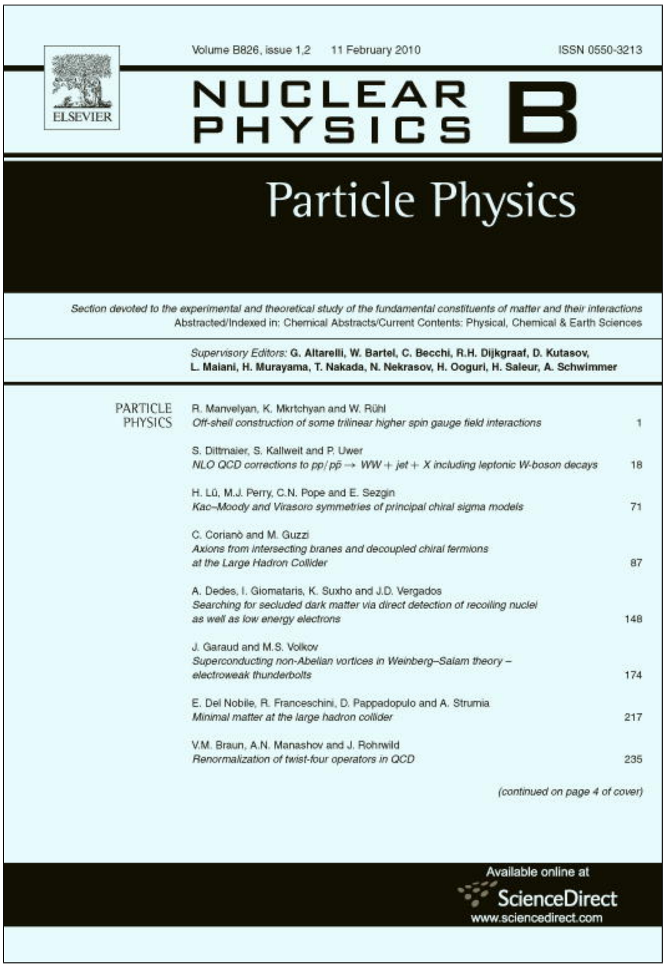

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Particle Physics

Section devoted to the experimental and theoretical study of the fundamental constituents of matter and their interactions
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Superconducting non-Abelian vortices in Weinberg–Salam theory – electroweak thunderbolts

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Abstract

We present a detailed analysis of classical solutions in the bosonic sector of the electroweak theory which describe vortices carrying a constant electric current $\mathcal{I}$. These vortices exist for any value of the Higgs boson mass and for any weak mixing angle and in the zero current limit they reduce to Z strings. Their current is produced by the condensate of vector W bosons and typically it can attain billions of amperes. For large $\mathcal{I}$ the vortices show a compact condensate core of size $\sim 1/\mathcal{I}$, embedded into a region of size $\sim \mathcal{I}$ where the electroweak gauge symmetry is completely restored, followed by a transition zone where the Higgs field interpolates between the symmetric and broken phases. Outside this zone the fields are the same as for the ordinary electric wire. An analytic approximation of the large $\mathcal{I}$ solutions suggests that the current can be arbitrarily large, due to the scale invariance of the vector boson condensate. Finite vortex segments are likely to be perturbatively stable. This suggests that they can transfer electric charge between different regions of space, similarly to thunderbolts. It is also possible that they can form loops stabilized by the centrifugal force – electroweak vortons.

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1. Introduction

More than 20 years ago Witten proposed a field theory model that admits classical solutions describing vortices carrying a constant current – 'superconducting strings' [1]. This model has a

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local U(1) × U(1) invariance and consists of two copies of the Abelian Higgs model for fields $(A_\mu^{(1)}, \phi_1)$ and $(A_\mu^{(2)}, \phi_2)$. It also includes an interaction between the two complex scalars chosen such that in vacuum one has $\phi_1 \neq 0$ but $\phi_2 = 0$ so that the vector field $A_\mu^{(1)}$ is massive while $A_\mu^{(2)}$ is massless.

The model admits as a solution the Abrikosov–Nielsen–Olesen (ANO) vortex [2] made of the 'vortex fields' $(A_\mu^{(1)}, \phi_1)$, with vanishing 'condensate fields' $(A_\mu^{(2)}, \phi_2)$. This embedded vortex is however unstable, but being topological it does not unwind into vacuum and relaxes to a 'dressed vortex' which contains a condensate of charged scalar bosons in the core described by $\phi_2 \neq 0$. Giving then a non-trivial phase to the condensate field ϕ_2 produces a current and promotes the 'dressed' vortex to the superconducting string supporting the long-range Biot–Savart field represented by $A_\mu^{(2)} \neq 0$.

The Witten string superconductivity has been much studied [3,4], mainly in the cosmological context [5,6], since Witten's model can be viewed as sector of some high energy Grand Unification Theory (GUT) [1] that could perhaps be relevant at the early stages of the cosmological evolutions. Using the typical values of the GUT parameters for estimates gives for the string current enormous values of order 10^{20} A, which suggests interesting applications [5,6]. String superconductivity in the GUT-related non-Abelian models has also been studied [7], in which case the string current is produced by a condensate of charged vector bosons.

Although the GUT physics could be important, one may wonder whether a similar string superconductivity could exist also in a less exotic context, at lower energies, as for example in the electroweak sector of Standard Model. In fact, the U(1) × U(1) symmetry of Witten's model is contained in the SU(2) × U(1) electroweak gauge symmetry. In addition, the electroweak theory contains a pair of complex Higgs scalars, one of which could well be responsible for the formation of the vortex while the other one – for the condensate. Since the ANO vortices can be embedded into the electroweak theory in the form of Z strings [8], one might expect that current-carrying generalizations of the latter could exist. These would be the electroweak analogues of Witten's superconducting strings.

However, no attempts to construct such solutions have ever been undertaken. This can probably be explained by the following reasons. The current-carrying Witten strings are usually viewed as excitations over the 'dressed' currentless vortex obtained by minimizing the energy of the 'bare' embedded ANO vortex. This explains their essential properties, as for example the value of the critical current [5]. Now, the 'bare' electroweak Z strings are also unstable [9], and it was conjectured [10,11] that they could similarly relax to 'W-dressed Z strings'. However, a systematic search for such solutions gave no result [12], probably due to the fact that Z strings are non-topological and can unwind into vacuum [13], so that it is less likely that they could be stabilized by the condensate. But if one does not find currentless 'dressed' Z strings, it is then natural to think that current-carrying strings in the electroweak theory do not exist either. This presumably explains why there has been almost no activity on electroweak vortices during the last 10–15 years.

At the same time, it is difficult to believe that Z strings are the only possible vortex solutions in the electroweak theory. The theory admits rather non-trivial solutions, such as sphalerons [14], periodic BPS solutions [15], spinning dumbbells [16,17], oscillons [18], spinning sphalerons [19], and others (see [20] for a review). In addition, in the semilocal limit where the SU(2) field decouples the theory admits current-carrying vortices [21], so that it is plausible that they could exist also for generic values of the weak mixing angle.

We have therefore analysed the problem and found that superconducting vortices indeed exist in the electroweak theory, despite the non-existence of the 'W-dressed Z strings'. In other words, these two types of solutions are not necessarily related. To construct the solutions, we essentially reverse the standard 'engineering' procedure used in Witten's model. There one starts from the 'dressed' currentless vortex and *increases* its 'winding number density' that determines the phase of the condensate; in what follows we shall call this parameter 'twist'. This produces a current that first increases with the twist but then starts to quench and finally vanishes when the solution reduces to the 'bare' ANO vortex [5]. The superconducting strings thus comprise a one-parameter family that interpolates between the 'dressed' vortex and the 'bare' ANO vortex.

We construct this family in the opposite direction, by starting from the 'bare' vortex and then *decreasing* its twist. Within Witten's model this gives of course the same solutions but in the reversed order – their current first increases, then starts to quench and vanishes for zero twist when the solution reduces to the 'dressed' string. However, the advantage of our method is that it can be applied also within the Weinberg–Salam theory, where there are no 'dressed' currentless vortices but only the 'bare' ones – Z strings. Their twist is determined by the eigenvalue of the second variation of the energy functional. Decreasing the twist gives us solutions with a non-zero current. Further decreasing it shows that the current always grows and tends to infinity when the twist approaches zero. As a result, contrary to what one would normally expect, and presumably because of the vector character of their current-carriers, the electroweak vortices do not (generically) exhibit the current quenching, so that they do not need to admit the 'dressed' currentless limit.

The stability analysis of these vortices shows that their finite segments are perturbatively stable. More precisely, so far this property has been demonstrated only in the semilocal limit [22], but it is likely that the result applies also for generic values of the weak mixing angle. The length of the stable segments increases with the current and can in principle attain any value, since there is no upper bound for the current.

This may have interesting consequences. First of all, this suggests that loops made of the stable segments and balanced against contraction by the centrifugal force could perhaps be stable as well. Although studying this would go far beyond the scope of the present paper, the possibility to have stable solitonic objects in the Standard Model could be very important, since if such *electroweak vortons* exist, they could perhaps contribute to the dark matter.

Another, perhaps a more direct consequence, is related to the well-know fact that, since Z strings are non-topological, they can have finite length. This suggests that their current-carrying generalizations could also exist in the form of finite (and stable) segments connecting oppositely charged regions of space. They would then be similar to thunderbolts. For the solutions we could explicitly construct the current can be as large as 10^{10} A, which exceeds by several orders of magnitudes the power of the strongest thunderbolts in the Earth atmosphere. This suggests that superconducting vortices could be important for the dynamics of the Standard Model, although analysing this issue in detail would again lead us too far away from our main subject.

The present paper is devoted to a systematic analysis of the string/vortex-type solutions in the electroweak theory (terms string and vortex are assumed to be synonyms). In what follows we shall show that every Z string admits a three-parameter family of non-Abelian, current carrying generalizations. In the gauge where the radial components of the gauge fields vanish, the upper and lower components of the Higgs field doublet behave quite analogously to the vortex field and condensate field in Witten's model. The new vortices exist for (almost) any value of the weak mixing angle and for any Higgs boson mass. They show a compact, regular core containing the W-condensate that carries a constant electric current, producing the Biot–Savart electromagnetic field outside the vortex. In the comoving reference frame the vortex is characterized by the current $\mathcal{I}$ and by the electromagnetic and Z fluxes through its cross section. After performing a Lorentz boost, it develops also a non-zero electric charge density, as well as momentum and angular momentum directed along the vortex.

It seems that the vortex current can be as large as one wants, at least we could not find an upper bound for it. We have also managed to construct a simple approximation in the large current limit which suggests that the current can be arbitrarily large, due to the fact that it is carried by *vectors*, whose condensate exhibits the scale invariance. To the best of our knowledge, a similar effect has never been reported before. For large currents the charged W boson condensate is contained in the vortex core of size $\sim 1/\mathcal{I}$, which is surrounded by a large region of size $\sim \mathcal{I}$ where the Higgs field is driven to zero by the strong magnetic field so that the electroweak gauge symmetry is completely restored. However, this does not destroy the vector boson superconductivity, since the scalar Higgs field is not the relevant order parameter in this case. Outside the symmetric phase region there is a transition zone where the Higgs field relaxes to the broken phase and the theory reduces to the ordinary electromagnetism. This reminds somewhat of the electroweak vacuum polarization scenario discussed by Ambjorn and Olesen [15], in which the system stays in the Higgs vacuum if only the magnetic field is not too strong, while very strong fields restore the full gauge symmetry.

The rest of the paper is organized as follows. In Section 2 the essential elements of the Weinberg–Salam theory are introduced. Section 3 presents the symmetry reduction, the basic field ansatz (3.10), and the field equations (3.16)–(3.24). In Section 4 the boundary conditions are discussed, while Section 5 describes the conserved quantities and the known solutions. The new solutions are first considered in Section 6 in the small current limit, when they can be treated as small deformations of Z strings. In Section 7 they are presented at the full non-perturbative level for generic values of the current. Section 8 describes the large current limit. Solutions for special parameter values are considered in Section 9, while Section 10 contains concluding remarks. Appendices A and B contain the derivation of the local solutions at the symmetry axis and at infinity. Appendix C describes the superconducting strings in Witten's model.

A very brief, preliminary summary of our results has been announced in Ref. [23].

Recently there has been an intense activity on the non-Abelian strings with gauge group $\mathrm{SU}(N)\times\mathrm{U}(1)$ [24]. These strings have nothing to do with ours, since they are obtained within the context of a supersymmetric theory that is different from the electroweak theory even for $N=2$. They are currentless and are supposed to be relevant in the QCD context, as models of gluon tubes. There has also been a report on current-carrying strings [25], but again in a model with a different Higgs sector in which solutions taking values in the Cartan subalgebra of $\mathrm{SU}(N)\times\mathrm{U}(1)$ are possible.

2. Weinberg–Salam theory

The bosonic part of the Weinberg–Salam (WS) theory is described by the action density

$$\mathcal{L} = -\frac{1}{4g^2}\mathrm{W}^a_{\mu\nu}\mathrm{W}^{a\mu\nu} - \frac{1}{4g'^2}B_{\mu\nu}B^{\mu\nu} + (D_\mu\Phi)^\dagger D^\mu\Phi - \frac{\beta}{8}\left(\Phi^\dagger\Phi - 1\right)^2. \tag{2.1}$$

Here Φ is in the fundamental representation of SU(2) and

$$\mathrm{W}^a_{\mu\nu} = \partial_\mu\mathrm{W}^a_\nu - \partial_\nu\mathrm{W}^a_\mu + \epsilon_{abc}\mathrm{W}^b_\mu\mathrm{W}^c_\nu, \qquad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu,$$

$$D_\mu\Phi = \left(\partial_\mu - \frac{i}{2}B_\mu - \frac{i}{2}\tau^a\mathrm{W}^a_\mu\right)\Phi, \tag{2.2}$$

where τ^a are the Pauli matrices. The two coupling constants are $g = \cos\theta_W$ and $g' = \sin\theta_W$ where the physical value of the Weinberg angle is $\sin^2\theta_W = 0.23$. All quantities in Eqs. (2.1), (2.2) are rendered dimensionless by rescaling, while their dimensionfull analogues will be denoted by boldfaced symbols, for example $\mathbf{B}_\mu = \boldsymbol{\Phi}_0 B_\mu$, $\mathbf{W}^a_\mu = \boldsymbol{\Phi}_0 \mathrm{W}^a_\mu$, $\boldsymbol{\Phi} = \boldsymbol{\Phi}_0 \Phi$ and spacetime coordinates $\mathbf{x}^\mu = x^\mu/\mathbf{g}_0\boldsymbol{\Phi}_0$. Here $\boldsymbol{\Phi}_0$ is the Higgs field vacuum expectation value and $\mathbf{g}_0$ relates to the electron charge via $\mathbf{e} = gg'\boldsymbol{\hbar}\mathbf{c}\mathbf{g}_0$. The dimensionfull action is

$$\mathbf{S} = \frac{1}{\mathbf{c}\mathbf{g}_0^2}\int \mathcal{L}\, d^4x. \tag{2.3}$$

The theory is invariant under the SU(2) × U(1) gauge transformations

$$\Phi \to \mathrm{U}\Phi, \qquad \mathcal{W} \to \mathrm{U}\mathcal{W}\mathrm{U}^{-1} + 2i\mathrm{U}\partial_\mu\mathrm{U}^{-1}\,dx^\mu, \tag{2.4}$$

with

$$\mathrm{U} = \exp\left(\frac{i}{2}\Theta + \frac{i}{2}\tau^a\theta^a\right) \tag{2.5}$$

where Θ and θ^a are functions of x^μ and

$$\mathcal{W} = \left(B_\mu + \tau^a \mathrm{W}^a_\mu\right) dx^\mu \tag{2.6}$$

is the SU(2) × U(1) Lie-algebra valued gauge field. Varying the action with respect to the fields gives the field equations,

$$\partial^\mu B_{\mu\nu} = g'^2\frac{i}{2}\left((D_\nu\Phi)^\dagger\Phi - \Phi^\dagger D_\nu\Phi\right) \equiv g'^2 J^0_\nu, \tag{2.7}$$

$$D^\mu \mathrm{W}^a_{\mu\nu} = g^2\frac{i}{2}\left((D_\nu\Phi)^\dagger\tau^a\Phi - \Phi^\dagger\tau^a D_\nu\Phi\right) \equiv g^2 J^a_\nu, \tag{2.8}$$

$$D_\mu D^\mu\Phi + \frac{\beta}{4}\left(\Phi^\dagger\Phi - 1\right)\Phi = 0, \tag{2.9}$$

with $D_\mu \mathrm{W}^a_{\alpha\beta} = \partial_\mu \mathrm{W}^a_{\alpha\beta} + \epsilon_{abc}\mathrm{W}^b_\mu \mathrm{W}^c_{\alpha\beta}$. Varying the action with respect to the spacetime metric gives the energy–momentum tensor $T^\mu{}_\nu = 2g^{\mu\sigma}\partial\mathcal{L}/\partial g^{\sigma\nu} - \delta^\mu{}_\nu\mathcal{L}$ which evaluates to

$$T^\mu{}_\nu = -\frac{1}{g^2}\mathrm{W}^{a\mu\sigma}\mathrm{W}^a_{\nu\sigma} - \frac{1}{g'^2}B^{\mu\sigma}B_{\nu\sigma} + \left(D^\mu\Phi\right)^\dagger D_\nu\Phi + (D_\nu\Phi)^\dagger D^\mu\Phi - \delta^\mu{}_\nu\mathcal{L}. \tag{2.10}$$

One has $T^\mu{}_\nu = 0$ for fields which are gauge copies of the *vacuum* $\mathrm{W}^a_\mu = B_\mu = 0$, $\Phi = \binom{1}{0}$. Linearising the field equations (2.7)–(2.9) with respect to small fluctuations around the vacuum gives the perturbative mass spectrum containing the photon and the massive Z, W and Higgs bosons with the masses

$$m_\mathrm{Z} = \frac{1}{\sqrt{2}}, \qquad m_\mathrm{W} = g m_\mathrm{Z}, \qquad m_\mathrm{H} = \sqrt{\beta}\, m_\mathrm{Z}. \tag{2.11}$$

Multiplying by $\mathbf{e}\boldsymbol{\Phi}_0/(gg')$ gives the dimensionfull masses, for example $\mathbf{m}_\mathrm{Z}\mathbf{c}^2 = \mathbf{e}\boldsymbol{\Phi}_0/(\sqrt{2}gg') \approx$ 91 GeV. The exact value of the parameter β defining the Higgs mass is currently unknown, but it is likely that it belongs to the interval $1.5 \leqslant \beta \leqslant 3.5$ [26].

We shall adopt the definition of Nambu for the electromagnetic and Z fields [16],

$$F_{\mu\nu} = \frac{g}{g'}B_{\mu\nu} - \frac{g'}{g}n^a \mathrm{W}^a_{\mu\nu}, \qquad Z_{\mu\nu} = B_{\mu\nu} + n^a \mathrm{W}^a_{\mu\nu}, \tag{2.12}$$

where $n^a = \Phi^\dagger \tau^a \Phi/(\Phi^\dagger \Phi)$. This definition is not unique [27], but it gives more satisfactory results [28] than the other known definitions [29]. It should be noted that the 2-forms (2.12) are not closed in general, so that there are no field potentials. However, this cannot be considered as a drawback, since there is no reason why the Maxwell equations should hold off the Higgs vacuum. Having defined the electromagnetic field, the electric current density is given by

$$J_\mu = \partial^\nu F_{\nu\mu}. \tag{2.13}$$

3. Symmetry reduction

In what follows we shall be constructing solutions of the field equations (2.7)–(2.9) describing a vortex oriented along the x^3 axis. The spacetime coordinates x^μ then split naturally into two groups: $x^k = (x^1, x^2)$ spanning the 2-planes orthogonal to the vortex and $x^\alpha = (x^0, x^3)$ parametrizing the 'vortex worldsheet'. We want the vortex to be stationary and invariant under translations along and rotations around the x^3 axis. It should therefore respect the spacetime symmetries generated by three Killing vectors

$$K_{(0)} = \frac{\partial}{\partial x^0}, \qquad K_{(3)} = \frac{\partial}{\partial x^3}, \qquad K_{(\varphi)} = \frac{\partial}{\partial \varphi}. \tag{3.1}$$

Here φ is the azimuthal angle in the x^1, x^2 plane: $x^1 = \rho\cos\varphi$, $x^2 = \rho\sin\varphi$.

Associated to the symmetries there will be conserved Noether charges carried by the vortex,

$$\mathcal{Q}[K] = \int T^0{}_\nu K^\nu \, d^2x, \tag{3.2}$$

where the integration is performed over the x^1, x^2 plane. These charges are the energy $E = \mathcal{Q}[K_{(0)}]$, momentum $P = \mathcal{Q}[K_{(3)}]$, and the angular momentum $M = \mathcal{Q}[K_{(\varphi)}]$ per unit vortex length. Next, integrating the electric current density (2.13) gives

$$I_\alpha = \int J_\alpha \, d^2x = \int \partial^\nu F_{\nu\alpha} \, d^2x. \tag{3.3}$$

Here I_0 is the electric charge per unit vortex length and I_3 is the total electric current along the vortex. Since the integrand in (3.3) is a total derivative, the integration is actually performed over the boundary at infinity, in the region where all fields approach the vacuum, in which case all definitions of $F_{\mu\nu}$ coincide [27].

The vortex can also be characterized by values of the electromagnetic and Z fluxes through its orthogonal cross section,

$$\Psi_F = \frac{1}{2}\int \epsilon_{ik} F_{ik} \, d^2x, \qquad \Psi_Z = \frac{1}{2}\int \epsilon_{ik} Z_{ik} \, d^2x. \tag{3.4}$$

In Abelian theories the fluxes are integer multiples of the elementary 'flux quantum', which is the consequence of the holonomy condition requiring that the parallel transport along a large circle around the vortex does not change Φ. In the non-Abelian case the holonomy condition does not enforce such a 'flux quantisation' and fluxes can assume any value.

It is instructive to restore the physical dimensions for a moment. The dimensionfull electric current density is $\frac{1}{\mathbf{c}}\mathbf{J}_\nu = \partial^\mu \mathbf{F}_{\mu\nu}$ where $\mathbf{F}_{\mu\nu} = \mathbf{g}_0 \boldsymbol{\Phi}_0^2 F_{\mu\nu}$ and the derivative is taken with respect to $\mathbf{x}^\mu = x^\mu/\mathbf{g}_0\boldsymbol{\Phi}_0$. Therefore

$$\int \mathbf{J}_\alpha \, d^2\mathbf{x} = \mathbf{c}\int \boldsymbol{\partial}^\mu \mathbf{F}_{\mu\alpha} \, d^2\mathbf{x} = \mathbf{c}\boldsymbol{\Phi}_0 \int \partial^\mu F_{\mu\alpha} \, d^2x = \mathbf{c}\boldsymbol{\Phi}_0 I_\alpha, \tag{3.5}$$

so that I_3 gives the current in units of

$$\mathbf{c}\boldsymbol{\Phi}_0 = \mathbf{c} \times 54.26 \times 10^9\ \text{V} = 1.8 \times 10^9\ \text{A}. \tag{3.6}$$

This value is quite large. Below we shall construct vortices with typical current $I_3 \sim 1$–10, which looks modest but corresponds in fact to billions of amperes! Very large currents are typical for superconducting strings. In the GUT-related model of Witten the current can be as large as 10^{20} A [1,3] – because the GUT Higgs field vacuum expectation value is much larger than $\boldsymbol{\Phi}_0$.

