21.3}$$

This quantity is in fact small compared with x, by the assumed condition $Ux/\nu \gg 1$. Thus the width of the laminar wake increases as the square root of the distance from the body.

In order to determine how the velocity decreases with increasing x in the wake, we return to formula (21.1). The region of integration has an area of the order of Y^2. Hence the integral can be estimated as $F_x \sim \rho U v Y^2$, and by using the relation (21.3) we obtain

$$v \sim F_x/\rho\nu x. \tag{21.4}$$

Having thus elucidated the qualitative features of laminar flow far from the body, we will now derive some quantitative formulae describing the flow pattern inside and outside the wake.

Flow inside the wake

In the Navier–Stokes equation for steady flow,

$$(\mathbf{v}\cdot\mathbf{grad})\,\mathbf{v} = -\mathbf{grad}\,(p/\rho) + \nu\triangle\mathbf{v}, \tag{21.5}$$

we use far from the body Oseen's approximation, replacing the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ by $(\mathbf{U}\cdot\mathbf{grad})\mathbf{v}$; cf. (20.17). Furthermore, inside the wake the derivative with respect to the longitudinal coordinate x in $\triangle v$ can be neglected in comparison with the transverse derivatives. We thus start from the equation

$$U\frac{\partial \mathbf{v}}{\partial x} = -\mathbf{grad}\,(p/\rho) + \nu\left(\frac{\partial^2 \mathbf{v}}{\partial y^2} + \frac{\partial^2 \mathbf{v}}{\partial z^2}\right). \tag{21.6}$$

We seek the solution of this in the form $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1$ is the solution of

$$U\frac{\partial \mathbf{v}_1}{\partial x} = \nu\left(\frac{\partial^2 \mathbf{v}_1}{\partial y^2} + \frac{\partial^2 \mathbf{v}_1}{\partial z^2}\right). \tag{21.7}$$

The quantity $\mathbf{v}_2$ arising from the term $-\mathbf{grad}\,(p/\rho)$ in the initial equation (21.6) may be sought as the gradient of a scalar Φ.† Since, far from the body, the derivatives with respect to x are small in comparison with those with respect to y and z, in the approximation considered we may neglect the term $\partial\Phi/\partial x$, i.e. take $v_x = v_{1x}$. We thus have for v_x the equation

$$U\frac{\partial v_x}{\partial x} = \nu\left(\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right). \tag{21.8}$$

This is formally the same as the two-dimensional equation of heat conduction, with x/U in place of the time, and the viscosity ν in place of the thermometric conductivity. The solution which decreases with increasing y and z (for fixed x) and gives an infinitely narrow wake as $x \to 0$ (in this approximation the dimensions of the body are regarded as small) is (cf. §51)

$$v_x = -\frac{F_x}{4\pi\rho\nu x}\exp\{-U(y^2+z^2)/4\nu x\}. \tag{21.9}$$

The constant coefficient in this formula is expressed in terms of the drag by means of formula (21.1), in which the integration may be extended over the whole yz-plane because of the rapid convergence. If the Cartesian coordinates are replaced by spherical polar coordinates r, θ, ϕ with the polar axis along the x-axis, then the region of the wake, $\sqrt{(y^2+z^2)} \ll x$, corresponds to $\theta \ll 1$. In these coordinates, formula (21.9) becomes

$$v_x = -\frac{F_x}{4\pi\rho\nu r}\exp\{-Ur\theta^2/4\nu\}. \tag{21.10}$$

The term in $\partial\Phi/\partial x$ (with Φ given by formula (21.12) below), which we have omitted, would give a term in v_x which contains an additional small factor θ.

The form of v_{1y} and v_{1z} must be the same as (21.9) but with different coefficients. We take the direction of the lift as the y-axis (so that $F_z = 0$). According to (21.2) we have, since $\Phi = 0$ at infinity,

$$\iint v_y\,dy\,dz = \iint (v_{1y} + \partial\Phi/\partial y)\,dy\,dz$$

$$= \iint v_{1y}\,dy\,dz = -F_y/\rho U,$$

$$\iint v_{1z}\,dy\,dz = 0.$$

† The velocity potential will be denoted in the rest of this section by Φ, so as to distinguish it from the azimuthal angle ϕ in spherical polar coordinates.

It is therefore clear that v_{1y} difffers from (21.9) in that F_x is replaced by F_y, and $v_{1z} = 0$. Thus we find

$$v_y = -\frac{F_x}{4\pi\rho\nu x}\exp\{-U(y^2+z^2)/4\nu x\} + \partial\Phi/\partial y, \quad v_z = \partial\Phi/\partial z. \tag{21.11}$$

To determine the function Φ, we proceed as follows. We write the equation of continuity, neglecting the longitudinal derivative:

$$\operatorname{div}\mathbf{v} \cong \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi + \frac{\partial v_{1y}}{\partial y} = 0.$$

Differentiating this equation with respect to x and using equation (21.7) for v_{1y}, we obtain

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{\partial\Phi}{\partial x} = -\frac{\partial}{\partial y}\left(\frac{\partial v_{1y}}{\partial x}\right)$$

$$= -\frac{\nu}{U}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{\partial v_{1y}}{\partial y}.$$

Hence

$$\partial\Phi/\partial x = -(\nu/U)\partial v_{1y}/\partial y.$$

Finally, substituting the expression for v_{1y} (the first term in (21.11)) and integrating with respect to x, we have

$$\Phi = -\frac{F_y}{2\pi\rho U}\frac{y}{y^2+z^2}\{\exp[-U(y^2+z^2)/4\nu x] - 1\}; \tag{21.12}$$

the constant of integration is chosen so that Φ remains finite when $y = z = 0$. In spherical polar coordinates (with the azimuthal angle ϕ measured from the xy-plane)

$$\Phi = -\frac{F_y}{2\pi\rho U}\frac{\cos\phi}{r\theta}\{\exp[-Ur\theta^2/4\nu] - 1\}. \tag{21.13}$$

It is seen from (21.11)–(21.13) that v_y and v_z, unlike v_x, contain terms which decrease only as $1/\theta^2$ when we move away from the axis of the wake, as well as those which decrease exponentially with increasing θ (for a given r).

If there is no lift, the flow in the wake is axially symmetrical, and $\Phi \equiv 0$.‡

FLOW OUTSIDE THE WAKE

Outside the wake, potential flow may be assumed. Since we are interested only in the terms in the potential Φ which decrease least rapidly at large distances, we seek a solution of Laplace's equation

$$\triangle\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

‡ This is true, in particular, for the wake behind a sphere. In this connection it may be noted that the formulae obtained, like (21.16) below, are in agreement with the velocity distribution (20.24) for flow at very low Reynolds numbers. In this case, the whole of the flow pattern described is moved to very large distances $r \gg l/R$, where l is the size of the body.

as a sum of two terms:

$$\Phi = \frac{a}{r} + \frac{\cos\phi}{r} f(\theta), \tag{21.14}$$

of which the first is spherically symmetrical and belongs to the force F_x, while the second is symmetrical about the xy-plane and belongs to the force F_y.

We obtain for the function $f(\theta)$ the equation

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}f}{\mathrm{d}\theta}\right) - \frac{f}{\sin\theta} = 0.$$

The solution of this equation finite as $\theta \to \pi$ is

$$f = b\cot\tfrac{1}{2}\theta. \tag{21.15}$$

The coefficient b must be determined from the condition for joining the solution to that inside the wake. The reason is that (21.13) relates to the angle range $\theta \ll 1$, and (21.14) to $\theta \gg \sqrt{(\nu/Ur)}$. These ranges overlap when $\sqrt{(\nu/Ur)} \ll \theta \ll 1$, and (21.13) then becomes

$$\Phi = \frac{F_y}{2\pi\rho U}\frac{\cos\phi}{r\theta},$$

and the second term in (21.14) is $(2b/r\theta)\cos\phi$. Comparison of these expressions shows that we must take $b = F_y/4\pi\rho U$.

To determine the coefficient a in (21.14), we notice that the total mass flux through a sphere S with large radius r equals zero, as for any closed surface. The rate of inflow through the part S_0 of S intercepted by the wake is

$$-\iint_{S_0} v_x\,\mathrm{d}y\,\mathrm{d}z = F_x/\rho U.$$

Hence the same quantity must flow out through the rest of the surface of the sphere, i.e. we must have

$$\oint_{S-S_0} \mathbf{v}\cdot\mathbf{df} = F_x/\rho U.$$

Since S_0 is small compared with S, we can put

$$\oint_S \mathbf{v}\cdot\mathbf{df} = \int_S \mathbf{grad}\,\Phi\cdot\mathbf{df} = -4\pi a = F_x/\rho U, \tag{21.16}$$

whence $a = -F_x/4\pi\rho U$.

The complete expression for the velocity potential is thus

$$\Phi = \frac{1}{4\pi\rho U r}(-F_x + F_y\cos\phi\cot\tfrac{1}{2}\theta), \tag{21.17}$$

which gives the flow everywhere outside the wake far from the body. The potential diminishes with increasing distance as $1/r$; the velocity accordingly decreases as $1/r^2$. If there is no lift, the flow outside the wake is axially symmetrical.

§22. The viscosity of suspensions

A fluid in which numerous fine solid particles are suspended (forming a *suspension*) may be regarded as a homogeneous medium if we are concerned with phenomena whose characteristic lengths are large compared with the dimensions of the particles. Such a medium has an effective viscosity η which is different from the viscosity η_0 of the original fluid. The value of η can be calculated for the case where the concentration of the suspended particles is small (i.e. their total volume is small in comparison with that of the fluid). The calculations are relatively simple for the case of spherical particles (A. Einstein 1906).

It is necessary to consider first the effect of a single solid globule, immersed in a fluid, on flow having a constant velocity gradient. Let the unperturbed flow be described by a linear velocity distribution

$$v_{0i} = \alpha_{ik} x_k, \tag{22.1}$$

where α_{ik} is a constant symmetrical tensor. The fluid pressure is constant:

$$p_0 = \text{constant},$$

and in future we shall take p_0 to be zero, i.e. measure only the deviation from this constant value. If the fluid is incompressible (div $\mathbf{v}_0 = 0$), the sum of the diagonal elements, or trace, of the tensor α_{ik} must be zero:

$$\alpha_{ii} = 0. \tag{22.2}$$

Now let a small sphere with radius R be placed at the origin. We denote the altered fluid velocity by $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$; $\mathbf{v}_1$ must vanish at infinity, but near the sphere $\mathbf{v}_1$ is not small compared with $\mathbf{v}_0$. It is clear from the symmetry of the flow that the sphere remains at rest, so that the boundary condition is $\mathbf{v} = 0$ for $r = R$.

The required solution of the equations of motion (20.1) to (20.3) may be obtained at once from the solution (20.4), with the function f given by (20.6), if we notice that the space derivatives of this solution are themselves solutions. In the present case we desire a solution depending on the components of the tensor α_{ik} as parameters (and not on the vector $\mathbf{u}$ as in §20). Such a solution is

$$\mathbf{v}_1 = \mathbf{curl\, curl}\,[(\boldsymbol{\alpha}\cdot\mathbf{grad})f], \qquad p = \eta_0\alpha_{ik}\partial^2\triangle f/\partial x_i\partial x_k,$$

where $(\boldsymbol{\alpha}\cdot\mathbf{grad})f$ denotes a vector whose components are $\alpha_{ik}\partial f/\partial x_k$. Expanding these expressions and determining the constants a and b in the function $f = ar + b/r$ so as to satisfy the boundary conditions at the surface of the sphere, we obtain the following formulae for the velocity and pressure:

$$v_{1i} = \frac{5}{2}\left(\frac{R^5}{r^4} - \frac{R^3}{r^2}\right)\alpha_{kl}n_i n_k n_l - \frac{R^5}{r^4}\alpha_{ik}n_k, \tag{22.3}$$

$$p = -5\eta_0\frac{R^3}{r^3}\alpha_{ik}n_i n_k, \tag{22.4}$$

where $\mathbf{n}$ is a unit vector in the direction of the position vector.

Returning now to the problem of determining the effective viscosity of a suspension, we calculate the mean value (over the volume) of the momentum flux density tensor Π_{ik}, which, in the linear approximation with respect to the velocity, is the same as the stress tensor $-\sigma_{ik}$:

$$\bar{\sigma}_{ik} = (1/V)\int \sigma_{ik}\,\mathrm{d}V.$$

The integration here may be taken over the volume V of a sphere with large radius, which is then extended to infinity.

First of all, we have the identity

$$\bar{\sigma}_{ik} = \eta_0\left(\overline{\frac{\partial v_i}{\partial x_k}}+\overline{\frac{\partial v_k}{\partial x_i}}\right) - \bar{p}\delta_{ik} + \frac{1}{V}\int\left\{\sigma_{ik} - \eta_0\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right) + p\,\delta_{ik}\right\}\mathrm{d}V. \tag{22.5}$$

The integrand on the right is zero except within the solid spheres; since the concentration of the suspension is supposed small, the integral may be calculated for a single sphere as if the others were absent, and then multiplied by the concentration n of the suspension (the number of spheres per unit volume). The direct calculation of this integral would require an investigation of internal stresses in the spheres. We can circumvent this difficulty, however, by transforming the volume integral into a surface integral over an infinitely distant sphere, which lies entirely in the fluid. To do so, we note that the equation of motion $\partial\sigma_{il}/\partial x_l = 0$ leads to the identity

$$\sigma_{ik} = \partial(\sigma_{il}x_k)/\partial x_l;$$

hence the transformation of the volume integral into a surface integral gives

$$\bar{\sigma}_{ik} = \eta_0\left(\overline{\frac{\partial v_i}{\partial x_k}}+\overline{\frac{\partial v_k}{\partial x_i}}\right) + n\oint\{\sigma_{il}x_k\,\mathrm{d}f_l - \eta_0(v_i\,\mathrm{d}f_k + v_k\,\mathrm{d}f_i)\}.$$

We have omitted the term in $\bar{p}$, since the mean pressure is necessarily zero; $\bar{p}$ is a scalar, which must be given by a linear combination of the components α_{ik}, and the only such scalar is $\alpha_{ii} = 0$.

In calculating the integral over a sphere with very large radius, only the terms of order $1/r^2$ need be retained in the expression (22.3) for the velocity. A simple calculation gives the value of the integral as

$$n\eta_0\cdot 20\pi R^3\{5\alpha_{lm}\overline{n_i n_k n_l n_m} - \alpha_{il}\overline{n_k n_l}\},$$

where the bar denotes an average with respect to directions of the unit vector $\mathbf{n}$. Effecting the averaging,† we finally have

$$\bar{\sigma}_{ik} = \eta_0\left(\overline{\frac{\partial v_i}{\partial x_k}}+\overline{\frac{\partial v_k}{\partial x_i}}\right) + 5\eta_0\alpha_{ik}\cdot\tfrac{4}{3}\pi R^3 n. \tag{22.6}$$

The first term in (22.6), on substitution of v_0 from (22.1), gives $2\eta_0\alpha_{ik}$; the first-order small component is identically zero after averaging with respect to the directions of $\mathbf{n}$, as it should be, since the effect resides entirely in the integral separated in (22.5). Hence the required relative correction to the effective viscosity η of the suspension is determined by the ratio of the second and first terms in (22.6). We thus obtain

$$\eta = \eta_0(1 + \tfrac{5}{2}\phi), \quad \phi = 4\pi R^3 n/3, \tag{22.7}$$

† The required mean values of products of components of the unit vector are symmetrical tensors, which can be formed only from the unit tensor δ_{ik}. We then easily find

$$\overline{n_i n_k} = \tfrac{1}{3}\delta_{ik},$$

$$\overline{n_i n_k n_l n_m} = \tfrac{1}{15}(\delta_{ik}\delta_{lm} + \delta_{il}\delta_{km} + \delta_{im}\delta_{kl}).$$

where ϕ is the small ratio of the total volume of the spheres to the total volume of the suspension.

The corresponding calculations and results become very lengthy even for a suspension of spheroidal particles.† As an illustration, we give the numerical values of the correction factor A in the formula

$$\eta = \eta_0(1 + A\phi), \qquad \phi = 4\pi ab^2 n/3,$$

for various values of a/b, where a and $b = c$ are the semi-axes of the spheroids:

a/b	0·1	0·2	0·5	1·0	2	5	10
A	8·04	4·71	2·85	2·5	2·91	5·81	13·6

The correction increases on either side of the value $a/b = 1$ which corresponds to spherical particles.

§23. Exact solutions of the equations of motion for a viscous fluid

If the non-linear terms in the equations of motion of a viscous fluid do not vanish identically, the solving of these equations offers great difficulties, and exact solutions can be obtained only in a very small number of cases. Such solutions are of considerable methodological interest, if not always of physical interest (because in practice turbulence occurs when the Reynolds number is sufficiently large).

We give below examples of exact solutions of the equations of motion for a viscous fluid.

ENTRAINMENT OF FLUID BY A ROTATING DISK

An infinite plane disk immersed in a viscous fluid rotates uniformly about its axis. Determine the motion of the fluid caused by this motion of the disk (T. von Kármán 1921).

We take cylindrical polar coordinates, with the plane of the disk as the plane $z = 0$. Let the disk rotate about the z-axis with angular velocity Ω. We consider the unbounded volume of fluid on the side $z > 0$. The boundary conditions are

$$v_r = 0, \quad v_\phi = \Omega r, \quad v_z = 0 \quad \text{for} \quad z = 0,$$
$$v_r = 0, \quad v_\phi = 0 \qquad\qquad \text{for} \quad z = \infty.$$

The axial velocity v_z does not vanish as $z \to \infty$, but tends to a constant negative value determined by the equations of motion. The reason is that, since the fluid moves radially away from the axis of rotation, especially near the disk, there must be a constant vertical flow from infinity in order to satisfy the equation of continuity. We seek a solution of the equations of motion in the form

$$\left.\begin{aligned} v_r = r\Omega F(z_1); \quad v_\phi = r\Omega G(z_1); \quad v_z = \sqrt{(\nu\Omega)}H(z_1); \\ p = -\rho\nu\Omega P(z_1), \quad \text{where} \quad z_1 = \sqrt{(\Omega/\nu)}z. \end{aligned}\right\} \tag{23.1}$$

In this velocity distribution, the radial and azimuthal velocities are proportional to the distance from the axis of rotation, while v_z is constant on each horizontal plane.

† In the flow of a suspension of non-spherical particles, the presence of velocity gradients has an orienting effect on them. The simultaneous action of orienting hydrodynamic forces and disorienting rotary Brownian motion gives rise to an anisotropic distribution of the particles as regards their orientation in space. This, however, need not be considered when calculating the correction to the viscosity η: the anisotropy of the orientation distribution is itself dependent on the velocity gradients (linearly in the first approximation), and including it would give stress tensor terms non-linear in the gradients.

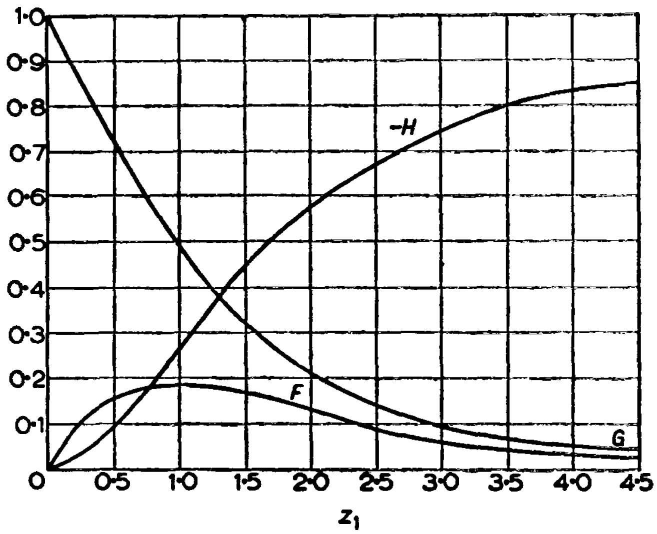

FIG. 7

Substituting in the Navier–Stokes equation and in the equation of continuity, we obtain the following equations for the functions F, G, H and P:

$$\left.\begin{aligned} F^2-G^2+F'H &= F'', & 2FG+G'H &= G'', \\ HH' &= P'+H'', & 2F+H' &= 0; \end{aligned}\right\} \tag{23.2}$$

the prime denotes differentiation with respect to z_1. The boundary conditions are

$$\left.\begin{aligned} &F = 0, \quad G = 1, \quad H = 0 \quad \text{for} \quad z_1 = 0. \\ &F = 0, \quad G = 0 \qquad\qquad\;\; \text{for} \quad z_1 = \infty. \end{aligned}\right\} \tag{23.3}$$

We have therefore reduced the solution of the problem to the integration of a system of ordinary differential equations in one variable; this can be achieved numerically. Figure 7 shows the functions F, G and $-H$ thus obtained. The limiting value of H as $z_1 \to \infty$ is $-0{\cdot}886$; in other words, the fluid velocity at infinity is $v_z(\infty) = -0{\cdot}886\sqrt{(\nu\Omega)}$.

The frictional force acting on unit area of the disk perpendicularly to the radius is $\sigma_{z\phi} = \eta(\partial v_\phi/\partial z)_{z=0}$. Neglecting edge effects, we may write the moment of the frictional forces acting on a disk with large but finite radius R as

$$M = 2\int_0^R 2\pi r^2 \sigma_{z\phi}\,\mathrm{d}r = \pi R^4 \rho\sqrt{(\nu\Omega^3)}G'(0).$$

The factor 2 in front of the integral appears because the disk has two sides exposed to the fluid. A numerical calculation of the function G leads to the formula

$$M = -1{\cdot}94\,R^4\rho\sqrt{(\nu\Omega^3)}. \tag{23.4}$$

FLOW IN DIVERGING AND CONVERGING CHANNELS

Determine the steady flow between two plane walls meeting at an angle α (Fig. 8 shows a cross-section of the two planes); the fluid flows out from the line of intersection of the planes (G. Hamel 1917).

We take cylindrical polar coordinates r, z, ϕ, with the z-axis along the line of intersection of the planes (the point O in Fig. 8), and the angle ϕ measured as shown in Fig. 8. The flow is

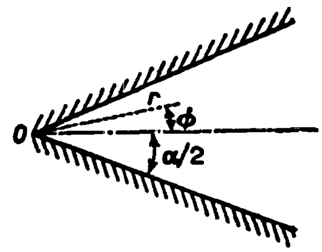

FIG. 8

uniform in the z-direction, and we naturally assume it to be entirely radial, i.e.

$$v_\phi = v_z = 0, \qquad v_r = v(r, \phi).$$

The equations (15.18) give

$$v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \phi^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right), \tag{23.5}$$

$$-\frac{1}{\rho r}\frac{\partial p}{\partial \phi} + \frac{2\nu}{r^2}\frac{\partial v}{\partial \phi} = 0, \tag{23.6}$$

$$\partial(rv)/\partial r = 0.$$

It is seen from the last of these that rv is a function of ϕ only. Introducing the function

$$u(\phi) = rv/6\nu, \tag{23.7}$$

we obtain from (23.6)

$$\frac{1}{\rho}\frac{\partial p}{\partial \phi} = \frac{12\nu^2}{r^2}\frac{\mathrm{d}u}{\mathrm{d}\phi},$$

whence

$$\frac{p}{\rho} = \frac{12\nu^2}{r^2}u(\phi) + f(r).$$

Substituting this expression in (23.5), we have

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + 4u + 6u^2 = \frac{1}{6\nu^2}r^3 f'(r),$$

from which we see that, since the left-hand side depends only on ϕ and the right-hand side only on r, each must be a constant, which we denote by $2C_1$. Thus $f'(r) = 12\nu^2 C_1/r^3$, whence $f(r) = -6\nu^2 C_1/r^2 + \text{constant}$, and we have for the pressure

$$\frac{p}{\rho} = \frac{6\nu^2}{r^2}(2u - C_1) + \text{constant}. \tag{23.8}$$

For $u(\phi)$ we have the equation

$$u'' + 4u + 6u^2 = 2C_1,$$

which, on multiplication by u' and one integration, gives

$$\tfrac{1}{2}u'^2 + 2u^2 + 2u^3 - 2C_1 u - 2C_2 = 0.$$

Hence we have

$$2\phi = \pm \int \frac{\mathrm{d}u}{\sqrt{(-u^3 - u^2 + C_1 u + C_2)}} + C_3, \tag{23.9}$$

which gives the required dependence of the velocity on ϕ; the function $u(\phi)$ can be expressed in terms of elliptic functions. The three constants C_1, C_2, C_3 are determined from the boundary conditions at the walls

$$u(\pm\tfrac{1}{2}\alpha) = 0 \tag{23.10}$$

and from the condition that the same mass Q of fluid passes in unit time through any cross-section r = constant:

$$Q = \rho \int_{-\alpha/2}^{\alpha/2} vr\,\mathrm{d}\phi = 6\nu\rho \int_{-\alpha/2}^{\alpha/2} u\,\mathrm{d}\phi. \tag{23.11}$$

Q may be either positive or negative. If $Q > 0$, the line of intersection of the planes is a source, i.e. the fluid emerges from the vertex of the angle: this is called *flow in a diverging channel*. If $Q < 0$, the line of intersection is a sink, and we have *flow in a converging channel*. The ratio $|Q|/\nu\rho$ is dimensionless and plays the part of the Reynolds number in the problem considered.

Let us first discuss converging flow ($Q < 0$). To investigate the solution (23.9)–(23.11) we make the assumptions, which will be justified later, that the flow is symmetrical about the plane $\phi = 0$ (i.e. $u(\phi) = u(-\phi)$), and that the function $u(\phi)$ is everywhere negative (i.e. the velocity is everywhere towards the vertex) and decreases monotonically from $u = 0$ at $\phi = \pm\frac{1}{2}\alpha$ to $u = -u_0 < 0$ at $\phi = 0$, so that u_0 is the maximum value of $|u|$. Then for $u = -u_0$ we must have $\mathrm{d}u/\mathrm{d}\phi = 0$, whence it follows that $u = -u_0$ is a zero of the cubic expression under the radical in the integrand of (23.9). We can therefore write

$$-u^3 - u^2 + C_1 u + C_2 = (u+u_0)\{-u^2 - (1-u_0)u + q\},$$

where q is another constant. Thus

$$2\phi = \pm \int_{-u_0}^{u} \frac{\mathrm{d}u}{\sqrt{[(u+u_0)\{-u^2-(1-u_0)u+q\}]}}, \tag{23.12}$$

the constants u_0 and q being determined from the conditions

$$\left.\begin{aligned} \alpha &= \int_{-u_0}^{0} \frac{\mathrm{d}u}{\sqrt{[(u+u_0)\{-u^2-(1-u_0)u+q\}]}}, \\ \tfrac{1}{6}\mathrm{R} &= \int_{-u_0}^{0} \frac{u\,\mathrm{d}u}{\sqrt{[(u+u_0)\{-u^2-(1-u_0)u+q\}]}} \end{aligned}\right\} \tag{23.13}$$

($\mathrm{R} = |Q|/\nu\rho$); the constant q must be positive, since otherwise these integrals would be complex. The two equations just given may be shown to have solutions u_0 and q for any R and $\alpha < \pi$. In other words, convergent symmetrical flow (Fig. 9) is possible for any aperture

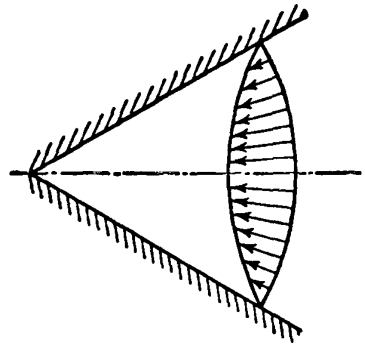

FIG. 9

angle $\alpha < \pi$ and any Reynolds number. Let us consider in more detail the flow for very large R. This corresponds to large u_0. Writing (23.12) (for $\phi > 0$) as

$$2(\tfrac{1}{2}\alpha - \phi) = \int\limits_u^0 \frac{\mathrm{d}u}{\sqrt{[(u+u_0)\{-u^2-(1-u_0)u+q\}]}},$$

we see that the integrand is small throughout the range of integration if $|u|$ is not close to u_0. This means that $|u|$ can differ appreciably from u_0 only for ϕ close to $\pm\frac{1}{2}\alpha$, i.e. in the immediate neighbourhood of the walls.† In other words, we have $u \cong$ constant $= -u_0$ for almost all angles ϕ, and in addition $u_0 = \mathrm{R}/6\alpha$, as we see from equations (23.13). The velocity v itself is $|Q|/\rho\alpha r$, giving a non-viscous potential flow with velocity independent of angle and inversely proportional to r. Thus, for large Reynolds numbers, the flow in a converging channel differs very little from potential flow of an ideal fluid. The effect of the viscosity appears only in a very narrow layer near the walls, where the velocity falls rapidly to zero from the value corresponding to the potential flow (Fig. 10).

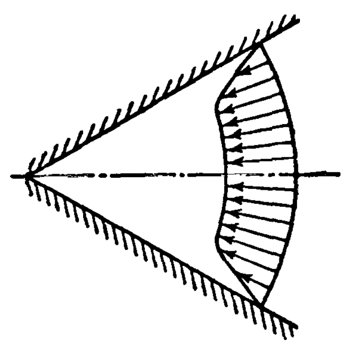

FIG. 10

Now let $Q > 0$, so that we have divergent flow. At first we again suppose that the flow is symmetrical about the plane $\phi = 0$, and that $u(\phi)$ (where now $u > 0$) varies monotonically from zero at $\phi = \pm\frac{1}{2}\alpha$ to $u_0 > 0$ at $\phi = 0$. Instead of (23.13) we now have

† The question may be asked how the integral can cease to be small, even if $u \cong -u_0$. The answer is that, for u_0 very large, one of the roots of $-u^2-(1-u_0)u+q = 0$ is close to $-u_0$, so that the radicand has two almost coincident zeros. the whole integral therefore being "almost divergent" at $u = -u_0$.

$$\left.\begin{aligned} \alpha &= \int\limits_0^{u_0} \frac{du}{\sqrt{[(u_0-u)\{u^2+(1+u_0)u+q\}]}}, \\ \tfrac{1}{6}\mathrm{R} &= \int\limits_0^{u_0} \frac{u\,du}{\sqrt{[(u_0-u)\{u^2+(1+u_0)u+q\}]}}. \end{aligned}\right\} \quad (23.14)$$

If we regard u_0 as given, then α increases monotonically as q decreases, and takes its greatest value for $q = 0$:

$$\alpha_{\max} = \int\limits_0^{u_0} \frac{du}{\sqrt{[u(u_0-u)(u+u_0+1)]}}.$$

It is easy to see that for given q, on the other hand, α is a monotonically decreasing function of u_0. Hence it follows that u_0 is a monotonically decreasing function of q for given α, so that its greatest value is for $q = 0$ and is given by the above equation. The maximum $\mathrm{R} = \mathrm{R}_{\max}$ corresponds to the maximum u_0. Using the substitutions $k^2 = u_0/(1+2u_0)$, $u = u_0 \cos^2 x$, we can write the dependence of $\mathrm{R}_{\max}$ on α in the parametric form

$$\left.\begin{aligned} \alpha &= 2\sqrt{(1-2k^2)} \int\limits_0^{\pi/2} \frac{dx}{\sqrt{(1-k^2\sin^2 x)}}, \\ \mathrm{R}_{\max} &= -6\alpha\frac{1-k^2}{1-2k^2} + \frac{12}{\sqrt{(1-2k^2)}} \int\limits_0^{\pi/2} \sqrt{(1-k^2\sin^2 x)}dx. \end{aligned}\right\} \quad (23.15)$$

Thus symmetrical flow, everywhere divergent (Fig. 11a), is possible for a given aperture angle only for Reynolds numbers not exceeding a definite value. As $\alpha \to \pi$ $(k \to 0)$, $\mathrm{R}_{\max} \to 0$; as $\alpha \to 0$ $(k \to 1/\sqrt{2})$, $\mathrm{R}_{\max}$ tends to infinity as $18{\cdot}8/\alpha$.

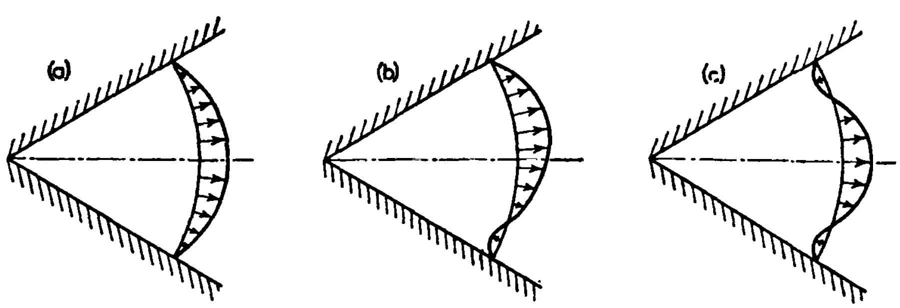

FIG. 11

For $\mathrm{R} > \mathrm{R}_{\max}$ the assumption of symmetrical flow, everywhere divergent, is unjustified, since the conditions (23.14) cannot be satisfied. In the range of angles $-\frac{1}{2}\alpha \leqslant \phi \leqslant \frac{1}{2}\alpha$ the function $u(\phi)$ must now have maxima or minima. The values of $u(\phi)$ corresponding to these extrema must again be zeros of the polynomial under the radical sign. It is therefore clear that the trinomial $u^2 + (1+u_0)u + q$ (with $u_0 > 0, q > 0$) must have two real negative roots in the range mentioned, so that the radicand can be written

$(u_0-u)\ (u+u_0')\ (u+u_0'')$, where $u_0>0$, $u_0'>0$, $u_0''>0$; we suppose $u_0'<u_0''$. The function $u(\phi)$ can evidently vary in the range $u_0 \geqslant u \geqslant -u_0'$, $u=u_0$ corresponding to a positive maximum of $u(\phi)$, and $u=-u_0'$ to a negative minimum. Without pausing to make a detailed investigation of the solutions obtained in this way, we may mention that for $\mathrm{R}>\mathrm{R}_{\max}$ a solution appears in which the velocity has one maximum and one minimum, the flow being asymmetric about the plane $\phi=0$ (Fig. 11b). When R increases further, a symmetrical solution with one maximum and two minima appears (Fig. 11c), and so on. In all these solutions, therefore, there are regions of both outward and inward flow (though of course the total discharge Q is positive). As $\mathrm{R}\to\infty$ the number of alternating minima and maxima increases without limit, so that there is no definite limiting solution. We may emphasize that in divergent flow as $\mathrm{R}\to\infty$ the solution does not, therefore, tend to the solution of Euler's equations as it does for convergent flow. Finally, it may be mentioned that, as R increases, the steady divergent flow of the kind described becomes unstable soon after R exceeds $\mathrm{R}_{\max}$, and in practice a non-steady or *turbulent* flow occurs (Chapter III).

SUBMERGED JET

Determine the flow in a jet emerging from the end of a narrow tube into an infinite space filled with the fluid—the *submerged jet* (L. Landau 1943).