Similarly, $\mathbf{E} = \boldsymbol{\Phi}_0^2 E$, $\mathbf{P} = (\boldsymbol{\Phi}_0^2/\mathbf{c})P$, $\mathbf{M} = (\boldsymbol{\Phi}_0^2/\mathbf{c}\mathbf{g}_0)M$ and $\boldsymbol{\Psi}_F = \Psi_F/\mathbf{g}_0$, $\boldsymbol{\Psi}_Z = \Psi_Z/\mathbf{g}_0$.

3.1. Field ansatz

Since the symmetry generators (3.1) commute between themselves and all internal symmetries of the theory are gauged, there exists a gauge where the symmetric fields do not depend on x^0, x^3, φ [30]. They can therefore depend only on ρ, so that

$$\mathcal{W} = \mathcal{W}_\mu(\rho)\,dx^\mu, \qquad \Phi = \Phi(\rho). \tag{3.7}$$

This ansatz contains 20 arbitrary real functions of ρ and it keeps its form under gauge transformations (2.4), (2.5) with Θ, θ^a depending only on ρ. This residual gauge symmetry can be used to set to zero the radial components of the gauge fields,

$$\mathcal{W}_\rho = 0, \tag{3.8}$$

which reduces the number of independent functions to 16. Now, it turns out that one can consistently truncate a half of them.

Let $\sigma_\alpha = (\sigma_0, \sigma_3)$ be a constant (co)vector in the (x^0, x^3) plane and let $\tilde{\sigma}_\alpha$ be its orthogonal, $\tilde{\sigma}_\alpha\sigma^\alpha = 0$. A direct inspection of the field equations reveals that one can consistently set to zero the projection of the gauge field on $\tilde{\sigma}_\alpha$, as well as the imaginary components of the fields. The latter include the τ^2 component of the Yang–Mills field and also the imaginary part of the Higgs field. One can therefore set on-shell

$$\tilde{\sigma}^\alpha\mathcal{W}_\alpha = 0, \qquad \mathrm{W}_\mu^2 = 0, \qquad \Im(\Phi) = 0, \tag{3.9}$$

which eliminates 8 real functions of ρ. The remaining non-trivial field components can be parametrized as

$$\begin{aligned}\mathcal{W} &= u(\rho)\sigma_\alpha\,dx^\alpha - v(\rho)\,d\varphi + \tau^1\big[u_1(\rho)\sigma_\alpha\,dx^\alpha - v_1(\rho)\,d\varphi\big] \\ &\quad + \tau^3\big[u_3(\rho)\sigma_\alpha\,dx^\alpha - v_3(\rho)\,d\varphi\big], \\ \Phi &= \begin{pmatrix} f_1(\rho) \\ f_2(\rho)\end{pmatrix},\end{aligned} \tag{3.10}$$

where $\alpha = 0, 3$ and f_1, f_2 are real. This ansatz has the following properties.

a. It is invariant under spacetime symmetries generated by $\partial/\partial x^0$, $\partial/\partial x^3$, $\partial/\partial\varphi$.
b. It is invariant under complex conjugation, $\mathcal{W} = \mathcal{W}^*$, $\Phi = \Phi^*$.
c. It keeps its form under Lorentz rotations in the x^0, x^3 plane, whose effect on (3.10) is to Lorentz-transform the components of the (co)vector σ_α:

$$\sigma_0 \to \sigma_0\cosh B - \sigma_3\sinh B, \qquad \sigma_3 \to \sigma_3\cosh B - \sigma_0\sinh B, \tag{3.11}$$

where B is the boost parameter. The norm $\sigma^2 \equiv \sigma_3^2 - \sigma_0^2$ is Lorentz-invariant, and we shall call the vortex magnetic if $\sigma^2 > 0$, electric for $\sigma^2 < 0$, and chiral if $\sigma^2 = 0$ [4].

d. It keeps its form under gauge transformations (2.4) generated by $\mathrm{U} = \exp\{-\frac{i}{2}\Gamma\tau^2\}$ with constant Γ, whose effect is to rotate the field amplitudes,

$$f_1 \to f_1 \cos\frac{\Gamma}{2} - f_2 \sin\frac{\Gamma}{2}, \qquad f_2 \to f_2 \cos\frac{\Gamma}{2} + f_1 \sin\frac{\Gamma}{2},$$
$$u_1 \to u_1 \cos\Gamma + u_3 \sin\Gamma, \qquad u_3 \to u_3 \cos\Gamma - u_1 \sin\Gamma,$$
$$v_1 \to v_1 \cos\Gamma + v_3 \sin\Gamma, \qquad v_3 \to v_3 \cos\Gamma - v_1 \sin\Gamma, \tag{3.12}$$

whereas u, v rest invariant. These transformations can also be written in a compact form using the complex variables:

$$(f_1 + i f_2) \to e^{\frac{i}{2}\Gamma}(f_1 + i f_2), \qquad (u_1 + i u_3) \to e^{-i\Gamma}(u_1 + i u_3),$$
$$(v_1 + i v_3) \to e^{-i\Gamma}(v_1 + i v_3). \tag{3.13}$$

This symmetry can be fixed by requiring that

$$v_1(0) = 0. \tag{3.14}$$

e. The ansatz does not change if we multiply the amplitudes u, u_1, u_3 by a constant dividing at the same time σ_α by the same constant. To fix this symmetry, we impose the condition

$$u_3(0) = 1, \tag{3.15}$$

which is possible if $u_3(0) \neq 0$.

It should be said that the gauge (3.10) is very convenient for calculations, since everything depends only on ρ, but it is not completely satisfactory, because, as we shall see below, the functions v, v_1, v_3 do not vanish at $\rho = 0$ and so the vector fields are not globally defined in this gauge. This problem can be cured by passing to another gauge (Eq. (4.4)).

3.2. *Field equations*

With the parametrization (3.10) the U(1) equations (2.7) reduce to

$$\frac{1}{\rho}(\rho u')' = \frac{g'^2}{2}\left\{(u + u_3) f_1^2 + 2 u_1 f_1 f_2 + (u - u_3) f_2^2\right\}, \tag{3.16}$$

$$\rho\left(\frac{v'}{\rho}\right)' = \frac{g'^2}{2}\left\{(v + v_3) f_1^2 + 2 v_1 f_1 f_2 + (v - v_3) f_2^2\right\}, \tag{3.17}$$

where the prime denotes differentiation with respect to ρ. The Higgs equations (2.9) become

$$\frac{1}{\rho}(\rho f_1')' = \left\{\frac{\sigma^2}{4}\left[(u + u_3)^2 + u_1^2\right] + \frac{1}{4\rho^2}\left[(v + v_3)^2 + v_1^2\right] + \frac{\beta}{4}\left(f_1^2 + f_2^2 - 1\right)\right\} f_1$$
$$+ \left(\frac{\sigma^2}{2} u u_1 + \frac{1}{2\rho^2} v v_1\right) f_2, \tag{3.18}$$

$$\frac{1}{\rho}(\rho f_2')' = \left\{\frac{\sigma^2}{4}\left[(u - u_3)^2 + u_1^2\right] + \frac{1}{4\rho^2}\left[(v - v_3)^2 + v_1^2\right] + \frac{\beta}{4}\left(f_1^2 + f_2^2 - 1\right)\right\} f_2$$
$$+ \left(\frac{\sigma^2}{2} u u_1 + \frac{1}{2\rho^2} v v_1\right) f_1. \tag{3.19}$$

The Yang–Mills equations (2.8) reduce to

$$\frac{1}{\rho}\left(\rho u_1'\right)' = -\frac{1}{\rho^2}(v_1 u_3 - v_3 u_1) v_3 + \frac{g^2}{2}\left[u_1\left(f_1^2 + f_2^2\right) + 2u f_1 f_2\right], \tag{3.20}$$

$$\frac{1}{\rho}\left(\rho u_3'\right)' = +\frac{1}{\rho^2}(v_1 u_3 - v_3 u_1) v_1 + \frac{g^2}{2}\left[(u_3 + u) f_1^2 + (u_3 - u) f_2^2\right], \tag{3.21}$$

$$\rho\left(\frac{v_1'}{\rho}\right)' = +\sigma^2(v_1 u_3 - v_3 u_1) u_3 + \frac{g^2}{2}\left[v_1\left(f_1^2 + f_2^2\right) + 2v f_1 f_2\right], \tag{3.22}$$

$$\rho\left(\frac{v_3'}{\rho}\right)' = -\sigma^2(v_1 u_3 - v_3 u_1) u_1 + \frac{g^2}{2}\left[(v_3 + v) f_1^2 + (v_3 - v) f_2^2\right]. \tag{3.23}$$

In addition, a careful inspection reveals that, although one has $\mathrm{W}_\rho^2 = 0$, the Yang–Mills equation (2.8) with $a = 2$ and $\nu = \rho$ is not satisfied identically but gives the condition

$$\Lambda = 0, \tag{3.24}$$

where

$$\Lambda \equiv \sigma^2\left(u_1 u_3' - u_3 u_1'\right) + \frac{1}{\rho^2}\left(v_1 v_3' - v_3 v_1'\right) - g^2\left(f_1 f_2' - f_2 f_1'\right). \tag{3.25}$$

Differentiating and using Eqs. (3.16)–(3.23) one finds that

$$\Lambda' + \frac{\Lambda}{\rho} = 0 \tag{3.26}$$

so that

$$\Lambda = \frac{C}{\rho}, \tag{3.27}$$

where C is an integration constant. This is a first integral for the above second order equations, so that replacing one of them by Eq. (3.27) would give a completely equivalent system. Eq. (3.24) requires that $C = 0$.

Let us also write down the energy density. Using Eq. (3.2) the total energy is

$$E = \int T_0^0\, d^2x \equiv 2\pi \int_0^\infty \rho(\mathcal{E}_1 + \mathcal{E}_2)\, d\rho \equiv E_1 + E_2 \tag{3.28}$$

with

$$\mathcal{E}_1 = \frac{\sigma_0^2 + \sigma_3^2}{2}\left\{\frac{1}{g'^2}u'^2 + \frac{1}{g^2}\left(u_1'^2 + u_3'^2\right) + \frac{1}{g^2\rho^2}(v_1 u_3 - v_3 u_1)^2 + \frac{1}{2}\left[\left((u + u_3) f_1 + u_1 f_2\right)^2 + \left((u - u_3) f_2 + u_1 f_1\right)^2\right]\right\} \tag{3.29}$$

and

$$\mathcal{E}_2 = \frac{1}{2\rho^2}\left(\frac{1}{g'^2}v'^2 + \frac{1}{g^2}\left(v_1'^2 + v_3'^2\right)\right) + f_1'^2 + f_2'^2 + \frac{1}{4\rho^2}\left[\left((v + v_3) f_1 + v_1 f_2\right)^2 + \left((v - v_3) f_2 + v_1 f_1\right)^2\right] + \frac{\beta}{8}\left(f_1^2 + f_2^2 - 1\right)^2. \tag{3.30}$$

4. Boundary conditions

In order to construct global solutions of Eqs. (3.16)–(3.24) in the interval $\rho \in [0, \infty)$ we shall need their local solutions in the vicinity of the singular points, $\rho = 0$ and $\rho = \infty$. We shall be considering fields that are regular at $\rho = 0$ and approach the vacuum for $\rho \to \infty$. Let us first consider local solutions at small ρ.

4.1. Boundary conditions at the symmetry axis

Expressions (3.29), (3.30) for the energy density can be used to derive the regularity conditions at $\rho = 0$. For regular fields the energy density must be bounded, and so the coefficients in front of the negative powers of ρ in (3.29), (3.30) should vanish at $\rho = 0$. This implies that at $\rho = 0$ one should have $v' = v_1' = v_3' = 0$ and also

$$v_1 u_3 - v_3 u_1 = 0, \qquad (v + v_3) f_1 + v_1 f_2 = 0, \qquad (v - v_3) f_2 + v_1 f_1 = 0. \tag{4.1}$$

In view of (3.14) these conditions reduce to

$$v_3 u_1 = 0, \qquad (v + v_3) f_1 = 0, \qquad (v - v_3) f_2 = 0. \tag{4.2}$$

Let us now remember that the azimuthal components of the vectors should vanish at the symmetry axis for the fields to be defined there. In the gauge (3.10) this would require that $v(0) = v_3(0) = 0$, but imposing such a condition would be much too restrictive. A more general possibility is first to perform a gauge transformation that gives φ-dependent phases to the scalars and shifts the azimuthal components of the vectors, and only then to impose the regularity condition at the axis. Since the two Higgs field components should be single-valued in the new gauge, their phase factors should contain integers. We therefore apply to the ansatz (3.10) the gauge transformation generated by

$$\mathrm{U} = \exp\left\{\frac{i}{2}\left(\eta + \psi \tau^3\right)\right\}, \tag{4.3}$$

with $\eta = (2n - \nu)\varphi + \sigma_\alpha x^\alpha$ and $\psi = \nu\varphi - \sigma_\alpha x^\alpha$ where $n, \nu \in \mathbb{Z}$. This gives

$$\begin{aligned}
\mathcal{W} &= \left\{u(\rho) + 1 + \tau^1_\psi u_1(\rho) + \tau^3\left[u_3(\rho) - 1\right]\right\}\sigma_\alpha\, dx^\alpha \\
&\quad + \left\{2n - \nu - v(\rho) - \tau^1_\psi v_1(\rho) + \tau^3\left[\nu - v_3(\rho)\right]\right\} d\varphi, \\
\Phi &= \begin{bmatrix} e^{in\varphi} f_1(\rho) \\ e^{i(n-\nu)\varphi + i\sigma_\alpha x^\alpha} f_2(\rho) \end{bmatrix}
\end{aligned} \tag{4.4}$$

with $\tau^1_\psi = \mathrm{U}\tau^1\mathrm{U}^{-1} = \tau^1 \cos\psi - \tau^2 \sin\psi$. We can assume without loss of generality that $n \geqslant 1$ and we shall see below that $1 \leqslant \nu \leqslant 2n$. The dependence of the fields on x^α will be convenient in what follows but not essential for the regularity issue.

The azimuthal components of the vectors in (4.4) will vanish at the axis provided that

$$v(0) = 2n - \nu, \qquad v_3(0) = \nu, \tag{4.5}$$

and also if $v_1(0) = 0$, as required by Eq. (3.14). This gives a larger set of the allowed boundary values than in the gauge (3.10). In addition, the regularity of the scalar fields requires that $f_1(0) = 0$ and, unless for $\nu = n$, that $f_2(0) = 0$. The conditions (4.2) then reduce to

$$\nu u_1(0) = 0, \qquad n f_1(0) = 0, \qquad (n - \nu) f_2(0) = 0. \tag{4.6}$$

Summarizing, the boundary conditions at $\rho = 0$ are given by

$$u_1(0) = 0, \qquad u_3(0) = 1, \qquad v_1(0) = 0, \qquad v_3(0) = \nu,$$
$$v(0) = 2n - \nu, \qquad f_1(0) = 0, \tag{4.7}$$

while $u(0)$ and $f_2(0)$ can be arbitrary if $\nu = n$, whereas for $\nu \neq n$ one should have $f_2(0) = 0$.

We can now work out the most general local series solutions of the equations for small ρ. The corresponding analysis is described in Appendix A, the result is

$$u = a_1 + \cdots, \qquad u_1 = a_2\rho^\nu + \cdots, \qquad u_3 = 1 + \cdots,$$
$$v = 2n - \nu + a_4\rho^2 + \cdots, \qquad v_1 = O(\rho^{\nu+2}), \qquad v_3 = \nu + a_3\rho^2 + \cdots,$$
$$f_1 = a_5\rho^n + \cdots, \qquad f_2 = q\rho^{|n-\nu|} + \cdots, \tag{4.8}$$

where a_1, a_2, a_3, a_4, a_5 and q are six integration constants. The dots here stand for the subleading term. As explained in Appendix A, all such terms will be taken into account in our numerical scheme.

4.2. Boundary conditions at infinity

We want the fields for $\rho \to \infty$ to approach vacuum configurations with zero energy density. The energy density $\mathcal{E}_1 + \mathcal{E}_2$ is given by (3.29), (3.30) and can be decomposed into the 'kinetic energy' containing the derivatives and the 'potential energy' that contains the rest. The potential energy is a sum of perfect squares, and it will vanish if only each term in the sum vanishes. For example, the last term in (3.30) will vanish if only $f_1^2 + f_2^2 = 1$ and so

$$f_1(\infty) = \cos\frac{\gamma}{2}, \qquad f_2(\infty) = \sin\frac{\gamma}{2}. \tag{4.9}$$

The value of γ is an essential parameter, let us call it vacuum angle. Notice that the global symmetry (3.12) acts as $\gamma \to \gamma + \Gamma$, but this symmetry has already been fixed by the condition (3.14) at the origin.

At the same time, just for discussing the local solutions at large ρ, it is convenient to temporarily set $\gamma = 0$ to simplify the analysis, since afterwords one can apply the transformation (3.12) to restore the generic value of γ. We therefore assume for the time being that at infinity one has $f_1 = 1$, $f_2 = 0$. With this, the potential energy part of Eqs. (3.29), (3.30) will vanish if only $u_1 = v_1 = 0$ and $u = -u_3$, $v = -v_3$. Under these conditions the field equations (3.16)–(3.23) reduce to $(\rho u')' = 0$, $(v'/\rho)' = 0$, which implies that $u = c_1 + Q\ln(\rho)$ and $v = c_2 + A\rho^2$ where c_1, c_2, Q, A are integration constants. One should set $A = 0$ since otherwise the kinetic part of the energy density diverges at infinity. However, one should keep the term $Q\ln(\rho)$, since the energy density approaches zero for $\rho \to \infty$ even if $Q \neq 0$. The conclusion is that for large ρ the fields approach the following exact solution of the field equations,

$$\mathcal{W} = (1 - \tau^3)\big((c_1 + Q\ln\rho)\sigma_\alpha\,dx^\alpha - c_2\,d\varphi\big), \qquad \Phi = \begin{bmatrix}1\\0\end{bmatrix}. \tag{4.10}$$

For this solutions the electromagnetic and Z field 2-forms (2.12) are closed and admit potentials

$$A_\mu\,dx^\mu = \frac{1}{gg'}\big((c_1 + Q\ln\rho)\sigma_\alpha\,dx^\alpha - c_2\,d\varphi\big), \qquad Z_\mu = 0. \tag{4.11}$$

This is the electromagnetic Biot–Savart solution describing fields outside an uniformly charged electric wire. The charge and current are obtained with (3.3),

$$I_\alpha = \int \partial^\nu F_{\nu\alpha}\, d^2x = -\oint \epsilon_{ks}\partial_k A_\alpha\, dx^s = -2\pi \frac{Q}{gg'}\sigma_\alpha, \tag{4.12}$$

assuming that the charge density is smooth inside the wire so that the Gauss theorem applies. In addition, there is also the magnetic flux along the wire,

$$\Psi_F = \oint A_k\, dx^k = -\frac{2\pi}{gg'} c_2. \tag{4.13}$$

In what follows we shall construct smooth vortex solutions in which the wire is represented by a regular distribution of massive non-linear fields, while in the far field zone the massive modes die out and everything reduces to the Biot–Savart configuration (4.10). The energy density at large ρ will then be proportional to Q^2/ρ^2 and the total energy per unit vortex length will be logarithmically divergent, just as for the ordinary infinitely long electric wire. The energy of finite vortex pieces, as for example vortex loops, will be finite.

Let us now consider the deviations from the electromagnetic configuration (4.10) within the full system of Eqs. (3.16)–(3.23) in order to determine how the solutions approach their asymptotic form. The corresponding analysis in the linear approximation is described in Appendix B. Applying finally the phase rotation (3.12) to restore the generic value of γ in Eq. (4.9), gives the large ρ behaviour of the solutions,

$$\begin{aligned}
&u = Q\ln\rho + c_1 + \frac{c_3 g'^2}{\sqrt{\rho}} e^{-m_Z\rho} + \cdots, \\
&v = c_2 + c_4 g'^2 \sqrt{\rho} e^{-m_Z\rho} + \cdots, \\
&u_1 + iu_3 = e^{-i\gamma}\left\{\frac{c_7}{\sqrt{\rho}} e^{-\int m_\sigma\, d\rho} + i\left[-Q\ln\rho - c_1 + \frac{c_3 g^2}{\sqrt{\rho}} e^{-m_Z\rho}\right]\right\} + \cdots, \\
&v_1 + iv_3 = e^{-i\gamma}\left\{c_8\sqrt{\rho} e^{-\int m_\sigma\, d\rho} + i\left[-c_2 + c_4 g^2\sqrt{\rho} e^{-m_Z\rho}\right]\right\} + \cdots, \\
&f_1 + if_2 = e^{\frac{i}{2}\gamma}\left\{1 + \frac{c_5}{\sqrt{\rho}} e^{-m_H\rho} + i\frac{c_6}{\sqrt{\rho}} e^{-\int m_\sigma\, d\rho}\right\} + \cdots.
\end{aligned} \tag{4.14}$$

These local solutions contain 10 independent parameters $c_1, \ldots, c_8$, γ, Q and they approach (modulo the γ-phases) the Abelian configuration (4.10) exponentially fast as $\rho \to \infty$, with the rates determined by the masses m_Z, m_H and m_σ. Here

$$m_\sigma = \sqrt{m_W^2 + \sigma^2(Q\ln\rho + c_1)^2} \tag{4.15}$$

reduces to the W boson mass $m_W = g/\sqrt{2}$ if $\sigma^2 = 0$. For $\sigma^2 \neq 0$ this could be viewed as the W boson mass 'screened' ($\sigma^2 < 0$) or 'dressed' ($\sigma^2 > 0$) by the interaction of charged W bosons with the long-ranged Biot–Savart field (4.10). We also note that the solutions (4.14) satisfy the field equations (3.16)–(3.23) but not the constraint (3.24). In fact, the latter has already been imposed on the local solutions (4.8) at *small* ρ, and since it 'propagates', it will be automatically enforced by extending the local solution (4.8) to large values of ρ.