We take spherical polar coordinates r, θ, ϕ, with the polar axis in the direction of the jet at its point of emergence, and with this point as origin. The flow is symmetrical about the polar axis, so that $v_\phi=0$ and v_θ, v_r are functions of r and θ only. The same total momentum flux (the "momentum of the jet") must pass through any closed surface surrounding the origin (in particular, through an infinitely distant surface). For this to be so, the velocity must be inversely proportional to r, so that

$$v_r = F(\theta)/r, \qquad v_\theta = f(\theta)/r, \tag{23.16}$$

where F and f are some functions of θ only. The equation of continuity is

$$\frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)=0.$$

Hence we find that

$$F(\theta) = -\mathrm{d}f/\mathrm{d}\theta - f\cot\theta. \tag{23.17}$$

The components $\Pi_{r\phi}$, $\Pi_{\theta\phi}$ of the momentum flux density tensor in the jet vanish identically by symmetry. We assume that the components $\Pi_{\theta\theta}$ and $\Pi_{\phi\phi}$ also vanish; this assumption is justified when we obtain a solution satisfying all the necessary conditions. Using the expressions (15.20) for the components of the tensor σ_{ik}, and formulae (23.16), (23.17), we easily see that the relation

$$\sin^2\theta\,\Pi_{r\theta} = \frac{1}{2}\frac{\partial}{\partial\theta}[\sin^2\theta(\Pi_{\phi\phi}-\Pi_{\theta\theta})]$$

holds between the components of the momentum flux density tensor in the jet. Hence it follows that $\Pi_{r\theta}=0$. Thus only the component Π_{rr} is non-zero, and it varies as $1/r^2$. It is easy to see that the equations of motion $\partial\Pi_{ik}/\partial x_k=0$ are automatically satisfied.

Next, we write

$$(\Pi_{\theta\theta}-\Pi_{\phi\phi})/\rho = (f^2+2\nu f\cot\theta-2\nu f')/r^2 = 0,$$

or

$$d(1/f)/d\theta + (1/f)\cot\theta + 1/2\nu = 0.$$

The solution of this equation is

$$f = -2\nu \sin\theta/(A - \cos\theta), \tag{23.18}$$

and then we have from (23.17)

$$F = 2\nu\left\{\frac{A^2-1}{(A-\cos\theta)^2} - 1\right\}. \tag{23.19}$$

The pressure distribution is found from the equation

$$\Pi_{\theta\theta}/\rho = p/\rho + f(f + 2\nu\cot\theta)/r^2 = 0,$$

which gives

$$p - p_0 = -\frac{4\rho\nu^2(A\cos\theta - 1)}{r^2(A-\cos\theta)^2}, \tag{23.20}$$

with p_0 the pressure at infinity. The constant A can be found in terms of the momentum of the jet, i.e. the total momentum flux in it. This flux is equal to the integral over the surface of a sphere

$$P = \oint \Pi_{rr}\cos\theta\, df = 2\pi\int_0^\pi r^2\Pi_{rr}\cos\theta\sin\theta\, d\theta.$$

The value of Π_{rr} is given by

$$\frac{1}{\rho}\Pi_{rr} = \frac{4\nu^2}{r^2}\left\{\frac{(A^2-1)^2}{(A-\cos\theta)^4} - \frac{A}{A-\cos\theta}\right\},$$

and a calculation of the integral gives

$$P = 16\pi\nu^2\rho A\left\{1 + \frac{4}{3(A^2-1)} - \tfrac{1}{2}A\log\frac{A+1}{A-1}\right\}. \tag{23.21}$$

Formulae (23.16)–(23.21) give the solution of the problem. When A varies from 1 to ∞, the jet momentum P takes all values between ∞ and 0.

The streamlines are determined by the equation $dr/v_r = r\,d\theta/v_\theta$, integration of which gives

$$\frac{r\sin^2\theta}{A-\cos\theta} = \text{constant}. \tag{23.22}$$

Figure 12 shows the characteristic form of the streamlines. The flow is a jet which comes from the origin and sucks in the surrounding fluid. If we arbitrarily regard as the boundary of the jet the surface where the streamlines have the least distance ($r\sin\theta$) from the axis, it is a cone with angle $2\theta_0$, where $\cos\theta_0 = 1/A$.

In the limiting case of a weak jet (small P, corresponding to large A), we have from (23.21)

$$P = 16\pi\nu^2\rho/A.$$

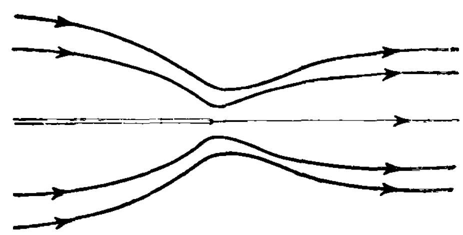

FIG. 12

In this case, the velocity is

$$v_\theta = -\frac{P}{8\pi\nu\rho}\frac{\sin\theta}{r}, \qquad v_r = \frac{P}{4\pi\nu\rho}\frac{\cos\theta}{r}. \tag{23.23}$$

In the opposite limit of a strong jet (large P, corresponding to $A \to 1$)†, we have

$$A = 1 + \tfrac{1}{2}\theta_0^2, \qquad \theta_0^2 = 64\pi\nu^2\rho/3P.$$

For large angles ($\theta \cong 1$), the velocity distribution is given by

$$v_\theta = -(2\nu/r)\cot\tfrac{1}{2}\theta, \qquad v_r = -2\nu/r; \tag{23.24}$$

for small angles ($\theta \cong \theta_0$),

$$v_\theta = -\frac{4\nu\theta}{(\theta_0^2+\theta^2)r}, \qquad v_r = \frac{8\nu\theta_0^2}{(\theta_0^2+\theta^2)^2 r}. \tag{23.25}$$

The solution here obtained is exact for a jet regarded as emerging from a point source. If the finite dimensions of the tube mouth are taken into account, the solution becomes the first term of an expansion in powers of the ratio of these dimensions to the distance r from the mouth of the tube. This is why, if we calculate from the above solution the total mass flux through a closed surface surrounding the origin, the result is zero. A non-zero total mass flux is obtained when further terms in the above-mentioned expansion are considered.‡

§24. Oscillatory motion in a viscous fluid

When a solid body immersed in a viscous fluid oscillates, the flow thereby set up has a number of characteristic properties. In order to study these, it is convenient to begin with a simple but typical example (G. G. Stokes 1851). Let us suppose that an incompressible fluid is bounded by an infinite plane surface which executes a simple harmonic oscillation in its own plane, with frequency ω. We require the resulting motion of the fluid. We take the

† However, the flow in a sufficiently strong jet is actually turbulent (§36). The Reynolds number for the jet considered is represented by the dimensionless parameter $\sqrt{(P/\rho\nu^2)}$.

‡ See Yu. B. Rumer, *Prikladnaya matematika i mekhanika* **16**, 255, 1952.

The submerged laminar jet with a non-zero angular momentum has been discussed by L. G. Loĭtsyanskiĭ (*ibid.* **17**, 3, 1953).

The hydrodynamic equations for any steady axially symmetrical flow of an incompressible viscous fluid with the velocity decreasing as $1/r$ can be reduced to a single second-order ordinary linear differential equation; see N. A. Slezkin, *Uchenye zapiski Moskovskogo gosudarstvennogo universiteta*, No. 2, 1934; *Prikladnaya matematika i mekhanika* **18**, 764, 1954.

solid surface as the yz-plane, and the fluid region as $x > 0$; the y-axis is taken in the direction of the oscillation. The velocity u of the oscillating surface is a function of time, of the form $A\cos(\omega t + \alpha)$. It is convenient to write this as the real part of a complex quantity:

$$u = \mathrm{re}(u_0 e^{-i\omega t}),$$

where the constant $u_0 = Ae^{-i\alpha}$ is in general complex, but can always be made real by a proper choice of the origin of time.

So long as the calculations involve only linear operations on the velocity u, we may omit the sign re and proceed as if u were complex, taking the real part of the final result. Thus we write

$$u_y = u = u_0 e^{-i\omega t}. \tag{24.1}$$

The fluid velocity must satisfy the boundary condition $\mathbf{v} = \mathbf{u}$ for $x = 0$, i.e. $v_x = v_z = 0$, $v_y = u$.

It is evident from symmetry that all quantities will depend only on the coordinate x and the time t. From the equation of continuity $\operatorname{div} \mathbf{v} = 0$ we therefore have $\partial v_x/\partial x = 0$, whence $v_x = \text{constant} = \text{zero}$, from the boundary condition. Since all quantities are independent of the coordinates y and z, and since v_x is zero, it follows that $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v} = 0$ identically. The equation of motion (15.7) becomes

$$\partial\mathbf{v}/\partial t = -(1/\rho)\,\mathbf{grad}\,p + \nu\triangle\mathbf{v}. \tag{24.2}$$

This is a linear equation. Its x-component is $\partial p/\partial x = 0$, i.e. $p = \text{constant}$.

It is further evident from symmetry that the velocity $\mathbf{v}$ is everywhere in the y-direction. For $v_y = v$ we have by (24.2)

$$\partial v/\partial t = \nu\partial^2 v/\partial x^2, \tag{24.3}$$

that is, a (one-dimensional) heat conduction equation. We shall look for a solution of this equation which is periodic in x and t, of the form

$$v = u_0 e^{i(kx - \omega t)},$$

so that $v = u$ for $x = 0$. Substituting in (24.3), we find

$$i\omega = \nu k^2, \qquad k = (1+i)/\delta, \qquad \delta = \sqrt{(2\nu/\omega)}, \tag{24.4}$$

so that the velocity is

$$v = u_0 e^{-x/\delta} e^{i(x/\delta - \omega t)}; \tag{24.5}$$

the choice of the sign of $\sqrt{i}$ in (24.4) is determined by the need for the velocity to decrease into the fluid.

Thus transverse waves can occur in a viscous fluid, with the velocity $v_y = v$ perpendicular to the direction of propagation. They are, however, rapidly damped as we move away from the solid surface whose motion generates the waves. The amplitude damping is exponential, the *depth of penetration* being δ.† This depth decreases with increasing frequency of the wave, but increases with the kinematic viscosity of the fluid.

The frictional force on the solid surface is evidently in the y-direction. The force per unit area is

$$\sigma_{xy} = \eta(\partial v_y/\partial x)_{x=0} = \sqrt{(\tfrac{1}{2}\omega\eta\rho)}\,(i-1)u. \tag{24.6}$$

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† Over a distance δ, the wave amplitude decreases by a factor of e; over one wavelength, it decreases by a factor of $e^{2\pi} \cong 540$.

Supposing u_0 real and taking the real part of (24.6), we have

$$\sigma_{xy} = -\sqrt{(\omega\eta\rho)}u_0 \cos(\omega t + \tfrac{1}{4}\pi).$$

The velocity of the oscillating surface, however, is $u = u_0 \cos \omega t$. There is therefore a phase difference between the velocity and the frictional force.†

It is easy to calculate also the (time) average of the energy dissipation in the above problem. This may be done by means of the general formula (16.3); in this particular case, however, it is simpler to calculate the required dissipation directly as the work done by the frictional forces. The energy dissipated per unit time per unit area of the oscillating plane is equal to the mean value of the product of the force σ_{xy} and the velocity $u_y = u$:

$$-\overline{\sigma_{xy}u} = \tfrac{1}{2}u_0{}^2\sqrt{(\tfrac{1}{2}\omega\eta\rho)}. \tag{24.7}$$

It is proportional to the square root of the frequency of the oscillations, and to the square root of the viscosity.

An explicit solution can also be given of the problem of a fluid set in motion by a plane surface moving in its plane according to any law $u = u(t)$. We shall not pause to give the corresponding calculations here, since the required solution of equation (24.3) is formally identical with that of an analogous problem in the theory of thermal conduction, which we shall discuss in §52 (the solution is formula (52.15)). In particular, the frictional force on unit area of the surface is given by

$$\sigma_{xy} = -\sqrt{\frac{\eta\rho}{\pi}} \int_{-\infty}^{t} \frac{\mathrm{d}u(\tau)}{\mathrm{d}\tau} \frac{\mathrm{d}\tau}{\sqrt{(t-\tau)}}; \tag{24.8}$$

cf. (52.14).

Let us now consider the general case of an oscillating body with any shape. In the case of an oscillating plane considered above, the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ in the equation of motion of the fluid was identically zero. This does not happen, of course, for a surface with arbitrary shape. We shall assume, however, that this term is small in comparison with the other terms, so that it may be neglected. The conditions necessary for this procedure to be valid will be examined below.

We shall therefore begin, as before, from the linear equation (24.2). We take the curl of both sides; the term **curl grad** p vanishes identically, giving

$$\partial(\mathbf{curl}\,\mathbf{v})/\partial t = \nu \triangle\, \mathbf{curl}\,\mathbf{v}, \tag{24.9}$$

i.e. **curl v** satisfies a heat conduction equation. We have seen above, however, that such an equation gives an exponential decrease of the quantity which satisfies it. We can therefore say that the vorticity decreases towards the interior of the fluid. In other words, the motion of the fluid caused by the oscillations of the body is rotational in a certain layer round the

† For oscillations of a half-plane (parallel to its edge) there is an additional frictional force due to edge effects. The problem of the motion of a viscous fluid caused by oscillations of a half-plane, and also the more general problem of the oscillations of a wedge with any angle, can be solved by a class of solutions of the equation $\triangle f + k^2 f = 0$, used in the theory of diffraction by a wedge. We give here, for reference, only one result: the increase in the frictional force on a half-plane, arising from the edge effect, can be regarded as the result of increasing the area of the half-plane by moving the edge a distance $\tfrac{1}{2}\delta$, with δ as in (24.4) (L. D. Landau 1947).

body, while at larger distances it rapidly changes to potential flow. The depth of penetration of the rotational flow is of the order of δ.

Two important limiting cases are possible here: the quantity δ may be either large or small compared with the dimension of the oscillating body. Let l be the order of magnitude of this dimension. We first consider the case $\delta \gg l$; this implies that $l^2\omega \ll \nu$. Besides this condition, we shall also suppose that the Reynolds number is small. If a is the amplitude of the oscillations, the velocity of the body is of the order of $a\omega$. The Reynolds number for the flow in question is therefore $\omega a l/\nu$. We thus suppose that

$$l^2\omega \ll \nu, \qquad \omega a l/\nu \ll 1. \tag{24.10}$$

This is the case of low frequencies of oscillation, which in turn means that the velocity varies only slowly with time, and therefore that we can neglect the derivative $\partial \mathbf{v}/\partial t$ in the general equation of motion $\partial \mathbf{v}/\partial t + (\mathbf{v}\cdot\mathbf{grad})\mathbf{v} = -(1/\rho)\mathbf{grad}\, p + \nu\Delta\mathbf{v}$. The term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$, on the other hand, can be neglected because the Reynolds number is small.

The absence of the term $\partial \mathbf{v}/\partial t$ from the equation of motion means that the flow is steady. Thus, for $\delta \gg l$, the flow can be regarded as steady at any given instant. This means that the flow at any given instant is what it would be if the body were moving uniformly with its instantaneous velocity. If, for example, we are considering the oscillations of a sphere immersed in the fluid, with a frequency satisfying the inequalities (24.10) (l being now the radius of the sphere), then we can say that the drag on the sphere will be that given by Stokes' formula (20.14) for uniform motion of the sphere at small Reynolds numbers.

Let us now consider the opposite case, where $l \gg \delta$. In order that the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ should again be negligible, it is necessary that the amplitude of the oscillations should be small in comparison with the dimensions of the body:

$$l^2\omega \gg \nu, \qquad a \ll l; \tag{24.11}$$

in this case, it should be noticed, the Reynolds number need not be small. The above inequality is obtained by estimating the magnitude of $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$. The operator $(\mathbf{v}\cdot\mathbf{grad})$ denotes differentiation in the direction of the velocity. Near the surface of the body, however, the velocity is nearly tangential. In the tangential direction the velocity changes appreciably only over distances of the order of the dimension of the body. Hence

$$(\mathbf{v}\cdot\mathbf{grad})\mathbf{v} \sim v^2/l \sim a^2\omega^2/l,$$

since the velocity itself is of the order of $a\omega$. The derivative $\partial \mathbf{v}/\partial t$, however, is of the order of $v\omega \sim a\omega^2$. Comparing these, we see that

$$(\mathbf{v}\cdot\mathbf{grad})\mathbf{v} \ll \partial \mathbf{v}/\partial t$$

if $a \ll l$. The terms $\partial \mathbf{v}/\partial t$ and $\nu\triangle\mathbf{v}$ are then easily seen to be of the same order.

We may now discuss the nature of the flow round an oscillating body when the conditions (24.11) hold. In a thin layer near the surface of the body the flow is rotational, but in the rest of the fluid we have potential flow.† Hence the flow everywhere except in the layer adjoining the body is given by the equations

$$\mathbf{curl}\,\mathbf{v} = 0, \qquad \operatorname{div}\mathbf{v} = 0. \tag{24.12}$$

† For oscillations of a plane surface not only **curl** $\mathbf{v}$ but also $\mathbf{v}$ itself decreases exponentially with characteristic distance δ. This is because the oscillating plane does not displace the fluid, and therefore the fluid remote from it remains at rest. For oscillations of bodies with other shapes the fluid is displaced, and therefore executes a motion where the velocity decreases appreciably only over distances of the order of the dimension of the body.

Hence it follows that $\triangle \mathbf{v} = 0$, and the Navier–Stokes equation reduces to Euler's equation. The flow is therefore ideal everywhere except in the surface layer. Since this layer is thin, in solving equations (24.12) to determine the flow of the rest of the fluid we should take as boundary condtions those which must be satisfied at the surface of the body, i.e. that the fluid velocity be equal to that of the body. The solutions of the equations of motion for an ideal fluid cannot satisfy these conditions, however. We can require only the fulfilment of the corresponding condition for the fluid velocity component normal to the surface.

Although equations (24.12) are inapplicable in the surface layer of fluid, the velocity distribution obtained by solving them satisfies the necessary boundary condition for the normal velocity component, and the actual variation of this component near the surface therefore has no significant properties. The tangential component would be found, by solving the equations (24.12), to have some value different from the corresponding velocity component of the body, whereas these velocity components should be equal also. Hence the tangential velocity component must change rapidly in the surface layer. The nature of this variation is easily determined. Let us consider any portion of the surface of the body, with dimension large compared with δ, but small compared with the dimension of the body. Such a portion may be regarded as approximately plane, and therefore we can use the results obtained above for a plane surface. Let the x-axis be directed along the normal to the portion considered, and the y-axis parallel to the tangential velocity component of the surface there. We denote by v_y the tangential component of the fluid velocity relative to the body; v_y must vanish on the surface. Lastly, let $v_0 e^{-i\omega t}$ be the value of v_y found by solving equations (24.12). From the results obtained at the beginning of this section, we can say that in the surface layer the quantiy v_y will fall off towards the surface according to the law†

$$v_y = v_0 e^{-i\omega t}[1 - e^{-(1-i)x\sqrt{(\omega/2\nu)}}]. \tag{24.13}$$

Finally, the total amount of energy dissipated in unit time will be given by the integral

$$\bar{\dot{E}}_{\text{kin}} = -\tfrac{1}{2}\sqrt{(\tfrac{1}{2}\omega\eta\rho)}\oint |v_0|^2 \, \mathrm{d}f \tag{24.14}$$

taken over the surface of the oscillating body.

In the Problems at the end of this section we calculate the drag on various bodies oscillating in a viscous fluid. Here we shall make the following general remark regarding these forces. Writing the velocity of the body in the complex form $u = u_0 e^{-i\omega t}$, we obtain a drag F proportional to the velocity u, and also complex: $F = \beta u$, where $\beta = \beta_1 + i\beta_2$ is a complex constant. This expression can be written as the sum of two terms with real coefficients:

$$F = (\beta_1 + i\beta_2)u = \beta_1 u - \beta_2 \dot{u}/\omega, \tag{24.15}$$

one proportional to the velocity u and the other to the acceleration $\dot{u}$.

The (time) average of the energy dissipation is given by the mean product of the drag and the velocity, where of course we must first take the real parts of the expressions given above, i.e. $u = \frac{1}{2}(u_0 e^{-i\omega t} + u_0{}^* e^{i\omega t})$, $F = \frac{1}{2}(u_0 \beta e^{-i\omega t} + u_0{}^* \beta^* e^{i\omega t})$. Noticing that the mean values of $e^{\pm 2i\omega t}$ are zero, we have

$$\bar{\dot{E}}_{\text{kin}} = \overline{Fu} = \tfrac{1}{4}(\beta + \beta^*)|u_0|^2 = \tfrac{1}{2}\beta_1 |u_0|^2. \tag{24.16}$$

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† The velocity distribution (24.13) is written in a frame where the solid body is at rest ($v_y = 0$ when $x = 0$). Hence v_0 must be taken as the solution of the problem of potential flow past a body at rest.

Thus we see that the energy dissipation arises only from the real part of β; the corresponding part of the drag (24.15), proportional to the velocity, may be called the *dissipative part*. The other part of the drag, proportional to the acceleration and determined by the imaginary part of β, does not involve the dissipation of energy and may be called the *inertial part*.

Similar considerations hold for the moment of the forces on a body executing rotary oscillations in a viscous fluid.

PROBLEMS

PROBLEM 1. Determine the frictional force on each of two parallel solid planes, between which is a layer of viscous fluid, when one of the planes oscillates in its own plane.

SOLUTION. We seek a solution of equation (24.3) in the form†

$$v = (A \sin kx + B \cos kx)e^{-i\omega t},$$

and determine A and B from the conditions $v = u = u_0 e^{-i\omega t}$ for $x = 0$ and $v = 0$ for $x = h$, where h is the distance between the planes. The result is

$$v = u\frac{\sin k(h-x)}{\sin kh}.$$

The frictional force per unit area on the moving plane is

$$P_{1y} = \eta(\partial v/\partial x)_{x=0} = -\eta ku \cot kh,$$

while that on the fixed plane is

$$P_{2y} = -\eta(\partial v/\partial x)_{x=h} = \eta ku \operatorname{cosec} kh,$$

the real parts of all quantities being understood.

PROBLEM 2. Determine the frictional force on an oscillating plane covered by a layer of fluid with thickness h, the upper surface being free.

SOLUTION. The boundary condition at the solid plane is $v = u$ for $x = 0$, and that at the free surface is $\sigma_{xy} = \eta\partial v/\partial x = 0$ for $x = h$. We find the velocity

$$v = u\frac{\cos k(h-x)}{\cos kh}.$$

The frictional force is

$$P_y = \eta(\partial v/\partial x)_{x=0} = \eta ku \tan kh.$$

PROBLEM 3. A plane disk with large radius R executes rotary oscillations with small amplitude about its axis, the angle of rotation being $\theta = \theta_0 \cos \omega t$, where $\theta_0 \ll 1$. Determine the moment of the frictional forces acting on the disk.

SOLUTION. For oscillations with small amplitude the term $(\mathbf{v} \cdot \mathbf{grad})\mathbf{v}$ in the equation of motion is always small compared with $\partial\mathbf{v}/\partial t$, whatever the frequency ω. If $R \gg \delta$, the disk may be regarded as infinite in determining the velocity distribution. We take cylindrical polar coordinates, with the z-axis along the axis of rotation, and seek a solution such that $v_r = v_z = 0$, $v_\phi = v = r\Omega(z, t)$. For the angular velocity $\Omega(z, t)$ of the fluid we obtain the equation

$$\partial\Omega/\partial t = \nu\partial^2\Omega/\partial z^2.$$

The solution of this equation which is $-\omega\theta_0 \sin \omega t$ for $z = 0$ and zero for $z = \infty$ is

$$\Omega = -\omega\theta_0 e^{-z/\delta} \sin(\omega t - z/\delta).$$

† In all the Problems to this section k and δ are defined as in (24.4).

The moment of the frictional forces on both sides of the disk is

$$M = 2\int_0^R r\cdot 2\pi r\eta(\partial v/\partial z)_{z=0}\,\mathrm{d}r = \omega\theta_0\pi\sqrt{(\omega\rho\eta)}R^4\cos(\omega t - \tfrac{1}{4}\pi).$$

PROBLEM 4. Determine the flow between two parallel planes when there is a pressure gradient which varies harmonically with time.

SOLUTION. We take the xz-plane half-way between the two planes, with the x-axis parallel to the pressure gradient, which we write in the form

$$-(1/\rho)\partial p/\partial x = ae^{-i\omega t}.$$

The velocity is everywhere in the x-direction, and is determined by the equation

$$\partial v/\partial t = ae^{-i\omega t} + \nu\partial^2 v/\partial y^2.$$

The solution of this equation which satisfies the conditions $v = 0$ for $y = \pm\frac{1}{2}h$ is

$$v = \frac{ia}{\omega}e^{-i\omega t}\left[1 - \frac{\cos ky}{\cos\frac{1}{2}kh}\right].$$

The mean value of the velocity over a cross-section is

$$\bar{v} = \frac{ia}{\omega}e^{-i\omega t}\left(1 - \frac{2}{kh}\tan\tfrac{1}{2}kh\right).$$

For $h/\delta \ll 1$ this becomes

$$\bar{v} \cong ae^{-i\omega t}h^2/12\nu,$$

in agreement with (17.5), while for $h/\delta \gg 1$ we have

$$\bar{v} \cong (ia/\omega)e^{-i\omega t},$$

in accordance with the fact that in this case the velocity must be almost constant over the cross-section, varying only in a thin surface layer.

PROBLEM 5. Determine the drag on a sphere with radius R which executes translatory oscillations in a fluid.

SOLUTION. We write the velocity of the sphere in the form $\mathbf{u} = \mathbf{u}_0 e^{-i\omega t}$. As in §20, we seek the fluid velocity in the form $\mathbf{v} = e^{-i\omega t}\,\mathbf{curl\,curl}\,f\mathbf{u}_0$, where f is a function of r only (the origin is taken at the instantaneous position of the centre of the sphere). Substituting in (24.9) and effecting transformations similar to those in §20, we obtain the equation

$$\triangle^2 f + (i\omega/\nu)\triangle f = 0$$

(instead of the equation $\triangle^2 f = 0$ in §20). Hence we have

$$\triangle f = \text{constant}\times e^{ikr}/r,$$

the solution being chosen which decreases exponentially with r. Integrating, we have

$$\mathrm{d}f/\mathrm{d}r = [ae^{ikr}(r - 1/ik) + b]/r^2; \tag{1}$$

the function f itself is not needed, since only the derivatives f' and f'' appear in the velocity. The constants a and b are determined from the condition that $\mathbf{v} = \mathbf{u}$ for $r = R$, and are found to be

$$a = -\frac{3R}{2ik}e^{-ikR}, \qquad b = -\tfrac{1}{2}R^3\left(1 - \frac{3}{ikR} - \frac{3}{k^2R^2}\right). \tag{2}$$

It may be pointed out that, at high frequencies ($R \gg \delta$), $a \to 0$ and $b \to -\frac{1}{2}R^3$, the values for potential flow obtained in §10, Problem 2; this is in accordance with what was said in §24.

The drag is calculated from formula (20.13), in which the integration is over the surface of the sphere. The result is

$$F = 6\pi\eta R\left(1 + \frac{R}{\delta}\right)u + 3\pi R^2\sqrt{(2\eta\rho/\omega)}\left(1 + \frac{2R}{9\delta}\right)\frac{\mathrm{d}u}{\mathrm{d}t}. \tag{3}$$

For $\omega = 0$ this becomes Stokes' formula, while for large frequencies we have

$$F = \tfrac{2}{3}\pi\rho R^3\frac{\mathrm{d}u}{\mathrm{d}t} + 3\pi R^2\sqrt{(2\eta\rho\omega)}u.$$

The first term in this expression corresponds to the inertial force in potential flow past a sphere (see §11, Problem 1), while the second gives the limit of the dissipative force. This second term could also have been found by calculating the energy dissipation according to (24.14); see Problem 6.

PROBLEM 6. Find the expression, in the limit of high frequencies ($\delta \ll R$), for the dissipative drag on an infinite cylinder with radius R oscillating at right angles to its axis.

SOLUTION. The velocity distribution round a cylinder at rest in a transverse flow is

$$v = (R^2/r^2)[2\mathbf{n}(\mathbf{u}\cdot\mathbf{n}) - \mathbf{u}] - \mathbf{u};$$

see §10, Problem 3. From this, we find as the tangential velocity at the surface of the cylinder

$$v_0 = -2u \sin\phi,$$

where r and ϕ are polar coordinates in the transverse plane, with ϕ measured from the direction of $\mathbf{u}$. From (24.14) we find the energy dissipated per unit length of the cylinder:

$$\bar{\dot{E}}_{\text{kin}} = \pi u^2 R\sqrt{(2\eta\rho\omega)}.$$

Comparison with (24.15) and (24.16) gives the result

$$F_{\text{dis}} = 2\pi u R\sqrt{(2\eta\rho\omega)}.$$

PROBLEM 7. Determine the drag on a sphere moving in an arbitrary manner, the velocity being given by a function $u(t)$.

SOLUTION. We represent $u(t)$ as a Fourier integral:

$$u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_\omega e^{-i\omega t}\,\mathrm{d}\omega, \qquad u_\omega = \int_{-\infty}^{\infty} u(\tau)e^{i\omega\tau}\,\mathrm{d}\tau.$$

Since the equations are linear, the total drag may be written as the integral of the drag forces for velocities which are the separate Fourier components $u_\omega e^{-i\omega t}$; these forces are given by (3) of Problem 5, and are

$$\pi\rho R^3 u_\omega e^{-i\omega t}\left\{\frac{6\nu}{R^2} - \frac{2i\omega}{3} + \frac{3\sqrt{(2\nu)}}{R}(1-i)\sqrt{\omega}\right\}.$$

Noticing that $(\mathrm{d}u/\mathrm{d}t)_\omega = -i\omega u_\omega$, we can rewrite this as

$$\pi\rho R^3 e^{-i\omega t}\left\{\frac{6\nu}{R^2}u_\omega + \tfrac{2}{3}(\dot{u})_\omega + \frac{3\sqrt{(2\nu)}}{R}(\dot{u})_\omega\frac{1+i}{\sqrt{\omega}}\right\}.$$

On integration over $\omega/2\pi$, the first and second terms give respectively $u(t)$ and $\dot{u}(t)$. To integrate the third term, we notice first of all that for negative ω this term must be written in the complex conjugate form, $(1+i)\sqrt{\omega}$ being replaced by $(1-i)/\sqrt{|\omega|}$; this is because formula (3) of Problem 5 was derived for a velocity $u = u_0 e^{-i\omega t}$ with $\omega > 0$, and for a velocity $u_0 e^{i\omega t}$ we should obtain the complex conjugate. Instead of an integral over ω from $-\infty$ to $+\infty$, we can therefore take twice the real part of the integral from 0 to ∞. We write

$$\begin{aligned}
\frac{1}{\pi}\mathrm{re}\left\{(1+i)\int_0^\infty \frac{(\dot{u})_\omega e^{-i\omega t}}{\sqrt{\omega}}\mathrm{d}\omega\right\} &= \frac{1}{\pi}\mathrm{re}\left\{(1+i)\int_{-\infty}^{\infty}\int_0^\infty \frac{\dot{u}(\tau)e^{i\omega(\tau-t)}}{\sqrt{\omega}}\mathrm{d}\omega\,\mathrm{d}\tau\right\}\\
&= \frac{1}{\pi}\mathrm{re}\left\{(1+i)\int_{-\infty}^{t}\int_0^\infty \frac{\dot{u}(\tau)e^{-i\omega(t-\tau)}}{\sqrt{\omega}}\mathrm{d}\omega\,\mathrm{d}\tau + (1+i)\int_t^\infty\int_0^\infty \frac{\dot{u}(\tau)e^{i\omega(\tau-t)}}{\sqrt{\omega}}\mathrm{d}\omega\,\mathrm{d}\tau\right\}\\
&= \sqrt{\frac{2}{\pi}}\mathrm{re}\left\{\int_{-\infty}^{t}\frac{\dot{u}(\tau)}{\sqrt{(t-\tau)}}\mathrm{d}\tau + i\int_t^\infty \frac{\dot{u}(\tau)}{\sqrt{(\tau-t)}}\mathrm{d}\tau\right\}\\
&= \sqrt{\frac{2}{\pi}}\int_{-\infty}^{t}\frac{\dot{u}(\tau)}{\sqrt{(t-\tau)}}\mathrm{d}\tau.
\end{aligned}$$

Thus we have finally for the drag

$$F = 2\pi\rho R^3 \left\{ \frac{1}{3}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{3\nu u}{R^2} + \frac{3}{R}\sqrt{\frac{\nu}{\pi}} \int_{-\infty}^{t} \frac{\mathrm{d}u}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\sqrt{(t-\tau)}} \right\}. \tag{4}$$

PROBLEM 8. Determine the drag on a sphere which at time $t = 0$ begins to move with a uniform acceleration, $u = \alpha t$.

SOLUTION. Putting, in formula (4) of Problem 7, $u = 0$ for $t < 0$ and $u = \alpha t$ for $t > 0$ we have for $t > 0$

$$F = 2\pi\rho R^3 \alpha \left[\frac{1}{3} + \frac{3\nu t}{R^2} + \frac{6}{R}\sqrt{\frac{t\nu}{\pi}} \right].$$

PROBLEM 9. The same as Problem 8, but for a sphere brought instantaneously into uniform motion.

SOLUTION. We have $u = 0$ for $t < 0$ and $u = u_0$ for $t > 0$. The derivative $\mathrm{d}u/\mathrm{d}t$ is zero except at the instant $t = 0$, when it is infinite, but the time integral of $\mathrm{d}u/\mathrm{d}t$ is finite, and equals u_0. As a result, we have for all $t > 0$

$$F = 6\pi\rho\nu R u_0 \left[1 + \frac{R}{\sqrt{(\pi\nu t)}} \right] + \tfrac{2}{3}\pi\rho R^3 u_0 \delta(t),$$

where $\delta(t)$ is the delta function. For $t \to \infty$ this expression tends asymptotically to the value given by Stokes' formula. The impulsive drag on the sphere at $t = 0$ is obtained by integrating the last term and is $\frac{2}{3}\pi\rho R^3 u_0$.

PROBLEM 10. Determine the moment of the forces on a sphere executing rotary oscillations about a diameter in a viscous fluid.

SOLUTION. For the same reasons as in §20, Problem 1, the pressure-gradient term can be omitted from the equation of motion, so that we have $\partial\mathbf{v}/\partial t = \nu \triangle \mathbf{v}$. We seek a solution in the form $\mathbf{v} = \mathbf{curl}\, f\mathbf{\Omega}_0 e^{-i\omega t}$, where $\mathbf{\Omega} = \mathbf{\Omega}_0 e^{-i\omega t}$ is the angular velocity of rotation of the sphere. We then obtain for f, instead of the equation $\triangle f = \text{constant}$,

$$\triangle f + k^2 f = \text{constant}.$$

Omitting an unimportant constant term in the solution of this equation, we find $f = ae^{ikr}/r$, taking the solution which vanishes at infinity. The constant a is determined from the boundary condition that $\mathbf{v} = \mathbf{\Omega}\times\mathbf{r}$ at the surface of the sphere. The result is

$$f = \frac{R^3}{r(1-ikR)} e^{ik(r-R)}, \qquad \mathbf{v} = (\mathbf{\Omega}\times\mathbf{r}) \left(\frac{R}{r}\right)^3 \frac{1-ikr}{1-ikR} e^{ik(r-R)},$$

where R is the radius of the sphere. A calculation like that in §20, Problem 1, gives the following expression for the moment of the forces exerted on the sphere by the fluid:

$$M = -\frac{8\pi}{3}\eta R^3 \Omega \frac{3 + 6R/\delta + 6(R/\delta)^2 + 2(R/\delta)^3 - 2i(R/\delta)^2(1 + R/\delta)}{1 + 2R/\delta + 2(R/\delta)^2}.$$

For $\omega \to 0$ (i.e. $\delta \to \infty$), we obtain $M = -8\pi\eta R^3 \Omega$, corresponding to uniform rotation of the sphere (see §20, Problem 1). In the opposite limiting case $R/\delta \gg 1$, we find

$$M = \frac{4\sqrt{2}}{3}\pi R^4 \sqrt{(\eta\rho\omega)}\,(i-1)\Omega.$$

This expression can also be obtained directly: for $\delta \ll R$ each element of the surface of the sphere may be regarded as plane, and the frictional force acting on it is found by substituting $u = \Omega R \sin\theta$ in formula (24.6).

PROBLEM 11. Determine the moment of the forces on a hollow sphere filled with viscous fluid and executing rotary oscillations about a diameter.

SOLUTION. We seek the velocity in the same form as in Problem 10. For f we take the solution $(a/r)\sin kr$, which is finite everywhere within the sphere, including the centre. Determining a from the boundary condition, we have

$$\mathbf{v} = (\mathbf{\Omega}\times\mathbf{r}) \left(\frac{R}{r}\right)^3 \frac{kr\cos kr - \sin kr}{kR\cos kR - \sin kR}.$$

A calculation of the moment of the frictional forces gives the expression

$$M = \tfrac{8}{3}\pi\eta R^3\Omega\frac{k^2R^2\sin kR + 3kR\cos kR - 3\sin kR}{kR\cos kR - \sin kR}.$$

The limiting value for $R/\delta \gg 1$ is of course the same as in the preceding problem. If $R/\delta \ll 1$ we have

$$M = \tfrac{8}{15}\pi\rho\omega R^5\Omega\left(i - \frac{R^2\omega}{35\nu}\right).$$

The first term corresponds to the inertial forces occurring in the rigid rotation of the whole fluid.

§25. Damping of gravity waves

Arguments similar to those given above can be advanced concerning the velocity distribution near the free surface of a fluid. Let us consider oscillatory motion occurring near the surface (for example, gravity waves). We suppose that the conditions (24.11) hold, the dimension l being now replaced by the wavelength λ:

$$\lambda^2\omega \gg \nu, \qquad a \ll \lambda; \tag{25.1}$$

a is the amplitude of the wave, and ω its frequency. Then we can say that the flow is rotational only in a thin surface layer, while throughout the rest of the fluid we have potential flow, just as we should for an ideal fluid.

The motion of a viscous fluid must satisfy the boundary conditions (15.16) at the free surface; these require that certain combinations of the space derivatives of the velocity should vanish. The flow obtained by solving the equations of ideal-fluid dynamics does not satisfy these conditions, however. As in the discussion of v_y in the previous section, we may conclude that the corresponding velocity derivatives decrease rapidly in a thin surface layer. It is important to notice that this does not imply a large velocity gradient as it does near a solid surface.

Let us calculate the energy dissipation in a gravity wave. Here we must consider the dissipation, not of the kinetic energy alone, but of the mechanical energy E_{mech}, which includes both the kinetic energy and the potential energy in the gravitational field. It is clear, however, that the presence or absence of a gravitational field cannot affect the energy dissipation due to processes of internal friction in the fluid. Hence $\dot{E}_{\text{mech}}$ is given by the same formula (16.3):

$$\dot{E}_{\text{mech}} = -\tfrac{1}{2}\eta\int\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)^2 \mathrm{d}V.$$

In calculating this integral for a gravity wave, it is to be noticed that, since the volume of the surface region of rotational flow is small, while the velocity gradient there is not large, the existence of this region may be ignored, unlike what was possible for oscillations of a solid surface. In other words, the integration is to be taken over the whole volume of fluid, which, as we have seen, moves as if it were an ideal fluid.

The flow in a gravity wave for an ideal fluid, however, has already been determined in §12. Since we have potential flow,

$$\partial v_i/\partial x_k = \partial^2\phi/\partial x_k\partial x_i = \partial v_k/\partial x_i,$$

so that

$$\dot{E}_{\text{mech}} = -2\eta\int\left(\frac{\partial^2\phi}{\partial x_i\partial x_k}\right)^2 \mathrm{d}V.$$

The potential ϕ has the form

$$\phi = \phi_0 \cos(kx - \omega t + \alpha) e^{kz}.$$

We are interested, of course, not in the instantaneous value of the energy dissipation, but in its mean value with respect to time. Noticing that the mean values of the squared sine and cosine are the same, we find

$$\bar{\dot{E}}_{\text{mech}} = -8\eta k^4 \int \overline{\phi^2} \, \mathrm{d}V. \tag{25.2}$$

The energy E_{mech} itself may be calculated for a gravity wave by using a theorem of mechanics that, in any system executing small oscillations (with small amplitude, that is), the mean kinetic and potential energies are equal. We can therefore write $\bar{E}_{\text{mech}}$ simply as twice the kinetic energy:

$$\bar{E}_{\text{mech}} = \rho \int \overline{v^2} \, \mathrm{d}V = \rho \int \overline{(\partial \phi / \partial x_i)^2} \, \mathrm{d}V,$$

whence

$$\bar{E}_{\text{mech}} = 2\rho k^2 \int \overline{\phi^2} \, \mathrm{d}V. \tag{25.3}$$

The damping of the waves is conveniently characterized by the *damping coefficient* γ, defined as

$$\gamma = |\bar{\dot{E}}_{\text{mech}}| / 2\bar{E}_{\text{mech}}. \tag{25.4}$$

In the course of time, the energy of the wave decreases according to the law $\bar{E}_{\text{mech}} = \text{constant} \times e^{-2\gamma t}$; since the energy is proportional to the square of the amplitude, the latter decreases with time as $e^{-\gamma t}$.

Using (25.2), (25.3), we find

$$\gamma = 2\nu k^2. \tag{25.5}$$

Substituting here (12.7), we obtain the damping coefficient for gravity waves in the form

$$\gamma = 2\nu\omega^4 / g^2. \tag{25.6}$$

PROBLEMS

PROBLEM 1. Determine the damping coefficient for long gravity waves propagated in a channel with constant cross-section; the frequency is supposed so large that $\sqrt{(\nu/\omega)}$ is small compared with the depth of the fluid in the channel and the width of the channel.

SOLUTION. The principal dissipation of energy occurs in the surface layer of fluid, where the velocity changes from zero at the boundary to the value $v = v_0 e^{-i\omega t}$ which it has in the wave. The mean energy dissipation per unit length of the channel is by (24.14) $l |v_0|^2 \sqrt{(\eta\rho\omega/8)}$, where l is the perimeter of the part of the channel cross-section occupied by the fluid. The mean energy of the fluid (again per unit length) is $S\rho\overline{v^2} = \frac{1}{2} S\rho |v_0|^2$, where S is the cross-sectional area of the fluid in the channel. The damping coefficient is $\gamma = l\sqrt{(\nu\omega/8S^2)}$. For a channel with rectangular section, therefore,

$$\gamma = \frac{2h + a}{2\sqrt{2ah}} \sqrt{(\nu\omega)},$$

where a is the width and h the depth of the fluid.

PROBLEM 2. Determine the flow in a gravity wave on a very viscous fluid ($\nu \gtrsim \omega\lambda^2$).

SOLUTION. The calculation of the damping coefficient as shown above is valid only when this coefficient is small ($\gamma \ll \omega$), so that the motion may be regarded as that of an ideal fluid to a first approximation. For arbitrary viscosity we seek a solution of the equations of motion

$$\frac{\partial v_x}{\partial t} = \nu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial z^2}\right) - \frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\frac{\partial v_z}{\partial t} = \nu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial z^2}\right) - \frac{1}{\rho}\frac{\partial p}{\partial z} - g,$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0$$

which depends on t and x as $e^{-i\omega t + ikx}$, and diminishes in the interior of the fluid ($z > 0$). We find

$$v_x = e^{-i\omega t + ikx}(Ae^{kz} + Be^{mz}), \qquad v_z = e^{-i\omega t + ikx}\left(-iAe^{kz} - \frac{ik}{m}Be^{mz}\right),$$

$$p/\rho = e^{-i\omega t + ikx}\omega Ae^{kz}/k - gz, \quad \text{where} \quad m = \sqrt{(k^2 - i\omega/\nu)}.$$

The boundary conditions at the fluid surface are

$$\sigma_{zz} = -p + 2\eta \partial v_z/\partial z = 0, \qquad \sigma_{xz} = \eta\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) = 0 \text{ for } z = \zeta.$$

In the second condition we can immediately put $z = 0$ instead of $z = \zeta$. The first condition, however, should be differentiated with respect to t, after which we replace $g\partial\zeta/\partial t$ by gv_z and then put $z = 0$. The condition that the resulting two homogeneous equations for A and B be compatible gives

$$\left(2 - \frac{i\omega}{\nu k^2}\right)^2 + \frac{g}{\nu^2 k^3} = 4\sqrt{\left(1 - \frac{i\omega}{\nu k^2}\right)}. \tag{1}$$

This equation gives ω as a function of the wave number k; ω is complex, its real part giving the frequency of the oscillations and its imaginary part the damping coefficient. The solutions of equation (1) that have a physical meaning are those whose imaginary parts are negative (corresponding to damping of the wave); only two roots of (1) meet this requirement. If $\nu k^2 \ll \sqrt{(gk)}$ (the condition (25.1)), then the damping coefficient is small, and (1) gives approximately $\omega = \pm\sqrt{(gk)} - i.2\nu k^2$, a result which we already know. In the opposite limiting case $\nu k^2 \gg \sqrt{(gk)}$, equation (1) has two purely imaginary roots, corresponding to damped aperiodic flow. One root is $\omega = -ig/2\nu k$, while the other is much larger (of order νk^2), and therefore of no interest, since the corresponding motion is strongly damped.

CHAPTER III

TURBULENCE

§26. Stability of steady flow

For any problem of viscous flow under given steady conditions there must in principle exist an exact steady solution of the equations of fluid dynamics. These solutions formally exist for all Reynolds numbers. Yet not every solution of the equations of motion, even if it is exact, can actually occur in Nature. Those which do must not only obey the equations of fluid dynamics, but also be stable. Any small perturbations which arise must decrease in the course of time. If, on the contrary, the small perturbations which inevitably occur in the flow tend to increase with time, the flow is unstable and cannot actually exist.†

The mathematical investigation of the stability of a given flow with respect to infinitely small perturbations will proceed as follows. On the steady solution concerned (whose velocity distribution is $\mathbf{v}_0(\mathbf{r})$, say), we superpose a non-steady small perturbation $\mathbf{v}_1(\mathbf{r}, t)$, which must be such that the resulting velocity $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$ satisfies the equations of motion. The equation for $\mathbf{v}_1$ is obtained by substituting in the equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \mathbf{grad})\mathbf{v} = -\frac{\mathbf{grad}\, p}{\rho} + \nu \triangle \mathbf{v}, \qquad \operatorname{div} \mathbf{v} = 0 \tag{26.1}$$

the velocity and pressure

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1, \qquad p = p_0 + p_1, \tag{26.2}$$

where the known functions $\mathbf{v}_0$ and p_0 satisfy the unperturbed equations

$$(\mathbf{v}_0 \cdot \mathbf{grad})\mathbf{v}_0 = -\frac{\mathbf{grad}\, p_0}{\rho} + \nu \triangle \mathbf{v}_0, \qquad \operatorname{div} \mathbf{v}_0 = 0. \tag{26.3}$$

Omitting terms above the first order in $\mathbf{v}_1$, we obtain

$$\frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_0 \cdot \mathbf{grad})\mathbf{v}_1 + (\mathbf{v}_1 \cdot \mathbf{grad})\mathbf{v}_0 = -\frac{\mathbf{grad}\, p_1}{\rho} + \nu \triangle \mathbf{v}_1, \qquad \operatorname{div} \mathbf{v}_1 = 0. \tag{26.4}$$

The boundary condition is that $\mathbf{v}_1$ vanish on fixed solid surfaces.

Thus $\mathbf{v}_1$ satisfies a system of homogeneous linear differential equations, with coefficients that are functions of the coordinates only, and not of the time. The general solution of such equations can be represented as a sum of particular solutions in which $\mathbf{v}_1$ depends on time

† In the previous edition, instability with respect to infinitesimal perturbations was called *absolute instability*. This adjective will not now be used in the present context, but will serve (in accordance with more customary terminology) as a contrast to *convected* (§28).

as $e^{-i\omega t}$. The frequencies ω of the perturbations are not arbitrary, but are determined by solving the equations (26.4) with the appropriate boundary conditions. The frequencies are in general complex. If there are ω whose imaginary parts are positive, $e^{-i\omega t}$ will increase indefinitely with time. In other words, such perturbations, once having arisen, will increase, i.e. the flow is unstable with respect to such perturbations. For the flow to be stable it is necessary that the imaginary part of any possible frequency ω be negative. The perturbations that arise will then decrease exponentially with time.

Such a mathematical investigation of stability is extremely complicated, however. The theoretical problem of the stability of steady flow past bodies with finite dimensions has not yet been solved. It is certain that steady flow is stable for sufficiently small Reynolds numbers. The experimental data seem to indicate that, when R increases, it eventually reaches a value R_{cr} (the *critical Reynolds number*) beyond which the flow is unstable with respect to infinitesimal disturbances. For sufficiently large Reynolds numbers ($R > R_{cr}$), steady flow past solid bodies is therefore impossible. The critical Reynolds number is not, of course, a universal constant, but takes a different value for each type of flow. These values appear to be of the order of 10 to 100; for example, in flow across a cylinder undamped non-steady flow has been observed for $R = ud/\nu \cong 30$, d being the diameter of the cylinder.

Let us now consider the nature of the non-steady flow which is established as a result of the instability of steady flow at large Reynolds numbers (L. D. Landau 1944). We begin by examining the properties of this flow at Reynolds numbers only slightly greater than R_{cr}. For $R < R_{cr}$ the imaginary parts of the complex frequencies $\omega = \omega_1 + i\gamma_1$ for all possible small perturbations are negative ($\gamma_1 < 0$). For $R = R_{cr}$ there is one frequency whose imaginary part is zero. For $R > R_{cr}$ the imaginary part of this frequency is positive, but, when R is close to R_{cr}, γ_1 is small in comparison with the real part ω_1.† The function $\mathbf{v}_1$ corresponding to this frequency is of the form

$$\mathbf{v}_1 = A(t)\mathbf{f}(x, y, z), \tag{26.5}$$

where **f** is some complex function of the coordinates, and the complex amplitude $A(t)$ is‡

$$A(t) = \text{constant} \times e^{\gamma_1 t} e^{-i\omega_1 t}. \tag{26.6}$$

This expression for $A(t)$ is actually valid, however, only during a short interval of time after the disruption of the steady flow; the factor $e^{\gamma_1 t}$ increases rapidly with time, whereas the method of determining $\mathbf{v}_1$ given above, which leads to expressions like (26.5) and (26.6), applies only when $|\mathbf{v}_1|$ is small. In reality, of course, the modulus $|A|$ of the amplitude of the non-steady flow does not increase without limit, but tends to a finite value. For R close to R_{cr} (we always mean, of course, $R > R_{cr}$), this finite value is small, and can be determined as follows.

Let us find the time derivative of the squared amplitude $|A|^2$. For very small values of t, when (26.6) is still valid, we have $\mathrm{d}|A|^2/\mathrm{d}t = 2\gamma_1 |A|^2$. This expression is really just the first term in an expansion in series of powers of A and A^*. As the modulus $|A|$ increases (still remaining small), subsequent terms in this expansion must be taken into account. The

---

† The set (or *spectrum*) of all possible perturbation frequencies for a given type of flow includes both separate isolated values (the *discrete spectrum*) and the whole of various frequency ranges (the *continuous spectrum*). It seems that for flow past finite bodies the frequencies with $\gamma_1 > 0$ can occur only in the discrete spectrum. The reason is that the perturbations corresponding to the frequencies in the continuous spectrum are in general not zero at infinity, but the unperturbed flow there is certainly a stable homogeneous plane-parallel flow.

‡ As usual, we understand the real part of (26.6).

next terms are those of the third order in A. However, we are not interested in the exact value of the derivative $\mathrm{d}\,|A|^2/\mathrm{d}t$, but in its time average, taken over times large compared with the period $2\pi/\omega_1$ of the factor $e^{-i\omega_1 t}$; we recall that, since $\omega_1 \gg \gamma_1$, this period is small compared with the time $1/\gamma_1$ required for the amplitude modulus $|A|$ to change appreciably. The third-order terms, however, must contain the periodic factor, and therefore vanish on averaging.† The fourth-order terms include one which is proportional to $A^2 A^{*2} = |A|^4$ and which does not vanish on averaging. Thus we have as far as fourth-order terms

$$\overline{\mathrm{d}\,|A|^2/\mathrm{d}t} = 2\gamma_1 |A|^2 - \alpha |A|^4, \tag{26.7}$$

where α (the *Landau constant*) may be either positive or negative.

We are interested in the case where an infinitesimal perturbation (superimposed on the original flow) first becomes unstable for $R > R_{cr}$. This corresponds to $\alpha > 0$. We have not put bars above $|A|^2$ and $|A|^4$ in (26.7), since the averaging is only over time intervals short compared with $1/\gamma_1$. For the same reason, in solving the equation we proceed as if the bar were omitted above the derivative also. The solution of equation (26.7) is

$$1/|A|^2 = \alpha/2\gamma_1 + \text{constant} \times e^{-2\gamma_1 t}.$$

Hence it is clear that $|A|^2$ tends asymptotically to a finite limit:

$$|A|^2_{\max} = 2\gamma_1/\alpha. \tag{26.8}$$

The quantity γ_1 is some function of the Reynolds number. Near R_{cr} it can be expanded as a series of powers of $\mathrm{R} - \mathrm{R}_{cr}$. But $\gamma_1(\mathrm{R}_{cr}) = 0$, by the definition of the critical Reynolds number. Hence we have to the first order

$$\gamma_1 = \text{constant} \times (\mathrm{R} - \mathrm{R}_{cr}). \tag{26.9}$$

Substituting this in (26.8), we see that the modulus $|A|$ of the amplitude is proportional to the square root of $\mathrm{R} - \mathrm{R}_{cr}$:

$$|A|_{\max} \propto \sqrt{(\mathrm{R} - \mathrm{R}_{cr})}. \tag{26.10}$$

Let us now briefly discuss the case where $\alpha < 0$ in (26.7). The two terms in that expansion are then insufficient to determine the limiting amplitude of the perturbation, and we have to include a negative term of higher order; let this be $-\beta|A|^6$ with $\beta > 0$, which gives

$$|A|^2_{\max} = \frac{|\alpha|}{2\beta} \pm \sqrt{\left(\frac{\alpha^2}{4\beta^2} + \frac{2|\alpha|}{\beta}\gamma_1\right)}, \tag{26.11}$$

with γ_1 as in (26.9). The dependence is shown in Fig. 13b; Fig. 13a corresponds to $\alpha > 0$, (26.10). When $\mathrm{R} > \mathrm{R}_{cr}$, there can be no steady flow; when $\mathrm{R} = \mathrm{R}_{cr}$, the perturbation discontinuously reaches a non-zero amplitude, though this is still assumed so small that the expansion in powers of $|A|^2$ is valid.‡ In the range $\mathrm{R}_{cr}' < \mathrm{R} < \mathrm{R}_{cr}$, the unperturbed flow is *metastable*, being stable with respect to infinitesimal perturbations but unstable with respect to those with finite amplitude (the continuous curve; the broken curve shows the unstable branch).

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† Strictly speaking, the third-order terms give, on averaging, not zero, but fourth-order terms, which we suppose included among the fourth-order terms in the expansion.

‡ Such systems are said to have *hard* self-excitation, in contrast to those with *soft* self-excitation, which are unstable with respect to infinitesimal perturbations.

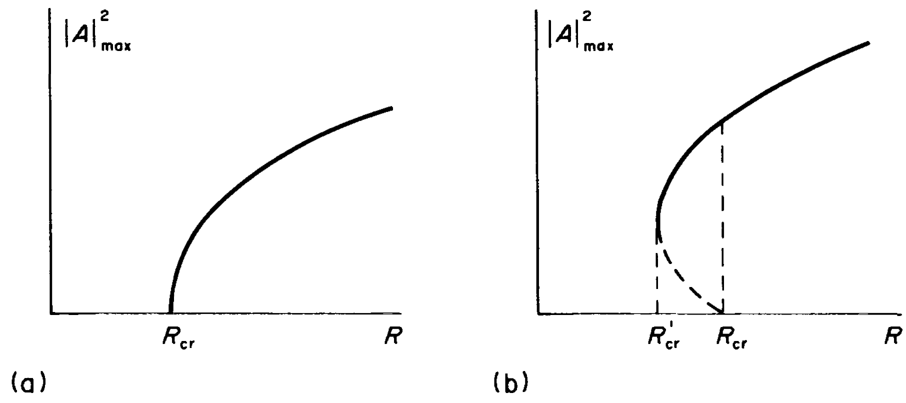

FIG. 13

Let us now return to the non-steady flow which occurs when $R > R_{cr}$, as a result of the instability with respect to small perturbations. For R close to R_{cr} the latter flow can be represented by superposing on the steady flow $\mathbf{v}_0(\mathbf{r})$ a periodic flow $\mathbf{v}_1(\mathbf{r}, t)$, with a small but finite amplitude which increases with R as in (26.10). The velocity distribution in this flow is of the form

$$\mathbf{v}_1 = \mathbf{f}(\mathbf{r})e^{-i(\omega_1 t+\beta_1)}, \tag{26.12}$$

where $\mathbf{f}$ is a complex function of the coordinates, and β_1 is some initial phase. For large $R - R_{cr}$, the separation of the velocity into $\mathbf{v}_0$ and $\mathbf{v}_1$ is no longer meaningful. We then have simply some periodic flow with frequency ω_1. If, instead of the time, we use as an independent variable the phase $\phi_1 \equiv \omega_1 t + \beta_1$, then we can say that the function $\mathbf{v}(\mathbf{r}, \phi_1)$ is a periodic function of ϕ_1, with period 2π. This function, however, is no longer a simple trigonometrical function. Its expansion in Fourier series

$$\mathbf{v} = \sum_p \mathbf{A}_p(\mathbf{r})e^{-i\phi_1 p} \tag{26.13}$$

(where the summation is over all integers p, positive and negative) includes not only terms with the fundamental frequency ω_1, but also terms whose frequencies are integral multiples of ω_1.

Equation (26.7) determines only the modulus of the time factor $A(t)$, and not its phase ϕ_1, which remains essentially indeterminate, and depends on the particular initial conditions which happen to occur at the instant when the flow begins. The initial phase β_1 can have any value, depending on these conditions. Thus the periodic flow under consideration is not uniquely determined by the given steady external conditions in which the flow takes place. One quantity—the initial phase of the velocity—remains arbitrary. We may say that the flow has one degree of freedom, whereas steady flow, which is entirely determined by the external conditions, has no degrees of freedom.

PROBLEM

Derive the equation for the energy balance between the unperturbed flow and a superimposed perturbation, without assuming that the latter is weak.

SOLUTION. Substituting (26.2) in (26.1), but not omitting the term of the second order in $\mathbf{v}_1$, we have

$$\partial\mathbf{v}_1/\partial t + (\mathbf{v}_0\cdot\mathbf{grad})\mathbf{v}_1 + (\mathbf{v}_1\cdot\mathbf{grad})\mathbf{v}_0 + (\mathbf{v}_1\cdot\mathbf{grad})\mathbf{v}_1 = -\mathbf{grad}\, p_1 + (1/R)\triangle\mathbf{v}_1; \tag{1}$$

all quantities are assumed to be brought to dimensionless form, as described in §19. Taking the scalar product of this equation with $\mathbf{v}_1$ and using the equations div $\mathbf{v}_0 = 0$, div $\mathbf{v}_1 = 0$, we obtain

$$\frac{\partial}{\partial t}(\tfrac{1}{2}v_1^2) = -v_{1i}v_{1k}\frac{\partial v_{0i}}{\partial x_k} - \frac{1}{\mathrm{R}}\frac{\partial v_{1i}}{\partial x_k}\frac{\partial v_{1i}}{\partial x_k} + \frac{\partial}{\partial x_k}\left\{-\tfrac{1}{2}v_1^2(v_{0k}+v_{1k}) - p_1 v_{1k} + \frac{1}{\mathrm{R}}v_{1i}\frac{\partial v_{1i}}{\partial x_k}\right\}.$$

The last term on the right gives zero on integration over the whole region of the flow, since $\mathbf{v}_0 = \mathbf{v}_1 = 0$ on the boundary surfaces of the region or at infinity. This gives as the required relation

$$\dot{E}_1 = T - D/\mathrm{R}, \tag{2}$$

$$E_1 = \int \tfrac{1}{2}v_1^2\,\mathrm{d}V, \qquad T = -\int v_{1i}v_{1k}\frac{\partial v_{0i}}{\partial x_k}\mathrm{d}V, \qquad D = \int\left(\frac{\partial v_{1i}}{\partial x_k}\right)^2\mathrm{d}V. \tag{3}$$

The functional T represents the energy exchange between the unperturbed flow and the perturbation, and may have either sign. The functional D is the dissipative energy loss, and $D > 0$ always. Note that the term in (1) non-linear in $\mathbf{v}_1$ does not contribute to the relation (2).

The relation (2) provides a lower limit of R_{cr} (O. Reynolds 1894; W. M'F. Orr 1907): the derivative $\mathrm{d}E_1/\mathrm{d}t$ must be negative, i.e. the perturbation decreases with time, if $\mathrm{R} < \mathrm{R}_E$, where

$$\mathrm{R}_E = \min(D/T), \tag{4}$$

the minimum of the functional being taken with respect to functions $\mathbf{v}_1(\mathbf{r})$ which satisfy the boundary conditions and the equation div $\mathbf{v}_1 = 0$. The existence of a finite minimum arises mathematically from the fact that T and D are both second-order homogeneous functionals. This proves the existence of a lower limit of R for metastability, below which the unperturbed flow is stable with respect to any perturbations. The "energy estimate" given by (4) is, however, much too low in the majority of cases.

§27. Stability of rotary flow

To investigate the stability of steady flow between two rotating cylinders (§18) in the limit of very large Reynolds numbers, we can use a simple method like that used in §4 to derive the condition for mechanical stability of a fluid at rest in a gravitational field (Rayleigh 1916). The principle of the method is to consider any small element of the fluid and to suppose that this element is displaced from the path which it follows in the flow concerned. As a result of this displacement, forces appear which act on the displaced element. If the original flow is stable, these forces must tend to return the element to its original position.

Each fluid element in the unperturbed flow moves in a circle r = constant about the axis of the cylinders. Let $\mu(r) = mr^2\dot{\phi}$ be the angular momentum of an element with mass m, $\dot{\phi}$ being the angular velocity. The centrifugal force acting on it is μ^2/mr^3; this force is balanced by the radial pressure gradient in the rotating fluid. Let us now suppose that a fluid element at a distance r_0 from the axis is slightly displaced from its path, being moved to a distance $r > r_0$ from the axis. The angular momentum of the element remains equal to its original value $\mu_0 = \mu(r_0)$. The centrifugal force acting on the element in its new position is therefore μ_0^2/mr^3. In order that the element should tend to return to its initial position, this force must be less than the equilibrium value μ^2/mr^3 which is balanced by the pressure gradient at the distance r. Thus the necessary condition for stability is $\mu^2 - \mu_0^2 > 0$. Expanding $\mu(r)$ in powers of the positive difference $r - r_0$, we can write this condition in the form

$$\mu\,\mathrm{d}\mu/\mathrm{d}r > 0. \tag{27.1}$$

According to formula (18.3), the angular velocity $\dot{\phi}$ of the moving fluid particles is

$$\dot{\phi} = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} + \frac{(\Omega_1 - \Omega_2)R_1^2 R_2^2}{R_2^2 - R_1^2}\frac{1}{r^2}.$$

Calculating $\mu = mr^2\dot{\phi}$ and omitting factors which are certainly positive, we can write the condition (27.1) as

$$(\Omega_2 R_2^2 - \Omega_1 R_1^2)\dot{\phi} > 0. \tag{27.2}$$

The angular velocity $\dot{\phi}$ varies monotonically from Ω_1 on the inner cylinder to Ω_2 on the outer cylinder. If the two cylinders rotate in opposite directions, i.e. if Ω_1 and Ω_2 have opposite signs, the function $\dot{\phi}$ changes sign between the cylinders, and its product with the constant number $\Omega_2 R_2^2 - \Omega_1 R_1^2$ cannot be everywhere positive. Thus in this case (27.2) does not hold at all points in the fluid, and the flow is unstable.

Now let the two cylinders be rotating in the same direction; taking this direction of rotation as positive, we have $\Omega_1 > 0$, $\Omega_2 > 0$. Then $\dot{\phi}$ is everywhere positive, and for the condition (27.2) to be fulfilled it is necessary that

$$\Omega_2 R_2^2 > \Omega_1 R_1^2. \tag{27.3}$$

If $\Omega_2 R_2^2 < \Omega_1 R_1^2$ the flow is unstable. For example, if the outer cylinder is at rest ($\Omega_2 = 0$), while the inner one rotates, then the flow is unstable. If, on the other hand, the inner cylinder is at rest ($\Omega_1 = 0$), the flow is stable.

It must be emphasized that no account has been taken, in the above arguments, of the effect of the viscous forces when the fluid element is displaced. The method is therefore applicable only for small viscosities, i.e. for large R.

To investigate the stability of the flow for any R, it is necessary to follow the general method, starting from equations (26.4); for flow between rotating cylinders, this was first done by G. I. Taylor (1924). In the present case the unperturbed velocity distribution $\mathbf{v}_0$ depends only on the (cylindrical) radial coordinate r, and not on the angle ϕ or the axial coordinate z. The complete set of independent solutions of equations (26.4) may therefore be sought in the form

$$\mathbf{v}_1(r, \phi, z) = e^{i(n\phi + kz - \omega t)}\mathbf{f}(r), \tag{27.4}$$

the direction of the vector $\mathbf{f}(r)$ being arbitrary. The wave number k, which takes a continuous range of values, determines the periodicity of the perturbation in the z-direction. The number n takes only integral values 0, 1, 2, . . . , as follows from the condition for the function to be single-valued with respect to the variable ϕ; the value $n = 0$ corresponds to axially symmetrical perturbations. The permissible values of the frequency ω are found by solving the equations with the necessary boundary conditions ($\mathbf{v}_1 = 0$ for $r = R_1$ and $r = R_2$). The problem thus formulated yields in general, for given n and k, a discrete series of eigenfrequencies $\omega = \omega_n^{(j)}(k)$, where j labels the branches of the function $\omega_n(k)$; these frequencies are in general complex.

The role of the Reynolds number in this case may be taken by $\Omega_1 R_1^2/\nu$ or $\Omega_2 R_2^2/\nu$ for given values of the ratios R_1/R_2 and Ω_1/Ω_2 which determine the type of flow. Let us follow the change of some eigenfrequency $\omega = \omega_n^{(j)}(k)$ as the Reynolds number gradually increases. The point where instability appears (for a particular form of perturbation) is determined by the value of R for which the function $\gamma(k) = \operatorname{im}\omega$ first becomes zero for some k. For $\mathrm{R} < \mathrm{R}_{cr}$, the function $\gamma(k)$ is always negative, but for $\mathrm{R} > \mathrm{R}_{cr}$ we have $\gamma > 0$ in some range of k. Let k_{cr} be the value of k for which $\gamma(k) = 0$ when $\mathrm{R} = \mathrm{R}_{cr}$. The corresponding function (27.4) gives the nature of the flow which occurs (superimposed on the original flow) in the fluid at the instant when the original flow ceases to be stable; it is periodic along the axis of the cylinders, with period $2\pi/k_{cr}$. The actual limit of stability is, of course, determined by the form of the perturbation, i.e. the function $\omega_n^{(j)}(k)$, for which R_{cr}

is least, and it is these "most dangerous" perturbations that are of interest here. As a rule (see below) they are axially symmetrical. Because of the great complexity of the calculation, a fairly complete study of them has been made only in the case where the space between the cylinders is narrow: $h \equiv R_2 - R_1 \ll R = \frac{1}{2}(R_1 + R_2)$. The results are as follows.†

It is found that a purely imaginary function $\omega(k)$ corresponds to the solution which gives the smallest R_{cr}. Hence, when $k = k_{cr}$, not only im ω but ω itself is zero. This means that the first instability of steady rotary flow leads to the appearance of another flow which is also steady.‡ It consists of toroidal *Taylor vortices* arranged in a regular manner along the cylinders. For the case where the two cylinders rotate in the same direction, Fig. 14 shows schematically the projections of the streamlines of these vortices on the meridional cross-

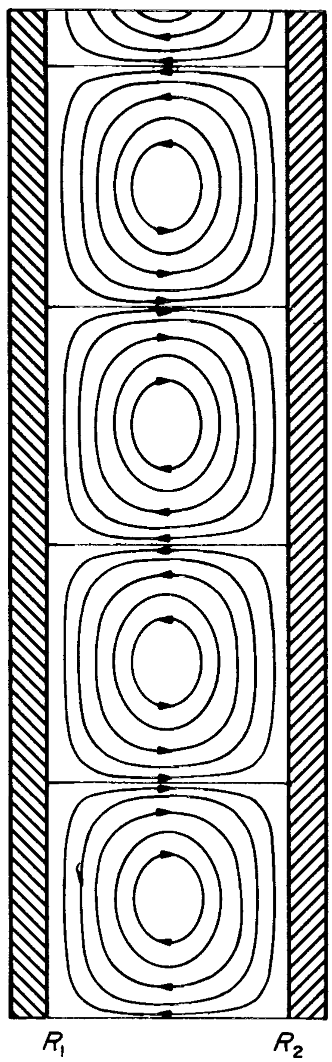

FIG. 14

† A detailed treatment is given by N. E. Kochin, I. A. Kibel' and N. V. Roze, *Theoretical Hydromechanics* (*Teoreticheskaya gidromekhanika*), Part 2, Moscow 1963; S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Oxford 1961; P. G. Drazin and W. H. Reid, *Hydrodynamic Stability*, Cambridge 1981.

‡ In such cases there is said to be *exchange of stabilities*. The experimental and numerical results for several particular cases suggest that this property is a general one for the flow considered and does not depend on h being small.

section plane of the cylinders; the velocity $\mathbf{v}_1$ actually has an azimuthal component also. The length $2\pi/k_{cr}$ of each period contains two vortices with opposite directions of rotation.

For R slightly greater than R_{cr} there is not one value of k but a whole range, for which im $\omega > 0$. However, it should not be thought that the resulting flow will be a superposition of flows with various periodicities. In reality, for each R a flow with a definite periodicity occurs which stabilizes the total flow. This periodicity, however, cannot be determined from the linearized equation (26.4).