We can now outline our strategy for solving the field equations. The local solutions at small and large ρ are given, respectively, by Eqs. (4.8) and (4.14). Within the numerical shooting to the fitting point method [31] we extend these asymptotic solutions to the intermediate values of ρ and match them. We use the multi-zone version of the method [32], with many zones whose number and sizes are adjusted to make sure that for large ρ the asymptotic solution (4.14) is a good approximation. In order to illustrate the procedure, let us describe the simplest case where there are only two zones and one matching point. To match the solutions at this point we have to fulfil the 16 matching conditions for the 8 field amplitudes and for their first derivatives via adjusting the free parameters in (4.8), (4.14). It is therefore essential to have enough of parameters, since otherwise the matching conditions would be incompatible. Counting the parameters we discover that there are 17 of them: 6 integration constants $a_1, \dots, a_5$ and q in (4.8), then 10 integration constants $c_1, \dots, c_8$, Q, γ in (4.14), and finally σ^2. This is enough to fulfill the 16 matching conditions and to have one parameter left free after the matching. This parameter will label the resulting global solutions in the interval $\rho \in [0, \infty)$. It is convenient to choose it to be q – the coefficient in front of the amplitude f_2 in (4.8).

Before integrating the equations, some preliminary considerations are in order.

5. Conserved quantities and the known solutions

Keeping only the leading terms, the boundary conditions for the field amplitudes for $0 \leftarrow \rho \rightarrow \infty$ are

$$\begin{aligned}
&a_1 \leftarrow u \rightarrow c_1 + Q\ln\rho, \qquad 2n-\nu \leftarrow v \rightarrow c_2,\\
&0 \leftarrow u_1 \rightarrow -(c_1 + Q\ln\rho)\sin\gamma, \qquad 0 \leftarrow v_1 \rightarrow -c_2\sin\gamma,\\
&1 \leftarrow u_3 \rightarrow -(c_1 + Q\ln\rho)\cos\gamma, \qquad \nu \leftarrow v_3 \rightarrow -c_2\cos\gamma,\\
&a_5\rho^n \leftarrow f_1 \rightarrow \cos\frac{\gamma}{2}, \qquad q\rho^{|n-\nu|} \leftarrow f_2 \rightarrow \sin\frac{\gamma}{2}.
\end{aligned} \tag{5.1}$$

The knowledge of these boundary conditions is sufficient to calculate some of the integral parameters associated to the solutions.

5.1. Fluxes and current

Since the fluxes and currents are gauge invariant, one can compute them in the gauge (3.10). The non-zero components of the field tensors read

$$\begin{aligned}
&B_{\rho\alpha} = \sigma_\alpha u', \qquad B_{\rho\varphi} = -v', \qquad \mathrm{W}^a_{\rho\alpha} = \sigma_\alpha u'_a,\\
&\mathrm{W}^a_{\rho\varphi} = -v'_a, \qquad \mathrm{W}^2_{\alpha\varphi} = (u_1 v_3 - u_3 v_1)\sigma_\alpha
\end{aligned} \tag{5.2}$$

where $a = 1, 3$. Introducing the function $\Omega(\rho)$ defined by

$$\cos\Omega = \frac{f_1^2 - f_2^2}{f_1^2 + f_2^2}, \qquad \sin\Omega = \frac{2f_1 f_2}{f_1^2 + f_2^2} \tag{5.3}$$

such that $\Omega(\infty) = \gamma$, one can define three orthogonal unit isovectors

$$n^a = (\sin\Omega, 0, \cos\Omega), \qquad k^a = (\cos\Omega, 0, -\sin\Omega), \qquad l^a = (0, 1, 0). \tag{5.4}$$

Here n^a corresponds to the definition in (2.12). The electromagnetic and Z fields read

$$\begin{aligned}
&F_{\rho\alpha} = \left(\frac{g}{g'}u' - \frac{g'}{g}\big(u'_1\sin\Omega + u'_3\cos\Omega\big)\right)\sigma_\alpha,\\
&F_{\rho\varphi} = -\frac{g}{g'}v' + \frac{g'}{g}\big(v'_1\sin\Omega + v'_3\cos\Omega\big),\\
&Z_{\rho\alpha} = \big(u' + u'_1\sin\Omega + u'_3\cos\Omega\big)\sigma_\alpha, \qquad Z_{\rho\varphi} = -v' - v'_1\sin\Omega - v'_3\cos\Omega.
\end{aligned} \tag{5.5}$$

The electromagnetic current density is $J^\alpha = \partial_\mu F^{\mu\alpha} = \frac{1}{\rho}(\rho F^{\rho\alpha})'$ integrating which over the x^1, x^2 plane gives

$$I_\alpha = -2\pi \int_0^\infty (\rho F_{\rho\alpha})'\, d\rho = -\frac{2\pi Q}{gg'}\sigma_\alpha. \tag{5.6}$$

For most of the solutions considered below the vector $\sigma_\alpha = (\sigma_0, \sigma_3)$ is spacelike, so that there is the rest frame where $\sigma_\alpha = \sigma\delta^3_\alpha$. The restframe value of the current is $I_\alpha = \delta^3_\alpha \mathcal{I}$ with $\mathcal{I} = -2\pi Q\sigma/(gg')$. The restframe components of the electromagnetic and Z field strengths are

$$B_{\hat{z}} = \frac{1}{\rho}F_{\rho\varphi}, \qquad B_{\hat{\varphi}} = -F_{\rho z}, \qquad H_{\hat{z}} = \frac{1}{\rho}Z_{\rho\varphi}, \qquad H_{\hat{\varphi}} = -Z_{\rho z}, \tag{5.7}$$

such that the magnetic fluxes are

$$\Psi_F = 2\pi \int_0^\infty \rho B_{\hat{z}}\, d\rho, \qquad \Psi_Z = 2\pi \int_0^\infty \rho H_{\hat{z}}\, d\rho. \tag{5.8}$$

One can also define the W-condensate components by projecting the restframe values of $\mathrm{W}^a_{\mu\nu}$ onto the unit isovectors k^a, l^a defined in (5.4),

$$w_{\hat{z}} = \frac{1}{\rho}k^a \mathrm{W}^a_{\rho\varphi}, \qquad w_{\hat{\varphi}} = -k^a \mathrm{W}^a_{\rho z}, \qquad w_{\hat{\rho}} = -\frac{1}{\rho}l^a \mathrm{W}^a_{z\varphi}. \tag{5.9}$$

5.2. Energy, momentum, angular momentum

Using the field equations one finds that the $\mathcal{E}_1$ part of the energy density (3.29) is actually a total derivative,

$$\rho\mathcal{E}_1 = \frac{\sigma_0^2 + \sigma_3^2}{2}\left(\frac{\rho}{g'^2}uu' + \frac{\rho}{g^2}\big(u_1u_1' + u_3u_3'\big)\right)', \tag{5.10}$$

so that using the boundary conditions (5.1) the energy is

$$E = \pi\frac{\sigma_0^2 + \sigma_3^2}{g^2g'^2}Qu(\infty) + 2\pi\int_0^\infty \rho\mathcal{E}_2\, d\rho \equiv E_1 + E_2. \tag{5.11}$$

A direct calculation shows that the momentum and angular momentum densities are also total derivatives [33],

$$\rho T^0_3 = \sigma^0\sigma_3\left(\frac{\rho}{g'^2}uu' + \frac{\rho}{g^2}\big(u_1u_1' + u_3u_3'\big)\right)',$$
$$\rho T^0_\varphi = \sigma^0\left(\frac{\rho}{g'^2}vu' + \frac{\rho}{g'^2}\big(v_1u_1' + v_3u_3'\big)\right)' \tag{5.12}$$

so that

$$P = \int T^0_3\, d^2x = 2\pi\frac{\sigma_0\sigma_3}{g^2g'^2}Qu(\infty),$$
$$M = \int T^0_\varphi\, d^2x = 2\pi\frac{\sigma_0}{g^2g'^2}Qv(\infty) = \frac{2\pi Q\sigma_0c_2}{g^2g'^2}. \tag{5.13}$$

5.3. Z and W strings

The only known solutions of Eqs. (3.16)–(3.24) for generic values of g, g' are the embedded ANO vortices. These exist in two different versions, called Z strings [8] and W strings [34], corresponding to two non-equivalent embeddings of U(1) to SU(2) × U(1).

Z strings are obtained by setting

$$\begin{aligned} &u=-1, \qquad v=2g'^2(v_{\rm ANO}-n)+2n-\nu\equiv v_{\rm Z}, \qquad u_1=0, \qquad u_3=1, \\ &v_1=0, \qquad v_3=2g^2(v_{\rm ANO}-n)+\nu\equiv v_{\rm Z3}, \qquad f_1=f_{\rm ANO}\equiv f_{\rm Z}, \qquad f_2=0 \end{aligned} \tag{5.14}$$

after which Eqs. (3.16)–(3.24) reduce to the ANO system

$$\begin{aligned} &\frac{1}{\rho}(\rho f'_{\rm ANO})'=\left(\frac{v^2_{\rm ANO}}{\rho^2}+\frac{\beta}{4}(f^2_{\rm ANO}-1)\right)f_{\rm ANO}, \\ &\rho\left(\frac{v'_{\rm ANO}}{\rho}\right)'=\frac{1}{2}f^2_{\rm ANO}v_{\rm ANO}, \end{aligned} \tag{5.15}$$

whose solutions satisfy the boundary conditions $0\leftarrow f_{\rm ANO}\rightarrow 1$, $n\leftarrow v_{\rm ANO}\rightarrow 0$ for $0\leftarrow \rho\rightarrow\infty$. The fluxes are $\Psi_Z=4\pi n$, $\Psi_F=0$ while the current, momentum and angular momentum vanish. The dependence of these solutions on the second winding number ν is pure gauge, since in the gauge (4.4) it disappears:

$$\mathcal{W}=2\big(g'^2+g^2\tau^3\big)\big(n-v_{\rm ANO}(\rho)\big)\,d\varphi, \qquad \Phi=\begin{pmatrix} e^{in\varphi}f_{\rm ANO}(\rho) \\ 0 \end{pmatrix}. \tag{5.16}$$

W strings are obtained by choosing $u=v=u_1=u_3=v_1=0$, $v_3=2v_{\rm ANO}(\rho)$, $f_1=-f_2=\frac{1}{\sqrt{2}}f_{\rm ANO}(\rho)$. Their fluxes are not the same as for Z strings, $\Psi_Z=\frac{4\pi n}{\sqrt{2}}$, $\Psi_F=-\frac{4\pi n}{\sqrt{2}}\frac{g'}{g}$, so that these solutions are physically different.

6. Small current limit – bound states around Z strings

Our goal is to construct solutions of Eqs. (3.16)–(3.24) more general that the embedded ANO vortices. Our strategy was briefly summarized above: we numerically extend the local solutions (4.8) and (4.14) to the intermediate region and impose there the 16 matching conditions, which can be fulfilled by adjusting the 17 free parameters in the local solutions. This leaves one extra parameter, q, which determines the value of the lower component of the Higgs field at the origin.

The matching conditions are resolved iteratively, within the standard method described in [32]. A good choice of the initial configuration is important for this, since otherwise the iterations will not converge. The idea then is to start in the vicinity of an already known solution, for which we can only choose the Z string (it is unclear whether W strings can also be used). Let us therefore choose the Z string solution (5.14) as the starting point of our analysis. One has in this case $f_2(\rho)=0$ so that $q=0$.

Suppose now that $0<q\ll 1$. It is then natural to expect the solution to be a slightly deformed Z string. It can therefore be represented in the form

$$\begin{aligned} &u=-1+\delta u, \qquad v=v_{\rm Z}+\delta v, \qquad u_1=\delta u_1, \qquad u_3=1+\delta u_3, \\ &v_1=\delta v_1, \qquad v_3=v_{\rm Z3}+\delta v_3, \qquad f_1=f_{\rm Z}+\delta f_1, \qquad f_2=\delta f_2, \end{aligned} \tag{6.1}$$

where $\delta u,\ldots,\delta f_2$ are small deformations. Inserting this to Eqs. (3.16)–(3.23) and linearising with respect to the deformations, one can consistently set $\delta u=\delta u_3=\delta f_1=\delta v=\delta v_3=0$.

The remaining three equations describe a slightly deformed Z string: $\mathcal{W} + \delta\mathcal{W}$, $\Phi + \delta\Phi$, where $\mathcal{W}$ and Φ are given by Eq. (5.16) and

$$\delta\mathcal{W} = \frac{\tau^+}{2} e^{i\psi}\big[\delta u_1(\rho)\sigma_\alpha\, dx^\alpha - \delta v_1(\rho)\, d\varphi\big] + \text{h.c.}, \qquad \delta\Phi = \begin{pmatrix} 0 \\ e^{i(n\varphi-\psi)}\delta f_2(\rho) \end{pmatrix}, \tag{6.2}$$

with $\psi = \nu\varphi - \sigma_\alpha x^\alpha$ and $\tau^+ = \tau^1 + i\tau^2$. Here h.c. stands for Hermitian conjugation. The amplitudes δf_2, δv_1, δu_1 fulfil the equations

$$\frac{1}{\rho}\big(\rho\delta f_2'\big)' = \left(\frac{(v_Z - v_{Z3})^2}{4\rho^2} + \frac{\beta}{4}\big(f_Z^2 - 1\big) + \sigma^2\right)\delta f_2 + \frac{v_Z f_Z}{2\rho^2}\delta v_1 - \frac{\sigma^2}{2} f_Z \delta u_1, \tag{6.3a}$$

$$\frac{1}{\rho}\big(\rho\delta u_1'\big)' = \left(\frac{v_{Z3}^2}{\rho^2} + \frac{g^2}{2} f_Z^2\right)\delta u_1 - \frac{v_{Z3}}{\rho^2}\delta v_1 - g^2 f_Z \delta f_2, \tag{6.3b}$$

$$\rho\left(\frac{\delta v_1'}{\rho}\right)' = \left(\frac{g^2}{2} f_Z^2 + \sigma^2\right)\delta v_1 - \sigma^2 v_{Z3}\delta u_1 + g^2 v_Z f_Z \delta f_2. \tag{6.3c}$$

By suitably redefining the variables one can rewrite these equations in the form of a spectral problem

$$\Psi'' = \big(\sigma^2 + V[\beta, \theta_W, n, \nu, \rho]\big)\Psi, \tag{6.4}$$

where Ψ is a 3 component vector and V is a symmetric potential energy matrix depending on the parameters of the background Z string solution, β, θ_W, n, and also on ν. Although the dependence on ν for the Z string background is pure gauge, this is not so for the deformations of this background, since for different values of ν in (6.3) one obtains different results.

We are interested in localized, bound state solutions of Eqs. (6.3). These equations admit a global symmetry

$$\delta f_2 \to \delta f_2 + \frac{\Gamma}{2} f_Z, \qquad \delta u_1 \to \delta u_1 + \Gamma, \qquad \delta v_1 \to \delta v_1 + \Gamma v_{Z3}, \tag{6.5}$$

which is merely the linearised version of (3.12). Using this symmetry one can impose the gauge condition $\delta f_2(\infty) = 0$ and then the local behaviour of solutions at large ρ can be read off from Eq. (4.14),

$$\delta u_1 = \frac{c_7}{\sqrt{\rho}} e^{-m_\sigma \rho} + \cdots, \qquad \delta v_1 = c_8\sqrt{\rho}\, e^{-m_\sigma \rho} + \cdots,$$
$$\delta f_2 = \frac{c_6}{\sqrt{\rho}} e^{-m_\sigma \rho} + \cdots. \tag{6.6}$$

Here $m_\sigma^2 = m_W^2 + \sigma^2$ while the logarithm present in Eq. (4.15) does not appear in the linear theory. The local behaviour at small ρ can be obtained from Eq. (4.8),

$$\delta u_1 = a_4\rho^\nu + \Gamma + \cdots, \qquad \delta v_1 = \Gamma v_3 + \cdots, \qquad \delta f_2 = q\rho^{|n-\nu|} + \frac{\Gamma}{2} f_Z + \cdots, \tag{6.7}$$

where we have included the free parameter Γ, since the symmetry (6.5) is now fixed at large ρ and not at $\rho = 0$. The gauge condition $\delta f_2(\infty) = 0$ generically implies that $\delta v_1(0) \neq 0$, which does not agree with the previously adopted in (3.14) condition $v_1(0) = 0$, but the advantage is that the field deformations taken in this gauge are clearly localized around the string core (see Fig. 1). One can use the symmetry (6.5) to return to the gauge where $\delta v_1(0) = 0$, but then the amplitudes δu_1, δv_1, δf_2 will not vanish at infinity.

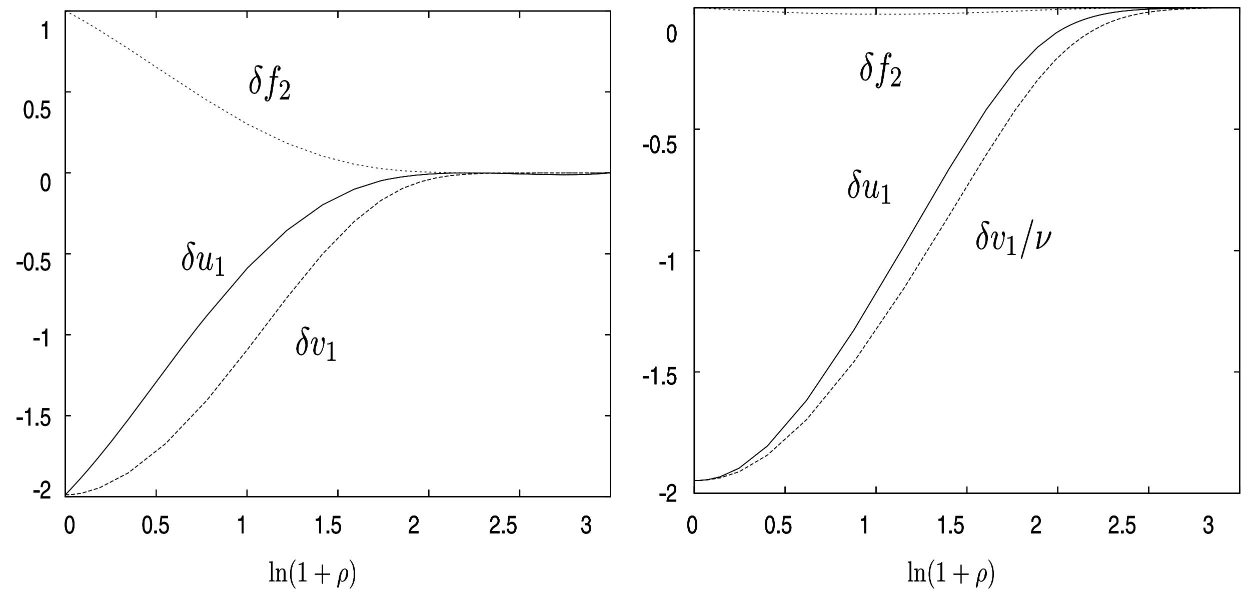

Fig. 1. The bound state solutions of Eqs. (6.3) with $\sin^2\theta_{\mathrm{W}} = 0.23$, $\beta = 2$ for $n = \nu = 1$, $\sigma^2 = 0.5036$ (left) and for $n = 1$, $\nu = 2$, $\sigma^2 = -0.0876$ (right).

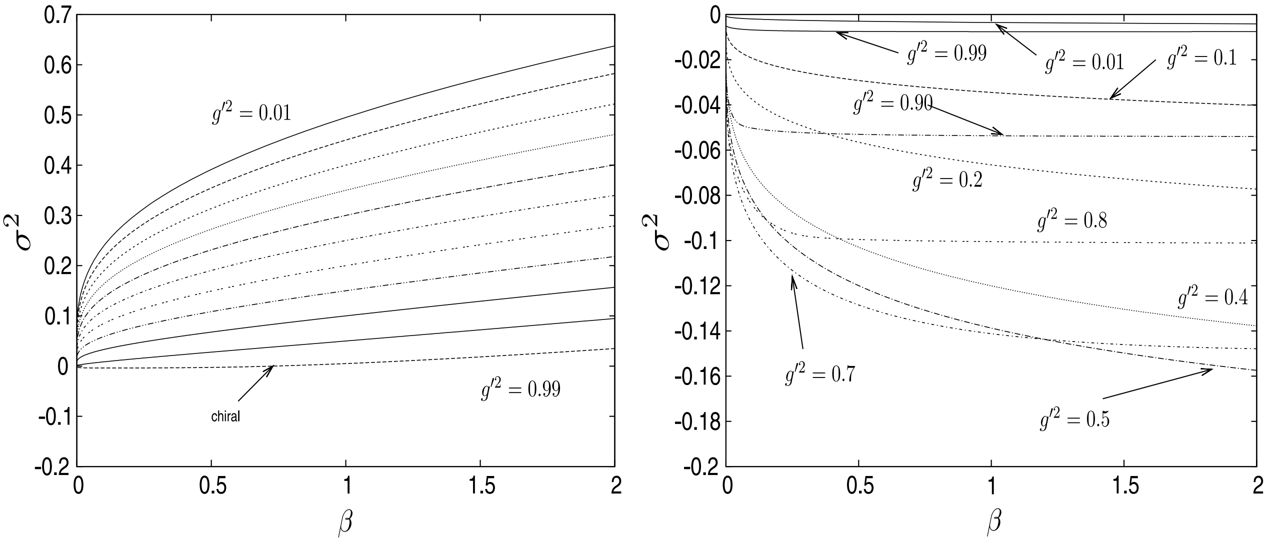

Fig. 2. The eigenvalue σ^2 of the spectral problem (6.3) against β for $n = \nu = 1$ (left) and for $n = 1$, $\nu = 2$ (right) for several values of $g'^2 = \sin^2\theta_{\mathrm{W}}$.

The local solutions (6.6), (6.7) contain 6 free parameters: $c_5, c_6, c_8, a_4, \Gamma, q$. One of them can be absorbed in the overall normalization and so there remain only 5, but together with σ^2 their number is again 6, which is just enough to fulfil the 6 matching conditions to construct global solutions of three second order equations (6.3). The solutions obtained are characterized by the following properties.