Figure 15 shows the approximate form of the curve separating the regions of unstable (shaded) and stable flow for a given value of R_1/R_2. The right-hand branch of the curve, corresponding to rotation of the two cylinders in the same direction, is asymptotic to the line $\Omega_2 R_2{}^2 = \Omega_1 R_1{}^2$; this property is in fact a general one, not dependent on the smallness of h. When the Reynolds number increases, for a given type of flow, we move upwards along a line through the origin which corresponds to the given value of Ω_1/Ω_2. In the right-hand part of the diagram, such lines for which $\Omega_2 R_2{}^2/\Omega_1 R_1{}^2 > 1$ do not meet the curve which bounds the region of instability. If, on the other hand, $\Omega_2 R_2{}^2/\Omega_1 R_1{}^2 < 1$, then for sufficiently large Reynolds numbers we enter the region of instability, in accordance with the condition (27.3). In the left-hand part of the diagram (Ω_1 and Ω_2 with opposite signs), any line through the origin meets the boundary of the shaded region; that is, when the Reynolds number is sufficiently large steady flow ultimately becomes unstable for any ratio $|\Omega_2/\Omega_1|$, again in agreement with the previous results. For $\Omega_2 = 0$ (when only the inner cylinder rotates), instability sets in when the Reynolds number, defined as $R = h\Omega_1 R_1/\nu$, is

$$R_{cr} = 41{\cdot}2\sqrt{(R/h)}. \tag{27.5}$$

In the flow under consideration, the viscosity has a stabilizing effect: a flow stable when $\nu = 0$ remains stable when the viscosity is taken into account, and one that is unstable may become stable for a viscous fluid.

There have been no systematic studies of perturbations without axial symmetry in flow between rotating cylinders. The results of calculations for particular cases suggest that the axially symmetrical perturbations always remain the most dangerous on the right-hand side of Fig. 15. On the left-hand side, however, when $|\Omega_2/\Omega_1|$ is sufficiently large, the form of the boundary curve may be somewhat changed when perturbations without axial symmetry are taken into account. The real part of the perturbation frequency then does not tend to zero, and so the resulting flow is not steady, which considerably alters the nature of the instability.

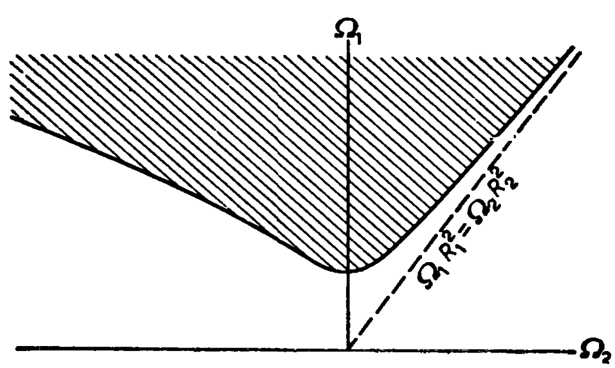

Fig. 15

The limiting case (as $h \to 0$) of flow between rotating cylinders is flow between two parallel planes in relative motion (see §17). This flow is stable with respect to infinitely small perturbations for any value of $R = uh/\nu$, where u is the relative velocity of the planes.

§28. Stability of flow in a pipe

The steady flow in a pipe discussed in §17 loses its stability in an unusual manner. Since the flow is uniform in the x-direction (along the pipe), the unperturbed velocity distribution $\mathbf{v}_0$ is independent of x. Similarly to the procedure in §27, we can therefore seek solutions of equations (26.4) in the form

$$\mathbf{v}_1 = e^{i(kx-\omega t)}\mathbf{f}(y, z). \tag{28.1}$$

Here also there is a value $R = R_{cr}$ for which $\gamma = \text{im}\,\omega$ first becomes zero for some value of k. It is of importance, however, that the real part of the function $\omega(k)$ is not now zero.

For values of R only slightly exceeding R_{cr}, the range of values of k for which $\gamma(k) > 0$ is small and lies near the point for which $\gamma(k)$ is a maximum, i.e. $d\gamma/dk = 0$ (as seen from Fig. 16). Let a slight perturbation occur in some part of the flow; it is a wave packet obtained by superposing a series of components with the form (28.1). In the course of time, the components for which $\gamma(k) > 0$ will be amplified, while the remainder will be damped. The amplified wave packet thus formed will also be carried downstream with a velocity equal to the group velocity $d\omega/dk$ of the packet (§67); since we are now considering waves whose wave numbers lie in a small range near the point where $d\gamma/dk = 0$, the quantity

$$d\omega/dk \cong d(\text{re}\,\omega)/dk \tag{28.2}$$

is real, and is therefore the actual velocity of propagation of the packet.

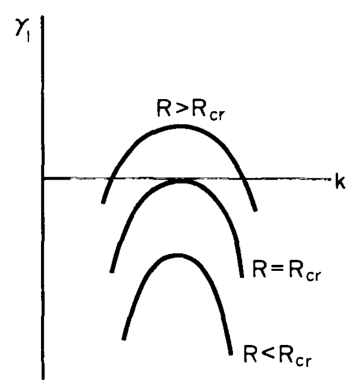

FIG. 16

This downstream displacement of the perturbations is very important, and causes the loss of stability to be totally different from that described in §27.

Since the positiveness of im ω now implies only an amplification of the perturbation as it moves downstream, there are two possibilities. In one case, despite the movement of the wave packet, the perturbation increases without limit in the course of time at any point fixed in space; this kind of instability with respect to any infinitesimal perturbations will be called *absolute instability*. In the other case, the packet is carried away so swiftly that at any point fixed in space the perturbation tends to zero as $t \to \infty$; this kind will be called *convected instability.*† For Poiseuille flow, it appears that the second kind occurs; see the next footnote but four.

The difference between the two cases is a relative one, in the sense that it depends on the choice of the frame of reference with respect to which the instability is considered: an instability convected in one frame becomes absolute in another frame moving with the packet, and an absolute instability becomes convected in a frame that moves away from the packet with sufficient speed. In the present case, however, the physical significance of the difference is given by the existence of a preferred frame of reference in which the instability should be regarded, namely that in which the pipe walls are at rest. Moreover, since actual pipes have a large but finite length, a perturbation arising anywhere may in principle be carried out of the pipe before it actually disrupts the laminar flow.

Since the perturbations increase with the coordinate x (downstream), and not with time at a given point, it is reasonable to investigate this type of instability as follows. Let us suppose that, at a given point, a continuously acting perturbation with a given frequency ω is applied to the flow, and examine what will happen to this perturbation as it is carried downstream. Inverting the function $\omega(k)$, we find what wave number k corresponds to the given (real) frequency ω. If im $k < 0$, the factor e^{ikx} increases with x, i.e. the perturbation is amplified downstream. The curve in the ωR-plane given by the equation im $k(\omega, \mathrm{R}) = 0$, called the *neutral stability curve* or *neutral curve*, defines the region of stability, and separates, for each R, the frequencies of perturbations which are amplified and damped downstream.

The actual calculations are extremely complicated. A complete analytical investigation has been made only for plane Poiseuille flow (between two parallel planes; C. C. Lin 1945). We shall give the results here.‡

The (unperturbed) flow between the planes is uniform not only in the direction of flow (along the x-axis) but throughout the xz-plane (the y-axis being perpendicular to the planes). We can therefore seek solutions of equations (26.4) in the form

$$\mathbf{v}_1 = e^{i(k_x x + k_z z - \omega t)}\mathbf{f}(y) \qquad (28.3)$$

with the wave vector $\mathbf{k}$ having any direction in the xz-plane. We are interested, however, only in the growing perturbations that are the first to appear as R increases, since these govern the limit of stability. It can be shown that, for a given value of the wave number, the first perturbation not damped has $\mathbf{k}$ in the x-direction, with $f_z = 0$. It is therefore sufficient to consider only perturbations in the xy-plane, independent of z and two-dimensional (like the unperturbed flow).††

The neutral curve for flow between planes is schematically shown in Fig. 17. The shaded area within the curve is the region of instability.§ The smallest value of R at which

† The general method of establishing the type of instability is described in *PK*, §62.

‡ See C. C. Lin, *The Theory of Hydrodynamic Stability*, Cambridge 1955. A discussion of these and later studies of the topic is to be found in the book by Drazin and Reid mentioned in a previous footnote.

†† The proof of this statement (H. B. Squire 1933) is that the equations (26.4) with a perturbation having the form (28.3) can be brought to a form in which they differ from the equations for two-dimensional perturbations only in that R is replaced by R $\cos\phi$, ϕ being the angle between $\mathbf{k}$ and $\mathbf{v}_0$ in the xz-plane. The critical number $\tilde{\mathrm{R}}_{\mathrm{cr}}$ for three-dimensional perturbations with a given k is therefore $\tilde{\mathrm{R}}_{\mathrm{cr}} = \mathrm{R}_{\mathrm{cr}} \sec\phi > \mathrm{R}_{\mathrm{cr}}$, where R_{cr} is calculated for two-dimensional perturbations.

§ The neutral curve in the kR-plane has a similar form. Since both ω and k are real on the neutral curve, the curves in the two planes represent the same dependence expressed in terms of different variables.

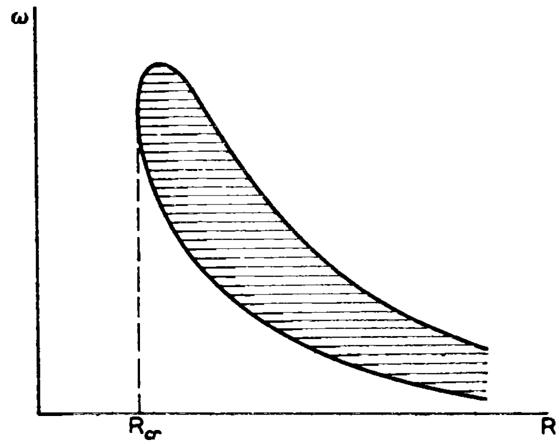

FIG. 17

undamped perturbations are possible is found to be $R_{cr} = 5772$ according to later and more accurate calculations by S. A. Orszag (1971); the Reynolds number is here defined as

$$R = U_{max} h/2\nu, \tag{28.4}$$

where U_{max} is the maximum flow velocity and $\frac{1}{2}h$ is half the distance between the planes, i.e. the distance over which the velocity increases from zero to its maximum value.† The value $R = R_{cr}$ corresponds to a perturbation wave number $k_{cr} = 2{\cdot}04/h$. As $R \to \infty$, the two branches of the neutral curve approach the R-axis asymptotically, with $\omega h/U_{max} \cong R^{-3/11}$ and $R^{-3/7}$ for the upper and lower branches respectively; on each branch, ω and k are related by $\omega h/U \cong (kh)^3$.

Thus, for any non-zero frequency ω that does not exceed a certain maximum value ($\sim U/h$), there is a finite range of R values in which the perturbations are amplified.‡ It is noteworthy that in this case a small but finite viscosity of the fluid has, in a sense, a destabilizing effect in comparison with the situation for a strictly ideal fluid.†† For, when $R \to \infty$, perturbations with any finite frequency are damped, but when a finite viscosity is introduced we eventually reach a region of instability; a further increase in the viscosity (decrease in R) finally brings us out of this region.

For flow in a pipe with circular cross-section, no complete theoretical study of the stability has yet been made, but the available results give good reason to suppose that the flow has stability (both absolute and convected) with respect to infinitesimal perturbations at any Reynolds number. When the unperturbed flow is axially symmetrical, the perturbations may be sought in the form

$$\mathbf{v}_1 = e^{i(n\phi + kz - \omega t)} \mathbf{f}(r), \tag{28.5}$$

as in (27.4). It may be regarded as proved that axially symmetrical perturbations ($n = 0$) are always damped. No undamped perturbations have been found, either, among those

† Another definition of R for two-dimensional Poiseuille flow is also used in the literature: $R = \bar{U}h/\nu$, where $\bar{U}$ is the fluid velocity averaged over the cross-section. Since $\bar{U} = \frac{2}{3} U_{max}$, we have $\bar{U}h/\nu = 4R/3$ when R is defined according to (28.4).

‡ The proof that the instability of two-dimensional Poiseuille flow is convected has been given by S. V. Iordanskiĭ and A. G. Kulikovskiĭ, *Soviet Physics JETP* **22**, 915, 1966. The proof relates, however, only to the range of very large R, where the two branches of the neutral curve are close to the abscissa axis; that is, $kh \ll 1$ on each branch. The problem remains unresolved for R values such that $kh \sim 1$ on the neutral curve.

†† This property was discovered by W. Heisenberg (1924).

without axial symmetry that have been studied (with particular values of n and in particular Reynolds number ranges). The stability of flow in a pipe is also suggested by the fact that, when perturbations at the entrance to the pipe are very carefully prevented, laminar flow can be maintained up to very large values of R, in practice up to $\mathrm{R} \cong 10^5$, where

$$\mathrm{R} = U_{\max} d/2\nu = \bar{U} d/\nu, \qquad (28.6)$$

d being the pipe diameter and $U_{\max}$ the fluid velocity on the pipe axis.

Flow between planes and in a circular pipe may be regarded as limiting cases of flow in an annular pipe between two coaxial cylindrical surfaces with radii R_1 and R_2 ($R_2 > R_1$). When $R_1 = 0$ we have a circular pipe, and the limit $R_1 \to R_2$ corresponds to flow between planes. There appears to be a critical R_{cr} for all non-zero values of $R_1/R_2 < 1$; when $R_1/R_2 \to 0$, $\mathrm{R}_{\mathrm{cr}} \to \infty$.

For each of these Poiseuille flows there is also a critical number $\mathrm{R}_{\mathrm{cr}}'$ which determines the limit of stability with respect to perturbations with finite amplitude. When $\mathrm{R} < \mathrm{R}_{\mathrm{cr}}'$, undamped non-steady flow in the pipe is impossible. If turbulent flow occurs in any section of the pipe, then for $\mathrm{R} < \mathrm{R}_{\mathrm{cr}}'$ the turbulent region will be carried downstream and will diminish in size until it disappears completely; if, on the other hand, $\mathrm{R} > \mathrm{R}_{\mathrm{cr}}'$, the turbulent region will enlarge in the course of time to include more and more of the flow. If perturbations of the flow occur continually at the entrance to the pipe, then for $\mathrm{R} < \mathrm{R}_{\mathrm{cr}}'$ they will be damped out at some distance down the pipe, no matter how strong they are initially. If, on the other hand, $\mathrm{R} > \mathrm{R}_{\mathrm{cr}}'$, the flow becomes turbulent throughout the pipe, and this can be achieved by perturbations that are weaker, if R is greater. In the range between $\mathrm{R}_{\mathrm{cr}}'$ and R_{cr}, laminar flow is metastable. For a pipe with circular cross-section, undamped turbulence has been observed for $\mathrm{R} \cong 1800$, and for flow between parallel planes for $\mathrm{R} \cong 1000$ and upwards.

Since the disruption of laminar flow in a pipe is "hard", it is accompanied by a discontinuous change in the drag force. For flow in a pipe with $\mathrm{R} > \mathrm{R}_{\mathrm{cr}}'$ there are essentially two different dependences of the drag on R, one for laminar and the other for turbulent flow (see §43). The drag has a discontinuity, whatever the value of R at which the change from one to the other occurs.

One further remark may be made, to complete this section. The limit of stability (neutral curve) obtained for flow in an infinitely long pipe has also another significance. Let us consider flow in a pipe whose length is very great (in comparison with its width) but finite. Let certain boundary conditions be imposed at each end, by specifying the velocity profile (for example, we can imagine the ends of the pipe to be closed with porous seals which create a uniform profile); everywhere except near the ends of the pipe, the unperturbed velocity profile may be taken to have the Poiseuille form independent of x. For a finite system thus defined, we can propose the problem of stability with respect to infinitesimal perturbations; the general procedure for establishing the condition for such *global stability* is described in *PK*, §65. It can be shown that the above-mentioned neutral curve for an infinite pipe is also the limit of global stability in a finite pipe, whatever the specific boundary conditions at its ends.†

§29. Instability of tangential discontinuities

Flows in which two layers of incompressible fluid move relative to each other, one "sliding" on the other, are unstable if the fluid is ideal; the surface of separation between

† See A. G. Kulikovskiĭ, *Journal of Applied Mathematics and Mechanics* 32, 100, 1968.

these two fluid layers would be a *surface of tangential discontinuity*, on which the fluid velocity tangential to the surface is discontinuous (H. Helmholtz 1868, W. Kelvin 1871). We shall see below (§35) what is the actual nature of the flow resulting from this instability; here we shall prove the above statement.

If we consider a small portion of the surface of discontinuity and the flow near it, we may regard this portion as plane, and the fluid velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ on each side of it as constants. Without loss of generality we can suppose that one of these velocities is zero; this can always be achieved by a suitable choice of the coordinate system. Let $\mathbf{v}_2 = 0$, and $\mathbf{v}_1$ be denoted by $\mathbf{v}$ simply; we take the direction of $\mathbf{v}$ as the x-axis, and the z-axis along the normal to the surface.

Let the surface of discontinuity receive a slight perturbation, in which all quantities—the coordinates of points on the surface, the pressure, and the fluid velocity—are periodic functions, proportional to $e^{i(kx-\omega t)}$. We consider the fluid on the side where its velocity is $\mathbf{v}$, and denote by $\mathbf{v}'$ the small change in the velocity due to the perturbation. According to the equations (26.4) (with constant $\mathbf{v}_0 = \mathbf{v}$ and $\nu = 0$), we have the following system of equations for the perturbation $\mathbf{v}'$:

$$\operatorname{div}\mathbf{v}' = 0, \qquad \frac{\partial \mathbf{v}'}{\partial t} + (\mathbf{v}\cdot\mathbf{grad})\mathbf{v}' = -\frac{\mathbf{grad}\, p'}{\rho}.$$

Since $\mathbf{v}$ is along the x-axis, the second equation can be rewritten as

$$\frac{\partial \mathbf{v}'}{\partial t} + v\frac{\partial \mathbf{v}'}{\partial x} = -\frac{\mathbf{grad}\, p'}{\rho}. \tag{29.1}$$

If we take the divergence of both sides, then the left-hand side gives zero by virtue of $\operatorname{div}\mathbf{v}' = 0$, so that p' must satisfy Laplace's equation:

$$\triangle p' = 0. \tag{29.2}$$

Let $\zeta = \zeta(x,t)$ be the displacement in the z-direction of points on the surface of discontinuity, due to the perturbation. The derivative $\partial\zeta/\partial t$ is the rate of change of the surface coordinate ζ for a given value of x. Since the fluid velocity component normal to the surface of discontinuity is equal to the rate of displacement of the surface itself, we have to the necessary approximation

$$\partial\zeta/\partial t = v'_z - v\partial\zeta/\partial x, \tag{29.3}$$

where, of course, the value of v'_z on the surface must be taken.

We seek p' in the form $p' = f(z)\, e^{i(kx-\omega t)}$. Substituting in (29.2), we have for $f(z)$ the equation $\mathrm{d}^2 f/\mathrm{d}z^2 - k^2 f = 0$, whence $f = \text{constant} \times e^{\pm kz}$. Suppose that the space on the side under consideration (side 1) corresponds to positive values of z. Then we must take $f = \text{constant} \times e^{-kz}$, so that

$$p'_1 = \text{constant} \times e^{i(kx-\omega t)}e^{-kz}. \tag{29.4}$$

Substituting this expression in the z-component of equation (29.1), we find†

$$v'_z = kp'_1/i\rho_1(kv-\omega). \tag{29.5}$$

† The case $kv = \omega$, though possible in principle, is not of interest here, since instability can arise only from complex frequencies ω, not from real ω.

The displacement ζ may also be sought in a form proportional to the same exponential factor $e^{i(kx-\omega t)}$, and we obtain from (29.3) $v'_z = i\zeta(kv-\omega)$. This gives, instead of (29.5),

$$p'_1 = -\zeta\rho_1(kv-\omega)^2/k. \tag{29.6}$$

The pressure p'_2 on the other side of the surface is given by a similar formula, where now $v = 0$ and the sign is changed (since in this region $z < 0$, and all quantities must be proportional to e^{kz}, not e^{-kz}). Thus

$$p'_2 = \zeta\rho_2\omega^2/k. \tag{29.7}$$

We have written different densities ρ_1 and ρ_2 in order to include the case where we have a boundary separating two different immiscible fluids.

Finally, from the condition that the pressures p'_1 and p'_2 be equal on the surface of discontinuity, we obtain $\rho_1(kv-\omega)^2 = -\rho_2\omega^2$, from which the desired relation between ω and k is found to be

$$\omega = kv\frac{\rho_1 \pm i\sqrt{(\rho_1\rho_2)}}{\rho_1+\rho_2}. \tag{29.8}$$

We see that ω is complex, and there are always ω having a positive imaginary part. Thus tangential discontinuities are unstable, even with respect to infinitely small perturbations.† In this form, the result is true for very small viscosities. In that case, it is meaningless to distinguish convected and absolute instability, since as k increases the imaginary part of ω increases without limit, and hence the amplification coefficient of the perturbation as it is carried along may be as large as we please.

When finite viscosity is taken into account, the tangential discontinuity is no longer sharp; the velocity changes from one value to another across a layer with finite thickness. The problem of the stability of such a flow is mathematically entirely similar to that of the stability of flow in a laminar boundary layer with a point of inflexion in the velocity profile (§41). The experimental and numerical results indicate that instability sets in very soon, and perhaps is always present.‡

§30. Quasi-periodic flow and frequency locking††

In the following discussion (§§30–32) it will be convenient to use certain geometrical representations. To do so, we define the mathematical concept of the *space of states* for the fluid, each point in which corresponds to a particular velocity distribution or velocity field in the fluid. States at adjacent instants then correspond to adjacent points.§

A steady flow is represented by a point, and a periodic flow by a closed curve in the space of states; these are called respectively a *limit point* or *critical point*, and a *limit cycle*. If the

† If the direction of the wave vector **k** (in the xy-plane) is not the same as that of **v** but is at an angle ϕ to it, v in (29.8) is replaced by $v\cos\phi$, as is clear from the fact that the unperturbed velocity occurs in the initial linearized Euler's equation only in the combination $\mathbf{v}\cdot\mathbf{grad}$. Such perturbations also are evidently unstable.

‡ Numerical calculations of the stability have been made for plane-parallel flows whose velocities vary between $\pm v_0$ according to a law such as $v = v_0 \tanh(z/h)$; the Reynolds number is then $\mathrm{R} = v_0 h/\nu$. The neutral curve in the kR-plane starts from the origin, so that for each R value there is a range of k values (increasing with R) for which the flow is stable.

††§§ 30–32 were written jointly with M. I. Rabinovich.

§ In the mathematical literature, this functional space with an infinity of dimensions (or the spaces with a finite number of dimensions which may replace it in some cases; see below) is often called *phase space*. We shall avoid this term here, in order to prevent confusion with its more specific usual meaning in physics.

flows are stable, then adjacent curves representing the establishment of the flow tend to a limit point or cycle as $t \to \infty$.

A limit cycle (or point) has in the space of states a certain *domain of attraction*, and paths which begin in that region will eventually reach the limit cycle. In this connection, the limit cycle is called an *attractor*. It should be emphasized that for flow in a given volume with given boundary conditions (and a given value of R) there may be more than one attractor. Cases can occur where the space of states contains various attractors, each with its own domain of attraction. That is, when $R > R_{cr}$ there may be more than one stable flow regime, and the different regimes occur in accordance with the way in which the R value is reached. It should be emphasized that these various stable regimes are solutions of a *non-linear* set of equations of motion.†

Let us now consider the phenomena which occur when the Reynolds number is further increased beyond the critical value at which the periodic flow discussed in §26 is established. As R increases, a point is eventually reached where this flow in its turn becomes unstable. The instability should in principle be examined similarly to the procedure in §26 for determining the instability of the original steady flow. The unperturbed flow is now the periodic flow $\mathbf{v}_0(\mathbf{r}, t)$ with frequency ω_1, and in the equations of motion we substitute $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_2$, where $\mathbf{v}_2$ is a small correction. For $\mathbf{v}_2$ we again obtain a linear equation, but the coefficients are now functions of time as well as of the coordinates, and are periodic functions of time, with period $T_1 = 2\pi/\omega_1$. The solution of such an equation is to be sought in the form

$$\mathbf{v}_2 = \Pi(\mathbf{r}, t)\, e^{-i\omega t}, \tag{30.1}$$

where $\Pi(\mathbf{r}, t)$ is a periodic function of time, with the same period T_1. The instability again occurs when there is a frequency $\omega = \omega_2 + i\gamma_2$ whose imaginary part $\gamma_2 > 0$; the real part ω_2 gives the new frequency which appears.

During the period T_1, the perturbation (30.1) changes by a factor $\mu \equiv e^{-i\omega T_1}$. This factor is called the *multiplier* of the periodic flow, and is a convenient characteristic of the amplification or damping of perturbations in that flow. A periodic flow of a continuous medium (a fluid) corresponds to an infinity of multipliers and an infinity of possible independent perturbations. It ceases to be stable at the value $R_{cr,2}$ for which one or more multipliers reach unit modulus, i.e. μ crosses the unit circle in the complex plane. Since the equations are real, the multipliers must cross this circle in complex conjugate pairs, or singly with real values $+1$ or -1. The loss of stability of the periodic flow is accompanied by a particular qualitative change in the path pattern in the space of states near the now unstable limit cycle; this change is called a local *bifurcation*. The nature of the bifurcation is largely determined by the points at which the multipliers cross the unit circle.‡

Let us consider the bifurcation when the unit circle is crossed by a pair of complex conjugate multipliers having the form $\mu = \exp(\mp 2\pi\alpha i)$ where α is irrational. This causes the occurrence of a secondary flow with a new independent frequency $\omega_2 = \alpha\omega_1$, leading to a quasi-periodic flow with two incommensurate frequencies. The counterpart of this flow in the space of states is a path in the form of an open winding on a two-dimensional

† This is the situation, for example, when Couette flow ceases to be stable; the new flow pattern that is established depends in fact on the history of the process whereby the cylinders are caused to rotate with particular angular velocities.

‡ A multiplier cannot be zero, since a perturbation cannot disappear in a finite time (one period T_1).

torus†, the now unstable limit cycle being the generator of the torus; the frequency ω_1 corresponds to rotation round the generator, and ω_2 to rotation round the torus (Fig. 18). Just as, when the first periodic flow appeared, there was one degree of freedom, we now have two arbitrary quantities (phases), so that the flow has two degrees of freedom. The loss of stability of a periodic motion, accompanied by the creation of a two-dimensional torus, is a typical phenomenon in fluid dynamics.

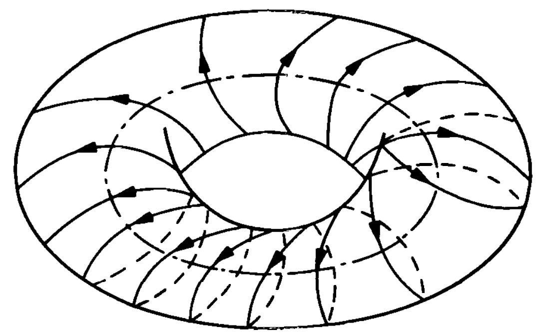

FIG. 18

Let us consider a hypothetical complication of the flow resulting from such a bifurcation, when the Reynolds number increases further ($R > R_{cr,2}$). It would be reasonable to suppose that, as R goes on increasing, new periods will successively appear. In terms of geometrical representations, this would signify loss of stability of the two-dimensional torus and the formation near it of a three-dimensional one, followed by a further bifurcation and its replacement by a four-dimensional one, and so on. The intervals between the Reynolds numbers corresponding to the successive appearance of new frequencies rapidly become shorter, and the flows are on smaller and smaller scales. The flow thus rapidly acquires a complicated and confused form, and is said to be *turbulent*, in contrast to the regular *laminar* flow, in which the fluid moves, as it were, in layers having different velocities.

Assuming now that this way or *scenario* of development of turbulence is in fact possible,‡ we write the general form of the function $\mathbf{v}(\mathbf{r}, t)$, whose time dependence is governed by some number N of different frequencies ω_i. It may be regarded as a function of N different phases $\phi_i = \omega_i t + \beta_i$ (and of the coordinates), periodic in each with period 2π. Such a function may be expressed as a series

$$\mathbf{v}(\mathbf{r}, t) = \sum \mathbf{A}_{p_1 p_2 \dots p_N}(\mathbf{r}) \exp \left\{ -i \sum_{i=1}^{n} p_i \phi_i \right\}, \tag{30.2}$$

which is a generalization of (26.13), the summation being over all integers $p_1, p_2, \dots, p_N$. The flow described by this formula involves N arbitrary initial phases β_i and has N degrees of freedom.††

† We use the mathematical terminology, in which *torus* denotes a surface without the enclosed volume. Thus a two-dimensional torus is the two-dimensional surface of a three-dimensional "doughnut".

‡ It was proposed by L. D. Landau (1944) and independently by E. Hopf (1948).

†† If we take the phases ϕ_i as coordinates representing the path on an N-dimensional torus, the corresponding velocities are constants $\dot{\phi}_i = \omega_i$. For this reason, quasi-periodic flow can be described as movement on a torus with constant velocity.

States whose phases differ only by an integral multiple of 2π are physically identical. Thus the essentially different values of each phase lie in the range $0 \leqslant \phi_i \leqslant 2\pi$. Let us consider a pair of phases, $\phi_1 = \omega_1 t + \beta_1$ and $\phi_2 = \omega_2 t + \beta_2$. At some instant, let $\phi_1 = \alpha$. Then ϕ_1 will have the "same" value as α at every time

$$t = \frac{\alpha - \beta_1}{\omega_1} + 2\pi s \frac{1}{\omega_1},$$

where s is any integer. At these times,

$$\phi_2 = \beta_2 + (\omega_2/\omega_1)\,(\alpha - \beta_1 + 2\pi s).$$

The different frequencies are incommensurate, and therefore ω_2/ω_1 is irrational. If we reduce each value of ϕ_2 to a value in the range from 0 to 2π by subtracting an appropriate integral multiple of 2π, we therefore find that, when s varies from 0 to ∞, ϕ_2 takes values indefinitely close to any given number in that range. That is, in the course of a sufficiently long time ϕ_1 and ϕ_2 simultaneously take values indefinitely close to any specified pair. The same is true of every phase. In this turbulence model, therefore, in the course of a sufficiently long time, the fluid passes through states indefinitely close to any specified state defined by any possible set of simultaneous values of the phases ϕ_i. The time to do so, however, increases very rapidly with N and becomes so great that in practice no trace of any periodicity remains.†

It should be emphasized here that the path of turbulence development discussed above is essentially based on linear treatments. It has in fact been assumed that, when new periodic solutions appear through the evolution of secondary instabilities, the already existing periodic solutions do not disappear, but on the contrary remain almost unchanged. In this model, turbulent flow is just a superposition of a large number of such unchanged solutions. In general, however, the nature of the solutions changes when the Reynolds number increases and they cease to be stable. The perturbations interact, and this may either simplify or complicate the flow. Here is an illustration of the first possibility.

Let us take a simple case by supposing that the perturbed solution contains only two independent frequencies. As already mentioned, the geometrical representation of such a flow is an open winding on a two-dimensional torus. A perturbation with frequency ω_1 arising at $R = R_{cr,1}$ may naturally be assumed to be stronger near $R = R_{cr,2}$ (where the perturbation with frequency ω_2 arises) and therefore taken as unchanged for relatively small changes in R in that neighbourhood. Then, to describe the evolution of the perturbation with frequency ω_2 against the background of the periodic flow with frequency ω_1, we use a new variable

$$a_2(t) = |a_2(t)| e^{-i\phi_2(t)}; \tag{30.3}$$

$|a_2|$ is the shortest distance to the torus generator (the now unstable limit cycle for frequency ω_1), i.e. the relative amplitude of the secondary periodic flow, and ϕ_2 is the phase of the latter. Let us consider the behaviour of $a_2(t)$ at discrete instants that are multiples of

† In established turbulent flow of this type, the probability for the system (fluid) to be in a given small volume near a chosen point in the space of phases $\phi_1, \phi_2, \ldots, \phi_N$ is the ratio of this volume $(\delta\phi)^N$ to the total volume $(2\pi)^N$. We can therefore say that in the course of a sufficiently long time the system will be in the neighbourhood of a given point only for a fraction $e^{-\kappa N}$ of the time, where $\kappa = \log(2\pi/\delta\phi)$.

the period $T_1 = 2\pi/\omega_1$. During one period, the perturbation with frequency ω_2 changes by a factor μ, where

$$\mu = |\mu| \exp(-2\pi i \omega_2/\omega_1)$$

is its multiplier; after an integral number τ of such periods, a_2 is multiplied by μ^τ. We assume that $\mathrm{R} - \mathrm{R}_{cr}$ is small; the growth factor of the perturbation is then also small, and $|\mu| - 1$ is positive but small, so that a_2 changes only slightly during the period T_1; the phase ϕ_2 varies simply in proportion to τ. We can thus treat the discrete variable τ as if it were continuous and represent the variation of $a_2(\tau)$ by a differential equation in τ.

The concept of the multiplier relates to very short time intervals after the onset of instability, when the perturbation is still describable by linear equations. In this range, $a_2(\tau)$ varies as μ^τ according to the above discussion, and

$$\mathrm{d}a_2/\mathrm{d}\tau = a_2(\tau) \log \mu;$$

just above the critical Reynolds number,

$$\begin{aligned} \log \mu &= \log |\mu| - 2\pi i \omega_2/\omega_1 \\ &\cong |\mu| - 1 - 2\pi i \omega_2/\omega_1. \end{aligned} \tag{30.4}$$

This is the first term in an expansion of $\mathrm{d}a_2/\mathrm{d}\tau$ in powers of a_2 and $a_2{}^*$, and when $|a_2|$ increases (still remaining small) the next term has to be taken into account. The term containing the same oscillatory factor is the third-order one $\propto a_2|a_2|^2$. We thus have

$$\mathrm{d}a_2/\mathrm{d}\tau = a_2 \log \mu - \beta_2 a_2 |a_2|^2, \tag{30.5}$$

where β_2, like μ, is a complex parameter depending on R, with re $\beta_2 > 0$; compare the corresponding discussion relating to (26.7). The real part of this equation gives immediately the steady value of the modulus:

$$|a_2{}^{(0)}|^2 = (|\mu| - 1)/\mathrm{re}\, \beta_2.$$

The imaginary part gives an equation for the phase $\phi_2(\tau)$; with the above steady value of the modulus, it is

$$\mathrm{d}\phi_2/\mathrm{d}\tau = 2\pi\omega_2/\omega_1 + |a_2{}^{(0)}|^2 \,\mathrm{im}\, \beta_2. \tag{30.6}$$

According to this, ϕ_2 rotates at a constant rate, a property which is, however, valid only in the approximation considered: as $\mathrm{R} - \mathrm{R}_{cr}$ increases, the rotation is no longer uniform, and the rate of rotation on the torus is itself a function of ϕ_2. To take account of this, we add on the right-hand side of (30.6) a small perturbation $\Phi(\phi_2)$; since all the physically different values of ϕ_2 lie in the range from 0 to 2π, $\Phi(\phi_2)$ is periodic with period 2π. Next, we approximate the irrational ratio ω_2/ω_1 by a rational fraction (which can be done with any desired degree of accuracy): $\omega_2/\omega_1 = m_2/m_1 + \Delta/2\pi$, where m_1 and m_2 are integers. The equation then becomes

$$\mathrm{d}\phi_2/\mathrm{d}\tau = 2\pi m_2/m_1 + \Delta + |a_2{}^{(0)}|^2 \,\mathrm{im}\, \beta_2 + \Phi(\phi_2). \tag{30.7}$$

We shall now consider phase values only at times that are a multiple of $m_1 T_1$, i.e. for values of $\tau = m_1 \bar{\tau}$, where $\bar{\tau}$ is an integer. The first term on the right of (30.7) causes in a time $m_1 T_1$ a change in phase by $2\pi m_2$, that is, by an integral multiple of 2π, which can simply be omitted. The whole right-hand side is then a small quantity, so that the change in the function $\phi_2(\bar{\tau})$ can be described by a differential equation in the continuous variable $\bar{\tau}$:

$$\frac{1}{m_1}\frac{\mathrm{d}\phi_2}{\mathrm{d}\bar{\tau}} = \Delta + |a_2^{(0)}|^2 \operatorname{im} \beta + \Phi(\phi_2); \tag{30.8}$$

in one step of the discrete variable $\bar{\tau}$, ϕ_2/m_1 changes only slightly.

In the general case, (30.8) has steady solutions $\phi_2 = \phi_2^{(0)}$ for which the right-hand side of the equation is zero. The fact that ϕ_2 is constant for times that are multiples of $m_1 T_1$ means that there is a limit cycle on the torus: the path is closed after m_1 turns. Since $\Phi(\phi_2)$ is periodic, such solutions occur in pairs (one pair in the simplest case): one on the ascending and one on the descending part of $\Phi(\phi_2)$. Of these two, only the latter is stable, for which (30.8) has near $\phi_2 = \phi_2^{(0)}$ the form

$$\mathrm{d}\phi_2/\mathrm{d}\bar{\tau} = -\text{constant} \times (\phi_2 - \phi_2^{(0)})$$

with the constant positive, and there is in fact a solution tending to $\phi_2 = \phi_2^{(0)}$; the second solution is unstable, and the constant is negative.

The formation of a stable limit cycle on the torus is equivalent to *frequency locking* – the disappearance of the quasi-periodic flow and the establishment of a new periodic one. This phenomenon, which in a system with many degrees of freedom can occur in many ways, prevents the occurrence of a flow that is a superposition of flows having a large number of incommensurate frequencies. In this sense, we can say that the probability of the actual occurrence of the Landau–Hopf scenario is very small; this, of course, does not mean that in particular cases several incommensurate frequencies may not appear before locking occurs.

§31. Strange attractors

There is as yet no complete theory of the origin of turbulence in various types of hydrodynamic flow. Various scenarios have, however, been proposed for the process whereby the flow becomes disordered, based mainly on computer studies of model systems of differential equations, partly supported by experiments. The purpose of the discussion in §§31 and 32 will be merely to give some account of these ideas, without going into the relevant results of such studies. It should only be noted that the experimental results relate to hydrodynamic flows in restricted volumes, and these are the flows to be considered in what follows.†

First of all, the following important general remark is to be made. In the analysis of the stability of periodic flow, only those multipliers are of interest whose moduli are close to 1 and which can cross the unit circle when R changes slightly. In viscous flow, the number of these "dangerous" multipliers is always finite, for the following reason. The various types (modes) of perturbation allowed by the equations of motion have different spatial scales, i.e. distances over which $\mathbf{v}_2$ varies significantly. As the scale of the motion decreases, the velocity gradients in it increase and it is retarded to a greater extent by the viscosity. If the allowed modes are arranged in order of decreasing scale, only a finite number at the

† We shall in fact be concerned with thermal convection in restricted volumes, and with Couette flow between coaxial cylinders with finite length. The theoretical ideas on the mechanism of turbulence formation in the boundary layer and in the wake in flow past finite bodies have not so far been much developed, despite the existence of a considerable quantity of experimental results.

beginning can be dangerous; those sufficiently far along the sequence are certain to be strongly damped and correspond to multipliers with small modulus. This enables us to suppose that the possible types of instability of periodic viscous flow can be analysed in essentially the same way as for a dissipative discrete mechanical system described by a finite number of variables; hydrodynamically, these may be, for example, the amplitudes of the Fourier components of the velocity field with respect to the coordinates. The space of states correspondingly has a finite number of dimensions.

Mathematically, we have to consider the time variation of a system that is represented by equations having the form

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}), \tag{31.1}$$

where $\mathbf{x}(t)$ is a vector in the space of n quantities $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$, which describe the system; the function $\mathbf{F}$ depends on a parameter whose variation may alter the nature of the flow.† For a dissipative system, the divergence of $\dot{\mathbf{x}}$ in $\mathbf{x}$-space is negative; this expresses the contraction of the volumes in that space during the motion:‡

$$\operatorname{div} \dot{\mathbf{x}} = \operatorname{div} \mathbf{F} \equiv \partial F^{(i)}/\partial x^{(i)} < 0. \tag{31.2}$$

Let us now return to the possible results of interaction between different periodic flows. Frequency locking simplifies the flow, but the interaction may also eliminate the quasi-periodicity in such a way as to complicate the picture significantly. So far, it has been tacitly assumed that when the periodic flow becomes unstable an additional periodic flow occurs. This is not logically necessary, however. If the velocity fluctuation amplitudes are limited, this means only that there is a limited volume in the space of states which contains the paths corresponding to steady viscous flow, but we cannot say in advance what the pattern of paths in that volume will be. They may tend to a limit cycle or to an open winding on the torus (corresponding to periodic and quasi-periodic flow), or they may behave quite differently, taking a complicated and confused form. This possibility is extremely important for our understanding of the mathematical nature of turbulence formation and the elucidation of its mechanism.

One can get an idea of the complicated and confused form of the paths within the limited volume containing them, by assuming that all the paths in the volume are unstable. They may include not only unstable cycles but also open paths which wind indefinitely through the limited region, without leaving it. The instability signifies that two points very close together in the space of states will move far apart as they continue along their respective paths; points initially close together may also belong to the same path, since the volume is limited and an open path can pass indefinitely close to itself. This complicated and irregular behaviour of the paths is associated with turbulent flow.

This picture has a further feature: the sensitivity of the flow to small changes in the initial conditions. If the flow is stable, a slight uncertainty in specifying these conditions causes only a similar uncertainty in the determination of the final state. If the flow is unstable, the initial uncertainty increases with time and the ultimate state of the system cannot be predicted (N. S. Krylov 1944; M. Born 1952).

† In mathematical terms, $\mathbf{F}$ is the *vector field* of the system. If it does not depend explicitly on the time, as in (31.1), the system is said to be *autonomous*.

‡ For a Hamiltonian mechanical system, the divergence is zero by Liouville's theorem; the components of $\mathbf{x}$ are in that case the generalized coordinates q and momenta p of the system.

An attracting set of unstable paths in the space of states of a dissipative system can in fact exist (E. N. Lorenz 1963), and it is usually called a *stochastic attractor* or *strange attractor.*†

At first sight, the requirement that all paths belonging to the attractor be unstable appears incompatible with the requirement that all adjacent paths tend to it as $t \to \infty$, since the instability implies that the paths move apart. The apparent contradiction is eliminated if we note that the paths can be unstable in some directions in the space of states and stable (that is, attractive) in other directions. In an n-dimensional space of states, the paths belonging to a strange attractor cannot be unstable in all $n-1$ directions (one direction being along the path), since this would mean a continuous increase in the initial volume in the space of states, which is not possible for a dissipative system. Consequently, adjacent paths tend towards the attractor paths in some directions and away from them in other (unstable) directions; see Fig. 19. These are called *saddle paths*, and it is the set of saddle paths that forms the strange attractor.

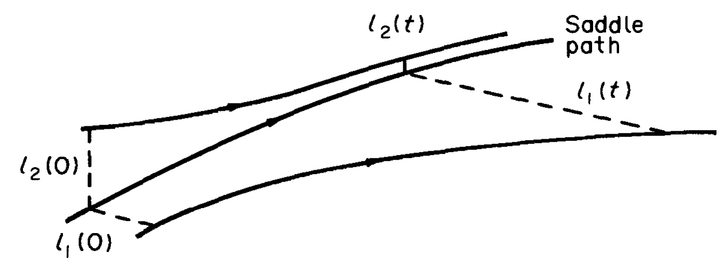

FIG. 19

The strange attractor may appear after only a few bifurcations forming new periods: even an infinitesimal non-linearity can eliminate a quasi-periodic regime (an open winding on the torus) and form a strange attractor on the torus (D. Ruelle and F. Takens 1971). This cannot occur, however, at the second bifurcation (from the end of the steady regime). Here, an open winding on the two-dimensional torus is formed. When the small non-linearity is taken into account, the torus continues to exist, so that the strange attractor could be accommodated on it. But a two-dimensional surface cannot carry an attracting set of unstable paths. The reason is that paths in the space of states cannot intersect one another (or themselves), since this would contradict the causality principle in the behaviour of classical systems, whereby the state of the system at any instant uniquely determines its behaviour at subsequent instants. On a two-dimensional surface, the impossibility of intersections makes the paths so orderly that they cannot become sufficiently random.

Even at the third bifurcation, however, a strange attractor can (but need not) be formed. This attractor, which replaces the three-frequency quasi-periodic regime, lies on a three-dimensional torus (S. Newhouse, D. Ruelle and F. Takens 1978).

The complicated and confused paths in a strange attractors lie in a limited volume in the space of states. There is not yet a known classification of the possible types of strange attractor that can occur in actual problems of fluid dynamics, nor even a set of criteria on

† In contrast to ordinary attractors (stable limit cycles, limit points, and so on); the word "strange" reflects the complexity of its structure, to be discussed later. In the physics literature, "strange attractor" also denotes more complicated attracting manifolds containing stable as well as unstable paths, but having such small domains of attraction as to be undetectable in either physical or numerical experiments.

which such a classification should be based. The available information as to the structure of strange attractors is derived essentially only from a study of instances arising in the computer solution of model systems of ordinary differential equations, which are quite different from the actual equations of fluid dynamics. It is, however, possible to draw some general conclusions about the structure of strange attractors from the saddle-type instability of the paths and the dissipative property of the system.

For clarity, we will refer to a three-dimensional space of states and imagine the attractor inside a two-dimensional torus. Let us consider a set of paths on the way to the attractor, which describe transient flow regimes leading to the establishment of "steady" turbulence. In a transverse cross-section the paths, or rather their traces, occupy a certain area; let us see how this area varies in size and shape along the paths. We note that the volume element near a saddle path expands in one transverse direction and contracts in the other; since the system is dissipative, the latter effect is the stronger, and volumes must decrease. These directions must vary along the paths, since otherwise the latter would get too far away and there would be too great a change in the fluid velocity. The net result is that the cross-section becomes smaller, flattened, and curved. This should apply not only to the whole cross-section but to every area element in it. It thus separates into nested zones separated by voids. In the course of time (i.e. along the paths) the number of zones rapidly increases, and they become narrower. The attractor formed as $t \to \infty$ consists of an uncountable manifold of layers not in contact, whose surfaces carry the saddle paths (with their attracting directions "outwards"). These layers are joined in a complicated manner at their sides and ends; each path belonging to the attractor wanders through all the layers and in the course of a sufficiently long time passes indefinitely close to any point of the attractor – the *ergodic* property. The total volume of the layers and their total cross-sectional area are zero.

In mathematical language, such manifolds in one direction are Cantorian sets. The Cantorian structure is the most characteristic property of the attractor and more generally of an n-dimensional ($n > 3$) space of states.

The volume of the strange attractor in its space of states is always zero. It may, however, be non-zero in another space with fewer dimensions. The latter is found as follows. We divide the whole of n-dimensional space into small cubes with edge ε and volume ε^n. Let $N(\varepsilon)$ be the least number of cubes which completely cover the attractor. We define the attractor dimension D as the limit†

$$D = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log (1/\varepsilon)}. \tag{31.3}$$

The existence of this limit signifies that the volume of the attractor in D-dimensional space is finite: when ε is small, $N(\varepsilon) \cong V\varepsilon^{-D}$ (where V is a constant), and $N(\varepsilon)$ may therefore be regarded as the number of D-dimensional cubes covering the volume V in D-dimensional space. When defined in accordance with (31.3), the dimension evidently cannot exceed the total dimension n of the space of states, but may be less, and unlike the ordinary dimension it may be non-integral, as happens for Cantorian sets.‡

† This is known in mathematics as the limiting capacity of the manifold. Its definition is similar to that of Hausdorff or fractal dimensions.

‡ The n-dimensional cubes covering the set may be "almost empty", and for this reason we can have $D < n$. For ordinary sets, the definition (31.3) gives obvious results. For example, with a set of N isolated points, $N(\varepsilon) = N$ and $D = 0$; for a line segment with length L, $N(\varepsilon) = L/\varepsilon$ and $D = 1$; for a two-dimensional surface area A, $N(\varepsilon) = S/\varepsilon^2$ and $D = 2$; and so on.

The following point is important. If turbulent flow is already established (the strange attractor has been reached), then the flow in a dissipative system (a viscous fluid) is the same in principle as stochastic flow of a non-dissipative system with a space of states having fewer dimensions. This is because, for steady flow, the viscous dissipation of energy is compensated on the average over a long time by the energy coming from the average flow (or from some other source of disequilibrium). Consequently, if we trace the development in time of a "volume" element belonging to the attractor (in some space whose dimension is determined by that of the attractor), it will be conserved on average, the compression in some directions being compensated by the extension due to the divergence of adjacent paths in other directions. This property can be used to obtain a different estimate of the attractor dimension.

Because the motion on the strange attractor is ergodic, as mentioned above, its average properties can be established by analysing the motion along one unstable path belonging to the attractor in the space of states. That is, we assume that an individual path reproduces the properties of the attractor if the motion along it lasts for a sufficient time.

Let $\mathbf{x} = \mathbf{x}_0(t)$ be the equation of such a path, a solution of (31.1). Let us consider the deformation of a "spherical" volume element as it moves along this path. The deformation is given by the equations (31.1) linearized with respect to the difference $\boldsymbol{\xi} = \mathbf{x} - \mathbf{x}_0(t)$, i.e. the deviation of paths adjacent to the one considered. These equations, written in components, are

$$\dot{\xi}^{(i)} = A_{ik}(t)\,\xi^{(k)}, \qquad A_{ik}(t) = [\partial F^{(i)}/\partial x^{(k)}]_{\mathbf{x} = \mathbf{x}_0(t)}. \tag{31.4}$$

In the movement along the path, the volume element is compressed in some directions and stretched in others, the sphere becoming an ellipsoid. Both the directions and the lengths of the semi-axes vary; let the latter be $l_s(t)$, where s labels the directions. The *Lyapunov characteristic indices* are

$$L_s = \lim_{t \to \infty} \frac{1}{t} \log \frac{l_s(t)}{l(0)}, \tag{31.5}$$

where $l(0)$ is the radius of the original sphere, at a time arbitrarily chosen as $t = 0$. The quantities thus determined are real, and equal in number to the dimension n of the space. One of them (corresponding to the direction along the path) is zero.†

The sum of the Lyapunov indices gives the mean change, along the path, in the volume element in the space of states. The relative local change in volume at any point on the path is given by the divergence $\operatorname{div} \dot{\mathbf{x}} = \operatorname{div} \dot{\boldsymbol{\xi}} = A_{ii}(t)$. It can be shown that the mean divergence along the path is‡

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \operatorname{div} \dot{\boldsymbol{\xi}}\, dt = \sum_{s=1}^{n} L_s. \tag{31.6}$$

For a dissipative system, this sum is negative; volumes in an n-dimensional space of states are compressed. The dimension of the strange attractor is defined so that volumes are

† Of course, the solution of (31.4), with specified initial conditions at $t = 0$, actually represents an adjacent path only if all the distances $l_s(t)$ remain small. This, however, does not make meaningless the definition (31.5), which involves indefinitely long times: for any large t, we can choose $l(0)$ so small that the linearized equations remain valid throughout the time concerned.

‡ See V. I. Oseledets, *Transactions of the Moscow Mathematical Society* **19**, 197, 1969.

conserved on average in "its" space. To do so, we arrange the Lyapunov indices in the order $L_1 \geqslant L_2 \geqslant \ldots \geqslant L_n$, and take account of as many stable directions as is necessary to compensate the stretching, by means of compression. The attractor dimension D_L thus defined is between m and $m+1$, where m is the number of indices, in the sequence, whose sum is still positive but becomes negative when L_{m+1} is included.† The fractional part of $D_L = m + d$ $(d < 1)$ is found from

$$\sum_{s=1}^{m} L_s + L_{m+1} d = 0 \tag{31.7}$$

(F. Ledrappier 1981). Since, in calculating d, we take into account only the least stable directions (omitting the negative L_s that are largest in modulus, at the end of the sequence), the estimate D_L of the dimension is in general too high. This estimate offers in principle a way of determining the dimension of the attractor from measurements of the time dependence of the velocity fluctuations in the turbulent flow.

§32. Transition to turbulence by period doubling

Let us now consider the loss of stability of a periodic flow when the multiplier passes through -1 or $+1$.

In an n-dimensional space of states, $n-1$ multipliers determine the behaviour of the paths in $n-1$ different directions near the periodic path considered (which are not the same as the direction of the tangent at each point of that path). Let a multiplier near ± 1 correspond to the lth direction, say. The other $n-2$ multipliers are small in modulus, and therefore all the paths in the corresponding $n-2$ directions will in the course of time come close to a two-dimensional surface Σ containing the lth direction and the direction of the tangents. One can say that near the limit cycle the space of states is almost two-dimensional as $t \to \infty$ (it cannot be strictly two-dimensional, since the paths can lie on either side of Σ and go from one side to the other). Let the flux of paths near Σ be cut by a surface σ. Each path, on repeatedly passing through σ, determines in accordance with the initial point of intersection $\mathbf{x}_j$ the next point of intersection $\mathbf{x}_{j+1}$. The relation $\mathbf{x}_{j+1} = f(\mathbf{x}_j; \mathrm{R})$ is called a *Poincaré mapping* or *sequence mapping*; it depends on R, in this case the Reynolds number,‡ whose value determines the closeness to the bifurcation where the periodic flow ceases to be stable. Since all paths are close to Σ, the set of points where they meet σ is almost one-dimensional and can be approximated by a line; the Poincaré mapping becomes the one-dimensional transformation

$$x_{j+1} = f(x_j; \mathrm{R}), \tag{32.1}$$

with x simply a coordinate along the line.†† The discrete variable j acts as the time measured in units of the period.

The mapping (32.1) affords an alternative method of determining the nature of the flow near the bifurcation. The periodic flow itself corresponds to a *fixed point* of the transformation (32.1) – the value $x_j = x_*$ which is unchanged by the mapping, i.e. for which $x_{j+1} = x_j$. The multiplier is the derivative $\mu = \mathrm{d}x_{j+1}/\mathrm{d}x_j$ taken at $x_j = x_*$. The points $x_j = x_* + \xi$ near x_* are mapped into $x_{j+1} \cong x_* + \mu\xi$. The fixed point is stable (and

† Including the zero Lyapunov index adds one to D_L, corresponding to the dimension along the path.
‡ Or the Rayleigh number in the case of thermal convection (§56).
†† In this section x has of course nothing to do with the coordinate in physical space.

is an attractor of the mapping) if $|\mu| < 1$: by iterating the mapping and starting from some point near x_*, we asymptotically approach the latter, as $|\mu|^r$, where r is the number of iterations. If $|\mu| > 1$, however, the fixed point is unstable.

Let us consider the loss of stability of periodic flow when the multiplier passes through -1. The equation $\mu = -1$ signifies that the initial perturbation changes sign after a time T_0, remaining the same in magnitude: after a further time T_0 it returns to its original value. Thus a passage of μ through -1 near a limit cycle with period T_0 creates a new limit cycle with period $2T_0$ (a *period-doubling bifurcation*).† Figure 20 gives a conventional representation of two successive such bifurcations; the continuous curves in diagrams *a* and *b* show the stable limit cycles $2T_0$ and $4T_0$, the broken curves the limit cycles that have become unstable.

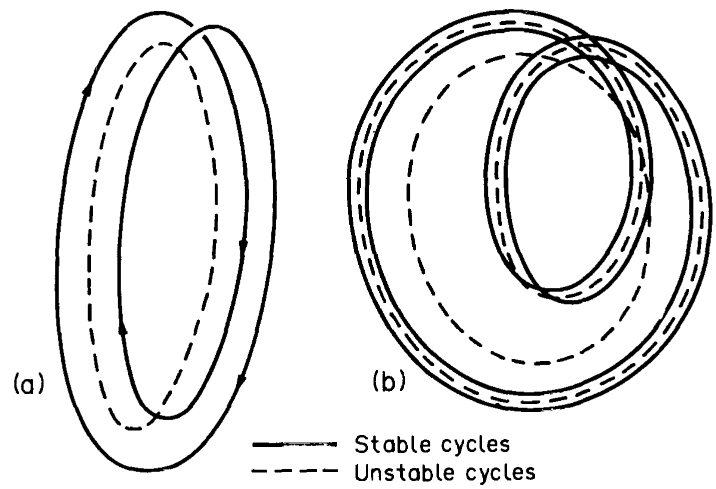

FIG. 20

If we arbitrarily take the fixed point of the Poincaré mapping as $x = 0$, the mapping near it which describes the period-doubling bifurcation may be expressed as the expansion

$$x_{j+1} = -[1+(\mathrm{R}-\mathrm{R}_1)]x_j + x_j^2 + \beta x_j^3, \tag{32.2}$$

where $\beta > 0$.‡ For $\mathrm{R} < \mathrm{R}_1$ the fixed point $x_* = 0$ is stable; for $\mathrm{R} > \mathrm{R}_1$ it is unstable. In order to see how the period-doubling occurs, we have to iterate the mapping (32.2) twice, i.e. consider it after two steps (two time units) and determine the fixed points of the re-formed mapping; if these exist and are stable, they correspond to the period-doubling cycle.

The twofold iteration of the transformation (32.2) gives (with the necessary accuracy in respect of the small quantities x_j and $\mathrm{R}-\mathrm{R}_1$) the mapping

$$x_{j+2} = x_j + 2(\mathrm{R}-\mathrm{R}_1)x_j - 2(1+\beta)x_j^3. \tag{32.3}$$

This always has the fixed point $x_* = 0$. When $\mathrm{R} < \mathrm{R}_1$, that point is the only one and is stable, with the multiplier $|dx_{j+2}/dx_j| < 1$; for flow with period 1 (in units of T_0) the time

† In this section the basic period (that of the first periodic flow) is denoted by T_0, not by T_1. The critical Reynolds numbers corresponding to successive period-doubling bifurcations will be denoted here by $\mathrm{R}_1, \mathrm{R}_2, \ldots$, without the suffix cr (R_1 replaces the previous $\mathrm{R}_{\mathrm{cr},2}$).

‡ The coefficient of $\mathrm{R}-\mathrm{R}_1$ can be made equal to unity by appropriately redefining R, and that of x_j^2 can be made $+1$ by redefining x_j; we assume in (32.2) that this has been done.

interval 2 is also a period. When $R = R_1$, the multiplier is $+1$, and when $R > R_1$ the point $x_* = 0$ becomes unstable. At that stage, a pair of stable fixed points are formed,

$$x_*^{(1),(2)} = \pm\sqrt{\left[\frac{R-R_1}{1+\beta}\right]}, \tag{32.4}$$

corresponding to a stable limit cycle of the double period†; the transformation (32.3) leaves each of these points in position, while (32.2) changes each into the other. It must be emphasized that the single-period cycle does not disappear at this bifurcation, but remains a solution (unstable) of the equations of motion.

Near the bifurcation, the motion is still "almost periodic" with period unity: the points $x_*^{(1)}$ and $x_*^{(2)}$ at which the paths return are close together. The interval $x_*^{(1)} - x_*^{(2)}$ between them is a measure of the amplitude of the oscillations with period 2; it increases as $\sqrt{(R-R_1)}$, similarly to the increase (26.10) in the amplitude of periodic flow after it begins at the point where the steady flow becomes unstable.

The repetition of period-doubling bifurcations is one route to the formation of turbulence. In this scenario the number of bifurcations is infinite, and they follow one another (as R increases) at ever decreasing intervals; the sequence of critical values R_1, R_2, . . . tends to a finite limit beyond which the periodicity disappears altogether and a complex aperiodic attractor is created in the space, associated in this scenario with the formation of turbulence. We shall see that the scenario has noteworthy properties of universality and scale invariance (M. J. Feigenbaum 1978).‡

The quantitative theory given below starts from the hypothesis that the bifurcations follow one another (as R increases) so quickly that even in the intervals between them the region occupied by the set of paths in the space of states remains almost two-dimensional, and the whole sequence of bifurcations can be described by a one-dimensional Poincaré mapping dependent on a single parameter.

The choice of mapping used below can be justified as follows. In a considerable part of the range of variation of x, the mapping must be a stretching one with $|\mathrm{d}f(x;\lambda)/\mathrm{d}x| > 1$; this allows instabilities to occur. The mapping must also bring back to a given range the paths that have left it, since otherwise the velocity fluctuations would increase without limit, which is impossible. The two requirements can be simultaneously satisfied only by non-monotonic functions $f(x;\lambda)$, that is, mappings (32.1) that are not one-to-one: the x_{j+1} values are uniquely determined by the preceding x_j, but not conversely. The simplest form of such a function has a single maximum, near which we put

$$x_{j+1} = f(x_j;\lambda) = 1 - \lambda x_j^2, \tag{32.5}$$

with λ a positive parameter which is to be regarded (in terms of fluid mechanics) as an increasing function of R.†† We shall arbitrarily take the segment $[-1, +1]$ as the range of

† To be called for brevity a *2-cycle*. The relevant fixed points will be called *cycle elements*.

‡ The sequence of period-doubling bifurcations (numbered below as 1, 2, . . .) need not begin with the first bifurcation of the periodic flow. It may in principle begin after the first few bifurcations with the appearance of incommensurate frequencies, when these have been locked by the mechanism discussed in §30.

†† The admissibility of mappings that are not one-to-one depends on the approximateness of the one-dimensional treatment. If all the paths were exactly on one surface Σ, so that the Poincaré mapping would be strictly one-dimensional, this non-uniqueness would be impossible, since it would imply that two paths with different x_j intersected at x_{j+1}. In the same sense, the approximateness is responsible for the possibility of a zero multiplier if the fixed point of the mapping is at an extremum of the mapping function; such a point may be described as "superstable", and is approached more rapidly than according to the above relationship.

variation of x; when λ is between 0 and 2, all iterations of the mapping (32.5) leave x in that range.

The transformation (32.5) has a fixed point at the root of $x_* = 1 - \lambda x_*^2$. This becomes unstable when $\lambda > \Lambda_1$, where Λ_1 is the value of λ for which the multiplier $\mu = -2\lambda x_* = -1$; from the two equations, we find $\Lambda_1 = 3/4$. This is the first critical value of λ, which determines the position of the first period-doubling bifurcation and the appearance of the 2-cycle. Let us now trace the appearance of subsequent bifurcations by means of an approximate technique of determining some qualitative features of the process, though this does not give exact values of the characteristic constants; exact statements will then be formulated.

Repetition of the transformation (32.5) gives

$$x_{j+2} = 1 - \lambda + 2\lambda^2 x_j^2 - \lambda^3 x_j^4. \tag{32.6}$$

Here we will neglect the term in x_j^4. The remaining equation is converted by the scale transformation†

$$x_j \to x_j/\alpha_0, \qquad \alpha_0 = 1/(1-\lambda)$$

to the form

$$x_{j+2} = 1 - \lambda_1 x_j^2,$$

which differs from (32.5) only in that λ is replaced by

$$\lambda_1 = \phi(\lambda) \equiv 2\lambda^2(\lambda - 1). \tag{32.7}$$

Repeating this operation with the scale factors $\alpha_1 = 1/(1-\lambda_1)$, etc., gives a sequence of mappings having the same form:

$$x_{j+2^m} = 1 - \lambda_m x_j^2, \qquad \lambda_m = \phi(\lambda_{m-1}). \tag{32.8}$$

The fixed points of the mappings (32.8) correspond to 2^m-cycles.‡ Since they all have the same form as (32.5), we can deduce at once that the 2^m-cycles ($m = 1, 2, 3, \ldots$) become unstable when $\lambda_m = \Lambda_1 = 3/4$. The corresponding critical values Λ_m of the initial parameter λ are found by solving the coupled equations

$$\Lambda_1 = \phi(\Lambda_2), \quad \Lambda_2 = \phi(\Lambda_3), \quad \ldots, \quad \Lambda_{m-1} = \phi(\Lambda_m);$$

they are obtained graphically by the construction shown in Fig. 21. Evidently, as $m \to \infty$ the sequence of numbers converges to a finite limit Λ_∞, the root of $\Lambda_\infty = \phi(\Lambda_\infty)$; this is $\Lambda_\infty = (1+\sqrt{3})/2 = 1{\cdot}37$. The scale factors also tend to a finite limit: $\alpha_m \to \alpha$, where $\alpha = 1/(1-\Lambda_\infty) = -2{\cdot}8$.

It is easy to find how Λ_m approaches Λ_∞ when m is large. From the equation $\Lambda_m = \phi(\Lambda_{m+1})$ when $\Lambda_\infty - \Lambda_m$ is small, we find

$$\Lambda_\infty - \Lambda_{m+1} = (\Lambda_\infty - \Lambda_m)/\Delta, \tag{32.9}$$

† This is not possible when $\lambda = 1$ (and the fixed point of the mapping (32.6) coincides with the central extremum: $x_* = 0$). The value $\lambda = 1$ is, however, certainly not the next critical value λ_2 that is needed here.

‡ To avoid misunderstanding, it should be emphasized that after the scale transformations the mappings (32.8) must be defined over extended ranges $|x| \leqslant |\alpha_0 \alpha_1 \ldots \alpha_{m-1}|$, not $|x| \leqslant 1$ as in (32.5) and (32.6). However, in view of the terms neglected, the expressions (32.8) can in practice give a description only of the range near the central extrema of the mapping functions.

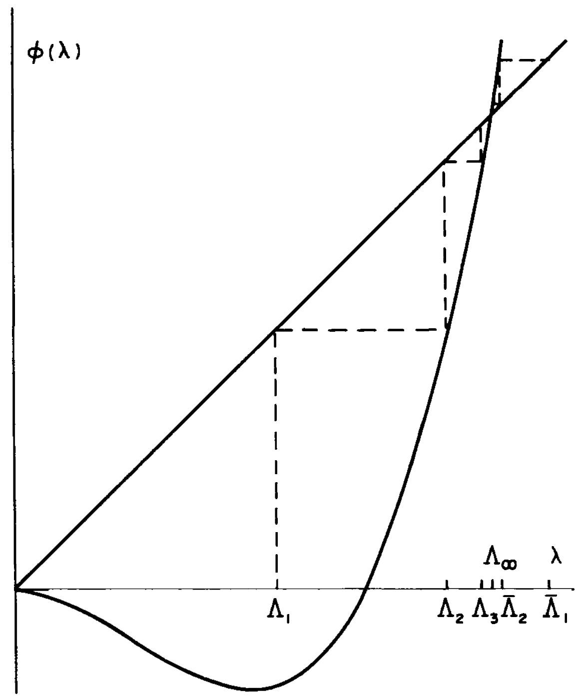

FIG. 21

where $\delta = \phi'(\Lambda_\infty) = 4+\sqrt{3} = 5{\cdot}73$. Thus $\Lambda_\infty - \Lambda_m \propto \delta^m$, that is, Λ_m approaches the limit in geometrical progression. The same relation applies to the intervals between successive critical numbers: equation (32.9) can be written in the equivalent form

$$\Lambda_{m+2} - \Lambda_{m+1} = (\Lambda_{m+1} - \Lambda_m)/\delta. \tag{32.10}$$

As regards fluid dynamics, it has already been mentioned that λ is to be regarded as a function of the Reynolds number, and accordingly the latter has critical values which correspond to successive period-doubling bifurcations and tend to a finite limit R_∞. It is evident that for these values we have the same limiting relations (32.9), (32.10), with the same constant δ, as for Λ_m.

The above arguments illustrate the origin of the basic features of the process, namely the infinity of bifurcations, whose times of appearance converge to the limit Λ_∞ according to (32.9) and (32.10), and the existence of the scale factor α. The values thus found for the characteristic constants are not exact, however. The exact values (found by repeated computer iterations of the mapping (32.5)) of the convergence factor (*Feigenbaum number*) δ and the scale factor α are

$$\delta = 4{\cdot}6692\ldots, \quad \alpha = -2{\cdot}5029\ldots, \tag{32.11}$$

and the limiting value $\Lambda_\infty = 1{\cdot}401$.† The value of δ is comparatively large; the rapid convergence has the result that the limiting relations are very nearly satisfied after only a small number of period doublings.

A deficiency of the above derivation is that, when all powers of x_j^2 above the first are neglected, the mapping (32.8) yields only the fact that the next bifurcation occurs; it does

† The value of Λ_∞ is somewhat arbitrary, since it depends on how the parameter is used in the initial mapping, i.e. the function $f(x;\lambda)$ (the values of δ and α, however, do not depend on this).

not allow all the elements of the 2^m-cycle described by this mapping to be determined.† In reality, the iterated mappings (32.5) are polynomials in x_j^2 whose degree is doubled at each iteration. They are complicated functions of x_j with a rapidly increasing number of extrema lying symmetrically about $x_j = 0$ (which also always remains an extremum).

It is noteworthy that not only the values of δ and α but also the limiting form of the infinitely iterated mapping are in a certain sense independent of the form of the initial mapping $x_{j+1} = f(x_j; \lambda)$; it is sufficient that the function $f(x; \lambda)$ of one parameter be smooth with a single quadratic maximum (let this be at $x = 0$) – it need not even be symmetrical about the maximum at great distances from it. This *universality* increases considerably the degree of generality of the theory described. The exact formulation of the property is as follows.

Let us consider the mapping specified by $f(x)$, i.e. $f(x; \lambda)$ with a particular choice of λ (see below), normalized by the condition $f(0) = 1$. By applying this twice, we get the function $f[f(x)]$. We change the scale of this function and of x by a factor $\alpha_0 = 1/f(1)$, obtaining a new function

$$f_1(x) = \alpha_0 f[f(x/\alpha_0)],$$

for which again $f_1(0) = 1$. Repeating this operation, we find a sequence of functions connected by the recurrence formula‡

$$f_{m+1}(x) = \alpha_m f_m[f_m(x/\alpha_m)] \equiv \hat{T} f_m, \quad \alpha_m = 1/f_m(1). \tag{32.12}$$

If this sequence tends, as $m \to \infty$, to a definite limiting function $f_\infty(x) \equiv g(x)$, then the latter must be a "fixed function" of the operator $\hat{T}$ defined in (32.12), i.e. must satisfy the functional relation

$$g(x) = \hat{T} g \equiv \alpha g[g(x/\alpha)], \quad \alpha = 1/g(1), \quad g(0) = 1. \tag{32.13}$$

According to the assumed properties of the admissible functions $f(x)$, $g(x)$ must be smooth and have a quadratic extremum at $x = 0$; the specific form of $f(x)$ has no other influence on equation (32.13) or on the conditions imposed on its solution. We should emphasize that, after the scale transformations used in the derivation (with $|\alpha_m| > 1$), the solution of the equation is determined for all values of the variable x in it, from $-\infty$ to $+\infty$, and not only in the range $-1 \leqslant x \leqslant 1$. The function $g(x)$ is necessarily even, since the admissible functions $f(x)$ include even ones, and an even mapping certainly remains even, after any number of iterations.