For given β and for small enough $\sin\theta_{\mathrm{W}}$ one finds $2n$ different bound states labeled by $\nu = 1, 2, \ldots, 2n$ with the eigenvalue $\sigma^2 = \sigma^2(\beta, \theta_{\mathrm{W}}, n, \nu)$ (see Fig. 1). These solutions can be interpreted as small deformations of Z strings by a current $I_\alpha \sim \sigma_\alpha$, although the current itself appears only in the next order of perturbation theory. If $\beta > 1$ then n of these solutions always have $\sigma^2 > 0$. For $n - 1$ solutions the eigenvalue σ^2 changes sign when θ_{W} increases, and there is

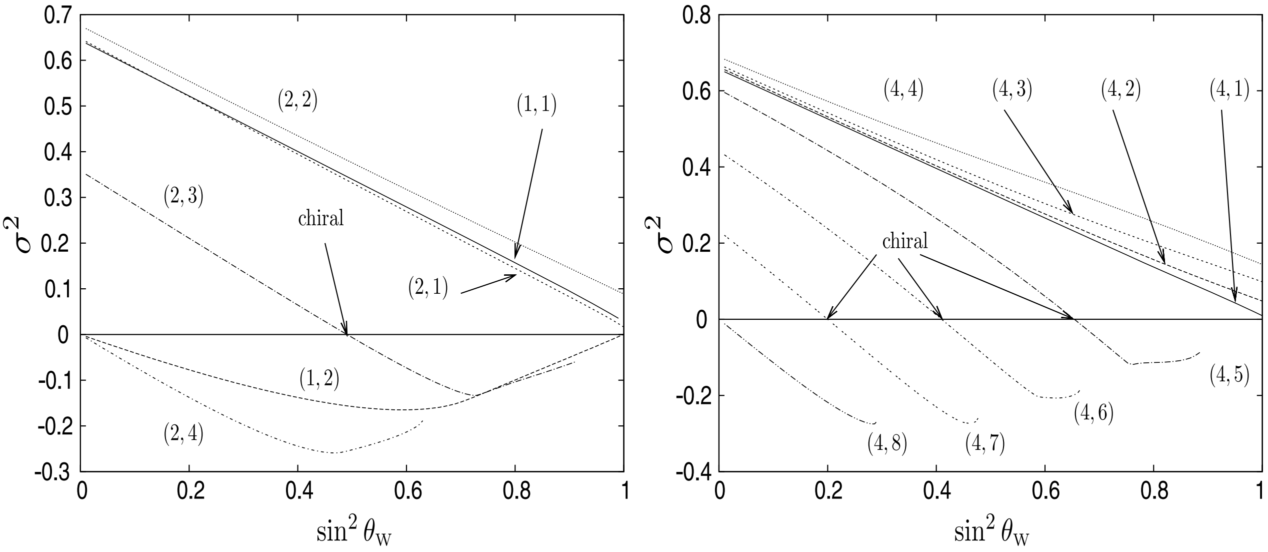

Fig. 3. The eigenvalue σ^2 of the spectral problem (6.3) against $\sin^2\theta_{\rm W}$ for $\beta = 2$. The values of n and ν are shown as (n,ν). In the $\sigma^2 < 0$ region the curves terminate when the condition (6.8) is violated. The chiral solutions with $\sigma^2 = 0$ are possible only for special values of $\theta_{\rm W}$.

one solution with $\sigma^2 < 0$ for any $\theta_{\rm W} > 0$ (see Figs. 2, 3). If σ^2 is negative then it cannot be too large, since solutions are localized if only $m_\sigma^2 = m_{\rm W}^2 + \sigma^2 > 0$ so that

$$\sigma^2 > -m_{\rm W}^2. \tag{6.8}$$

Every Z string admits therefore 'magnetic' ($\sigma^2 > 0$) and, unless $\theta_{\rm W}$ is close to $\pi/2$, 'electric' ($\sigma^2 < 0$) linear deformations. The 'chiral' ($\sigma^2 = 0$) deformations are not generic and possible only for special values of β, $\theta_{\rm W}$ such that

$$\sigma^2(\beta, \theta_{\rm W}, n, \nu) = 0. \tag{6.9}$$

This condition determines a set of curves in the β, $\theta_{\rm W}$ plane (see Fig. 4), let us call them chiral curves. They coincide with the curves delimiting the parameter regions of Z string stability [9]. This can be explained by noting that a simple reinterpretation of the above considerations allows us to reproduce the results of the Z string stability analysis [9]. Let

$$(\delta\mathcal{W}, \delta\Phi) \sim \exp\bigl\{\pm i\bigl(\sigma_0 x^0 + \sigma_3 x^3\bigr)\bigr\} \tag{6.10}$$

be the Z string perturbation eigenvector (6.2) for an eigenvalue $\sigma^2 = \sigma_3^2 - \sigma_0^2$. One has

$$\sigma_0 = \sqrt{\sigma_3^2 - \sigma^2} \tag{6.11}$$

from where it follows that if $\sigma^2 > 0$ then choosing $\sigma_3 < \sigma$ gives exponentially growing in time solutions, that is unstable modes, since the frequency σ_0 is then imaginary, while choosing $\sigma_3 > \sigma$ (as was done above) gives stationary Z string deformations, since σ_0 is then real. If $\sigma^2 < 0$ then σ_0 is always real. The sign of σ^2 therefore determines if the negative modes exist or not. As a result, the $\sigma^2(\beta, \theta_{\rm W}, n, \nu) = 0$ curves separate the parameter regions in the $\beta, \theta_{\rm W}$ plane in which the ν-th deformation of the n-th Z string is of the electric/magnetic type, and at the same time the regions where the ν-th Z string perturbation mode is of stable/unstable type.

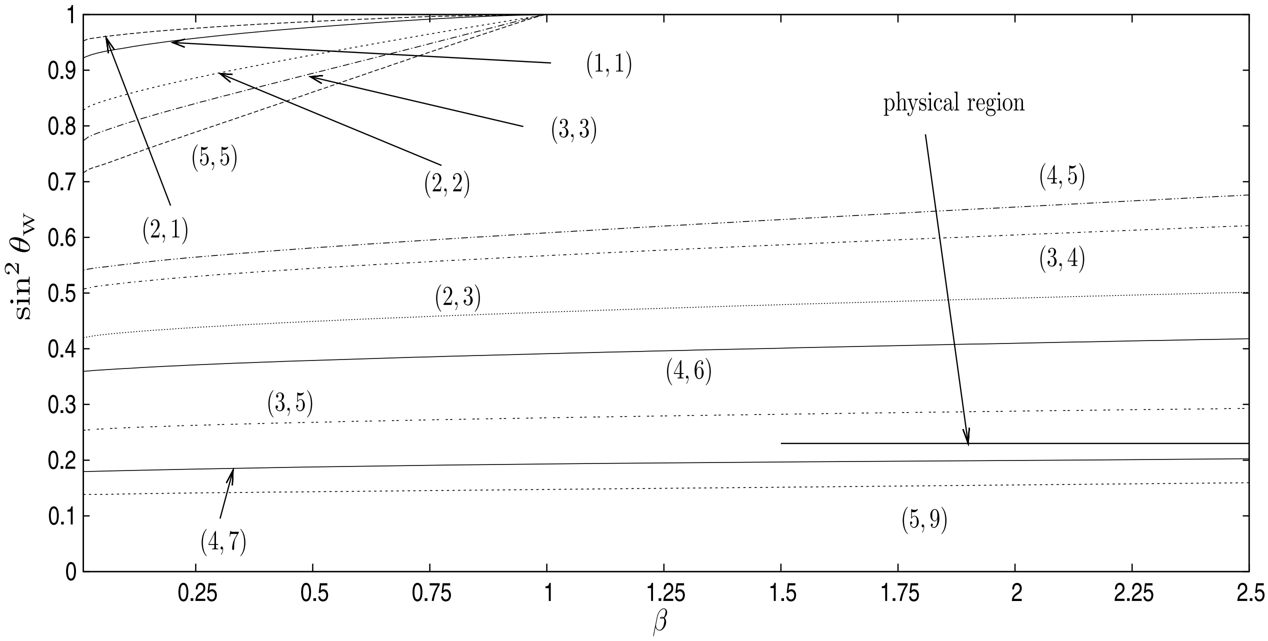

Fig. 4. The $\sigma^2(\beta, \theta_W, n, \nu) = 0$ curves for several values of (n, ν). Curves with $\nu \leqslant n$ are confined in the $\beta \leqslant 1$ region, while those with $\nu > n$ extend to larger values of β.

7. Generic superconducting vortices

Having constructed the slightly deformed Z strings in the linear theory, we can promote them to solutions of the full system of Eqs. (3.16)–(3.23), first for small q. In the fully non-linear theory all 8 field amplitudes deviate from their Z string values, and not only u_1, v_1, f_2. However, if the deviations δu_1, δv_1, δf_2 scale as q for $q \ll 1$, those for the remaining 5 amplitudes scale as q^2.

Starting from small values of q, we increase q iteratively, thereby obtaining fully non-linear superconducting vortices. Eqs. (3.16)–(3.23) can be viewed in this case as a non-linear boundary value problem with the eigenvalue σ^2. It turns out that only the magnetic ($\sigma^2 > 0$) and chiral ($\sigma^2 = 0$) solutions are compatible with the boundary conditions at infinity. The main difference with the linearised case is that, once all fields amplitudes are taken into account, the logarithmic term appears in the expression (4.15) for the effective mass,

$$m_\sigma^2 = m_W^2 + \sigma^2(Q \ln \rho + c_1)^2. \tag{7.1}$$

This implies that for $\sigma^2 < 0$ one has the 'tachyonic mass' $m_\sigma^2 < 0$ at large ρ, in which case the asymptotic solutions (4.14) oscillate and do not approach the vacuum at infinity (this was not realised in Ref. [23]). The energy then diverges faster than $\ln \rho$ and the interpretation of this is not so clear, since such solutions cannot be considered as field-theoretic models of electric wires or charge distributions. We shall therefore not consider the $\sigma^2 < 0$ solutions in what follows.

For the magnetic solutions with $\sigma^2 > 0$ the problem does not arise, since the logarithmic term in the effective mass (7.1) only improves their localization. As a result, the magnetic solutions do generalize within the full, non-linear theory. The described above multiplet structure of the perturbative solutions persists also at the non-linear level, up to the fact that only solutions with $\sigma^2 \geqslant 0$ are now allowed. For $n = 1$ one finds one such solution, with $\nu = 1$, while for $n = 2$ there are already three: with $\nu = 1, 2$ and, provided that θ_W is not too large, also with $\nu = 3$ (see Fig. 3). The general rule seems to be such that for a given n there are $2n - 1$ solutions, of which

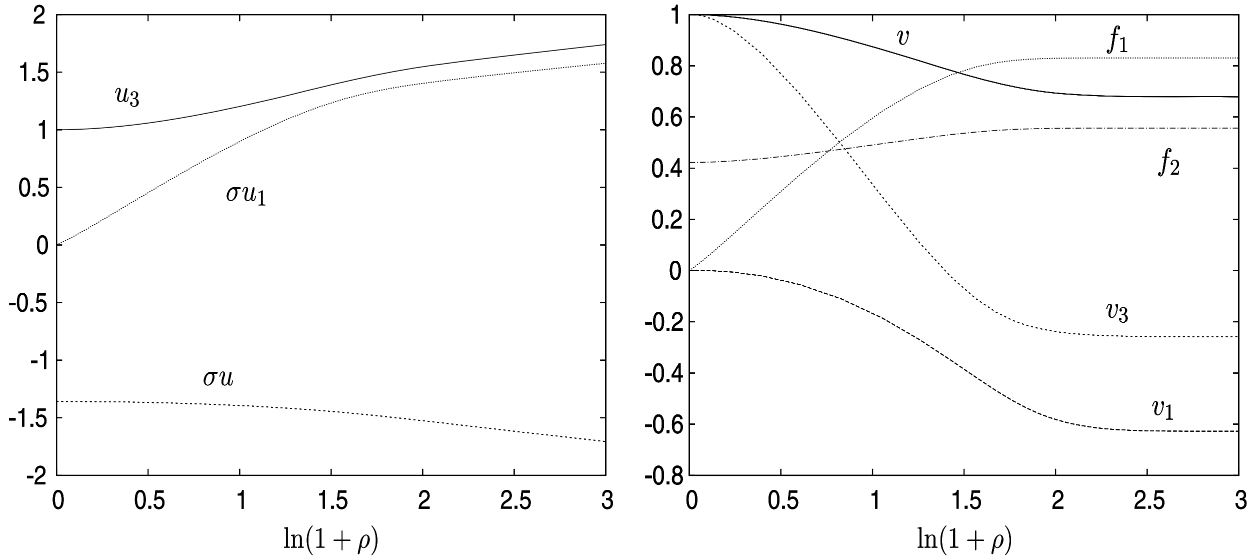

Fig. 5. Profiles of the superconducting vortex solution for $\beta = 2$, $g'^2 = 0.23$, $n = \nu = 1$ and $\sigma = 0.373$. The 'electric' amplitudes (left) show at large ρ the logarithmic growth typical for the Biot–Savart field. The remaining amplitudes (right) are everywhere bounded.

those with $\nu = 1, 2, \ldots, n$ exist for any θ_W while those with $\nu = n + 1, \ldots, 2n - 1$ exist if only θ_W is not too close to $\pi/2$, as shown in Figs. 2, 3.

The regions of their existence are delimited by the chiral curves in Fig. 4. For each given curve one has $\sigma^2(\beta, \theta_W, n, \nu) > 0$ in the region *below* the curve. Let us call it *allowed region*, it corresponds to magnetic solutions that generalize within the full non-linear theory. The region *above* the curve corresponds to the electric solutions with $\sigma^2 < 0$, we therefore call it *forbidden region*. The chiral curves with $\nu \leqslant n$ are all contained in the upper left corner of the diagram where $\beta \leqslant 1$, while those for $\nu > n$ extend to the region where $\beta > 1$. It follows that if $\nu \leqslant n$ then the allowed region corresponds to any θ_W if $\beta > 1$, and to θ_W that is not too close to $\pi/2$ if $\beta < 1$. If $\nu > n$ then the allowed region corresponds to the lower part of the diagram located below the (n, ν) chiral curve.

The chiral curves in Fig. 4 are obtained in the limit of vanishing current, while for finite currents they remain qualitatively similar but shift *upward*. For large currents they seem to approach the upper boundary of the diagram, such that the allowed regions increase. Therefore, without entering too much into details, one can simply say that the superconducting vortices exist for *almost all* values of β, θ_W.

The typical solution is shown in Fig. 5. These solutions describe vortices carrying an electric current – smooth, globally regular field-theoretic analogues of electric wires. At small ρ they exhibit a completely regular core filled with massive fields creating the electric charge and current. At large ρ the massive fields decay and there remains only the Biot–Savart electromagnetic field. Integrating the charge and current densities over the vortex cross section gives the vortex 'worldsheet current' $I_\alpha \sim \sigma_\alpha$ whose components are the vortex electric charge (per unit length), I_0, and the total electric current through the vortex cross section, I_3. Since $\sigma^2 \equiv \sigma_3^2 - \sigma_0^2 > 0$, the vector I_α is spacelike, which means that there is a comoving reference frame where $\sigma_0 = I_0 = 0$. However, the current I_3 cannot be boosted away, so that this is an essential parameter, which suggests the term 'superconductivity'. In what follows we shall denote by $\mathcal{I}$ the value of I_3 in the comoving frame. We shall also call σ *twist*, since in the comoving frame one has $\sigma_3 = \sigma$, which determines the z-dependent relative phase of the two Higgs field components in the ansatz (4.4).

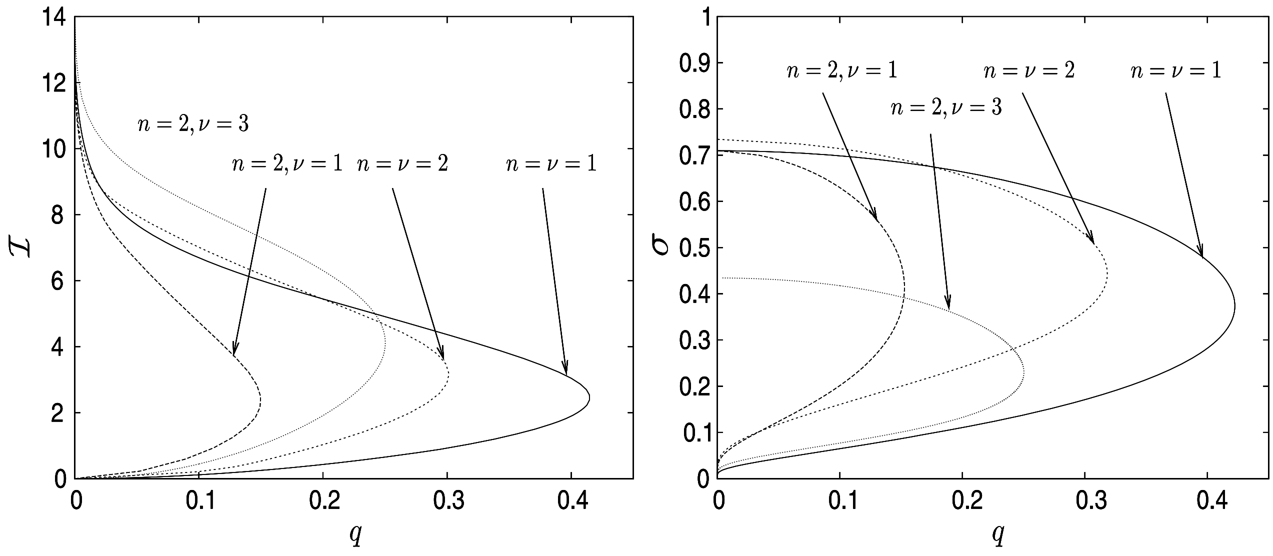

Fig. 6. The restframe current $\mathcal{I}$ (left) and twist σ (right) as functions of the parameter q for the solutions with $\beta=2$, $\sin^2\theta_{\mathrm{W}}=0.23$.

The solutions exist for $\beta>0$ and for θ_{W} belonging to the region below the (n,ν)-chiral curve in Fig. 4. Solutions depend on $\beta,\theta_{\mathrm{W}}$ and also on q,n,ν so that σ^2 is in fact a function of five arguments, $\sigma^2=\sigma^2(\beta,\theta_{\mathrm{W}},n,\nu,q)$. In addition, reconstructing the fields (4.4), every solution of the differential equations determines actually a whole family of vortices with fixed $\sigma^2=\sigma_3^2-\sigma_0^2$ but with different values of (σ_0,σ_3) related to each other by Lorentz boosts. For given $\beta,\theta_{\mathrm{W}}$ the superconducting vortices therefore comprise a four parameter family that can be labeled by n,ν,σ_0 and q. These parameters determine physical quantities associated to the vortex: the electromagnetic and Z fluxes Ψ_F and Ψ_Z, the charge I_0 per unit vortex length and the current I_3, as well as the momentum along the vortex, P, and the angular momentum M.

The parameter q measures the deviation from the Z string limit. Increasing q starting from zero increases $\mathcal{I}$. It turns out, however, that the dependence $\mathcal{I}(q)$ is not one-to-one, since for q not exceeding a certain maximal value $q_\star(\beta,\theta_{\mathrm{W}},n,\nu)$ one finds *two* different solutions with different currents, as shown in Fig. 6, while for $q>q_\star$ there are no solutions at all. The solutions thus show two different branches that merge for $q\to q_\star$ but have different behaviour for $q\to 0$, when the lower branch reduces to the currentless Z string, while the current of the upper branch solutions grows seemingly without bounds. It is worth noting that this behaviour is drastically different from that found in the model of superconducting strings of Witten (see Fig. 22 in Appendix C), where the maximal value of q corresponds to the 'dressed' currentless string.

A similar two-branch structure emerges also for other solution parameters traced against q, as for example for $\sigma(q)$ (see Fig. 6). This suggests that solutions should be labeled not by q but by a parameter that changes monotonously, as for example the current $\mathcal{I}$ or twist σ (see Figs. 7, 8).

The electromagnetic and Z fluxes are shown in Fig. 9 as functions of the current. In the Z string limit $\mathcal{I}\to 0$ the Z flux reduces to $4\pi n$ while the electromagnetic flux vanishes. As was said above, in the fully non-Abelian theory there is no reason for the fluxes to be quantized, and so they vary continuously with $\mathcal{I}$.

As the vortices support the long-ranged electromagnetic field, their energy per unit length diverges logarithmically. It is the 'electric' part E_1 of the total energy (5.11) which diverges, since it contains $Qu(\infty)$ and one has $u(\rho)\sim Q\ln\rho$ for $\rho\to\infty$. The 'charge parameter' Q generically does not vanish, it is shown against the current in Fig. 10 and against θ_{W} in Fig. 16. The 'magnetic' energy E_2 is finite and reaches its minimal value for $q=q_\star$ as shown in Fig. 10.

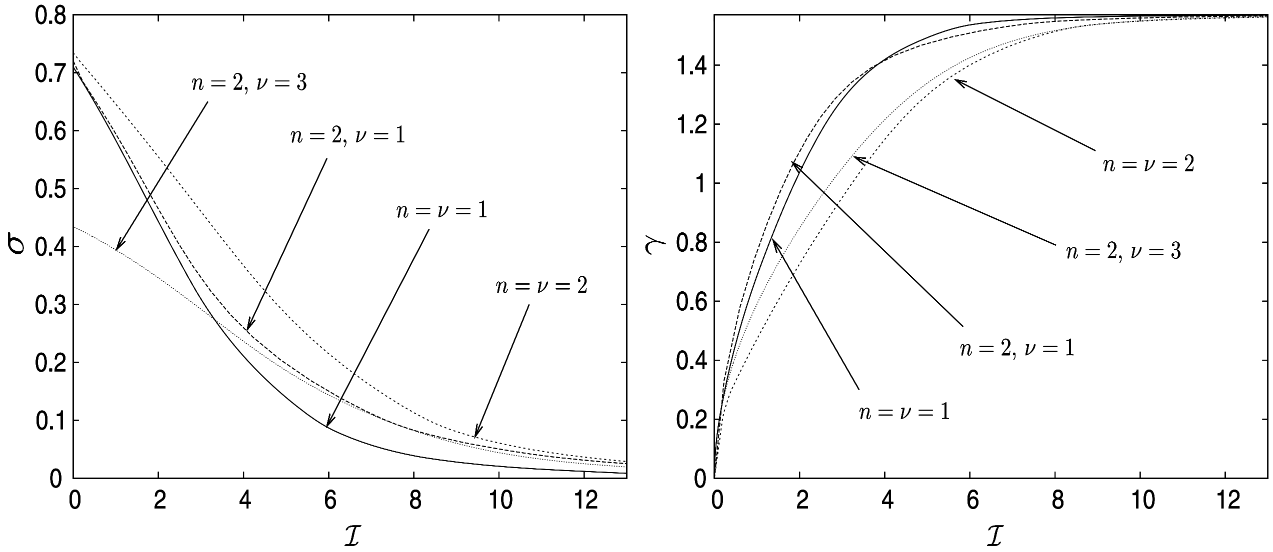

Fig. 7. The twist σ (left) and vacuum angle γ (right) against the current $\mathcal{I}$ for the solutions with $\beta = 2$, $\sin^2\theta_W = 0.23$. One has $\sigma \to 0$ and $\gamma \to \pi/2$ when $\mathcal{I}$ grows, while the product $\sigma\mathcal{I}^3$ approaches a non-zero value, so that $\sigma \sim \mathcal{I}^{-3}$.