Such a solution of equation (32.13) does in fact exist and is unique, although it cannot be derived in an analytical form; it is a function having an infinity of extrema and unlimited in magnitude, the constant α being determined along with $g(x)$. In practice, it is sufficient to derive the function in the range $[-1, 1]$, after which it can be continued outside the range by iterating the operation $\hat{T}$. Note that at each stage of iteration of $\hat{T}$ in (32.12) the values of $f_{m+1}(x)$ in the range $[-1, 1]$ are determined by those of $f_m(x)$ in a part of this range

† That is, all the 2^m points $x_*^{(1)}, x_*^{(2)}, \ldots$ which change successively into one another (and are periodic) when the mapping (31.5) is iterated, and are fixed (and stable) with respect to the 2^m-fold iterated mapping. To avoid any doubts, it may be noted that the derivatives dx_{j+2^m}/dx_j are necessarily the same at all points $x_*^{(1)}, x_*^{(2)}, \ldots$ (and therefore pass simultaneously through -1 at the next bifurcation); we shall not give here the proof of this property (which is evidently necessary).

‡ There is an obvious analogy between this procedure and the one used previously in deriving (32.8).

shortened by a factor $|\alpha_m| \cong |\alpha|$. This means that in the limit of many iterations, the determination of $g(x)$ in the range $[-1, 1]$ (and therefore on the whole of the x-axis) is governed by smaller and smaller parts of the initial function near its maximum, and herein lies the ultimate cause of the universality.†

The function $g(x)$ determines the structure of the aperiodic attractor formed by an infinite sequence of period doublings. This occurs at a parameter value $\lambda = \Lambda_\infty$ which is quite definite for a given function $f(x;\lambda)$. It is therefore clear that the functions formed from $f(x;\lambda)$ by repeated iteration of the transformation (32.12) do in fact converge to $g(x)$ only for this isolated value of λ. It follows in turn that the fixed function of the operator $\hat{T}$ is unstable with respect to small changes corresponding to small deviations of λ from the value Λ_∞. The study of this instability enables us to determine the universal constant δ, again independent of the specific form of $f(x)$.‡

The scale factor α determines the change (decrease) in the geometrical characteristics (in the space of states) of the attractor at each stage of period doubling; these characteristics are the distances between limit cycle elements on the x-axis. However, since each doubling is accompanied by a further increase in the number of cycle elements, this statement must be made more specific and precise. It is clear a priori that the scale cannot vary in the same way for the distances between every pair of points.†† For, if two adjacent points are transformed by an almost linear section of the mapping function, the distance between them is reduced by a factor $|\alpha|$; but if the transformation takes place by a section of the mapping function near its extremum, the distance is reduced by a factor α^2.

At the bifurcation ($\lambda = \Lambda_m$) each element (point) of the 2^m-cycle splits into two adjacent points, the distance between which gradually increases, but the points remain close over the whole range of variation of λ as far as the next bifurcation. If we follow the conversions of cycle elements into one another in the course of time, i.e. in successive mappings $x_{j+1} = f(x_j;\lambda)$, each component of the pair changes into the other after 2^m time units. This means that the distance between the points in the pair is a measure of the oscillation amplitude of the newly formed double period, and in this sense has especial physical interest.

Let us arrange all the elements of the 2^{m+1}-cycle in the order in which they are traversed in the course of time, and denote them by $x_{m+1}(t)$, where the time t, measured in units of the basic period T_0, takes integral values: $t/T_0 = 1, 2, \ldots, 2^{m+1}$. These elements are formed from those of the 2^m-cycle by splitting into pairs. The intervals between the points in each pair are

$$\xi_{m+1}(t) = x_{m+1}(t) - x_{m+1}(t+T_m), \tag{32.14}$$

where $T_m = 2^m T_0 = \frac{1}{2}T_{m+1}$ is the period of the 2^m-cycle, or half that of the 2^{m+1}-cycle. We

† The statement that there exists a unique solution of equation (32.13) is founded on computer simulation. The solution is sought, in the range $[-1, 1]$, as a polynomial of high degree in x^2; the accuracy of the simulation must increase with the width of the x value range (outside that mentioned) to which we wish to continue the function by iteration of $\hat{T}$. In the range $[-1, 1]$, $g(x)$ has one extremum, near which $g(x) = 1 - 1{\cdot}528x^2$ if it is taken to be a maximum, a choice which is arbitrary in view of the invariance of equation (32.13) under a change in the sign of g.

‡ See the original papers by M. J. Feigenbaum, *Journal of Statistical Physics* **19**, 25, 1978; **21**, 669, 1979.

†† These are distances in the unstretched range $[-1, 1]$ which is arbitrarily taken, from the start, as the range of x containing all cycle elements. Since α is negative, the bifurcations are accompanied by inversion of the positions of the elements relative to $x = 0$.

use the function $\sigma_m(t)$, a scale factor which determines the change in the intervals (32.14) from one cycle to the next†:

$$\xi_{m+1}(t)/\xi_m(t) = \sigma_m(t). \qquad (32.15)$$

Evidently

$$\xi_{m+1}(t+T_m) = -\xi_{m+1}(t), \qquad (32.16)$$

and therefore

$$\sigma_m(t+T_m) = -\sigma_m(t). \qquad (32.17)$$

The function $\sigma_m(t)$ has complicated properties, but it can be shown that its limiting form for large m is very closely approximated by the simple expressions

$$\left.\begin{aligned}\sigma_m(t) &= 1/\alpha \text{ for } 0 < t < \tfrac{1}{2}T_m,\\ &= 1/\alpha^2 \text{ for } \tfrac{1}{2}T_m < t < T_m,\end{aligned}\right\} \qquad (32.18)$$

with the appropriate choice of zero for t.‡

These formulae yield some conclusions as to the change in the flow frequency spectrum when period doubling occurs. In fluid dynamics terms, $x_m(t)$ is to be regarded as a characteristic of the fluid velocity. For a flow with period T, the spectrum of the function $x_m(t)$ of the continuous time t contains frequencies $k\omega_m$ ($k = 1, 2, 3, \ldots$), i.e. the fundamental frequency $\omega_m = 2\pi/T_m$ and its harmonics. After the period doubling, the flow is described by the function $x_{m+1}(t)$ with period $T_{m+1} = 2T_m$. Its spectrum contains not only the same frequencies $k\omega_m$ but also the subharmonics of ω_m, the frequencies $\frac{1}{2}l\omega_m$, $l = 1, 3, 5, \ldots$.

Let us write

$$x_{m+1}(t) = \tfrac{1}{2}\{\xi_{m+1}(t) + \eta_{m+1}(t)\},$$

where ξ_{m+1} is the difference (32.14) and

$$\eta_{m+1}(t) = x_{m+1}(t) + x_{m+1}(t+T_m).$$

The spectrum of $\eta_{m+1}(t)$ contains only the frequencies $k\omega_m$; the Fourier components for the subharmonics,

$$\frac{1}{T_{m+1}}\int_0^{T_{m+1}} \eta_{m+1}(t)e^{i\pi lt/T_m}\,dt = \frac{1}{2T_m}\int_0^{T_m} \{\eta_{m+1}(t) - \eta_{m+1}(t+T_m)\}\, e^{i\pi lt/T_m}\,dt$$

are zero, since $\eta_{m+1}(t+T_m) = \eta_{m+1}(t)$. On the other hand, in the first approximation the quantities $\eta_m(t)$ are unchanged in the bifurcation: $\eta_{m+1}(t) \cong \eta_m(t)$; this means that the strength of the oscillations with frequencies $k\omega_m$ also remains unchanged.

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† Since the two cycles exist in different ranges of values of λ, $(\Lambda_{m-1}, \Lambda_m)$ and $(\Lambda_m, \Lambda_{m+1})$, and the quantities (32.14) vary considerably in these ranges, their significance in the definition (32.15) needs to be made more precise. We shall take them for the values of λ where the cycles are superstable (see the footnote following (32.5)); one such value occurs in the range where each cycle exists.

‡ We shall not give here the study of the properties of $\sigma_m(t)$, which is simple in principle but laborious; see M. J. Feigenbaum, *Los Alamos Science* **1**, 4, 1980.

The spectrum of $\zeta_{m+1}(t)$, on the other hand, contains only the subharmonics $\frac{1}{2}l\omega_m$, the new frequencies which appear at doubling stage $m+1$. The total strength of these components is given by the integral

$$I_{m+1} = \frac{1}{T_{m+1}} \int_0^{T_{m+1}} \zeta_{m+1}{}^2(t)\,dt. \tag{32.19}$$

Expressing $\zeta_{m+1}(t)$ in terms of $\zeta_m(t)$, we can write

$$I_{m+1} = \frac{1}{2T_m} \cdot 2 \int_0^{T_m} \sigma_m{}^2(t)\,\zeta_m{}^2(t)\,dt.$$

With (32.16)–(32.18),

$$I_{m+1} = \tfrac{1}{2}\left(\frac{1}{\alpha^2} + \frac{1}{\alpha^4}\right)\frac{1}{T_m}\int_0^{T_m} \zeta_m{}^2(t)\,dt$$

$$= \tfrac{1}{2}\left(\frac{1}{\alpha^2} + \frac{1}{\alpha^4}\right) I_m,$$

and finally

$$I_m/I_{m+1} = 10{\cdot}8. \tag{32.20}$$

Thus the strength of the new components which appear after a period-doubling bifurcation exceeds the one for the next bifurcation by a definite factor independent of the bifurcation number (M. J. Feigenbaum 1979).†

Let us now consider the development of the flow properties when λ increases further beyond Λ_∞ (the Reynolds number $\mathrm{R} > \mathrm{R}_\infty$), in the turbulent range. Since, at the moment of its formation (at $\lambda = \Lambda_\infty$), the aperiodic attractor is described by a one-dimensional Poincaré mapping, we can suppose that even for values of λ slightly above Λ_∞ it is permissible to treat the properties of the attractor in terms of such a mapping.

The attractor formed by an infinite sequence of period doublings is at its appearance not a strange attractor as defined in §31: the 2^∞-cycle occurring as the limit of stable 2^m-cycles when $m \to \infty$ is also stable. The points of this attractor form on the interval $[-1, 1]$ an uncountable Cantorian set. Its measure on this interval, i.e. the total "length" of its elements, is zero; its dimension is between 0 and 1, and is found to be 0·54.‡

When $\lambda > \Lambda_\infty$, the attractor becomes a strange attractor, i.e. an attracting set of unstable paths. On the interval $[-1, 1]$, the points belonging to it occupy ranges whose total length is not zero. These ranges are the traces on the sectional plane σ of a continuous two-dimensional band which makes a large number of turns and is closed. In this connection, it should be remembered that the one-dimensional treatment is approximate. In reality, the

† This applies not only to the total strength of the subharmonics but also to each of them. For each subharmonic that appears after bifurcation m there are two (one to the right and one to the left) after bifurcation $m+1$. The ratio of strengths of the individual peaks that appear after two successive bifurcations is therefore twice (32.20). A more exact value of this quantity is 10·48. This is found by analysing the state at the point $\lambda = \Lambda_\infty$ itself by means of the universal function $g(x)$; at this point, all frequencies are already present, and the problem corresponding to that raised in the last footnote but one does not arise. See M. Nauenberg and J. Rudnick, *Physical Review* **B 24**, 493, 1981.

‡ See P. Grassberger, *Journal of Statistical Physics* **26**, 173, 1981.

band has a small but non-zero thickness. The segments forming its cross-section are therefore really strips with non-zero width. Across this width, the strange attractor has the layered Cantorian structure described in §31.† This structure will not be of interest, and we shall return to a discussion in terms of the one-dimensional Poincaré mapping.

The general development of the strange attractor as λ increases beyond Λ_∞ is as follows. For a given $\lambda > \Lambda_\infty$ the attractor occupies a number of ranges in the interval $[-1, 1]$; the spaces between these ranges are the attraction domains, and contain the elements of unstable cycles with periods not exceeding some 2^m. When λ increases, the rate of divergence of the paths on the strange attractor increases, and it "expands", successively absorbing the cycles with periods $2^m, 2^{m+1}, \dots$; the number of ranges occupied by the attractor decreases, and their lengths increase. Thus the number of turns of the band mentioned above is successively halved, while their widths increase. There is then a sort of reverse cascade of successive simplifications of the attractor. The absorption of an unstable 2^m-cycle by the attractor is called a *reverse doubling bifurcation.* Figure 22 illustrates this process for two successive reverse bifurcations. In Fig. 22a, the band makes four turns and the reverse bifurcation converts it into one with two (Fig. 22b); the final bifurcation gives a band with only one turn and closed after a twist (Fig. 22c).

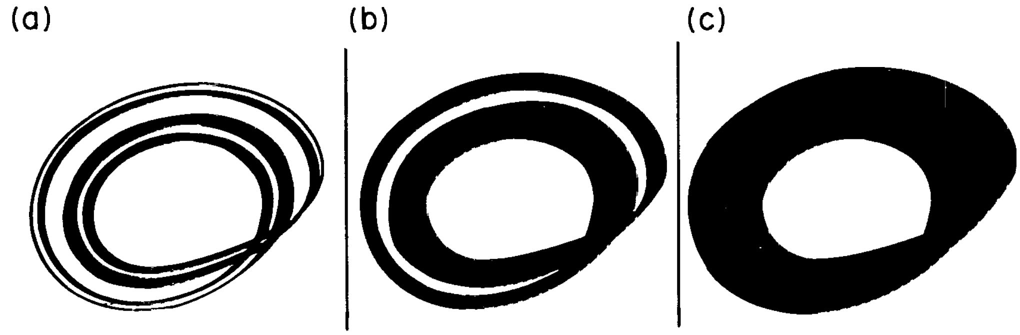

FIG. 22

Let the values of λ corresponding to successive reverse doubling bifurcations be denoted by $\bar{\Lambda}_{m+1}$, arranged in the order $\bar{\Lambda}_m > \bar{\Lambda}_{m+1}$. We shall show that they are in geometrical progression with the same universal factor δ as for forward bifurcations.

Before the final (as λ increases) reverse bifurcation, the attractor occupies two ranges separated by a gap containing the fixed point x_* of the mapping (32.5), which corresponds to an unstable cycle with period 1:

$$x_* = \frac{\sqrt{(1+4\lambda)}-1}{2\lambda}.$$

The bifurcation takes place at the value $\lambda = \bar{\Lambda}_1$, when this point is reached by the limits of the expanding attractor. Figure 22b shows that the outer limit of the attractor band becomes the inner limit after one loop and the boundary of the gap between turns after another. It follows that $\lambda = \bar{\Lambda}_1$ is given by the condition $x_{j+2} = x_*$, where

$$x_{j+2} = 1 - \lambda(1-\lambda)^2$$

† The dimension of the attractor in this direction is much less than unity, but it is not a universal property, and depends on the specific mapping.

is the result of twice iterating the mapping over the point $x_j = 1$, the limit of the attractor; $\bar{\Lambda}_1 = 1{\cdot}543$. The previous reverse bifurcations $\bar{\Lambda}_2, \bar{\Lambda}_3, \ldots$ can be approximately determined in succession by means of the recurrence relation between $\bar{\Lambda}_{m+1}$ and $\bar{\Lambda}_m$. This approximate relation is derived by the same method as was used above to deal with the sequence of forward doubling bifurcations, and has the form $\bar{\Lambda}_m = \phi(\bar{\Lambda}_{m+1})$ with the same function $\phi(\Lambda)$ from (32.7). The corresponding graphical construction is shown in the upper part of Fig. 21. Since $\phi(\Lambda)$ is the same for the forward and reverse bifurcation sequences, so is the expression governing the convergence of the sequences of numbers Λ_m and $\bar{\Lambda}_m$ (from below and above respectively) to their common limit $\Lambda_\infty \equiv \bar{\Lambda}_\infty$:

$$\bar{\Lambda}_{m+1} - \Lambda_\infty = (1/\delta)(\bar{\Lambda}_m - \Lambda_\infty). \tag{32.21}$$

The development of the strange attractor properties for $\lambda > \Lambda_\infty$ is accompanied by corresponding changes in the frequency spectrum. The randomness of the flow is represented in the spectrum by the presence of a "noise" component whose strength increases with the width of the attractor. Against this background there are discrete peaks corresponding to the fundamental frequency of the unstable cycles and their harmonics and subharmonics; at successive reverse bifurcations, the relevant subharmonics disappear, in the opposite order to that of their appearance in the sequence of forward bifurcations. The instability of the cycles which create these frequencies is shown by the broadening of the peaks.

Transition to turbulence by alternation

Let us consider finally the elimination of periodic flow when the multiplier passes through the value $\mu = +1$.

This type of bifurcation is described (in the one-dimensional Poincaré mapping) by a function $x_{j+1} = f(x_j; \mathrm{R})$, which for a certain value $\mathrm{R} = \mathrm{R}_{cr}$ of the Reynolds number touches the line $x_{j+1} = x_j$. Taking the point of contact as $x_j = 0$, we can write the expansion of the mapping function near it as†

$$x_{j+1} = (\mathrm{R} - \mathrm{R}_{cr}) + x_j + x_j^2. \tag{32.22}$$

When $\mathrm{R} < \mathrm{R}_{cr}$ (see Fig. 23), there are two fixed points

$$x_*^{(1),(2)} = \mp\sqrt{(\mathrm{R}_{cr} - \mathrm{R})},$$

of which $x_*^{(1)}$ corresponds to stable and $x_*^{(2)}$ to unstable periodic flow. When $\mathrm{R} = \mathrm{R}_{cr}$, the multiplier at both points is $+1$, the two periodic flows coalesce, and when $\mathrm{R} > \mathrm{R}_{cr}$ they disappear, the fixed points passing into the complex domain.

When $\mathrm{R} - \mathrm{R}_{cr}$ is small, the curve (32.22) and the straight line $x_{j+1} = x_j$ are close together (near $x_j = 0$). In this range of x, therefore, each iteration of the mapping (32.22) moves the trace of the path only slightly, and many steps are needed for it to cover the whole range. In other words, over a comparatively long interval of time the path is regular and almost periodic in the space of states. Such a path corresponds to regular laminar flow in physical space. This yields another theoretically possible scenario for the onset of turbulence (P. Manneville and Y. Pomeau 1980).

It can be imagined that the particular region of the mapping function is adjacent to regions which randomize the paths, corresponding in the space of states to a set of locally

† The coefficient of $\mathrm{R} - \mathrm{R}_{cr}$ and the positive coefficient of x_j^2 can be made equal to unity by appropriate definitions of R and x_j, and this is assumed in (32.22).

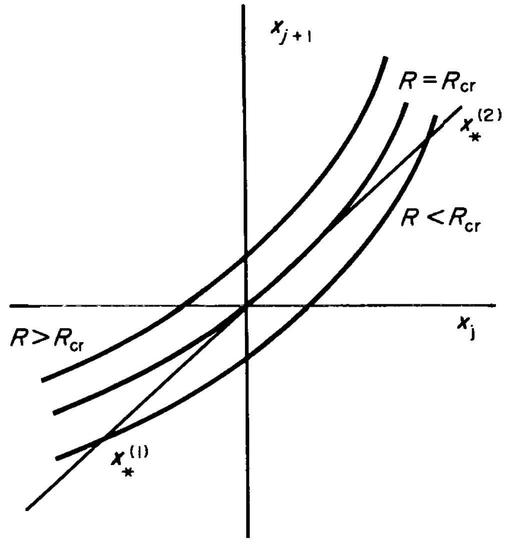

FIG. 23

unstable paths. This set is not itself an attractor, however, and in the course of time the point representing the system moves out of the set. When $\mathrm{R} < \mathrm{R}_{cr}$, the path reaches a stable cycle, and periodic laminar flow is established in physical space. When $\mathrm{R} > \mathrm{R}_{cr}$, there is no stable cycle, and a motion arises in which the turbulent periods alternate with laminar ones, the scenario therefore being called the transition to turbulence by alternation.

No general conclusions can be drawn as to the duration of the turbulent periods. The dependence of the laminar period duration on $\mathrm{R} - \mathrm{R}_{cr}$ is easily ascertained, however. To do so, we write the difference equation (32.22) as a differential equation. Since x_j changes only slightly in one mapping step, we replace $x_{j+1} - x_j$ by the derivative $\mathrm{d}x/\mathrm{d}t$ with respect to the continuous variable t:

$$\mathrm{d}x/\mathrm{d}t = \mathrm{R} - \mathrm{R}_{cr} + x^2. \tag{32.23}$$

Let us find the time τ needed to traverse the segment between the points x_1 and x_2 lying on either side of $x = 0$ at distances much greater than $\mathrm{R} - \mathrm{R}_{cr}$ but still within the range where the expansion (32.22) is valid. We have

$$\tau = \frac{1}{\sqrt{(\mathrm{R} - \mathrm{R}_{cr})}}[\tan^{-1}\{x/\sqrt{(\mathrm{R} - \mathrm{R}_{cr})}\}]_{x_1}^{x_2},$$

whence

$$\tau \propto 1/\sqrt{(\mathrm{R} - \mathrm{R}_{cr})}; \tag{32.24}$$

this gives the required dependence. Thus the duration of the laminar periods decreases as $\mathrm{R} - \mathrm{R}_{cr}$ increases.

This scenario leaves unresolved both the way in which its starting-point is approached and the nature of the turbulence that occurs.

§33. Fully developed turbulence

Turbulent flow at fairly large Reynolds numbers is characterized by the presence of an extremely irregular disordered variation of the velocity with time at each point. This is called *fully developed turbulence.* The velocity continually fluctuates about some mean value. A similar irregular variation of the velocity exists between points in the flow at a given instant. No complete quantitative theory of turbulence has yet been evolved. Nevertheless, several important qualitative results are known, and the present section gives an account of these.

We introduce the concept of the mean velocity, obtained by averaging over long intervals of time the actual velocity at each point. By such an averaging the irregular variation of the velocity is smoothed out, and the mean velocity varies smoothly from point to point. In what follows we shall denote the mean velocity by $\mathbf{u}$. The difference $\mathbf{v}' = \mathbf{v} - \mathbf{u}$ between the true velocity and the mean velocity varies irregularly in the manner characteristic of turbulence; we shall call it the *fluctuating part* of the velocity.

Let us consider in more detail the nature of this irregular motion which is superposed on the mean flow. This motion may in turn be qualitatively regarded as the superposition of *turbulent eddies* of different sizes; by the size of an eddy we mean the order of magnitude of the distances over which the velocity varies appreciably. As the Reynolds number increases, large eddies appear first; the smaller the eddies, the later they appear. For very large Reynolds numbers, eddies of every size from the largest to the smallest are present. An important part in any turbulent flow is played by the largest eddies, whose size (the *fundamental* or *external* scale of turbulence) is of the order of the dimensions of the region in which the flow takes place; in what follows we shall denote by l this order of magnitude for any given turbulent flow. These large eddies have the largest amplitudes. The velocity in them is comparable with the variation of the mean velocity over the distance l; we shall denote by Δu the order of magnitude of this variation. We are speaking here of the order of magnitude, not of the mean velocity itself, but of its variation, since it is this variation Δu which characterizes the velocity of the tubulent flow. The mean velocity itself can have any magnitude, depending on the frame of reference used.† The frequencies corresponding to these eddies are of the order of u/l, the ratio of the mean velocity u (and not its variation Δu) to the dimension l. For the frequency determines the period with which the flow pattern is repeated when observed in some fixed frame of reference. Relative to such a frame, however, the whole pattern moves with the fluid at a velocity of the order of u.

The small eddies, on the other hand, which correspond to large frequencies, participate in the turbulent flow with much smaller amplitudes. They may be regarded as a fine detailed structure superposed on the fundamental large turbulent eddies. Only a comparatively small part of the total kinetic energy of the fluid resides in the small eddies.

From the picture of turbulent flow given above, we can draw a conclusion regarding the manner of variation of the fluctuating velocity from point to point at any given instant. Over large distances (comparable with l), the variation of the fluctuating velocity is given by the variation in the velocity of the large eddies, and is therefore comparable with Δu. Over small distances (compared with l), it is determined by the small eddies, and is therefore small (compared with Δu) (but large compared with the variation of the mean velocity over these small distances). The same kind of picture is obtained if we observe the variation of the velocity with time at any given point. Over short time intervals (compared with $T \sim l/u$), the velocity does not vary appreciably; over long intervals, it varies by a quantity of the order of Δu.

The length l appears as a characteristic dimension in the Reynolds number R, which determines the properties of a given flow. Besides this Reynolds number, we can introduce

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† It seems that in fact the size of the largest eddies is actually somewhat less than l, and their velocity is somewhat less than Δu.

the qualitative concept of the Reynolds numbers for turbulent eddies of various sizes. If λ is the order of magnitude of the size of a given eddy, and v_λ the order of magnitude of its velocity, then the corresponding Reynolds number is defined as $R_\lambda \sim v_\lambda \lambda/\nu$. This number decreases with the size of the eddy.

For large Reynolds numbers R, the Reynolds numbers R_λ of the large eddies are also large. Large Reynolds numbers, however, are equivalent to small viscosities. We therefore conclude that, for the large eddies which are the basis of any turbulent flow, the viscosity is unimportant. It follows from this that there is no appreciable dissipation of energy in the large eddies.

The viscosity of the fluid becomes important only for the smallest eddies, whose Reynolds number is comparable with unity. We denote the size of these eddies by λ_0, which we shall determine later in this section. It is in these small eddies, which are unimportant as regards the general pattern of a turbulent flow, that the dissipation of energy occurs.

We thus arrive at the following conception of energy dissipation in turbulent flow (L. Richardson 1922). The energy passes from the large eddies to smaller ones, practically no dissipation occurring in this process. We may say that there is a continuous flow of energy from large to small eddies, i.e. from small to large frequencies. This flow of energy is dissipated in the smallest eddies, where the kinetic energy is transformed into heat. For a steady state to be maintained, it is of course necessary that external energy sources should be present which continually supply energy to the large eddies.

Since the viscosity of the fluid is important only for the smallest eddies, we may say that none of the quantities pertaining to eddies of sizes $\lambda \gg \lambda_0$ can depend on ν (more exactly, these quantities cannot be changed if ν varies but the other conditions of the motion are unchanged). This circumstance reduces the number of quantities which determine the properties of turbulent flow, and the result is that similarity arguments, involving the dimensions of the available quantities, become very important in the investigation of turbulence.

Let us apply these arguments to determine the order of magnitude of the energy dissipation in turbulent flow. Let ε be the mean dissipation of energy per unit time per unit mass of fluid.† We have seen that this energy is derived from the large eddies, whence it is gradually transferred to smaller eddies until it is dissipated in eddies of size $\sim \lambda_0$. Hence, although the dissipation is ultimately due to the viscosity, the order of magnitude of ε can be determined only by those quantities which characterize the large eddies. These are the fluid density ρ, the dimension l and the velocity Δu. From these three quantities we can form only one having the dimensions of ε, namely erg/g sec = cm^2/sec^3. Thus we find

$$\varepsilon \sim (\Delta u)^3/l, \tag{33.1}$$

and this determines the order of magnitude of the energy dissipation in turbulent flow.

In some respects a fluid in turbulent motion may be qualitatively described as having a "turbulent viscosity" ν_{turb} which differs from the true kinematic viscosity ν. Since ν_{turb} characterizes the properties of the turbulent flow, its order of magnitude must be determined by ρ, Δu and l. The only quantity that can be formed from these and has the dimensions of kinematic viscosity is $l\Delta u$, and therefore

$$\nu_{\text{turb}} \sim l\Delta u. \tag{33.2}$$

† In this chapter ε denotes the mean dissipation of energy, and not the internal energy of the fluid.

The ratio of the turbulent viscosity to the ordinary viscosity is consequently

$$\nu_{\text{turb}}/\nu \sim R \tag{33.3}$$

i.e. it increases with the Reynolds number.†

The energy dissipation ε is expressed in terms of ν_{turb} by

$$\varepsilon \sim \nu_{\text{turb}}(\Delta u/l)^2 \tag{33.4}$$

in accordance with the usual definition of viscosity. Whereas ν determines the energy dissipation in terms of the space derivatives of the true velocity, ν_{turb} relates it to the gradient ($\sim \Delta u/l$) of the mean velocity.

We may also apply similarity arguments to determine the order of magnitude Δp of the variation of pressure over the region of turbulent flow. The only quantity having the dimensions of pressure which can be formed from ρ, l and Δu is $\rho(\Delta u)^2$. Hence we must have

$$\Delta p \sim \rho(\Delta u)^2. \tag{33.5}$$

Let us now consider the properties of the turbulence as regards eddy sizes λ which are small compared with the fundamental eddy size l. We shall refer to these properties as *local* properties of the turbulence. We shall consider fluid that is far from all solid surfaces (more precisely, that is at distances from them large compared with λ).

It is natural to assume that such small-scale turbulence, far from solid bodies, is homogeneous and isotropic. The latter property means that, over regions whose dimensions are small compared with l, the properties of the turbulent flow are independent of direction; in particular, they do not depend on the direction of the mean velocity. It must be emphasized that here, and everywhere in the present section, when we speak of the properties of the turbulent flow in a small region of the fluid, we mean the relative motion of the fluid particles in that region, and not the absolute motion of the region as a whole, which is due to the larger eddies.

It is found that several important results concerning the local properties of turbulence can be obtained immediately from similarity arguments. These results are due to A. N. Kolmogorov and to A. M. Obukhov (1941). To obtain them, we shall first determine which parameters can be involved in the properties of turbulent flow over regions small compared with l but large compared with the distances λ_0 at which the viscosity of the fluid begins to be important. It is these intermediate distances which we shall discuss below. The parameters in question are the fluid density ρ and another quantity characterizing any turbulent flow, the energy ε dissipated per unit time per unit mass of fluid. We have seen that ε is the energy flux which continually passes from larger to smaller eddies. Hence, although the energy dissipation is ultimately due to the viscosity of the fluid and occurs in the smallest eddies, the quantity ε determine the properties of larger eddies. It is natural to suppose that (for given ρ and ε) the local properties of the turbulence are independent of the dimension l and velocity Δu of the flow as a whole. The fluid viscosity ν also cannot appear in any of the quantities in which we are at present interested (we recall that we are concerned with distances $\lambda \gg \lambda_0$).

† In reality, however, a fairly large numerical coefficient should be included. This is because, as mentioned above, l and Δu may differ quite considerably from the actual scale and velocity of the turbulent flow. The ratio ν_{turb}/ν may be more accurately written $\nu_{\text{turb}}/\nu \sim \mathbf{R}/\mathbf{R}_{\text{cr}}$, which formula takes into account the fact that ν_{turb} and ν must in reality be comparable in magnitude not for $\mathbf{R} \sim 1$, but for $\mathbf{R} \sim \mathbf{R}_{\text{cr}}$.

Let us determine the order of magnitude v_λ of the turbulent velocity variation over distances of the order of λ. It must be determined only by ε and, of course, the distance λ itself.† From these two quantities we can form only one having the dimensions of velocity, namely $(\varepsilon\lambda)^{\frac{1}{3}}$. Hence we can say that the relation

$$v_\lambda \propto (\varepsilon\lambda)^{\frac{1}{3}} \tag{33.6}$$

must hold. We thus find that the velocity variation over a small distance is proportional to the cube root of the distance (*Kolmogorov and Obukhov's law*). The quantity v_λ may also be regarded as the velocity of turbulent eddies whose size is of the order of λ: the variation of the mean velocity over small distances is small compared with the variation of the fluctuating velocity over those distances, and may be neglected.

The relation (33.6) may be obtained in another way by expressing a constant quantity, the dissipation ε, in terms of quantities characterizing the eddies of size λ; ε must be proportional to the squared gradient of the velocity v_λ and to the appropriate turbulent viscosity coefficient $\nu_{\text{turb},\lambda} \propto v_\lambda\lambda$:

$$\varepsilon \propto \nu_{\text{turb},\lambda}(v_\lambda/\lambda)^2 \propto v_\lambda{}^3/\lambda,$$

whence we obtain (33.6).

Let us now put the problem somewhat differently, and determine the order of magnitude v_τ of the velocity variation at a given point over a time interval τ which is short compared with the time $T \sim l/u$ characterizing the flow as a whole. To do this, we notice that, since there is a net mean flow, any given portion of the fluid is displaced, during the interval τ, over a distance of the order of τu, u being the mean velocity. Hence the portion of fluid which is at a given point at time τ will have been at a distance τu from that point at the initial instant. We can therefore obtain the required quantity v_τ by direct substitution of τu for λ in (33.6):

$$v_\tau \propto (\varepsilon\tau u)^{\frac{1}{3}}. \tag{33.7}$$

The quantity v_τ must be distinguished from v_τ', the variation in velocity of a portion of fluid as it moves about. This variation can evidently depend only on ε, which determines the local properties of the turbulence, and of course on τ itself. Forming the only combination of ε and τ that has the dimensions of velocity, we obtain

$$v_\tau' \propto (\varepsilon\tau)^{\frac{1}{2}}. \tag{33.8}$$

Unlike the velocity variation at a given point, it is proportional to the square root of τ, not to the cube root. It is easy to see that, for τ small compared with T, v_τ' is always less than v_τ.‡

Using the expression (33.1) for ε, we can rewrite (33.6) and (33.7) as

$$\left.\begin{aligned} v_\lambda &\propto \Delta u(\lambda/l)^{\frac{1}{3}}, \\ v_\tau &\propto \Delta u(\tau/T)^{\frac{1}{3}}. \end{aligned}\right\} \tag{33.9}$$

This form shows clearly the similarity property of local turbulence: the small-scale characteristics of different turbulent flows are the same apart from the scale of measurement of lengths and velocities (or, equivalently, lengths and times).††

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† The dimensions of ε are erg/g sec = cm^2/sec^3, and do not include mass; the only quantity involving the mass dimension is the density ρ. The latter is therefore not involved in quantities whose dimensions do not include mass.

‡ The inequality $v_\tau' \ll v_\tau$ has in essence been assumed in the derivation of (33.7).

†† In this connection, the term *self-similarity* is often used in recent literature.

Let us now find at what distances the fluid viscosity begins to be important. These distances λ_0 also determine the order of magnitude of the size of the smallest eddies in the turbulent flow (called the "internal scale" of the turbulence, in contradistinction to the "external scale" l). To determine λ_0, we form the local Reynolds number $R_\lambda \sim v_\lambda \lambda/\nu \sim \Delta u \cdot \lambda^{4/3}/\nu l^{1/3} \sim R(\lambda/l)^{4/3}$, with the Reynolds number $R \sim l\Delta u/\nu$ for the flow as a whole. The order of magnitude of λ_0 is that for which $R_{\lambda_0} \sim 1$. Hence we find

$$\lambda_0 \sim l/R^{\frac{3}{4}}. \tag{33.10}$$

The same expression can be obtained by forming from ε and ν the only combination having the dimensions of length, namely

$$\lambda_0 \sim (\nu^3/\varepsilon)^{\frac{1}{4}}. \tag{33.11}$$

Thus the internal scale of the turbulence decreases rapidly with increasing R. For the corresponding velocity we have

$$v_{\lambda_0} \sim \Delta u/R^{\frac{1}{4}}; \tag{33.12}$$

this also decreases when R increases.†

The range of scales $\lambda \sim l$ is called the *energy range*; the majority of the kinetic energy of the fluid is concentrated there. Values $\lambda \lesssim \lambda_0$ form the *dissipation range*, where the kinetic energy is dissipated. For very large values of R, these two ranges are quite far apart, and between them lies the *inertial range*, in which $\lambda_0 \ll \lambda \ll l$; the results derived in this section are valid there.