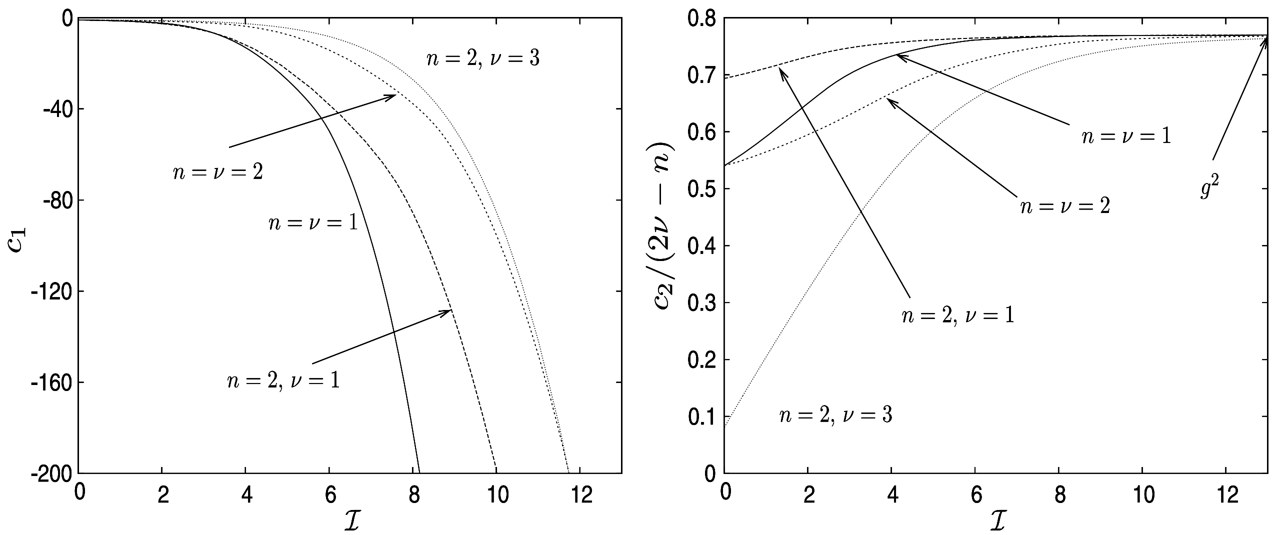

Fig. 8. The parameters c_1 (left) and $c_2/(2n - \nu)$ (right) against $\mathcal{I}$ for the $n = 1, 2$ solutions with $\beta = 2$, $\sin^2\theta_W = 0.23$. One has $c_2 \to (2\nu - n)g^2$ for $\mathcal{I} \to \infty$.

This suggests that the value $q = q_\star$ is in some sense 'energetically preferable'. Since this value determines the 'bifurcation point' between the two solution branches, one can expect that the stability of the solutions may change at this point.

8. Large current limit

Witten's superconducting strings exhibit the current quenching – there is an upper bound for their current [1,3,5] (see Appendix C). In the Weinberg–Salam theory we *generically* do not observe a similar phenomenon. Large $\mathcal{I}$ corresponds to small σ, and we were able to descend down to $\sigma \sim 10^{-3}$ (after this the numerics become somewhat involved) always finding larger and larger $\mathcal{I}$, without any tendency for this increase to stop.

It seems that the reason for which the current can be unbounded is related to the spin of the current carries. The existence of the critical current in superconductivity models is usually

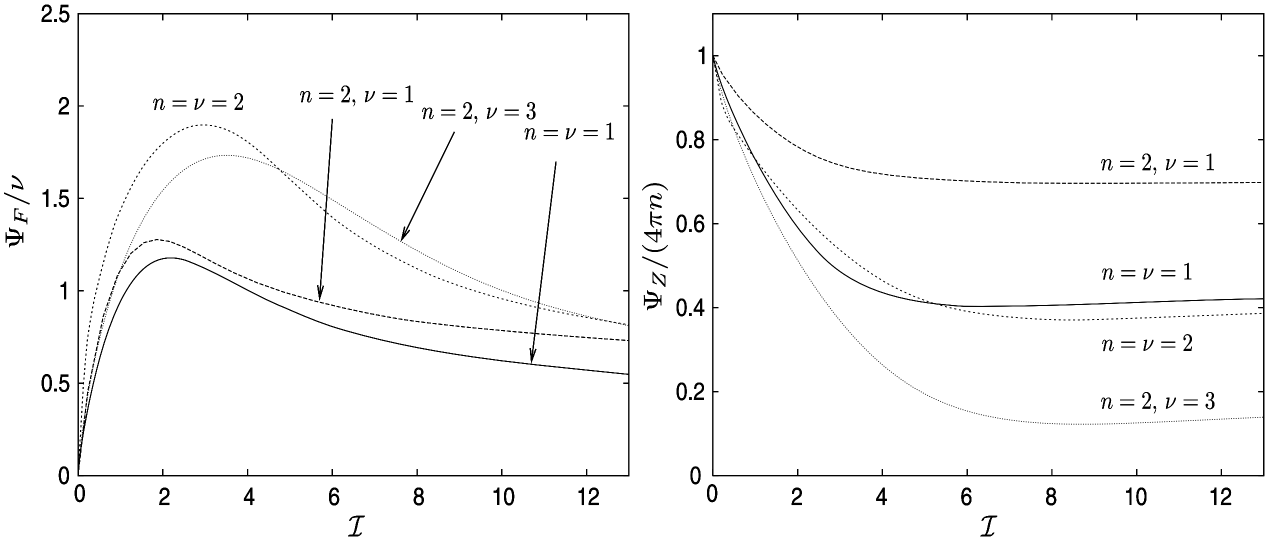

Fig. 9. The electromagnetic flux Ψ_F/ν (left) and Z flux $\Psi_Z/(4\pi n)$ (right) against the current for the vortices with $\beta=2$, $\sin^2\theta_W=0.23$.

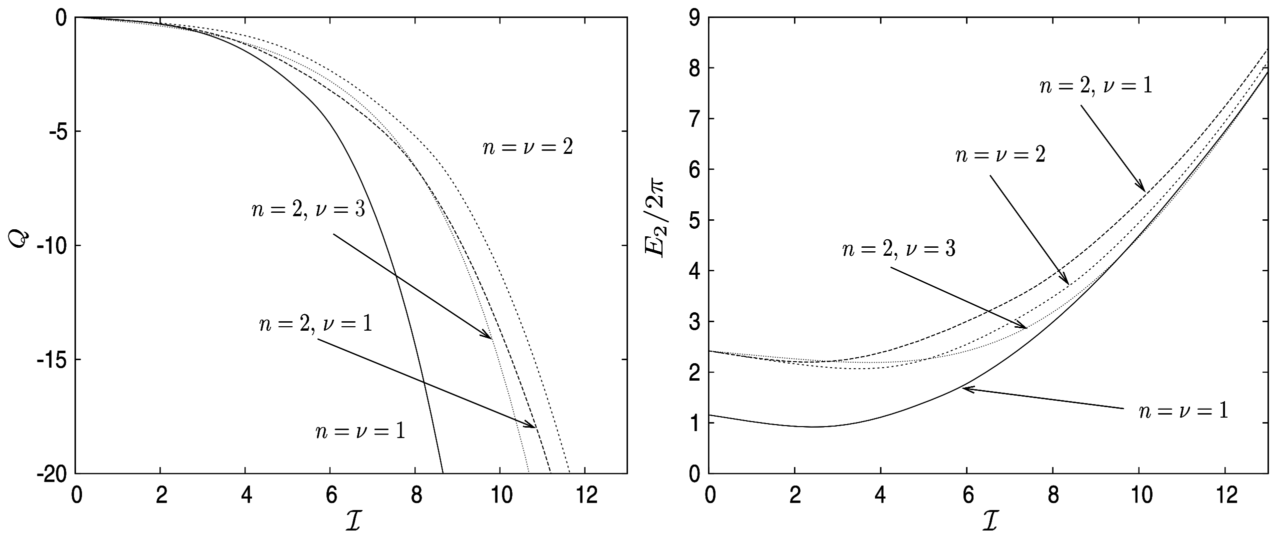

Fig. 10. The charge parameter Q (left) and the magnetic energy (right) against $\mathcal{I}$ for the solutions with $\beta=2$, $\sin^2\theta_W=0.23$.

related to the fact that too large currents produce strong magnetic fields which quench the scalar condensate and destroy the superconductivity. This is what happens in Witten's model where the current is carried by scalars, and so quenching the condensate quenches the current. Specifically, the current defined by Eq. (C.3) of Appendix C vanishes for $\phi_2\to 0$. In the Weinberg–Salam theory, as we shall see below, very strong magnetic fields also quench the Higgs field inside the vortex. However, this does not destroy superconductivity, since the current is carried by *vector* particles, and so the scalar Higgs field is not the relevant order parameter. In fact, the current defined by Eqs. (2.12), (2.13) does not vanish for $\Phi\to 0$. For large currents the vector boson condensate can be described by the pure Yang–Mills theory, whose scale invariance implies that the current can be as large as one wants. In the Witten model this mechanism does not work because the current carriers are scalars, which break the scale invariance.

At a more technical level the current quenching can be related to the 'dressed' currentless solutions. If they exist, then changing the twist σ makes the system interpolate between the

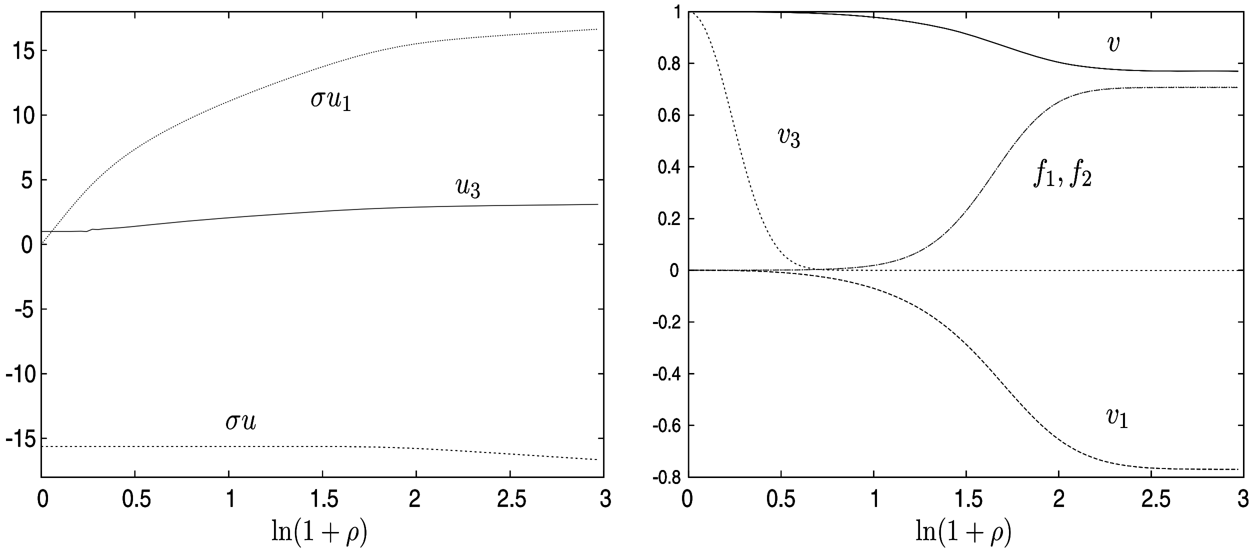

Fig. 11. Solutions profiles for $\beta = 2$, $g'^2 = 0.23$, $n = \nu = 1$ and $\sigma = 0.008$.

'bare' and 'dressed' strings, and since both of them are currentless, the current passes through a maximum in between (see Appendix C). As we shall see in the next section, 'dressed' solutions exist in the electroweak theory only for special values of β, θ_W. This implies that the current quenching can exist for these values in the electroweak case too, but for generic parameter values there are no 'dressed' solutions, so that when one starts from the 'bare' Z string and decreases the twist, the current always grows. For large currents a simple approximation for the solutions can be found that agrees very well with the numerics and suggests that the current can indeed be arbitrarily large.

The large current limit is characterized by the presence of two different length scales. One of them does not depend on the current and is determined by the Z boson or Higgs boson masses (which are both of the same order of magnitude for physical values of β), $R_Z \sim m_Z^{-1} \sim m_H^{-1} \sim 1$. Another scale is related to the 'dressed' W boson mass (4.15), which *increases* with current and becomes large for large currents, $m_\sigma \sim \sigma Q \sim \mathcal{I}$, so that the corresponding length is small, $R_\sigma \sim m_\sigma^{-1} \sim \mathcal{I}^{-1}$. One can therefore expect the solutions to show a small central region where $\rho \leqslant R_\sigma$ that accommodates the very heavy field modes of mass m_σ. For $R_\sigma \leqslant \rho \leqslant R_Z$ these 'supermassive' modes die out but the other massive modes remain, while the asymptotic region where $R_Z \leqslant \rho$ contains only the massless modes.

The numerical profiles of the large $\mathcal{I}$ solutions (see Fig. 11) essentially confirm these expectations, but also reveal additional interesting features. For large currents the vacuum angle γ tends to $\pi/2$ (see Fig. 7) so that one has $f_1 = f_2 = 1/\sqrt{2}$ for $\rho \to \infty$. However, as is seen in Fig. 11, one has $f_1 \approx f_2$ not only for large ρ but everywhere. In particular, there is a region where $f_1 \approx f_2 \approx 0$, which means that the gauge symmetry is completely restored to SU(2) × U(1). In fact, at the origin one still has $f_1 = a_5\rho$ and $f_2 = q$ (see Eq. (4.8)) but the values of q, a_5 are extremely small. In addition, one has in this region $v_1 \approx \sigma u_3 \approx 0$, $v \approx 1$, $\sigma u \approx$ const, while v_3 and u_1 change very quickly, and in particular v_3 vanishes very rapidly. The latter corresponds to freezing away of the very massive modes.

One can wonder how the existence of the very massive modes can be compatible with the complete symmetry restoration. However, these modes arise not via the Higgs mechanism but rather due to the screening by the current, which is somewhat similar to the colour screening in pure Yang–Mills systems. When v_3 vanishes, the Higgs field starts to change and approaches its vacuum value, after which all massive degrees of freedom die out and the solutions reduce to the U(1) electric wire (4.10).

It turns out to be possible to find a relatively simple approximation for the large current solutions that explains all their empirically observed features. Specifically, since the vacuum angle γ approaches $\pi/2$ for large currents, let us set $\gamma=\pi/2$. Let us also restrict ourselves to the simplest case where $n=\nu=1$. The boundary conditions (5.1) then become

$$\begin{aligned} &a_1 \leftarrow u \rightarrow c_1+Q\ln\rho, \qquad 0 \leftarrow u_1 \rightarrow -(c_1+Q\ln\rho), \qquad 1 \leftarrow u_3 \rightarrow 0,\\ &1 \leftarrow v \rightarrow c_2, \qquad 0 \leftarrow v_1 \rightarrow -c_2, \qquad 1 \leftarrow v_3 \rightarrow 0,\\ &a_5\rho \leftarrow f_1 \rightarrow 1/\sqrt{2}, \qquad q \leftarrow f_2 \rightarrow 1/\sqrt{2}. \end{aligned} \tag{8.1}$$

8.1. *W-condensate region* – SU(2) *Yang–Mills string*

Let us approximate the solution at small ρ by

$$\begin{aligned} &f_1=f_2=\sigma u_3=v_1=0, \qquad \sigma u=\sigma a_1, \qquad v=1,\\ &\sigma u_1(\rho)=\lambda U_1(\lambda\rho), \qquad v_3=V_3(\lambda\rho), \end{aligned} \tag{8.2}$$

where λ is a scale parameter. Inserting this to (3.16)–(3.24) the equations reduce to

$$\frac{1}{x}\left(xU_1'\right)'=\frac{V_3^2}{x^2}U_1, \qquad x\left(\frac{V_3'}{x}\right)'=U_1^2V_3, \tag{8.3}$$

with $x=\lambda\rho$. They admit a solution with the following boundary conditions for $0\leftarrow x\rightarrow\infty$,

$$\begin{aligned} &x+\cdots \leftarrow U_1(x) \rightarrow 0.85+0.91\ln(x)+\cdots,\\ &1-0.45x^2+\cdots \leftarrow V_3(x) \rightarrow 0.32\sqrt{x}e^{0.06x}x^{-0.91x}+\cdots \end{aligned} \tag{8.4}$$

for which V_3 rapidly approaches zero (see Fig. 12). Inserting $U_1(x)$, $V_3(x)$ to (8.2) gives a family of solutions distinguished from each other by the value of the scale parameter λ. This scaling symmetry arises due to the fact that, since the Higgs field is zero, the system is pure Yang–Mills. In fact, Eqs. (8.3) coincide with the Yang–Mills equations for the SU(2) theory

$$\mathcal{L}_{\mathrm{SU(2)}}=-\frac{1}{4}\mathrm{W}^a_{\mu\nu}\mathrm{W}^{a\mu\nu} \tag{8.5}$$

provided that the Yang–Mills field is cylindrically-symmetric

$$\tau^a\mathrm{W}^a_\nu\,dx^\mu=\tau^1\lambda U_1(\lambda\rho)\,dz+\tau^3V_3(\lambda\rho)\,d\varphi. \tag{8.6}$$

We shall therefore call the solution shown in Fig. 12 Yang–Mills string.

The Yang–Mills strings are made of the pure gauge field and describe the charged W-condensate trapped in the region where the amplitude V_3 is different from zero. Outside this region there remains the Biot–Savart field generated by the condensate and represented by U_1. The right-hand sides of Eqs. (8.3) can be viewed as conserved (although not gauge invariant) current densities. Integrating over the ρ,φ plane gives the current along the z-direction, $\mathcal{I}\sim\lambda$, localized in the region of the size $\sim 1/\lambda$ where V_3 is different from zero.

The Yang–Mills strings approximate the central current-carrying core of the electroweak vortex. Their scale invariance implies that there is no upper bound for the current $\mathcal{I}\sim\lambda$, since the scale parameter λ can assume any value.

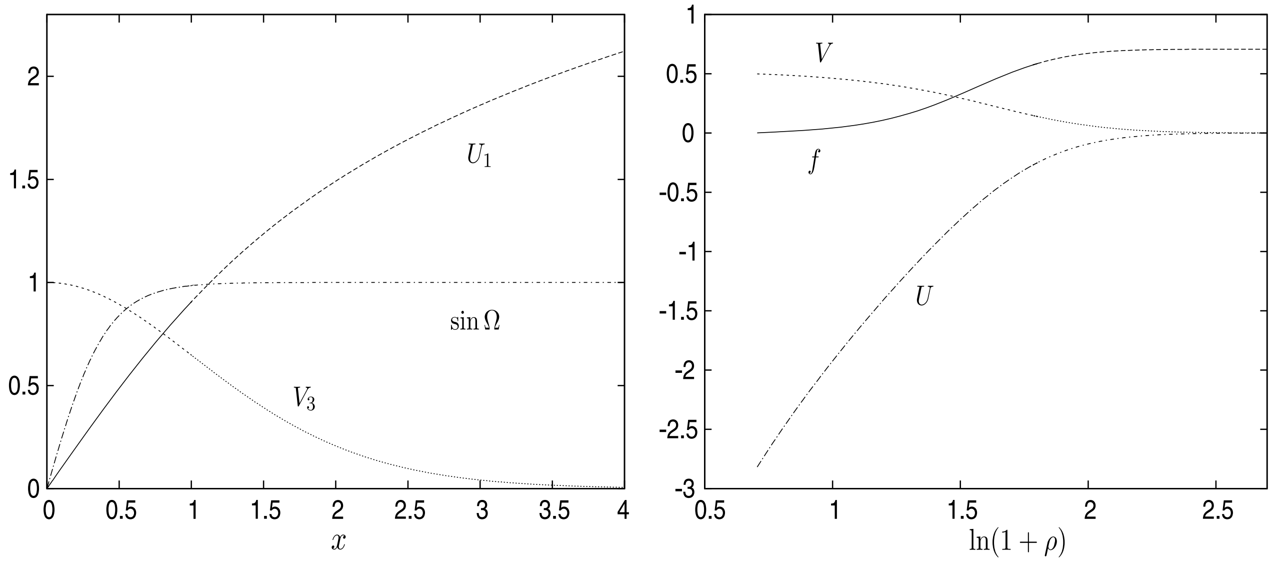

Fig. 12. Left: solution of Eqs. (8.3) with the boundary condition (8.4) and the function $\sin\Omega$ defined by Eqs. (5.3), (8.18). Right: the typical solution of Eqs. (8.11).

8.2. *External* U(1) × U(1) *region*

Let us now assume that the Yang–Mills string approximation is valid only in the central vortex core, for $x \leqslant x_0$, with x_0 chosen such that $V_3(x_0)$ is sufficiently close to zero, for example $x_0 = 4$ (see Fig. 12). This determines the upper value $\rho_0 = x_0/\lambda$. As we shall see, for large λ the precise knowledge of x_0 is unimportant. For $\rho > \rho_0$ we set

$$v_3 = \sigma u_3 = 0, \qquad f_1 = f_2 \equiv f/\sqrt{2}, \tag{8.7}$$

while the remaining field amplitudes can be combined into

$$U = \frac{\sigma}{2}(u + u_1), \qquad V = \frac{1}{2}(v + v_1),$$
$$U_A = \sigma\left(g^2 u - g'^2 u_1\right), \qquad V_A = g^2 v - g'^2 v_1. \tag{8.8}$$

Using formulas of Section 5.1 one can check that the electromagnetic and Z fields admit then potentials $gg' A_\mu\, dx^\mu = U_A \sigma_\alpha\, dx^\alpha + V_A\, d\varphi$ and $2Z_\mu\, dx^\mu = U\sigma_\alpha\, dx^\alpha + V\, d\varphi$ while the W-condensate components vanish. The original SU(2) × U(1) theory (2.1) reduces then to the U(1) × U(1) model for the variables A_μ, Z_μ and $\phi \equiv f$,

$$\mathcal{L}_{\mathrm{U(1)\times U(1)}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} + (\mathcal{D}_\mu\phi)^*\mathcal{D}^\mu\phi - \frac{\beta}{8}\left(|\phi|^2 - 1\right)^2 \tag{8.9}$$

with $\mathcal{D}_\mu\phi = (\partial_\mu - \frac{i}{2}Z_\mu)\phi$. This contains the free Maxwell theory and the Abelian Higgs model. Eqs. (3.16)–(3.24) reduce to the Maxwell equations

$$\left(\rho U_A'\right)' = 0, \qquad \left(\frac{V_A'}{\rho}\right)' = 0, \tag{8.10}$$

and to the equations of the Abelian Higgs model

$$\frac{1}{\rho}\left(\rho f'\right)' = \left(U^2 + \frac{V^2}{\rho^2} + \frac{\beta}{4}\left(f^2 - 1\right)\right)f, \tag{8.11a}$$

$$\rho\left(\frac{V'}{\rho}\right)' = \frac{1}{2}f^2V, \tag{8.11b}$$

$$\frac{1}{\rho}(\rho U')' = \frac{1}{2}f^2U. \tag{8.11c}$$

The solution of (8.10) is $U_A = A + B\ln\rho$, $V_A = C$. Eqs. (8.11) admit a one-parameter family of solutions with the boundary conditions

$$U(\rho_0) \leftarrow U \to 0, \qquad \frac{1}{2} \leftarrow V \to 0, \qquad 0 \leftarrow f \to 1 \tag{8.12}$$

for $\rho_0 \leftarrow \rho \to \infty$ (see Fig. 12). The numerics show that if $U'(\rho_0)$ is large and positive then $U(\rho_0)$ is large and negative and the derivatives $f'(\rho_0)$ and $V'(\rho_0)$ tend to zero such that there is a neighbourhood of ρ_0 where

$$V \approx 1/2, \qquad f \approx 0, \qquad U \approx a + b\ln\rho \tag{8.13}$$

with $a = a(b)$. As a result, $U(\rho_0) = a(b) + b\ln\rho_0$ and $U'(\rho_0) = b/\rho_0$.