Kolmogorov and Obukhov's law can be expressed in an equivalent spatial spectrum form. We replace the scales λ by corresponding wave numbers $k \sim 1/\lambda$ of the eddies; let $E(k)dk$ be the kinetic energy per unit mass of fluid in eddies with k values in the range dk. The function $E(k)$ has the dimensions cm³/sec²; the combination of ε and k having these dimensions gives

$$E(k) \propto \varepsilon^{2/3} k^{-5/3}. \tag{33.13}$$

The equivalence of this expression and (33.6) is easily seen by noting that $v_\lambda{}^2$ gives the order of magnitude of the total energy in eddies with all scales of the order of λ or less. The same result is reached by integration of (33.13):

$$\int_k^\infty E(k)\,dk \propto \varepsilon^{2/3}/k^{2/3} \sim (\varepsilon\lambda)^{2/3} \sim v_\lambda{}^2.$$

Together with the spatial scales of the turbulent eddies, we can also consider their time characteristics (frequencies). The lower end of the frequency spectrum of turbulent motion is at frequencies $\sim u/l$. The upper end is

$$\omega_0 \sim u/\lambda_0 \sim uR^{3/4}/l, \tag{33.14}$$

corresponding to the internal scale of turbulence. The inertial range corresponds to frequencies

$$u/l \ll \omega \ll (u/l)R^{3/4}.$$

† Formulae (33.10)–(33.12) give the manner of variation of the relevant quantities with R. Quantitatively, it would be more correct to replace R in them by R/R_{cr}.

The inequality $\omega \gg u/l$ signifies that as regards the local properties of turbulence the unperturbed flow may be treated as steady. The energy distribution in the frequency spectrum in the inertial range is found from (33.13) by substituting $k \sim \omega/u$:

$$E(\omega) \propto (u\varepsilon)^{2/3}\omega^{-5/3}, \tag{33.15}$$

where $E(\omega)\mathrm{d}\omega$ is the energy in the frequency range $\mathrm{d}\omega$.

The frequency ω gives the time repetition period in the region of space concerned, as observed from a fixed frame of reference. It is to be distinguished from the frequency ω' which gives the flow repetition period in a given portion of fluid moving in space. The energy distribution in this frequency spectrum cannot depend on u, and must be determined only by ε and the frequency ω' itself. Again using dimensional arguments, we find

$$E(\omega') \propto \varepsilon/\omega'^2. \tag{33.16}$$

This is in the same relationship to (33.15) as (33.8) is to (33.7).

Turbulent mixing causes a gradual separation of fluid particles that were originally close together. Let us consider two particles at a distance λ that is small (in the inertial range). Again, by dimensional arguments, the rate of change of this distance with time is

$$\mathrm{d}\lambda/\mathrm{d}t \propto (\varepsilon\lambda)^{1/3}. \tag{33.17}$$

Integration of this shows that the time τ over which two particles initially at a distance λ_1 move apart to a distance $\lambda_2 \gg \lambda_1$ is in order of magnitude

$$\tau \sim \lambda_2{}^{4/3}/\varepsilon^{1/3}. \tag{33.18}$$

Note that the process is self-accelerating: the rate of separation increases with λ. This occurs because only eddies with scales $\lesssim \lambda$ contribute to the separation of particles at a distance λ; the larger eddies carry both particles and do not cause them to separate.†

Finally, let us consider the properties of the flow in regions whose dimension λ is small compared with λ_0. In such regions the flow is regular and its velocity varies smoothly. Hence we can expand v_λ in a series of powers of λ and, retaining only the first term, obtain $v_\lambda = \text{constant} \times \lambda$. The order of magnitude of the constant is v_{λ_0}/λ_0, since for $\lambda \sim \lambda_0$ we must have $v_\lambda \sim v_{\lambda_0}$. Thus

$$v_\lambda \sim v_{\lambda_0}\lambda/\lambda_0 \sim \Delta u \cdot \mathrm{R}^{\frac{1}{4}}\lambda/l. \tag{33.19}$$

This formula may also be obtained by equating two expressions for the energy dissipation ε: the expression $(\Delta u)^3/l$ (33.1), which determines ε in terms of quantities characterizing the large eddies, and the expression $\nu(v_\lambda/\lambda)^2$, which determines ε in terms of the velocity gradient for the eddies in which the energy dissipation actually occurs.

§34. The velocity correlation functions

Formula (33.6) determines qualitatively the *correlation of velocities* in local turbulence, i.e. the relation between the velocities at two neighbouring points. Let us now introduce

† These results can be applied to particles suspended in the fluid, which are passively conveyed by its motion.

functions which will serve to characterize this correlation quantitatively.† One is the rank-two correlation tensor

$$B_{ik} = \langle (v_{2i} - v_{1i})(v_{2k} - v_{1k}) \rangle, \qquad (34.1)$$

where $\mathbf{v}_2$ and $\mathbf{v}_1$ are the fluid velocities at two neighbouring points, and the angle brackets denote an average with respect to time. The radius vector from point 1 to point 2 will be denoted by $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. In discussing local turbulence, we shall suppose this distance much less than the fundamental scale l, but not necessarily much greater than the internal scale λ_0 of the turbulence.

The velocity variation over short distances is due to the small eddies. The properties of local turbulence, however, do not depend on the averaged flow. We can therefore simplify the study of the correlation functions of local turbulence by considering instead an idealized case of turbulent flow in which there is isotropy and homogeneity not only on small scales (as in local turbulence) but also on every scale; the averaged velocity is then zero. This completely isotropic and homogeneous turbulence‡ can be regarded as occurring in a fluid subjected to vigorous shaking and then left to itself. Such a flow will, of course, necessarily decay in the course of time, and so the components of the correlation tensor are also time-dependent.†† The relations derived below between the various correlation functions apply to homogeneous isotropic turbulence on every scale, and to local turbulence at distances $r \ll l$.

Since local turbulence is isotropic, the tensor B_{ik} cannot depend on any direction in space. The only vector that can appear in the expression for B_{ik} is the radius vector $\mathbf{r}$. The general form of such a symmetrical tensor of rank two is

$$B_{ik} = A(r)\delta_{ik} + B(r)n_i n_k, \qquad (34.2)$$

where $\mathbf{n}$ is a unit vector in the direction of $\mathbf{r}$.

To see the significance of the functions A and B, we take the coordinate axes so that one of them is in the direction of $\mathbf{n}$, denoting the velocity component along this axis by v_r and the component perpendicular to $\mathbf{n}$ by v_t. The correlation tensor component B_{rr} is then the mean square relative velocity of two fluid particles along the line joining them. Similarly, B_{tt} is the mean square transverse velocity of one particle relative to the other. Since $n_r = 1$, $n_t = 0$, we have from (34.2)

$$B_{rr} = A + B, \qquad B_{tt} = A, \qquad B_{rt} = 0$$

The expression (34.2) may now be written as

$$B_{ik} = B_{tt}(r)(\delta_{ik} - n_i n_k) + B_{rr}(r) n_i n_k. \qquad (34.3)$$

Expanding the parentheses in the definition (34.1) gives

$$B_{ik} = \langle v_{1i} v_{1k} \rangle + \langle v_{2i} v_{2k} \rangle - \langle v_{1i} v_{2k} \rangle - \langle v_{1k} v_{2i} \rangle.$$

Because of the homogeneity, the mean values of $v_i v_k$ at points 1 and 2 are the same, and because of the isotropy, $\langle v_{1i} v_{2k} \rangle$ is unaltered when points 1 and 2 are interchanged (i.e. when $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ changes sign); thus

$$\langle v_{1i} v_{1k} \rangle = \langle v_{2i} v_{2k} \rangle = \tfrac{1}{3} \langle v^2 \rangle \delta_{ik}, \quad \langle v_{1i} v_{2k} \rangle = \langle v_{2i} v_{1k} \rangle.$$

† Correlation functions were first used in the dynamics of turbulence by L. V. Keller and A. A. Fridman (1924).

‡ The concept is due to G. I. Taylor (1935).

†† The averaging in the definition (34.1) must here, strictly speaking, be taken not as time averaging but as averaging over all possible positions of the points 1 and 2 (for a given distance between them) at a given instant.

Hence

$$B_{ik} = \tfrac{2}{3}\langle v^2 \rangle \delta_{ik} - 2b_{ik}, \quad b_{ik} = \langle v_{1i} v_{2k} \rangle. \tag{34.4}$$

The symmetrical auxiliary tensor b_{ik} tends to zero as $r \to \infty$; for the turbulent flow velocities at infinitely distant points may be regarded as statistically independent, so that the mean value of their product reduces to the product of the means of each factor separately, which are zero by hypothesis.

We differentiate (34.4) with respect to the coordinates of point 2:

$$\frac{\partial B_{ik}}{\partial x_{2k}} = -2\frac{\partial b_{ik}}{\partial x_{2k}} = -2\left\langle v_{1i}\frac{\partial v_{2k}}{\partial x_{2k}}\right\rangle.$$

By the equation of continuity, $\partial v_{2k}/\partial x_{2k} = 0$, and so

$$\partial B_{ik}/\partial x_{2k} = 0.$$

Since B_{ik} is a function only of $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$, differentiation with respect to x_{2k} is equivalent to that with respect to x_k. Substituting (34.3) for B_{ik}, we easily find

$$B_{rr}' + (2/r)(B_{rr} - B_{tt}) = 0,$$

where the prime denotes differentiation with respect to r. Thus the longitudinal and transverse correlation functions are related by

$$B_{tt} = \frac{1}{2r}\frac{\mathrm{d}}{\mathrm{d}r}(r^2 B_{rr}). \tag{34.5}$$

According to (33.6), the velocity difference over a distance r in the inertial range is proportional to $r^{1/3}$. Hence the correlation functions B_{rr} and B_{tt} are proportional to $r^{2/3}$ in that range. We then get from (34.5) the simple relation

$$B_{tt} = \tfrac{4}{3}B_{rr} \qquad (\lambda_0 \ll r \ll l). \tag{34.6}$$

For distances $r \ll \lambda_0$, the velocity difference is proportional to r, and therefore B_{rr} and B_{tt} are proportional to r^2. Formulas (34.5) then gives

$$B_{tt} = 2B_{rr} \qquad (r \ll \lambda_0). \tag{34.7}$$

For these distances, B_{tt} and B_{rr} can also be expressed in terms of the mean energy dissipation ε. We write $B_{rr} = ar^2$, where a is a constant, and combine (34.3), (34.4) and (34.7) to find

$$b_{ik} = \tfrac{1}{3}\langle v^2 \rangle \delta_{ik} - ar^2\delta_{ik} + \tfrac{1}{2}ax_i x_k.$$

Differentiating this relation, we have

$$\left\langle \frac{\partial v_{1i}}{\partial x_{1l}}\frac{\partial v_{2i}}{\partial x_{2l}}\right\rangle = 15a, \quad \left\langle \frac{\partial v_{1i}}{\partial x_{1l}}\frac{\partial v_{2l}}{\partial x_{2i}}\right\rangle = 0.$$

Since these hold for arbitrarily small r, we can put $\mathbf{r}_1 = \mathbf{r}_2$, obtaining

$$\langle (\partial v_i/\partial x_l)^2 \rangle = 15a, \langle (\partial v_i/\partial x_l)(\partial v_l/\partial x_i) \rangle = 0.$$

According to (16.3), we have for the mean energy dissipation

$$\varepsilon = \tfrac{1}{2}\nu\left\langle\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)^2\right\rangle = \nu\left\{\left\langle\left(\frac{\partial v_i}{\partial x_k}\right)^2\right\rangle + \left\langle\frac{\partial v_i}{\partial x_k}\frac{\partial v_k}{\partial x_i}\right\rangle\right\} = 15a\nu,$$

whence $a = \varepsilon/15\nu$.† We thus obtain the following final expressions for the correlation functions in terms of the energy dissipation:

$$B_{tt} = 2\varepsilon r^2/15\nu, \qquad B_{rr} = \varepsilon r^2/15\nu \tag{34.8}$$

(A. N. Kolmogorov 1941).

We next define the rank-three tensor

$$B_{ikl} = \langle (v_{2i} - v_{1i})(v_{2k} - v_{1k})(v_{2l} - v_{1l}) \rangle \tag{34.9}$$

and the auxiliary tensor

$$b_{ik,l} = \langle v_{1i}v_{1k}v_{2l} \rangle = -\langle v_{2i}v_{2k}v_{1l} \rangle. \tag{34.10}$$

The latter is symmetrical in the first pair of suffixes; the second equation (34.10) results from the fact that interchanging points 1 and 2 is equivalent to changing the sign of **r**, i.e. inverting the coordinates, and therefore changes the sign of the rank-three tensor. When $r = 0$ and the points 1 and 2 coincide, $b_{ik,l}(0) = 0$: the mean value of the product of an odd number of fluctuating velocity components is zero. Expanding the parentheses in the definition (34.9) gives B_{ikl} in terms of $b_{ik,l}$:

$$B_{ikl} = 2(b_{ik,l} + b_{il,k} + b_{lk,i}). \tag{34.11}$$

As $r \to \infty$, the tensor $b_{ik,l}$ and therefore B_{ikl} tend to zero.

Isotropy shows that $b_{ik,l}$ must be expressible in terms of the unit tensor δ_{ik} and the components of the unit tensor **n**. The general form of such a tensor symmetrical in the first pair of suffixes is

$$b_{ik,l} = C(r)\delta_{ik}n_l + D(r)(\delta_{il}n_k + \delta_{kl}n_i) + F(r)n_in_kn_l. \tag{34.12}$$

Differentiating this with respect to the coordinates of point 2 and using the equation of continuity, we find

$$\partial b_{ik,l}/\partial x_{2l} = \langle v_{1i}v_{1k}\partial v_{2l}/\partial x_{2l} \rangle = 0.$$

Substitution of (34.12) leads, after a simple calculation, to the two equations

$$[r^2(3C + 2D + F')]' = 0, \qquad C' + 2(C + D)/r = 0.$$

Integration of the former gives

$$3C + 2D + F = \text{constant}/r^2.$$

When $r = 0$, C, D and F must be zero; the constant is therefore zero, and $3C + 2D + F = 0$. The two equations found then give

$$D = -C - \tfrac{1}{2}rC', \qquad F = rC' - C. \tag{34.13}$$

Substitution of these in (34.12) and thence in (34.11) gives $B_{ikl} = -2(rC' + C)(\delta_{ik}n_l + \delta_{il}n_k + \delta_{kl}n_i) + 6(rC' - C)n_in_kn_l$. Again taking one of the coordinate axes to be parallel to **n**, we find as the components of B_{ikl}

$$B_{rrr} = -12C, \quad B_{rtt} = -2(C + rC'), \quad B_{rrt} = B_{ttt} = 0. \tag{34.14}$$

† For isotropic turbulence, the mean dissipation is related to the mean square vorticity by the simple formula

$$\langle (\mathbf{curl}\, \mathbf{v})^2 \rangle = \tfrac{1}{2}\left\langle \left(\frac{\partial v_i}{\partial x_k} - \frac{\partial v_k}{\partial x_i}\right)^2 \right\rangle = \varepsilon/\nu.$$

From this, we see that the non-zero correlation functions B_{rtt} and B_{rrr} are related by

$$B_{rtt} = \frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}r}(rB_{rrr}). \tag{34.15}$$

We shall also need an expression for $b_{ik,l}$ in terms of the components of B_{ikl}. From (34.12)–(34.14),

$$b_{ik,l} = -\frac{1}{12}B_{rrr}\delta_{ik}n_l + \frac{1}{24}(rB_{rrr}' + 2B_{rrr})(\delta_{il}n_k + \delta_{kl}n_i) - \frac{1}{12}(rB_{rrr}' - B_{rrr})n_in_kn_l. \tag{34.16}$$

The relations (34.5) and (34.15) follow from the continuity equation alone. With the Navier–Stokes equation, we can derive a relation between the correlation tensors B_{ik} and B_{ikl} (T. von Kármán and L. Howarth 1938; A. N. Kolmogorov 1941).

To do so, we calculate the derivative $\partial b_{ik}/\partial t$ (a completely homogeneous and isotropic turbulent flow, it will be remembered, decays in the course of time). Expressing the derivatives $\partial v_{1i}/\partial t$ and $\partial v_{2k}/\partial t$ by means of the Navier–Stokes equation, we find

$$\frac{\partial}{\partial t}\langle v_{1i}v_{2k}\rangle = -\frac{\partial}{\partial x_{1l}}\langle v_{1i}v_{1l}v_{2k}\rangle - \frac{\partial}{\partial x_{2l}}\langle v_{1i}v_{2k}v_{2l}\rangle - \frac{1}{\rho}\frac{\partial}{\partial x_{1i}}\langle p_1 v_{2k}\rangle -$$
$$-\frac{1}{\rho}\frac{\partial}{\partial x_{2k}}\langle p_2 v_{1i}\rangle + \nu\triangle_1\langle v_{1i}v_{2k}\rangle + \nu\triangle_2\langle v_{1i}v_{2k}\rangle. \tag{34.17}$$

The correlation function for the pressure and the velocity is

$$\langle p_1 \mathbf{v}_2\rangle = 0. \tag{34.18}$$

For isotropy implies that this function must have the form $f(r)\mathbf{n}$. And, from the equation of continuity,

$$\mathrm{div}_2\langle p_1\mathbf{v}_2\rangle = \langle p_1\mathrm{div}_2\mathbf{v}_2\rangle = 0.$$

The only vector having the form $f(r)\mathbf{n}$ and zero divergence is constant $\times\,\mathbf{n}/r^2$, and this would not be finite at $r = 0$; the constant must therefore be zero.

Now replacing the derivatives with respect to x_{1i} and x_{2i} in (34.17) by those with respect to $-x_i$ and x_i, we get

$$\frac{\partial}{\partial t}b_{ik} = \frac{\partial}{\partial x_l}(b_{il,k} + b_{kl,i}) + 2\nu\triangle b_{ik}. \tag{34.19}$$

Here we have to substitute b_{ik} and $b_{ik,l}$ from (34.4) and (34.16). The time derivative of the kinetic energy per unit mass, $\frac{1}{2}\langle v^2\rangle$, is just the energy dissipation $-\varepsilon$. Hence

$$\frac{\partial}{\partial t}(\tfrac{1}{3}\langle v^2\rangle) = -\tfrac{2}{3}\varepsilon.$$

A straightforward but lengthy calculation gives†

$$-\tfrac{2}{3}\varepsilon - \tfrac{1}{2}\frac{\partial B_{rr}}{\partial t} = \frac{1}{6r^4}\frac{\partial}{\partial r}(r^4B_{rrr}) - \frac{\nu}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial B_{rr}}{\partial r}\right). \tag{34.20}$$

† The result of the calculation corresponds to (34.20) with the operator $1 + \frac{1}{5}r\partial/\partial r$ applied to each side, but since the only solution of $f + \frac{1}{5}r\partial f/\partial r = 0$ finite when $r = 0$ is $f = 0$, the operator may be omitted.

The value of B_{rr} varies considerably with time only over an interval corresponding to the fundamental scale of turbulence ($\sim l/u$). In relation to local turbulence the unperturbed flow may be regarded as steady, as already mentioned in §33. This means that for local turbulence it is sufficiently accurate to neglect the derivative $\partial B_{rr}/\partial t$ on the left-hand side of (34.20) in comparison with ε. Multiplying the resulting equation by r^4 and integrating with respect to r we find, since the correlation functions are zero when $r = 0$, the following relation between B_{rr} and B_{rrr}:

$$B_{rrr} = -\tfrac{4}{5}\varepsilon r + 6\nu \frac{dB_{rr}}{dr} \tag{34.21}$$

(A. N. Kolmogorov 1941). This is valid when r is either greater or less than λ_0. When $r \gg \lambda_0$, the viscosity term is small, and we have simply

$$B_{rrr} = -\tfrac{4}{5}\varepsilon r. \tag{34.22}$$

If we substitute in (34.21) for $r \ll \lambda_0$ the expression (34.8) for B_{rr}, the result is zero, because in this case we must have $B_{rrr} \propto r^3$, and so the first-order terms must cancel.

The one equation (34.20) relates two independent functions B_{rr} and B_{rrr}, and therefore does not by itself enable us to find these. The presence of the correlation functions of two orders is due to the non-linearity of the Navier–Stokes equation. For the same reason, calculating the time derivative of the third-order correlation function would give an equation containing also a fourth-order one, and so on. This leads to an infinite sequence of equations. It is not possible to arrive in this way at a closed system of equations without making some additional assumptions.

One further general remark† should be made. It might be thought that the possibility exists in principle of obtaining a universal formula, applicable to any turbulent flow, which should give B_{rr} and B_{tt} for all distances r that are small compared with l. In fact, however, there can be no such formula, as we see from the following argument. The instantaneous value of $(v_{2i} - v_{1i})(v_{2k} - v_{1k})$ might in principle be expressed as a universal function of the energy dissipation ε at the instant considered. When we average these expressions, however, an important part will be played by the manner of variation of ε over times of the order of the periods of the large eddies (with size $\sim l$), and this variation is different for different flows. The result of the averaging therefore cannot be universal.‡

Loĭtsyanskiĭ's integral

We can rewrite equation (34.20) with b_{rr} and $b_{rr,r}$ in place of B_{rr} and B_{rrr}:

$$\frac{\partial b_{rr}}{\partial t} = \frac{1}{r^4}\frac{\partial}{\partial r}\left[2\nu r^4 \frac{\partial b_{rr}}{\partial r} + r^4 b_{rr,r}\right]. \tag{34.23}$$

We multiply this by r^4 and integrate over r from 0 to ∞. The expression in square brackets is zero when $r = 0$. Assuming that it tends also to zero as $r \to \infty$, we find

$$\Lambda \equiv \int_0^\infty r^4 b_{rr} dr = \text{constant} \tag{34.24}$$

† Due to L. D. Landau (1944).

‡ The question whether fluctuations of ε should be reflected in the form of the correlation functions in the inertial range can scarcely be resolved with certainty until we have a consistent theory of turbulence; it has been posed by A. N. Kolmogorov (*Journal of Fluid Mechanics* 13, 82, 1962) and A. M. Obukhov (*ibid.* 77). Existing attempts to apply relevant corrections to Kolmogorov and Obukhov's law are based on hypotheses about the statistical properties of the dissipation, whose correctness it is difficult to assess.

(L. G. Loĭtsyanskiĭ 1939). The integral converges if b_{rr} decreases at infinity faster than r^{-5}, and is in fact constant if $b_{rr,r}$ decreases faster than r^{-4}.

The functions b_{rr} and b_{tt} are related by a formula similar to (34.5) for B_{rr} and B_{tt}. We therefore have (under the same conditions)

$$\int_0^\infty b_{tt} r^4 \, dr = -\frac{3}{2} \int_0^\infty b_{rr} r^4 \, dr.$$

Since $b_{rr} + 2b_{tt} = \langle \mathbf{v}_1 \cdot \mathbf{v}_2 \rangle$, the integral (34.24) can be put in the form

$$\Lambda = -\frac{1}{4\pi} \int r^2 \langle \mathbf{v}_1 \cdot \mathbf{v}_2 \rangle \, dV, \tag{34.25}$$

where $dV = d^3(x_1 - x_2)$. This integral is closely related to the angular momentum of a fluid in a state of homogeneous and isotropic turbulence. It can be shown (though we shall not pause to do so) that the square of the total angular momentum $\mathbf{M}$ of the fluid in some large volume V within an infinite fluid is $M^2 = 4\pi\rho^2 \Lambda V$; the increase of $\mathbf{M}$ as $\sqrt{V}$ and not as V occurs because $\mathbf{M}$ is the sum of a large number of statistically independent terms (the angular momenta of separate small portions of fluid) with zero mean values.

The value of M^2 in a given volume V may vary because of the interaction with surrounding regions of the fluid. If this interaction decreased sufficiently rapidly with increasing distance, it would be a surface effect for the part of the fluid considered. The times during which M^2 could change considerably would then increase with the dimensions of V; these times and dimensions are to be regarded as very large, and in this sense M^2 would be conserved.

The condition stated is closely related to the conditions, formulated in deriving (34.24) from (34.23), for a sufficiently rapid decrease in the correlation functions. In incompressible fluid theory, however, it is doubtful whether they are satisfied. The physical point lies in the infinite speed of propagation of perturbations in an incompressible fluid. Mathematically, this is shown by the integral form of the fluid pressure dependence on the velocity distribution: if the right-hand side of (15.11) is regarded as given, the solution is

$$p(\mathbf{r}) = \frac{\rho}{4\pi} \int \frac{\partial^2 v_i(\mathbf{r}') v_k(\mathbf{r}')}{\partial x'_i \partial x'_k} \frac{dV'}{|\mathbf{r} - \mathbf{r}'|}.$$

As a result, any local perturbation of the velocity instantaneously affects the pressure in all space, and the pressure affects the acceleration of the fluid, and therefore the subsequent change in the velocity.

A natural way of formulating the problem is as follows. At the initial instant ($t = 0$), let an isotropic turbulent flow be set up, in which the functions $b_{ik}(r, t)$ and $b_{ik,l}(r, t)$ decrease exponentially with increasing distance. Expressing the pressure in terms of the velocities by means of the above formula, we can then use the equations of motion of the fluid in an attempt to determine the dependence of the time derivatives of the correlation functions at $t = 0$ on the distance as $r \to \infty$. This determines also the dependence of the correlation functions themselves on r for $t > 0$. The investigation yields the following results.†

† See I. Proudman and W. H. Reid, *Philosophical Transactions of the Royal Society* A **247**, 163, 1954; G. K. Batchelor and I. Proudman, *ibid.* **248**, 369, 1956. These researches have also been described by A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics: Mechanics of Turbulence*, Vol. 2, §15.5, 15.6, Cambridge (Mass.) 1975.

For $t > 0$, $b_{rr}(r,t)$ decreases at infinity at least as r^{-6}, and perhaps exponentially. Loĭtsyanskiĭ's integral is therefore convergent. The decrease of $b_{rr,r}$ is only as r^{-4}, and Λ is therefore not conserved. Its time derivative is some non-zero negative (since $b_{rr,r}$ is found empirically to be negative) function of time. This function is entirely governed by inertial forces. It is reasonable to suppose that, as the turbulence decays, these forces become less important, and in the final stage they may be neglected in comparison with the viscous forces. Thus Λ decreases (the angular momentum "spreads" uniformly through infinite space), tending to a constant limit which it reaches in the final stage of turbulence.

It is therefore possible to determine for this stage the law of time variation of the fundamental scale l and characteristic velocity v of the turbulence. An estimate of the integral (34.25) gives $\Lambda \sim v^2 l^5 = \text{constant}$. Another relation is obtained by estimating the rate of energy decrease by viscous dissipation. The dissipation ε is proportional to the square of the velocity gradients; estimating these as v/l, we find $\varepsilon \sim \nu(v/l)^2$. Equating it to the derivative $\partial(v^2)/\partial t \sim v^2/t$, where t is reckoned from the start of the final stage of turbulence, we have $l \sim (\nu t)^{1/2}$ and so

$$v = \text{constant} \times t^{-5/4} \tag{34.26}$$

(M. D. Millionshchikov 1939).

Correlation function spectrum

As well as the coordinate representation of the correlation functions discussed above, there is a spectral (wave vector) representation of these functions that has methodological and physical interest. It is obtained by expansion as a Fourier space integral:

$$B_{ik}(\mathbf{r}) = \int B_{ik}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} \mathrm{d}^3 k/(2\pi)^3,$$

$$B_{ik}(\mathbf{k}) = \int B_{ik}(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} \mathrm{d}^3 x;$$

the spectral correlation function is denoted by the same symbol B_{ik} with a different independent variable, the wave vector $\mathbf{k}$. Since in isotropic turbulence $B_{ik}(-\mathbf{r}) = B_{ik}(\mathbf{r})$, we have $B_{ik}(\mathbf{k}) = B_{ik}(-\mathbf{k}) = B_{ik}{}^*(\mathbf{k})$, and the spectral functions $B_{ik}(\mathbf{k})$ are therefore real.

As $r \to \infty$, the functions $B_{ik}(\mathbf{r})$ tend to a finite limit given by the first term in (34.4). Accordingly, their Fourier components contain a delta function:

$$B_{ik}(\mathbf{k}) = \tfrac{2}{3}(2\pi)^3 \delta(\mathbf{k}) \langle v^2 \rangle - 2b_{ik}(\mathbf{k}). \tag{34.27}$$

The components with $\mathbf{k} \neq 0$ are the same for the functions B_{ik} and $-2b_{ik}$.

Differentiation with respect to the coordinates x_l in the coordinate representation is equivalent to multiplication by ik_l in the spectral representation. The continuity equation $\partial b_{ik}(\mathbf{r})/\partial x_i = 0$ therefore reduces in the spectral representation to the condition that the tensor $b_{ik}(\mathbf{k})$ be transverse to the wave vector:

$$k_i b_{ik}(\mathbf{k}) = 0. \tag{34.28}$$

Because of the isotropy, the tensor $b_{ik}(\mathbf{k})$ must be expressible in terms of $\mathbf{k}$ and the unit tensor δ_{ik} only. The general form of such a symmetrical tensor satisfying the condition (34.28) is

$$b_{ik}(\mathbf{k}) = F^{(2)}(k)\,(\delta_{ik} - k_i k_k/k^2), \tag{34.29}$$

where $F^{(2)}(k)$ is a real function of the wave number.

The spectral representation of the rank-three correlation tensor is found similarly, $B_{ikl}(\mathbf{k})$ being expressed in terms of $b_{ik,l}(\mathbf{k})$ by (34.11); these tensors do not contain a delta function. The continuity equation $\partial b_{ik,l}(\mathbf{r})/\partial x_l = 0$ gives the condition that $b_{ik,l}(\mathbf{k})$ be transverse as regards the third suffix:

$$k_l b_{ik,l}(\mathbf{k}) = 0. \tag{34.30}$$

The general form of such a tensor is

$$b_{ik,l}(\mathbf{k}) = iF^{(3)}(k)\{\delta_{il}k_k/k + \delta_{kl}k_l/k - 2k_ik_kk_l/k^3\}. \tag{34.31}$$

Since $b_{ik,l}(-\mathbf{r}) = -b_{ik,l}(\mathbf{r})$, the spectral functions $b_{ik,l}(\mathbf{k})$ are imaginary; a factor i has been included in (34.31), so as to make $F^{(3)}(k)$ real.

Equation (34.19) in the spectral representation is

$$\frac{\partial}{\partial t}b_{ik}(\mathbf{k}) = ik_l[b_{il,k}(\mathbf{k}) + b_{kl,i}(\mathbf{k})] - 2\nu k^2 b_{ik}(\mathbf{k}).$$

Substitution of (34.29) and (34.31) gives

$$\partial F^{(2)}(k,t)/\partial t = -2kF^{(3)}(k,t) - 2\nu k^2 F^{(2)}(k,t). \tag{34.32}$$

The function $F^{(2)}(\mathbf{k})$ has an important physical significance. To understand this, let us approach the definition of the spectral correlation function at a somewhat earlier stage.†

We use the customary Fourier expansion of the fluctuating velocity $\mathbf{v}(\mathbf{r})$ itself:

$$\mathbf{v}(\mathbf{r}) = \int \mathbf{v}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} d^3k/(2\pi)^3, \qquad \mathbf{v}_{\mathbf{k}} = \int \mathbf{v}(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^3x.$$

The latter integral is in fact divergent, since $\mathbf{v}(\mathbf{r})$ does not tend to zero at infinity. This is unimportant, however, in the formal derivations below, whose purpose is to calculate the mean squares, which are certainly finite.

The correlation tensor $b_{ik}(\mathbf{r})$ is expressed in terms of the velocity Fourier components by the integral

$$b_{il}(\mathbf{r}) = \iint \langle v_{i\mathbf{k}} v_{l\mathbf{k}'} \rangle e^{i(\mathbf{k}\cdot\mathbf{r}_2 + \mathbf{k}'\cdot\mathbf{r}_1)} d^3k\, d^3k'/(2\pi)^6. \tag{34.33}$$

For this to be a function only of $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$, the integrand must contain a delta function of $\mathbf{k} + \mathbf{k}'$, i.e. must be

$$\langle v_{i\mathbf{k}} v_{l\mathbf{k}'} \rangle = (2\pi)^3 (v_iv_l)_{\mathbf{k}} \delta(\mathbf{k} + \mathbf{k}'). \tag{34.34}$$

This relation is to be regarded as a definition of a quantity here symbolically denoted by $(v_iv_l)_{\mathbf{k}}$. Substituting (34.34) in (34.33) and eliminating the delta function by integration over d^3k', we find

$$b_{il}(\mathbf{r}) = \int (v_iv_k)_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} d^3k(2\pi)^3;$$

that is, the $(v_iv_l)_{\mathbf{k}}$ are the Fourier components of $b_{il}(\mathbf{r})$, and are therefore symmetrical in i and l, and real. In particular, $b_{ii}(\mathbf{k}) = (\mathbf{v}^2)_{\mathbf{k}}$, and we can now say that this quantity is

† The following arguments are a paraphrase of the proof given in *SP* 1, §122.

positive, as is evident from its relation (34.34) to the positive quantity $\langle \mathbf{v}_{\mathbf{k}}\mathbf{v}_{\mathbf{k}'} \rangle = \langle |\mathbf{v}_{\mathbf{k}}|^2 \rangle$, the mean square modulus of the fluctuating velocity Fourier component.

The value of the correlation function $b_{ii}(\mathbf{r})$ for $\mathbf{r} = 0$ determines the mean square velocity of the fluid at any point in space. It is expressed in terms of the spectral function by

$$\langle \mathbf{v}^2 \rangle = b_{ii}(\mathbf{r} = 0) = \int b_{ii}(\mathbf{k})\,\mathrm{d}^3 k/(2\pi)^3$$

or, substituting $b_{ii}(\mathbf{k})$ from (34.29),

$$\tfrac{1}{2}\langle \mathbf{v}^2 \rangle = \int F^{(2)}(k)\,\mathrm{d}^3 k/(2\pi)^3 = \int_0^\infty F^{(2)}(k)\cdot 4\pi k^2\,\mathrm{d}k/(2\pi)^3. \tag{34.35}$$

The meaning of this expression is clear from the foregoing: the positive quantity $F^{(2)}(k)/(2\pi)^3$ is the spectral density of the kinetic energy per unit mass of the fluid in $\mathbf{k}$-space. The energy in the fluctuations whose wave number is in the range $\mathrm{d}k$ is $E(k)\mathrm{d}k$, where

$$E(k) = k^2 F^{(2)}(k)/2\pi^2. \tag{34.36}$$

The first term on the right of (34.32) arises as the Fourier component of the first term on the right of (34.19). When $r \to 0$, the latter reduces to the derivative

$$\langle v_{1k}\frac{\partial}{\partial x_{1l}} v_{1i}v_{1l}\rangle + \langle v_{1i}\frac{\partial}{\partial x_{1l}} v_{1k}v_{1l}\rangle = \frac{\partial}{\partial x_{1l}}\langle v_{1i}v_{1k}v_{1l}\rangle$$

and is zero on account of the homogeneity. In the spectral representation, this means that

$$\int \mathbf{k}\, F^{(3)}(k)\,\mathrm{d}^3 k = 0, \tag{34.37}$$

so that $F^{(3)}(k)$ has variable sign.