8.3. Matching the solutions

The solutions obtained for $\rho \leqslant \rho_0$ and for $\rho \geqslant \rho_0$ should agree at $\rho = \rho_0 = x_0/\lambda$ where the functions and their first derivatives should match. Matching u, u_1 and their derivatives gives four conditions that determine the free parameters a_1, b, A, B in the above solutions,

$$A = B\ln\lambda + g^2a_1 - 0.85g'^2\lambda, \qquad B = -0.91g'^2\lambda,$$
$$b = 0.45\lambda, \qquad \sigma a_1 = 2a(b) - 0.85\lambda - 2b\ln\lambda.$$

It is worth noting that x_0 has dropped from these relations. Matching similarly v, v_1, f gives only one condition, $C = g^2$, but the derivatives v', v_1', f' do not match precisely, since they vanish for $\rho \leqslant \rho_0$ but not for $\rho \geqslant \rho_0$. However, this discrepancy tends to zero when λ grows, since $U'(\rho_0) \sim \lambda^2$ then increases thus rendering the approximation (8.13) better and better. Similarly, although there is a discrepancy in value of v_3, since it vanishes for $\rho \geqslant \rho_0$ but not for $\rho \leqslant \rho_0$, increasing λ reduces this discrepancy as well.

As a result, by matching the two solutions we can approximate the global solution in the interval $0 \leqslant \rho < \infty$. The approximate solution depends on λ and approaches the true solution when λ increases. For $\lambda \to \infty$ the discrepancy of values of v_3, v', v_1', f' at the matching point vanishes and so the approximate solution becomes exact. There only remains to relate λ to the current. Comparing with (8.1) we see that $\sigma Q = B$ (also $\sigma c_1 = A$ and, as noticed in Fig. 8, $c_2 = g^2$), and since the current $\mathcal{I} = -2\pi\sigma Q/(gg')$, it follows that

$$\lambda = 0.17\frac{g}{g'}\mathcal{I}. \tag{8.14}$$

The approximate solution shown in Fig. 13 clearly exhibit the same structure as the true one in Fig. 11. Although it corresponds not to a very large value of the scale parameter, $\lambda = 7.6$, its current is twice as large as compared to that in Fig. 11. The true solution in Fig. 11 shows a slowly growing amplitude u_3 whereas in Fig. 13 one has $\sigma u_3 = 0$. This difference is a subleading effect, since in Fig. 11 the vacuum angle is not exactly $\pi/2$ and so there is a growing mode $u_3 \sim \cos\gamma\ln\rho$. For $\mathcal{I} \to \infty$ one has $\gamma \to \pi/2$ and this mode disappears, so that u_3 becomes everywhere bounded, and multiplying it by $\sigma \to 0$ will give $\sigma u_3 = 0$, as shown in Fig. 13. With some more analysis such subleading effects can be taken into account to determine, for example, $\gamma(\mathcal{I})$, $\sigma(\mathcal{I})$, $q(\mathcal{I})$.

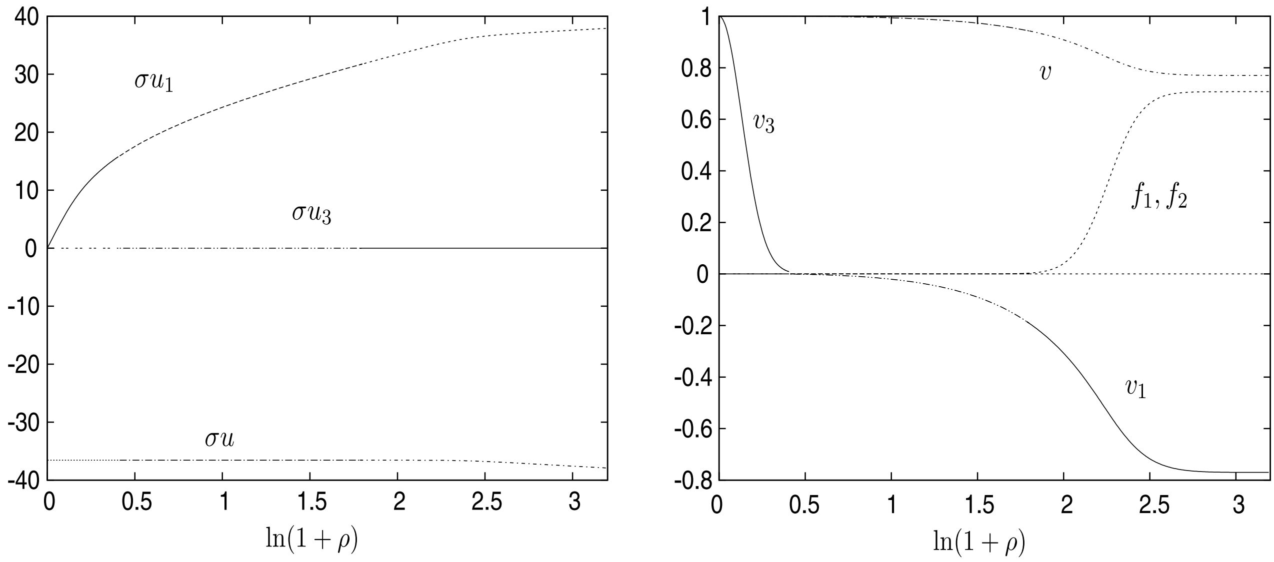

Fig. 13. The approximate solution for $\beta = 2$, $g'^2 = 0.23$, $n = \nu = 1$ and $\lambda = 7.6$, $\mathcal{I} = 24.43$.

8.4. Symmetry restoration inside the vortex

Comparing Fig. 11 and Fig. 13 reveals that the zero Higgs field region expands when the current increases. The inner structure of the large current vortex can be schematically represented by Fig. 14. The vortex shows a large symmetric phase region of size $\sim \mathcal{I}$ where the Higgs field is very close to zero. In the centre of this region there is a compact core of size $\sim \mathcal{I}^{-1}$ containing the charged W-condensate and approximated by the Yang–Mills string. Outside the core there live the electromagnetic and Z fields. The symmetric phase is surrounded by a 'crust' of thickness $\sim R_Z$ where the Higgs field approaches the constant vacuum value while the Z field becomes massive and dies away. The region outside the 'crust' is described by the Maxwell electrodynamics and is dominated by the Biot–Savart magnetic field generated by the current in the core.

This picture reminds somewhat the 'W-dressed superconducting string' scenario of Ambjorn and Olesen [15,35]. This scenario is based on the observation [15] that applying a very strong external magnetic field B,

$$m_{\mathrm{W}}^2/e < B < m_{\mathrm{H}}^2/e, \tag{8.15}$$

makes the Higgs vacuum unstable with respect to a condensate formation. For undercritical fields, $B < m_{\mathrm{W}}^2/e$, the ground state is in the Higgs vacuum, while for supercritical fields, $B > m_{\mathrm{H}}^2/e$, the Higgs field vanishes in the ground state so that the symmetry is restored [15]. This suggests that if Witten's superconducting string is considered as the source of external magnetic field, then the structure of the electroweak vacuum in its vicinity should change. There should be a cylindrical layer around the string where the field falls within the limits Eq. (8.15) and where the Higgs field should be non-trivial. Inside this layer the field is supercritical and the Higgs field is zero, while outside the field is undercritical and the Higgs field is in vacuum. All this is very similar indeed to what is shown in Fig. 14. This suggests that we have a self-consistent electroweak realization of the Ambjorn–Olesen scenario in which the Witten string is replaced by the Yang–Mills string of completely electroweak origin. The following estimates confirm this.

First of all, the radius of the layer where the Higgs field is non-trivial should scale as $\mathcal{I}$, and we shall now see that the crust radius scales precisely in this way. Indeed, inside the crust

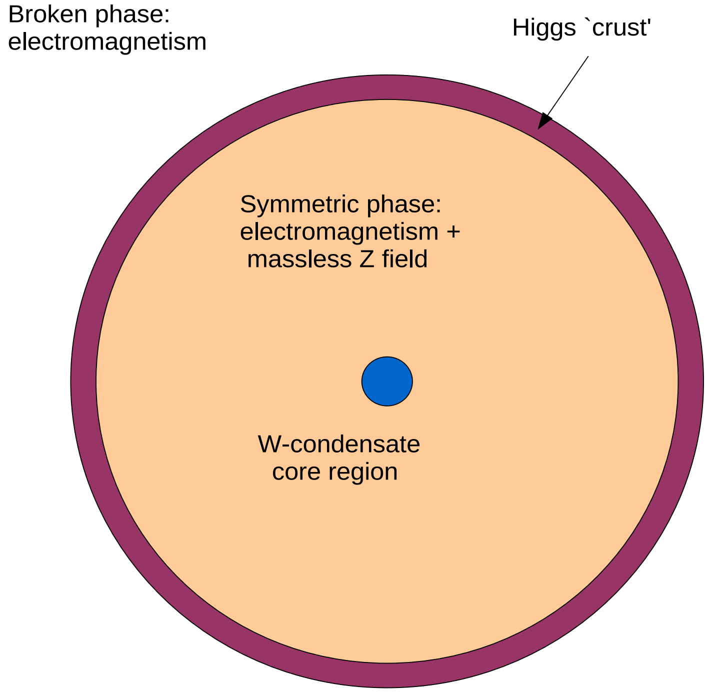

Fig. 14. The schematic structure of the large current vortex.

f is very small and one can neglect the right-hand sides in (8.11c) to obtain $U = b\ln(\rho/\rho_\star)$, where $\rho_\star$ is an integration constant and $b = 0.45\lambda$. As long as $\rho \ll \rho_\star$ the U^2 term in (8.11a) is large and makes f stay very close to zero, since f can increase and pass through the inflection point if only U^2 is small. The latter follows from the fact that the second derivative of f with respect to $\ln\rho$ vanishes if only the right-hand side of (8.11a) vanishes, which is only possible if U^2 is small enough to be canceled by the negative term $\frac{\beta}{4}(f^2-1)$. Therefore, f can vary if only $U \sim \sqrt{\beta}$, but as soon as f deviates from zero, it acts as the mass term for U to drive it to zero. As a result, f interpolates between zero and unity close to the place where U vanishes, for $\rho \sim \rho_*$, in an interval of size $\delta\rho \sim R_Z$. Multiplying (8.11c) by ρ and integrating from ρ_0 to infinity gives $2b = -\int_{\rho_0}^{\infty} \rho f^2 U\, d\rho$ and since the integrand is almost everywhere zero apart from a small vicinity of $\rho_\star$, it follows that $b \sim \sqrt{\beta}\rho_\star$. This gives the size of the symmetric phase region (the crust radius)

$$\rho_* = 0.28 \times \frac{g}{g'}\frac{\mathcal{I}}{\sqrt{\beta}}, \tag{8.16}$$

where the coefficient is found numerically. As a result, one can approximate the solution of Eqs. (8.11) by

$$f \approx \Theta(\rho-\rho_\star), \qquad U \approx b(1-f)\ln(\rho/\rho_\star), \qquad V \approx \frac{1}{2}(1-f)\big(1-(\rho/\rho_\star)^2\big), \tag{8.17}$$

where $\Theta(\rho-\rho_\star)$ is the step-function smoothed over a finite interval $\delta\rho \sim R_Z$.

Let us now consider the distribution of the fields inside the vortex. The fields are defined by Eqs. (5.5), (5.9) where there is a hitherto undetermined function Ω. Using Eqs. (5.3), (8.7) one finds that $\Omega = \pi/2$ in the external region, while in the core Ω is not yet defined. Let us calculate the first order correction to the core configuration (8.2) assuming that f_1, f_2 are small functions of $x = \lambda\rho$ and that $f_1 \sim x$ and $f_2 = O(1)$ for $x \to 0$. Linearising Eqs. (3.18), (3.19) gives for large λ, with $f_\pm = f_1 \pm f_2$,

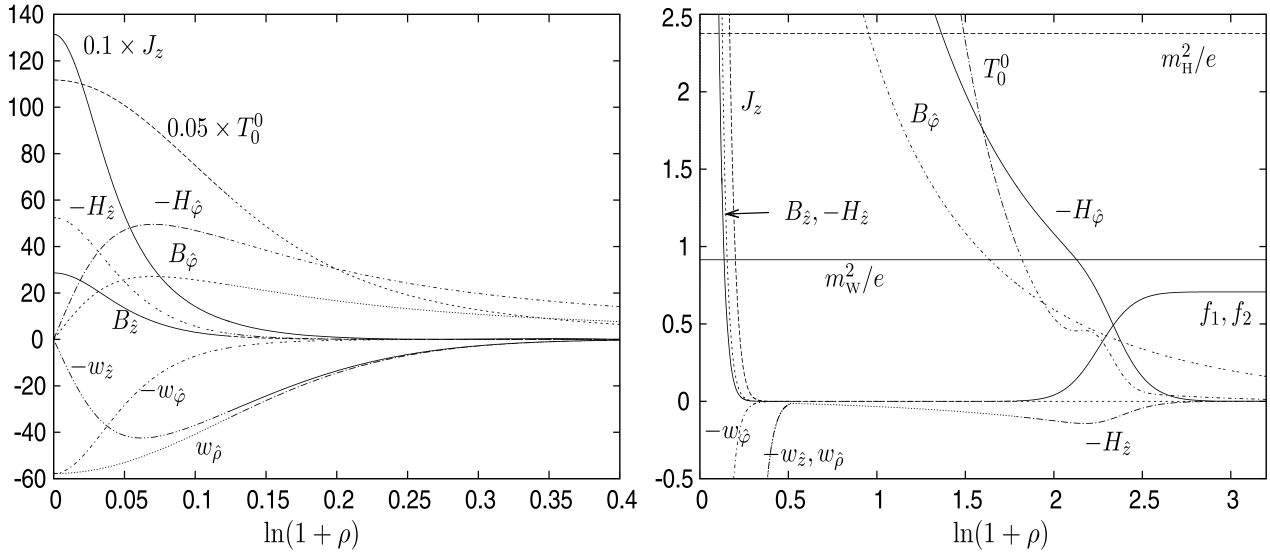

Fig. 15. The field distributions inside (left) and outside (right) the current-carrying condensate core of the vortex, for the same solution as in Fig. 13.

$$\frac{1}{x}(xf'_\pm)' = \left(\left(\frac{U_1}{2} \mp p\right)^2 + \frac{1+V_3^2}{4x^2}\right)f_\pm + \frac{V_3}{2x^2}f_\mp, \tag{8.18}$$

where $p = -\sigma a_1/(2\lambda) \sim \ln\lambda$ and U_1, V_3 are given by Eqs. (8.3). Integrating these equations and using Eq. (5.3) determines Ω inside the core and shows that it approaches $\pi/2$ very rapidly (see Fig. 12).

We have now all the ingredients to compute the fields (5.5), (5.9) and the electric current density $J_z = \partial^\nu F_{\nu z}$. The result presented in Fig. 15 shows that the current, W-condensate, and the z-component of the magnetic field are trapped in the core, where the energy density is maximal. Outside the core there extend only the Biot–Savart magnetic field and the Z field, which are both massless as long as the Higgs field stays close to zero. At a distance $\sim\mathcal{I}$ away from the core the system enters the 'crust' region where the Higgs field approaches the vacuum value. This gives rise to a local maximum in the energy density and drives the Z field very rapidly to zero. Outside the 'crust' there remains only the Biot–Savart magnetic field.

The most interesting feature is that the 'crust' region where the Higgs field is non-trivial exists only under certain values of the magnetic field. Outside this region, where the field is either too weak or too strong, the Higgs field is either in vacuum or vanishes, in agreement with the predictions of [15]. At the same time, the magnetic field inside the crust does not quite fall within the limits $(m_W^2/e, m_H^2/e)$, which were predicted in [15] for the *homogeneous* magnetic field and for $\beta = 1$. The electroweak condensate studied in [15] contains the W, Z and Higgs fields. In our case the crust region does not contain the W field, since it develops a very large effective mass m_σ due to the interaction with the Biot–Savart field, so that it can be different form zero only in the central core region.

We also notice in Fig. 15 that, since $w_{\hat\rho}$, $w_{\hat\varphi}$ do not vanish at $\rho = 0$, the W-condensate field strength is actually singular at the axis. However, this singularity is mild, since the energy density and the field potentials are globally regular. This singularity is absent for solutions with $\nu > 1$.

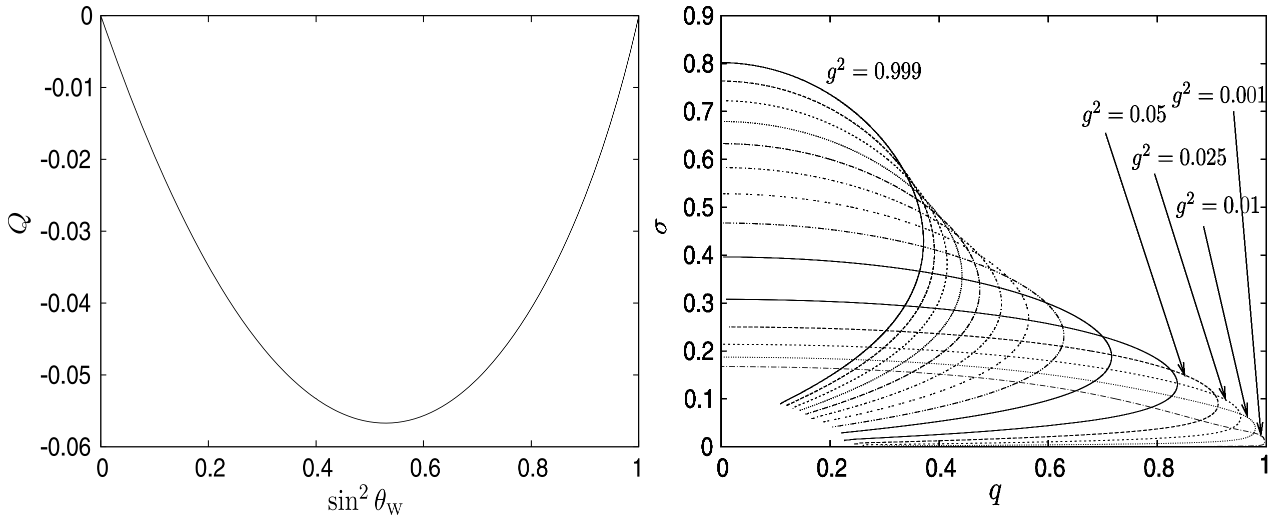

Fig. 16. Left: the charge parameter Q against θ_{W} for the solutions with $\beta = 2$, $n = \nu = 1$ and $q = 0.2$. Right: the twist σ against q for solutions with $n = \nu = 1$, $\beta = 2$. For $g \neq 0$ the curves $\sigma(q)$ start at $q = 0$ at their maximum, then go down towards larger values of q, but then always bend back to $q = 0$. If $g = 0$ then σ decreases monotonously to zero at $q = 1$.

9. Special parameter limits

The described above properties of the solutions are generic. However, new features can arise for special values of β, θ_{W}. Such values do not belong to the physical region, where $1.5 \leqslant \beta \leqslant 3.5$ and $\sin^2\theta_{\mathrm{W}} = 0.23$, but it is instructive to consider them, since this helps to understand the structure of the solution space. For example, if $\theta_{\mathrm{W}} \to 0$ or $\theta_{\mathrm{W}} \to \pi/2$ then B_μ or W^a_μ, respectively, become massless and decouple. The long-range mode $Q \ln\rho$ then disappears since Q vanishes (see Fig. 16), which renders the energy *finite*. Another interesting case corresponds to the $\sigma^2 = 0$ chiral solutions, which exist only for special values of $\theta_{\mathrm{W}} = \theta_{\mathrm{W}}(\beta, n, \nu, q)$. Finally, we shall consider infinite Higgs mass limit, $\beta \to \infty$, in which case the theory reduces to the gauged $\mathbb{CP}^1$ sigma-model.

9.1. Semilocal limit, $\theta_{\mathrm{W}} = \pi/2$

Since the right-hand side of the Yang–Mills equations (2.8) is proportional to g^2, one has $\mathrm{W}^a_\mu \sim g^2$ for small g so that it vanishes for $g \to 0$ and the theory (2.1) reduces to

$$\mathcal{L} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} + (D_\mu \Phi)^\dagger D^\mu \Phi - \frac{\beta}{8}\big(\Phi^\dagger \Phi - 1\big)^2, \tag{9.1}$$

with $D_\mu\Phi = (\partial_\mu - \frac{i}{2} B_\mu)\Phi$. This reduced theory is sometimes called semilocal, since it has the local U(1) invariance while the SU(2) now becomes global [20]. Its spectrum contains a massive vector, a Higgs boson, and two massless scalars (which acquire a finite mass $m_\sigma = \sigma$ in the vortex). Setting $u_1 = v_1 = 0$, $u_3 = 1$, $v_3 = \nu$ the Yang–Mills field vanishes (in the gauge (4.4)) and Eqs. (3.16)–(3.24) reduce to four coupled equations for u, v, f_1, f_2. The boundary conditions are $a_1 \leftarrow u \to 0$, $2n - \nu \leftarrow v \to -\nu$, $0 \leftarrow f_1 \to 1$, $q \leftarrow f_2 \to 0$ for $0 \leftarrow \rho \to \infty$. Since the logarithmic term is now absent, all field amplitudes are bounded. The solutions are called in this case twisted semilocal strings, they were studied in detail in Ref. [21]. They exist only for $\beta > 1$ and for $\nu = 1, \ldots, n$, since for $\theta_{\mathrm{W}} = \pi/2$ the allowed region in Fig. 4 is $\beta > 1$ if $\nu \leqslant n$ and is empty for $\nu > n$. For given β, n, ν the solutions comprise a family that can be labeled by $q \in [0, 1)$.

Their current is due to the global SU(2) symmetry, it is determined by J^a_μ in Eq. (2.8),

$$I^{\mathrm{SU(2)}}_\alpha = \int J^3_\alpha\, d^2x = \pi\sigma_\alpha \int_0^\infty \left\{(u+1)f_1^2 + (1-u)f_2^2\right\}\rho\, d\rho. \tag{9.2}$$

The energy E is finite and *decreases* with current up to the lower bound $2\pi n$ [36], as does σ, while q increases and approaches unity, whereas for solutions with $g \neq 0$ it always passes through a maximum and then tends to zero for large currents (see Fig. 6). As a result, the curves $q(\sigma)$ for fixed values of g approach the $g = 0$ curve pointwise but not uniformly for $g \to 0$ (see Fig. 16).