Equation (34.32) has a simple meaning: it represents the energy balance of the various spectral components in the turbulent flow. The second term on the right is negative; it gives the energy loss due to dissipation. The first term (due to the non-linear term in the Navier–Stokes equation) describes the energy redistribution in the spectrum, i.e. the energy transfer from the components with smaller k to those with larger k. The energy density $E(k)$ has a maximum at $k \sim 1/l$; the majority of the total energy of the turbulent flow is concentrated near the maximum (in the energy range, §33). The energy dissipation density $2\nu k^2 E(k)$ is greatest for $k \sim 1/\lambda_0$; the majority of the total dissipation is concentrated in the dissipation range. At very high Reynolds numbers, these two regions are far apart and the inertial range lies between them.

Integrating (34.32) over $\mathrm{d}^3 k/(2\pi)^3$ gives on the left the time derivative of the total kinetic energy of the fluid; this is equal to the total energy dissipation $-\varepsilon$. We thus find the following "normalization condition" for $E(k)$:

$$2\nu \int_0^\infty k^2 E(k,t)\,\mathrm{d}k = \varepsilon. \tag{34.38}$$

In the inertial range of wave numbers ($1/l \ll k \ll 1/\lambda_0$), the spectral functions, like the correlation functions in the coordinate representation, may be regarded as time-independent. According to (33.13), we have in this range

$$E(k) = C_1 \varepsilon^{2/3} k^{-5/3}, \tag{34.39}$$

where C_1 is a constant coefficient, related to the coefficient C in the correlation function

$$B_{rr}(r) = C(\varepsilon r)^{2/3} \tag{34.40}$$

by $C_1 = 0{\cdot}76C$; see the Problem. The empirical values are $C \cong 2$, $C_1 \cong 1{\cdot}5$.† Then

$$|B_{rrr}|/B_{rr}{}^{3/2} = 4/5C^{3/2} \cong 0{\cdot}3.$$

PROBLEM

Relate the coefficients C_1 and C in formulae (34.39) and (34.40) for the correlation function and the spectral density of energy in the inertial range.

SOLUTION. The functions

$$B_{ii}(r) = 2B_{tt}(r) + B_{rr}(r) = (11/3)B_{rr}(r)$$

(from (34.6)) and

$$B_{ii}(k) = -2b_{ii}(k) = -4F^{(2)}(k) = -8\pi^2 E(k)/k^2$$

($k \neq 0$) are related through the Fourier integral

$$B_{ii}(k) = \int B_{ii}(r) e^{-i\mathbf{k}\cdot\mathbf{r}}\,\mathrm{d}^3x.$$

If the wave number is in the inertial range ($1/l \ll k \ll 1/\lambda_0$), the oscillatory factor cuts off the integral at an upper limit $r \sim 1/k \ll l$. At small distances, the integral converges, since $B_{ii}(r) \to 0$ as $r \to 0$. In practice, therefore, the integral is governed by distances that lie in the inertial range ($\lambda_0 \ll r \ll l$), and we can substitute in it $B_{rr}(r)$ from (34.40), at the same time extending the integration to all space. In the integral

$$I = \int r^{3/2} e^{-i\mathbf{k}\cdot\mathbf{r}}\,\mathrm{d}^3x,$$

we first integrate over the directions of $\mathbf{r}$, obtaining

$$I = \frac{4\pi}{k}\,\mathrm{im} \int_0^\infty r^{5/3} e^{ikr}\,\mathrm{d}r = \frac{4\pi}{k^{11/3}} \int_0^\infty \xi^{5/3} e^{i\xi}\,\mathrm{d}\xi.$$

The remaining integral is found by rotating the contour of integration in the complex ξ-plane from the right-hand half of the real axis to the upper half of the imaginary axis. The result is

$$I = -\frac{4\pi}{k^{11/3}} \frac{10\pi}{9\Gamma(1/3)}.$$

Combining these expressions, we have finally

$$C_1 = \frac{55}{27\Gamma(1/3)} C = 0{\cdot}76C.$$

† The majority of the experiments relate to turbulence in the atmosphere or the ocean. The Reynolds numbers in these measurements were as high as 3×10^8.

§35. The turbulent region and the phenomenon of separation

Turbulent flow is in general rotational. However, the distribution of the vorticity in the fluid has certain peculiarities in turbulent flow (for very large R): in "steady" turbulent flow past bodies, the whole volume of the fluid can usually be divided into two separate regions. In one of these the flow is rotational, while in the other the vorticity is zero, and we have potential flow. Thus the vorticity is non-zero only in a part of the fluid (though not in general only in a finite part).

That such a limited region of rotational flow can exist is a consequence of the fact that turbulent flow may be regarded as the motion of an ideal fluid, satisfying Euler's equations.† We have seen (§8) that, for the motion of an ideal fluid, the law of conservation of circulation holds. In particular, if at any point on a streamline the curl of the velocity is zero, then the same is true at every point on that streamline. Conversely, if at any point on a streamline $\mathbf{curl\ v} \neq 0$, then it does not vanish anywhere on the streamline. Hence it is clear that the existence of limited regions of rotational and irrotational flow is compatible with the equations of motion if the region of rotational flow is such that the streamlines within it do not penetrate into the region outside it. Such a distribution of the vorticity will be stable, and it will remain zero beyond the surface of separation.

One of the properties of the region of rotational turbulent flow is that the exchange of fluid between this region and the surrounding space can occur only in one direction. The fluid can enter this region from the region of potential flow, but can never leave it.

We should emphasize that the arguments given here cannot, of course, be regarded as affording a rigorous proof of the statements made. However, the existence of limited regions of rotational turbulent flow seems to be confirmed by experiment.

The flow is turbulent both in the rotational and in the irrotational region. The nature of the turbulence, however, is totally different in the two regions. To elucidate the reason for this difference, we may point out the following general property of potential flow, which obeys Laplace's equation $\triangle \phi = 0$. Let us suppose that the flow is periodic in the xy-plane, so that ϕ involves x and y through a factor having the form $e^{ik_1x+ik_2y}$. Then

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = -(k_1{}^2 + k_2{}^2)\phi = -k^2\phi,$$

and, since the sum of the second derivatives must be zero, the second derivative of ϕ with respect to z must equal ϕ multiplied by a positive coefficient: $\partial^2\phi/\partial z^2 = k^2\phi$. The dependence of ϕ on z is then given by a damping factor of the form e^{-kz} for $z > 0$ (the unlimited increase given by e^{kz} is clearly impossible). Thus, if the potential flow is periodic in some plane, it must be damped in the direction perpendicular to that plane. Moreover, the greater k_1 and k_2 (i.e. the smaller the period of the flow in the xy-plane), the more rapidly the flow is damped along the z-axis. All these arguments remain qualitatively valid in cases where the motion is not strictly periodic, but has only some periodic quality.

From this the following result is obtained. Outside the region of rotational flow, the turbulent eddies must be damped, and must be so more rapidly for the smaller eddies. In other words, the small eddies do not penetrate very far into the region of potential flow. Consequently, only the largest eddies are important in this region; they are damped at distances of the order of the (transverse) dimension of the rotational region, which is just the fundamental scale of turbulence in this case. At distances greater than this dimension there is practically no turbulence, and the flow may be regarded as laminar.

† The applicability of these equations to turbulent flow ends at distances of the order of λ_0. The sharp boundary between rotational and irrotational flow is therefore defined only to within such distances.

We have seen that the energy dissipation in turbulent flow occurs in the smallest eddies; the large eddies do not involve appreciable dissipation, which is why Euler's equation is applicable to them. From what has been said above, we reach the important result that the energy dissipation occurs mainly in the region of rotational turbulent flow, and hardly at all outside that region.

Bearing in mind all these properties of the rotational and irrotational turbulent flow, we shall henceforward, for brevity, call the region of rotational turbulent flow simply the *region of turbulent flow* or the *turbulent region*. In the following sections we shall discuss the form of this region in various cases.

The turbulent region must be bounded in some direction by part of the surface of the body past which the flow takes place. The line bounding this part of the surface is called the *line of separation*. From it begins the surface of separation between the turbulent fluid and the remainder. The formation of a turbulent region in flow past a body is called the *phenomenon of separation*.

The form of the turbulent region is determined by the properties of the flow in the main body of the fluid (i.e. not in the immediate neighbourhood of the surface). A complete theory of turbulence (which does not yet exist) would have to make it possible, in principle, to determine the form of this region by using the equations of motion for an ideal fluid, given the position of the line of separation on the surface of the body. The actual position of the line of separation, however, is determined by the properties of the flow in the immediate neighbourhood of the surface (known as the *boundary layer*), where the viscosity plays a vital part (see §40).

In referring (in subsequent sections) to a free boundary of the turbulent region, we shall of course mean its time-averaged position. The instantaneous position of the boundary is a highly irregular surface; these irregular distortions and their time variation are due mainly to the large eddies and accordingly extend to depths comparable with the fundamental scale of the turbulence. The irregular movement of the boundary surface has the result that a point in the flow fixed in space and not too far from the average position of the surface is alternately on opposite sides of the boundary. When the flow pattern is observed at such a point, there will be alternate periods where small-scale turbulence is present and absent.†

§36. The turbulent jet

The form of the turbulent region, and some other basic properties of it, can be established in certain cases by simple similarity arguments. These cases include, among others, various kinds of free turbulent jet in a space filled with fluid (L. Prandtl 1925).

As a first example, let us consider the turbulent region formed when a flow is separated at an angle formed by two infinite intersecting planes (shown in cross-section in Fig. 24). For laminar flow (Fig. 3, §10), the flow along one side of the angle (AO, say) would turn smoothly and flow along the other side away from the angle (OB). In turbulent flow, the pattern is totally different.

The flow along one side of the angle now does not turn on reaching the vertex, but continues in its former direction. A flow appears along the other side in the direction BO.

† This is called the alternation (or intermittency) of turbulence. It is to be distinguished from the similar property of the flow structure within a turbulent region, called by the same name. The available models of such phenomena will not be discussed here.

The two flows mix in the turbulent region;† the boundaries of this region are shown, dashed, in cross-section in Fig. 24. The origin of this region can be seen as follows. Let us imagine a flow in which a uniform stream along AO continues in the same direction, occupying the whole space above the plane AO and its continuation into the fluid to the right, while the fluid below this plane is at rest. In other words, we have a surface of separation (the plane AO produced) between fluid moving with constant velocity and stationary fluid. Such a surface of discontinuity, however, is unstable, and cannot exist in practice (see §29). This instability leads to mixing and the formation of a turbulent region. The flow along BO arises because fluid must enter the turbulent region from outside.

Let us determine the form of the turbulent region. We take the x-axis in the direction shown in Fig. 24, the origin being at O. We denote by Y_1 and Y_2 the distances from the xz-plane to the upper and lower boundaries of the turbulent region, and require to determine Y_1 and Y_2 as functions of x. This can easily be done from similarity considerations. Since the planes are infinite in all directions, there are no constant parameters at our disposal having the dimensions of length. Hence it follows that Y_1, Y_2 can only be directly proportional to the distance x:

$$Y_1 = x \tan \alpha_1, \qquad Y_2 = x \tan \alpha_2. \tag{36.1}$$

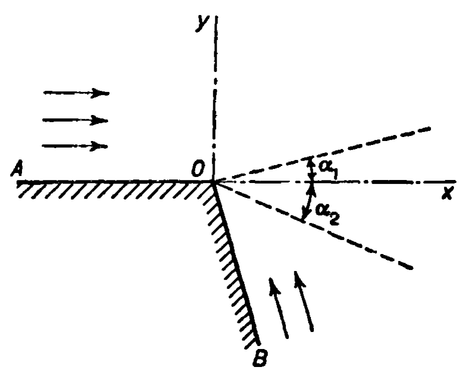

FIG. 24

The proportionality coefficients are simply numerical constants; we write them as $\tan \alpha_1$, $\tan \alpha_2$, so that α_1 and α_2 are the angles between the two boundaries of the turbulent region and the x-axis. Thus the turbulent region is bounded by two planes intersecting along the vertex of the angle.

The values of α_1, α_2 depend only on the size of the angle, and not, for example, on the velocity of the main stream. They cannot be calculated theoretically; the experimental results for flow round a right angle are $\alpha_1 = 5°$, $\alpha_2 = 10°$.‡

The velocities of the flows along the two sides of the angle are not the same; their ratio is a definite number, again depending only on the size of the angle. When the angle is not close to zero, one of the velocities is considerably the greater, namely that of the main stream, which is in the same direction (AO) as the turbulent region. For example, in flow round a right angle, the velocity along the plane AO is thirty times that along BO.

† We recall that, outside the turbulent region, there is irrotational turbulent flow which gradually becomes laminar as we move away from the boundaries of this region.

‡ Here, and elsewhere, we refer to experimental results on the velocity distribution in a transverse cross-section of the turbulent jet, reduced by means of calculations based on a semi-empirical theory (see the final note to the present section).

We may also mention that the difference between the fluid pressures on the two sides of the turbulent region is very small. For example, in flow round a right angle it is found that $p_1 - p_2 = 0{\cdot}003\rho U_1^2$, where U_1 is the velocity of the main stream (along AO), p_1 the pressure in that stream, and p_2 the pressure in the stream along BO.

In the limiting case of flow round an angle of zero, we have simply the edge of a plate with fluid moving along both sides. The angle $\alpha_1 + \alpha_2$ of the turbulent region is zero, i.e. there is no turbulent region; the velocities of the flows along the two sides of the plate become equal. As the angle AOB increases, a point is reached when the plane BO forms the lower boundary of the turbulent region; the angle AOB is by then obtuse. As the angle increases further, the turbulent region continues to be bounded by the plane BO on one side. Here we have simply a separation, with the line of separation along the vertex of the angle. The angle of the turbulent region remains finite.

As a second example, let us consider the problem of a turbulent jet of fluid issuing from the end of a narrow tube into an infinite space filled with the same fluid. The problem of laminar flow in such a "submerged jet" has been solved in §23. At distances (the only ones we shall consider) large compared with the dimensions of the mouth of the tube, the jet is axially symmetrical, whatever the actual shape of the opening.

Let us determine the form of the turbulent region in the jet. We take the axis of the jet as the x-axis, and denote by R the radius of the turbulent region; we require to determine R as a function of x (which is measured from the end of the tube). As in the previous example, this function is easily determined directly from dimensional considerations. At distances large compared with the dimensions of the mouth of the tube, the actual shape and size of the opening cannot affect the form of the jet. Hence we have at our disposal no characteristic parameters with the dimensions of length. It therefore follows as before that R must be proportional to x:

$$R = x \tan \alpha, \tag{36.2}$$

where the numerical constant $\tan \alpha$ is the same for all jets. Thus the turbulent region is a cone; the experimental value of the angle 2α is about 25 degrees (Fig. 25).†

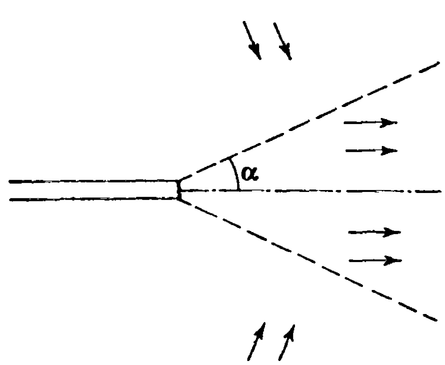

FIG. 25

† Formula (36.2) gives $R = 0$ for $x = 0$; that is, the coordinate x is measured from the point where the jet would start from a point source. This need not coincide with the actual position of the outlet aperture, but may be behind it by a distance of the same order of magnitude as is needed to establish the dependence (36.2). Since we are interested in the asymptotic form for large x, this difference may be neglected.

The flow in the jet is mainly axial. Because there are no parameters having the dimensions of length or velocity which could describe the flow in the jet,† the longitudinal velocity u_x (time-averaged) in it must have a distribution

$$u_x(r, x) = u_0(x) f[r/R(x)], \tag{36.3}$$

where r is the distance from the jet axis and u_0 is the velocity on the axis. Thus the velocity profiles in different cross-sections of the jet differ only as regards the scales of measurement of distance and speed; the jet structure is said to be *self-similar*. The function $f(\xi)$, equal to 1 when $\xi = 0$, decreases rapidly as the argument increases. It is equal to $\frac{1}{2}$ for $\xi = 0{\cdot}4$, and reaches $\sim 0{\cdot}01$ at the boundary of the turbulent region. The transverse velocity has about the same order of magnitude over the cross-section of the turbulent region, and at the boundary of the region is about $-0{\cdot}025\, u_0$ (being directed into the jet there). This transverse velocity is responsible for the inflow into the turbulent region. The flow outside the turbulent region can be found theoretically; see Problem 1.

The dependence of the velocity in the jet on the distance x can be determined from the following simple arguments. The total momentum flux in the jet through a spherical surface centred at its point of emergence must remain constant when the radius of the surface is varied. The momentum flux density in the jet is of the order of ρu^2, where u is of the order of some mean velocity in the jet. The area of the part of the jet cross-section where the velocity is appreciably different from zero is of the order of R^2. Hence the total momentum flux is $P \sim \rho u^2 R^2$. Substituting (36.2), we get

$$u \sim \sqrt{(P/\rho)}\,(1/x), \tag{36.4}$$

that is, the velocity diminishes inversely as the distance from the point of emergence.

The mass Q of fluid which passes per unit time through a cross-section of the turbulent region of the jet is of the order of $\rho u R^2$. Substituting (36.2) and (36.4), we find that $Q = \text{constant} \times x$: we write an equals sign because, if two quantities which vary within wide limits are always of the same order of magnitude, they must be proportional. The proportionality factor is conveniently expressed not in terms of the momentum flux P but in terms of the mass Q_0 of fluid which issues from the tube per unit time. At distances of the order of the linear dimensions a of the tube aperture, we must have $Q \sim Q_0$. Thus the constant is $\sim Q_0/a$, and

$$Q = \beta Q_0 x/a, \tag{36.5}$$

where β is a numerical coefficient which depends only on the form of the aperture. If the latter is circular with radius a, the empirical value is $\beta \cong 1{\cdot}5$. Thus the discharge through the cross-section of the turbulent region increases with x, and fluid is drawn into the turbulent region.‡

The flow in any section of the length of the jet is characterized by the Reynolds number for that section, defined as uR/ν. By virtue of (36.2) and (36.4), however, the product uR is constant along the jet, so that the Reynolds number is the same for all such sections. It can be taken, for instance, as $Q_0/\rho a \nu$. The constant Q_0/a which appears here is the only parameter which determines the flow in the jet. When the "strength" Q_0 of the jet increases

† Note once more that we are considering fully developed turbulence in the jet, and the viscosity therefore should not appear in the formulae concerned.

‡ The total flux through any infinite plane across the jet is infinite, i.e. a jet issuing into an infinite space carries with it an infinite amount of fluid.

(the value of a remaining constant), the Reynolds number eventually reaches a critical value, after which the flow simultaneously becomes turbulent along the whole length of the jet.†

PROBLEMS

PROBLEM 1. Determine the mean flow in the jet outside the turbulent region.

SOLUTION. We take spherical polar coordinates r, θ, ϕ, with the polar axis along the axis of the jet, and the origin at its point of emergence. Because the jet is axially symmetrical, the component u_ϕ of the mean velocity is zero, while u_θ and u_r are functions only of r and θ. The same arguments as were used in the problem of the laminar jet (§23) show that u_θ and u_r must have the forms $u_\theta = f(\theta)/r$, $u_r = F(\theta)/r$. Outside the turbulent region we have potential flow, i.e. **curl u** $= 0$, so that $\partial u_r/\partial\theta - \partial(ru_\theta)/\partial r = 0$. But ru_θ is independent of r, so that $\partial u_r/\partial\theta = (1/r)\mathrm{d}F/\mathrm{d}\theta = 0$, whence $F = \text{constant} = -b$, say, or

$$u_r = -b/r. \tag{1}$$

From the equation of continuity,

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta \sin\theta) = 0,$$

we then obtain

$$f = \frac{\text{constant} - b\cos\theta}{\sin\theta}.$$

The constant of integration must be $-b$ if the velocity is not infinite for $\theta = \pi$ (it does not matter that f is infinite for $\theta = 0$, since the solution in question refers only to the space outside the turbulent region, whereas $\theta = 0$ lies inside that region). Thus

$$u_\theta = -\frac{b(1+\cos\theta)}{r\sin\theta} = -\frac{b}{r}\cot\tfrac{1}{2}\theta. \tag{2}$$

The component of the velocity in the direction of the jet (u_x) and its absolute magnitude are

$$u_x = \frac{b}{r} = \frac{b\cos\theta}{x}, \qquad u = \frac{b}{r\sin\frac{1}{2}\theta}. \tag{3}$$

The constant b can be related to the constant $B = \beta Q_0/a$ in (36.5). Let us consider a segment of the cone formed by the turbulent region, bounded by two infinitely close cross-sections of the cone. The mass of fluid entering this segment per unit time is $\mathrm{d}Q = -2\pi r\rho\sin\alpha\cdot u_\theta \mathrm{d}r = 2\pi b\rho(1+\cos\alpha)\mathrm{d}r$, while from formula (36.5) we have $\mathrm{d}Q = B\,\mathrm{d}x = B\cos\alpha\,\mathrm{d}r$. Comparing the two expressions, we obtain

$$b = \frac{B\cos\alpha}{2\pi\rho(1+\cos\alpha)}. \tag{4}$$

At the boundary of the turbulent region, the velocity **u** is directed into this region, making an angle $\frac{1}{2}(\pi - \alpha)$ with the positive direction of the x-axis.

† In order to make more detailed calculations for various kinds of turbulent flow, it is customary to employ certain "semi-empirical" theories, based on assumptions concerning the dependence of the turbulent viscosity coefficient on the gradient of the mean velocity. For example, in Prandtl's theory it is assumed that (for plane flow)

$$\nu_{\text{turb}} = l^2|\partial u_x/\partial y|,$$

where the dependence of l (called the *mixing length*) on the coordinates is chosen in accordance with the results of similarity arguments; for instance, in free turbulent jets we put $l = cx$, c being an empirical constant. Such theories usually give good agreement with experiment, and are therefore useful for interpolatory calculations. However, it is not possible to give universal values to the empirical constants which characterize each theory; for example, the value of the ratio of the mixing length l to the transverse dimension of the turbulent region has to be chosen diffierently in various particular cases. It should also be mentioned that good agreement with experimental results can be obtained with various expressions for the turbulent viscosity.

Let us compare the mean velocity $\bar{u}_x$ inside the turbulent region (defined as $\bar{u}_x = Q/\pi\rho R^2 = B/\pi\rho x \tan^2 \alpha$) with the velocity $(u_x)_{pot}$ at the boundary of the region. Taking the first equation (3) with $\theta = \alpha$, we find

$$(u_x)_{pot}/\bar{u}_x = \tfrac{1}{2}(1-\cos\alpha).$$

For $\alpha = 12°$, this ratio is 0·011, i.e. the velocity at the boundary of the turbulent region is small compared with the mean velocity inside the region.

PROBLEM 2. Determine the law of variation of size and velocity in a submerged turbulent jet issuing from an infinitely long thin slit.

SOLUTION. By the same reasoning as for the axial jet, we conclude that the turbulent region is bounded by two planes intersecting along the slit, i.e. the half-width of the jet is $Y = x \tan \alpha$. The momentum flux in the jet (per unit length of the slit) is of the order of $\rho u^2 Y$. The dependence of the mean velocity u on x is therefore given by $u \sim \text{constant}/\sqrt{x}$. The discharge through a cross-section of the turbulent region is $Q \sim \rho u Y$, whence $Q = \text{constant} \times \sqrt{x}$. The local Reynolds number $R = uY/\nu$ increases in the same way with x. The experimental data give a value $2\alpha \cong 25°$ for the angle of a plane jet, about the same as for a circular jet.

§37. The turbulent wake

For Reynolds numbers considerably above the critical value, in flow past a solid body, a long region of turbulent flow is formed behind the body. This is called the *turbulent wake*. At distances large compared with the dimension of the body, simple arguments enable us to determine the form of this wake and the way in which the fluid velocity decreases there (L. Prandtl 1926).

As in the investigation of the laminar wake in §21, we denote by $\mathbf{U}$ the velocity of the incident stream, and take the direction of $\mathbf{U}$ as the x-axis. The fluid velocity at any point, averaged over the turbulent fluctuations, is written as $\mathbf{U}+\mathbf{u}$. Denoting by a some mean width of the wake, we shall find a as a function of x. If there is no lift, then at large distances from the body the wake is axially symmetrical and circular in cross-section; in this case, a may be the radius of the wake. If a lift force is present, a direction is selected in the yz-plane, and the wake is not axially symmetrical at any distance from the body.

The longitudinal fluid velocity component in the wake is of the order of U, while the transverse component is of the order of some mean value u of the turbulent velocity. The angle between the streamlines and the x-axis is therefore of the order of u/U. The boundary of the wake is, as we know, the boundary beyond which the streamlines of the rotational turbulent flow cannot pass. Hence it follows that the angle between the boundary of the longitudinal cross-section of the wake and the x-axis is also of the order of u/U. This means that we can write

$$\mathrm{d}a/\mathrm{d}x \sim u/U. \tag{37.1}$$

Next we use formulae (21.1), (21.2), which determine the forces on the body in terms of integrals of the fluid velocity in the wake (the velocity now being interpreted as its mean value). The region of integration in these integrals is of the order of a^2. Hence an estimate of the integral gives $F \sim \rho U u a^2$, where F is of the order of the drag or the lift. Thus

$$u \sim F/\rho U a^2. \tag{37.2}$$

Substituting in (37.1), we find $\mathrm{d}a/\mathrm{d}x \sim F/\rho U^2 a^2$, from which we have by integration

$$a \sim (Fx/\rho U^2)^{\frac{1}{3}}. \tag{37.3}$$

Thus the width of the wake increases as the cube root of the distance from the body. For the velocity u, we have from (37.2) and (37.3)

$$u \sim (FU/\rho x^2)^{\frac{1}{3}}, \tag{37.4}$$

i.e. the mean fluid velocity in the wake is inversely proportional to $x^{\frac{2}{3}}$.

The flow in any section of the wake is characterized by the Reynolds number $\mathrm{R} \sim au/\nu$. Substituting (37.3) and (37.4), we obtain

$$\mathrm{R} \sim F/\nu\rho U a \sim (F^2/\rho^2 U x \nu^3)^{\frac{1}{3}}.$$

We see that this number is not constant along the wake, unlike what we found for the turbulent jet. At sufficiently large distances from the body, R becomes so small that the flow in the wake is no longer turbulent. Beyond this point we have the laminar wake, whose properties have been investigated in §21.

In §21 formulae have been obtained which describe the flow outside the wake and far from the body. These formulae hold for flow outside the turbulent wake as well as outside the laminar wake.

We may mention here some general properties of the velocity distribution round the body. Both inside and outside the turbulent wake, the velocity (by which we always mean $\mathbf{u}$) decreases away from the body. However, the longitudinal velocity u_x falls off more rapidly ($\sim 1/x^2$) outside the wake than inside it. Far from the body, therefore, we may suppose u_x to be zero outside the wake. We may say that u_x falls from some maximum value on the axis of the wake to zero at the boundary of the wake. The transverse components u_y, u_z at the boundary are of the same order of magnitude as they are inside the wake, diminishing rapidly as we move away from the wake at a given distance from the body.

§38. Zhukovskiĭ's theorem

The velocity distribution round a body, described at the end of the last section, does not hold for exceptional cases where the thickness of the wake formed behind the body is very small compared with its width. A wake of this kind is formed in flow past bodies whose thickness (in the y-direction) is small compared with their width (in the z-direction); the length (in the direction of flow, the x-direction) may be of any magnitude. That is, we are considering flow past bodies whose cross-section transverse to the flow is very elongated. These bodies include, in particular, *wings*, i.e. bodies whose width, or *span*, is large in comparison with their other dimensions.

It is clear that, in such a case, there is no reason why the velocity component u_y perpendicular to the plane of the turbulent wake should fall off appreciably at distances of the order of the thickness of the wake. On the contrary, this component will now be of the same order of magnitude inside the wake and at considerable distances from it, of the order of the span. Here, of course, we assume that the lift is not zero, since otherwise the transverse velocity practically vanishes.

Let us consider the vertical lift force F_y resulting from such a flow. According to formula (21.2), it is given by the integral

$$F_y = -\rho U \iint u_y \,\mathrm{d}y\,\mathrm{d}z, \tag{38.1}$$

where, on account of the nature of the distribution of u_y, the integration must now be taken over the whole transverse plane. Furthermore, since the thickness of the wake (in the y-direction) is small, while the velocity u_y inside the wake is not large compared with its value outside, we can with sufficient accuracy take the integration over y to be over the region outside the wake, writing

$$\int_{-\infty}^{\infty} u_y \,\mathrm{d}y \cong \int_{y_1}^{\infty} u_y \,\mathrm{d}y + \int_{-\infty}^{y_2} u_y \,\mathrm{d}y,$$

where y_1 and y_2 are the coordinates of the boundaries of the wake (Fig. 26).

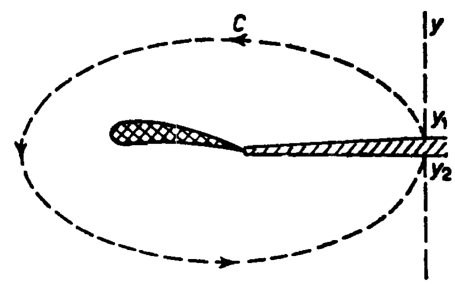

FIG. 26

Outside the wake, however, we have potential flow, and $u_y = \partial\phi/\partial y$; bearing in mind that $\phi = 0$ at infinity, we therefore obtain

$$\int u_y \,\mathrm{d}y = \phi_2 - \phi_1,$$

where ϕ_2 and ϕ_1 are the values of the potential on the two sides of the wake. We may say that $\phi_2 - \phi_1$ is the discontinuity of the potential at the surface of discontinuity which may be substituted for a thin wake. The derivative $u_y = \partial\phi/\partial y$ must remain continuous. A discontinuity in the velocity component normal to the surface of the wake would mean that some quantity of fluid flows into the wake; in the approximation in which the thickness of the wake is neglected, however, this inflow must be zero. Thus we replace the wake by a surface of tangential discontinuity. Next, in the same approximation, the pressure also must be continuous at the wake. Since the variation of the pressure is given in the first approximation, according to Bernoulli's equation, by $\rho U u_x = \rho U \partial\phi/\partial x$, it follows that the derivative $\partial\phi/\partial x$ must also be continuous. The derivative $\partial\phi/\partial z$ (the velocity along the wing) is in general discontinuous, however.

Since the derivative $\partial\phi/\partial x$ is continuous, the discontinuity $\phi_2 - \phi_1$ depends only on z, and not on the coordinate x along the wake. Thus we have the following formula for the lift:

$$F_y = -\rho U \int (\phi_2 - \phi_1)\,\mathrm{d}z. \qquad (38.2)$$

The integration over z may be taken over the width of the wake (of course, $\phi_2 - \phi_1 \equiv 0$ outside the wake).

This formula can be put in a somewhat different form. To do so, we notice that, using well-known properties of an integral of the gradient of a scalar, we can write the difference $\phi_2 - \phi_1$ as a contour integral

$$\oint \mathbf{grad}\, \phi \cdot \mathbf{dl} = \oint (u_y \mathrm{d}y + u_x \mathrm{d}x),$$

taken along a contour which starts from the point y_1, encircles the body, and ends at the point y_2, thus passing at every point through the region of potential flow. Since the wake is thin we can, without changing the integral except by quantities of higher order, close this contour by means of the short segment from y_2 to y_1. Denoting by Γ the velocity circulation round the closed contour C enclosing the body (Fig. 26), we have

$$\Gamma = \oint \mathbf{u} \cdot \mathbf{dl} = \phi_2 - \phi_1, \tag{38.3}$$

and for the lift force the formula

$$F_y = -\rho U \int \Gamma \, \mathrm{d}z. \tag{38.4}$$

The sign of the velocity circulation is always chosen to be that obtained for a counter-clockwise path. The sign in formula (38.3) also depends on the chosen direction of flow. We always suppose that the flow is in the positive direction of the x-axis (from left to right).

The relation between the lift and the circulation given by formula (38.4) constitutes *Zhukovskiĭ's theorem*, first derived by N. E. Zhukovskiĭ in 1906. Cf. §46 for the application of this theorem to streamlined wings.

PROBLEMS

PROBLEM 1. Determine the manner of widening of the turbulent wake formed in transverse flow past a cylinder with infinite length.

SOLUTION. The drag f_x per unit length of the cylinder is of the order of $\rho U u Y$. Combining this with the relation (37.1), we find the width Y of the wake to be

$$Y = A\sqrt{(x f_x / \rho U^2)}, \tag{1}$$

where A is a constant. The mean velocity u in the wake falls off in accordance with $u \sim \sqrt{(f_x/\rho x)}$. The Reynolds number $\mathbf{R} \sim Yu/\nu \sim f_x/\rho U \nu$ is independent of x, and there is therefore no laminar wake.

We may mention that, according to experimental results, the constant coefficient in (1) is $A = 0{\cdot}9$ (Y being the half-width of the wake; if Y is taken as the distance at which the velocity u_x falls to half its maximum value (at the centre of the wake), then $A = 0{\cdot}4$).

PROBLEM 2. Determine the flow outside the wake formed in transverse flow past a body of infinite length.

SOLUTION. Outside the wake we have potential flow; we shall denote the potential by Φ to distinguish it from the angle ϕ in the system of cylindrical polar coordinates which we take, with the z-axis along the length of the body. As in (21.16), we conclude that we must have

$$\oint \mathbf{u} \cdot \mathbf{df} = \oint \mathbf{grad}\, \Phi \cdot \mathbf{df} = f_x/\rho U,$$

where now the integration is over the surface of a cylinder with large radius and unit length with its axis in the x-direction, and f_x is the drag per unit length of the body. The solution of the two-dimensional Laplace's equation $\Delta\Phi = 0$ that satisfies this condition is $\Phi = (f_x/2\pi\rho U)\log r$. Next, we have for the lift, by formula (38.2), $f_y = \rho U(\Phi_1 - \Phi_2)$. The solution of Laplace's equation that diminishes least rapidly with increasing distance and has a discontinuity of the plane $\phi = 0$ is $\Phi = \text{constant} \times \phi = -\phi f_y/2\pi\rho U$, the constant being determined by $\phi_2 - \phi_1 = 2\pi$. The flow is given by the sum of these two solutions, i.e.

$$\Phi = \frac{1}{2\pi\rho U}(f_x \log r - \phi f_y). \tag{2}$$

The cylindrical components of the velocity $\mathbf{u}$ are

$$u_r = \partial\Phi/\partial r = f_x/2\pi\rho U r, \qquad u_\phi = (1/r)\partial\Phi/\partial\phi = -f_y/2\pi\rho U r. \tag{3}$$

The velocity $\mathbf{u}$ is at a constant angle $\tan^{-1}(f_y/f_x)$ to the r-direction.

PROBLEM 3. Determine the manner of bending of the wake behind a body with infinite length when there is a lift force.

SOLUTION. If there is a lift force, the wake (regarded as a surface of discontinuity) is curved in the xy-plane. The function $y = y(x)$ which determines this is given by the equation $\mathrm{d}x/(u_x + U) = \mathrm{d}y/u_y$. Substituting, by (3), $u_y \cong -f_y/2\pi\rho U x$ and neglecting u_x in comparison with U, we obtain

$$\mathrm{d}y/\mathrm{d}x = -f_y/2\pi\rho U^2 x,$$

whence

$$y = \text{constant} - (f_y/2\pi\rho U^2)\log x.$$