9.2. *Isospint limit, $\theta_\mathrm{W} = 0$*

If $g' = \sin\theta_\mathrm{W}$ is small then $B_\mu \sim g'^2$ so that for $g' \to 0$ the theory (2.1) reduces to

$$\mathcal{L} = -\frac{1}{4}\mathrm{W}^a_{\mu\nu}\mathrm{W}^{a\mu\nu} + (D_\mu\Phi)^\dagger D^\mu\Phi - \frac{\beta}{8}\left(\Phi^\dagger\Phi - 1\right)^2, \tag{9.3}$$

with $D_\mu\Phi = (\partial_\mu - \frac{i}{2}\tau^a\mathrm{W}^a_\mu)\Phi$. The SU(2) is now local while the U(1) is global. The spectrum contains three real vectors of mass m_Z and a Higgs boson with the mass m_H. The field equations are obtained by setting in (3.18)–(3.24) $u(\rho) = \text{const}$, $v(\rho) = 2n - \nu$, which implies that the U(1) part of the field (4.4) vanishes and that one has in Eqs. (4.8), (4.14), (5.1) $a_1 = c_1 = u$, $c_2 = 2n - \nu$ and $Q = c_3 = c_4 = 0$. Since $Q = 0$, the logarithmic term at large ρ is absent. The typical solution is shown in Fig. 17. These solutions relate to the lower boundary of the chiral diagram in Fig. 4, so that they exist for $\beta > 0$ and for $\nu = 1, \ldots, 2n - 1$. Their current is global, related to the global U(1) invariance of the theory (9.3), it can be expressed in terms of J^0_μ in Eq. (2.7),

$$I^{\mathrm{U(1)}}_\alpha = \int J^0_\alpha\, d^2x = \pi\sigma_\alpha \int_0^\infty \left\{(u_3 + u)f_1^2 + 2u_1 f_1 f_2 - (u_3 - u)f_2^2\right\}\rho\, d\rho. \tag{9.4}$$

As for all solutions described above, there are no apparent restrictions on its values. The energy is finite and *increases* with current (see Fig. 17).

These solutions can be obtained from the generic electroweak vortices in the limit $\theta_\mathrm{W} \to 0$. The semilocal solutions can similarly be obtained in the limit $\theta_\mathrm{W} \to \pi/2$. Both limits are pointwise but not uniform at large ρ, since the generic solutions carry a long-range field $\sim Q\ln\rho$ while in the limits one has $Q = 0$. Since the electromagnetic current vanishes for $Q = 0$, one can wonder if the global currents (9.2) and (9.4) can be obtained via the limiting procedure. The answer is that they can be obtained as limits of the rescaled electromagnetic current $I_\alpha/(gg')$. The following relations take place,

$$\begin{aligned}\frac{1}{gg'}J_\alpha &= \frac{1}{gg'}\partial^\mu F_{\mu\alpha} = \frac{1}{g'^2}\partial^\mu B_{\mu\alpha} - \frac{1}{g^2}\partial^\mu\left(n^a\mathrm{W}^a_{\mu\alpha}\right)\\ &= J^0_\alpha - \frac{1}{g^2}\partial^\mu\left(n^a\mathrm{W}^a_{\mu\alpha}\right)\\ &= \frac{1}{g'^2}\partial^\mu B_{\mu\alpha} - n^a J^a_\alpha - \frac{1}{g^2}\left(D^\mu n^a\right)W^a_{\mu\alpha}.\end{aligned} \tag{9.5}$$

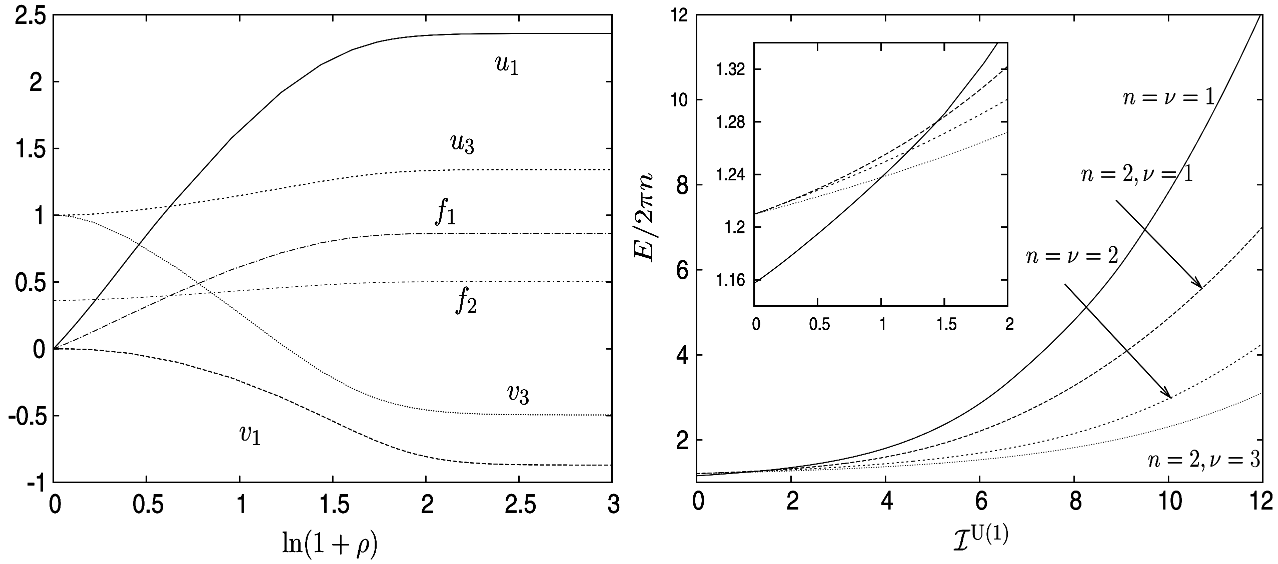

Fig. 17. Features of the $n=\nu=1$, $\beta=2$ solutions in the isospin limit, $g=1$. The profiles for $\sigma=0.5$ (left) and the energy against the restframe current (right). The insertion shows that E/n increases with $\mathcal{I}$ faster for $n=1$ than for $n=2$, so that the vortices repel each other for small currents and attract otherwise.

Here we have used Eqs. (2.7), (2.8) and the identity $\partial_\sigma(n^a \mathrm{W}^a_{\mu\alpha}) = (D_\sigma n^a)W^a_{\mu\alpha} + n^a D_\sigma \mathrm{W}^a_{\mu\alpha}$ where $D_\sigma n^a = \partial_\sigma n^a + \epsilon_{abc}\mathrm{W}^b_\sigma n^c$. Integrating the first line of these relations over the vortex cross section gives $\frac{1}{gg'}I_\alpha$. Integral of the second line reduces in the limit $g'\to 0$ to the global current (9.4), since the total derivative term in this line gives no contribution, because the field $\mathrm{W}^a_{\mu\nu}$ becomes short-ranged in this limit.

Let us now notice that the whole expression in (9.5) is the total derivative, so that its integral will not change upon replacing $n^a = (\Phi^\dagger\tau^a\Phi)/(\Phi^\dagger\Phi)$ by any other vector with the same value at infinity. Replacing n^a by $\tilde{n}^a = \delta^a_3$ one will have $D_\mu\tilde{n}^a = \epsilon_{ab3}\mathrm{W}^b_\mu$ which scales as g^2 for small g, so that the last term in the third line vanishes when $g\to 0$. The second term gives upon integration the global current (9.2) (up to the sign), while the first term gives zero, since the field B_μ becomes massive in the limit. We therefore conclude that

$$I^{\mathrm{U}(1)}_\alpha \leftarrow \frac{1}{gg'}J_\alpha \to -I^{\mathrm{SU}(2)}_\alpha \tag{9.6}$$

for $g'\to 0$ and $g\to 0$, respectively, so that the global currents can indeed be obtained from the local electromagnetic current via the limiting procedure.

9.3. Special chiral solutions

Witten's superconducting strings reduce to the 'bare' ANO vortex when $q\to 0$ and to the chiral 'dressed' solutions for $\sigma^2\to 0$ (see Appendix C). Superconducting vortices in the Weinberg–Salam theory also reduce to the 'bare' ANO vortex for $q\to 0$ (Z string), but the limit $\sigma^2\to 0$ corresponds to infinite and not zero restframe current. Therefore, one does not generically find the chiral solutions, although they can be obtained for special parameter values. As was discussed in Section 6, for $q\ll 1$ they exist if $\beta, \theta_{\mathrm{W}}, n, \nu$ belong to the chiral curves shown in Fig. 4. These curves are obtained by solving the linear eigenvalue problem (6.3). If q is not small, then one should solve the full system (3.16)–(3.24) under the condition

$$\sigma^2(\beta, \theta_{\mathrm{W}}, n, \nu, q) = 0. \tag{9.7}$$

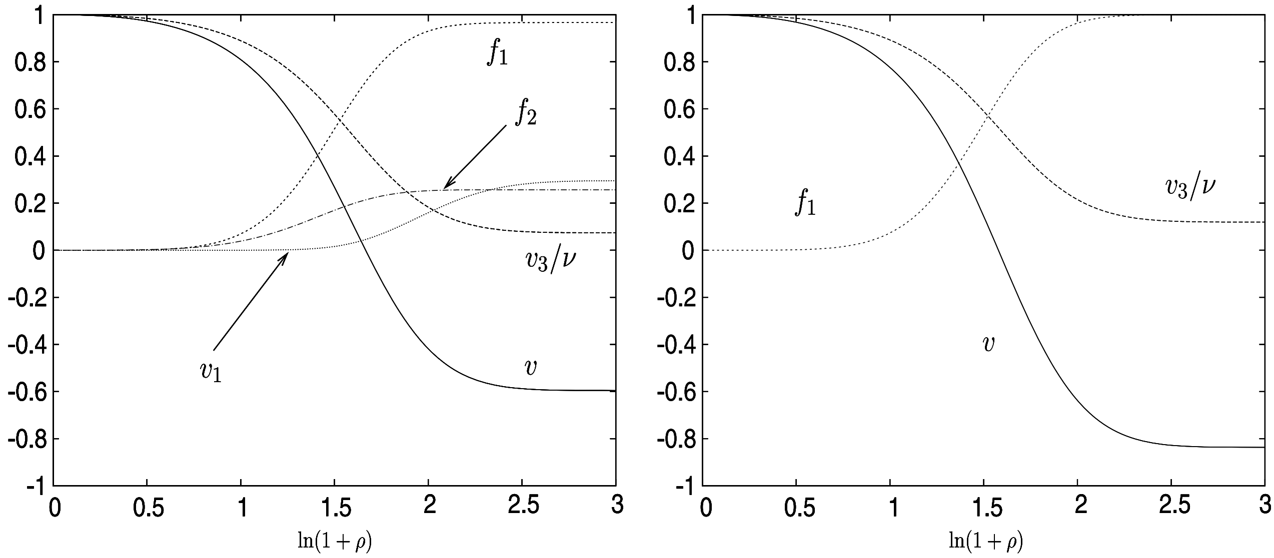

Fig. 18. Profiles of the special chiral solution (left) and Z string (right) for $n = 4$, $\nu = 7$, $\beta = 2$, $g'^2 = 0.23$.

This condition reduces by one the number of the free parameters in the local solutions (4.8), (4.14), so that in order to perform the matching one has to consider θ_W as one of the shooting parameters. For given n, ν, q this determines curves in the (β, θ_W) plane. For $q \to 0$ these curves reduce to those shown in Fig. 4, while for $q \neq 0$ they look qualitatively similar but shift *upwards*. As a result, given a point β, θ_W in an upper vicinity of a curve in Fig. 4 one can adjust q such that the shifted curve will pass through this point. This fine tuning determines the values of β, θ_W, q for which there is a solution of Eqs. (3.16)–(3.23) with $\sigma^2 = 0$. The chiral solutions are therefore not generic but exist only for values of β, θ_W which are close to the chiral curves in Fig. 4.

Reconstructing the fields (4.4), a given solution with $\sigma^2 = \sigma_3^2 - \sigma_0^2 = 0$ determines a family of chiral vortices labeled by $\sigma_0 = \pm\sigma_3$, different members of this family being related by Lorentz boosts. These vortices carry a chiral current and support a long-range field so that their energy is infinite. However, there is a distinguished solution with $\sigma_0 = \sigma_3 = 0$ for which the current is zero and the energy is finite. This special chiral solution is not a Z string or its gauge copy, since it has $\Psi_A \neq 0$ (see Fig. 18).

Such solutions remind of the 'W-dressed Z strings' discussed some time ago [10,11]. By analogy with the 'dressed' vortices in the Witten model, these were supposed to be Z strings stabilized by the W-condensate in the core. However, the numerical search for them has given negative result [12]. It seems therefore that Z strings cannot be stabilized by the condensate, probably because they are non-topological and can unwind into vacuum [13].

The special chiral solutions cannot be considered as the 'dressed' Z strings, first because they do not exist for generic values of the parameters, and secondly because they are not always less energetic than the Z string with the same β, θ_W, n. Specifically, for $n = 1$ they indeed have lower energy than the corresponding Z strings, but their parameters β, θ_W lie in this case in the non-physical region (see Fig. 4), while if they were the true 'W-dressed Z strings' they would exist for any parameter values. The special chiral solutions can also be found in the physical region (see Fig. 18), but only for higher values of n, ν starting from $n = 4$, $\nu = 7$ (see Fig. 4). In this case they turn out to be more energetic than the corresponding Z strings.

Although the special chiral solutions are not 'dressed' Z strings, one can relate them to Z strings very much in the same way as one interpolates between the 'bare' and 'dressed' vortices in the Witten model. Given a special chiral solution one can iteratively decrease q relaxing

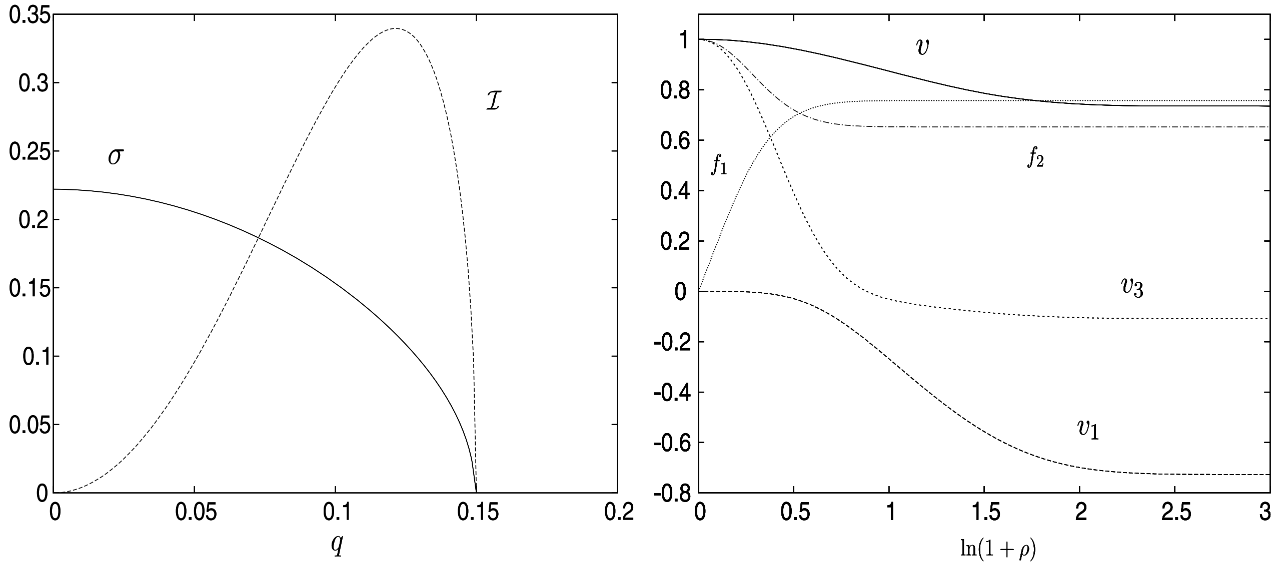

Fig. 19. Left: the current $\mathcal{I}$ and twist σ against q for the family of the $n=2$, $\nu=3$ electroweak vortices for $\beta=2$ and $\sin^2\theta_W=0.56$ that interpolates between the Z string for $q=0$ and the special chiral solution for $q=q_\star=0.15$. Right: the $n=\nu=1$, $\beta=\infty$ solution for $\sigma=0.3$. The electric amplitudes are not shown.

at the same time the $\sigma^2=0$ condition. This gives solutions with $\sigma^2>0$ which in the $q\to 0$ limit reduce to the Z string. The current $\mathcal{I}(q)$ then shows (see Fig. 19) the quenched behaviour very similar for that for the Witten strings (see Fig. 22), but it should be stressed again that this type of behaviour is not generic in the electroweak theory, since it requires the fine-tuning of θ_W.

9.4. *Infinite Higgs mass limit, $\beta\to\infty$*

In this limit the constraint is enforced, $\Phi^\dagger\Phi=f_1^2+f_2^2=1$, so that $f_1=\cos\Theta(\rho)$, $f_2=\sin\Theta(\rho)$. The equations for u, u_1, u_3, v, v_1, v_3 are given by (3.16), (3.17), (3.20)–(3.23), while the equation for Θ reads

$$\frac{1}{\rho}\left(\rho\Theta'\right)'=\frac{\sigma^2}{2}(u_1\cos 2\Theta-u_3\sin 2\Theta)u+\frac{1}{2\rho^2}(v_1\cos 2\Theta-v_3\sin 2\Theta)v. \tag{9.8}$$

The boundary conditions for Θ are

$$\frac{\pi}{2}+a_5\rho^n+\cdots\leftarrow\Theta\to\frac{\gamma}{2}+\frac{c_5}{\sqrt{\rho}}e^{-\int m_\sigma\,d\rho}+\cdots \tag{9.9}$$

so that one always has $q=f_2(0)=1$. One cannot therefore use q to label the solutions, but one can use σ. It follows that for $\nu\neq n$ the energy density diverges at the origin – due to the term $(v-v_3)^2f_2^2/\rho^2$ in (3.30). Therefore, solutions with $\nu\neq n$ become singular for $\beta\to\infty$. The solutions for $n=\nu$ look qualitatively similar to those with finite β (see Fig. 19). As usual, the limit where σ is small corresponds to large currents. However, since $\Phi^\dagger\Phi=1$, the solutions no longer show a region of vanishing Higgs field. This can be seen already from Eq. (8.16), since the size of the $\Phi=0$ region scales as $\rho_\star\sim\mathcal{I}/\sqrt{\beta}$ and shrinks to zero when $\beta\to\infty$.

It seems that for $\beta=\infty$ there is no upper bound for σ. Large σ's correspond to small currents, the solutions then approaching Z strings. The vacuum angle γ then tends to zero, while the mass term (4.15) approaches infinity (since $\sigma Q\sim\mathcal{I}\to 0$ and $m_\sigma\approx\sigma$) in which case Θ changes very fast at small ρ to approach its asymptotic value. For $\sigma=\infty$ one should set $\Theta(\rho)=0$, and then the solutions reduce to Z strings 'in the London limit'.

10. Summary and concluding remarks

In summary, we have presented new solutions in the bosonic sector of the Weinberg–Salam theory which describe straight vortices (strings) carrying a constant electric current. Such vortices contain a regular central core filled with the condensate of massive W bosons creating the electric current. The current produces the long-range electromagnetic field. The solutions exist for any value of the Higgs boson mass and for (almost) any weak mixing angle, in particular for the physical values $\beta \in (1.5, 3.5)$ and $\sin^2\theta_W = 0.23$. They comprise a family that can be labeled by the four parameters in the ansatz (4.4): σ_α, n, ν. These parameters determine the vortex electric charge density $I_0 \sim \sigma_0$, the electric current $I_3 \sim \sigma_3$, the electromagnetic and Z fluxes Ψ_F and Ψ_Z, as well as the vortex momentum $P \sim \sigma_0\sigma_3$ and angular momentum $M \sim \sigma_0$.

The spacetime vector $I_\alpha = (I_0, I_3) \sim \sigma_\alpha$ is generically spacelike, so that its temporal component can be boosted away to give $I_\alpha = \delta^3_\alpha \mathcal{I}$ and $\sigma_\alpha = \delta^3_\alpha \sigma$. The restframe current $\mathcal{I}$ can assume any value. In the $\mathcal{I} \to 0$ limit the solutions reduce to Z strings, while the twist σ reduces to the eigenvalue of the linear fluctuation operator on the Z string background.

For large currents the solutions show a symmetric phase region of size $\sim \mathcal{I}$ where the magnetic field is so strong that it drives the Higgs field to zero. However, this does not destroy the vector boson superconductivity, since the scalar Higgs field is not the relevant order parameter in this case. The current-carrying W-condensate is confined in the very centre of the symmetric phase, in a core of size $\sim 1/\mathcal{I}$, while the rest of this region is dominated by the massless electromagnetic and Z fields. The symmetric phase is surrounded by the 'crust' layer where the Higgs field relaxes to the broken phase while the Z field becomes massive and dies away. Outside the crust there remains only the long-range Biot–Savart magnetic field, which produces a mild, logarithmic energy divergence at large distances for their core. However, finite vortex pieces, as for example vortex loops, will have finite energy.

Straight, infinite vortices can have finite energy in special cases: either for $\theta_W = 0, \pi/2$, when the massless fields decouple, or for $\sigma_\alpha = 0$, when the current vanishes. The latter case includes Z strings and also the currentless limit of the chiral solutions with $\sigma_3 = \pm\sigma_0$. However, the chiral solutions are not generic and exist only for special values of θ_W.

The W boson condensate producing the vortex current can be visualized as a superposition of two oppositely charged fluids made of W^+ and W^-, respectively, flowing in the opposite directions. In the vortex restframe the densities of both fluids are the same, which is why the total momentum through the vortex cross section vanishes while the current does not. Passing to a different Lorentz frame the fluid densities will no longer be equal, which will produce a net momentum. The vortex angular momentum can be explained in a similar way if one assumes that the fluids perform the spiral motions in the opposite directions.

Stability of the new solutions is a very important issue. At first one may think that they should be unstable, since in the zero current limit they reduce to unstable Z strings. However, it seems that current can stabilize them. So far this has been checked only in the $\theta_W = \pi/2$ limit, where the complete stability analysis has been carried out [22]. It turns out then that all negative modes of current-carrying vortices can be removed by imposing the periodic boundary conditions, while one cannot do the same in the zero current limit. It seems that the same conclusion applies also for $\theta_W \neq \pi/2$, although the detailed verification of this is still in progress. However, the main argument is simple enough and goes as follows.

As was mentioned above, the Z string perturbation eigenmodes are proportional to $e^{\pm i(\sigma_0 t + \sigma_3 z)}$ (see Eq. (6.2)) with $\sigma_0 = \sqrt{\sigma_3^2 - \sigma^2}$ where $\sigma^2 > 0$ is the eigenvalue of the spectral problem (6.3). It follows that all modes with $\sigma_3 < \sigma$ are unstable, since σ_0 is imaginary. Since

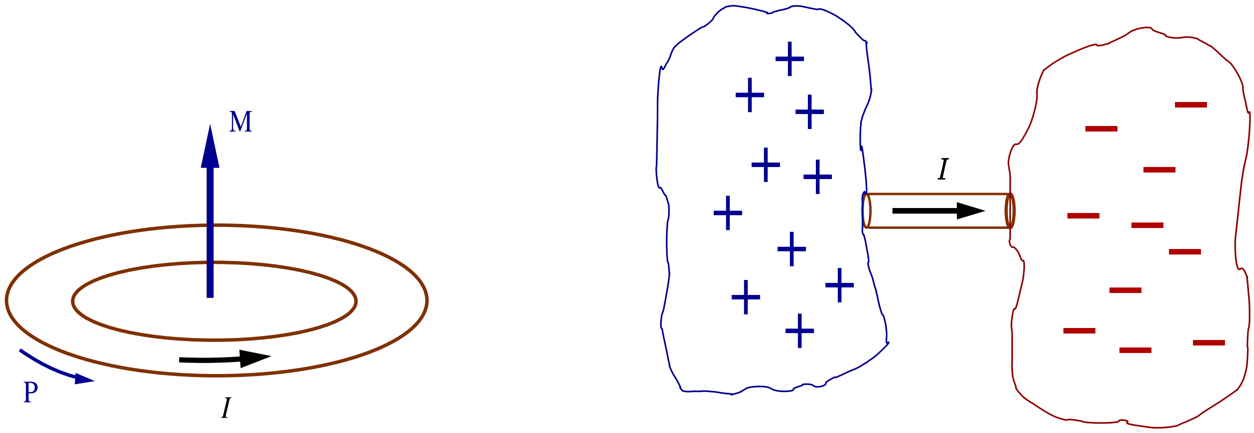

Fig. 20. Vortex loops balanced by the centrifugal force (left) and vortex segments in a polarized medium (right).

$$\lambda = \frac{2\pi}{\sigma_3} > \lambda_{\text{min}} = \frac{2\pi}{\sigma}, \tag{10.1}$$

it follows that all unstable modes are longer than λ_{min}. This suggests that one could eliminate them by imposing z-periodic boundary conditions with a period $L \leqslant \lambda_{\text{min}}$, which would leave 'no room' for such modes to exist. However, this would not remove one particular *homogeneous* mode with $\sigma_3 = 0$, since it is independent of z and so can be considered as periodic with any period.

Now, when passing to current-carrying solutions the situation will be essentially the same. It is clear on continuity grounds that, at least for small currents, the solutions will still have negative modes with the wavelength bounded from below by a non-zero value λ_{min}. However, the crucial point is that the homogeneous mode will disappear from the spectrum – simply because the background solutions will now have a non-trivial z-dependence, implying that all perturbation modes will depend on z as well. This was checked in [22] for $\theta_W = \pi/2$ and it is very plausible that for generic θ_W the situation will be the same.

Since the homogeneous mode is absent, imposing periodic boundary conditions will remove all negative modes (a more detailed consideration shows that, at least in the semilocal limit, the period should be $L = \pi/\sigma$ [22]). This is somewhat similar to the hydrodynamical Plateau–Rayleigh instability of a water column [37] or to the gravitational Gregory–Laflamme instability of black strings in the theory of gravity in higher dimensions [38], which manifest themselves only starting from a certain minimal length. Since σ decreases with current, L then *grows*. For large currents one has $\sigma \sim \mathcal{I}^{-3}$ (see the caption to Fig. 7) so that the length of the stable vortex segment scales as $\mathcal{I}^3$ while its thickness grows as $\mathcal{I}$ (see Eq. (8.16)).

There could be different ways of imposing periodic boundary conditions on the vortex segment. One possibility is to bend it and identify the extremities to make a loop. If the vortex carries a momentum P then the loop will have an angular momentum M which may balance it against contraction (see Fig. 20). One may therefore conjecture that such electroweak analogues of the 'cosmic vortons' [39] could exist and could perhaps be stable – if they are made of stable vortex segments. Of course, verification of this conjecture requires serious efforts, since so far vortons have been constructed explicitly only in the global limit of the Witten model [40]. However, even a remote possibility to have stable solitons in the Standard Model could be very important, since if electroweak vortons exist, they could be a dark matter candidate.

Another possibility to impose periodic boundary conditions is to attach the vortex ends to something. It is known that Z strings, since they are not topological, do not have to be al-

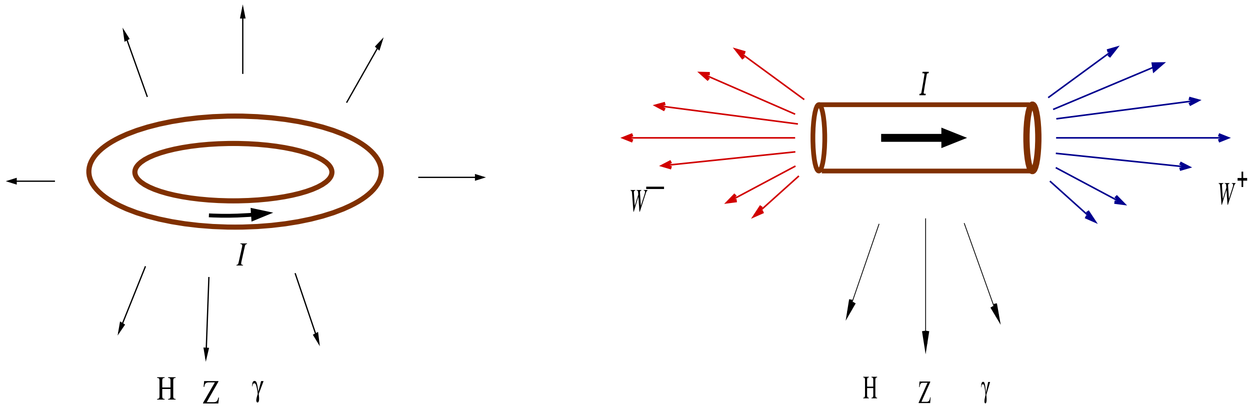

Fig. 21. Disintegration of an unstable vortex loop (left) and of an isolated vortex segment (right).

ways infinitely long but can exist in the form of finite segments whose extremities look like a monopole–antimonopole pair [16,17]. This suggests that their current-carrying generalizations could perhaps also exist in the form of finite segments joining oppositely polarized regions of space, similar to thunderbolts between clouds. If some processes during the electroweak phase transition polarize the medium, then the 'vortex thunderbolts' could provide a very efficient discharge mechanism – in view of the very large currents they can carry. They could be stable in this case and exist as long as the 'clouds' they join are not completely discharged.

There is however an important difference with the ordinary atmospheric thunderbolts, which exist only in a polarized medium and emerge when the large electric field between the clouds creates the plasma channel and works against the resistance to drive the charges through. In superconducting vortices on the other hand the current flows without any resistance and no electric field is needed. In fact, inside the (restframe) vortex the electric field is zero. The vortices do not therefore need any polarization of the surrounding medium to exist.

Finite segments of superconducting vortices can probably be created at high temperatures or in high energy particle collisions. Once created, the current will start to leave the segment through its extremities, so that the latter will emit positive and negative charges (see Fig. 21). Since the current is carried by W bosons, it follows that one vortex end will emit the W^+ while the other the W^-, until the segment radiates away all its energy. In addition, if the initial current is large enough, then the vortex will contain a zero Higgs field region, whose shrinking will be accompanied by a Higgs boson emission. The vortex segment will therefore end up in a blast of radiation, creating two jets of W^+ and W^- and a shower of Higgs bosons. One can similarly argue that if the vortex segment is created in the form of an unstable loop (for example without angular momentum) then it will shrink emitting a shower of neutral bosons – Higgs, Z and photons.

The creation of superconducting vortex segments or loops with their subsequent disintegration can presumably be observed in the LHC experiments. It is also possible that processes of this type could be accompanied by a fermion number non-conservation, since already in the zero current limit the vortices (Z strings) admit the sphaleron interpretation [13]. However, a special analysis is needed to work out details of these processes and their experimental signatures. We leave this and other possible applications of the superconducting vortices for a separate study.

Acknowledgements

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Appendix A. Local solutions near the symmetry axis

The boundary conditions at the origin (4.7) imply that at small ρ one has

$$u = u(0) + \delta u, \qquad u_3 = 1 + \delta u_3, \qquad u_1 = \delta u_1,$$
$$v = 2n - \nu + \delta v, \qquad v_3 = \nu + \delta v_3, \qquad v_1 = \delta v_1,$$
$$f_1 = \delta f_1, \qquad f_2 = f_2(0) + \delta f_2, \tag{A.1}$$

where the deviations $\delta u, \ldots, \delta f_2$ vanish for $\rho \to 0$ and $u(0)$ is a free parameter, while $f_2(0)$ is also free if $\nu = n$ but for $\nu \neq n$ one has $f_2(0) = 0$. Inserting this to Eqs. (3.16)–(3.24) and linearising gives eight second order and one first order linear equations for the deviations. If $f_2(0) = 0$ then most of them decouple one from another and can be easily solved. For $f_2(0) \neq 0$ the equations split into several subsystems which can be solved in series. The most general solution contains 15 integration constants, but 9 out of the 15 independent solutions do not vanish for $\rho \to 0$ and should therefore be excluded from the analysis. As a result, the most general solution that vanishes at the origin is expressed by Eqs. (4.8) in the main text, it contains 6 integration constants.

These local solutions are obtained in the linear approximation. However, they can be used to take into account all non-linear terms in the equations when starting the numerical integration at $\rho = 0$. This is achieved by rewriting the second order field equations (3.16)–(3.23) in the first order form, $y_r'(\rho) = \mathcal{F}_r(\rho, y_s)$, with $r, s = 1, \ldots, 16$, where y_r are chosen such that $y_r'(0) = 0$ by virtue of the above linear analysis. The functions $\mathcal{F}_r(\rho, y_s)$ will be defined for $\rho > 0$ but not in general at $\rho = 0$. For example, choosing $y_1 = u$, $y_2 = \rho u'$ gives $y_1' = y_2/\rho \equiv \mathcal{F}_1$ which is undefined at $\rho = 0$. However, since we know that $y_r'(0) = 0$, we can set $\mathcal{F}_r(\rho = 0, y_s) = 0$ *by definition*, and then we shall be able to integrate the equations starting *exactly* at $\rho = 0$, thereby avoiding any approximations. Such a procedure is sometimes called desingularization of the equations at a singular point.

Appendix B. Solutions in the asymptotic region

Solutions of Eqs. (3.16)–(3.23) have to approach the Biot–Savart field (4.10) far away from the vortex core. Therefore,

$$u = \underline{u} + \delta u, \qquad u_3 = -\underline{u} + \delta u_3, \qquad u_1 = \delta u_1,$$
$$v = c_2 + \delta v, \qquad v_3 = -c_2 + \delta v_3, \qquad v_1 = \delta v_1,$$
$$f_1 = 1 + \delta f_1, \qquad f_2 = \delta f_2, \tag{B.1}$$

where $\underline{u} = Q \ln \rho + c_1$ and the deviations $\delta u, \ldots, \delta f_2$ tend to zero as $\rho \to \infty$. Here c_1, c_2, Q are the same integration constants as in (4.10). Inserting (B.1) to (3.16)–(3.23) and linearising with respect to the deviations, the resulting linear system reduces to five decoupled second order equations for $\delta u_Z = \delta u + \delta u_3$, $\delta v_Z = \delta v + \delta v_3$, $\delta u_A = g^2 \delta u - g'^2 \delta u_3$, $\delta v_A = g^2 \delta v - g'^2 \delta v_3$ and δf_1 plus a system of three coupled second order equations. The five decoupled equations describe the photon and Z and Higgs bosons and can be easily solved. Their most general solution that vanishes at infinity is

$$\delta u_Z = \frac{c_3}{\sqrt{\rho}} e^{-m_Z \rho} + \cdots, \qquad \delta v_Z = c_4 \sqrt{\rho}\, e^{-m_Z \rho} + \cdots,$$
$$\delta f_1 = \frac{c_5}{\sqrt{\rho}} e^{-m_H \rho} + \cdots, \tag{B.2}$$

while the equations for $\delta u_A, \delta v_A$ do not admit solutions that vanish at infinity and so one sets $\delta u_A = \delta v_A = 0$. In fact, the electromagnetic degrees of freedom are already incorporated into the background amplitude $\underline{u}$.

The remaining three equations are obtained by linearising Eqs. (3.19), (3.20), (3.22) with respect to $\delta u_1, \delta v_1, \delta f_2$ and describe charged W bosons interacting with the Biot–Savart field. It is convenient to replace one of them by the first integral (3.27), which gives an equivalent system. Linearising and defining $\delta f_2 = \sigma^2 \underline{u} X + \frac{1}{2} Y$, $\delta u_1 = g^2 X - \underline{u} Y$ the resulting equations read

$$X'' + \left(\frac{1}{\rho} + \frac{\chi'}{\chi}\right) X' - \left(\frac{c_2^2}{\rho^2} + \chi\right) X = \frac{Q}{\rho \chi} Y', \tag{B.3a}$$

$$(\delta v_1)'' - \frac{1}{\rho}(\delta v_1)' - \chi \delta v_1 = c_2 \chi Y, \tag{B.3b}$$

$$\frac{c_2}{\rho^2}(\delta v_1)' - \chi Y' - \frac{2g^2\sigma^2 Q}{\rho} X = \frac{C}{\rho}, \tag{B.3c}$$

where C is the same integration constant as in (3.27) and $\chi = m_\sigma^2 = \frac{g^2}{2} + \sigma^2 \underline{u}^2$. These three equations are equivalent to one fourth order equation

$$Z'''' - 2\left(\mu^2 Z'\right)' + \left(\mu^4 - 3\left(\mu^2\right)'' - \frac{6}{\rho}\left(\mu^2\right)' + \frac{4(c_2^2 - 1)}{\rho^4}\right) Z = C \frac{c_2(5 + 4\rho^2\mu^2)}{4\rho^{5/2}} \tag{B.4}$$

provided that $\delta v_1 = \int \frac{d\rho}{\sqrt{\rho}} Z$ and $\mu^2 = \chi + \frac{4c_2^2 - 5}{4\rho^2}$. Let us first consider the solutions with $C = 0$. The asymptotic form of the two independent solutions that vanish at infinity is

$$Z = \left(A_1 \mu^{-5/2}(1 + \cdots) + A_2 \rho \sqrt{\mu}\left(1 + \frac{1}{2\rho\mu} + \cdots\right)\right) \exp\left(-\int \mu \, d\rho\right) \tag{B.5}$$

which determines

$$\delta f_2 = \frac{c_6}{\sqrt{\rho}} e^{-\int \mu \, d\rho} + \cdots, \qquad \delta u_1 = \frac{c_7}{\sqrt{\rho}} e^{-\int \mu \, d\rho} + \cdots,$$
$$\delta v_1 = c_8 \sqrt{\rho} e^{-\int \mu \, d\rho} + \cdots, \tag{B.6}$$

where c_6, c_7, c_8 are expressed in terms of the two integration constants A_1, A_2. Eqs. (B.3) should have fore more solutions, since they contain five derivatives and one integration constant, C. Two of them are obtained by choosing the plus sign in the exponent in Eq. (B.5), but they are unbounded and should be excluded. Yet one more solution is given by $X = 0$, $\delta v_1 = -c_2 Y =$ const, but it does not vanish at infinity and should be excluded as well.

The last solution is obtained by setting $C \neq 0$, which leads to $\delta v_1 \sim C \ln\ln\rho$. This solution is unbounded but it will be eventually removed by the $C = 0$ constraint (3.24), which is imposed at small ρ but propagates everywhere when constructing the global solutions. We should therefore keep this solution to increase the number of independent local solutions. However, since replacing it by any other field mode would have the same effect, we omit this solution but treat all three coefficients c_6, c_7, c_8 in (B.6) as *independent* parameters. This gives a good numerical convergence. In addition, without changing the leading asymptotic terms, one can replace $\int \mu \, d\rho$ in Eq. (B.6) by $\int m_\sigma \, d\rho$.

These considerations lead finally to Eqs. (4.14) in the main text.

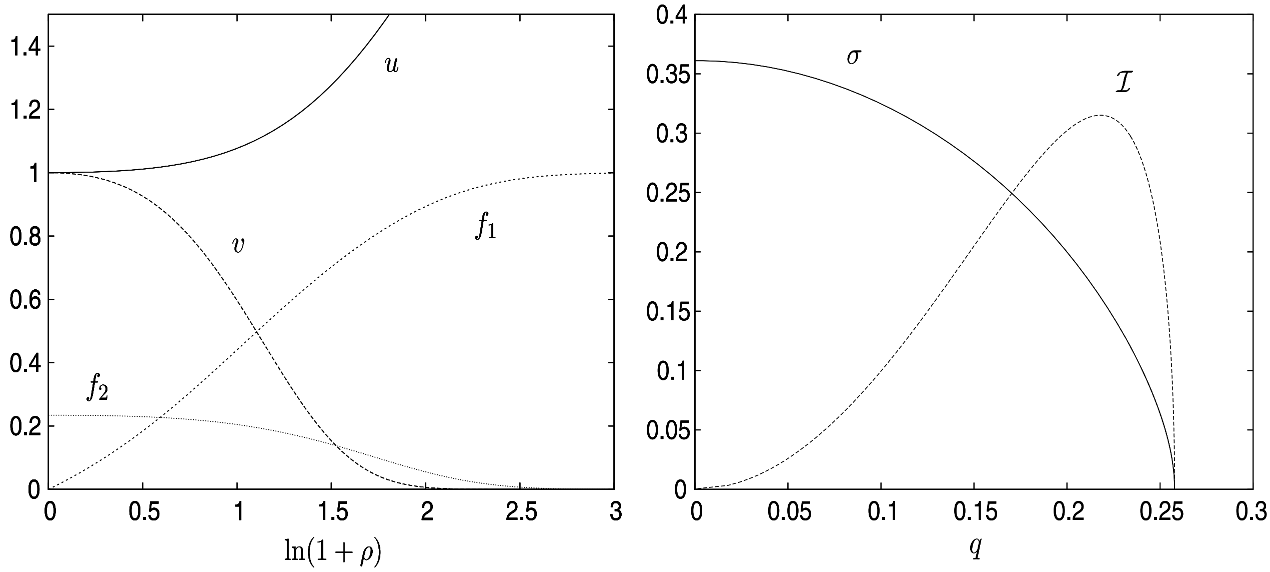

Fig. 22. Left: profiles of the Witten string for $\lambda_1 = 0.1$, $\lambda_2 = 10$, $\eta_1 = 1$, $\eta_2 = 0.31$, $g_1 = g_2 = 1$, $\gamma = 0.6$ and $\sigma = 0.12$. Right: the restframe current $\mathcal{I}$ and the twist σ against the parameter q for the same parameter values as in the left panel.

Appendix C. Superconducting strings in the Witten model

The Witten model is defined by the Lagrangian [1]

$$\mathcal{L}_W = -\frac{1}{4}\sum_{a=1,2} F^{(a)}_{\mu\nu} F^{(a)\mu\nu} + \sum_{a=1,2} (D_\mu \phi_a)^* D^\mu \phi_a - \mathrm{U}, \tag{C.1}$$

with $F^{(a)}_{\mu\nu} = \partial_\mu A^{(a)}_\nu - \partial_\nu A^{(a)}_\mu$ and $D_\mu \phi_a = (\partial_\mu - i g_a A^{(a)}_\mu)\phi_a$ and with

$$\mathrm{U} = \frac{\lambda_1}{4}\big(|\phi_1|^2 - \eta_1^2\big)^2 + \frac{\lambda_2}{4}\big(|\phi_2|^2 - \eta_2^2\big)^2 + \gamma |\phi_1|^2 |\phi_2|^2 - \frac{\lambda_2}{4}\eta_2^4.$$

The spectrum contains two scalars with masses $m_1^2 = \lambda_1 \eta_1^2$ and $m_2^2 = \gamma \eta_1^2 - \frac{1}{2}\lambda_2 \eta_2^2$ as well as a vector of mass $m_v^2 = 2 g_1^2 \eta_1^2$ and a massless vector. One chooses the fields in the form

$$g_1 A^{(1)}_\mu \, dx^\mu = \big(n - v(\rho)\big)\, d\varphi, \qquad \phi_1 = \eta_1 f_1(\rho) e^{in\varphi}, \tag{C.2a}$$

$$g_2 A^{(2)}_\mu \, dx^\mu = \big(\sigma_0 \, dx^0 + \sigma_3 \, dx^3\big)\big(1 - u(\rho)\big), \qquad \phi_2 = \eta_2 f_2(\rho) e^{i\sigma_0 x^0 + i\sigma_3 x^3}. \tag{C.2b}$$

They are regular at the origin where $u \to 1$, $v \to n$, $f_1 \to 0$, $f_2 \to q$ while at infinity they approach the Biot–Savart field, $u \to c_1 + Q \ln \rho$, $v \sim e^{-m_v \rho}$, $f_1 - 1 \sim e^{-m_1 \rho}$, $f_2 \sim e^{-\int m_\sigma \, d\rho}$ where the mass of the second scalar is dressed, $m_\sigma^2 = m_2^2 + \sigma^2 (c_1 + Q \ln \rho)^2$ (as in Eq. (7.1)). The current density is

$$J_\nu = \partial^\mu F^{(2)}_{\mu\nu} = 2 g_2 \Re\big(i \phi_2^* D_\nu \phi_2\big), \tag{C.3}$$

and the string current $I_\alpha = \int J_\alpha \, d^2x = -2\pi Q \sigma_\alpha / g_2$. Setting $f_2 = 0$, $u = 1$ gives the currentless ANO solution, while perturbing it gives a linear bound state problem (as in Section 6 above) with the eigenvalue $\sigma^2 > 0$ (for suitably chosen model parameters). The bound states with $\sigma_3 > \sigma$ describe small deformations of the ANO vortex by the current. Finally, increasing iteratively $q = f_2(0)$ gives the generic solutions (see Fig. 22). One finds that the current $\mathcal{I} = \sqrt{I_3^2 - I_0^2}$ first increases with q but then starts to quench, till both $\mathcal{I}$ and σ (but not Q) vanish at $q = q_\star$ as shown in Fig. 22. The $q = q_\star$ solution can be viewed either as the 'dressed' currentless string with $\sigma_\alpha = 0$, in which case it turns out to be less energetic that the initial 'bare' ANO vortex, or as a chiral solution with $\sigma_0 = \pm\sigma_3$.

Summarizing, changing the condensate parameter q, the Witten strings interpolate between the 'bare' ANO vortex for $q = 0$ and the 'dressed'/chiral vortex for $q = q_\star$.

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